



On the regularity of axisymmetric, swirl-free solutions of the Euler equation in four and higher dimensions

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Abstract. In this paper, we consider axisymmetric, swirl-free solutions of the Euler equation in four and higher dimensions. We show that in dimension $d \geq 4$, axisymmetric, swirl-free solutions of the Euler equation have properties which could allow finite-time singularity formation of a form that is excluded when $d = 3$, and we prove a conditional blowup result for axisymmetric, swirl-free solutions of the Euler equation in dimension $d \geq 4$. The condition which must be imposed on the solution in order to imply blowup becomes weaker as $d \rightarrow +\infty$, suggesting the dynamics are becoming much more singular as the dimension increases.

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1. Introduction

The incompressible Euler equation is one of the fundamental equations of fluid dynamics, and is given by

$$\partial_t u + (u \cdot \nabla)u + \nabla p = 0, \quad \nabla \cdot u = 0,$$

where $u \in \mathbb{R}^d$ is the fluid velocity, and p is the pressure, which can be determined from the velocity using the divergence free constraint, yielding

$$-\Delta p = \sum_{i,j=1}^d \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j}.$$

This means that the Euler equation can be expressed in terms of the Helmholtz projection as

$$\partial_t u + P_{\text{df}}((u \cdot \nabla)u) = 0.$$

It is a classical result that for sufficiently smooth initial data, the Euler equation has global smooth solutions in two dimensions. In three dimensions, it is a major open question in nonlinear PDE whether smooth solutions of the Euler equation can blow up in finite-time. One of the main partial regularity results is the Beale–Kato–Majda criterion, which states that the $L_t^1 L_x^\infty$ norm of the vorticity controls regularity [2], and so if a smooth solution of the Euler equation blows up in finite-time $T_{\max} < +\infty$, then

$$\int_0^{T_{\max}} \|\vec{\omega}(\cdot, t)\|_{L^\infty} dt = +\infty.$$

A special case of the Euler equation in three dimensions is that of axisymmetric, swirl-free solutions. Axisymmetric, swirl free vector fields have the form

$$u(x) = u_r(r, z)e_r + u_z(r, z)e_z,$$

where

$$r = \sqrt{x_1^2 + x_2^2}, \quad z = x_3, \quad e_r = \frac{(x_1, x_2, 0)}{r}, \quad e_z = e_3,$$

and this class of vector fields is preserved by the dynamics of the Euler equation. The scalar vorticity for axisymmetric, swirl-free solutions is $\omega = \partial_r u_z - \partial_z u_r$, and satisfies the evolution equation

$$\partial_t \omega + (u \cdot \nabla) \omega - \frac{u_r}{r} \omega = 0.$$

This implies that the quantity ω/r is transported by the flow with

$$(\partial_t + u \cdot \nabla) \frac{\omega}{r} = 0,$$

and this is enough to guarantee global regularity subject to reasonable hypotheses on the initial data. Ladyzhenskaya proved the first global regularity result for axisymmetric, swirl-free fluids in three dimensions, proving global regularity for axisymmetric, swirl-free solutions of the Navier–Stokes equation [17]. Ukhovskii and Yudovich proved global regularity [23] for axisymmetric, swirl-free solutions of the Euler equation in three dimensions with initial vorticity satisfying $\omega^0, \omega^0/r \in L^2 \cap L^\infty$, which was later generalised by Serfati [21] and Saint Raymond [20]. Danchin relaxed this requirement [6, 7] to initial data satisfying $\omega^0 \in L^{3,1} \cap L^\infty$, $\omega^0/r \in L^{3,1}$. We will note that in the standard regularity class for well-posedness of strong solutions to the three-dimensional Euler equation, $H^s(\mathbb{R}^3)$, $s > 5/2$, the conditions by Danchin hold automatically, and so these conditions do not involve any assumptions beyond sufficient regularity of the initial data.

In this paper, we will consider a generalisation of axisymmetric, swirl-free solutions of the Euler equation to four and higher dimensions. For $d \geq 4$, we will consider solutions of the form

$$u(x, t) = u_r(r, z, t)e_r + u_z(r, z, t)e_z,$$

where

$$r = \sqrt{x_1^2 + \cdots + x_{d-1}^2}, \quad z = x_d, \quad e_r = \frac{(x_1, \dots, x_{d-1}, 0)}{r}, \quad e_z = e_d,$$

The scalar vorticity is again given by $\omega = \partial_r u_z - \partial_z u_r$, and now satisfies the evolution equation

$$\partial_t \omega + (u \cdot \nabla) \omega - k \frac{u_r}{r} \omega = 0,$$

where $k = d - 2$. This implies that the quantity ω/r^k is transported by the flow with

$$(\partial_t + u \cdot \nabla) \frac{\omega}{r^k} = 0.$$

Remark 1.1. This immediately opens the possibility of finite-time singularity formation when $d \geq 4$, because the transported quantity ω/r^k is not necessarily bounded when $k \geq 2$. In general, for sufficiently smooth, axisymmetric, swirl-free velocities, the vorticity must vanish linearly at the axis. To be more precise, we will show that if $u \in H^s(\mathbb{R}^d)$, $s > 2 + d/2$, then $\omega/r \in L^\infty$. For $d \geq 4$, by contrast, ω/r^k may be unbounded for even a Schwartz class vector field.

The fact that there is some control on the advected quantity ω/r of the kind imposed by Ukhovskii and Yudovich [23] or Danchin [6, 7] is essential to the guaranteeing global regularity. In fact, for solutions in which the vorticity is only C^ε , for a small $\varepsilon > 0$, Elgindi recently proved finite-time blowup for solutions of the axisymmetric, swirl-free Euler equation in three dimensions [9]. Note that for these solutions, ω/r is singular at the z -axis, $r = 0$, even before the blowup time. When $d \geq 4$, the advected quantity ω/r^k may be singular on the z -axis even for smooth solutions, so this gives us a good reason to be optimistic that there will be examples of finite-time blowup for smooth, axisymmetric, swirl-free solutions of the Euler equation in four and higher dimensions. If the quantity ω/r^k being advected by the flow does not have any reasonable control, then there is no barrier to blowup from axisymmetry, and we can expect blowup for smooth solutions in $d \geq 4$ analogous to the blowup for strong – but not smooth – solutions proven by Elgindi when $d = 3$.

It would seem natural to expect that for $d \geq 4$, in the case where $\omega/r^k \in L^\infty$, there will be global regularity by similar arguments guaranteeing global regularity in $d = 3$, when $\omega^0/r \in L^\infty$. This is the case when $d = 4$, and we have the following result.

Theorem 1.2. *Suppose the initial data $u^0 \in H_{\text{df}}^s(\mathbb{R}^4)$, $s > 4$, is axisymmetric and swirl-free, and assume $\omega^0/r^2 \in L^1 \cap L^\infty$. Then there exists a global smooth solution of the Euler equation $u \in C([0, +\infty), H_{\text{df}}^s(\mathbb{R}^4)) \cap C^1([0, +\infty), H_{\text{df}}^{s-1}(\mathbb{R}^4))$. Furthermore, we have a bound on vorticity, with for all $R > 0$ and for all $0 \leq t < +\infty$,*

$$\|\omega(\cdot, t)\|_{L^\infty} \leq \max\left(\|\omega^0\|_{L^\infty(\mathcal{C}_R^c)}, R^2 \left\| \frac{\omega^0}{r^2} \right\|_{L^\infty(\mathcal{C}_R)}\right) \exp(2\mu t),$$

where

$$\mu = \frac{C}{R} \left\| \frac{\omega}{r^k} \right\|_{L^\infty}^{1/2} \left(\|\omega^0\|_{L^1(\mathcal{C}_R^c)} + R^2 \left\| \frac{\omega^0}{r^2} \right\|_{L^1(\mathcal{C}_R)} \right)^{1/2},$$

with the cylinder $\mathcal{C}_R \in \mathbb{R}^d$ given by

$$\mathcal{C}_R = \{x \in \mathbb{R}^d : |x'| < R\}.$$

Remark 1.3. This paper replaces an earlier version by the first author that did not include this result. In the meantime, Theorem 1.2 was proven independently and first published by Choi, Jeong, and Lim [5]. We include it in this paper nonetheless, because it fits naturally into the analysis of upper and lower bounds on the singular behaviour of axisymmetric solutions the Euler equation in higher dimensions.

Lim and Jeong further refined the analysis in [18], proving global regularity for axisymmetric swirl free solutions of the Euler equation whenever $d \leq 6$ and $\omega^0/r^{d-2} \in L^\infty$. For $3 \leq d \leq 5$, they proved an upper bound on the growth rate of the form

$$\|\omega(\cdot, t)\|_{L^\infty} \leq C(1+t)^{4/(6-d)},$$

and for $d = 6$, the bound

$$\|\omega(\cdot, t)\|_{L^\infty} \leq Ce^{\kappa t},$$

in all cases under the assumption that $\omega^0/r^{d-2} \in L^\infty$. These results supersede Theorem 1.4 below for $d = 5, 6$, but the regularity criterion in Theorem 1.4 remains relevant when $d \geq 7$, in which case the finite-time blowup problem is open.

Theorem 1.4. Suppose that $u \in C([0, T_{\max}), H_{\text{df}}^s(\mathbb{R}^d)) \cap C^1([0, T_{\max}), H_{\text{df}}^{s-1}(\mathbb{R}^d))$, with $d \geq 5$ and $s > 2 + d/2$, is an axisymmetric, swirl-free solution of the Euler equation, with finite-time blowup at $T_{\max} < +\infty$, and assume that $\omega/r^k \in L^1 \cap L^\infty$. Then for all $0 \leq t < T_{\max}$,

$$\|\omega(\cdot, t)\|_{L^1(\mathbb{R}^d)} \geq \left(\frac{M_d}{\|\omega/r^k\|_{L^\infty}^{(d-2)/(d-4)} \|\omega/r^k\|_{L^1}^{2/(d-4)}} \right) \frac{1}{(T_{\max} - t)^{2\frac{d-2}{d-4}}},$$

where $M_d > 0$ is a constant depending only on the dimension. Furthermore,

$$\int_0^{T_{\max}} \|u_r^+(\cdot, t)\|_{L^\infty} dt = +\infty,$$

where $u_r^+ = \max(u_r, 0)$.

We will note that in terms of scaling these regularity criteria are substantially stronger than the Beale–Kato–Majda criterion, and so any blowup of this kind must happen in the bulk of the flow, at least to a larger degree than Beale–Kato–Majda on its own would require. These regularity criteria follow from the fact that when $\omega/r^k \in L^\infty$, there can only be finite-time blowup at some time $T_{\max} < +\infty$ if a fluid trajectory runs off to spatial infinity with $r(t) \rightarrow \infty$ as $t \rightarrow T_{\max}$.

As further evidence of potential singularity formation in very high dimensions, we prove that any solution of the Euler equation in $\mathbb{R}^d$ with a particular geometry – that is, axisymmetric, swirl-free, with a scalar vorticity $\omega^0(r, z)$ that is odd in z and satisfying $\omega^0(r, z) \geq 0$ for all $r, z > 0$ – blows up in finite time as long as a large enough proportion of energy remains in the vertical velocity component. Furthermore, the proportion of energy remaining in the vertical component necessary to guarantee finite-time blowup shrinks at a rate $1/d$, making the requirement very weak in high dimensions.

Definition 1.5. For an axisymmetric, swirl-free solution of the incompressible Euler equation $u \in C([0, T_{\max}); H_*^s(\mathbb{R}^d)) \cap C^1([0, T_{\max}), H_*^{s-1}(\mathbb{R}^d))$, with $d \geq 3$ and $s > 2 + d/2$, for all $0 \leq t < T_{\max}$, let

$$K_r(t) = \int_0^\infty \int_0^\infty r^{d-2} u_r(r, z, t)^2 dr dz, \quad K_z(t) = \int_0^\infty \int_0^\infty r^{d-2} u_z(r, z, t)^2 dr dz,$$

and let the total energy of the system be given by

$$K_0 = K_r(0) + K_z(0).$$

Note that for all $0 \leq t < T_{\max}$,

$$K_r(t) + K_z(t) = \frac{1}{m_{d-2}} \|u(\cdot, t)\|_{L^2}^2 = \frac{1}{m_{d-2}} \|u^0\|_{L^2}^2 = K_0,$$

due to conservation of energy, so the radial and vertical components of energy always add to the initial total energy, although energy can be exchanged between these two components.

Theorem 1.6. Suppose that $u \in C([0, T_{\max}); H_*^s(\mathbb{R}^d)) \cap C^1([0, T_{\max}), H_*^{s-1}(\mathbb{R}^d))$, with $d \geq 4$ and $s > 2 + d/2$, is a solution of the Euler equation, and assume that $\omega^0(r, z)$ is odd in z , with, for all $r, z > 0$,

$$\omega^0(r, z) \geq 0,$$

and ω^0 is not identically zero. Further suppose that there exists $\varepsilon > 0$ such that for all $0 \leq t < T_{\max}$,

$$K_z(t) \geq \frac{1 + \varepsilon}{d} K_0.$$

Then this solution of the Euler equation blows up in finite-time with

$$T_{\max} \leq \frac{1}{\varepsilon K_0} \int_0^\infty \int_0^\infty r^{d-1} z \omega^0(r, z) dr dz.$$

Remark 1.7. Note that when the dimension is very large, the amount of energy that needs to remain in the vertical component in order to guarantee blowup becomes extremely small, which suggests increasingly singular behaviour in very high dimensions. This is consistent with the work of Drivas and Elgindi [8], which established, in a very different geometry, finite-time blowup in the infinite-dimensional formal limit $d \rightarrow +\infty$ for the Euler equation on the torus $\mathbb{T}^d$. The relationship between the finite-time blowup problem for the axisymmetric Euler equation and the dimension was also studied numerically by Hou and Zhang [12] and, in the viscous case, by Hou [11].

The paper will be structured as follows. In Section 2, we will go over some key definitions as well as the notation used in the paper. In Section 3, we will show that the class of axisymmetric, swirl-free vector fields is preserved in four and higher dimensions by the dynamics of the Euler equation and derive the evolution equation for the vorticity. In Section 4, we will give a Biot–Savart law for recovering the velocity from the vorticity. In Section 5, we will discuss the regularity theory in four and higher dimensions when ω/r^k

is bounded, establishing regularity criteria and proving Theorems 1.2 and 1.4. In Section 6, we will consider the case of a vorticity with certain symmetry and sign conditions that corresponds to colliding vortex tubes, and prove some preliminary results that suggest the possibility of growth in very large dimensions, proving Theorem 1.6. In Appendix A, we will express the Biot–Savart law in terms of a stream function.

Remark 1.8. It must be stated at the outset that the Cauchy problem for the Euler equation in four and higher dimensions is completely unphysical. Real, physical fluid flows are three-dimensional. In some cases, the flow of actual physical fluids is close enough to being perfectly two-dimensional for the two-dimensional equation to be a reasonable model, but it is almost inconceivable that the Euler equation in four or higher dimensions could be a reasonable model for any physical fluids. Even in the case of relativistic fluids, where time is not neatly separable from space, the Lorentzian structure means the problem is still fundamentally $(3 + 1)$ -dimensional, as it is in the classical case if time is treated as a dimension.

Nonetheless, the question of finite-time blowup of the Euler equation in four and higher dimensions is scientifically interesting beyond purely abstract mathematical curiosity. The dramatic qualitative differences between two and three-dimensional fluid mechanics show that turbulence has a very fundamental dependence on dimension. For this reason, any advance on the finite-time blowup of the Euler (or Navier–Stokes) equation in four and higher dimensions – which seems much more within reach than in the three-dimensional case – could shed significant light on the possibility of blowup in three dimensions, by allowing the study of the dependence of mechanisms for finite-time blowup on the dimension.

2. Definitions and notation

Throughout this paper,

$$3 \leq d = \text{the spatial dimension,} \quad \text{and} \quad k = d - 2.$$

For axisymmetric solutions, we shall use the following notation for the decomposition $\mathbb{R}^d = \mathbb{R}^{d-1} \times \mathbb{R}$:

$$\mathbb{R}^d \ni x = (x', x_d), \quad x' = (x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1}.$$

We can define the generalised cylindrical coordinates (r, z) for this decomposition by

$$r = |x'|, \quad z = x_d,$$

with associated unit vectors

$$e_r = \left(\frac{x'}{|x'|}, 0 \right), \quad e_z = e_d,$$

where $\{e_1, \dots, e_d\}$ denote the standard basis for $\mathbb{R}^d$. We will say that a vector field $u: \mathbb{R}^d \rightarrow \mathbb{R}^d$ is axisymmetric and swirl-free if

$$u(x) = u_r(r, z)e_r + u_z(r, z)e_z,$$

and we will say that a scalar function $f: \mathbb{R}^d \rightarrow \mathbb{R}$ is axisymmetric if

$$f(x) = \tilde{f}(r, z).$$

We will sometimes slightly abuse notation by equating f with $\tilde{f}$. For derivatives in terms of the spatial variables, we will often use the shorthand

$$\partial_i = \frac{\partial}{\partial x_i},$$

and likewise for ∂_r and ∂_z .

The Lebesgue spaces $L^p(\mathbb{R}^d)$ will be given the standard definition and norm. For axisymmetric functions, we will take $L^p = L^p(m_{d-2} r^{d-2} dr dz)$, where m_d is the surface area of the d -sphere embedded in $\mathbb{R}^{d+1}$. So, for $1 \leq p < \infty$,

$$\|f\|_{L^p} = \|\tilde{f}\|_{L^p} = \left(m_{d-2} \int_0^\infty \int_{-\infty}^\infty |\tilde{f}(r, z)|^p r^{d-2} dz dr \right)^{1/p}.$$

For $1 \leq p, q < +\infty$, we will take the Lorentz spaces $L^{p,q}(\mathbb{R}^d)$ to be the spaces with the quasinorm

$$\|f\|_{L^{p,q}} = \left(p \int_0^\infty \alpha^{q-1} |\{x \in \mathbb{R}^d : |f(x)| > \alpha\}|^{q/p} d\alpha \right)^{1/q}.$$

In the case $q = \infty$, we will take the quasinorm to be

$$\|f\|_{L^{p,\infty}} = \left(\sup_{\alpha>0} \alpha^p |\{x \in \mathbb{R}^d : |f(x)| > \alpha\}| \right)^{1/p}.$$

Note that $L^{p,p} = L^p$, and that for axisymmetric functions, we will again use the measure $m_{d-2} r^{d-2} dr dz$ when working with cylindrical coordinates, so that

$$\|f\|_{L^{p,q}} = \|\tilde{f}\|_{L^{p,q}}.$$

We will define both homogeneous and inhomogeneous Sobolev spaces, $\dot{H}^s$ and H^s as follows: for $s > -d/2$,

$$\|f\|_{H^s} = \left(\int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2}, \quad \|f\|_{\dot{H}^s} = \left(\int_{\mathbb{R}^d} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi \right)^{1/2},$$

where $\hat{f}$ denotes the Fourier transform of f .

For all $s \in \mathbb{R}$, let $H_{\text{df}}^s(\mathbb{R}^d)$ be the space of divergence free vector fields:

$$H_{\text{df}}^s = H_{\text{df}}^s(\mathbb{R}^d) = \{v \in H^s(\mathbb{R}^d; \mathbb{R}^d) : \nabla \cdot v = 0\}.$$

Note that the divergence is only a continuous function when $s > 1 + d/2$, so in general we will say that $\nabla \cdot v = 0$ if

$$\xi \cdot \hat{v}(\xi) = 0$$

for almost every $\xi \in \mathbb{R}^d$. We will also take $H_{\text{gr}}^s(\mathbb{R}^d)$ to be the space of gradients,

$$H_{\text{gr}}^s = H_{\text{gr}}^s(\mathbb{R}^d) = \{\nabla f : f \in \dot{H}^1(\mathbb{R}^d) \cap \dot{H}^{s+1}(\mathbb{R}^d)\}.$$

Remark 2.1. Note that $v \in H_{\text{gr}}^s$ if and only if $\hat{v}(\xi) \in \text{span}(\xi)$ for almost every $\xi \in \mathbb{R}^d$, and $v \in H_{\text{df}}^s$ if and only if $\hat{v}(\xi) \in \text{span}(\xi)^\perp$ for almost every $\xi \in \mathbb{R}^d$. This leads to the Helmholtz decomposition

$$H^s = H_{\text{df}}^s \oplus H_{\text{gr}}^s.$$

Denote the Sobolev space of axisymmetric, swirl-free vector fields as

$$H_{\text{as}}^s = H_{\text{as}}^s(\mathbb{R}^d; \mathbb{R}^d) = \{v \in H^s(\mathbb{R}^d; \mathbb{R}^d) : v(x) = v_r(r, z)e_r + v_z(r, z)e_z\},$$

and the spaces of axisymmetric, swirl-free, divergence free vector fields and axisymmetric, swirl-free gradients as

$$H_*^s = H_{\text{as}}^s \cap H_{\text{df}}^s, \quad H_{\text{as \& gr}}^s = H_{\text{as}}^s \cap H_{\text{gr}}^s.$$

Let $C^k(\mathbb{R}^d)$ denote the space of k times continuously differentiable functions, with bounded derivatives up to order k , with norm

$$\|f\|_{C^k} = \sup_{\substack{\alpha_1, \dots, \alpha_d \geq 0 \\ \alpha_1 + \dots + \alpha_d \leq k}} \|\partial_1^{\alpha_1} \dots \partial_d^{\alpha_d} f\|_{L^\infty}.$$

We will take the analogous definition when dealing with C^k functions that also depend on a time variable.

Remark 2.2. Note that for all $s > k + d/2$, we have the continuous embedding

$$H^s(\mathbb{R}^d) \hookrightarrow C^k(\mathbb{R}^d), \quad \|f\|_{C^k} \leq C_{s,k,d} \|f\|_{H^s}.$$

We will refer to the anti-symmetric and symmetric parts of the gradient of a vector field u as A and S , with

$$A_{ij} = \frac{1}{2}(\partial_i u_j - \partial_j u_i) \quad \text{and} \quad S_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i),$$

and define the anti-symmetric gradient operator and symmetric gradient operator likewise:

$$\nabla_{\text{asym}} u = A \quad \text{and} \quad \nabla_{\text{sym}} u = S.$$

For $d = 3$, A can be represented in terms of the vorticity vector $\vec{\omega} = \nabla \times u$ by

$$A = \frac{1}{2} \begin{pmatrix} 0 & \vec{\omega}_3 & -\vec{\omega}_2 \\ -\vec{\omega}_3 & 0 & \vec{\omega}_1 \\ \vec{\omega}_2 & -\vec{\omega}_1 & 0 \end{pmatrix}.$$

For axisymmetric, swirl-free vector fields u ,

$$A(x) = \frac{1}{2} \omega(r, z) (e_r \otimes e_z - e_z \otimes e_r),$$

where the scalar vorticity is

$$\omega(r, z) = \partial_r u_z - \partial_z u_r.$$

For $d = 3$,

$$\vec{\omega}(x) = -\omega(r, z)e_\theta.$$

3. The axisymmetric Euler equation in higher dimensions

In this section, we will show that the axisymmetric, swirl-free Euler equation in four and higher dimensions can be expressed in terms of the evolution of the scalar vorticity, just as it can in three dimensions.

3.1. Preliminaries

We will begin by going over some of the vector calculus results necessary for the analysis of axisymmetric, swirl-free solutions. The computations are mostly routine exercises, however, they are very important to the analysis.

Proposition 3.1. *Suppose $u \in H_*^s(\mathbb{R}^d)$, with $s > 1 + d/2$, is axisymmetric and swirl-free. Then the gradient of u can be expressed as*

$$\begin{aligned} \nabla u(x) = & \partial_r u_r(r, z) e_r \otimes e_r + \frac{u_r(r, z)}{r} \tilde{I}_d \\ & + \partial_z u_z(r, z) e_z \otimes e_z + \partial_r u_z(r, z) e_r \otimes e_z + \partial_z u_r(r, z) e_z \otimes e_r, \end{aligned}$$

where

$$\tilde{I}_d = I_d - e_r \otimes e_r - e_z \otimes e_z.$$

Proof. First we know that

$$u(x) = u_r(r, z) e_r + u_z(r, z) e_z \quad \text{and} \quad \nabla r = e_r,$$

and therefore we can see that

$$\begin{aligned} \nabla u(x) = & \partial_r u_r(r, z) e_r \otimes e_r + \partial_z u_z(r, z) e_z \otimes e_z \\ & + \partial_r u_z(r, z) e_r \otimes e_z + \partial_z u_r(r, z) e_z \otimes e_r + u_r(r, z) \nabla e_r. \end{aligned}$$

It remains only to show that

$$\nabla e_r = \frac{1}{r} \tilde{I}_d.$$

Recall that $e_r = x'/|x'|$, where $x' = (x_1, \dots, x_{d-1}, 0)$. Clearly we have

$$\nabla x' = I_d - e_z \otimes e_z \quad \text{and} \quad \nabla \frac{1}{|x'|} = \nabla \frac{1}{r} = -\frac{1}{r^2} e_r.$$

Therefore, we can conclude that

$$\nabla e_r = \frac{1}{r} \nabla x' + \nabla \frac{1}{|x'|} \otimes x' = \frac{1}{r} (I_d - e_z \otimes e_z) - \frac{1}{r^2} e_r \otimes x' = \frac{1}{r} (I_d - e_z \otimes e_z - e_r \otimes e_r).$$

This completes the proof. ■

Proposition 3.2. *Suppose $u \in H^s(\mathbb{R}^d)$, with $s > 1 + d/2$, is axisymmetric and swirl-free. Then the divergence of u can be expressed as*

$$\nabla \cdot u(x) = \partial_r u_r(r, z) + k \frac{u_r(r, z)}{r} + \partial_z u_z(r, z),$$

where $k = d - 2$.

Proof. Taking the identity from Proposition 3.1, we can see that

$$\begin{aligned}\nabla \cdot u(x) &= \text{tr}(\nabla u(x)) = \partial_r u_r + \partial_z u_z + \frac{u_r(r, z)}{r} \text{tr}(\tilde{I}_d) \\ &= \partial_r u_r(r, z) + \partial_z u_z(r, z) + (d-2) \frac{u_r(r, z)}{r}.\end{aligned}$$

This completes the proof. $\blacksquare$

Proposition 3.3. Suppose $u \in H^s(\mathbb{R}^d)$, with $s > 1 + d/2$, is axisymmetric and swirl-free. Then $(u \cdot \nabla)u$ is axisymmetric and swirl-free.

Proof. We know by hypothesis that

$$u(x) = u_r(r, z)e_r + u_z(r, z)e_z,$$

and by Proposition 3.1 that

$$\begin{aligned}\nabla u(x) &= \partial_r u_r(r, z)e_r \otimes e_r + \frac{u_r(r, z)}{r} \tilde{I}_d + \partial_z u_z(r, z)e_z \otimes e_z \\ &\quad + \partial_r u_z(r, z)e_r \otimes e_z + \partial_z u_r(r, z)e_z \otimes e_r.\end{aligned}$$

We can see that $\tilde{I}_d$ is the matrix for the projection onto the subspace $\text{span}(e_r, e_z)^\perp$, and so

$$\begin{aligned}(u \cdot \nabla)u &= (\nabla u)^{\text{tr}} u = u_r \partial_r u_r e_r + u_z \partial_z u_z e_z + u_r \partial_r u_z e_z + u_z \partial_z u_r e_r \\ &= (u \cdot \nabla u_r) e_r + (u \cdot \nabla u_z) e_z.\end{aligned}$$

Recalling the definition of an axisymmetric vector field, this completes the proof. $\blacksquare$

Proposition 3.4. Suppose $u \in H^s(\mathbb{R}^d)$, with $s > 1 + d/2$, is axisymmetric and swirl-free. Then

$$A(x) = \frac{1}{2} \omega(r, z)(e_r \otimes e_z - e_z \otimes e_r),$$

where $\omega(r, z) = \partial_r u_z - \partial_z u_r$.

Proof. Taking the formula for ∇u from Proposition 3.1, we can see that

$$\begin{aligned}A(x) &= \frac{1}{2} (\partial_r u_z e_r \otimes e_z + \partial_z u_r e_z \otimes e_r - \partial_r u_z e_z \otimes e_r - \partial_z u_r e_r \otimes e_z) \\ &= \frac{1}{2} (\partial_r u_z - \partial_z u_r)(e_r \otimes e_z - e_z \otimes e_r).\end{aligned}$$

This completes the proof. $\blacksquare$

Proposition 3.5. For all $d \geq 3$, the divergence of the radial direction $e_r \in \mathbb{R}^d$ is

$$\nabla \cdot e_r = \frac{k}{r}.$$

Proof. Recall that

$$e_r = \frac{x'}{|x'|}.$$

It then follows that

$$\nabla \cdot e_r = \frac{\nabla \cdot x'}{|x'|} + x' \cdot \nabla \frac{1}{|x'|} = \frac{d-1}{r} - x' \cdot \frac{x'}{|x'|^3} = \frac{d-2}{r}.$$

This completes the proof. ■

Proposition 3.6. *Suppose that $u \in H^s(\mathbb{R}^d; \mathbb{R}^d)$, with $s > 2 + d/2$, is an axisymmetric, swirl-free vector field. Then the vector Laplacian can be expressed as*

$$-\Delta u(x) = \left(-\partial_r^2 - \frac{k}{r} \partial_r + \frac{k}{r^2} - \partial_z^2\right) u_r(r, z) e_r + \left(-\partial_r^2 - \frac{k}{r} \partial_r - \partial_z^2\right) u_z(r, z) e_z.$$

Proof. Taking the divergence of the gradient formula in Proposition 3.1, using the fact that $-\Delta u = -\operatorname{div} \nabla u$, and applying Proposition 3.5, we find that

$$-\Delta u = -\partial_z^2 u_r e_r - \partial_z^2 u_z e_z - \partial_r^2 u_z e_z - \frac{k}{r} \partial_r u_z e_z - \operatorname{div}(\partial_r u_r e_r \otimes e_r) - \operatorname{div}\left(\frac{u_r}{r} \tilde{I}_d\right).$$

Again applying Proposition 3.5, we can see that

$$-\operatorname{div}(\partial_r u_r (e_r \otimes e_r)) = \left(-\partial_r^2 u_r - \frac{k}{r} \partial_r u_r\right) e_r,$$

and that

$$-\operatorname{div}\left(\frac{u_r}{r} \tilde{I}_d\right) = -\left(\partial_r \frac{u_r}{r}\right) \tilde{I}_d e_r - \frac{u_r}{r} \operatorname{div}(I_d - e_r \otimes e_r - e_z \otimes e_z) = \frac{k}{r^2} u_r e_r.$$

This completes the proof. ■

Proposition 3.7. *Suppose that $f \in H^s$, with $s > 2 + d/2$, is a scalar axisymmetric function with $f(x) = \tilde{f}(r, z)$. Then*

$$\nabla f(x) = \partial_r \tilde{f}(r, z) e_r + \partial_z \tilde{f}(r, z) e_z$$

and

$$-\Delta f(x) = -\partial_r^2 \tilde{f}(r, z) - \frac{k}{r} \partial_r \tilde{f}(r, z) - \partial_z^2 \tilde{f}(r, z).$$

Proof. Observe that

$$\nabla r = e_r,$$

and so, applying the chain rule,

$$\nabla f(x) = \partial_r \tilde{f}(r, z) e_r + \partial_z \tilde{f}(r, z) e_z.$$

By taking the divergence of this equation and applying Proposition 3.2, we complete the proof. ■

Proposition 3.8. Suppose that $u \in H_{\text{as}}^s(\mathbb{R}^d)$, with $s > 2 + d/2$. Then the vector Laplacian can be expressed as

$$-\Delta u = -2 \operatorname{div} \nabla_{\text{asym}} u - \nabla \nabla \cdot u.$$

Proof. First note that for $s > 2 + d/2$, $H_{\text{as}}^s(\mathbb{R}^d) \hookrightarrow C^2(\mathbb{R}^d)$, so the both sides of the expression are well defined for all $x \in \mathbb{R}^d$. Let

$$u(x) = u_r(r, z)e_r + u_z(r, z)e_z.$$

Recalling that

$$\nabla \cdot u = \partial_r u_r + \frac{k}{r} u_r + \partial_z u_z,$$

we can compute that

$$(3.1) \quad \nabla \nabla \cdot u = \left(\partial_r^2 + \frac{k}{r} \partial_r - \frac{k}{r^2} \right) u_r e_r + \partial_z^2 u_z e_z + \partial_z \left(\partial_r u_r + \frac{k}{r} u_r \right) e_z + \partial_r \partial_z u_z e_r.$$

Likewise, recalling that

$$2 \nabla_{\text{asym}} u = (\partial_r u_z - \partial_z u_r)(e_r \otimes e_z - e_z \otimes e_r),$$

and applying the product rule

$$\operatorname{div}(fM) = f \operatorname{div}(M) + M^{\text{tr}} \nabla f,$$

we find that

$$(3.2) \quad \begin{aligned} 2 \operatorname{div} \nabla_{\text{asym}} u &= \frac{k}{r} (\partial_r u_z - \partial_z u_r) e_z + \partial_r (\partial_r u_z - \partial_z u_r) e_z - \partial_z (\partial_r u_z - \partial_z u_r) e_r \\ &= \partial_z^2 u_r e_r + \left(\partial_r^2 + \frac{k}{r} \partial_r \right) u_z e_z - \frac{k}{r} \partial_z u_r e_z - \partial_r \partial_z u_r e_z - \partial_z \partial_r u_z e_r. \end{aligned}$$

Adding together (3.1) and (3.2), this completes the proof. $\blacksquare$

Remark 3.9. Note that the operators $-\Delta$, $\operatorname{div} \nabla_{\text{asym}}$, $\nabla \nabla \cdot$, $(-\Delta)^{-1}$ all preserve the class of axisymmetric, swirl-free vector fields. For the first three operators, this can be observed directly from the computations above. For $(-\Delta)^{-1}$, the result follows in any space where the kernel of the Laplace operator is trivial from the fact that $-\Delta$ preserves this class.

Proposition 3.10. Axisymmetric, swirl free vector fields in dimension $d \geq 3$ have the following Helmholtz decomposition:

$$H_{\text{as}}^s(\mathbb{R}^d) = H_*^s(\mathbb{R}^d) \oplus H_{\text{as \& gr}}^s(\mathbb{R}^d),$$

for all $s \in \mathbb{R}$. In particular, for all $v \in H_{\text{as}}^s$,

$$(3.3) \quad v = -2 \operatorname{div} \nabla_{\text{asym}} (-\Delta)^{-1} v - \nabla \nabla \cdot (-\Delta)^{-1} v,$$

with

$$-2 \operatorname{div} \nabla_{\text{asym}} (-\Delta)^{-1} v \in H_*^s(\mathbb{R}^d) \quad \text{and} \quad -\nabla \nabla \cdot (-\Delta)^{-1} v \in H_{\text{as \& gr}}^s(\mathbb{R}^d).$$

Proof. We know immediately from the classical Helmholtz decomposition that for all $u \in H_*^s$ and $\nabla f \in H_{\text{as \& gr}}^s$,

$$\langle u, \nabla f \rangle_s = 0,$$

and so the spaces are orthogonal. It remains only to show that

$$H_{\text{as}}^s(\mathbb{R}^d) = H_*^s(\mathbb{R}^d) + H_{\text{as \& gr}}^s(\mathbb{R}^d).$$

We will first consider this result for $s > d/2$. Applying Proposition 3.8 to $(-\Delta)^{-1}v$, we find that

$$v = -2 \operatorname{div} \nabla_{\text{asym}}(-\Delta)^{-1}v - \nabla \nabla \cdot (-\Delta)^{-1}v.$$

It is obvious that $-\nabla \nabla \cdot (-\Delta)^{-1}v \in H_{\text{as \& gr}}^s$, because $\nabla \cdot v$ is an axisymmetric, scalar function, and thus it remains only to show that $-2 \operatorname{div} \nabla_{\text{asym}}(-\Delta)^{-1}v \in H_*^s$. Letting $A = \nabla_{\text{asym}}(-\Delta)^{-1}v$, we can see that

$$\nabla \cdot (-2 \operatorname{div} A) = -2 \operatorname{div}^2 A = -2 \sum_{i,j=1}^d \partial_i \partial_j A_{ij} = 0,$$

due to the anti-symmetry of A . This completes the proof for $s > d/2$.

We should note that the formula for the Laplacian in Proposition 3.8 holds pointwise only for C^2 vector fields, so $(-\Delta)^{-1}v$ must be at least C^2 for this identity to hold classically. This is why we require that $s > d/2$ for the identity to hold pointwise. It remains true, however, that $-2 \operatorname{div} \nabla_{\text{asym}}(-\Delta)^{-1}$ and $-\nabla \nabla \cdot (-\Delta)^{-1}$ are well defined, bounded, linear operators mapping $H^s \rightarrow H^s$, for all $s \in \mathbb{R}$, and so the decomposition is still valid and well defined for $s \leq d/2$.

In order to prove that the identity

$$v = -2 \operatorname{div} \nabla_{\text{asym}}(-\Delta)^{-1}v - \nabla \nabla \cdot (-\Delta)^{-1}v$$

holds in H_{as}^s for all $s \in \mathbb{R}$, we will need to work in Fourier space. We will show that

$$\mathcal{F}(-2 \operatorname{div} \nabla_{\text{asym}}(-\Delta)^{-1}v)(\xi) = P_{\operatorname{span}\{\xi\}^\perp} \hat{v}(\xi),$$

$$\mathcal{F}(-\nabla \nabla \cdot (-\Delta)^{-1}v)(\xi) = P_{\operatorname{span}\{\xi\}} \hat{v}(\xi),$$

for almost every $\xi \in \mathbb{R}^d$. This establishes the decomposition, because it proves both the identity (3.3) and also shows that $-2 \operatorname{div} \nabla_{\text{asym}}(-\Delta)^{-1}v \in H_*^s$, $-\nabla \nabla \cdot (-\Delta)^{-1}v \in H_{\text{as \& gr}}^s$. It is simple to compute that

$$\mathcal{F}(-\nabla \nabla \cdot (-\Delta)^{-1}v)(\xi) = \left(\frac{\xi \otimes \xi}{|\xi|^2} \right) \hat{v}(\xi) = P_{\operatorname{span}\{\xi\}} \hat{v}(\xi),$$

for almost every $\xi \in \mathbb{R}^d$. We also compute that

$$\mathcal{F}(-2 \operatorname{div} \nabla_{\text{asym}}(-\Delta)^{-1}v)(\xi) = \frac{1}{|\xi|^2} \xi \cdot (\xi \otimes \hat{v}(\xi) - \hat{v}(\xi) \otimes \xi).$$

For all $\xi \in \mathbb{R}^d$, let

$$P_{\operatorname{span}\{\xi\}^\perp} \hat{v}(\xi) = w(\xi) \quad \text{and} \quad P_{\operatorname{span}\{\xi\}} \hat{v}(\xi) = \lambda(\xi) \xi.$$

We can immediately see that for almost every $\xi \in \mathbb{R}^d$,

$$\xi \otimes \hat{v}(\xi) - \hat{v}(\xi) \otimes \xi = \xi \otimes w(\xi) - w(\xi) \otimes \xi.$$

By definition, we have $\xi \cdot w(\xi) = 0$, for almost every $\xi \in \mathbb{R}^d$, and so we can conclude that

$$\xi \cdot (\xi \otimes \hat{v}(\xi) - \hat{v}(\xi) \otimes \xi) = \xi \cdot (\xi \otimes w(\xi) - w(\xi) \otimes \xi) = |\xi|^2 w(\xi),$$

and therefore,

$$\mathcal{F}(-2 \operatorname{div} \nabla_{\text{asym}}(-\Delta)^{-1} v)(\xi) = P_{\operatorname{span}\{\xi\}^\perp} \hat{v}(\xi).$$

This completes the proof. ■

Remark 3.11. Note that the above proposition shows that for all $v \in H_{\text{as}}^s$, $P_{\text{df}}(v) \in H_{\text{as}}^s$.

3.2. Local existence of solutions

Theorem 3.12. *For all $u^0 \in H_{\text{df}}^s(\mathbb{R}^d)$, with $s > 1 + d/2$ and $d \geq 3$, there exists a unique solution of the Euler equation $u \in C([0, T_{\max}), H_{\text{df}}^s(\mathbb{R}^d)) \cap C^1([0, T_{\max}), H_{\text{df}}^{s-1}(\mathbb{R}^d))$, where*

$$T_{\max} > \frac{C_{s,d}}{\|u^0\|_{H^s}}.$$

Furthermore, if $T_{\max} < +\infty$,

$$\int_0^{T_{\max}} \|A(\cdot, t)\|_{L^\infty} dt = +\infty.$$

Remark 3.13. This is the classic Beale–Kato–Majda criterion, and was proven in the case $d = 3, s \geq 3$ in [2]. Kato and Ponce proved the general case where $d \geq 3$ and $s > 1 + d/2$ in [15]. They also proved an analogous regularity criterion in terms of the strain, showing that if $T_{\max} < +\infty$, then

$$\int_0^{T_{\max}} \|S(\cdot, t)\|_{L^\infty} dt = +\infty.$$

See also the work of Kato [14] for details on the local well-posedness theory. For a thorough overview, see Section 3.2 of [19]. Note that this gives a definition of the finite-time blowup $T_{\max}$, as follows:

$$\lim_{t \rightarrow T_{\max}} \|u(\cdot, t)\|_{H^s} = +\infty,$$

as it is clear that $T_{\max} < +\infty$ implies

$$\|u(\cdot, t)\|_{H^s} \geq \frac{C_{s,d}}{T_{\max} - t}.$$

Theorem 3.14. *For all $u^0 \in H_*^s(\mathbb{R}^d)$, with $s > 1 + d/2$, there exists a unique solution of the Euler equation $u \in C([0, T_{\max}), H_*^s(\mathbb{R}^d)) \cap C^1([0, T_{\max}), H_*^{s-1}(\mathbb{R}^d))$. Furthermore, if $T_{\max} < +\infty$, then*

$$\int_0^{T_{\max}} \|\omega(\cdot, t)\|_{L^\infty} dt = +\infty.$$

Proof. We know (Theorem 3.12) that there must be a smooth solution of the Euler equation $u \in C([0, T_{\max}), H_{\text{df}}^s(\mathbb{R}^d)) \cap C^1([0, T_{\max}), H_{\text{df}}^{s-1}(\mathbb{R}^d))$. Applying Propositions 3.3 and 3.10, we can see that if $u(\cdot, t) \in H_*^s$, then $P_{\text{df}}((u \cdot \nabla)u)(\cdot, t) \in H_*^{s-1}$. Recalling that

$\partial_t u + P_{\text{df}}((u \cdot \nabla)u) = 0$, we can conclude that if $u(\cdot, t) \in H_*^s$, then $\partial_t u(\cdot, t) \in H_*^{s-1}$. This implies the subspace of axisymmetric, swirl-free vector fields is preserved by the dynamics of the Euler equation, although this deals only with the formal level.

To make this rigorous, and not only formal, it is necessary to consider how the local existence for the Euler equation (Theorem 3.12) is proven. In Section 3.2 of [19], this local existence theorem is proven, first by proving local existence for a mollified Euler equation

$$\partial_t u^\varepsilon + P_{\text{df}} \mathcal{J}_\varepsilon (\mathcal{J}_\varepsilon u^\varepsilon \cdot \nabla \mathcal{J}_\varepsilon u^\varepsilon) = 0,$$

and then by proving strong convergence $u^\varepsilon \rightarrow u$ in the space $C([0, T_{\max}), H_{\text{df}}^s(\mathbb{R}^d)) \cap C^1([0, T_{\max}), H_{\text{df}}^{s-1}(\mathbb{R}^d))$. Note that $\mathcal{J}_\varepsilon$ is a regularisation operator given by convolution with a smooth, positive, compactly supported function of mass 1. As long as $\mathcal{J}_\varepsilon$ involves convolution with a radial function, it will preserve the class of axisymmetric, swirl-free vector fields. The Picard iteration then guarantees that u^ε must be axisymmetric and swirl-free. Finally, u must be axisymmetric and swirl-free, because it is trivial that this property is preserved by strong convergence in H^s .

Finally, we observe that

$$\|A\|_{L^\infty} = \frac{1}{\sqrt{2}} \|\omega\|_{L^\infty},$$

and this completes the proof. ■

3.3. The vorticity equation

Now that the necessary preliminaries are out of the way, we can derive the evolution equation for the vorticity in four and higher dimensions.

Proposition 3.15. *Suppose that $u \in C([0, T_{\max}); H_*^s(\mathbb{R}^d)) \cap C^1([0, T_{\max}), H_*^{s-1}(\mathbb{R}^d))$, with $d \geq 3$ and $s > 2 + d/2$, is a solution of the Euler equation. Then for all $0 \leq t < T_{\max}$, the vorticity satisfies the evolution equation*

$$\partial_t \omega + (u \cdot \nabla) \omega - k \frac{u_r}{r} \omega = 0.$$

Proof. We begin by observing that due to Sobolev embedding, we have $u(\cdot, t) \in C^2$ and consequently $\omega(\cdot, t) \in C^1$, for all $0 \leq t < T_{\max}$. Next we write the axisymmetric Euler equation in terms of the components u_r and u_z , yielding

$$\partial_t u_r + (u \cdot \nabla) u_r + \partial_r p = 0,$$

$$\partial_t u_z + (u \cdot \nabla) u_z + \partial_z p = 0.$$

Recalling that $\omega = \partial_r u_z - \partial_z u_r$; we differentiate the equation for u_r with respect to z and the equation for u_z with respect to r , yielding

$$(3.4) \quad \partial_t \partial_z u_r + (u \cdot \nabla) \partial_z u_r + (\partial_z u \cdot \nabla) u_r + \partial_r \partial_z p = 0,$$

$$(3.5) \quad \partial_t \partial_r u_z + (u \cdot \nabla) \partial_r u_z + (\partial_r u \cdot \nabla) u_z + \partial_r \partial_z p = 0.$$

Subtracting (3.4) from (3.5), we find that

$$\partial_t \omega + (u \cdot \nabla) \omega - (\partial_z u \cdot \nabla) u_r + (\partial_r u \cdot \nabla) u_z = 0.$$

It remains only to compute the difference term. Plugging in we find that

$$\begin{aligned}
 (3.6) \quad & -(\partial_z u \cdot \nabla) u_r + (\partial_r u \cdot \nabla) u_z \\
 & = -\partial_z u_r \partial_r u_r - \partial_z u_z \partial_z u_r + \partial_r u_r \partial_r u_z + \partial_r u_z \partial_z u_z \\
 & = (\partial_r u_r + \partial_z u_z)(\partial_r u_z - \partial_z u_r) = -k \frac{u_r}{r} \omega,
 \end{aligned}$$

where we have used the divergence free constraint

$$\partial_r u_r + \partial_z u_z + k \frac{u_r}{r} = 0.$$

This completes the proof. $\blacksquare$

Remark 3.16. We will show in the next section that for axisymmetric, swirl-free vector fields, the velocity u is uniquely determined by the scalar vorticity ω , so the vorticity equation completely determines the dynamics of the axisymmetric, swirl-free Euler equation in four and higher dimensions, just as it does in three dimensions.

One of the most important features of the axisymmetric Euler equation in three dimensions is that the quantity ω/r is advected by the flow, and therefore remains bounded in a number of important spaces. There is an analogous result in four and higher dimensions: the quantity ω/r^k is advected by the flow.

Proposition 3.17. Suppose that $u \in C([0, T_{\max}); H_*^s(\mathbb{R}^d)) \cap C^1([0, T_{\max}), H_*^{s-1}(\mathbb{R}^d))$, with $d \geq 3$ and $s > 2 + d/2$, is a solution of the Euler equation. Then for all $0 \leq t < T_{\max}$,

$$(\partial_t + u \cdot \nabla) \frac{\omega}{r^k} = 0.$$

If we let $\xi = \omega/r^k$, then this can be expressed as

$$(\partial_t + u \cdot \nabla) \xi = 0.$$

Proof. Applying the product rule and plugging into the vorticity equation, we can see that

$$\begin{aligned}
 (\partial_t + u \cdot \nabla) \frac{\omega}{r^k} &= \frac{1}{r^k} \partial_t \omega + \frac{1}{r^k} (u \cdot \nabla) \omega - u_r k \frac{\omega}{r^{k+1}} \\
 &= \frac{1}{r^k} \left(\partial_t \omega + (u \cdot \nabla) \omega - k \frac{u_r}{r} \omega \right) = 0.
 \end{aligned}$$

This completes the proof. $\blacksquare$

Proposition 3.18. Suppose that $u \in C([0, T_{\max}); H_*^s(\mathbb{R}^d)) \cap C^1([0, T_{\max}), H_*^{s-1}(\mathbb{R}^d))$, with $d \geq 3$ and $s > 2 + d/2$, is a solution of the Euler equation, and let $X(r, z, t)$ be the associated Lagrangian flow map satisfying

$$\begin{aligned}
 \frac{dX}{dt}(r, z, t) &= u(X(r, z, t), t), \\
 X(r, z, 0) &= r e_r + z e_z.
 \end{aligned}$$

Then for all $0 \leq t < T_{\max}$, $r \in \mathbb{R}^+$ and $z \in \mathbb{R}$,

$$\frac{\omega(X(r, z, t), t)}{X_r(r, z, t)^k} = \frac{\omega^0(r, z)}{r^k}.$$

Proof. We will begin by letting

$$\xi(r, z, t) = \frac{\omega(r, z, t)}{r^k}.$$

We have already shown that ξ is advected by the flow and so

$$(\partial_t + u \cdot \nabla)\xi = 0.$$

Define $\tilde{\xi}$ by

$$\tilde{\xi}(r, z, t) = \xi(X(r, z, t), t).$$

Applying the chain rule, we can see that for all $0 \leq t < T_{\max}$, $r \in \mathbb{R}^+$ and $z \in \mathbb{R}$,

$$\partial_t \tilde{\xi}(r, z, t) = \left(\partial_t \xi + \frac{dX}{dt} \cdot \nabla \xi \right)(X(r, z, t), t) = (\partial_t \xi + u \cdot \nabla \xi)(X(r, z, t), t) = 0.$$

Therefore, we can conclude that for all $0 \leq t < T_{\max}$, $r \in \mathbb{R}^+$ and $z \in \mathbb{R}$,

$$\xi(X(r, z, t), t) = \xi(X(r, z, 0), 0) = \xi^0(r, z),$$

and that consequently,

$$\frac{\omega(X(r, z, t), t)}{X_r(r, z, t)^k} = \frac{\omega^0(r, z)}{r^k}.$$

This completes the proof. ■

Corollary 3.19. *Suppose that $u \in C([0, T_{\max}); H_*^s(\mathbb{R}^d)) \cap C^1([0, T_{\max}), H_*^{s-1}(\mathbb{R}^d))$, with $d \geq 3$ and $s > 2 + d/2$, is a solution of the Euler equation, and assume that $\omega^0/r^k \in L^\infty$. Then for all $r \in \mathbb{R}^+$, $z \in \mathbb{R}$, $0 \leq t < T_{\max}$,*

$$|\omega(r, z, t)| \leq \left\| \frac{\omega}{r^k} \right\|_{L^\infty} r^k.$$

Proof. We can see immediately from Proposition 3.17 that for all $0 \leq t < T_{\max}$,

$$\|\xi(\cdot, t)\|_{L^\infty} = \|\xi^0\|_{L^\infty},$$

and therefore for all $r \in \mathbb{R}^+$, $z \in \mathbb{R}$ and $0 \leq t < T_{\max}$,

$$\frac{\omega(r, z, t)}{r^k} \leq \|\xi^0\|_{L^\infty}.$$

This completes the proof. ■

Corollary 3.20. *Suppose that $u \in C([0, T_{\max}); H_*^s(\mathbb{R}^d)) \cap C^1([0, T_{\max}), H_*^{s-1}(\mathbb{R}^d))$, with $d \geq 3$ and $s > 2 + d/2$, is a solution of the Euler equation. Then the Lorentz quasinorms of ω/r^k are all preserved by the flow: for all $1 \leq p, q \leq +\infty$ and for all $0 \leq t < T_{\max}$,*

$$\left\| \frac{\omega}{r^k}(\cdot, t) \right\|_{L^{p,q}(\mathbb{R}^d)} = \left\| \frac{\omega}{r^k} \right\|_{L^{p,q}(\mathbb{R}^d)}.$$

Proof. First we observe that by Proposition 3.18, for all $0 \leq t < T_{\max}$,

$$\|\xi \circ X(\cdot, t)\|_{L^{p,q}(\mathbb{R}^d)} = \|\xi^0\|_{L^{p,q}(\mathbb{R}^d)}.$$

Because the transport velocity is divergence free, the flow map $X(\cdot, t)$ is volume preserving, and therefore

$$\|\xi \circ X(\cdot, t)\|_{L^{p,q}(\mathbb{R}^d)} = \|\xi(\cdot, t)\|_{L^{p,q}(\mathbb{R}^d)},$$

and this completes the proof. $\blacksquare$

Remark 3.21. We will note that there is a fundamental difference between the case $d = 3$ and the case $d \geq 4$. When we take $d = 3$, we must have $\omega^0/r \in L^\infty$ for sufficiently smooth initial data, while in $d \geq 4$, there are Schwartz class initial data where $\omega/r^k \notin L^\infty$. We will show this now.

Proposition 3.22. Suppose $u \in H_*^s(\mathbb{R}^d)$, with $s > 2 + d/2$. Then $\omega/r \in L^\infty$ and

$$\left\| \frac{\omega}{r} \right\|_{L^\infty} \leq C_{s,d} \|u\|_{H^s}.$$

Proof. We will begin by observing that if $u \in H^s(\mathbb{R}^d)$, then clearly $\nabla A \in H^{s-2}(\mathbb{R}^d)$, $s - 2 > d/2$, so by Sobolev embedding,

$$\|\nabla A\|_{L^\infty} \leq C \|\nabla A\|_{H^{s-2}} \leq C \|u\|_{H^s}.$$

Now it remains only to control magnitude of ω/r by the magnitude of ∇A . By definition, ∇A is a three tensor with

$$(\nabla A)_{ijk} = \partial_i A_{jk}.$$

Recall that

$$A(x) = \frac{1}{2} \omega(r, z)(e_r \otimes e_z - e_z \otimes e_r),$$

and we can see that

$$\begin{aligned} (3.7) \quad \nabla A &= \frac{1}{2} \nabla \omega \otimes (e_r \otimes e_z - e_z \otimes e_r) + \frac{1}{2} \omega (\nabla e_r \otimes e_z - (\nabla e_r \otimes e_z)^*) \\ &= \frac{1}{2} \nabla \omega \otimes (e_r \otimes e_z - e_z \otimes e_r) + \frac{1}{2} \frac{\omega}{r} (\tilde{I}_d \otimes e_z - (\tilde{I}_d \otimes e_z)^*), \end{aligned}$$

where M^* for a three tensor is transpose of the last two elements

$$M_{ijk}^* = M_{ikj}.$$

Recalling that $\nabla \omega = \partial_r \omega e_r + \partial_z \omega e_z$, and observing that the two terms in (3.7) are orthogonal, we can see that

$$|\nabla A|^2 = \frac{1}{2} |\nabla \omega|^2 + \frac{d-2}{2} \left| \frac{\omega}{r} \right|^2.$$

This clearly implies that

$$\left\| \frac{\omega}{r} \right\|_{L^\infty} \leq \sqrt{\frac{2}{d-2}} \|\nabla A\|_{L^\infty},$$

and this completes the proof. $\blacksquare$

Proposition 3.23. *For all $d \geq 4$, there exists a Schwartz class divergence free vector field $u \in \mathcal{S}(\mathbb{R}^d)_{\text{df}}$ such that $\omega/r^k \notin L^\infty$. In particular,*

$$u(x) = r(1 - 2z^2) \exp(-r^2 - z^2) e_r - z(k + 1 - 2r^2) \exp(-r^2 - z^2) e_z$$

satisfies these conditions.

Proof. We will begin by showing that u is Schwartz class and divergence free. Using that $x' = r e_r$, we can see that

$$u(x) = (x'(1 - 2x_d^2) - x_d(k + 1 - 2|x'|^2) e_d) \exp(-|x|^2),$$

and so we can see that $u \in \mathcal{S}(\mathbb{R}^d)$. Next we compute that

$$\begin{aligned} \nabla \cdot u &= \partial_r u_r + \frac{k}{r} u_r + \partial_z u_z \\ &= ((k + 1 - 2r^2)(1 - 2z^2) + (-1 + 2z^2)(k + 1 - 2r^2)) \exp(-r^2 - z^2) = 0. \end{aligned}$$

Now that we have shown that $u \in \mathcal{S}_{\text{df}}$, it remains only to show that ω/r^k is unbounded. Computing the vorticity we find that

$$\begin{aligned} \omega &= \partial_r u_z - \partial_z u_r \\ &= (4rz + 2(k + 1)rz - 4r^3z + 4rz + 2rz - 4rz^3) \exp(-r^2 - z^2) \\ &= rz(2k + 12 - 4r^2 - 4z^2) \exp(-r^2 - z^2), \end{aligned}$$

and so we can conclude that

$$\frac{\omega}{r^k} = \frac{z}{r^{d-3}} (2k + 12 - 4r^2 - 4z^2) \exp(-r^2 - z^2).$$

By hypothesis, $d \geq 4$, and so $\omega/r^k \notin L^\infty$, and this completes the proof. $\blacksquare$

We should note that the vorticity equation can also be expressed in “divergence form”, by making use of the coordinate divergence $\tilde{\nabla} \cdot v = \partial_r v_r + \partial_z v_z$.

Proposition 3.24. *Suppose that $u \in C([0, T_{\max}); H_*^s(\mathbb{R}^d)) \cap C^1([0, T_{\max}), H_*^{s-1}(\mathbb{R}^d))$, with $d \geq 3$ and $s > 2 + d/2$, is a solution of the Euler equation. Then for all $0 \leq t < T_{\max}$,*

$$\partial_t \omega + \tilde{\nabla} \cdot (\omega u) = 0,$$

that is,

$$\partial_t \omega + \partial_r(\omega u_r) + \partial_z(\omega u_z) = 0.$$

Proof. We begin by applying the product rule, and observing that

$$\tilde{\nabla} \cdot (\omega u) = (u \cdot \nabla) \omega + (\tilde{\nabla} \cdot u) \omega.$$

Recalling the identity (3.6) from Proposition 3.15, we already shown that

$$\partial_t \omega + (u \cdot \nabla) \omega + (\tilde{\nabla} \cdot u) \omega = 0,$$

and this completes the proof. $\blacksquare$

4. The Biot–Savart law in higher dimensions

The scalar vorticity uniquely determines the velocity for axisymmetric, swirl-free vector fields. In this section, we will derive integral formulae for recovering the velocity and velocity component functions from the vorticity.

Proposition 4.1. *Suppose that $u \in H_*^s(\mathbb{R}^d)$, with $s > d/2$. Then u can be recovered from its scalar vorticity as*

$$(4.1) \quad u(x) = -\alpha_d \int_{\mathbb{R}^d} \omega(\bar{r}, \bar{z})(e_{\bar{r}} \otimes e_z - e_z \otimes e_{\bar{r}}) \frac{x - \bar{x}}{|x - \bar{x}|^d} d\bar{x},$$

where $\bar{r} = |\bar{x}'|$, $\bar{z} = \bar{x}_d$, and

$$\alpha_d = \frac{(d-2)\Gamma(d/2-1)}{4\pi^{d/2}}.$$

Proof. Recalling that $A = \frac{1}{2}(\nabla u - (\nabla u)^{\text{tr}})$, we can see that

$$-2 \operatorname{div}(A) = -\Delta u + \nabla(\nabla \cdot u) = -\Delta u,$$

and so u can be obtained from A by convolution with the Poisson kernel:

$$\begin{aligned} u &= -2 \operatorname{div}(-\Delta)^{-1} A = -2 \operatorname{div} \frac{\alpha_d}{d-2} \int_{\mathbb{R}^d} \frac{1}{|x - \bar{x}|^{d-2}} A(\bar{x}) d\bar{x} \\ &= -2 \frac{\alpha_d}{d-2} \sum_{i=1}^d \int_{\mathbb{R}^d} \partial_{x_i} \frac{1}{|x - \bar{x}|^{d-2}} A_{ij}(\bar{x}) d\bar{x} \\ &= -2 \frac{\alpha_d}{d-2} \int_{\mathbb{R}^d} A^{\text{tr}}(\bar{x}) \nabla_x \frac{1}{|x - \bar{x}|^{d-2}} \\ &= -\alpha_d \int_{\mathbb{R}^d} \omega(\bar{r}, \bar{z})(e_{\bar{r}} \otimes e_z - e_z \otimes e_{\bar{r}}) \frac{x - \bar{x}}{|x - \bar{x}|^d} d\bar{x}. \quad \blacksquare \end{aligned}$$

Lemma 4.2. *For any $f \in L^1([-1, 1]; (1 - y_1^2)^{(d-4)/2} dy_1)$,*

$$(4.2) \quad \begin{aligned} \int_{\mathbb{S}^{d-2}} f(y_1) dS(y) &= m_{d-3} \int_{-1}^1 (1 - y_1^2)^{(d-4)/2} f(y_1) dy_1 \\ &= m_{d-3} \int_0^\pi f(\cos \theta) \sin^{d-3} \theta d\theta, \end{aligned}$$

where for $n \geq 0$,

$$m_n = \frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)}$$

is the surface area of the n -sphere embedded in $\mathbb{R}^{n+1}$.

Note that $m_0 = 2$, corresponding to the fact that S^0 has two points. The second line of (4.2) follows from change of variables $y_1 = \cos \theta$.

Proof. Write $\mathbb{R}^{d-1} \ni y = (y_1, \tilde{y})$. Observing that for $y \in \mathbb{S}^{d-2} \subset \mathbb{R}^{d-1}$,

$$y_1^2 + |\tilde{y}|^2 = |y|^2 = 1,$$

and also that

$$dS(y) = \frac{1}{\sqrt{1-y_1^2}} dy_1 dS(\tilde{y}),$$

we arrive at

$$\begin{aligned} \int_{\mathbb{S}^{d-2}} f(y_1) dS(y) &= \int_{-1}^1 \int_{|\tilde{y}|=\sqrt{1-y_1^2}} f(y_1) \frac{1}{\sqrt{1-y_1^2}} dS(\tilde{y}) dy_1 \\ &= \int_{-1}^1 f(y_1) \frac{1}{\sqrt{1-y_1^2}} \left| \mathbb{S}_{\sqrt{1-y_1^2}}^{d-3} \right| dy_1. \end{aligned}$$

The first equality in (4.2) then follows from

$$\left| \mathbb{S}_{\sqrt{1-y_1^2}}^{d-3} \right| = m_{d-3} (\sqrt{1-y_1^2})^{d-3},$$

and the second follows from the substitution $y_1 = \cos \theta$. ■

Proposition 4.3. *Suppose that $u \in H_{\text{df}}^s(\mathbb{R}^d)$, with $s > d/2$ and $d \geq 3$, is axisymmetric and swirl-free. Then the components u_r and u_z can be recovered from the scalar vorticity using the formulas*

$$(4.3) \quad u_r(r, z) = -\frac{d-2}{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} \bar{r}^{d-2} (z - \bar{z}) \omega(\bar{r}, \bar{z}) \\ \times \int_{-1}^1 \frac{y_1 (1-y_1^2)^{(d-4)/2}}{(r^2 + \bar{r}^2 - 2r\bar{r}y_1 + (z - \bar{z})^2)^{d/2}} dy_1 d\bar{r} d\bar{z}$$

or

$$(4.4) \quad u_r(r, z) = -\frac{d-2}{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} \bar{r}^{d-2} (z - \bar{z}) \omega(\bar{r}, \bar{z}) \\ \times \int_0^{\pi} \frac{\sin^{d-3} \theta \cos \theta}{(r^2 + \bar{r}^2 - 2r\bar{r} \cos \theta + (z - \bar{z})^2)^{d/2}} d\theta d\bar{r} d\bar{z}$$

and

$$(4.5) \quad u_z(r, z) = \frac{d-2}{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} \bar{r}^{d-2} \omega(\bar{r}, \bar{z}) \\ \times \int_{-1}^1 \frac{(ry_1 - \bar{r})(1-y_1^2)^{(d-4)/2}}{(r^2 + \bar{r}^2 - 2r\bar{r}y_1 + (z - \bar{z})^2)^{d/2}} dy_1 d\bar{r} d\bar{z}$$

or

$$(4.6) \quad u_z(r, z) = \frac{d-2}{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} \bar{r}^{d-2} \omega(\bar{r}, \bar{z}) \\ \times \int_0^{\pi} \frac{\sin^{d-3} \theta (r \cos \theta - \bar{r})}{(r^2 + \bar{r}^2 - 2r\bar{r} \cos \theta + (z - \bar{z})^2)^{d/2}} d\theta d\bar{r} d\bar{z}.$$

Proof. We apply (4.1). Using axisymmetry, we may set $x = re_1 + ze_d$, and so $e_r = e_1$. Then

$$\begin{aligned} u_r(r, z) &= e_1 \cdot u(re_1 + ze_d) \\ &= -\alpha_d e_1 \cdot \int_{\mathbb{R}^d} \omega(\bar{r}, \bar{z})(e_{\bar{r}} \otimes e_z - e_z \otimes e_{\bar{r}}) \frac{re_1 - \bar{x}' + (z - \bar{z})e_z}{(r^2 + |\bar{x}'|^2 - 2r\bar{x}_1 + (z - \bar{z})^2)^{d/2}} d\bar{x} \\ &= -\alpha_d \int_{\mathbb{R}^d} \omega(\bar{r}, \bar{z}) \frac{\bar{x}_1}{\bar{r}} \frac{z - \bar{z}}{(r^2 + \bar{r}^2 - 2r\bar{x}_1 + (z - \bar{z})^2)^{d/2}} d\bar{x}. \end{aligned}$$

Letting $y = \bar{x}'/\bar{r}$ parameterise $\mathbb{S}^{d-2} \subset \mathbb{R}^{d-1}$, we find that

$$\begin{aligned} (4.7) \quad u_r(r, z) &= -\alpha_d \int_{-\infty}^{\infty} \int_0^{\infty} \bar{r}^{d-2} (z - \bar{z}) \omega(\bar{r}, \bar{z}) \\ &\quad \times \int_{\mathbb{S}^{d-2}} \frac{y_1}{(r^2 + \bar{r}^2 - 2r\bar{r}y_1 + (z - \bar{z})^2)^{d/2}} dS(y) d\bar{r} d\bar{z}. \end{aligned}$$

Expressions (4.3) and (4.4) then follow from applying Lemma 4.2 on spherical integration, and observing that

$$\alpha_d m_{d-3} = \frac{d-2}{2\pi},$$

where m_{d-3} is taken as in Lemma 4.2.

The computation of u_z is similar:

$$\begin{aligned} u_z(r, z) &= e_d \cdot u(re_1 + ze_d) \\ &= -\alpha_d e_z \cdot \int_{\mathbb{R}^d} \omega(\bar{r}, \bar{z})(e_{\bar{r}} \otimes e_z - e_z \otimes e_{\bar{r}}) \frac{re_1 - \bar{x}' + (z - \bar{z})e_z}{(r^2 + \bar{r}^2 - 2r\bar{x}'_1 + (z - \bar{z})^2)^{d/2}} d\bar{x} \\ &= \alpha_d \int_{\mathbb{R}^d} \omega(\bar{r}, \bar{z}) \frac{ry_1 - \bar{r}}{(r^2 + \bar{r}^2 - 2r\bar{r}y_1 + (z - \bar{z})^2)^{d/2}} d\bar{x} \\ &= \alpha_d \int_{-\infty}^{\infty} \int_0^{\infty} m_{d-2} \bar{r}^{d-2} \omega(\bar{r}, \bar{z}) \\ &\quad \times \int_{\mathbb{S}^{d-2}} \frac{ry_1 - \bar{r}}{(r^2 + \bar{r}^2 - 2r\bar{r}y_1 + (z - \bar{z})^2)^{d/2}} dS(y) d\bar{r} d\bar{z}, \end{aligned}$$

and again applying Lemma 4.2 produces expressions (4.5) and (4.6). $\blacksquare$

Next we extend an upper bound of [10] on the radial velocity from $d = 3$ to all $d \geq 3$.

Lemma 4.4. For all $d \geq 3$, define $F_d: [0, +\infty] \rightarrow (0, +\infty)$ by

$$F_d(s) = \int_0^\pi \frac{\theta^{d-3}}{(1 + 2s(1 - \cos(\theta)))^{d/2}} d\theta.$$

Then, for all $s > 0$,

$$(4.8) \quad F_d(s) \leq \frac{\pi^{d-2}}{d-2}, \quad F_d(s) \leq \frac{C}{s^{d/2-1}}, \quad \text{with } C = \int_0^\infty \frac{\tau^{d-3}}{(1 + \frac{4}{\pi^2} \tau^2)^{d/2}} d\tau.$$

Proof. For the first inequality, observe that $1 - \cos(\theta) \geq 0$, and so F_d is clearly a decreasing function in s , and therefore

$$F_d(s) \leq F_d(0) = \frac{\pi^{d-2}}{d-2}.$$

For the second inequality, observe that for all $0 \leq \theta \leq \pi$, we have $1 - \cos(\theta) \geq \frac{2}{\pi^2} \theta^2$. Therefore,

$$F_d(s) \leq \int_0^\pi \frac{\theta^{d-3}}{(1 + \frac{4s}{\pi^2} \theta^2)^{d/2}} d\theta.$$

Now take the change of variables $\tau = \sqrt{s} \theta$, and we can see that

$$F_d(s) \leq \left(\frac{1}{\sqrt{s}}\right)^{d-2} \int_0^{\pi\sqrt{s}} \frac{\tau^{d-3}}{(1 + \frac{4}{\pi^2} \tau^2)^{d/2}} d\tau \leq \frac{1}{s^{d/2-1}} \int_0^\infty \frac{\tau^{d-3}}{(1 + \frac{4}{\pi^2} \tau^2)^{d/2}} d\tau. \quad \blacksquare$$

Proposition 4.5. Suppose that $d \geq 3$, $s > d/2$, $0 \leq b < 1/2$. Let $q = (d-2)b/(1/2-b)$. Then for all $u \in H_{\text{df \& as}}^s$, such that $\omega/r^k \in L^1 \cap L^\infty$ and $r^q \omega \in L^1$,

$$(4.9) \quad \|u_r\|_{L^\infty} \leq C_d \left\| \frac{\omega}{r^k} \right\|_{L^\infty}^{1/2} \left\| \frac{\omega}{r^k} \right\|_{L^1}^b \|r^q \omega\|_{L^1}^{1/2-b},$$

where C_d depends only on d . In particular, when $b = 0$,

$$(4.10) \quad \|u_r\|_{L^\infty} \leq C_d \left\| \frac{\omega}{r^k} \right\|_{L^\infty}^{1/2} \|\omega\|_{L^1}^{1/2}.$$

Proof. We will prove (4.10). Then (4.9) follows from writing

$$\omega = \left(\frac{\omega}{r^k}\right)^{2b} (r^q \omega)^{1-2b}$$

and applying Hölder's inequality. Note that the inequality in (4.10) is invariant under translation in z , and under the rescaling (we leave this to the reader to check)

$$u^\lambda(r, z) = u(\lambda r, \lambda z).$$

Therefore it suffices to prove the inequality (4.10) for $r = 1$, $z = 0$, and the result then follows for all $r > 0$, $z \in \mathbb{R}$ by scaling and translation invariance (note that $u_r(0, z) = 0$, so we get the case $r = 0$ for free). Fixing $r = 1$, $z = 0$, we can apply Proposition 4.3 to observe

$$\begin{aligned} u_r(1, 0) &= \frac{d-2}{2\pi} \int_{-\infty}^\infty \int_0^\infty \int_0^\pi \bar{r}^{d-2} \bar{z} \omega(\bar{r}, \bar{z}) \frac{\sin^{d-3} \theta \cos \theta}{(1 + \bar{r}^2 - 2\bar{r} \cos \theta + \bar{z}^2)^{d/2}} d\theta d\bar{r} d\bar{z} \\ &= \frac{d-2}{2\pi} \int_{-\infty}^\infty \int_0^\infty \frac{\bar{r}^{d-2} \bar{z} \omega(\bar{r}, \bar{z})}{((\bar{r}-1)^2 + \bar{z}^2)^{d/2}} \int_0^\pi \frac{\sin^{d-3} \theta \cos \theta}{(1 + \frac{2\bar{r}}{(\bar{r}-1)^2 + \bar{z}^2} (1 - \cos \theta))^{d/2}} d\theta d\bar{r} d\bar{z}. \end{aligned}$$

Using the fact that $0 \leq \sin(\theta) \leq \theta$ for all $0 \leq \theta \leq \pi$, we can see that

$$|u_r(1, 0)| \leq \frac{d-2}{2\pi} \int_{-\infty}^\infty \int_0^\infty \frac{\bar{r}^{d-2} |\omega(\bar{r}, \bar{z})|}{((\bar{r}-1)^2 + \bar{z}^2)^{d/2-1/2}} F_d\left(\frac{\bar{r}}{(\bar{r}-1)^2 + \bar{z}^2}\right) d\bar{r} d\bar{z}$$

for F_d defined in Lemma 4.4. Note that we have a singular integral kernel at $\bar{r} = 1$, $\bar{z} = 0$, but there will be some cancellation, because

$$s = \frac{2\bar{r}}{(\bar{r} - 1)^2 + \bar{z}^2} \rightarrow +\infty \quad \text{as } (\bar{r}, \bar{z}) \rightarrow (1, 0)$$

and

$$F_d(s) \leq \frac{C}{s^{d/2-1}}.$$

We will now break the domain into three pieces to deal with the singular integral. We have

$$|u_r(1, 0)| \leq \frac{d-2}{2\pi} \sum_{j=1}^3 I_j, \quad I_j = \iint_{A_j} \frac{\bar{r}^k |\omega(\bar{r}, \bar{z})|}{((\bar{r} - 1)^2 + \bar{z}^2)^{k/2+1/2}} F_d\left(\frac{\bar{r}}{(\bar{r} - 1)^2 + \bar{z}^2}\right) d\bar{r} d\bar{z},$$

where

$$A_1 = \{(\bar{r} - 1)^2 + \bar{z}^2 \leq \tfrac{1}{4}\}, \quad A_2 = \{\tfrac{1}{4} \leq (\bar{r} - 1)^2 + \bar{z}^2 \leq 4\}, \quad A_3 = \{4 \leq (\bar{r} - 1)^2 + \bar{z}^2\}.$$

We begin by bounding I_2 . Applying (4.8) from Lemma 4.4, we find that

$$\begin{aligned} (4.11) \quad I_2 &\leq C \iint_{A_2} \frac{\bar{r}^{k/2} |\omega(\bar{r}, \bar{z})|}{((\bar{r} - 1)^2 + \bar{z}^2)^{1/2}} d\bar{r} d\bar{z} \\ &\leq C \left\| \frac{\omega}{r^k} \right\|_{L^\infty}^{1/2} \iint_{A_2} \frac{|\omega(\bar{r}, \bar{z})|^{1/2}}{((\bar{r} - 1)^2 + \bar{z}^2)^{1/2}} \bar{r}^k d\bar{r} d\bar{z} \\ &\leq C \left\| \frac{\omega}{r^k} \right\|_{L^\infty}^{1/2} \left(\iint_{A_2} |\omega(\bar{r}, \bar{z})| \bar{r}^k d\bar{r} d\bar{z} \right)^{1/2} \left(\iint_{A_2} \frac{1}{(\bar{r} - 1)^2 + \bar{z}^2} \bar{r}^k d\bar{r} d\bar{z} \right)^{1/2} \\ &\leq C \left\| \frac{\omega}{r^k} \right\|_{L^\infty}^{1/2} \|\omega\|_{L^1}^{1/2}, \end{aligned}$$

where we applied Hölder's inequality in the last step with respect to the measure $\bar{r}^k d\bar{r} d\bar{z}$.

Now we will bound I_1 , considering two cases. First suppose that $\left\| \frac{\omega}{r^k} \right\|_{L^\infty} \leq \|\omega\|_{L^1}$. Then again applying Lemma 4.4, we find that

$$\begin{aligned} I_1 &\leq C \iint_{(\bar{r}-1)^2 + \bar{z}^2 \leq 1/4} \frac{\bar{r}^{k/2} |\omega(\bar{r}, \bar{z})|}{((\bar{r} - 1)^2 + \bar{z}^2)^{1/2}} d\bar{r} d\bar{z} \\ &\leq C \left\| \frac{\omega}{r^k} \right\|_{L^\infty} \iint_{(\bar{r}-1)^2 + \bar{z}^2 \leq 1/4} \frac{1}{((\bar{r} - 1)^2 + \bar{z}^2)^{1/2}} d\bar{r} d\bar{z} \\ &\leq C \left\| \frac{\omega}{r^k} \right\|_{L^\infty} \leq C \left\| \frac{\omega}{r^k} \right\|_{L^\infty}^{1/2} \|\omega\|_{L^1}^{1/2}. \end{aligned}$$

Now suppose that $\|\omega/r^k\|_{L^\infty} > \|\omega\|_{L^1}$, and let

$$R = \frac{1}{2} \|\omega\|_{L^1}^{1/2} \left\| \frac{\omega}{r^k} \right\|_{L^\infty}^{-1/2}.$$

Then

$$\begin{aligned}
 I_1 &\leq \iint_{(\bar{r}-1)^2 + \bar{z}^2 \leq R^2} \frac{\bar{r}^{k/2} |\omega(\bar{r}, \bar{z})|}{((\bar{r}-1)^2 + \bar{z}^2)^{1/2}} d\bar{r} d\bar{z} \\
 &\quad + \iint_{R^2 \leq (\bar{r}-1)^2 + \bar{z}^2 \leq 1/4} \frac{\bar{r}^{k/2} |\omega(\bar{r}, \bar{z})|}{((\bar{r}-1)^2 + \bar{z}^2)^{1/2}} d\bar{r} d\bar{z} \\
 &\leq C \left\| \frac{\omega}{r^k} \right\|_{L^\infty} \iint_{(\bar{r}-1)^2 + \bar{z}^2 \leq R^2} \frac{1}{((\bar{r}-1)^2 + \bar{z}^2)^{1/2}} d\bar{r} d\bar{z} \\
 &\quad + \frac{C}{R} \iint_{R^2 \leq (\bar{r}-1)^2 + \bar{z}^2 \leq 1/4} |\omega(\bar{r}, \bar{z})| \bar{r}^k d\bar{r} d\bar{z} \\
 &\leq CR \left\| \frac{\omega}{r^k} \right\|_{L^\infty} + \frac{C}{R} \|\omega\|_{L^1} = C \left\| \frac{\omega}{r^k} \right\|_{L^\infty}^{1/2} \|\omega\|_{L^1}^{1/2}.
 \end{aligned}$$

It remains now only to bound I_3 . First suppose that $\|\omega\|_{L^1} \leq \|\omega/r^k\|_{L^\infty}$. Then we can see from the bound (4.8) from Lemma 4.4 that

$$\begin{aligned}
 I_3 &\leq C \iint_{(\bar{r}-1)^2 + \bar{z}^2 \geq 4} \frac{\bar{r}^k |\omega(\bar{r}, \bar{z})|}{((\bar{r}-1)^2 + \bar{z}^2)^{k/2+1/2}} d\bar{r} d\bar{z} \\
 &\leq C \iint_{(\bar{r}-1)^2 + \bar{z}^2 \geq 4} \bar{r}^k |\omega(\bar{r}, \bar{z})| d\bar{r} d\bar{z} \leq C \|\omega\|_{L^1} \leq C \|\omega\|_{L^1}^{1/2} \left\| \frac{\omega}{r^k} \right\|_{L^\infty}^{1/2}.
 \end{aligned}$$

Next suppose that $\|\omega\|_{L^1} > \|\omega/r^k\|_{L^\infty}$. Let $R = 2(\|\omega\|_{L^1} \|\omega/r^k\|_{L^\infty}^{-1})^{1/(2(k+1))}$. Then

$$\begin{aligned}
 I_3 &\leq C \iint_{4 \leq (\bar{r}-1)^2 + \bar{z}^2 \leq R^2} \frac{\bar{r}^k |\omega(\bar{r}, \bar{z})|}{((\bar{r}-1)^2 + \bar{z}^2)^{k/2+1/2}} d\bar{r} d\bar{z} \\
 &\quad + C \iint_{(\bar{r}-1)^2 + \bar{z}^2 \geq R^2} \frac{\bar{r}^k |\omega(\bar{r}, \bar{z})|}{((\bar{r}-1)^2 + \bar{z}^2)^{k/2+1/2}} d\bar{r} d\bar{z} \\
 &\leq C \left\| \frac{\omega}{r^k} \right\|_{L^\infty} \iint_{4 \leq (\bar{r}-1)^2 + \bar{z}^2 \leq R^2} \frac{\bar{r}^{2k}}{((\bar{r}-1)^2 + \bar{z}^2)^{k/2+1/2}} d\bar{r} d\bar{z} \\
 &\quad + \frac{C}{R^{k+1}} \iint_{(\bar{r}-1)^2 + \bar{z}^2 \geq R^2} |\omega(\bar{r}, \bar{z})| \bar{r}^k d\bar{r} d\bar{z} \\
 &\leq CR^{k+1} \left\| \frac{\omega}{r^k} \right\|_{L^\infty} + \frac{C}{R^{k+1}} \|\omega\|_{L^1} = C \|\omega\|_{L^1}^{1/2} \left\| \frac{\omega}{r^k} \right\|_{L^\infty}^{1/2}.
 \end{aligned}$$

We have now shown, by bounding I_1 , I_2 , and I_3 , that

$$|u_r(1, 0)| \leq C_d \|\omega\|_{L^1}^{1/2} \left\| \frac{\omega}{r^k} \right\|_{L^\infty}^{1/2},$$

which completes the proof. Note that the dependence of the constant C_d on the dimension comes from the factor of $(d-2)/(2\pi)$ from the Biot–Savart law, the constants in Lemma 4.4 that are expressed explicitly in terms of d , as well as in the bounds on certain definite integrals (e.g., (4.11)). ■

5. Conditional regularity in four and higher dimensions

In this section, we will prove conditional regularity results in four and higher dimensions when the advected quantity, ω/r^{d-2} , is bounded for the initial data.

Proposition 5.1. *Suppose that $u \in C([0, T_{\max}), H_{\text{df}}^s(\mathbb{R}^d)) \cap C^1([0, T_{\max}), H_{\text{df}}^{s-1}(\mathbb{R}^d))$, with $d \geq 3$ and $s > 2 + d/2$, is an axisymmetric, swirl-free solution of the Euler equation, and assume that $\omega^0/r^k \in L^\infty$. Then for all $R > 0$ and for all $0 \leq t < T_{\max}$,*

$$\|\omega(\cdot, t)\|_{L^\infty} \leq \max(\|\omega^0\|_{L^\infty(\mathcal{C}_R^c)}, R^k \|\omega/r^k\|_{L^\infty(\mathcal{C}_R)}) \left(1 + \frac{1}{R} \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} d\tau\right)^k.$$

Recall that $\mathcal{C}_R$ is the cylinder $\{(r, z) : r < R\}$.

Proof. Again letting $X(r, z, t)$ be the associated flow map, and recalling Proposition 3.18, we let

$$\omega(X(r, z, t), t) = \frac{\omega^0(r, z)}{r^k} X_r(r, z, t)^k.$$

We know that

$$X_r(r, z, t) = r + \int_0^t u_r(X(r, z, \tau), \tau) d\tau,$$

and so we can conclude that for all $0 \leq t < T_{\max}$, $r \in \mathbb{R}^+$, $z \in \mathbb{R}$,

$$X_r(r, z, t) \leq r + \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} d\tau.$$

This clearly implies that

$$|\omega(X(r, z, t), t)| \leq \frac{|\omega^0(r, z)|}{r^k} \left(r + \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} d\tau\right)^k.$$

Now fix $R > 0$. We can clearly see that for all $0 \leq r \leq R$,

$$\begin{aligned} |\omega(X(r, z, t), t)| &\leq \frac{|\omega^0(r, z)|}{r^k} \left(R + \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} d\tau\right)^k \\ &= \frac{|\omega^0(r, z)|}{r^k} R^k \left(1 + \frac{1}{R} \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} d\tau\right)^k. \end{aligned}$$

Likewise, for all $r \geq R$,

$$\begin{aligned} |\omega(X(r, z, t), t)| &\leq |\omega^0(r, z)| \left(1 + \frac{1}{r} \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} d\tau\right)^k \\ &\leq |\omega^0(r, z)| \left(1 + \frac{1}{R} \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} d\tau\right)^k. \end{aligned}$$

Therefore, we can clearly see that for all $0 \leq t < T_{\max}$,

$$\|\omega \circ X(\cdot, t)\|_{L^\infty} \leq \max\left(\|\omega^0\|_{L^\infty(\mathcal{C}_R^c)}, R^k \left\|\frac{\omega}{r^k}\right\|_{L^\infty(\mathcal{C}_R)}\right) \left(1 + \frac{1}{R} \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} d\tau\right)^k.$$

Recalling that $X(\cdot, t)$ is a volume-preserving diffeomorphism, we see that

$$\|\omega \circ X(\cdot, t)\|_{L^\infty} = \|\omega(\cdot, t)\|_{L^\infty},$$

and this completes the proof. $\blacksquare$

Corollary 5.2. *Suppose that $u \in C([0, T_{\max}), H_{\text{df}}^s(\mathbb{R}^d)) \cap C^1([0, T_{\max}), H_{\text{df}}^{s-1}(\mathbb{R}^d))$, with $d \geq 4$ and $s > 2 + d/2$, is an axisymmetric, swirl-free solution of the Euler equation, and assume that $\omega^0/r^k \in L^\infty$. If $T_{\max} < +\infty$, then*

$$\int_0^{T_{\max}} \|u_r^+(\cdot, t)\|_{L^\infty} dt = +\infty.$$

Proof. This follows from Proposition 5.1 and the Beale–Kato–Majda theorem. $\blacksquare$

Proposition 5.3. *Suppose that $u \in C([0, T_{\max}), H_{\text{df}}^s(\mathbb{R}^d)) \cap C^1([0, T_{\max}), H_{\text{df}}^{s-1}(\mathbb{R}^d))$, with $d \geq 3$ and $s > 2 + d/2$, is an axisymmetric, swirl-free solution of the Euler equation, and assume that $\omega^0/r^k \in L^\infty$. Then for all $1 \leq q < +\infty$, for all $R > 0$, and for all $0 \leq t < T_{\max}$,*

$$\|\omega(\cdot, t)\|_{L^q} \leq \left(\|\omega^0\|_{L^q(\mathcal{C}_R^c)}^q + R^{kq} \left\| \frac{\omega}{r^k} \right\|_{L^q(\mathcal{C}_R)}^q \right)^{1/q} \left(1 + \frac{1}{R} \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} d\tau \right)^k.$$

Proof. Begin by fixing $R > 0$. As we have already shown above, for all $0 \leq r \leq R$,

$$\begin{aligned} |\omega(X(r, z, t), t)| &\leq \frac{|\omega^0(r, z)|}{r^k} \left(R + \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} d\tau \right)^k \\ &= \frac{|\omega^0(r, z)|}{r^k} R^k \left(1 + \frac{1}{R} \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} d\tau \right)^k, \end{aligned}$$

and likewise, for all $r \geq R$,

$$\begin{aligned} |\omega(X(r, z, t), t)| &\leq |\omega^0(r, z)| \left(1 + \frac{1}{r} \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} d\tau \right)^k \\ &\leq |\omega^0(r, z)| \left(1 + \frac{1}{R} \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} d\tau \right)^k. \end{aligned}$$

Therefore, we can clearly see that for all $0 \leq t < T_{\max}$,

$$\|\omega \circ X(\cdot, t)\|_{L^q(\mathcal{C}_R)} \leq \left\| \frac{\omega}{r^k} \right\|_{L^q(\mathcal{C}_R)} R^k \left(1 + \frac{1}{R} \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} d\tau \right)^k,$$

and that

$$\|\omega \circ X(\cdot, t)\|_{L^q(\mathcal{C}_R^c)} \leq \|\omega^0\|_{L^q(\mathcal{C}_R^c)} \left(1 + \frac{1}{R} \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} d\tau \right)^k.$$

We can compute that

$$\begin{aligned} \|\omega \circ X(\cdot, t)\|_{L^q(\mathbb{R}^d)}^q &= \|\omega \circ X(\cdot, t)\|_{L^q(\mathcal{C}_R)}^q + \|\omega \circ X(\cdot, t)\|_{L^q(\mathcal{C}_R^c)}^q \\ &\leq \left(\left\| \frac{\omega}{r^k} \right\|_{L^q(\mathcal{C}_R)}^{kq} + \|\omega^0\|_{L^q(\mathcal{C}_R^c)}^q \right) \left(1 + \frac{1}{R} \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} d\tau \right)^{kq}. \end{aligned}$$

Recalling that $X(\cdot, t)$ is a volume-preserving diffeomorphism, we see that

$$\|\omega \circ X(\cdot, t)\|_{L^q} = \|\omega(\cdot, t)\|_{L^q},$$

and this completes the proof. $\blacksquare$

Proposition 5.4. *Suppose that $u \in C([0, T_{\max}), H_{\text{df}}^s(\mathbb{R}^d)) \cap C^1([0, T_{\max}), H_{\text{df}}^{s-1}(\mathbb{R}^d))$, with $d \geq 3$, $s > 2 + d/2$, is an axisymmetric, swirl-free solution of the Euler equation, and assume that $\omega^0/r^k \in L^1 \cap L^\infty$. Then for all $R > 0$ and for all $0 \leq t < T_{\max}$,*

$$\begin{aligned} \|u_r^+(\cdot, t)\|_{L^\infty} &\leq C_d \left\| \frac{\omega}{r^k} \right\|_{L^\infty}^{1/2} \left(\|\omega^0\|_{L^1(\mathcal{C}_R^c)} + R^k \left\| \frac{\omega}{r^k} \right\|_{L^1(\mathcal{C}_R)} \right)^{1/2} \\ &\quad \times \left(1 + \frac{1}{R} \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} d\tau \right)^{k/2}. \end{aligned}$$

Furthermore, letting

$$f(t) = 1 + \frac{1}{R} \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} d\tau,$$

we can conclude that for all $0 \leq t < T_{\max}$,

$$\frac{df}{dt} \leq \mu f^{k/2},$$

where

$$\mu = \frac{C_d}{R} \left\| \frac{\omega}{r^k} \right\|_{L^\infty}^{1/2} \left(\|\omega^0\|_{L^1(\mathcal{C}_R^c)} + R^k \left\| \frac{\omega}{r^k} \right\|_{L^1(\mathcal{C}_R)} \right)^{1/2}.$$

Proof. Fix $R > 0$. Applying Propositions 4.5 and 5.3, we find that

$$\begin{aligned} \|u_r^+(\cdot, t)\|_{L^\infty} &\leq C_d \left\| \frac{\omega(\cdot, t)}{r^k} \right\|_{L^\infty}^{1/2} \|\omega(\cdot, t)\|_{L^1}^{1/2} \\ &\leq \left\| \frac{\omega}{r^k} \right\|_{L^\infty}^{1/2} \left(\|\omega^0\|_{L^1(\mathcal{C}_R^c)} + R^k \left\| \frac{\omega}{r^k} \right\|_{L^1(\mathcal{C}_R)} \right)^{1/2} \left(1 + \frac{1}{R} \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} d\tau \right)^{k/2}. \end{aligned}$$

Finally, we can use the fundamental theorem of calculus to compute that

$$\frac{df}{dt} = \frac{1}{R} \|u_r^+(\cdot, t)\|_{L^\infty} \leq \mu f^{k/2}.$$

This completes the proof. $\blacksquare$

Theorem 5.5. *Suppose that the initial data $u^0 \in H_{\text{df}}^s(\mathbb{R}^4)$, $s > 4$, is axisymmetric and swirl-free, and assume that $\omega^0/r^2 \in L^1 \cap L^\infty$. Then there exists a global smooth solution of the Euler equation $u \in C([0, +\infty), H_{\text{df}}^s(\mathbb{R}^4)) \cap C^1([0, +\infty), H_{\text{df}}^{s-1}(\mathbb{R}^4))$. Furthermore, we have a bound on vorticity, with for all $R > 0$ and for all $0 \leq t < +\infty$,*

$$(5.1) \quad \|\omega(\cdot, t)\|_{L^\infty} \leq \max \left(\|\omega^0\|_{L^\infty(\mathcal{C}_R^c)}, R^2 \left\| \frac{\omega^0}{r^2} \right\|_{L^\infty(\mathcal{C}_R)} \right) \exp(2\mu t),$$

where

$$\mu = \frac{C_d}{R} \left\| \frac{\omega}{r^k} \right\|_{L^\infty}^{1/2} \left(\|\omega^0\|_{L^1(\mathcal{C}_R^c)} + R^2 \left\| \frac{\omega^0}{r^2} \right\|_{L^1(\mathcal{C}_R)} \right)^{1/2}.$$

Proof. We know that such a solution must exist locally-in-time up until some maximal time $T_{\max}$. By the Beale–Kato–Majda criterion, we know that if the bound (5.1) holds for all times $0 \leq t < T_{\max}$, then we must have $T_{\max} = +\infty$, and therefore it suffices to prove this bound holds up until the maximal time of existence.

We will begin by fixing $R > 0$ and letting

$$f(t) = 1 + \frac{1}{R} \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} d\tau.$$

We can see from Proposition 5.1 that

$$\|\omega(\cdot, t)\|_{L^\infty} \leq \max\left(\|\omega^0\|_{L^\infty(\mathcal{C}_R^c)}, R^2 \left\|\frac{\omega^0}{r^2}\right\|_{L^\infty(\mathcal{C}_R)}\right) f(t)^2.$$

We know from Proposition 4.5 that

$$\frac{df}{dt} \leq \mu f,$$

and we can see by definition that $f(0) = 1$. Applying Grönwall's inequality, we find that for all $0 \leq t < T_{\max}$,

$$f(t) \leq \exp(\mu t),$$

and consequently

$$\|\omega(\cdot, t)\|_{L^\infty} \leq \max\left(\|\omega^0\|_{L^\infty(\mathcal{C}_R^c)}, R^2 \left\|\frac{\omega^0}{r^2}\right\|_{L^\infty(\mathcal{C}_R)}\right) \exp(2\mu t).$$

This completes the proof. ■

Theorem 5.6. *If the initial data $u^0 \in H_{\text{df}}^s(\mathbb{R}^3)$, $s > 7/2$, is axisymmetric and swirl-free, then there exists a global smooth solution of the Euler equation $u \in C([0, +\infty), H_{\text{df}}^s(\mathbb{R}^3)) \cap C^1([0, +\infty), H_{\text{df}}^{s-1}(\mathbb{R}^3))$. Furthermore, we have a bound on vorticity, with for all $R > 0$ and for all $0 \leq t < +\infty$,*

$$(5.2) \quad \|\omega(\cdot, t)\|_{L^\infty} \leq \max\left(\|\omega^0\|_{L^\infty(\mathcal{C}_R^c)}, R \left\|\frac{\omega^0}{r}\right\|_{L^\infty(\mathcal{C}_R)}\right) \left(1 + \frac{1}{2}\mu t\right)^2,$$

where

$$\mu = \frac{C_d}{R} \left\|\frac{\omega^0}{r}\right\|_{L^\infty}^{1/2} \left(\|\omega^0\|_{L^1(\mathcal{C}_R^c)} + R \left\|\frac{\omega^0}{r}\right\|_{L^1(\mathcal{C}_R)}\right)^{1/2}.$$

Proof. We know that such a solution must exist locally-in-time up until some maximal time $T_{\max}$. By the Beale–Kato–Majda criterion, we know that if the bound (5.2) holds for all times $0 \leq t < T_{\max}$, then we must have $T_{\max} = +\infty$, and therefore it suffices to prove this bound holds up until the maximal time of existence.

We will begin by fixing $R > 0$ and letting

$$f(t) = 1 + \frac{1}{R} \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} d\tau.$$

We can see from Proposition 5.1 that

$$\|\omega(\cdot, t)\|_{L^\infty} \leq \max\left(\|\omega^0\|_{L^\infty(\mathcal{C}_R^c)}, R \left\|\frac{\omega^0}{r}\right\|_{L^\infty(\mathcal{C}_R)}\right) f(t).$$

We know from Proposition 4.5 that

$$\frac{df}{dt} \leq \mu f^{1/2},$$

and we can see by definition that $f(0) = 1$. Integrating this differential inequality, we find that for all $0 \leq t < T_{\max}$,

$$f(t) \leq \left(1 + \frac{1}{2}\mu t\right)^2,$$

and this completes the proof. $\blacksquare$

Theorem 5.7. Suppose that $u \in C([0, T_{\max}), H_{\text{df}}^s(\mathbb{R}^d)) \cap C^1([0, T_{\max}), H_{\text{df}}^{s-1}(\mathbb{R}^d))$, with $d \geq 5$ and $s > 2 + d/2$, is an axisymmetric, swirl-free solution of the Euler equation, and assume that $\omega^0/r^k \in L^1 \cap L^\infty$. Then for all $R > 0$ and for all $0 \leq t < T_{\max}$,

$$(5.3) \quad \|\omega(\cdot, t)\|_{L^\infty} \leq \frac{\max(\|\omega^0\|_{L^\infty}, R^k \|\omega/r^k\|_{L^\infty})}{(1 - \alpha t)^{\frac{2(d-2)}{d-4}}},$$

where

$$\alpha = \frac{(d-4)C_d}{2R} \left\| \frac{\omega}{r^k} \right\|_{L^\infty}^{1/2} \left(\|\omega^0\|_{L^1(\mathcal{C}_R^c)} + R^k \left\| \frac{\omega}{r^k} \right\|_{L^1(\mathcal{C}_R)} \right)^{1/2}.$$

In particular, this implies that

$$T_{\max} \geq \frac{2R}{(d-4)C_d \|\omega/r^k\|_{L^\infty}^{1/2} (\|\omega^0\|_{L^1(\mathcal{C}_R^c)} + R^k \|\omega/r^k\|_{L^1(\mathcal{C}_R)})^{1/2}}.$$

Proof. We know from the Beale–Kato–Majda theorem that if $T_{\max} < +\infty$, then

$$\int_0^{T_{\max}} \|\omega(\cdot, t)\|_{L^\infty} dt = +\infty,$$

and so it suffices to prove the bound (5.3), which immediately then implies the lower bound on $T_{\max}$. We will begin by defining f as in Proposition 5.4, letting

$$f(t) = 1 + \frac{1}{R} \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} d\tau.$$

Applying Proposition 5.1, we have for all $0 \leq t < T_{\max}$,

$$\|\omega(\cdot, t)\|_{L^\infty} \leq \max\left(\|\omega^0\|_{L^\infty}, R^k \left\| \frac{\omega}{r^k} \right\|_{L^\infty}\right) f(t)^k,$$

therefore it suffices to show that for all $0 \leq t < T_{\max}$,

$$f(t)^k \leq \frac{1}{(1 - \alpha t)^{\frac{2(d-2)}{d-4}}}.$$

Observing that $\alpha = \frac{d-4}{2}\mu$, and applying Proposition 5.4, we find that

$$\frac{df}{dt} \leq \frac{\alpha}{k/2 - 1} f^{k/2},$$

and that consequently

$$\frac{d}{dt}(f^{1-k/2}) \geq -\alpha.$$

Integrating this differential inequality and using the fact that $f(0) = 1$ by definition, we find that for all $0 \leq t < T_{\max}$,

$$f(t)^{1-k/2} \geq 1 - \alpha t,$$

and therefore

$$f(t)^{\frac{1}{2}(d-4)} \leq \frac{1}{1 - \alpha t}.$$

Finally, we observe that for all $0 \leq t < T_{\max}$,

$$f(t) \leq \frac{1}{(1 - \alpha t)^{2/(d-4)}} \quad \text{and} \quad f(t)^k \leq \frac{1}{(1 - \alpha t)^{2(d-2)/(d-4)}}.$$

This completes the proof. ■

Corollary 5.8. *Suppose that $u \in C([0, T_{\max}), H_{\text{df}}^s(\mathbb{R}^d)) \cap C^1([0, T_{\max}), H_{\text{df}}^{s-1}(\mathbb{R}^d))$, with $d \geq 5$ and $s > 2 + d/2$, is an axisymmetric, swirl-free solution of the Euler equation, with finite-time blowup at $T_{\max} < +\infty$, and assume that $\omega^0/r^k \in L^1 \cap L^\infty$. Then for all $R > 0$ and all $0 \leq t < T_{\max}$,*

$$\|\omega(\cdot, t)\|_{L^1(\mathcal{C}_R^c)} + R^k \left\| \frac{\omega}{r^k} \right\|_{L^1(\mathbb{R}^d)} \geq \frac{4R^2}{(d-4)^2 C_d^2 \|\omega/r^k\|_{L^\infty(\mathbb{R}^d)} (T_{\max} - t)^2}.$$

Note, in particular, that this implies that

$$\lim_{t \rightarrow T_{\max}} (T_{\max} - t)^2 \|\omega(\cdot, t)\|_{L^1(\mathcal{C}_R^c)} \geq \frac{4R^2}{(d-4)^2 C_d^2 \|\omega/r^k\|_{L^\infty(\mathbb{R}^d)}}.$$

Proof. Fix $R > 0$. Applying Theorem 5.7 to our solution at time t , we can see that for all $0 \leq t < T_{\max}$,

$$T_{\max} - t \geq \frac{2R}{(d-4)C_d \left\| \frac{\omega(\cdot, t)}{r^k} \right\|_{L^\infty}^{1/2} (\|\omega(\cdot, t)\|_{L^1(\mathcal{C}_R^c)} + R^k \|\omega(\cdot, t)/r^k\|_{L^1(\mathcal{C}_R)})^{1/2}}.$$

Observe that for all $0 \leq t < T_{\max}$,

$$\left\| \frac{\omega(\cdot, t)}{r^k} \right\|_{L^\infty} = \left\| \frac{\omega}{r^k} \right\|_{L^\infty},$$

and likewise

$$\left\| \frac{\omega(\cdot, t)}{r^k} \right\|_{L^1(\mathcal{C}_R)} \leq \left\| \frac{\omega(\cdot, t)}{r^k} \right\|_{L^1(\mathbb{R}^d)} = \left\| \frac{\omega}{r^k} \right\|_{L^1(\mathbb{R}^d)}.$$

Therefore, we may conclude that for all $0 \leq t < T_{\max}$,

$$T_{\max} - t \geq \frac{2R}{(d-4)C_d \|\omega/r^k\|_{L^\infty}^{1/2} (\|\omega(\cdot, t)\|_{L^1(\mathcal{C}_R^c)} + R^k \|\omega/r^k\|_{L^1})^{1/2}}.$$

Rearranging this inequality, we find that for all $0 \leq t < T_{\max}$,

$$\|\omega(\cdot, t)\|_{L^1(\mathcal{C}_R^c)} + R^k \left\| \frac{\omega}{r^k} \right\|_{L^1} \geq \frac{4R^2}{(d-4)^2 C_d^2 \|\omega/r^k\|_{L^\infty} (T_{\max} - t)^2},$$

and this completes the proof. ■

Corollary 5.9. *Suppose that $u \in C([0, T_{\max}), H_{\text{df}}^s(\mathbb{R}^d)) \cap C^1([0, T_{\max}), H_{\text{df}}^{s-1}(\mathbb{R}^d))$, with $d \geq 5$ and $s > 2 + d/2$, is an axisymmetric, swirl-free solution of the Euler equation, with finite-time blowup at $T_{\max} < +\infty$, and assume that $\omega^0/r^k \in L^1 \cap L^\infty$. Then for all $0 \leq t < T_{\max}$,*

$$\|\omega(\cdot, t)\|_{L^1(\mathbb{R}^d)} \geq \left(\frac{M_d}{\|\omega/r^k\|_{L^\infty}^{(d-2)/(d-4)} \|\omega/r^k\|_{L^1}^{2/(d-4)}} \right) \frac{1}{(T_{\max} - t)^{2\frac{d-2}{d-4}}},$$

where

$$M_d = \sup_{M>0} \left(\frac{4M^2}{(d-4)^2 C_d^2} - M^k \right).$$

Proof. For all $R > 0$, we can see from Corollary 5.8 that

$$\begin{aligned} \|\omega(\cdot, t)\|_{L^1(\mathbb{R}^d)} &\geq \|\omega(\cdot, t)\|_{L^1(\mathcal{C}_R^c)} \\ &\geq \frac{4R^2}{(d-4)^2 C_d^2 \|\omega/r^k\|_{L^\infty}(T_{\max} - t)^2} - R^k \left\| \frac{\omega}{r^k} \right\|_{L^1}. \end{aligned}$$

Now we will introduce the parameter

$$M = \frac{R(T_{\max} - t)^{2/(d-4)}}{\|\omega/r^k\|_{L^\infty}^{1/(k-2)} \|\omega/r^k\|_{L^1}^{1/(k-2)}}.$$

Substituting in M , we can see that for all $M > 0$ and for all $0 \leq t < T_{\max}$,

$$\begin{aligned} (5.4) \quad &\|\omega(\cdot, t)\|_{L^1(\mathbb{R}^d)} \\ &\geq \left(\frac{4M^2}{(d-4)^2 C_d^2} - M^k \right) \left(\frac{1}{\|\omega/r^k\|_{L^\infty}^{k/(k-2)} \|\omega/r^k\|_{L^1}^{2/(k-2)}} \right) \frac{1}{(T_{\max} - t)^{2\frac{d-2}{d-4}}}. \end{aligned}$$

We know that $k \geq 3$, and so we can observe that

$$0 < \sup_{M>0} \left(\frac{4M^2}{(d-4)^2 C_d^2} - M^k \right) < +\infty,$$

and so by taking the supremum over $M > 0$ of (5.4), we complete the proof. $\blacksquare$

6. Anti-parallel vortex tubes

In this section, we will discuss the special case of colliding anti-parallel vortex tubes. This scenario will involve vorticities which are odd with respect to z and positive for $z > 0$. We will begin by deriving a special form of the Biot–Savart law that applies to vorticities which are odd with respect to z .

Proposition 6.1. *Suppose that $u \in H_{\text{df}}^s(\mathbb{R}^d)$, with $s > d/2$, $d \geq 3$, is axisymmetric and swirl-free, and assume that $\omega(r, z)$ is odd in z , with*

$$\omega(r, -z) = -\omega(r, z), \quad \text{for all } r \geq 0, z \in \mathbb{R}.$$

Then the velocity can be recovered from the vorticity in the upper half plane using the following formulas:

$$(6.1) \quad u_r(r, z) = \frac{d-2}{2\pi} \int_0^\infty \int_0^\infty H(r, z, \bar{r}, \bar{z}) \bar{r}^{d-2} \omega(\bar{r}, \bar{z}) d\bar{r} d\bar{z},$$

where

$$\begin{aligned} H(r, z, \bar{r}, \bar{z}) = & \int_0^1 \left(\frac{z + \bar{z}}{(r^2 + \bar{r}^2 - 2r\bar{r}\tau + (z + \bar{z})^2)^{d/2}} + \frac{z - \bar{z}}{(r^2 + \bar{r}^2 + 2r\bar{r}\tau + (z - \bar{z})^2)^{d/2}} \right. \\ & - \frac{z + \bar{z}}{(r^2 + \bar{r}^2 + 2r\bar{r}\tau + (z + \bar{z})^2)^{d/2}} \\ & \left. - \frac{z - \bar{z}}{(r^2 + \bar{r}^2 - 2r\bar{r}\tau + (z - \bar{z})^2)^{d/2}} \right) \tau (1 - \tau^2)^{(d-4)/2} d\tau, \end{aligned}$$

and

$$(6.2) \quad u_z(r, z) = -\frac{d-2}{2\pi} \int_0^\infty \int_0^\infty G(r, z, \bar{r}, \bar{z}) \bar{r}^{d-2} \omega(\bar{r}, \bar{z}) d\bar{r} d\bar{z},$$

where

$$\begin{aligned} G(r, z, \bar{r}, \bar{z}) = & \int_0^1 \left(\frac{\bar{r} - r\tau}{(r^2 + \bar{r}^2 - 2r\bar{r}\tau + (z - \bar{z})^2)^{d/2}} + \frac{\bar{r} + r\tau}{(r^2 + \bar{r}^2 + 2r\bar{r}\tau + (z - \bar{z})^2)^{d/2}} \right. \\ & - \frac{\bar{r} - r\tau}{(r^2 + \bar{r}^2 - 2r\bar{r}\tau + (z + \bar{z})^2)^{d/2}} \\ & \left. - \frac{\bar{r} + r\tau}{(r^2 + \bar{r}^2 + 2r\bar{r}\tau + (z + \bar{z})^2)^{d/2}} \right) (1 - \tau^2)^{(d-4)/2} d\tau. \end{aligned}$$

Further note that $u_r(r, z)$ is even in z , while $u_z(r, z)$ is odd in z , and for all $r, z > 0$,

$$u_r(0, z) = 0, \quad u_z(r, 0) = 0.$$

Proof. We begin by recalling the Biot–Savart law in Proposition 4.3:

$$\begin{aligned} u_r(r, z) = & -\frac{d-2}{2\pi} \int_{-\infty}^\infty \int_0^\infty \bar{r}^{d-2} (z - \bar{z}) \omega(\bar{r}, \bar{z}) \\ & \times \int_{-1}^1 \frac{\tilde{y}_1(1 - \tilde{y}_1^2)^{(d-4)/2}}{(r^2 + \bar{r}^2 - 2r\bar{r}\tilde{y}_1 + (z - \bar{z})^2)^{d/2}} d\tilde{y}_1 d\bar{r} d\bar{z}. \end{aligned}$$

Using the odd symmetry of ω in $\bar{z}$, and the change of variables $\tilde{z} = -\bar{z}$,

$$\begin{aligned} & - \int_{-\infty}^0 \int_0^\infty \bar{r}^{d-2} (z - \bar{z}) \omega(\bar{r}, \bar{z}) \int_{-1}^1 \frac{\tilde{y}_1(1 - \tilde{y}_1^2)^{(d-4)/2}}{(r^2 + \bar{r}^2 - 2r\bar{r}\tilde{y}_1 + (z - \bar{z})^2)^{d/2}} d\tilde{y}_1 d\bar{r} d\bar{z} \\ & = \int_0^\infty \int_0^\infty \bar{r}^{d-2} (z + \tilde{z}) \omega(\bar{r}, \tilde{z}) \int_{-1}^1 \frac{\tilde{y}_1(1 - \tilde{y}_1^2)^{(d-4)/2}}{(r^2 + \bar{r}^2 - 2r\bar{r}\tilde{y}_1 + (z + \tilde{z})^2)^{d/2}} d\tilde{y}_1 d\bar{r} d\tilde{z}. \end{aligned}$$

Therefore, the identity for u_r can be rewritten as

$$u_r(r, z) = \frac{d-2}{2\pi} \int_0^\infty \int_0^\infty \bar{r}^{d-2} \omega(\bar{r}, \bar{z}) \\ \times \int_{-1}^1 \left(\frac{(z+\bar{z})\tilde{y}_1(1-\tilde{y}_1^2)^{(d-4)/2}}{(r^2+\bar{r}^2-2r\bar{r}\tilde{y}_1+(z+\bar{z})^2)^{d/2}} - \frac{(z-\bar{z})\tilde{y}_1(1-\tilde{y}_1^2)^{(d-4)/2}}{(r^2+\bar{r}^2-2r\bar{r}\tilde{y}_1+(z-\bar{z})^2)^{d/2}} \right) d\tilde{y}_1 d\bar{r} d\bar{z}.$$

Making the substitution $\tau = -\tilde{y}_1$, for $-1 \leq \tilde{y} \leq 0$, we can see that

$$\int_{-1}^0 \left(\frac{(z+\bar{z})\tilde{y}_1(1-\tilde{y}_1^2)^{(d-4)/2}}{(r^2+\bar{r}^2-2r\bar{r}\tilde{y}_1+(z+\bar{z})^2)^{d/2}} - \frac{(z-\bar{z})\tilde{y}_1(1-\tilde{y}_1^2)^{(d-4)/2}}{(r^2+\bar{r}^2-2r\bar{r}\tilde{y}_1+(z-\bar{z})^2)^{d/2}} \right) d\tilde{y}_1 \\ = \int_0^1 \left(-\frac{(z+\bar{z})\tau(1-\tau^2)^{(d-2)/4}}{(r^2+\bar{r}^2+2r\bar{r}\tau+(z+\bar{z})^2)^{d/2}} + \frac{(z-\bar{z})\tau(1-\tau^2)^{(d-2)/4}}{(r^2+\bar{r}^2+2r\bar{r}\tau+(z-\bar{z})^2)^{d/2}} \right) d\tau.$$

By making the trivial substitution $\tau = \tilde{y}$ for $0 \leq \tilde{y} \leq 1$, we complete the proof of the identity (6.1).

We proceed similarly for u_z , starting with the identity from Proposition 4.3,

$$u_z(r, z) = \frac{d-2}{2\pi} \int_{-\infty}^\infty \int_0^\infty \bar{r}^{d-2} \omega(\bar{r}, \bar{z}) \int_{-1}^1 \frac{(r\tilde{y}_1 - \bar{r})(1-\tilde{y}_1^2)^{(d-4)/2}}{(r^2+\bar{r}^2-2r\bar{r}\tilde{y}_1+(z-\bar{z})^2)^{d/2}} d\tilde{y}_1 d\bar{r} d\bar{z}.$$

Again applying the odd symmetry of ω in z , we find that

$$u_z(r, z) = -\frac{d-2}{2\pi} \int_0^\infty \int_0^\infty \bar{r}^{d-2} \omega(\bar{r}, \bar{z}) \\ \times \int_{-1}^1 \left(\frac{(\bar{r}-r\tilde{y}_1)(1-\tilde{y}_1^2)^{(d-4)/2}}{(r^2+\bar{r}^2-2r\bar{r}\tilde{y}_1+(z-\bar{z})^2)^{d/2}} - \frac{(\bar{r}-r\tilde{y}_1)(1-\tilde{y}_1^2)^{(d-4)/2}}{(r^2+\bar{r}^2-2r\bar{r}\tilde{y}_1+(z+\bar{z})^2)^{d/2}} \right) d\tilde{y}_1 d\bar{r} d\bar{z}.$$

Again taking the substitution $\tau = -\tilde{y}_1$, we find that

$$\int_{-1}^0 \left(\frac{(\bar{r}-r\tilde{y}_1)(1-\tilde{y}_1^2)^{(d-4)/2}}{(r^2+\bar{r}^2-2r\bar{r}\tilde{y}_1+(z-\bar{z})^2)^{d/2}} - \frac{(\bar{r}-r\tilde{y}_1)(1-\tilde{y}_1^2)^{(d-4)/2}}{(r^2+\bar{r}^2-2r\bar{r}\tilde{y}_1+(z+\bar{z})^2)^{d/2}} \right) d\tilde{y}_1 \\ = \int_0^1 \left(\frac{(\bar{r}+r\tau)(1-\tau^2)^{(d-4)/2}}{(r^2+\bar{r}^2+2r\bar{r}\tau+(z-\bar{z})^2)^{d/2}} - \frac{(\bar{r}+r\tau)(1-\tau^2)^{(d-4)/2}}{(r^2+\bar{r}^2+2r\bar{r}\tau+(z+\bar{z})^2)^{d/2}} \right) d\tau.$$

By making the trivial substitution $\tau = \tilde{y}_1$ for $0 \leq \tilde{y}_1 \leq 1$, we complete the proof of the identity (6.2).

Finally, we observe that

$$H(r, -z, \bar{r}, \bar{z}) = H(r, z, \bar{r}, \bar{z}),$$

which implies that

$$u_r(r, z) = u_r(r, -z).$$

Likewise, we can observe that

$$G(r, -z, \bar{r}, \bar{z}) = -G(r, z, \bar{r}, \bar{z}),$$

which implies that $u_z(r, -z) = -u_z(r, z)$. This completes the proof. ■

It is clear, from the analysis in Section 5, that blowup for solutions of the axisymmetric, swirl-free Euler equation in four and higher dimensions requires compression along the z -axis and stretching in the r -hyperplane. We will show that this is precisely what is afforded to us by vorticities which are odd in z , and non-negative for $z > 0$, giving hyperbolic flow in the neighbourhood of a stagnation point at the origin.

Proposition 6.2. *Suppose that $u \in H_{\text{df}}^s(\mathbb{R}^d)$, with $s > 1 + d/2$ and $d \geq 3$, is axisymmetric and swirl-free. Further suppose that ω is odd in z , that for all $r, z > 0$, $\omega(r, z) \geq 0$, and that ω is not identically zero. Then for all $z > 0$,*

$$u_z(0, z) < 0,$$

and for all $r > 0$,

$$u_r(r, 0) > 0.$$

Proof. Taking the identity (6.2) from Proposition 6.1, we see that

$$\begin{aligned} u_z(0, z) = & -\frac{d-2}{\pi} \int_0^\infty \int_0^\infty \bar{r}^{d-1} \omega(\bar{r}, \bar{z}) \int_0^1 (1-\tau^2)^{(d-4)/2} \\ & \times \left(\frac{1}{(\bar{r}^2 + (z-\bar{z})^2)^{d/2}} - \frac{1}{(\bar{r}^2 + (z+\bar{z})^2)^{d/2}} \right) d\tau d\bar{r} d\bar{z}. \end{aligned}$$

Observe that for all $\bar{r}, \bar{z}, z > 0$,

$$\int_0^1 (1-\tau^2)^{(d-4)/2} \left(\frac{1}{(\bar{r}^2 + (z-\bar{z})^2)^{d/2}} - \frac{1}{(\bar{r}^2 + (z+\bar{z})^2)^{d/2}} \right) d\tau > 0.$$

This implies that for all $z > 0$,

$$u_z(0, z) < 0.$$

Likewise, applying the identity (6.1) from Proposition 6.1, we can see that for all $r > 0$,

$$\begin{aligned} u_r(r, 0) = & \frac{d-2}{\pi} \int_0^\infty \int_0^\infty \bar{z} \bar{r}^{d-2} \omega(\bar{r}, \bar{z}) \int_0^1 \tau (1-\tau^2)^{(d-4)/2} \\ & \times \left(\frac{1}{(r^2 + \bar{r}^2 - 2r\bar{r}\tau + \bar{z}^2)^{d/2}} - \frac{1}{(r^2 + \bar{r}^2 + 2r\bar{r}\tau + \bar{z}^2)^{d/2}} \right) d\tau d\bar{r} d\bar{z}. \end{aligned}$$

Observe that for all $\bar{r}, \bar{z}, r > 0$,

$$\int_0^1 \tau (1-\tau^2)^{(d-4)/2} \left(\frac{1}{(r^2 + \bar{r}^2 - 2r\bar{r}\tau + \bar{z}^2)^{d/2}} - \frac{1}{(r^2 + \bar{r}^2 + 2r\bar{r}\tau + \bar{z}^2)^{d/2}} \right) d\tau > 0,$$

and so for all $r > 0$,

$$u_r(r, 0) > 0. \quad \blacksquare$$

Proposition 6.3. *Suppose $u \in C([0, T_{\max}); H_*^s(\mathbb{R}^d)) \cap C^1([0, T_{\max}), H_*^{s-1}(\mathbb{R}^d))$, with $d \geq 3$, $s > 2 + d/2$, is a solution of the Euler equation. Then the velocity $\tilde{u} \in C([0, T_{\max}); H_*^s(\mathbb{R}^d)) \cap C^1([0, T_{\max}), H_*^{s-1}(\mathbb{R}^d))$ is also a solution of the Euler equation, where*

$$\begin{aligned} \tilde{u}_r(r, z, t) &= u_r(r, -z, t), \\ \tilde{u}_z(r, z, t) &= -u_z(r, -z, t), \\ \tilde{\omega}(r, z, t) &= -\omega(r, -z, t). \end{aligned}$$

Proof. First, we observe that the new velocity field is divergence free, with

$$\begin{aligned}\nabla \cdot \tilde{u}(x) &= \partial_r \tilde{u}_r(r, z, t) + \partial_z \tilde{u}_z(r, z) + \frac{k}{r} \tilde{u}_r(r, z, t) \\ &= \left(\partial_r u_r + \partial_z u_z + \frac{k}{r} u_r \right)(r, -z, t) = 0.\end{aligned}$$

Likewise, taking the scalar curl, we confirm that

$$\partial_r \tilde{u}_z(r, z) - \partial_z \tilde{u}_r(r, z) = -\partial_r u_z(r, -z) + \partial_z u_r(r, -z) = -\omega(r, -z) = \tilde{\omega}(r, z),$$

and so we have confirmed $\tilde{\omega}$ is the scalar vorticity associated with $\tilde{u}$. Finally, we compute

$$\begin{aligned}\left(\partial_t \tilde{\omega} + \tilde{u}_r \partial_r \omega + \tilde{u}_z \partial_z \omega + \frac{k}{r} \tilde{u}_r \tilde{\omega} \right)(r, z, t) \\ = \left(-\partial_t \tilde{\omega} - \tilde{u}_r \partial_r \omega - \tilde{u}_z \partial_z \omega - \frac{k}{r} \tilde{u}_r \tilde{\omega} \right)(r, -z, t) = 0.\end{aligned}\quad \blacksquare$$

Theorem 6.4. Suppose that $u \in C([0, T_{\max}); H_*^s(\mathbb{R}^d)) \cap C^1([0, T_{\max}), H_*^{s-1}(\mathbb{R}^d))$, with $d \geq 3$ and $s > 2 + d/2$, is a solution of the Euler equation, and assume that $\omega^0(r, z)$ is odd in z . Then for all $0 \leq t < T_{\max}$, the vorticity $\omega(r, z, t)$ is odd in z and

$$\int_0^\infty \int_0^\infty \omega(r, z, t) \, dr \, dz = \int_0^\infty \int_0^\infty \omega^0(r, z) \, dr \, dz.$$

Furthermore, if for all $r, z > 0$,

$$\omega^0(r, z) \geq 0,$$

then for all $0 \leq t < T_{\max}$, $r, z > 0$,

$$\omega(r, z, t) \geq 0.$$

Proof. We begin by defining $\tilde{u}$ and $\tilde{\omega}$ as in Proposition 6.3. We know that $\tilde{u}$ must also be a solution of the Euler equation and because $\omega^0(r, z) = -\omega^0(r, -z)$, we can see that $\omega^0 = \tilde{\omega}^0$. By uniqueness, this implies that for all $0 \leq t < T_{\max}$,

$$\omega(r, z, t) = \tilde{\omega}(r, z, t) = -\omega(r, -z, t),$$

and so we can conclude that ω being odd in z is preserved by the dynamics. It then follows from Proposition 6.1 that $u_r(r, z)$ is even in z and $u_z(r, z)$ is odd in z .

To prove the conservation of vorticity mass, we will consider the divergence form of the vorticity equation, $\partial_t \omega + \partial_r(\omega u_r) + \partial_z(\omega u_z) = 0$, computing that

$$\begin{aligned}\frac{d}{dt} \int_0^\infty \int_0^\infty \omega(r, z, t) \, dr \, dz &= - \int_0^\infty \int_0^\infty \partial_r(\omega(r, z, t) u_r(r, z, t)) \, dr \, dz \\ &\quad - \int_0^\infty \int_0^\infty \partial_z(\omega(r, z, t) u_z(r, z, t)) \, dz \, dr,\end{aligned}$$

and so

$$\begin{aligned}\frac{d}{dt} \int_0^\infty \int_0^\infty \omega(r, z, t) \, dr \, dz \\ = \int_0^\infty \omega(0, z, t) u_r(0, z, t) \, dz + \int_0^\infty \omega(r, 0, t) u_z(r, 0, t) \, dr = 0,\end{aligned}$$

where we have used the fact that ωu_r and ωu_z decay at infinity, and that

$$\omega(0, z, t) = 0, \quad \omega(r, 0, t) = 0,$$

due to regularity and oddness, respectively. Finally, we will observe that for all $r \in \mathbb{R}$,

$$u_z(r, 0) = 0,$$

because u_z is odd in z . This implies that stream lines cannot cross the plane $z = 0$, and therefore the set $\{(r, z) : r, z > 0\}$ is preserved by the flow map. We know that ω/r^k is transported by the velocity, so if for all $r, z > 0$,

$$\omega/r^k(r, z) \geq 0,$$

then for all $0 \leq t < T_{\max}$, $r, z > 0$,

$$\frac{\omega}{r^k}(r, z, t) \geq 0. \quad \blacksquare$$

We have now shown that in the geometric setting of odd scalar vorticities, non-negative when $z > 0$, we can view $\omega(r, z, t) dr dz$ as a finite measure on $[0, +\infty]^2$ with constant total measure. Up to rescaling, $\omega(r, z, t) dr dz$ is a probability measure. Therefore, it makes sense to consider the regularity problem in terms of the moments. We will show that a certain radial moment is increasing, and a certain vertical moment is decreasing.

Theorem 6.5. *Suppose that $u \in C([0, T_{\max}); H_*^s(\mathbb{R}^d)) \cap C^1([0, T_{\max}), H_*^{s-1}(\mathbb{R}^d))$, with $d \geq 3$ and $s > 2 + d/2$, is a solution of the Euler equation, and assume that $\omega(r, z, t)$ is odd in z , with for all $r, z > 0$,*

$$\omega(r, z, t) \geq 0.$$

Then for all $0 \leq t < T_{\max}$,

$$\frac{d}{dt} \int_0^\infty \int_0^\infty z \omega(r, z, t) dr dz = -\frac{1}{2} \int_0^\infty u_z(0, z)^2 dz - k \int_0^\infty \int_0^\infty \frac{1}{r} u_r(r, z)^2 dr dz.$$

Proof. We will use the divergence form of the vorticity equation,

$$\partial_t \omega + \partial_r(\omega u_r) + \partial_z(\omega u_z) = 0.$$

We can then compute that

$$\begin{aligned} \frac{d}{dt} \int_0^\infty \int_0^\infty z \omega(r, z, t) dr dz &= - \int_0^\infty \int_0^\infty z (\partial_r(\omega u_r) + \partial_z(\omega u_z)) \\ &= \int_0^\infty \int_0^\infty u_z \omega dr dz = \int_0^\infty \int_0^\infty u_z (\partial_r u_z - \partial_z u_r) dr dz, \end{aligned}$$

by integration by parts. Now calculate that

$$\int_0^\infty \int_0^\infty u_z \partial_r u_z dr dz = \int_0^\infty \int_0^\infty \frac{1}{2} \partial_r u_z^2 dr dz = - \int_0^\infty u_z(0, z)^2 dz.$$

Integrating by parts and applying the divergence free constraint, we find that

$$\begin{aligned} - \int_0^\infty \int_0^\infty u_z \partial_z u_r dr dz &= \int_0^\infty \int_0^\infty \partial_z u_z u_r dr dz \\ &= -k \int_0^\infty \int_0^\infty \frac{1}{r} u_r^2 dr dz - \int_0^\infty \int_0^\infty u_r \partial_r u_r dr dz. \end{aligned}$$

Finally, observe that

$$-\int_0^\infty \int_0^\infty u_r \partial_r u_r \, dr \, dz = -\frac{1}{2} \int_0^\infty \int_0^\infty \partial_r u_r^2 \, dr \, dz = \int_0^\infty u_r(0, z)^2 \, dz = 0. \quad \blacksquare$$

Theorem 6.6. Suppose that $u \in C([0, T_{\max}); H_*^s(\mathbb{R}^d)) \cap C^1([0, T_{\max}), H_*^{s-1}(\mathbb{R}^d))$, with $d \geq 3$ and $s > 2 + d/2$, is a solution of the Euler equation, and assume that $\omega(r, z, t)$ is odd in z , with

$$\omega(r, z, t) \geq 0 \quad \text{for all } r, z > 0.$$

Then for all $0 \leq t < T_{\max}$,

$$\frac{d}{dt} \int_0^\infty \int_0^\infty r^{d-1} \omega(r, z, t) \, dr \, dz = \frac{d-1}{2} \int_0^\infty r^{d-2} u_r(r, 0)^2 \, dr.$$

Proof. Following the same approach as above and integrating by parts, we compute that

$$\begin{aligned} \frac{d}{dt} \int_0^\infty \int_0^\infty r^{d-1} \omega(r, z, t) \, dr \, dz &= - \int_0^\infty \int_0^\infty r^{d-1} (\partial_r(\omega u_r) + \partial_z(\omega u_z)) \\ &= (d-1) \int_0^\infty \int_0^\infty r^{d-2} u_r \omega \, dr \, dz = (d-1) \int_0^\infty \int_0^\infty r^{d-2} u_r (\partial_r u_z - \partial_z u_r) \, dr \, dz. \end{aligned}$$

Next observe that

$$- \int_0^\infty \int_0^\infty r^{d-2} u_r \partial_z u_r \, dr \, dz = \frac{1}{2} \int_0^\infty r^{d-2} u_r(r, 0)^2 \, dr.$$

Using the form of the divergence free constraint

$$\partial_r(r^{d-2} u_r) + \partial_z(r^{d-2} u_z) = 0,$$

we can compute that

$$\begin{aligned} \int_0^\infty \int_0^\infty r^{d-2} u_r \partial_r u_z \, dr \, dz &= - \int_0^\infty \int_0^\infty u_z \partial_r(r^{d-2} u_r) \, dr \, dz \\ &= \int_0^\infty \int_0^\infty u_z \partial_z(r^{d-2} u_z) \, dr \, dz = -\frac{1}{2} \int_0^\infty r^{d-2} u_z(r, 0)^2 \, dr = 0. \end{aligned}$$

This completes the proof. $\blacksquare$

We will now consider a moment involving both r and z whose time derivative does not have a sign, but does relate in an interesting way to the partition of the energy between the radial and vertical directions.

Proposition 6.7. Suppose that $u \in C([0, T_{\max}); H_*^s(\mathbb{R}^d)) \cap C^1([0, T_{\max}), H_*^{s-1}(\mathbb{R}^d))$, with $d \geq 3$ and $s > 2 + d/2$, is a solution of the Euler equation, and assume that $\omega(r, z, t)$ is odd in z , with

$$\omega(r, z, t) \geq 0, \quad \text{for all } r, z > 0.$$

Then for all $0 \leq t < T_{\max}$,

$$\begin{aligned} \frac{d}{dt} \int_0^\infty \int_0^\infty r^{d-1} z \omega(r, z, t) \, dr \, dz \\ = -(d-1) \int_0^\infty \int_0^\infty r^{d-2} u_z(r, z, t)^2 \, dr \, dz + \int_0^\infty \int_0^\infty r^{d-2} u_r(r, z, t)^2 \, dr \, dz. \end{aligned}$$

This can also be expressed as

$$\frac{d}{dt} \int_0^\infty \int_0^\infty r^{d-1} z \omega(r, z, t) \, dr \, dz = -dK_z(t) + K_0,$$

where

$$\begin{aligned} K_z(t) &= \int_0^\infty \int_0^\infty r^{d-2} u_z(r, z, t)^2 \, dr \, dz, \\ K_0 &= \int_0^\infty \int_0^\infty r^{d-2} u_z^0(r, z)^2 \, dr \, dz + \int_0^\infty \int_0^\infty r^{d-2} u_r^0(r, z)^2 \, dr \, dz. \end{aligned}$$

Proof. Again starting with the divergence form of the vorticity equation and integrating by parts, we compute that

$$\begin{aligned} \frac{d}{dt} \int_0^\infty \int_0^\infty r^{d-1} z \omega(r, z, t) \, dr \, dz &= - \int_0^\infty \int_0^\infty r^{d-1} z (\partial_r(\omega u_r) + \partial_z(\omega u_z)) \, dr \, dz \\ &= \int_0^\infty \int_0^\infty ((d-1)r^{d-2} z \omega u_r + r^{d-1} \omega u_z) \, dr \, dz. \end{aligned}$$

We will deal with these two terms separately. Again integrating by parts, we find that

$$\begin{aligned} \int_0^\infty \int_0^\infty r^{d-2} z \omega u_r \, dr \, dz &= \int_0^\infty \int_0^\infty r^{d-2} z u_r (\partial_r u_z - \partial_z u_r) \, dr \, dz \\ &= \int_0^\infty \int_0^\infty \left(-z u_z \partial_r (r^{d-2} u_r) - \frac{1}{2} r^{d-2} z \partial_z u_r^2 \right) \, dr \, dz \\ &= \int_0^\infty \int_0^\infty \left(z u_z \partial_z (r^{d-2} u_z) + \frac{1}{2} r^{d-2} u_r^2 \right) \, dr \, dz \\ &= \frac{1}{2} \int_0^\infty \int_0^\infty (-r^{d-2} u_z^2 + r^{d-2} u_r^2) \, dr \, dz. \end{aligned}$$

Next we compute that

$$\begin{aligned} \int_0^\infty \int_0^\infty r^{d-1} \omega u_z \, dr \, dz &= \int_0^\infty \int_0^\infty r^{d-1} (\partial_r u_z - \partial_z u_r) u_z \, dr \, dz \\ &= -\frac{d-1}{2} \int_0^\infty \int_0^\infty r^{d-2} u_z^2 \, dr \, dz + \int_0^\infty \int_0^\infty r^{d-1} u_r \partial_z u_z \, dr \, dz. \end{aligned}$$

Putting these estimates together we find that

$$\begin{aligned} (6.3) \quad \frac{d}{dt} \int_0^\infty \int_0^\infty r^{d-1} z \omega(r, z, t) \, dr \, dz &= -(d-1) \int_0^\infty \int_0^\infty r^{d-2} u_z^2 \, dr \, dz \\ &\quad + \frac{d-1}{2} \int_0^\infty \int_0^\infty r^{d-2} u_r^2 \, dr \, dz + \int_0^\infty \int_0^\infty r^{d-1} u_r \partial_z u_z \, dr \, dz. \end{aligned}$$

We will use the divergence free constraint to simplify the last term, computing that

$$\begin{aligned} \int_0^\infty \int_0^\infty r^{d-1} u_r \partial_z u_z \, dr \, dz &= \int_0^\infty \int_0^\infty r^{d-1} u_r \left(-(d-2) \frac{u_r}{r} - \partial_r u_r \right) \, dr \, dz \\ &\quad \times \left(\frac{d-1}{2} - (d-2) \right) \int_0^\infty \int_0^\infty r^{d-2} u_r^2 \, dr \, dz. \end{aligned}$$

Plugging this back into (6.3), we find that

$$\begin{aligned} \frac{d}{dt} \int_0^\infty \int_0^\infty r^{d-1} z \omega(r, z, t) \, dr \, dz &= -(d-1) \int_0^\infty \int_0^\infty r^{d-2} u_z(r, z, t)^2 \, dr \, dz \\ &\quad + \int_0^\infty \int_0^\infty r^{d-2} u_r(r, z, t)^2 \, dr \, dz. \end{aligned}$$

The result then follows by conservation of energy by making the substitution

$$K_r(t) = K_0 - K_z(t). \quad \blacksquare$$

We will now use Proposition 6.7 to prove Theorem 1.6, our conditional blowup result for the Euler equation with the colliding vortex tube geometry, which is restated for the reader's convenience.

Theorem 6.8. *Suppose that $u \in C([0, T_{\max}); H_*^s(\mathbb{R}^d)) \cap C^1([0, T_{\max}), H_*^{s-1}(\mathbb{R}^d))$, with $d \geq 4$ and $s > 2 + d/2$, is a solution of the Euler equation, and assume that $\omega^0(r, z)$ is odd in z , with*

$$\omega^0(r, z) \geq 0, \quad \text{for all } r, z > 0,$$

and that ω^0 is not identically zero. Further, suppose that there exists $\varepsilon > 0$ such that for all $0 \leq t < T_{\max}$,

$$K_z(t) \geq \frac{1 + \varepsilon}{d} K_0.$$

Then this solution of the Euler equation blows up in finite-time, with

$$T_{\max} \leq \frac{1}{\varepsilon K_0} \int_0^\infty \int_0^\infty r^{d-1} z \omega^0(r, z) \, dr \, dz.$$

Proof. Applying Theorem 6.4, we can see that the oddness and positivity conditions on the vorticity are both preserved by the dynamics, and applying Proposition 6.7, we find that for all $0 \leq t < T_{\max}$,

$$\frac{d}{dt} \int_0^\infty \int_0^\infty r^{d-1} z \omega(r, z, t) \, dr \, dz = -d K_z(t) + K_0 \leq -\varepsilon K_0.$$

Integrating this differential inequality we find that for all $0 \leq t < T_{\max}$,

$$\int_0^\infty \int_0^\infty r^{d-1} z \omega(r, z, t) \, dr \, dz \leq \int_0^\infty \int_0^\infty r^{d-1} z \omega^0(r, z) \, dr \, dz - \varepsilon K_0 t.$$

Due to non-negativity, we can see that for all $0 \leq t < T_{\max}$,

$$\int_0^\infty \int_0^\infty r^{d-1} z \omega(r, z, t) \, dr \, dz > 0.$$

Taking the limit $t \rightarrow T_{\max}$, we can see that

$$\varepsilon K_0 T_{\max} \leq \int_0^\infty \int_0^\infty r^{d-1} z \omega^0(r, z) \, dr \, dz,$$

and this completes the proof. $\blacksquare$

Remark 6.9. The most natural blowup scenario to consider for axisymmetric, swirl-free solutions in four and higher dimensions are vorticities that are odd in z , positive on the upper half plane, and vanish only linearly at the axis of symmetry, leading to $\omega^0/r^{d-2} \notin L^\infty$. Theorem 6.8 suggests the possibility that there exists some $d_0 \geq 4$, such that for all $d \geq d_0$, there exists $u^0 \in H_*^s(\mathbb{R}^d)$, $s > 2 + d/2$, with ω^0 odd in z , for all $r, z > 0$, $\omega^0(r, z) > 0$, and $\omega^0/r^{d-2} \notin L^\infty$, such that the solution of the Euler equation with this initial data blows up in finite-time.

This is precisely the geometric setup given in Elgindi's recent blowup result [9] for axisymmetric, swirl-free $C^{1,\alpha}$ solutions of the Euler equation in three dimensions. Similar to the geometric setup proposed for finite-time blowup here, the vorticity Elgindi's blowup solutions are odd in z and positive in the upper half plane, and involve blowup driven by compression along the z axis, and stretching in the horizontal plane. Furthermore, a key aspect of Elgindi's blowup result is that $\omega^0/r \notin L^\infty$. This suggests that a similar approach may be able to yield smooth solutions of the Euler equation in four and higher dimensions that blowup in finite-time when $\omega/r^k \notin L^\infty$. This is also the geometry used by Choi and Jeong [4] to prove growth of the vorticity at infinity of the form

$$\|\omega(\cdot, t)\|_{L^\infty} > C(1+t)^{1/15-\varepsilon},$$

for smooth, axisymmetric, swirl-free solutions of the Euler equation in three dimensions. This class of data was also investigated by Childress in three dimensions as a potential case of vorticity growth [3].

Remark 6.10. Even in two dimensions, the geometric scenario considered in Remark 6.9 has a substantial precedent in the literature. There are a great number of papers in the literature of the two-dimensional Euler equation based on vorticities ω that are odd with respect to both x_1 and x_2 ,

$$\omega(-x_1, x_2) = -\omega(x_1, x_2) \quad \text{and} \quad \omega(x_1, -x_2) = -\omega(x_1, x_2),$$

and non-negative in the first quadrant, with $\omega(x_1, x_2) \geq 0$ for $x_1, x_2 > 0$. For examples, Iftimie, Gamblin, and Sideris used a vorticity with this structure to prove linear growth in the radius of support in two dimensions [13], showing that

$$\frac{dR}{dt} \geq \kappa(\omega^0),$$

where $\kappa(\omega^0) > 0$ is a constant depending on the initial vorticity. This is also the geometry of the log-Lipshitz stationary solution of the Euler equation considered by Bahouri and Chemin [1] that exhibits double exponential growth in the Lagrangian map. Kiselev and Šverák used a very similar geometry [16], but with a boundary, to prove double exponential growth of the vorticity gradient for solutions of the Euler equation in two dimensions.

We should note that this is precisely the two-dimensional analogue geometric setup proposed for finite-time blowup in four and higher dimensions in Remark 6.9. Note that in

two dimensions, $r = |x_1|$ and $e_r = \text{sgn}(x_1)e_1$, and so u is axisymmetric if and only if u_1 is odd in x_1 and u_2 is even in x_1 . Recall that $\omega = \partial_1 u_2 - \partial_2 u_1$, and we can see that if u is axisymmetric, then ω is odd with respect to x_1 . Using the Biot–Savart law, it can be seen that in fact u is axisymmetric in two dimensions if and only if ω is odd with respect to x_1 . Furthermore, in two dimensions any axisymmetric vector field is automatically swirl-free, as $\mathbb{R}^2 = \text{span}\{e_r, e_z\}$. Thus, the requirement that ω is odd with respect to x_1 in two dimensions is analogous to the requirement that u is axisymmetric, swirl-free in three and higher dimensions. Likewise, the requirement that ω is odd with respect to x_2 in two dimensions is analogous to the requirement that ω is odd with respect to z in three and higher dimensions. Finally, the requirement that ω is non-negative in the first quadrant in two dimensions is analogous to ω being non-negative for $z > 0$ in three and higher dimensions. This shows that the geometric setup proposed for finite-time blowup in four and higher dimensions in Remark 6.9 is entirely analogous to the geometric setup to previous works giving lower bounds based on a hyperbolic stagnation point at the origin.

A. A stream function approach to the Biot–Savart law

In this appendix, we will give a second derivation of the Biot–Savart law (Proposition 4.3) for axisymmetric no-swirl divergence-free vector fields in $\mathbb{R}^d$, $d \geq 3$, using the stream function. For the three-dimensional axisymmetric flows, the stream function is given by

$$\psi(r, z) = \frac{1}{2\pi} \iint_{\Pi} \int_0^\pi \frac{r \bar{r} \cos \theta}{(r^2 + \bar{r}^2 - 2r\bar{r} \cos \theta + (z - \bar{z})^2)^{1/2}} d\theta \omega_\theta(\bar{r}, \bar{z}) d\bar{r} d\bar{z},$$

see p. 97 of [10], and also formula (2.3) in [4]. Here $\Pi = (0, \infty) \times \mathbb{R}$ for $(\bar{r}, \bar{z})$, and $\omega_\theta = \partial_z u_r - \partial_r u_z$ has an opposite sign as ω . Another common choice of the stream function for $d = 3$ is $\psi_\theta = r^{-1}\psi$ with $-\Delta(\psi_\theta e_\theta) = \omega_\theta e_\theta$, see equation (10.12) in [22].

For general dimension $d = k + 2 \geq 3$, due to $\text{div } u = 0$,

$$\partial_r(r^k u_r) + \partial_z(r^k u_z) = 0,$$

there is $\psi(r, z)$ such that

$$(A.1) \quad r^k u_r = -\partial_z \psi, \quad r^k u_z = \partial_r \psi.$$

Thus,

$$\omega = \partial_r u_z - \partial_z u_r = \partial_r(r^{-k} \partial_r \psi) - \partial_z(-r^{-k} \partial_z \psi) = r^{-k} \left(\partial_r^2 + \partial_z^2 - \frac{k}{r} \partial_r \right) \psi.$$

Let $\psi = r^{k+1} \eta$. Then

$$\omega = r \left(\partial_r^2 + \frac{k+2}{r} \partial_r + \partial_z^2 \right) \eta = r \Delta_{\mathbb{R}^{k+3} \times \mathbb{R}} \eta.$$

Hence, we have $\eta = (\Delta_{\mathbb{R}^{k+3} \times \mathbb{R}})^{-1} \Omega$, $\Omega = \omega/r$. We can compute $\eta(r, z)$ at $re_1 + ze_z$, where e_z is the last standard basis vector in $\mathbb{R}^{k+3} \times \mathbb{R}$,

$$\eta(r, z) = \int_{\mathbb{R}^{k+3} \times \mathbb{R}} \frac{-C_0}{|re_1 + ze_z - x|^{k+2}} \Omega(x) dx,$$

where

$$C_0 = \frac{\Gamma(d/2)}{4\pi^{d/2+1}}$$

is the constant of the Newton kernel in $\mathbb{R}^{d+2}$. Write $x = (\bar{r}y, \bar{z}) \in \mathbb{R}^{k+3} \times \mathbb{R}$, with $(\bar{r}, \bar{z}) \in \Pi$ and $y = (y_1, y') \in \mathbb{S}^d$, $y_1 \in [-1, 1]$ being the first component of y . We have

$$|re_1 + ze_z - x|^2 = (r - \bar{r}y_1)^2 + |\bar{r}y'|^2 + (z - \bar{z})^2 = (r - \bar{r})^2 + 2r\bar{r}(1 - y_1) + (z - \bar{z})^2$$

and

$$\eta(r, z) = \iint_{\Pi} \int_{\mathbb{S}^d} \frac{-C_0}{[(r - \bar{r})^2 + 2r\bar{r}(1 - y_1) + (z - \bar{z})^2]^{d/2}} dS(y) \frac{\omega(\bar{r}, \bar{z})}{\bar{r}} d\bar{r} d\bar{z}.$$

By Lemma 4.2,

$$\eta(r, z) = -C_1 \iint_{\Pi} \int_0^\pi \frac{(\sin \theta)^{k+1}}{X^{d/2}} d\theta \omega(\bar{r}, \bar{z}) \bar{r}^{k+1} d\bar{r} d\bar{z},$$

where $C_1 = C_0 m_{k+1} = \frac{\Gamma(d/2)}{4\pi^{d/2+1}} \cdot \frac{2\pi^{d/2}}{\Gamma(d/2)} = \frac{1}{2\pi}$, and

$$X = (r - \bar{r})^2 + 2r\bar{r}(1 - \cos \theta) + (z - \bar{z})^2.$$

Thus,

$$(A.2) \quad \psi(r, z) = -\frac{1}{2\pi} \iint_{\Pi} \int_0^\pi \frac{(\sin \theta)^{k+1}}{X^{d/2}} d\theta \omega(\bar{r}, \bar{z}) (r\bar{r})^{k+1} d\bar{r} d\bar{z}.$$

From (A.2) and (A.1), since $u_r(r, z) = -\frac{1}{r^k} \partial_z \psi$, we have

$$u_r(r, z) = -\frac{d}{2\pi} \iint_{\Pi} \int_0^\pi \frac{(\sin \theta)^{k+1}}{X^{\frac{d}{2}+1}} d\theta (z - \bar{z}) \omega(\bar{r}, \bar{z}) r\bar{r}^{k+1} d\bar{r} d\bar{z}.$$

We can rewrite it using $\partial_\theta X = 2r\bar{r} \sin \theta$ and integrate by parts in θ ,

$$(A.3) \quad \begin{aligned} u_r(r, z) &= \frac{1}{2\pi} \iint_{\Pi} \int_0^\pi (\sin \theta)^k \partial_\theta \left(\frac{1}{X^{d/2}} \right) d\theta (z - \bar{z}) \omega(\bar{r}, \bar{z}) \bar{r}^k d\bar{r} d\bar{z} \\ &= -\frac{k}{2\pi} \iint_{\Pi} \int_0^\pi \frac{(\sin \theta)^{k-1} \cos \theta}{X^{d/2}} d\theta (z - \bar{z}) \omega(\bar{r}, \bar{z}) \bar{r}^k d\bar{r} d\bar{z}. \end{aligned}$$

This recovers (4.4).

We can also rewrite ψ in (A.2) as in (A.3) to get

$$(A.4) \quad \begin{aligned} \psi(r, z) &= \frac{1}{2\pi} \iint_{\Pi} \int_0^\pi (\sin \theta)^k \partial_\theta \left(\frac{1}{kX^{k/2}} \right) d\theta \omega(\bar{r}, \bar{z}) (r\bar{r})^k d\bar{r} d\bar{z} \\ &= -\frac{1}{2\pi} \iint_{\Pi} \int_0^\pi \frac{(\sin \theta)^{k-1} \cos \theta}{X^{k/2}} d\theta \omega(\bar{r}, \bar{z}) (r\bar{r})^k d\bar{r} d\bar{z}. \end{aligned}$$

From (A.4) and (A.1), since $u_z(r, z) = \frac{1}{r^k} \partial_r \psi$, and using $\partial_r X = 2r - 2\bar{r} \cos \theta$, we have

$$\begin{aligned} u_z(r, z) &= \frac{1}{2\pi} \iint_{\Pi} \int_0^\pi (\sin \theta)^{k-1} \cos \theta \left(\frac{k(r - \bar{r} \cos \theta)}{X^{k/2+1}} - \frac{k}{rX^{k/2}} \right) d\theta \omega(\bar{r}, \bar{z}) \bar{r}^k d\bar{r} d\bar{z} \\ &= I_1 + I_2. \end{aligned}$$

We rewrite I_2 and integrate by parts to get

$$\begin{aligned} I_2 &= -\frac{1}{2\pi} \iint_{\Pi} \int_0^\pi \frac{1}{rX^{k/2}} \partial_\theta ((\sin \theta)^k) d\theta \omega(\bar{r}, \bar{z}) \bar{r}^k d\bar{r} d\bar{z} \\ &= -\frac{k}{2\pi} \iint_{\Pi} \int_0^\pi \frac{(\sin \theta)^{k+1}}{X^{d/2}} d\theta \omega(\bar{r}, \bar{z}) \bar{r}^{k+1} d\bar{r} d\bar{z}. \end{aligned}$$

Thus,

$$\begin{aligned} u_z(r, z) &= I_1 + I_2 \\ &= \frac{k}{2\pi} \iint_{\Pi} \int_0^\pi \frac{(\sin \theta)^{k-1}}{X^{d/2}} [(\cos \theta)(r - \bar{r} \cos \theta) - (\sin \theta)^2 \bar{r}] d\theta \omega(\bar{r}, \bar{z}) \bar{r}^k d\bar{r} d\bar{z} \\ &= \frac{k}{2\pi} \iint_{\Pi} \int_0^\pi \frac{(\sin \theta)^{k-1}}{X^{d/2}} (r \cos \theta - \bar{r}) d\theta \omega(\bar{r}, \bar{z}) \bar{r}^k d\bar{r} d\bar{z}. \end{aligned}$$

This recovers (4.6).

We can rewrite the stream function formulas in (A.2) and (A.4) as

$$\psi(r, z, t) = -\frac{1}{2\pi} \iint_{\Pi} \mathcal{F}(S)(r\bar{r})^{d/2-1} \omega(\bar{r}, \bar{z}, t) d\bar{r} d\bar{z},$$

where

$$S = \frac{(r - \bar{r})^2 + (z - \bar{z})^2}{r\bar{r}}, \quad X = r\bar{r}(2 - 2\cos \theta + S),$$

and for $s > 0$,

$$\mathcal{F}(s) = \int_0^\pi \frac{(\sin \theta)^{k+1}}{[2(1 - \cos \theta) + s]^{d/2}} d\theta = \int_0^\pi \frac{(\sin \theta)^{k-1} \cos \theta}{[2(1 - \cos \theta) + s]^{k/2}} d\theta.$$

This form isolates the dependence on length variables r, r', z, z' in the angular integral. The velocity components are recovered from the stream function via (A.1), resulting in the expressions

$$u_r(r, z, t) = \frac{1}{\pi r^{d-2}} \iint_{\Pi} \mathcal{F}'(S)(z - \bar{z})(r\bar{r})^{d/2-2} \omega(\bar{r}, \bar{z}, t) d\bar{r} d\bar{z}$$

and

$$u_z(r, z, t) = -\frac{1}{2\pi r^{d-2}} \iint_{\Pi} \left(\mathcal{F}'(S) \partial_r S + \frac{d-2}{2r} \mathcal{F}(S) \right) (r\bar{r})^{d/2-1} \omega(\bar{r}, \bar{z}, t) d\bar{r} d\bar{z}.$$

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