Rev. Mat. Iberoam. 42 (2026), no. 5, 1727–1782
DOI 10.4171/RMI/1629

Haar wavelet characterization of Besov-type spaces with optimal indices and its application to pointwise multipliers

Winfried Sickel, Dachun Yang, Wen Yuan and Yirui Zhao

Abstract. In this article, we give a sufficient and necessary condition on the indices of Besov-type spaces on $\mathbb{R}$ under which the characterization of these spaces in terms of the Haar system holds. This confirms that the sufficient condition obtained by H. Triebel is optimal. Moreover, a special case of our results also negatively answers a question, asked by H. Triebel, about the characterization of lifted local bounded mean oscillation spaces $J^s\mathrm{bmo}$ on $\mathbb{R}$ with $s \in (-1, -1/2]$ in terms of the Haar system. As an application, we prove that characteristic functions of intervals are pointwise multipliers of Besov-type spaces with the aforementioned optimal indices.

Contents

1. Introduction

Within the last 40 years, wavelets have proven to be an essential and powerful tool for the understanding and the study of many operators and function spaces; see, for example, [27, 41, 61, 69]. Characterizations of function spaces in terms of wavelet systems, in particular, the Haar wavelet system, have found considerable applications. We refer to Coifman et al. [9], Nazarov et al. [44], and Hytönen [24] for the applications of the Haar system to

---

Mathematics Subject Classification 2020: 42C40 (primary); 42B15, 42B35, 46E35 (secondary).
Keywords: Besov space, Besov-type space, local bounded mean oscillation space, Haar wavelet, pointwise multiplier.

the estimations of singular integral operators, and to Ropela [48], Triebel [56,62], Oswald [45–47], and Garrigós et al. [16–19] for the applications in characterizing various function spaces in terms of the Haar system.

The main purpose of this article is to characterize Besov-type spaces on $\mathbb{R}$, with optimal indices, by means of the Haar wavelet system. Since throughout this article we always work in $\mathbb{R}$, for simplicity of presentation, we will not indicate this underlying space in the related notation. Recall that, as a generalization of classical Besov spaces, the Besov-type spaces, restricted to the Banach case, were first introduced by El Baraka [12–14], and the extension to the quasi-Banach case was later given in [50, 72, 76]. From then on, Besov-type spaces have received widely development and found plenty of applications in harmonic analysis and PDEs; see, for example, [23, 30, 32–34, 64–66, 70]. It should be mentioned that Besov-type spaces can also be considered as a generalization of bmo, the space of functions with local bounded mean oscillation, or, more generally, of the Triebel–Lizorkin spaces $F^s_{\infty,p}$. Recall that, for any $p \in (0,\infty)$ and $s \in \mathbb{R}$,

$$B^{s,1/p}_{p,p} = F^s_{\infty,p} \quad \text{and} \quad B^{0,1/2}_{2,2} = F^0_{\infty,2} = \mathrm{bmo} \tag{1.1}$$

(see, for example, [15,76]). Nowadays, it is known that Besov-type spaces $B^{s,\tau}_{p,p}$ are also of importance in connection with the description of the set of all pointwise multipliers for the classical Besov spaces $B^s_{p,p}$; see, for example, [31,36,78]. It is well known that pointwise multipliers of function spaces play an important role in harmonic analysis and PDEs (see, for example, the monographs [1, 6, 39, 40, 53]). In particular, we mention that, in recent years, the characterization of pointwise multipliers of $\mathrm{BMO}(\Omega)$, which was studied by Janson [25] for $\Omega = \mathbb{T}^n$ and by Nakai and Yabuta [43] for $\Omega = \mathbb{R}^n$ (see also the recent survey [42] of Nakai and the references therein), plays an irreplaceable role in solving the so-called optimal bilinear decomposition for multiplications of functions in the Hardy space $H^1(\mathbb{R}^n)$ and its dual space $\mathrm{BMO}(\mathbb{R}^n)$, which is precisely Conjecture 1.7 in [5]. The corresponding conjectures on Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0,1)$ and their dual spaces, as well as their local versions, were completely solved in [4,7,74,79] via first establishing the characterizations of pointwise multipliers of Campanato spaces.

The basis problem for the Haar system in Besov and Triebel–Lizorkin spaces has attracted widely attention and been intensively studied; see, for example, [16–19, 46, 56, 62, 67]. However, this basis problem for Besov-type spaces $B^{s,\tau}_{p,q}$ with $\tau \in (0,\infty)$ must be rephrased, because these spaces are not separable. But it is still meaningful to ask whether one can characterize Besov-type spaces in terms of the Haar system (for short, *Haar characterization*). Already in 2013, Triebel [63] investigated the Haar characterization of some Morrey-type smoothness spaces, proving a first version of the Haar characterization for Besov-type spaces. At that time, it was widely unknown whether the restrictions on the indices of the Haar characterization in [63] are optimal. In 2020, three of the authors studied the necessity of a part of the restrictions in [77]. This has been done as well by Haroske and Triebel in [23], where they proposed a conjecture about the optimal restriction of the Haar characterization of Morrey smoothness spaces (see Conjecture 5.38 in [23]).

In this article, we give a sufficient and necessary condition on the indices of Besov-type spaces $B^{s,\tau}_{p,q}$ on $\mathbb{R}$ under which their Haar characterization holds (see Theorem 2.2). This condition confirms the sufficient condition obtained by Triebel in Theorem 3.41 of [65]. This includes the cases where the Besov-type spaces reduce to the classical

Triebel–Lizorkin spaces with $p = \infty$ (see Corollary 2.4 and Remark 2.5); the latter complements the Haar characterization of Triebel–Lizorkin spaces in the limiting case. Moreover, this optimal condition indicates that the conjecture proposed by Haroske and Triebel in Conjecture 5.38 of [23] fails (see Remark 2.3 (i)). Furthermore, our results imply that the Haar characterization for the lifted local bounded mean oscillation space bmo^s with $s \le -1/2$ is no longer true, which can be also deduced from the argument in Remark 12.1 of [18]. This failure gives a negative answer to the open question proposed by Triebel in Remark 3.45 of [65] about the Haar characterization of bmo^s on $\mathbb{R}$ with $s \in (-1, -1/2]$ (see Remark 7.23).

As an application, we prove that characteristic functions of intervals are pointwise multipliers of Besov-type spaces under the same aforementioned optimal indices of their Haar characterizations (see Corollary 2.6), which is also sharp when Besov-type spaces reduce to Triebel–Lizorkin spaces $F^s_{\infty,p}$. Historically, pointwise multipliers generated by characteristic functions of domains play a key role in harmonic analysis and PDEs; we refer, for example, to [10, 15, 22, 49, 51, 54, 55, 59, 78].

The remainder of this article is organized as follows.

In Section 2, we describe our main results and their applications to pointwise multipliers. In Section 3, we recall some basic properties and the Daubechies wavelet characterization of Besov-type spaces. In Sections 4–6, we provide a different proof of the sufficient condition from that one given by Triebel in Theorem 3.41 of [65]. Section 4 is devoted to showing the boundedness of the analysis operator T, which maps a given distribution f into the sequence of its Fourier–Haar coefficients. In Section 5, we study the boundedness of the synthesis operator S, mapping a sequence of complex numbers into a Haar series. Next, in Section 6, we search for those situations where $T\colon B^{s,\tau}_{p,q} \to b^{s,\tau}_{p,q}$ is an isomorphism with the inverse S. In Section 7, we construct some subtle counterexamples, which are presented, respectively, in Subsections 7.1, 7.2, and 7.3, to show that the restriction on indices for T being an isomorphism is necessary. Moreover, in Subsection 7.4, we apply our results in Section 2 to lifted local bounded mean oscillation spaces $J^s\mathrm{bmo}$, which are of independent interest. Section 8 is devoted to the proof of applications of the main results in Section 2.

We end this introduction by making some conventions on the notation. We always let $\mathbb{N} := \{1, 2, \dots,\}$, $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$, and $\mathbb{N}_{-1} := \mathbb{Z}_+ \cup \{-1\}$, and let $\mathbb{Z}$ denote the set of all *integers*. We also use $\mathbb{R}$ and $\mathbb{C}$ to denote respectively the set of all *real numbers* and the set of all *complex numbers*. For any $p \in (0, \infty]$, let $p' := p/(p-1)$ if $p \in (1, \infty]$ and $p' := \infty$ if $p \in (0, 1]$ be the conjugate index of p. Let L^1_{loc} be the set of all locally integrable complex-valued functions on $\mathbb{R}$, C^∞_{c} the set of all infinitely differentiable complex-valued functions on $\mathbb{R}$ with compact support, $\mathcal{S}$ the set of all *Schwartz functions* on $\mathbb{R}$ equipped with the well-known topology determined by a countable family of norms, and $\mathcal{S}'$ the *topological dual* of $\mathcal{S}$; that is, the space of all bounded linear functionals on $\mathcal{S}$ equipped with the weak-$*$ topology. For any $j, k \in \mathbb{Z}$, we denote by $Q_{j,k}$ the *dyadic interval* $[2^{-j}k, 2^{-j}(k+1))$, and by $\ell(Q_{j,k})$ its edge length. Let

$$\mathcal{D} := \{Q_{j,k} : j, k \in \mathbb{Z}\},$$

let $j_Q := -\log_2 \ell(Q)$ for any $Q \in \mathcal{D}$, and define

$$\mathcal{D}_\ell := \{Q \in \mathcal{D} : j_Q = \ell\}$$

for any $\ell \in \mathbb{Z}$. For any set $E \subset \mathbb{R}$, we denote by $\mathbf{1}_E$ its *characteristic function* and by $|E|$ its 1-dimensional *Lebesgue measure* if E is measurable. Moreover, the *symbol* C denotes a positive constant which is independent of the main parameters involved, but may vary from line to line. The *symbol* $f \lesssim g$ always means that there exists a positive constant C such that $f \le Cg$. If $f \lesssim g$ and $g \lesssim f$, we then write $f \sim g$. If $f \le Cg$ and $g = h$ or $g \le h$, we then write $f \lesssim g = h$ or $f \lesssim g \le h$. In addition, given two quasi-Banach spaces X and Y, we denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators from X into Y, equipped with the *operator norm* denoted by $\|T\|_{X\to Y}$ for any $T \in \mathcal{L}(X, Y)$. For any $p \in (0, 1]$ and any sequence $\{a_k\}_{k\in I}$ of complex numbers with an at most countable set I of indices, we will use, without further references, the well-known inequality

$$\Big(\sum_{k\in I} |a_k|\Big)^p \le \sum_{k\in I} |a_k|^p.$$

Finally, in all proofs we consistently retain the notation introduced in the original theorem (or related statement).

2. Main results

We begin with recalling the definition of the Haar system on the real line. The *generator* of the Haar system, denoted by h, is defined by setting

$$h := \mathbf{1}_{[0,1/2)} - \mathbf{1}_{[1/2,1)}.$$

Consider the set

$$\mathcal{H} := \{h_{j,m} : j \in \mathbb{N}_{-1} \text{ and } m \in \mathbb{Z}\}, \tag{2.1}$$

where $h_{j,m}(\cdot) := 2^{j/2}h(2^j \cdot -m)$ for any $j \in \mathbb{Z}_+$ and $m \in \mathbb{Z}$, and where $h_{-1,m}(\cdot) := \mathbf{1}_{[0,1)}(\cdot - m)$ for any $m \in \mathbb{Z}$. Then the set $\mathcal{H}$ forms the well-known *orthonormal Haar wavelet system* (for short, *Haar system*) on $\mathbb{R}$. For any suitable distribution $f \in \mathcal{S}'$, its *Fourier–Haar coefficient* is defined to be

$$\delta(f) := \{\delta_{j,m}(f)\}_{j\in\mathbb{N}_{-1},m\in\mathbb{Z}} := \{\langle f, h_{j,m}\rangle\}_{j\in\mathbb{N}_{-1},m\in\mathbb{Z}}$$

whenever $\langle f, h_{j,m}\rangle$ for any $j \in \mathbb{N}_{-1}$ and $m \in \mathbb{Z}$ is well defined. Recall that, given quasi-normed linear spaces X and Y, an operator $T \in \mathcal{L}(X, Y)$ is called an *isomorphism* if it is bijective with bounded inverse.

We first recall what is known in the classical case $\tau = 0$, which is a part of Theorem 3.18 in [67].

Theorem A. Let $p, q \in (0, \infty]$ and $s \in \mathbb{R}$. Then the mapping $T : f \mapsto \delta(f)$ is an isomorphism from $B^s_{p,q}$ onto $b^s_{p,q}$ if

$$\frac{1}{p} - 1 < s < \min\Big\{1, \frac{1}{p}\Big\}. \tag{2.2}$$

Remark 2.1. (i) The necessity of the condition (2.2) follows from the necessity of the assertion that $\mathcal{H}$ is an unconditional basis of $B^s_{p,q}$ when $p, q \in (0, \infty)$ (see, for instance, Theorem 1.1 in [17]). Indeed, in this case, the canonical system is clearly an unconditional basis of $b^s_{p,q}$, and it is preserved in $B^s_{p,q}$ by the isomorphism of T. Therefore, the validity of the isomorphism implies that $\mathcal{H}$ is an unconditional basis in $B^s_{p,q}$, which, combined with Theorem 1.1 in [17], further implies (2.2) when $p, q \in (0, \infty)$. We thank the referee for pointing out this to us.

(ii) We should point out that our proofs in Section 7 also imply the *necessity* of the condition (2.2) in Theorem A.

Already in 2013, Triebel [63] published a first extension of Theorem A to Morrey-type spaces. Later, in 2014, he obtained the following result concerning Besov-type spaces, we refer to Theorem 3.41 in [65] with

$$L^{\tau-1/p} B^s_{p,q} = B^{s,\tau}_{p,q}.$$

Theorem B. Let $p, q \in (0, \infty]$, $s \in \mathbb{R}$, and $\tau \in [0, 1/p]$. Then the mapping $T : f \mapsto \delta(f)$ is an isomorphism from $B^{s,\tau}_{p,q}$ onto $b^{s,\tau}_{p,q}$ if

$$\frac{1}{p} - 1 < s < \min\left\{1, \frac{1}{p} - \tau\right\}. \tag{2.3}$$

Our main contribution is to demonstrate that the condition in Theorem B is necessary. Moreover, we also give a detailed proof for the sufficiency, different from that one given by Triebel [65].

Theorem 2.2. *Let $p, q \in (0, \infty]$, $s \in \mathbb{R}$, and $\tau \in [0, 1/p]$. Then the mapping $T : f \mapsto \delta(f)$ is an isomorphism from $B^{s,\tau}_{p,q}$ onto $b^{s,\tau}_{p,q}$ if and only if (2.3) holds.*

Remark 2.3. (i) Theorem 2.2 shows that the conjecture raised in [23] (see Conjecture 5.38) fails. Indeed, if Conjecture 5.38 in [23] with $\varrho := \tau p - 1 \in [-1, 0]$ holds, then, by equations (5.109) and (2.30) in [23], we find that $1/p - \tau - 1 < s < \min\{1, 1/p - \tau\}$ should be the optimal range for T being an isomorphism from $B^{s,\tau}_{p,q}$ onto $b^{s,\tau}_{p,q}$. But this contradicts Theorem 2.2.

(ii) Theorem 2.2 with $\tau = 0$ coincides with the known results of classical Besov spaces $B^s_{p,q}$; see, for instance, [56], Theorem 2.9 in [62], [17], and also Remark 2.1.

(iii) The a priori assumption $\tau \in [0, 1/p]$ of Theorem 2.2 is reasonable. Indeed, when $\tau \in (1/p, \infty)$, $B^{s,\tau}_{p,q}$ coincides with the classical Besov space $B^{s+\tau-1/p}_{\infty,\infty}$ (see Remark 3.2) and, if $\tau \in (-\infty, 0)$, then $B^{s,\tau}_{p,q}$ is trivial (see, for example, [76]).

(iv) Here we only deal with the situation that T is an isomorphism as a mapping of $B^{s,\tau}_{p,q}$ onto $b^{s,\tau}_{p,q}$. In the classical case where $\tau = 0$, it is well known that, in some situations, the range space can be quite different from $b^{s,\tau}_{p,q}$, e.g., a sequence space without any finite sequence. We refer to [19] for more details and see also [3] and Theorem 3.2.6 in [28].

Some special cases of Theorem 2.2 might be of interest. The following consequence follows directly from Theorem 2.2 and the fact that $B^{s,1/p}_{p,p} = F^s_{\infty,p}$.

Corollary 2.4. *Let $p \in (0,\infty)$ and $s \in \mathbb{R}$. Then the mapping $T: f \mapsto \delta(f)$ is an isomorphism from $F^s_{\infty,p}$ onto $b^{s,1/p}_{p,p}$ if and only if*

$$\frac{1}{p} - 1 < s < 0. \tag{2.4}$$

Remark 2.5. In Triebel's monograph, see Theorem 3.18 in [67], one can find the sufficiency part of this assertion under the restrictions $p \in (1,\infty]$, $1/p - 1 < s < 0$. As a service for the reader, we summarize and comment our findings with respect to lifted local bounded mean oscillation spaces $J^s\mathrm{bmo} = F^s_{\infty,2}$ in Subsection 7.4.

As an application of Theorem 2.2 and Corollary 2.4, we establish the following result concerning characteristic functions as pointwise multipliers for Besov-type spaces.

Corollary 2.6. *Let $p, q \in (0,\infty]$, $s \in \mathbb{R}$, $\tau \in [0, 1/p]$, and $I \subset \mathbb{R}$ be an interval. Then the following statements hold.*

(i) *If (2.3) holds, then the mapping $P_I: f \mapsto \mathbf{1}_I f$ is bounded on $B^{s,\tau}_{p,q}$. Moreover,*

$$\sup_J \|P_J\|_{B^{s,\tau}_{p,q} \to B^{s,\tau}_{p,q}} < \infty,$$

where the supremum is taken over all intervals $J \subset \mathbb{R}$.

(ii) *If $p \in (0,\infty)$ and (2.4) holds, then the mapping $P_I: f \mapsto \mathbf{1}_I f$ is bounded on $F^s_{\infty,p}$. Moreover,*

$$\sup_J \|P_J\|_{F^s_{\infty,p} \to F^s_{\infty,p}} < \infty,$$

where the supremum is taken over all intervals $J \subset \mathbb{R}$.

Remark 2.7. To the best of our knowledge, Corollary 2.6(i) is new. On the other hand, Corollary 2.6(ii) reduces to Theorem 2.48 in [67]. Moreover, the condition (2.4) is optimal for Corollary 2.6(ii); see, for instance, Theorem 2.48 in [67].

3. Besov-type spaces

In this section, we recall some basic properties of Besov-type spaces.

Given $u \in \mathcal{S}'$, we denote by $\widehat{u}$ its *Fourier transform*, and by $u^\vee$ its *inverse Fourier transform* in $\mathcal{S}'$. In particular, if $u \in \mathcal{S}$, then, for any $\xi \in \mathbb{R}$,

$$\widehat{u}(\xi) := (2\pi)^{-1/2} \int_{\mathbb{R}} u(x)\, e^{-ix\cdot\xi}\, dx$$

and $u^\vee(\xi) := \widehat{u}(-\xi)$. Let $\varphi_0 \in \mathcal{S}$ be a radial and real-valued function satisfying that $0 \le \varphi_0 \le 1$, $\varphi_0 \equiv 1$ on $[-1,1]$, and $\operatorname{supp}\varphi_0 \subset (-3/2, 3/2)$. For any $k \in \mathbb{Z}_+$ and $f \in \mathcal{S}'$, define

$$S_k f := (\varphi_k \widehat{f}\,)^\vee, \tag{3.1}$$

where $\varphi_1(\cdot) := \varphi_0(\cdot/2) - \varphi_0(\cdot)$ and, for any integer $k \ge 2$, $\varphi_k(\cdot) := \varphi_1(2^{-k+1}\cdot)$. Clearly, $\sum_{k\in\mathbb{Z}_+} \varphi_k = 1$ on $\mathbb{R}$.

Next, we recall the definition of Besov(-type) spaces as follows; see, for example, [76].

Definition 3.1. Let $p, q \in (0, \infty]$, $s \in \mathbb{R}$, and $\tau \in [0, \infty)$.

(i) The *Besov space* $B^s_{p,q}$ is defined to be the set of all $f \in \mathcal{S}'$ such that

$$\|f\|_{B^s_{p,q}} := \left\{ \sum_{j=0}^{\infty} 2^{jsq} \left[\int_{\mathbb{R}} |S_j f(x)|^p \, dx \right]^{q/p} \right\}^{1/q} < \infty,$$

with the usual modifications made if $p = \infty$ and/or $q = \infty$.

(ii) The *Besov-type space* $B^{s,\tau}_{p,q}$ is defined to be the set of all $f \in \mathcal{S}'$ such that

$$\|f\|_{B^{s,\tau}_{p,q}} := \sup_{P \in \mathcal{D}} \frac{1}{|P|^{\tau}} \left\{ \sum_{j=\max\{j_P, 0\}}^{\infty} 2^{jsq} \left[\int_{P} |S_j f(x)|^p \, dx \right]^{q/p} \right\}^{1/q} < \infty,$$

with the usual modifications made if $p = \infty$ and/or $q = \infty$.

It is known that, for any $p, q \in (0, \infty]$ and $s \in \mathbb{R}$, $B^{s,0}_{p,q} = B^s_{p,q}$ with equivalent quasi-norms. The following characterization of Besov-type spaces, due to Triebel (Theorem 3.64 in [65]), may be helpful for the reader to understand the philosophy of these spaces. Under some additional conditions on s, τ, p, and q, it holds that $f \in \mathcal{S}'$ belongs to $B^{s,\tau}_{p,q}$ if and only if

$$\|f\|^*_{B^{s,\tau}_{p,q}} := \sup_{P \in \mathcal{D}} |P|^{-\tau} \|f\|_{B^s_{p,q}(P)} < \infty;$$

that is, we control the quasi-norm $\|f\|_{B^s_{p,q}(P)}$ on all the dyadic intervals, hence, locally and globally. This relates the "new" spaces $B^{s,\tau}_{p,q}$ to the "old" spaces $B^s_{p,q}(P)$ in a rather transparent way.

Remark 3.2. Let the notation be as in Definition 3.1.

(i) It is known that $B^{s,\tau}_{p,q}$ are quasi-Banach spaces independent of the choices of φ_0 (see Lemma 2.1 and Corollary 2.1 in [76]).

(ii) We have the monotonicity with respect to s and with respect to q; that is, if $-\infty < s_1 < s_0 < \infty$ and $q_0, q_1 \in (0, \infty]$, then $B^{s_0,\tau}_{p,q_0} \hookrightarrow B^{s_1,\tau}_{p,q_1}$; moreover, if $0 < q_0 \le q_1 \le \infty$, then $B^{s,\tau}_{p,q_0} \hookrightarrow B^{s,\tau}_{p,q_1}$.

(iii) If either $q \in (0, \infty)$ and $\tau \in (1/p, \infty)$ or $q = \infty$ and $\tau \in [1/p, \infty)$, then $B^{s,\tau}_{p,q} = B^{s+\tau-1/p}_{\infty,\infty}$ (see, for instance, Theorem 2(ii) in [73]). If $s + \tau - 1/p > 0$, then the space $B^{s+\tau-1/p}_{\infty,\infty}$ is a Hölder–Zygmund space (see, for instance, Section 2.5.7 of [57]).

(iv) Since $B^{s,\tau}_{p,q}$ with $\tau \in (1/p, \infty)$ is a classical Besov space, we mainly consider the case $\tau \in [0, 1/p]$ in this article.

The concept of wavelet bases in Besov and Triebel–Lizorkin spaces is well developed (see, for example, Meyer [41], Wojtasczyk [69], and Triebel [60, 61]). Let ϕ be an *orthonormal scaling function* on $\mathbb{R}$ with compact support and of sufficiently high regularity. Let ψ be one *corresponding orthonormal wavelet*. We suppose $\phi \in C^{N_1}$ and $\operatorname{supp} \phi \subset [-N_2, N_2]$ for some $N_1, N_2 \in \mathbb{N}$. These further imply that

$$\psi \in C^{N_1} \quad \text{and} \quad \operatorname{supp} \psi \subset [-N_3, N_3] \tag{3.2}$$

for some $N_3 \in \mathbb{N}$. For any $j \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, we write

$$\phi_{j,k}(\cdot) := 2^{j/2} \phi(2^j \cdot - k) \quad \text{and} \quad \psi_{j,k}(\cdot) := 2^{j/2} \psi(2^j \cdot - k).$$

Furthermore, it is well known that, for any $|\gamma| \le N_1$, $j \in \mathbb{Z}_+$, and $k \in \mathbb{Z}$,

$$\int_{\mathbb{R}} \psi_{j,k}(x) x^\gamma \, dx = 0$$

(see, e.g., Proposition 3.1 in [69]). For convenience, for any $k \in \mathbb{Z}$, let $\psi_{-1,k} := \phi_{0,k}$. Thus,

$$\Psi := \{\psi_{j,k} : j \in \mathbb{N}_{-1} \text{ and } k \in \mathbb{Z}\} \tag{3.3}$$

yields an *orthonormal basis* of L^2 (see, for instance, Section 3.9 of [41], and see also Section 3.1 of [60]) and is called the *Daubechies wavelet system* in $\mathbb{R}$. Thus, for any $f \in L^2$,

$$f = \sum_{j=-1}^{\infty} \sum_{k \in \mathbb{Z}} \lambda_{j,k}(f) \psi_{j,k} \tag{3.4}$$

in L^2, where, for any $j \in \mathbb{N}_{-1}$ and $k \in \mathbb{Z}$, $\lambda_{j,k}(f) := \langle f, \psi_{j,k} \rangle$. Let

$$\lambda(f) := \{\lambda_{j,k}(f)\}_{j \in \mathbb{N}_{-1}, k \in \mathbb{Z}}.$$

Now, we need some associated sequence spaces (see, for instance, Definition 3.24 in [65]). Recall that, for any given dyadic interval P, $j_P := -\log_2 \ell(P)$.

Definition 3.3. Let $p, q \in (0, \infty]$, $s \in \mathbb{R}$, and $\tau \in [0, \infty)$. The *sequence space* $b^{s,\tau}_{p,q}$ is defined to be the set of all sequences $\lambda := \{\lambda_{j,k}\}_{j \in \mathbb{N}_{-1}, k \in \mathbb{Z}}$ in $\mathbb{C}$ such that $\|\lambda\|_{b^{s,\tau}_{p,q}} < \infty$, where

$$\begin{aligned} \|\lambda\|_{b^{s,\tau}_{p,q}} := \sup_{P \in \mathcal{D}} \Bigg\{ \frac{1}{|P|^\tau} \Bigg[\Bigg(\sum_{\{m:\, Q_{0,m} \subset P\}} |\lambda_{-1,m}|^p \Bigg)^{q/p} \\ + \sum_{j=\max\{j_P, 0\}}^{\infty} 2^{j(s+1/2-1/p)q} \Bigg(\sum_{\{m:\, Q_{j,m} \subset P\}} |\lambda_{j,m}|^p \Bigg)^{q/p} \Bigg]^{1/q} \Bigg\}. \end{aligned}$$

Remark 3.4. We should point out that $b^{s,0}_{p,q}$ coincides with $b^s_{p,q}$ in Definition 3.10 of [65] when $n = 1$; that is, for any $\lambda := \{\lambda_{j,k}\}_{j \in \mathbb{N}_{-1}, k \in \mathbb{Z}}$ in $\mathbb{C}$,

$$\begin{aligned} \|\lambda\|_{b^{s,0}_{p,q}} &= \|\lambda\|_{b^s_{p,q}} \\ &:= \Bigg[\Bigg(\sum_{m \in \mathbb{Z}} |\lambda_{-1,m}|^p \Bigg)^{q/p} + \sum_{j \in \mathbb{Z}_+} 2^{j(s+1/2-1/p)q} \Bigg(\sum_{m \in \mathbb{Z}} |\lambda_{j,m}|^p \Bigg)^{q/p} \Bigg]^{1/q}. \end{aligned}$$

As a special case of Theorem 3.26 in [65] with $r := \tau - 1/p$ (see also Theorem 4.12 in [37]), we have the following Daubechies wavelet characterization.

Proposition 3.5. *Let $p, q \in (0, \infty]$, $s \in \mathbb{R}$, and $\tau \in [0, \infty)$. Let Ψ be the Daubechies wavelet system as in (3.3) with $N_1 \in \mathbb{N}$ depending on p, q, s, and τ. Then $f \in B^{s,\tau}_{p,q}$ if and only if $f \in \mathcal{S}'$ and f can be represented as in (3.4) in $\mathcal{S}'$ such that $\|\lambda(f)\|_{b^{s,\tau}_{p,q}} < \infty$. Moreover,*

$$\|f\|_{B^{s,\tau}_{p,q}} \sim \|\lambda(f)\|_{b^{s,\tau}_{p,q}}$$

with the positive equivalence constants independent of f.

Combining Proposition 3.5 and Definition 3.3, we easily obtain the following conclusion, and hence we omit the details.

Lemma 3.6. *Let the notation be as in Proposition 3.5. Then there exists a positive constant C such that, for any $f \in B^{s,\tau}_{p,q}$,*

$$\sup_{j\in\mathbb{Z}_+} 2^{j(s+\tau+1/2-1/p)} \sup_{k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle| \le C\|f\|_{B^{s,\tau}_{p,q}}.$$

Moreover, $B^{s,\tau}_{p,q} \hookrightarrow B^{s+\tau-1/p}_{\infty,\infty}$.

4. The analysis operator

Our investigations are based on the study of two linear operators, namely the analysis operator T and the synthesis operator S. Recall that $\mathcal{H}$ denotes the Haar system in $\mathbb{R}$ as in (2.1). In this section, we concentrate on T, defined by setting, for any suitable distribution f,

$$T : f \mapsto \{\langle f, h_{j,m}\rangle\}_{j\in\mathbb{N}_{-1},\, m\in\mathbb{Z}}.$$

Clearly, the first problem we face is whether the operator T is well defined on $B^{s,\tau}_{p,q}$. To answer this, we need some information on the dual space of a Besov space; see, for instance, Section 2.11.2 of [57].

Lemma 4.1. *Let $p \in [1,\infty)$, $q \in (0,\infty)$, and $s \in \mathbb{R}$. Then $(B^s_{p,q})' = B^{-s}_{p',q'}$, where $(B^s_{p,q})'$ denotes the dual space of $B^s_{p,q}$.*

Proposition 4.2. *Let $p, q \in (0,\infty]$, $s \in \mathbb{R}$, and $\tau \in [0, 1/p]$ satisfy $s > 1/p - \tau - 1$, and let $f \in B^{s,\tau}_{p,q}$. Then the following two statements hold:*

(i) *the pair $\langle f, h\rangle$ is well defined for any $h \in \mathcal{H}$;*

(ii) *$f = 0$ if and only if $\langle f, h\rangle = 0$ for all $h \in \mathcal{H}$.*

Proof. We first show (i). Since $s > 1/p - \tau - 1$, it follows that there exists $\varepsilon \in (0,\infty)$ such that $s > 1/p - \tau - 1 + \varepsilon$. By this, Lemma 3.6, and Remark 3.2 (ii), we find that

$$B^{s,\tau}_{p,q} \hookrightarrow B^{s+\tau-1/p}_{\infty,\infty} \hookrightarrow B^{-1+\varepsilon}_{\infty,2}. \tag{4.1}$$

Fix $h \in \mathcal{H}$. Applying Theorem 2.1 (ii) in [77] and Remark 3.2 (ii), we obtain

$$h \in B^1_{1,\infty} \hookrightarrow B^{1-\varepsilon}_{1,2}. \tag{4.2}$$

From this and Lemma 4.1, we deduce that

$$|\langle f, h\rangle| \le \|f\|_{B^{-1+\varepsilon}_{\infty,2}} \|h\|_{B^{1-\varepsilon}_{1,2}} \lesssim \|f\|_{B^{s,\tau}_{p,q}} \|h\|_{B^1_{1,\infty}} < \infty. \tag{4.3}$$

Thus, $\langle f, h\rangle$ is well defined.

Item (ii) is a direct consequence of (4.1), $B^{-1+\varepsilon}_{\infty,2} \hookrightarrow B^{-1}_{\infty,1}$, and Proposition 1.1 (iii) in [17]. This then finishes the proof of Proposition 4.2. ■

Based on a standard method of the comparison of a smooth wavelet decomposition with the Haar decomposition (see, for example, [62]), we obtain the following conclusion.

Theorem 4.3. *Let $p \in (0, \infty)$, $q \in (0, \infty]$, $s \in \mathbb{R}$, and $\tau \in (0, 1/p]$. Then the analysis operator*

$$\begin{aligned} T : B^{s,\tau}_{p,q} &\longrightarrow b^{s,\tau}_{p,q}, \\ f &\longmapsto \delta(f), \end{aligned}$$

is a bounded linear mapping if $1/p - 1 < s < 1$.

To prove Theorem 4.3, we need the following Young inequality for Lebesgue sequence spaces, whose proof follows directly from the standard proof of the classical Young inequality. We omit the details.

Lemma 4.4. *Let $r \in [1, \infty]$ and $L \in \mathbb{Z}_+$. Then, for any nonnegative sequences $\{a_k\}_{k\in\mathbb{Z}}$ and $\{b_k\}_{k\in\mathbb{Z}_+}$, the following two inequalities hold*:

$$\text{(i)} \qquad \Bigg[\sum_{j=L}^{\infty}\Bigg(\sum_{k=L}^{j} a_{j-k}\, b_k\Bigg)^r\Bigg]^{1/r} \le \sum_{k=0}^{\infty} a_k \Bigg(\sum_{j=L}^{\infty} b_j^r\Bigg)^{1/r},$$

with the usual modification made when $r = \infty$.

$$\text{(ii)} \qquad \Bigg[\sum_{j=L}^{\infty}\Bigg(\sum_{k=j+1}^{\infty} a_{j-k}\, b_k\Bigg)^r\Bigg]^{1/r} \le \sum_{k=-\infty}^{-1} a_k \Bigg(\sum_{j=L}^{\infty} b_j^r\Bigg)^{1/r},$$

with the usual modification made when $r = \infty$.

As a consequence of Lemma 4.4 with $r = 1$ and the well-known inequality that, for any $p \in (0, 1]$,

$$\Bigg(\sum_{k\in I} |a_k|\Bigg)^p \le \sum_{k\in I} |a_k|^p, \quad \text{for } \{a_k\}_{k\in I} \subset \mathbb{C},\ I \text{ an at most countable set of indices,}$$

we obtain the following.

Corollary 4.5. *Let $r \in (0, 1]$ and $L \in \mathbb{Z}_+$. Then, for any nonnegative sequences $\{a_k\}_{k\in\mathbb{Z}}$ and $\{b_k\}_{k\in\mathbb{Z}_+}$, the following two inequalities hold*:

$$\text{(i)} \qquad \sum_{j=L}^{\infty}\Bigg(\sum_{k=L}^{j} a_{j-k}\, b_k\Bigg)^r \le \sum_{k=0}^{\infty} a_k^r \sum_{j=L}^{\infty} b_j^r.$$

$$\text{(ii)} \qquad \sum_{j=L}^{\infty}\Bigg(\sum_{k=j+1}^{\infty} a_{j-k}\, b_k\Bigg)^r \le \sum_{k=-\infty}^{-1} a_k^r \sum_{j=L}^{\infty} b_j^r.$$

Proof of Theorem 4.3. The proof is rather lengthy, but the idea is transparent. Mainly, our estimates below are based on careful investigations of relations of the supports between two wavelet families, namely the Daubechies wavelet system and the Haar system.

Step 1. *Preparations.*

Fix $f \in B^{s,\tau}_{p,q}$. Let $\{\psi_{k,\ell}\}_{k\in\mathbb{N}_{-1},\ell\in\mathbb{Z}}$ be a sufficiently smooth Daubechies wavelet system as in Proposition 3.5. Observe that, for any $j \in \mathbb{N}_{-1}$ and $m \in \mathbb{Z}$,

$$\langle f, h_{j,m}\rangle = \sum_{k=-1}^{\infty} \sum_{\ell\in\mathbb{Z}} \langle f, \psi_{k,\ell}\rangle \langle \psi_{k,\ell}, h_{j,m}\rangle, \tag{4.4}$$

at least formally. To investigate the coefficients $\langle \psi_{k,\ell}, h_{j,m}\rangle$, we need to employ the various support relations. First, we need some notation. For any $j \in \mathbb{Z}_+$ and $m \in \mathbb{Z}$, let

$$I_{j,m} := [2^{-j}m, 2^{-j}(m+1)],$$

and let

$$I^+_{j,m} := \left[2^{-j}m, 2^{-j}\left(m+\frac{1}{2}\right)\right] \quad \text{and} \quad I^-_{j,m} := \left[2^{-j}\left(m+\frac{1}{2}\right), 2^{-j}(m+1)\right]. \tag{4.5}$$

Let also $I_{-1,m} := I^+_{-1,m} := I^-_{-1,m} := I_{0,m}$. Moreover, for any $j,k \in \mathbb{N}_{-1}$ and $m \in \mathbb{Z}$, let

$$U_{k,m,j} := \{\ell \in \mathbb{Z} : |\operatorname{supp} h_{j,m} \cap \operatorname{supp} \psi_{k,\ell}| > 0\}.$$

Obviously, for any $j,k \in \mathbb{N}_{-1}$ and $m,\ell \in \mathbb{Z}$, we have $\langle \psi_{k,\ell}, h_{j,m}\rangle = 0$ if $\ell \notin U_{k,m,j}$. Next, choose $j,k \in \mathbb{N}_{-1}$ and $m \in \mathbb{Z}$. We consider the following two cases for j and k.

If $k > j$, then, in this case, the following situations may occur:

(i) $\operatorname{supp} \psi_{k,\ell} \subset I^+_{j,m}$;

(ii) $\operatorname{supp} \psi_{k,\ell} \subset I^-_{j,m}$;

(iii) $\operatorname{supp} \psi_{k,\ell}$ is neither a subset of $I^+_{j,m}$ nor of $I^-_{j,m}$.

If (i) or (ii) holds for some $\ell \in \mathbb{Z}$, then, from the moment condition of $\psi_{k,\ell}$, we infer that $\langle \psi_{k,\ell}, h_{j,m}\rangle = 0$. This motivates us to introduce the set

$$U^*_{k,m,j} := \{\ell \in U_{k,m,j} : \operatorname{supp} \psi_{k,\ell} \text{ is neither a subset of } I^+_{j,m} \text{ nor of } I^-_{j,m}\}. \tag{4.6}$$

Then, for any $\ell \in U^*_{k,m,j}$, we obtain

$$|\langle \psi_{k,\ell}, h_{j,m}\rangle| \le 2^{(j+k)/2} \int_{I_{j,m}} |\psi(2^k x - \ell)|\, dx \le 2^{(j+k)/2}\, 2^{-k} \|\psi\|_{L^1} \sim 2^{(j-k)/2}. \tag{4.7}$$

If $k \le j$, then we simply let $U^*_{k,m,j} := U_{k,m,j}$. For any $\ell \in U^*_{k,m,j}$, using the condition that $\int_{\mathbb{R}} h_{j,m}(x)\, dx = 0$, we conclude that

$$\begin{aligned} |\langle \psi_{k,\ell}, h_{j,m}\rangle| &= 2^{(j+k)/2} \left| \int_{I_{j,m}} h(2^j x - m) \right.\\ &\qquad \left. \times \left[\psi(2^k x - \ell) - \psi(2^k x_0 - \ell) + \psi(2^k x_0 - \ell)\right] dx \right| \\ &\le 2^{(j+k)/2} \int_{I_{j,m}} |\psi(2^k x - \ell) - \psi(2^k x_0 - \ell)|\, dx \\ &\le \|\psi'\|_{L^\infty} 2^{(j+k)/2}\, 2^{-j}\, 2^{k-j} \sim 2^{3(k-j)/2}, \end{aligned} \tag{4.8}$$

where we have chosen $x_0 \in I_{j,m}$. In addition, if $j = -1$ (or $k = -1$) in the left-hand sides of both (4.7) and (4.8), then the right-hand sides of both (4.7) and (4.8) still hold with j (or k) replaced by 0.

Therefore, at least formally, we have the following decomposition: for any $j \in \mathbb{N}_{-1}$ and $m \in \mathbb{Z}$,

$$\langle f, h_{j,m} \rangle =: F_{j,m}^{(1)} + F_{j,m}^{(2)},$$

where

$$F_{j,m}^{(1)} := \sum_{k=-1}^{j} \sum_{\ell \in U_{k,m,j}^*} \langle f, \psi_{k,\ell} \rangle \langle \psi_{k,\ell}, h_{j,m} \rangle \tag{4.9}$$

and

$$F_{j,m}^{(2)} := \sum_{k=j+1}^{\infty} \sum_{\ell \in U_{k,m,j}^*} \langle f, \psi_{k,\ell} \rangle \langle \psi_{k,\ell}, h_{j,m} \rangle. \tag{4.10}$$

Moreover, for any $j, k \in \mathbb{N}_{-1}$ and $m \in \mathbb{Z}$, by the definition of $U_{k,m,j}^*$, it is easy to verify that

$$|U_{k,m,j}^*| \sim 1, \tag{4.11}$$

where the implicit positive equivalence constants depend only on the support set of ψ.

Next, we aim to proceed by decomposing the desired estimates of $\delta(f)$ in $b_{p,q}^{s,\tau}$ into several steps. We only deal with the cases for $j, k \in \mathbb{Z}_+$ in the summations of (4.9) and (4.10) because the estimations of the cases for $j = -1$ or $k = -1$ are quite similar.

Step 2. *Estimations related to* $F_{j,m}^{(1)}$.

In this step, we mainly need to estimate

$$\mathrm{I} := \sum_{j=\max\{j_P,0\}}^{\infty} 2^{j(s+1/2-1/p)q} \left(\sum_{\{m:\, Q_{j,m} \subset P\}} \left| \sum_{k=0}^{j} \sum_{\ell \in U_{k,m,j}^*} \langle f, \psi_{k,\ell} \rangle \langle \psi_{k,\ell}, h_{j,m} \rangle \right|^p \right)^{q/p},$$

where P is a dyadic interval. Without loss of generality, we may assume that $P := [0, 2^{-L})$ for some $L \in \mathbb{Z}$.

Step 2.1. Assume that $p \in (0, 1]$. We first consider small dyadic intervals P with $|P| \le 1$. By (4.11), we dominate I by the two parts; that is,

$$\mathrm{I} \lesssim \mathrm{I}_1 + \mathrm{I}_2, \tag{4.12}$$

where

$$\mathrm{I}_1 := \sum_{j=\max\{L,0\}}^{\infty} 2^{j(s+1/2-1/p)q} \left(\sum_{\{m:\, Q_{j,m} \subset P\}} \left| \sum_{k=0}^{L} \sum_{\ell \in U_{k,m,j}^*} \langle f, \psi_{k,\ell} \rangle \langle \psi_{k,\ell}, h_{j,m} \rangle \right|^p \right)^{q/p}$$

and

$$
\text{(4.13)} \qquad \mathrm{I}_2 := \sum_{j=\max\{L,0\}}^{\infty} 2^{j(s+1/2-1/p)q} \times \Bigg(\sum_{\{m:\, Q_{j,m}\subset P\}} \Bigg| \sum_{k=L}^{j} \sum_{\ell\in U^*_{k,m,j}} \langle f, \psi_{k,\ell}\rangle \langle \psi_{k,\ell}, h_{j,m}\rangle \Bigg|^p \Bigg)^{q/p}.
$$

We first estimate I_1. Applying (4.8), we find that

$$
\mathrm{I}_1 \lesssim \sum_{j=L}^{\infty} 2^{j(s+1/2-1/p)q} \Bigg[\sum_{\{m:\, Q_{j,m}\subset P\}} \sum_{k=0}^{L} \sum_{\ell\in U^*_{k,m,j}} |\langle f, \psi_{k,\ell}\rangle|^p \, 2^{3(k-j)p/2} \Bigg]^{q/p}.
$$

Note that, when $j \ge L \ge k$, both $Q_{j,m} \subset P = [0, 2^{-L})$ and $\ell \in U^*_{k,m,j}$ imply $-N_3 \le \ell \le N_3 + 1$, where N_3 is as in (3.2). For simplicity, we only deal with the case where $\ell = 0$ because the proofs of the other cases where $\ell \in \{-N_3, \dots, N_3 + 1\} \setminus \{0\}$ are quite similar. From this, we infer that

$$
\text{(4.14)} \qquad \sum_{j=L}^{\infty} 2^{j(s+1/2-1/p)q} \Bigg[\sum_{\{m:\, Q_{j,m}\subset P\}} \sum_{k=0}^{L} |\langle f, \psi_{k,0}\rangle|^p \, 2^{3(k-j)p/2} \Bigg]^{q/p} = 2^{-Lq/p} \sum_{j=L}^{\infty} 2^{j(s-1)q} \Bigg(\sum_{k=0}^{L} 2^{3kp/2} \, |\langle f, \psi_{k,0}\rangle|^p \Bigg)^{q/p}.
$$

By Lemma 3.6, this is bounded by

$$
2^{-Lq/p} \sum_{j=L}^{\infty} 2^{j(s-1)q} \Bigg[\sum_{k=0}^{L} 2^{-k(s+\tau-1-1/p)p} \Bigg]^{q/p} \|f\|^q_{B^{s,\tau}_{p,q}} \lesssim 2^{-Lq/p} \, 2^{L(s-1)q} \, 2^{-L(s+\tau-1-1/p)q} \|f\|^q_{B^{s,\tau}_{p,q}} = 2^{-L\tau q} \|f\|^q_{B^{s,\tau}_{p,q}},
$$

where the first step used the assumptions $s < 1$ and $\tau \le 1/p$. Thus, we further obtain

$$
\mathrm{I}_1 \lesssim 2^{-L\tau q} \|f\|^q_{B^{s,\tau}_{p,q}},
$$

where the implicit positive constant depends only s, p, q, τ, and N_3.

Next, we estimate I_2, which is a little bit different from the estimation of I_1. Observe that, for any given $j \in \mathbb{N} \cap [L, \infty)$, $k \in \{L, \dots, j\}$, and $\ell \in \mathbb{Z}$,

$$
|\{m \in \mathbb{Z} : Q_{j,m} \subset P \text{ and } |Q_{j,m} \cap \operatorname{supp} \psi_{k,\ell}| > 0\}| \lesssim 2^{j-k},
$$

where the implicit positive constant depends only on N_3. Thus, for any given $L \le k \le j$,

$$
\text{(4.15)} \qquad \sum_{\{m:\, Q_{j,m}\subset P\}} \sum_{\ell\in U^*_{k,m,j}} |\langle f, \psi_{k,\ell}\rangle|^p \lesssim 2^{j-k} \sum_{\{\ell:\, Q_{k,\ell}\subset P\}} |\langle f, \psi_{k,\ell}\rangle|^p.
$$

By this and (4.8), we conclude that

$$\mathrm{I}_2 \lesssim \sum_{j=L}^{\infty} 2^{j(s+1/2-1/p)q}\Big[\sum_{\{m:\,Q_{j,m}\subset P\}}\sum_{k=L}^{j}\sum_{\ell\in U^*_{k,m,j}} |\langle f,\psi_{k,\ell}\rangle|^p 2^{3(k-j)p/2}\Big]^{q/p} \tag{4.16}$$

$$\lesssim \sum_{j=L}^{\infty} 2^{j(s+1/2-1/p)q}\Big[\sum_{k=L}^{j} 2^{j-k}\, 2^{3(k-j)p/2} \sum_{\{\ell:\,Q_{k,\ell}\subset P\}} |\langle f,\psi_{k,\ell}\rangle|^p\Big]^{q/p}$$

$$= \sum_{j=L}^{\infty}\Big[\sum_{k=L}^{j} 2^{(j-k)(s-1)p} \sum_{\{\ell:\,Q_{k,\ell}\subset P\}} 2^{k(s+1/2-1/p)p}|\langle f,\psi_{k,\ell}\rangle|^p\Big]^{q/p}$$

$$= \sum_{j=L}^{\infty}\Big(\sum_{k=L}^{j} a_{j-k}\, b_k\Big)^{q/p},$$

where

$$a_j := 2^{j(s-1)p}$$

and

$$b_j := \sum_{\{\ell:\,Q_{j,\ell}\subset P\}} 2^{j(s+1/2-1/p)p}\,|\langle f,\psi_{j,\ell}\rangle|^p \tag{4.17}$$

for any $j\in\mathbb{Z}_+$. Note that $\{a_j^{q/p}\}_{j\in\mathbb{Z}_+}\in\ell^1$ because $s<1$. Moreover, using Proposition 3.5, we obtain

$$\big\|\{b_j\}_{j\in\mathbb{Z}_+\cap[L,\infty)}\big\|_{\ell^{q/p}} \lesssim 2^{-L\tau p}\|f\|^p_{B^{s,\tau}_{p,q}}. \tag{4.18}$$

From these, (4.16), Lemma 4.4 (i), and Corollary 4.5 (i), we deduce that

$$\mathrm{I}_2 \lesssim 2^{-L\tau q}\|f\|^q_{B^{s,\tau}_{p,q}}. \tag{4.19}$$

Consequently, combining with the estimates of I_1 and I_2, we obtain

$$\mathrm{I} \lesssim 2^{-L\tau q}\|f\|^q_{B^{s,\tau}_{p,q}},$$

where the implicit positive constant is independent of L and f.

Step 2.2. Assume that $p\in(0,1]$. Now, we consider the large dyadic interval, namely $L\in\mathbb{Z}\setminus\mathbb{Z}_+$. In this case, we only need to estimate I_2 as in (4.13) with $L:=0$. Thus, repeating the proof of (4.16) with L replaced by 0 and the proof of (4.19), we then obtain

$$\mathrm{I} \lesssim 2^{-L\tau q}\|f\|^q_{B^{s,\tau}_{p,q}}.$$

Step 2.3. Assume that $p\in(1,\infty)$ and $L\in\mathbb{Z}_+$. In this case, we still dominate I by two parts as in (4.12). Let $\varepsilon\in(0,1-s)$. Such ε exists due to $s<1$.

We first estimate the sum I_1. Applying Hölder's inequality with $1 = 1/p + 1/p'$, (4.8), and (4.11), we find that

$$\mathrm{I}_1 \lesssim \sum_{j=L}^{\infty} 2^{j(s+1/2-1/p)q} \times \left\{ \sum_{\{m:\, Q_{j,m}\subset P\}} \left[\sum_{k=0}^{L} 2^{-k\varepsilon p} \sum_{\ell\in U^*_{k,m,j}} |\langle f, \psi_{k,\ell}\rangle|^p\, 2^{3(k-j)p/2}\right] \left(\sum_{k=0}^{L} 2^{k\varepsilon p'}\right)^{p/p'} \right\}^{q/p}$$
$$\lesssim 2^{L\varepsilon q} \sum_{j=L}^{\infty} 2^{j(s-1-1/p)q} \left[\sum_{k=0}^{L} 2^{k(3/2-\varepsilon)p} \sum_{\{m:\, Q_{j,m}\subset P\}} \sum_{\ell\in U^*_{k,m,j}} |\langle f, \psi_{k,\ell}\rangle|^p\right]^{q/p}.$$

Similarly to (4.14), we only need to deal with the case for $\ell = 0$. From this, Lemma 3.6, $s < 1$, and $\varepsilon < 1 - s \le 1 - s + 1/p - \tau$, we infer that

$$2^{L\varepsilon q} \sum_{j=L}^{\infty} 2^{j(s-1-1/p)q} \left[\sum_{k=0}^{L} 2^{k(3/2-\varepsilon)p} \sum_{\{m:\, Q_{j,m}\subset P\}} |\langle f, \psi_{k,0}\rangle|^p\right]^{q/p}$$
$$= 2^{L\varepsilon q} \sum_{j=L}^{\infty} 2^{j(s-1-1/p)q} \left[\sum_{k=0}^{L} 2^{k(3/2-\varepsilon)p}\, 2^{j-L}\, |\langle f, \psi_{k,0}\rangle|^p\right]^{q/p}$$
$$\lesssim 2^{L(s+\varepsilon-1-1/p)q} \left[\sum_{k=0}^{L} 2^{k(1-\tau-s+1/p-\varepsilon)p}\right]^{q/p} \|f\|^q_{B^{s,\tau}_{p,q}} \lesssim 2^{-L\tau q}\, \|f\|^q_{B^{s,\tau}_{p,q}}.$$

Thus, we have

$$\mathrm{I}_1 \lesssim 2^{-L\tau q} \|f\|^q_{B^{s,\tau}_{p,q}}.$$

Then we estimate the sum I_2. By Hölder's inequality, (4.8), and (4.11) again, we obtain

$$\mathrm{I}_2 \lesssim \sum_{j=L}^{\infty} 2^{j(s+\varepsilon-1-1/p)q} \left[\sum_{k=L}^{j} 2^{k(3/2-\varepsilon)p} \sum_{\{m:\, Q_{j,m}\subset P\}} \sum_{\ell\in U^*_{k,m,j}} |\langle f, \psi_{k,\ell}\rangle|^p\right]^{q/p}. \tag{4.20}$$

Using this and (4.15), we conclude that

$$\mathrm{I}_2 \lesssim \sum_{j=L}^{\infty} 2^{j(s+\varepsilon-1-1/p)q} \left[\sum_{k=L}^{j} 2^{k(3/2-\varepsilon)p} \sum_{\{\ell:\, Q_{k,\ell}\subset P\}} 2^{j-k} |\langle f, \psi_{k,\ell}\rangle|^p\right]^{q/p} \tag{4.21}$$
$$= \sum_{j=L}^{\infty} \left(\sum_{k=L}^{j} c_{j-k}\, b_k\right)^{q/p},$$

where $c_j := 2^{j(s-1+\varepsilon)p}$ and b_j is as in (4.17). Clearly, we have that $\{c_j^{q/p}\}_{j\in\mathbb{Z}_+} \in \ell^1$ because $\varepsilon \in (0, 1-s)$. Applying this, Lemma 4.4 (i), Corollary 4.5 (i), and (4.18), we obtain

$$\mathrm{I}_2 \lesssim 2^{-L\tau q} \|f\|^q_{B^{s,\tau}_{p,q}}.$$

Step 2.4. Assume that $p \in (1, \infty)$ and let $L \in \mathbb{Z} \setminus \mathbb{Z}_+$. In this case, we only need to consider I_2 with L replaced by 0. Repeating the proofs of (4.20) and (4.21) with L replaced by 0, we further obtain

$$\mathrm{I}_2 \lesssim 2^{-L\tau q} \|f\|^q_{B^{s,\tau}_{p,q}}$$

in this case.

Step 3. *Estimations related to* $F^{(2)}_{j,m}$.

This time we need to estimate

$$\mathrm{II} := \sum_{j=\max\{j_P, 0\}}^{\infty} 2^{j(s+1/2-1/p)q} \Big[\sum_{\{m:\, Q_{j,m} \subset P\}} |F^{(2)}_{j,m}|^p \Big]^{q/p}.$$

Here, without loss of generality again, we may assume that $P := [0, 2^{-L})$ for some $L \in \mathbb{Z}$.

Step 3.1. Assume that $p \in (0, 1]$. We observe that, for any given $j \in \mathbb{Z}_+ \cap [L, \infty)$, $k \in \mathbb{Z}_+ \cap (j, \infty)$, and $m \in \mathbb{Z}$ satisfying $Q_{j,m} \subset P$, if $\ell \in \mathbb{Z}$ satisfies that $|\operatorname{supp} \psi_{k,\ell} \cap Q_{j,m}| > 0$, then

$$Q_{k,\ell} \subset [-2^{-k} N_3, 2^{-L} + 2^{-k} N_3] \subset [-2^{-L} N_3, 2^{-L}(N_3 + 1)]. \tag{4.22}$$

To avoid more complicated notation, without loss of generality, we may replace (4.22) simply by $Q_{k,\ell} \subset P$ and further conclude that, for any given $j \in \mathbb{Z}_+ \cap [L, \infty)$ and $k \in \mathbb{Z}_+ \cap (j, \infty)$,

$$\sum_{\{m:\, Q_{j,m} \subset P\}} \sum_{\ell \in U^*_{k,m,j}} |\langle f, \psi_{k,\ell} \rangle|^p \lesssim \sum_{\{\ell:\, Q_{k,\ell} \subset P\}} |\langle f, \psi_{k,\ell} \rangle|^p, \tag{4.23}$$

where the implicit positive constant depends only on N_3. From this and (4.7), we deduce that

$$\begin{aligned}
\mathrm{II} &\lesssim \sum_{j=\max\{L,0\}}^{\infty} 2^{j(s+1-1/p)q} \Big(\sum_{k=j+1}^{\infty} 2^{-kp/2} \sum_{\{\ell:\, Q_{k,\ell} \subset P\}} |\langle f, \psi_{k,\ell} \rangle|^p \Big)^{q/p} \\
&= \sum_{j=\max\{L,0\}}^{\infty} \Big[\sum_{k=j+1}^{\infty} 2^{(j-k)(s+1-1/p)p} \sum_{\{\ell:\, Q_{k,\ell} \subset P\}} 2^{k(s+1/2-1/p)p} |\langle f, \psi_{k,\ell} \rangle|^p \Big]^{q/p} \\
&= \sum_{j=\max\{L,0\}}^{\infty} \Big(\sum_{k=j+1}^{\infty} d_{j-k}\, b_k \Big)^{q/p},
\end{aligned}$$

where $d_j := 2^{j(s+1-1/p)p}$, $j \in \mathbb{Z} \setminus \mathbb{Z}_+$, and b_j is as in (4.17).

Observe that $\{d_j^{q/p}\}_{j \in \mathbb{Z} \setminus \mathbb{Z}_+} \in \ell^1$ because $s > 1/p - 1$. From these, Lemma 4.4 (ii), Corollary 4.5 (ii), and (4.18), it follows that

$$\mathrm{II} \lesssim 2^{-L\tau q} \|f\|^q_{B^{s,\tau}_{p,q}}.$$

Step 3.2. Assume that $p \in (1,\infty)$ and let $\varepsilon \in (0,\infty)$ be sufficiently small such that $s+1-1/p>\varepsilon$. Such ε exists due to $s>1/p-1$. Using Hölder's inequality with $1=1/p+1/p'$, (4.7), (4.11), and (4.23), we conclude that

$$\begin{aligned}
\mathrm{II} &\lesssim \sum_{j=\max\{L,0\}}^{\infty} 2^{j(s+1/2-1/p)q} \Big\{ \sum_{\{m:\, Q_{j,m}\subset P\}} \Big[\sum_{k=j+1}^{\infty} 2^{k\varepsilon p} \\
&\quad \times \sum_{\ell\in U^*_{k,m,j}} |\langle f,\psi_{k,\ell}\rangle|^p\, 2^{(j-k)p/2}\Big] \Big(\sum_{k=j+1}^{\infty} 2^{-k\varepsilon p'}\Big)^{p/p'} \Big\}^{q/p} \\
&\lesssim \sum_{j=\max\{L,0\}}^{\infty} 2^{j(s-\varepsilon+1-1/p)q} \Big[\sum_{k=j+1}^{\infty} 2^{k(-1/2+\varepsilon)p} \sum_{\{\ell:\, Q_{k,\ell}\subset P\}} |\langle f,\psi_{k,\ell}\rangle|^p \Big]^{q/p} \\
&= \sum_{j=\max\{L,0\}}^{\infty} \Big[\sum_{k=j+1}^{\infty} 2^{(j-k)(s-\varepsilon+1-1/p)p} \sum_{\{\ell:\, Q_{k,\ell}\subset P\}} 2^{k(s+1/2-1/p)p} |\langle f,\psi_{k,\ell}\rangle|^p \Big]^{q/p} \\
&= \sum_{j=\max\{L,0\}}^{\infty} \Big(\sum_{k=j+1}^{\infty} e_{j-k}\, b_k \Big)^{q/p},
\end{aligned}$$

where $e_j := 2^{j(s-\varepsilon+1-1/p)p}$, $j\in\mathbb{Z}\setminus\mathbb{Z}_+$, and b_j is as in (4.17). Since $\varepsilon < s+1-1/p$, it follows that $\{e_j\}_{j\in\mathbb{Z}\setminus\mathbb{Z}_+}, \{e_j^{q/p}\}_{j\in\mathbb{Z}\setminus\mathbb{Z}_+} \in \ell^1$. By this, Lemma 4.4 (ii), Corollary 4.5 (ii), and (4.18), we conclude that

$$\mathrm{II} \lesssim 2^{-L\tau q} \|f\|^q_{B^{s,\tau}_{p,q}}.$$

Now, we can summarize all our findings. If (4.4) holds, then Steps 2 and 3 have proved

$$\|\delta(f)\|_{b^{s,\tau}_{p,q}} \lesssim \|f\|_{B^{s,\tau}_{p,q}}.$$

Thus, it remains to show (4.4). Indeed, for any $N\in\mathbb{N}$, let

$$f_N := \sum_{k=-1}^{N} \sum_{\ell\in\mathbb{Z}} \langle f,\psi_{k,\ell}\rangle\, \psi_{k,\ell}.$$

By Proposition 3.5, we obtain $f_N \to f$ in $\mathcal{S}'$ as $N\to\infty$ and $\lambda(f)\in b^{s,\tau}_{p,q}$. From this and Proposition 3.5 again, we infer that $f-f_N \in B^{s,\tau}_{p,q}$, which, combined with Lemma 4.1, (4.1), and (4.2), further implies that, for any $N\in\mathbb{N}$ and $h\in\mathcal{H}$,

$$|\langle f-f_N, h\rangle| \le \|f-f_N\|_{B^{-1+\varepsilon}_{\infty,2}} \|h\|_{B^{1-\varepsilon}_{1,2}} \lesssim \|\lambda(f-f_N)\|_{b^{-1+\varepsilon}_{\infty,2}} \|h\|_{B^1_{1,\infty}}.$$

Using (4.1) and Proposition 3.5, we find that $\lambda(f)\in b^{-1+\varepsilon}_{\infty,2}$. From this, we deduce that, when $N\to\infty$,

$$\|\lambda(f-f_N)\|_{b^{-1+\varepsilon}_{\infty,2}} \to 0.$$

This, together with the proved fact that $h\in B^1_{1,\infty}$ (see (4.2)), further implies that $|\langle f - f_N, h\rangle| \to 0$ as $N\to\infty$. This finishes the proof of (4.4) and hence Theorem 4.3. ∎

For convenience, in view of (1.1), as a special case of Theorem 4.3, we formulate the result for the Triebel–Lizorkin space $F^s_{\infty,p}$ with $p \in (0,\infty)$ as follows.

Corollary 4.6. *If $p \in (0,\infty)$ and $1/p - 1 < s < 1$, the analysis operator $T\colon f \mapsto \delta(f)$ is a bounded linear mapping from $F^s_{\infty,p}$ into $f^s_{\infty,p} = b^{s,1/p}_{p,p}$.*

Remark 4.7. The special case where $s = 0$ and $p = 2$ is of particular interest because, in this case, bmo $= F^0_{\infty,2}$ with the equivalent norms. Here bmo is the class of functions with local bounded mean oscillation in the sense of Goldberg [20]; see Section 7.4 for the precise definition. For the identity bmo $= F^0_{\infty,2}$, we refer to Triebel (Theorem 2.5.8 in [57] and p. 133 of [15]). Therefore, Corollary 4.6 establishes the boundedness of $T\colon \mathrm{bmo} \to b^{0,1/2}_{2,2}$.

5. The synthesis operator

In this section, we investigate the synthesis operator S given by setting

$$\begin{aligned} S : b^{s,\tau}_{p,q} &\longrightarrow B^{s,\tau}_{p,q}, \\ \mu := \{\mu_{j,m}\}_{j\in\mathbb{N}_{-1},m\in\mathbb{Z}} &\longmapsto f_\mu := \sum_{j=-1}^{\infty} \sum_{m\in\mathbb{Z}} \mu_{j,m}\, h_{j,m}. \end{aligned} \tag{5.1}$$

First, we need to show the synthesis operator S is well defined. To this end, we introduce the following concept: for any given $\mu := \{\mu_{j,m}\}_{j\in\mathbb{N}_{-1},m\in\mathbb{Z}}$ in $\mathbb{C}$ and $N \in \mathbb{Z}_+$, the *truncated sequence* $\mu_N := \{(\mu_N)_{j,m}\}_{j\in\mathbb{N}_{-1},m\in\mathbb{Z}}$ is defined by setting, for any $j \in \mathbb{N}_{-1}$ and $m \in \mathbb{Z}$,

$$(\mu_N)_{j,m} := \begin{cases} \mu_{j,m} & \text{if } -1 \le j \le N, \\ 0 & \text{if } j > N. \end{cases}$$

Observe that, for any given μ and the truncated sequence μ_N with $N \in \mathbb{Z}_+$, the sum f_{μ_N} is locally bounded and hence $f_{\mu_N} \in L^1_{\mathrm{loc}}$. Recall that, for any $p \in [1,\infty]$, the *Sobolev space* $W^{1,p}$ on $\mathbb{R}$ is defined to be the set of all $f \in L^p$ such that

$$\|f\|_{W^{1,p}} := \|f\|_{L^p} + \|f'\|_{L^p} < \infty,$$

where f' denotes the weak derivative of f.

We need the following inequality, which is precisely Corollary 1.5 in [19] (A corresponding estimate for $W^{1,p}(0,1)$ with $p \in [1,\infty)$ can also be obtained by using Theorem 1 in Section III.2 of [28]).

Lemma 5.1. *If $p \in [1,\infty]$, then there exists a positive constant C such that, for any $\varphi \in W^{1,p}$,*

$$\sup_{j\in\mathbb{N}_{-1}} 2^{j(3/2-1/p)} \Big(\sum_{m\in\mathbb{Z}} |\langle \varphi, h_{j,m}\rangle|^p \Big)^{1/p} \le C \|\varphi\|_{W^{1,p}}.$$

Lemma 5.2. *Let $\mu := \{\mu_{j,m}\}_{j\in\mathbb{N}_{-1},m\in\mathbb{Z}} \in b^{-1}_{\infty,1}$. Then f_μ defines a tempered distribution on $\mathbb{R}$ and f_{μ_N} converges to f_μ in $\mathcal{S}'$ as $N \to \infty$.*

Proof. For any $N \in \mathbb{Z}_+$, by Lemma 5.1 with $p := 1$ and the disjointness of the supports of $\{h_{j,m}\}_{m\in\mathbb{Z}}$ for any $j \in \{1, \dots, N\}$, we conclude that, for any $\varphi \in \mathcal{S} \subset W^{1,1}$,

$$|\langle f_{\mu_N}, \varphi\rangle| \le \sum_{j=-1}^{N} \sup_{m\in\mathbb{Z}} |\mu_{j,m}| \sum_{m\in\mathbb{Z}} |\langle h_{j,m}, \varphi\rangle|$$

$$\lesssim \|\varphi\|_{W^{1,1}} \sum_{j=-1}^{N} 2^{-j/2} \sup_{m\in\mathbb{Z}} |\mu_{j,m}| \lesssim \|\varphi\|_{W^{1,1}} \|\mu_N\|_{b^{-1}_{\infty,1}} \lesssim \|\varphi\|_{W^{1,1}} \|\mu\|_{b^{-1}_{\infty,1}}. \tag{5.2}$$

This yields the uniform boundedness of the sequence $\{|\langle f_{\mu_N}, \varphi\rangle|\}_{N\in\mathbb{Z}_+}$. Using (5.2), we find that, for any $N, M \in \mathbb{Z}_+$ with $N \ge M$ and for any $\varphi \in \mathcal{S}$,

$$|\langle f_{\mu_N} - f_{\mu_M}, \varphi\rangle| \lesssim \|\varphi\|_{W^{1,1}} \|\mu_N - \mu_M\|_{b^{-1}_{\infty,1}},$$

which further implies that $\{\langle f_{\mu_N}, \varphi\rangle\}_{N\in\mathbb{Z}_+}$ is a Cauchy sequence and hence a convergent sequence. For any $\varphi \in \mathcal{S}$, let

$$\langle f_\mu, \varphi\rangle := \lim_{N\to\infty} \langle f_{\mu_N}, \varphi\rangle.$$

From this and (5.2), it follows that $f_\mu \in \mathcal{S}'$ and f_{μ_N} converges to f_μ in $\mathcal{S}'$ as $N \to \infty$, which completes the proof of Lemma 5.2. ■

Lemma 5.3. *Let $p, q \in (0, \infty]$, $s \in \mathbb{R}$, and $\tau \in [0, 1/p]$ satisfy $s > 1/p - 1 - \tau$. Then, for any $\mu \in b^{s,\tau}_{p,q}$, f_μ defines a tempered distribution on $\mathbb{R}$ and f_{μ_N} converges to f_μ in $\mathcal{S}'$ as $N \to \infty$.*

Proof. Applying Lemma 3.6, Remark 3.2 (ii), and Proposition 3.5, we obtain

$$b^{s,\tau}_{p,q} \hookrightarrow b^{s+\tau-1/p}_{\infty,\infty} \hookrightarrow b^{-1}_{\infty,1}.$$

This, together with Lemma 5.2, further implies the desired conclusion, which completes the proof of Lemma 5.3. ■

Our main result of this section proves the Morrey variant of Proposition 2.6 in [62] under a stronger upper bound and a weaker lower bound.

Theorem 5.4. *Let $p \in (0, \infty)$, $q \in (0, \infty]$, $s \in \mathbb{R}$, and $\tau \in (0, 1/p]$. Then the synthesis operator S is a bounded linear mapping from $b^{s,\tau}_{p,q}$ into $B^{s,\tau}_{p,q}$ if*

$$\max\left\{\frac{1}{p} - 2, \frac{1}{p} - \tau - 1\right\} < s < \frac{1}{p} - \tau. \tag{5.3}$$

Proof. As in the previous section, the proof is rather lengthy but the idea is transparent.

Step 1. *Preparations.*

Fix $\mu := \{\mu_{j,m}\}_{j\in\mathbb{N}_{-1}, m\in\mathbb{Z}} \in b^{s,\tau}_{p,q}$. Now, we rewrite $S(\mu)$ by using sufficiently smooth Daubechies wavelets; see Proposition 3.5. To this end, for any $h \in \mathcal{H}$, since $h \in L^2$ and Ψ as in (3.3) is an orthonormal basis in L^2, it follows that

$$h = \sum_{k=-1}^{\infty} \sum_{\ell\in\mathbb{Z}} \langle h, \psi_{k,\ell}\rangle \psi_{k,\ell}$$

in L^2 and hence in $\mathcal{S}'$. By this, Lemma 5.3, and the assumption that $s > 1/p - \tau - 1$, we conclude that, for any $\phi \in \mathcal{S}$,

$$\langle f_\mu, \phi\rangle = \sum_{j=-1}^{\infty} \sum_{m\in\mathbb{Z}} \mu_{j,m} \langle h_{j,m}, \phi\rangle = \sum_{j=-1}^{\infty} \sum_{m\in\mathbb{Z}} \sum_{k=-1}^{\infty} \sum_{\ell\in\mathbb{Z}} \langle \mu_{j,m} \langle h_{j,m}, \psi_{k,\ell}\rangle \psi_{k,\ell}, \phi\rangle,$$

which further implies that

$$f_\mu = S(\mu) = \sum_{j=-1}^{\infty} \sum_{m\in\mathbb{Z}} \sum_{k=-1}^{\infty} \sum_{\ell\in\mathbb{Z}} \mu_{j,m} \langle h_{j,m}, \psi_{k,\ell}\rangle \psi_{k,\ell} \tag{5.4}$$

in $\mathcal{S}'$. As in the previous section, we need additional information about the coefficients $\langle h, \psi\rangle$ for any $h \in \mathcal{H}$ and $\psi \in \Psi$. For any given $k, j \in \mathbb{N}_{-1}$ and $\ell \in \mathbb{Z}$, let

$$J_{k,\ell,j} := \{m \in \mathbb{Z} : |\operatorname{supp} h_{j,m} \cap \operatorname{supp} \psi_{k,\ell}| > 0\}.$$

Obviously, for any $k, j \in \mathbb{N}_{-1}$ and $\ell \in \mathbb{Z}$, $\langle h_{j,m}, \psi_{k,\ell}\rangle = 0$ if $m \notin J_{k,\ell,j}$. Moreover, similarly to (4.6), for any $k, j \in \mathbb{N}_{-1}$ and $\ell \in \mathbb{Z}$, in the case where $k \geq j$, we let

$$J^*_{k,\ell,j} := \{m \in J_{k,\ell,j} : \operatorname{supp} \psi_{k,\ell} \text{ is neither a subset of } I^+_{j,m} \text{ nor of } I^-_{j,m}\},$$

where $I^+_{j,m}$ and $I^-_{j,m}$ for any $j \in \mathbb{N}_{-1}$ and $m \in \mathbb{Z}$ are the same as in (4.5); in the case where $k < j$, we simply write $J^*_{k,\ell,j} := J_{k,\ell,j}$. For any given $k, j \in \mathbb{N}_{-1}$ and $\ell \in \mathbb{Z}$, from the definition of $J^*_{k,\ell,j}$, we easily infer that

$$|J^*_{k,\ell,j}| \lesssim \begin{cases} 1 & \text{if } k \geq j, \\ 2^{j-k} & \text{if } k < j. \end{cases} \tag{5.5}$$

Next, we claim that the right-hand side of (5.4) unconditionally converges in $\mathcal{S}'$. In what follows, we only consider $k, j \in \mathbb{Z}_+$ in the summations of (5.4) because the proofs involving $k = -1$ or $j = -1$ are quite similar. By the definition of $b^{s,\tau}_{p,q}$, we find that, for any $j \in \mathbb{Z}_+$ and $m \in \mathbb{Z}$,

$$|\mu_{j,m}| \leq \|\mu\|_{b^{s,\tau}_{p,q}} 2^{-j(s+1/2+\tau-1/p)}. \tag{5.6}$$

Using this, (4.7), (5.5), Lemma 2.4 in [76] with $M > \max\{1/p - \tau - s - 1/2, 0\}$, and the assumption that $s < 1/p - \tau$, we find that, for any $\phi \in \mathcal{S}$,

$$\begin{aligned}
\mathrm{U}_1 &:= \sum_{k=0}^{\infty} \sum_{\ell\in\mathbb{Z}} \sum_{j=0}^{k} \sum_{m\in J^*_{k,\ell,j}} |\mu_{j,m}| |\langle h_{j,m}, \psi_{k,\ell}\rangle| |\langle \psi_{k,\ell}, \phi\rangle| \\
&\lesssim \|\mu\|_{b^{s,\tau}_{p,q}} \sum_{k=0}^{\infty} \sum_{\ell\in\mathbb{Z}} \sum_{j=0}^{k} \sum_{m\in J^*_{k,\ell,j}} 2^{-j(s+1/2+\tau-1/p)} 2^{(j-k)/2} 2^{-kM} \frac{1}{(1+|2^{-k}\ell|)^{1+M}} \\
&\lesssim \|\mu\|_{b^{s,\tau}_{p,q}} \sum_{k=0}^{\infty} 2^{-k(1/2+M)} \sum_{\ell\in\mathbb{Z}} \frac{1}{(1+|2^{-k}\ell|)^{1+M}} \sum_{j=0}^{k} 2^{-j(s+\tau-1/p)} \\
&\sim \|\mu\|_{b^{s,\tau}_{p,q}} \sum_{k=0}^{\infty} 2^{-k(1/2+M+s+\tau-1/p)} \sim \|\mu\|_{b^{s,\tau}_{p,q}}.
\end{aligned}$$

On the other hand, from (5.6), (4.8), (5.5), Lemma 2.4 in [76] with $M > \max\{1/p - \tau - s - 1/2, 0\}$, and the assumption that $s > 1/p - \tau - 1$, it follows that, for any $\phi \in \mathcal{S}$,

$$\begin{aligned}
\mathrm{U}_2 &:= \sum_{k=0}^{\infty} \sum_{\ell\in\mathbb{Z}} \sum_{j=k+1}^{\infty} \sum_{m\in J^*_{k,\ell,j}} |\mu_{j,m}| |\langle h_{j,m}, \psi_{k,\ell}\rangle| |\langle \psi_{k,\ell}, \phi\rangle| \\
&\lesssim \|\mu\|_{b^{s,\tau}_{p,q}} \sum_{k=0}^{\infty} \sum_{\ell\in\mathbb{Z}} \sum_{j=k+1}^{\infty} \sum_{m\in J^*_{k,\ell,j}} 2^{-j(s+1/2+\tau-1/p)}\, 2^{3(k-j)/2}\, 2^{-kM} \frac{1}{(1+|2^{-k}\ell|)^{1+M}} \\
&\lesssim \|\mu\|_{b^{s,\tau}_{p,q}} \sum_{k=0}^{\infty} 2^{k(1/2-M)} \sum_{\ell\in\mathbb{Z}} \frac{1}{(1+|2^{-k}\ell|)^{1+M}} \sum_{j=k+1}^{\infty} 2^{-j(s+1+\tau-1/p)} \\
&\sim \|\mu\|_{b^{s,\tau}_{p,q}} \sum_{k=0}^{\infty} 2^{k(1/2-M-s-\tau+1/p)} \sim \|\mu\|_{b^{s,\tau}_{p,q}}.
\end{aligned}$$

Thus, $\mathrm{U}_1 + \mathrm{U}_2 < \infty$ implies the claim holds. This, together with (5.4), further implies that, in $\mathcal{S}'$,

$$f_\mu = \sum_{k=-1}^{\infty} \sum_{\ell\in\mathbb{Z}} \sum_{j=-1}^{\infty} \sum_{m\in J^*_{k,\ell,j}} \mu_{j,m} \langle h_{j,m}, \psi_{k,\ell}\rangle \psi_{k,\ell} =: F_1 + F_2, \tag{5.7}$$

where

$$F_1 := \sum_{k=-1}^{\infty} \sum_{\ell\in\mathbb{Z}} \sum_{j=-1}^{k} \sum_{m\in J^*_{k,\ell,j}} \mu_{j,m} \langle h_{j,m}, \psi_{k,\ell}\rangle \psi_{k,\ell}$$

and

$$F_2 := \sum_{k=-1}^{\infty} \sum_{\ell\in\mathbb{Z}} \sum_{j=k+1}^{\infty} \sum_{m\in J^*_{k,\ell,j}} \mu_{j,m} \langle h_{j,m}, \psi_{k,\ell}\rangle \psi_{k,\ell}.$$

By the definition of $J^*_{k,\ell,j}$, we conclude that

$$\sup_{j,k\geq -1} \sup_{m\in\mathbb{Z}} |\{\ell\in\mathbb{Z} : m \in J^*_{k,\ell,j}\}| < \infty. \tag{5.8}$$

The value of this supremum only depends on the size of $\operatorname{supp}\psi$, i.e., the number N_3. Without loss of generality, we may assume that $P := [0, 2^{-L})$ with $L\in\mathbb{Z}$ below. Note that, for any given $j, k \in \mathbb{N}_{-1}$ and $\ell\in\mathbb{Z}$ satisfying $Q_{k,\ell} \subset P$, if $m\in\mathbb{Z}$ satisfies that $|\operatorname{supp}\psi_{k,\ell} \cap Q_{j,m}| > 0$, then

$$Q_{j,m} \subset [-2^{-k}N_3, 2^{-L} + 2^{-k}N_3] \subset [-2^{-L}N_3, 2^{-L}(N_3+1)]; \tag{5.9}$$

again, to avoid more complicated notation, we replace (5.9) simply by $Q_{j,m} \subset P$ and, using (5.8), further conclude that

$$\sum_{\{\ell:\, Q_{k,\ell}\subset P\}} \sum_{m\in J^*_{k,\ell,j}} |\mu_{j,m}|^p \lesssim \sum_{\{m:\, Q_{j,m}\subset P\}} |\mu_{j,m}|^p, \tag{5.10}$$

where the implicit positive constant depends only on N_3.

Step 2. *We estimate* $\|F_1\|_{B^{s,\tau}_{p,q}}$.

By Proposition 3.5, we only need to estimate $\|\lambda(F_1)\|_{b^{s,\tau}_{p,q}}$. To this end, let

$$\mathrm{I} := \sum_{k=\max\{L,0\}}^{\infty} 2^{k(s+1/2-1/p)q}\Big(\sum_{\{\ell:\,Q_{k,\ell}\subset P\}}\Big|\sum_{j=0}^{k}\sum_{m\in J^*_{k,\ell,j}}\mu_{j,m}\,\langle h_{j,m},\psi_{k,\ell}\rangle\Big|^p\Big)^{q/p}.$$

We estimate I by the following four steps.

Substep 2.1. Assume that $p\in(0,1]$ and $L\in\mathbb{N}$. In this case, we divide I into the following two parts:

$$\mathrm{I}\lesssim \mathrm{I}_1+\mathrm{I}_2,$$

where

$$\mathrm{I}_1 := \sum_{k=L}^{\infty} 2^{k(s+1/2-1/p)q}\Big(\sum_{\{\ell:\,Q_{k,\ell}\subset P\}}\Big|\sum_{j=0}^{L-1}\sum_{m\in J^*_{k,\ell,j}}\mu_{j,m}\,\langle h_{j,m},\psi_{k,\ell}\rangle\Big|^p\Big)^{q/p}$$

and

$$\mathrm{I}_2 := \sum_{k=L}^{\infty} 2^{k(s+1/2-1/p)q}\Big(\sum_{\{\ell:\,Q_{k,\ell}\subset P\}}\Big|\sum_{j=L}^{k}\sum_{m\in J^*_{k,\ell,j}}\mu_{j,m}\,\langle h_{j,m},\psi_{k,\ell}\rangle\Big|^p\Big)^{q/p}.$$

As we will see below, I_2 can be estimated in a straightforward manner, but I_1 needs some additional care. Therefore, we start with the estimation of I_2. Using (4.7) and (5.10), we obtain

$$\begin{aligned}
\mathrm{I}_2 &\lesssim \sum_{k=L}^{\infty} 2^{k(s+1/2-1/p)q}\Big[\sum_{\{\ell:\,Q_{k,\ell}\subset P\}}\sum_{j=L}^{k}\sum_{m\in J^*_{k,\ell,j}}|\mu_{j,m}|^p\,2^{(j-k)p/2}\Big]^{q/p}\\
&\lesssim \sum_{k=L}^{\infty}\Big[\sum_{j=L}^{k}2^{(k-j)(s-1/p)p}\sum_{\{m:\,Q_{j,m}\subset P\}}2^{j(s+1/2-1/p)p}\,|\mu_{j,m}|^p\Big]^{q/p}\\
&= \sum_{k=L}^{\infty}\Big(\sum_{j=L}^{k}\alpha_{k-j}\beta_j\Big)^{q/p},
\end{aligned}\tag{5.11}$$

where $\alpha_j := 2^{j(s-1/p)p}$ and

$$\beta_j := \sum_{\{m:\,Q_{j,m}\subset P\}}2^{j(s+1/2-1/p)p}|\mu_{j,m}|^p\tag{5.12}$$

for any $j\in\mathbb{Z}_+$. Clearly, $\{\alpha_j^{q/p}\}_{j\in\mathbb{Z}_+}\in\ell^1$ because $s<1/p-\tau$ and $\tau>0$. In addition, combining with the definition of $b^{s,\tau}_{p,q}$, we have the upper bound

$$\big\|\{\beta_j\}_{j\in\mathbb{Z}_+\cap[L,\infty)}\big\|_{\ell^{q/p}}\le 2^{-L\tau p}\|\mu\|^p_{b^{s,\tau}_{p,q}}.\tag{5.13}$$

Applying this, Lemma 4.4 (i), and Corollary 4.5 (i), we further obtain

$$\mathrm{I}_2 \lesssim 2^{-L\tau q} \|\mu\|^q_{b^{s,\tau}_{p,q}}.$$

Now, we turn to the estimation of I_1. Estimate (5.10) is too rough for the estimation of I_1. Note that, for any given $k \in \mathbb{N} \cap [L,\infty)$ and $j \in \{0,\dots,L-1\}$ and for any $\ell \in \mathbb{Z}$ satisfying $Q_{k,\ell} \subset P$, $|Q_{j,m} \cap \operatorname{supp} \psi_{k,\ell}| > 0$ implies that $2^{-j}(m+1) > -2^{-k}N_3$ and $2^{-j}m < 2^{-L} + 2^{-k}N_3$, which, combined with $j < L \le k$, further implies that

$$m \in (-N_3 - 1, N_3 + 1). \tag{5.14}$$

To avoid more complicated notation, we replace (5.14) simply by $m = 0$ because the argument for the cases where $m \in (-N_3-1, N_3+1) \setminus \{0\}$ is quite similar. In this sense, by (5.8), we find that, for any given $k \in \mathbb{N} \cap [L,\infty)$ and $j \in \{0,\dots,L-1\}$,

$$\sum_{\{\ell:\, Q_{k,\ell} \subset P\}} \sum_{m \in J^*_{k,\ell,j}} |\mu_{j,m}|^p \sim |\mu_{j,0}|^p. \tag{5.15}$$

Since $s < 1/p - \tau < 1/p$, from (4.7) and (5.15), we deduce that

$$\begin{aligned} \mathrm{I}_1 &\lesssim \sum_{k=L}^{\infty} 2^{k(s+1/2-1/p)q} \left[\sum_{j=0}^{L-1} |\mu_{j,0}|^p \, 2^{(j-k)p/2} \right]^{q/p} \\ &\lesssim 2^{L(s-1/p)q} \left(\sum_{j=0}^{L-1} 2^{jp/2} |\mu_{j,0}|^p \right)^{q/p}. \end{aligned} \tag{5.16}$$

Applying Lemma 3.6, we conclude that, for any $j \in \mathbb{Z}_+$,

$$2^{j(s+\tau+1/2-1/p)} |\mu_{j,0}| \le \|\mu\|_{b^{s+\tau-1/p}_{\infty,\infty}} \lesssim \|\mu\|_{b^{s,\tau}_{p,q}}. \tag{5.17}$$

From this, (5.16), and the assumption that $s < 1/p - \tau$, we infer that

$$\begin{aligned} \mathrm{I}_1 &\lesssim 2^{L(s-1/p)q} \left[\sum_{j=0}^{L-1} 2^{j(1/p-s-\tau)p} \|\mu\|^p_{b^{s,\tau}_{p,q}} \right]^{q/p} \\ &\lesssim 2^{L(s-1/p)q} \, 2^{L(1/p-s-\tau)q} \|\mu\|^q_{b^{s,\tau}_{p,q}} = 2^{-L\tau q} \|\mu\|^q_{b^{s,\tau}_{p,q}}. \end{aligned}$$

Substep 2.2. Assume that $p \in (0,1]$ and $L \in \mathbb{Z} \setminus \mathbb{N}$. In this case, we only need to consider I_2 with L replaced by 0. Repeating the proof of (5.11) with L replaced by 0, we then obtain

$$\mathrm{I} \lesssim 2^{-L\tau q} \|\mu\|^q_{b^{s,\tau}_{p,q}}.$$

This is the desired estimate.

Substep 2.3. Assume that $p \in (1,\infty)$ and $L \in \mathbb{N}$. Let $\varepsilon \in (0,\infty)$ be sufficiently small such that

$$\varepsilon < \frac{1}{p} - \tau - s. \tag{5.18}$$

We first estimate I_2. From (4.7), Hölder's inequality with $1=1/p+1/p'$, (5.5), and (5.10), we deduce that

$$
\begin{aligned}
\mathrm{I}_2 &\lesssim \sum_{k=L}^{\infty} 2^{k(s-1/p)q}\Bigg[\sum_{\{\ell:\,Q_{k,\ell}\subset P\}}\sum_{j=L}^{k}\sum_{m\in J^*_{k,\ell,j}} 2^{j(1/2-\varepsilon)p}|\mu_{j,m}|^p\Bigg(\sum_{j=L}^{k}2^{j\varepsilon p'}\Bigg)^{p/p'}\Bigg]^{q/p}\\
&\lesssim \sum_{k=L}^{\infty} 2^{k(s+\varepsilon-1/p)q}\Bigg[\sum_{j=L}^{k}\sum_{\{m:\,Q_{j,m}\subset P\}} 2^{j(1/2-\varepsilon)p}\,|\mu_{j,m}|^p\Bigg]^{q/p}\\
&= \sum_{k=L}^{\infty}\Bigg[\sum_{j=L}^{k} 2^{(k-j)(s+\varepsilon-1/p)p}\sum_{\{m:\,Q_{j,m}\subset P\}} 2^{j(s+1/2-1/p)p}\,|\mu_{j,m}|^p\Bigg]^{q/p}, \qquad (5.19)
\end{aligned}
$$

which is bounded by $2^{-L\tau q}\|\mu\|^q_{b^{s,\tau}_{p,q}}$ from Lemma 4.4(i), Corollary 4.5(i), (5.18), (5.12), and (5.13).

Now, we turn to the estimation of I_1. By Hölder's inequality, (5.5), (4.7), (5.15), the assumption $s<1/p-\tau<1/p$, (5.17), and (5.18), we find that

$$
\begin{aligned}
\mathrm{I}_1 &\lesssim \sum_{k=L}^{\infty} 2^{k(s-1/p)q}\Bigg[\sum_{\{\ell:\,Q_{k,\ell}\subset P\}}\sum_{j=0}^{L-1}\sum_{m\in J^*_{k,\ell,j}} 2^{j(1/2-\varepsilon)p}|\mu_{j,m}|^p\Bigg(\sum_{j=0}^{L-1}2^{j\varepsilon p'}\Bigg)^{p/p'}\Bigg]^{q/p}\\
&\lesssim 2^{L\varepsilon q}\sum_{k=L}^{\infty} 2^{k(s-1/p)q}\Bigg[\sum_{j=0}^{L-1} 2^{j(1/2-\varepsilon)p}\,|\mu_{j,0}|^p\Bigg]^{q/p}\\
&\lesssim 2^{L(s+\varepsilon-1/p)q}\Bigg[\sum_{j=0}^{L-1} 2^{j(1/2-\varepsilon)p}\,|\mu_{j,0}|^p\Bigg]^{q/p}\\
&\lesssim 2^{L(s+\varepsilon-1/p)q}\Bigg[\sum_{j=0}^{L-1} 2^{j(1/p-s-\tau-\varepsilon)p}\Bigg]^{q/p}\|\mu\|^q_{b^{s,\tau}_{p,q}}\\
&\lesssim 2^{L(s+\varepsilon-1/p)q}\,2^{L(1/p-s-\tau-\varepsilon)q}\,\|\mu\|^q_{b^{s,\tau}_{p,q}} = 2^{-L\tau q}\,\|\mu\|^q_{b^{s,\tau}_{p,q}}.
\end{aligned}
$$

Substep 2.4. Assume that $p\in(1,\infty)$ and $L\in\mathbb{Z}\setminus\mathbb{N}$. In this case, we only need to consider I_2 with L replaced by 0. Repeating the proof of (5.19) with L replaced by 0, we then obtain

$$\mathrm{I}\lesssim 2^{-L\tau q}\,\|\mu\|^q_{b^{s,\tau}_{p,q}}.$$

In conclusion, combining all the above estimates, by Definition 3.3 and Proposition 3.5, we conclude that

$$\|F_1\|_{B^{s,\tau}_{p,q}}\sim\|\lambda(F_1)\|_{b^{s,\tau}_{p,q}}\lesssim\|\mu\|_{b^{s,\tau}_{p,q}}. \qquad (5.20)$$

Step 3. *We estimate* $\|F_2\|_{B^{s,\tau}_{p,q}}$.

To this end, we only need to estimate $\|\lambda(F_2)\|_{b^{s,\tau}_{p,q}}$ due to Proposition 3.5. Recall that, when $k,j\in\mathbb{N}_{-1}$ with $k<j$, $|J^*_{k,\ell,j}|\lesssim 2^{j-k}$; see (5.5).

Substep 3.1. Assume that $p \in (0,1]$. Using (4.8) and (5.10), we conclude that

$$\mathrm{II} := \sum_{k=\max\{L,0\}}^{\infty} 2^{k(s+1/2-1/p)q} \Bigg(\sum_{\{\ell:\, Q_{k,\ell} \subset P\}} \Bigg| \sum_{j=k+1}^{\infty} \sum_{m \in J^*_{k,\ell,j}} \mu_{j,m} \langle h_{j,m}, \psi_{k,\ell} \rangle \Bigg|^p \Bigg)^{q/p}$$

$$\lesssim \sum_{k=\max\{L,0\}}^{\infty} \Bigg[\sum_{j=k+1}^{\infty} 2^{(k-j)(s+2-1/p)p} \sum_{\{m:\, Q_{j,m} \subset P\}} 2^{j(s+1/2-1/p)p} |\mu_{j,m}|^p \Bigg]^{q/p}.$$

This, together with Lemma 4.4(ii), Corollary 4.5(ii), (5.13), and $s > 1/p - 2$, further implies that

$$\mathrm{II} \lesssim 2^{-L\tau q} \|\mu\|^q_{b^{s,\tau}_{p,q}}.$$

This is the desired estimate.

Substep 3.2. Assume that $p \in (1,\infty)$. Since $1/p - \tau > s > 1/p - \tau - 1 \geq -1$, it follows that there exists $\varepsilon \in (0,\infty)$ such that

$$\frac{1}{p'} < \varepsilon < s + 2 - \frac{1}{p}. \tag{5.21}$$

By this, Hölder's inequality, (4.8), (5.5), and (5.10), we find that

$$\mathrm{II} \lesssim \sum_{k=\max\{L,0\}}^{\infty} 2^{k(s+1/2-1/p)q} \Bigg[\sum_{\{\ell:\, Q_{k,\ell} \subset P\}} \sum_{j=k+1}^{\infty} \sum_{m \in J^*_{k,\ell,j}} |\mu_{j,m}|^p \, 2^{3(k-j)p/2} \, 2^{j\varepsilon p}$$

$$\times \Bigg(\sum_{j=k+1}^{\infty} \sum_{m \in J^*_{k,\ell,j}} 2^{-j\varepsilon p'} \Bigg)^{p/p'} \Bigg]^{q/p}$$

$$\lesssim \sum_{k=\max\{L,0\}}^{\infty} 2^{k(s+2-1/p)q} \Bigg\{ \sum_{\{\ell:\, Q_{k,\ell} \subset P\}} \sum_{j=k+1}^{\infty} 2^{j(-3/2+\varepsilon)p} \sum_{m \in J^*_{k,\ell,j}} |\mu_{j,m}|^p$$

$$\times \Bigg[2^{-k} \sum_{j=k+1}^{\infty} 2^{j(1-\varepsilon p')} \Bigg]^{p/p'} \Bigg\}^{q/p}$$

$$\lesssim \sum_{k=\max\{L,0\}}^{\infty} 2^{k(s+2-1/p-\varepsilon)q} \Bigg[\sum_{j=k+1}^{\infty} 2^{j(-3/2+\varepsilon)p} \sum_{\{m:\, Q_{j,m} \subset P\}} |\mu_{j,m}|^p \Bigg]^{q/p}$$

$$= \sum_{k=\max\{L,0\}}^{\infty} \Bigg[\sum_{j=k+1}^{\infty} 2^{(k-j)(s+2-1/p-\varepsilon)p} \sum_{\{m:\, Q_{j,m} \subset P\}} 2^{j(s+1/2-1/p)p} |\mu_{j,m}|^p \Bigg]^{q/p},$$

which, combined with Lemma 4.4(ii), Corollary 4.5(ii), (5.21), and (5.13), further implies that

$$\mathrm{II} \lesssim 2^{-L\tau q} \|\mu\|^q_{b^{s,\tau}_{p,q}}.$$

This is the desired estimate.

In summary, applying Definition 3.3 and Proposition 3.5, we find that

$$\|F_2\|_{B^{s,\tau}_{p,q}} \sim \|\lambda(F_2)\|_{b^{s,\tau}_{p,q}} \lesssim \|\mu\|_{b^{s,\tau}_{p,q}}.$$

This, together with (5.20), further implies that, under the restriction of (5.3),

$$\|f_\mu\|_{B^{s,\tau}_{p,q}} \lesssim \|\mu\|_{b^{s,\tau}_{p,q}},$$

which completes the proof of Theorem 5.4. ∎

Again, as a special case of Theorem 5.4, we state the consequence for the space $F^s_{\infty,p}$.

Corollary 5.5. *Let $p \in (1/2, \infty)$ and $s \in \mathbb{R}$. Then the synthesis operator S is a bounded linear mapping from $b^{s,1/p}_{p,p} = f^s_{\infty,p}$ into $F^s_{\infty,p}$ if $\max\{1/p - 2, -1\} < s < 0$.*

6. Sufficiency for isomorphism

This section aims to complete the sufficiency proof of Theorem 2.2.

Obviously, we have $S \circ T = \mathrm{Id}$ and $T \circ S = \mathrm{id}$, where Id refers to the identity mapping on finite linear combinations of the elements of the Haar system, and id refers to the identity mapping on finite sequences. Now, we deal with extensions of these two assertions.

The following theorem is already known (see Theorem 3.41 in [65]). Here we provide a different proof based on Theorems 4.3 and 5.4.

Theorem 6.1. *Let $p \in (0, \infty)$, $q \in (0, \infty]$, $s \in \mathbb{R}$, and $\tau \in (0, 1/p]$. Then the mapping $T\colon f \mapsto \delta(f)$ is an isomorphism from $B^{s,\tau}_{p,q}$ onto $b^{s,\tau}_{p,q}$ (with the inverse S) if*

$$\frac{1}{p} - 1 < s < \min\left\{1, \frac{1}{p} - \tau\right\}. \tag{6.1}$$

Proof. The injectivity of $T\colon B^{s,\tau}_{p,q} \to b^{s,\tau}_{p,q}$ follows directly from Proposition 4.2(ii).

Next, we show that T is surjective. To do so, we choose $\mu \in b^{s,\tau}_{p,q}$. For any $N \in \mathbb{N}$, let μ_N denote the truncated sequence of μ. Choosing $\varepsilon \in (0, s - 1/p + \tau + 1)$ and applying (4.3), (4.2), and Theorem A with $p := \infty$, $q := 2$, and $s := -1 + \varepsilon$, we conclude that, for any $N \in \mathbb{N}$ and $h \in \mathcal{H}$,

$$|\langle f_\mu - f_{\mu_N}, h\rangle| \lesssim \|f_\mu - f_{\mu_N}\|_{B^{-1+\varepsilon}_{\infty,2}} \|h\|_{B^{1-\varepsilon}_{1,2}} \lesssim \|\mu - \mu_N\|_{b^{-1+\varepsilon}_{\infty,2}} \|h\|_{B^1_{1,\infty}} \to 0$$

as $N \to \infty$, where f_μ is the same as in (5.1). From this and the orthogonality of $\mathcal{H}$, we deduce that, for any $j \in \mathbb{N}_{-1}$ and $m \in \mathbb{Z}$,

$$\langle f_\mu, h_{j,m}\rangle = \lim_{N\to\infty} \langle f_{\mu_N}, h_{j,m}\rangle = \mu_{j,m}$$

and hence $Tf_\mu = \mu$. By Theorem 5.4, we obtain $f_\mu \in B^{s,\tau}_{p,q}$. Therefore, T is surjective. Consequently, T is an isomorphism of $B^{s,\tau}_{p,q}$ onto $b^{s,\tau}_{p,q}$, which completes the proof of Theorem 6.1. ∎

As an immediate consequence of Theorem 6.1, we obtain the following corollary.

Corollary 6.2. *Let the notation be the same as in Theorem 6.1 and let $f \in \mathcal{S}'$. If (6.1) holds, then $f \in B^{s,\tau}_{p,q}$ if and only if there exists $\mu := \{\mu_{j,m}\}_{j\in\mathbb{N}_{-1},m\in\mathbb{Z}} \in b^{s,\tau}_{p,q}$ such that*

$$f = f_\mu := \sum_{j\in\mathbb{N}_{-1}} \sum_{m\in\mathbb{Z}} \mu_{j,m} h_{j,m} \tag{6.2}$$

unconditionally converges in $\mathcal{S}'$. Moreover, for any $f \in B^{s,\tau}_{p,q}$, the Fourier–Haar series f_μ is the unique representation of f with $\mu = \{\langle f, h_{j,m}\rangle\}_{j\in\mathbb{N}_{-1},m\in\mathbb{Z}}$.

Proof. The sufficiency follows directly from Theorem 5.4, and the necessity can be concluded by the injectivity of T. Moreover, we have shown the unconditional convergence in the proof of Theorem 5.4; see (5.4) and (5.7). The uniqueness of f_μ is obvious because T is surjective. This finishes the proof of Corollary 6.2. ∎

Remark 6.3. In Corollary 6.2, we cannot expect the right-hand side of (6.2) converges to f in $B^{s,\tau}_{p,q}$ because the space $B^{s,\tau}_{p,q}$ is not separable when $\tau \in (0, \infty)$.

Because of $B^{s,1/p}_{p,p} = F^s_{\infty,p}$ with $s \in \mathbb{R}$ and $p \in (0, \infty)$, we find a characterization of $F^s_{\infty,p}$ in terms of Fourier–Haar coefficients as well.

Corollary 6.4. *Let $p \in (0, \infty)$ and $s \in \mathbb{R}$. Then the mapping $T: f \mapsto \delta(f)$ is an isomorphism from $F^s_{\infty,p}$ onto $b^{s,1/p}_{p,p}$ (with the inverse S) if $1/p - 1 < s < 0$.*

Remark 6.5. (i) To guarantee that $T: F^s_{\infty,p} \to b^{s,1/p}_{p,p}$ is an isomorphism, Triebel (Theorem 3.41 in [65]) proved the sufficiency of the restriction $1/p - 1 < s < 0$; see also Theorem 3.18 in [67] and Section 5.7 of [23] for further comments.

(ii) Garrigós et al. [18] investigated the boundedness of the dyadic average operators, defined by setting, for any $N \in \mathbb{Z}_+$ and $f \in \mathcal{S}$,

$$E_N f := \sum_{k\in\mathbb{Z}} \mathbf{1}_{Q_{N,k}} 2^N \int_{Q_{N,k}} f(y)\, dy. \tag{6.3}$$

Let $p \in (0, \infty]$ and $s \in \mathbb{R}$. Garrigós et al. (Theorem 1.4 in [18]) proved that the operators E_N admit an extension from $\mathcal{S}$ into $F^s_{\infty,p}$ such that

$$\sup_{N\in\mathbb{Z}_+} \|E_N\|_{F^s_{\infty,p}\to F^s_{\infty,p}} < \infty \tag{6.4}$$

if and only if $-1 < s \le 0$. Let us discuss the relation between (6.4) and the assertion that T is an isomorphism from $F^s_{\infty,p}$ onto $b^{s,1/p}_{p,p}$. Observe that, for any $N \in \mathbb{Z}_+$, if f is a linear combination of elements of the Haar system, then

$$E_N f = \sum_{j=-1}^{N-1} \sum_{k\in\mathbb{Z}} \langle f, h_{j,k}\rangle h_{j,k}. \tag{6.5}$$

Since it always holds that, for any $N \in \mathbb{Z}_+$,

$$\|\delta(E_N f)\|_{b^{s,1/p}_{p,p}} \le \|\delta(f)\|_{b^{s,1/p}_{p,p}},$$

it follows that, if T is an isomorphism from $F^s_{\infty,p}$ onto $b^{s,1/p}_{p,p}$, then, for any $f \in F^s_{\infty,p}$,

$$f = (S \circ T) f = f_\mu$$

with $\mu := \{\langle f, h_{j,k}\rangle\}_{j\in\mathbb{N}_{-1}, k\in\mathbb{Z}}$ and

$$\|E_N f\|_{F^s_{\infty,p}} \lesssim \|\delta(E_N f)\|_{b^{s,1/p}_{p,p}} \lesssim \|f\|_{F^s_{\infty,p}}.$$

This further implies (6.4) under the range $s \in (1/p - 1, 0)$, which is more restrictive than the range $s \in (-1, 0]$ stated in [18].

7. Necessity for isomorphism

In this section, we investigate the necessity of the restrictions in (2.3). Precisely, we obtain the necessity of the lower bound $s > 1/p - 1$, whose proof borrows some ideas from Sections 4-7 of [52], in Subsection 7.1, the necessity of the upper bound $s < 1/p - \tau$ in Subsection 7.2, and the necessity of the upper bound $s < 1$ in Subsection 7.3. Finally, in Subsection 7.4, we apply Theorem 2.2 to lifted local bounded mean oscillation spaces, which is of independent interest.

Now, we first recall the characterization of $B^{s,\tau}_{p,q}$ in terms of local means. Consider $\psi_0, \psi \in \mathcal{S}$ and an integer $M \geq -1$ such that, for some given $\varepsilon \in (0,\infty)$,

$$|\widehat{\psi_0}(\xi)| > 0 \quad \text{for any } |\xi| < 2\varepsilon, \tag{7.1}$$

$$|\widehat{\psi}(\xi)| > 0 \quad \text{for any } \varepsilon/2 < |\xi| < 2\varepsilon, \tag{7.2}$$

and, when $M \in \mathbb{Z}_+$,

$$\int_{\mathbb{R}} x^\alpha \psi(x)\, dx = 0 \quad \text{for any } |\alpha| \leq M. \tag{7.3}$$

Here the conditions (7.1) and (7.2) are called the *Tauberian conditions*. For any $j \in \mathbb{N}$, we let

$$\psi_j(\cdot) := 2^j \psi(2^j \cdot).$$

The following local mean characterization is originally from Theorem 4.1 in [11] for the space $\mathcal{L}^{\lambda,s}_{p,q} = B^{s,\lambda/(nq)}_{p,q}$, $p, q \in (0,\infty)$, $s \in \mathbb{R}$, and $\lambda \in [0,\infty)$. We also refer to Theorem 2 in [71], which treats the homogeneous version with full ranges of p, q. The inhomogeneous analogs were mentioned in p. 3809 of [71].

Lemma 7.1. *Let $s \in (-\infty, M+1)$, $\tau \in [0,\infty)$, and $p, q \in (0,\infty]$. Then the quasi-norms*

$$\sup_{P\in\mathcal{D}} \frac{1}{|P|^\tau} \left\{ \sum_{j=\max\{j_P,0\}}^{\infty} 2^{jsq} \left[\int_P |\psi_j * f(x)|^p\, dx \right]^{q/p} \right\}^{1/q} \tag{7.4}$$

and $\|f\|_{B^{s,\tau}_{p,q}}$ are equivalent in $B^{s,\tau}_{p,q}$.

Usually $\psi_j * f$ in (7.4) is called the *local mean*. Compared with Definition 3.1, the local mean characterization Lemma 7.1 allows us to choose ψ_0, ψ with compact supports, which is convenient as we are considering wavelets with compact supports. In what follows, we always choose functions $\psi_0, \psi \in C_c^\infty$ supported in $(-2^{-4}, 2^{-4})$ satisfying the Tauberian conditions with $\varepsilon = 1/2$. In addition, we assume that ψ satisfies (7.3) for some $M \in \mathbb{N}$. It is known that there exists a subinterval $J \subset [1/4, 3/4]$ and a positive constant c_0 such that $|J| \gtrsim 1$ and, for any $x \in J$,

$$|\psi * h(x)| \geq c_0 \tag{7.5}$$

(see, for instance, p. 110 of [52] or equation (88) in [17]). This is useful for showing certain lower bounds.

We also need the atomic characterization of Besov-type spaces in terms of the following atoms.

Definition 7.2. Let $d, N, L \in (0, \infty)$. A function a_Q is called a (d, N, L)-*atom* (for short, *atom*) on a cube $Q \subset \mathbb{R}$ if

(i) $\operatorname{supp}(a_Q) \subset dQ$;

(ii) $|D^\alpha a_Q(x)| \leq |Q|^{-1/2-|\alpha|}$ for any $x \in \mathbb{R}$ and $\alpha \in \mathbb{Z}_+$ with $\alpha \leq N$;

(iii) $\int_{\mathbb{R}} x^\beta a_Q(x)\, dx = 0$ for any $\beta \in \mathbb{Z}_+$ with $\beta \leq L$.

The following lemma is a part of Theorem 3.33 in [65] with $r := \tau - 1/p$ therein.

Lemma 7.3. *Let $s \in \mathbb{R}$, $p, q \in (0, \infty]$, and $\tau \in [0, \infty)$. Let $d \in (1, \infty)$ and $N, L \in (0, \infty)$ satisfy $N > s + \max\{\tau - 1/p, 0\}$ and $L > \max\{1/p, 1\} - 1 - s$.*

Then $f \in B^{s,\tau}_{p,q}$ if and only if $f \in \mathcal{S}'$ and f can be represented as $f = \sum_{Q \in \mathcal{D}} \lambda_Q a_Q$ in $\mathcal{S}'$ such that $\|\{\lambda_Q\}_{Q \in \mathcal{D}}\|_{b^{s,\tau}_{p,q}} < \infty$. Moreover,

$$\|f\|_{B^{s,\tau}_{p,q}} \sim \|\{\lambda_Q\}_{Q \in \mathcal{D}}\|_{b^{s,\tau}_{p,q}}$$

with the positive equivalence constants independent of f.

7.1. The necessity of the lower bound $s > 1/p - 1$

Theorem 7.4. *Let $p, q \in (0, \infty]$ and $\tau \in [0, 1/p]$. If $s \in (-\infty, 1/p - 1]$, then the operator $T : B^{s,\tau}_{p,q} \to b^{s,\tau}_{p,q}$ is not an isomorphism.*

The case $s < 1/p - 1$

We first consider the case for $s < 1/p - 1$ in Theorem 7.4. To this end, we now provide some test functions and give the upper bound for their norms.

First, we provide the definition of a family of test functions. Let $\eta \in C_c^\infty$ be an odd function supported in $(-2^{-4}, 2^{-4})$ such that η has the vanishing moment of order $M_0 \in \mathbb{N}$ and such that

$$\int_0^{1/2} \eta(x)\, dx \geq 1. \tag{7.6}$$

Fix $m\in\mathbb{N}$ and, for any $\ell\in\mathbb{N}\cap[m,\infty)$, let $\mathcal{P}_\ell^m$ be a set of $2^{m-\ell}$-*separated points* in $[0,1]$; that is,

$$\mathcal{P}_\ell^m := \{x_{\ell,1},\dots,x_{\ell,N_\ell}\}$$

with $N_\ell\le 2^{\ell-m}$ and, for any $\nu\in\{1,\dots,N_\ell-1\}$, $x_{\ell,\nu}<x_{\ell,\nu+1}$ and

$$x_{\ell,\nu+1}-x_{\ell,\nu}\ge 2^{m-\ell}. \tag{7.7}$$

Moreover, let $\mathcal{L}^m$ denote a finite collection of nonnegative integers greater than or equal to m, and let

$$\mathcal{U}^m := \{(\ell,\nu):\ell\in\mathcal{L}^m,\ x_{\ell,\nu}\in\mathcal{P}_\ell^m\}.$$

For any $\ell\in\mathcal{L}^m$, define

$$\mathcal{U}_\ell^m := \{\nu:(\ell,\nu)\in\mathcal{U}^m\}$$

and, for any $(\ell,\nu)\in\mathcal{U}^m$, define

$$\eta_{\ell,\nu}(\cdot) := 2^{\ell/2}\,\eta(2^\ell(\cdot-x_{\ell,\nu}))$$

and, for any $s\in\mathbb{R}$, define

$$g_{s,m} := \sum_{\ell\in\mathcal{L}^m} 2^{-\ell(s+1/2)}\sum_{\nu\in\mathcal{U}_\ell^m} a_{\ell,\nu}\,\eta_{\ell,\nu}, \tag{7.8}$$

where the sequence $\{a_{\ell,\nu}\}_{(\ell,\nu)\in\mathcal{U}^m}$ in $\mathbb{C}$ satisfies $\sup_{(\ell,\nu)\in\mathcal{U}^m}|a_{\ell,\nu}|\le 1$. Let us mention that functions of this type have been considered already in [52].

Proposition 7.5. *Let $s\in(-M_0,\infty)$, $p,q\in(0,\infty]$, and $\tau\in[0,1/p]$.*

(i) *For any $m\in\mathbb{N}$, let $g_{s,m}$ be as in (7.8). Then there exists a positive constant C, independent of m, such that*

$$\|g_{s,m}\|_{B^{s,\tau}_{p,q}}\le C2^{-m/p}\,[\sharp(\mathcal{L}^m)]^{1/q}.$$

(ii) *Assume that the sets $\{\mathcal{L}^m\}_{m\in\mathbb{N}}$ are disjoint and, for any sequence $\{\beta_m\}_{m\in\mathbb{N}}$ in $\mathbb{C}$, let*

$$g := \sum_{m\in\mathbb{N}}\beta_m\,g_{s,m}. \tag{7.9}$$

Then there exists a positive constant C, depending only on s, τ, p, and q, such that

$$\|g\|_{B^{s,\tau}_{p,q}}\le C\Big[\sum_{m\in\mathbb{N}}|\beta_m|^q\,2^{-mq/p}\,\sharp(\mathcal{L}^m)\Big]^{1/q}.$$

Proof. We only show (ii), because (i) follows directly from (ii). Fix $m\in\mathbb{N}$. Clearly, for any $(\ell,\nu)\in\mathcal{U}^m$, there exists a dyadic cube $Q_{\ell,k(\nu)}\in\mathcal{D}_\ell$ such that $x_{\ell,\nu}\in Q_{\ell,k(\nu)}$ and hence $\eta_{\ell,\nu}$ is an atom supported in $3Q_{\ell,k(\nu)}$. Furthermore, (7.7) guarantees that, for any $\ell\in\mathcal{L}^m$,

$$Q_{\ell,k(\nu)}\ne Q_{\ell,k(\nu')}\quad\text{if } \nu\ne\nu'.$$

By these, the assumption that the sets $\{\mathcal{L}^m\}_{m=1}^\infty$ are disjoint, Lemma 7.3, and the fact that $\sup_{(\ell,\nu)\in\mathcal{U}^m}|a_{\ell,\nu}|\le 1$, we obtain

$$\begin{aligned}
\|g\|_{B^{s,\tau}_{p,q}} &\lesssim \sup_{P\in\mathcal{D}}|P|^{-\tau}\Big\{\sum_{m\in\mathbb{N}}|\beta_m|^q\sum_{\{\ell\in\mathcal{L}^m,\ell\ge j_P\}}2^{-\ell q/p}\Big[\sum_{\{\nu\in\mathcal{U}^m_\ell,Q_{\ell,k(\nu)}\subset P\}}|a_{\ell,\nu}|^p\Big]^{q/p}\Big\}^{1/q}\\
&\le \sup_{L\ge0}2^{L\tau}\Big[\sum_{m\in\mathbb{N}}|\beta_m|^q\sum_{\ell\in\mathcal{L}^m}2^{-\ell q/p}\,2^{(\ell-m-L)q/p}\Big]^{1/q}\\
&\quad+\sup_{L\le0}\,2^{L\tau}\Big[\sum_{m\in\mathbb{N}}|\beta_m|^q\sum_{\ell\in\mathcal{L}^m}2^{-\ell q/p}\,2^{(\ell-m)q/p}\Big]^{1/q}\\
&=\sup_{L\ge0}\,2^{L(\tau-1/p)}\Big[\sum_{m\in\mathbb{N}}|\beta_m|^q\,2^{-mq/p}\,\sharp(\mathcal{L}^m)\Big]^{1/q}\\
&\quad+\sup_{L\le0}\,2^{L\tau}\Big[\sum_{m\in\mathbb{N}}|\beta_m|^q\,2^{-mq/p}\,\sharp(\mathcal{L}^m)\Big]^{1/q}\\
&\sim\Big[\sum_{m\in\mathbb{N}}|\beta_m|^q\,2^{-mq/p}\,\sharp(\mathcal{L}^m)\Big]^{1/q},
\end{aligned}$$

where the last step used the assumption $\tau\in[0,1/p]$. This proves (ii) and hence finishes the proof of Proposition 7.5. ∎

In what follows, fix $A\subset\{2^j : j\in\mathbb{Z}_+\}$ and some $N\in\mathbb{N}$ such that

$$2^N\le\sharp(A)<2^{N+1}. \tag{7.10}$$

For any given $n\in\{1,\dots,N\}$, $k\in\mathbb{Z}_+$, and $\mu\in\mathbb{Z}$ and for any $x\in\mathbb{R}$, define

$$\eta_{n,k,\mu}(x):=\eta(2^{k+n}(x-2^{-k}\mu-2^{-k-1})),$$

where η is as in (7.6). Define the *Rademacher function* r_k, $k\in\mathbb{Z}_+$, by setting, for any $t\in\mathbb{R}$,

$$r_0(t):=\begin{cases}1 & \text{if } 0\le t-\lfloor t\rfloor<1/2,\\ -1 & \text{if } 1/2\le t-\lfloor t\rfloor<1\end{cases}$$

and $r_k(t):=r_0(2^kt)$, where the *symbol* $\lfloor t\rfloor$ denotes the largest integer not greater than t.

Lemma 7.6. *Let $s\in\mathbb{R}$, $p,q\in(0,\infty]$, and $\tau\in[0,1/p]$. For any $n\in\{1,\dots,N\}$, $t\in[0,1]$, and $x\in[0,1]$, define*

$$f_{n,t}(x):=2^{n/p-N/q}\sum_{2^k\in A}r_k(t)\,2^{-(k+n)s}\sum_{\mu=0}^{2^k-1}\eta_{n,k,\mu}(x). \tag{7.11}$$

Then, for any $n\in\{1,\dots,N\}$ and $t\in[0,1]$,

$$\|f_{n,t}\|_{B^{s,\tau}_{p,q}}\lesssim1,$$

where the implicit positive constant is independent of n, t, and N.

Proof. Fix $n \in \{1, \dots, N\}$. Let

$$\mathcal{L}^n := \{\ell : 2^{\ell-n} \in A\}.$$

Then we can rewrite $f_{n,t}$ for any $t \in [0, 1]$ as

$$f_{n,t} = 2^{n/p-N/q} \sum_{\ell\in\mathcal{L}^n} r_{\ell-n}(t)\, 2^{-\ell(s+1/2)} \sum_{\mu=0}^{2^{\ell-n}-1} 2^{\ell/2}\, \eta(2^\ell(\cdot - x_{\ell,\mu})),$$

where

$$x_{\ell,\mu} := 2^{n-\ell}\mu + 2^{n-\ell-1} \in [0, 1].$$

Note that, for any $\ell \in \mathcal{L}^n$,

$$|x_{\ell,\mu} - x_{\ell,\mu'}| \ge 2^{n-\ell} \quad \text{if } \mu \ne \mu'.$$

Thus, from Proposition 7.5 (i) and (7.10), it follows that

$$\| f_{n,t}\|_{B^{s,\tau}_{p,q}} \lesssim 2^{n/p-N/q}\, 2^{-n/p}\, (\sharp\mathcal{L}^n)^{1/q} \sim 1.$$

This finishes the proof of Lemma 7.6. ∎

The case $q \le p$

For any $t \in [0, 1]$ and any suitable distribution g, we define

$$T_t g := \sum_{2^j\in A} r_j(t) \sum_{\mu=0}^{2^j-1} \langle h_{j,\mu}, g\rangle\, h_{j,\mu}, \tag{7.12}$$

where r_j denotes the Rademacher function on $[0, 1]$. Let $f_{n,t}$ be as in (7.11) for any $n \in \{1, \dots, N\}$ and $t \in [0, 1]$ and let $\{\alpha_n\}_{n=1}^N$ be given real numbers. For any $t \in [0, 1]$, define

$$f_t := \sum_{n=1}^N \alpha_n f_{n,t}.$$

The key part is the lower bound of $\|T_{t_1} f_{t_2}\|_{B^{s,\tau}_{p,q}}$ for most $t_1, t_2 \in [0, 1]$.

Lemma 7.7. *Let $s \in \mathbb{R}$, $0 < q \le p < \infty$, and $\tau \in [0,\infty)$. Let $f_{n,t}$ be as in (7.11) for any $n \in \{1, \dots, N\}$ and $t \in [0, 1]$. Then there exists a positive constant C, independent of N, such that the lower bounds*

$$\Big(\int_0^1 \int_0^1 \|T_{t_1} f_{N,t_2}\|^q_{B^{s,\tau}_{p,q}}\, dt_1\, dt_2\Big)^{1/q} \ge C\, 2^{N(1/p-1-s)} \tag{7.13}$$

and

$$\Big(\int_0^1 \int_0^1 \|T_{t_1} f_{t_2}\|^q_{B^{s,\tau}_{p,q}}\, dt_1\, dt_2\Big)^{1/q} \ge C\Big|\sum_{n=1}^N \alpha_n\, 2^{n(1/p-1-s)}\Big| \tag{7.14}$$

hold.

The following lemma is Khinchine's inequality; see, for instance, p. 336 of [38].

Lemma 7.8. *For any $p \in (0,\infty)$, there exist constants $0 < A_p \le B_p < \infty$ such that, for any $m \in \mathbb{N}$ and any complex numbers $\{c_i\}_{i=1}^m$,*

$$A_p\Big(\sum_{j=1}^m |c_j|^2\Big)^{p/2} \le \int_0^1 \Big|\sum_{j=1}^m c_j r_j(t)\Big|^p \, dt \le B_p\Big(\sum_{j=1}^m |c_j|^2\Big)^{p/2}.$$

Now, we are ready to show Lemma 7.7, whose proof borrows some ideas from the proof of Proposition 5.2 in [52].

Proof of Lemma 7.7. Observe that the inequality (7.13) follows directly from (7.14) by choosing $\alpha_N := 1$ and $\alpha_n := 0$ for $n < N$. Thus, we only need to prove (7.14).

To this end, note that, for any $k \in \mathbb{Z}_+$ and $t_1, t_2 \in [0,1]$, $\psi_k * T_{t_1} f_{t_2}$ is supported in the interval $[-1,2]$. From this, $q \le p$, and the Minkowski integral inequality for $L^{p/q}([0,1])$ with $q \le p$, it follows that

$$\begin{aligned}
\text{(7.15)}\quad &\Big(\int_0^1\int_0^1 \|T_{t_1} f_{t_2}\|^q_{B^{s,\tau}_{p,q}} \, dt_1\, dt_2\Big)^{1/q} \\
&\quad \ge \Big[\int_0^1\int_0^1 \sum_{2^k\in A} \|2^{ks}\psi_k * T_{t_1} f_{t_2}\|^q_{L^p([0,1])} \, dt_1\, dt_2\Big]^{1/q} \\
&\quad \ge \Big\{\sum_{2^k\in A}\Big(\int_{[0,1]}\Big[\int_0^1\int_0^1 |2^{ks}\psi_k * T_{t_1} f_{t_2}(x)|^q \, dt_1\, dt_2\Big]^{p/q} dx\Big)^{q/p}\Big\}^{1/q}.
\end{aligned}$$

For any $2^k \in A$ and $n \in \{1,\dots,N\}$, let

$$\Lambda_{k,n} := 2^{n(-s+1/p)} \sum_{\mu=0}^{2^k-1} \eta_{n,k,\mu}$$

and

$$\Lambda_k := \sum_{n=1}^N \alpha_n \Lambda_{k,n}.$$

Using these, for any $2^k \in A$ and $t_1, t_2 \in [0,1]$, we rewrite $\psi_k * T_{t_1} f_{t_2}$ as

$$\psi_k * T_{t_1} f_{t_2}(x) = \sum_{2^j\in A} \beta_{j,t_2}(x) r_j(t_1) \quad \text{for any } x \in \mathbb{R},$$

where

$$\beta_{j,t_2}(x) := 2^{-N/q} \sum_{2^\ell\in A} r_\ell(t_2)\, 2^{-\ell s} \sum_{\mu=0}^{2^j-1} \langle \Lambda_\ell, h_{j,\mu}\rangle \psi_k * h_{j,\mu}(x).$$

By Khinchine's inequality (see Lemma 7.8) and picking up only the term $j = k$, we find that, for any $2^k \in A$ and $x \in [0,1]$,

$$\left[\int_0^1 \int_0^1 |2^{ks}\psi_k * T_{t_1} f_{t_2}(x)|^q \, dt_1 \, dt_2\right]^{1/q} \sim \left\{\int_0^1 2^{ksq}\left[\sum_{2^j \in A} |\beta_{j,t_2}(x)|^2\right]^{q/2} dt_2\right\}^{1/q} \geq \left[\int_0^1 2^{ksq} |\beta_{k,t_2}(x)|^q \, dt_2\right]^{1/q}.$$

Using Khinchine's inequality again and picking up only the term $\ell = k$, we conclude that, for any $2^k \in A$ and $x \in [0,1]$,

$$\left[\int_0^1 \int_0^1 |2^{ks}\psi_k * T_{t_1} f_{t_2}(x)|^q \, dt_1 \, dt_2\right]^{1/q} \gtrsim 2^{-N/q}\left|\sum_{\mu=0}^{2^k-1} \langle \Lambda_k, h_{k,\mu}\rangle \psi_k * h_{k,\mu}(x)\right|. \tag{7.16}$$

For each $2^k \in A$ and $\mu \in \{0,\ldots,2^k-1\}$, since the functions $h_{k,\mu}$ and $\eta_{n,k,\mu'}$ have disjoint supports if $\mu \neq \mu'$, it follows that

$$\begin{aligned}
\langle \Lambda_k, h_{k,\mu}\rangle &= \sum_{n=1}^N \alpha_n \, 2^{n(1/p-s)} \sum_{\mu'=0}^{2^k-1} \langle \eta_{n,k,\mu'}, h_{k,\mu}\rangle = \sum_{n=1}^N \alpha_n 2^{n(1/p-s)} \langle \eta_{n,k,\mu}, h_{k,\mu}\rangle \\
&= \sum_{n=1}^N \alpha_n \, 2^{n(1/p-s)} \int_{\mathbb{R}} \eta(2^{k+n}(x - 2^{-k}\mu - 2^{-k-1})) \, 2^{k/2} \, h(2^k x - \mu) \, dx \\
&= \sum_{n=1}^N \alpha_n \, 2^{n(1/p-s)} \int_{\mathbb{R}} \eta(2^n(y - 2^{-1})) \, 2^{-k/2} \, h(y) \, dy \\
&= \sum_{n=1}^N \alpha_n 2^{n(1/p-s)} 2^{-k/2}\left[\int_{-1/2}^0 \eta(2^n y) \, dy - \int_0^{1/2} \eta(2^n y) \, dy\right] \\
&= -\sum_{n=1}^N \alpha_n \, 2^{n(1/p-1-s)} \, 2^{-k/2+1} \int_0^{1/2} \eta(z) \, dz.
\end{aligned}$$

This further implies that (7.16) is bounded from below by

$$\gtrsim 2^{-N/q-k/2}\left|\sum_{\mu=0}^{2^k-1} \psi_k * h_{k,\mu}(x) \sum_{n=1}^N \alpha_n \, 2^{n(1/p-1-s)} \int_0^{1/2} \eta(u) \, du\right|. \tag{7.17}$$

For any $2^k \in A$ and $\mu \in \{0,\ldots,2^k-1\}$, let

$$J_{k,\mu} := 2^{-k}\mu + 2^{-k} J, \tag{7.18}$$

where the interval J is as in (7.5). Clearly, for any $x \in J_{k,\mu}$,

$$\psi_k * h_{k,\mu'}(x) = 0 \quad \text{if } \mu \neq \mu'.$$

Using this and (7.5) and putting (7.17) into the rightmost side of (7.15), we obtain

$$\begin{aligned}
&\Big\{\sum_{2^k\in A}\Big(\int_0^1\Big[\int_0^1\int_0^1 |2^{ks}\psi_k * T_{t_1} f_{t_2}(x)|^q\,dt_1\,dt_2\Big]^{p/q}dx\Big)^{q/p}\Big\}^{1/q}\\
&\quad\gtrsim\Big\{\sum_{2^k\in A}\Big(\sum_{\mu=0}^{2^k-1}\int_{J_{k,\mu}}\Big[2^{-N/q-k/2}|\psi_k * h_{k,\mu}(x)|\\
&\qquad\times\Big|\sum_{n=1}^{N}\alpha_n 2^{n(1/p-1-s)}\int_0^{1/2}\eta(u)\,du\Big|\Big]^p dx\Big)^{q/p}\Big\}^{1/q}\\
&\quad=\Big|\sum_{n=1}^{N}\alpha_n\, 2^{n(1/p-1-s)}\int_0^{1/2}\eta(u)\,du\Big|\\
&\qquad\times\Big(\sum_{2^k\in A}2^{-N}\Big[\sum_{\mu=0}^{2^k-1}\int_{J_{k,\mu}}|2^{-k/2}\psi_k * h_{k,\mu}(x)|^p\,dx\Big]^{q/p}\Big)^{1/q}\\
&\quad\gtrsim\Big|\sum_{n=0}^{N}\alpha_n\, 2^{n(1/p-1-s)}\Big|,
\end{aligned}$$

where the last step used (7.10), (7.6), and the fact that $|J_{k,\mu}|\gtrsim 2^{-k}$. This finishes the proof of Lemma 7.7. ∎

Proof of Theorem 7.4: *Case $s<1/p-1$ and $q\le p$.* Suppose that the analysis operator $T\colon B^{s,\tau}_{p,q}\to b^{s,\tau}_{p,q}$ is an isomorphism for $s<1/p-1$. From Lemma 7.7, it follows that there exist $t_1,t_2\in[0,1]$ such that

$$\|T_{t_1} f_{N,t_2}\|_{B^{s,\tau}_{p,q}}\gtrsim 2^{N(1/p-1-s)}.$$

On the other hand, by Lemma 7.6, we conclude that $\|f_{N,t_2}\|_{B^{s,\tau}_{p,q}}\lesssim 1$. This implies that

$$\|T_{t_1}\|_{B^{s,\tau}_{p,q}\to B^{s,\tau}_{p,q}}\gtrsim 2^{N(1/p-1-s)}.$$

Let

$$E^{\pm}:=\{h_{j,\mu}: 2^j\in A,\ r_j(t_1)=\pm 1,\ \mu\in\{0,\dots,2^j-1\}\}. \tag{7.19}$$

By the definition of T_{t_1}, we obtain $T_{t_1}=P_{E^+}-P_{E^-}$ and hence the operator norm of either P_{E^+} or P_{E^-} is bounded from below by $2^{N(1/p-1-s)}$, which diverges as $N\to\infty$ because $s<1/p-1$. Without loss of generality, we may assume that

$$\|P_{E^+}\|_{B^{s,\tau}_{p,q}\to B^{s,\tau}_{p,q}}\gtrsim 2^{N(1/p-1-s)}.$$

However, since we assume that $T\colon B^{s,\tau}_{p,q}\to b^{s,\tau}_{p,q}$ is an isomorphism, it follows that, for any $f\in B^{s,\tau}_{p,q}$,

$$\|P_{E^+}f\|_{B^{s,\tau}_{p,q}}\lesssim\|\{\langle f,h\rangle\}_{h\in E^+}\|_{b^{s,\tau}_{p,q}}\le\|\delta(f)\|_{b^{s,\tau}_{p,q}}\lesssim\|f\|_{B^{s,\tau}_{p,q}},$$

where the implicit positive constants are independent of N. This further implies that P_{E^+} is uniformly bounded with respect to N, which contradicts the assumption that P_{E^+} is not uniformly bounded. Hence T is not an isomorphism from $B^{s,\tau}_{p,q}$ onto $b^{s,\tau}_{p,q}$ in this case. ∎

The case $q > p$

We need to recall some basic concepts on the complex interpolation of quasi-Banach spaces (see, for example, [2]). Let A_0, A_1 be a couple of quasi-Banach spaces, which are continuously embedding into a large Hausdorff topological vector space Y. The *space* $A_0 + A_1$ is defined by setting

$$A_0 + A_1 := \{h \in Y : \text{there exists } h_i \in A_i,\ i \in \{0,1\},\ \text{such that } h = h_0 + h_1\},$$

and its *quasi-norm* $\|\cdot\|_{A_0+A_1}$ is given by setting, for any $h \in A_0 + A_1$,

$$\|h\|_{A_0+A_1} := \inf\{\|h_0\|_{A_0} + \|h_1\|_{A_1} : h = h_0 + h_1,\ h_i \in A_i,\ i \in \{0,1\}\}.$$

A quasi-Banach space $(A, \|\cdot\|_A)$ of complex-valued sequences is called a *quasi-Banach sequence lattice* if, for any $a = \{a_k\}_{k\in\mathbb{Z}_+} \in A$ and for a complex-valued sequence $b = \{b_k\}_{k\in\mathbb{Z}_+}$ satisfying $|b_k| \le |a_k|$ for all $k \in \mathbb{Z}_+$, we have $b \in A$ and $\|b\|_A \le \|a\|_A$. The *Calderón product* of the couple A_0, A_1 of quasi-Banach sequence lattices is defined to be the set of all complex-valued sequences $a = \{a_k\}_{k\in\mathbb{Z}_+}$ satisfying that, for all $k \in \mathbb{Z}_+$,

$$|a_k| \le |a_k^{(0)}|^{1-\theta}\, |a_k^{(1)}|^{\theta}$$

for some $a^{(0)} = \{a_k^{(0)}\}_{k\in\mathbb{Z}_+} \in A_0$ and $a^{(1)} = \{a_k^{(1)}\}_{k\in\mathbb{Z}_+} \in A_1$. Its *quasi-norm* $\|a\|_{A_0^{1-\theta}A_1^{\theta}}$ is defined by setting

$$\|a\|_{A_0^{1-\theta}A_1^{\theta}} := \inf\left\{\|a^{(0)}\|_{A_0}^{1-\theta}\, \|a^{(1)}\|_{A_1}^{\theta}\right\},$$

where the infimum is taken over all couples $(a^{(0)}, a^{(1)})$ satisfying $|a_k| \le |a_k^{(0)}|^{1-\theta}|a_k^{(1)}|^{\theta}$, for $k \in \mathbb{Z}_+$, with $a^{(0)} \in A_0$ and $a^{(1)} \in A_1$.

Let

$$U := \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$$

be an open strip on the plane, and let $\bar{U}$ be its closure, where, for any $z \in \mathbb{C}$, we denote by $\mathrm{Re}(z)$ its *real part*. A quasi-Banach space A is said to be *analytically convex* if there exists a positive constant C such that, for any analytic function $f : U \to A$ which is continuous on the closed strip $\bar{U}$,

$$\max_{z\in U}\|f(z)\|_A \le C \max_{\mathrm{Re}(z)\in\{0,1\}}\|f(z)\|_A.$$

Assume that $A_0 + A_1$ is analytically convex. The set

$$\mathscr{F} := \mathscr{F}(A_0, A_1)$$

is defined to be the collection of all bounded continuous functions $f : \bar{U} \to A_0 + A_1$ on $\bar{U}$ that are analytic on U, $\mathbb{R} \ni t \mapsto f(j + it) \in A_j$ is a bounded continuous function for each $j \in \{0,1\}$, and is equipped with the quasi-norm

$$\|f\|_{\mathscr{F}} := \max_{j\in\{0,1\}} \sup_{t\in\mathbb{R}}\|f(j + it)\|_{A_j}.$$

The *outer complex interpolation space* $[A_0, A_1]_\theta$ with $\theta \in (0,1)$ is defined by setting

$$[A_0, A_1]_\theta := \{g \in A_0 + A_1 : g = f(\theta) \text{ for some } f \in \mathcal{F}\}$$

and its *norm* $\|\cdot\|_{[A_0,A_1]_\theta}$ is given by setting, for any $g \in [A_0, A_1]_\theta$,

$$\|g\|_{[A_0,A_1]_\theta} := \inf\{\|f\|_{\mathcal{F}} : g = f(\theta) \text{ for some } f \in \mathcal{F}\}.$$

The following conclusion is a part of Corollary 1.11 in [75].

Lemma 7.9. *Let $\theta \in (0,1)$, $s, s_0, s_1 \in \mathbb{R}$, $p, p_0, p_1, q, q_0, q_1 \in (0,\infty]$, and $\tau, \tau_0, \tau_1 \in [0,\infty)$ be such that*

$$s = (1-\theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \tau = (1-\theta)\tau_0 + \theta\tau_1.$$

Then

$$[B^{s_0,\tau_0}_{p_0,q_0}, B^{s_1,\tau_1}_{p_1,q_1}]_\theta \subset B^{s,\tau}_{p,q}.$$

Remark 7.10. The proof of Lemma 7.9 mainly relies on Hölder's inequality and both the definitions of the complex interpolation and the Calderón product. The proof of the converse embedding is much more involved, but we do not need to use the converse embedding for our purpose.

Let $\mathring{B}^{s,\tau}_{p,q}$ denote the closure of the Schwartz space in $B^{s,\tau}_{p,q}$. It is well known that the complex interpolation yields the corresponding operator interpolation (see, for instance, Theorem 4.4.1 in [2]).

Lemma 7.11. *Let, for any $i \in \{0,1\}$, $s_i \in \mathbb{R}$, $p_i, q_i \in (0,\infty]$, and $\tau_i \in [0,\infty)$. Assume that*

$$L : \mathring{B}^{s_0,\tau_0}_{p_0,q_0} \cap \mathring{B}^{s_1,\tau_1}_{p_1,q_1} \to B^{s_0,\tau_0}_{p_0,q_0} \cap B^{s_1,\tau_1}_{p_1,q_1}$$

is linear and there exist two positive constants M_0 and M_1 such that, for any $i \in \{0,1\}$ and $f \in \mathring{B}^{s_0,\tau_0}_{p_0,q_0} \cap \mathring{B}^{s_1,\tau_1}_{p_1,q_1}$,

$$\|L(f)\|_{B^{s_i,\tau_i}_{p_i,q_i}} \le M_i \|f\|_{B^{s_i,\tau_i}_{p_i,q_i}}.$$

Then L can be uniquely extended to a bounded linear mapping from $[\mathring{B}^{s_0,\tau_0}_{p_0,q_0}, \mathring{B}^{s_1,\tau_1}_{p_1,q_1}]_\theta$ to $[B^{s_0,\tau_0}_{p_0,q_0}, B^{s_1,\tau_1}_{p_1,q_1}]_\theta$ with the norm at most $M_0^{1-\theta} M_1^\theta$ for any $\theta \in [0,1]$.

Proof of Theorem 7.4: *Case $s < 1/p - 1$ and $q > p$.* In this case, suppose that $0 < p_0 < q_0 \le \infty$, $\tau_0 \in [0, 1/p_0]$, $s_0 \in (-\infty, 1/p_0 - 1)$, and $T: B^{s_0,\tau_0}_{p_0,q_0} \to b^{s_0,\tau_0}_{p_0,q_0}$ is an isomorphism. Now, we choose $\theta := 1/2$, $s_1 \in \mathbb{R}$, and $p_1 \in (0,\infty]$ satisfying

$$s_1 \in \left(\frac{1}{p_1} - 1, \min\left\{1, \frac{1}{p_1}\right\}\right) \quad \text{and} \quad s_0 - \frac{1}{p_0} + s_1 - \frac{1}{p_1} + 2 < 0.$$

Such s_1 and p_1 exist if s_1 is sufficiently close to $1/p_1 - 1$ because $s_0 \in (-\infty, 1/p_0 - 1)$. Then we take $q_1 \in (0,\infty)$ sufficiently small such that

$$\frac{1}{p_0} - \frac{1}{q_0} + \frac{1}{p_1} - \frac{1}{q_1} \le 0.$$

Finally, choose $\tau_1 \in [0, 1/p_1]$. The aforementioned choices ensure that $T\colon B^{s_1,\tau_1}_{p_1,q_1} \to b^{s_1,\tau_1}_{p_1,q_1}$ is an isomorphism due to Theorem B and that

$$s := \frac{1}{2}s_0 + \frac{1}{2}s_1, \quad \frac{1}{p} = \frac{1}{2p_0} + \frac{1}{2p_1}, \quad \frac{1}{q} := \frac{1}{2q_0} + \frac{1}{2q_1} \quad \text{and} \quad \tau := \frac{1}{2}\tau_0 + \frac{1}{2}\tau_1$$

satisfy $s < 1/p - 1$, $0 < q \le p < \infty$ and $0 \le \tau \le 1/p$. Since $T\colon B^{s_i,\tau_i}_{p_i,q_i} \to b^{s_i,\tau_i}_{p_i,q_i}$ is an isomorphism for any $i \in \{0,1\}$, it follows that T_t is uniformly bounded on $B^{s_i,\tau_i}_{p_i,q_i}$ for all $t \in [0,1]$, where T_t is as in (7.12). Using this and Lemma 7.11, we find that, for any $t \in [0,1]$ and $f \in \mathcal{S}$,

$$\|T_t f\|_{[B^{s_0,\tau_0}_{p_0,q_0}, B^{s_1,\tau_1}_{p_1,q_1}]_\theta} \lesssim \|f\|_{[\mathring{B}^{s_0,\tau_0}_{p_0,q_0}, \mathring{B}^{s_1,\tau_1}_{p_1,q_1}]_\theta}, \tag{7.20}$$

where the implicit positive constant is independent of t and f. Let $f_{N,t} \in \mathcal{S}$ be as in (7.11) and let $\mathring{b}^{s_i,\tau_i}_{p_i,q_i}$ be the closure of finite sequences in $b^{s_i,\tau_i}_{p_i,q_i}$ for any $i \in \{0,1\}$. Then, for any $N \in \mathbb{N}$, there exist $t_1, t_2 \in [0,1]$ such that

$$\begin{aligned} 2^{N(1/p-1-s)} &\lesssim \|T_{t_1} f_{N,t_2}\|_{B^{s,\tau}_{p,q}} \lesssim \|T_{t_1} f_{N,t_2}\|_{[B^{s_0,\tau_0}_{p_0,q_0}, B^{s_1,\tau_1}_{p_1,q_1}]_\theta} \\ &\lesssim \|f_{N,t_2}\|_{[\mathring{B}^{s_0,\tau_0}_{p_0,q_0}, \mathring{B}^{s_1,\tau_1}_{p_1,q_1}]_\theta} \sim \|S_\varphi(f_{N,t_2})\|_{[\mathring{b}^{s_0,\tau_0}_{p_0,q_0}, \mathring{b}^{s_1,\tau_1}_{p_1,q_1}]_\theta} \\ &\sim \|S_\varphi(f_{N,t_2})\|_{(\mathring{b}^{s_0,\tau_0}_{p_0,q_0})^{1-\theta} (\mathring{b}^{s_1,\tau_1}_{p_1,q_1})^\theta} \lesssim \|f_{N,t_2}\|^{1-\theta}_{B^{s_0,\tau_0}_{p_0,q_0}} \|f_{N,t_2}\|^{\theta}_{B^{s_1,\tau_1}_{p_1,q_1}} \lesssim 1, \end{aligned} \tag{7.21}$$

where the first step used Lemma 7.7, the second step used Lemma 7.9, the third step used (7.20), the fourth step used the well-known φ-transform characterization (see Theorem 2.1 in [76] for the precise definition of S_φ) with certain $\varphi \in \mathcal{S}$, the fifth step used the well-known fact that $[A_0, A_1]_\theta = A_0^{1-\theta} A_1^\theta$ if one of A_0, A_1 is analytically convex and separable, the next step used the definition of Calderón product and Theorem 2.1 in [76], and the last step used Lemma 7.6. This leads to an obvious contradiction when N is sufficiently large because $s < 1/p - 1$. Thus, $T\colon B^{s_0,\tau_0}_{p_0,q_0} \to b^{s_0,\tau_0}_{p_0,q_0}$ is not an isomorphism, which then completes the proof of Theorem 7.4. ■

The endpoint case $s = 1/p - 1$

Now, we consider the endpoint case $s = 1/p - 1$. The following proof borrows some ideas from the proof of Theorem 6.1 in [52]. Fix $N \in \mathbb{N}$ sufficiently large. Then we choose $A \subset \{2^n : n \in \mathbb{N}\}$ satisfying that both $\sharp(A) \ge 4^N$ and there exist 4^N disjoint intervals $\{I_\kappa\}_{\kappa=1}^{4^N}$ such that the length of I_κ is N and $\sharp(I_\kappa \cap \log_2(A)) \sim N$ for all $\kappa \in \{1,\dots,4^N\}$. Let

$$Z := \frac{1}{4^N} \sum_{\kappa=1}^{4^N} \sharp(I_\kappa \cap \log_2(A)). \tag{7.22}$$

Clearly, $Z \sim N$.

We consider the following test functions. For any $\kappa \in \{1,\dots,4^N\}$, let b_κ be the largest integer in I_κ,

$$U_\kappa := I_\kappa \cap \log_2(A),$$

and

$$E_\kappa := \{(j,\mu) : j \in U_\kappa,\ \mu \in 2^{j-b_\kappa+N+2}\mathbb{Z} \cap [1,2^j)\}. \tag{7.23}$$

Define

$$\mathcal{L} := \mathcal{L}^N := \{b_\kappa + N : \kappa \in \{1,\dots,4^N\}\}.$$

Let η be as in (7.6) and, for any $\ell \in \mathcal{L}$ and $x \in \mathbb{R}$, let

$$H_\ell(x) := \sum_{\sigma=1}^{N} 2^{-\sigma} \sum_{\{\rho\in\mathbb{N}:\,0<2^{2N+2-\ell}\rho<1\}} \eta(2^{\ell-\sigma}(x - 2^{2N+2-\ell}\rho)). \tag{7.24}$$

Then we have the following upper estimate.

Lemma 7.12. *Let $p,q \in (0,\infty]$ and $\tau \in [0,1/p]$. For any $t \in [0,1]$ and $x \in \mathbb{R}$, let*

$$f_t(x) := 2^{2N(1/p-1/q)} \sum_{\ell\in\mathcal{L}} r_\ell(t)\, 2^{\ell/p'} H_\ell(x),$$

where $\{r_j\}_{j\in\mathbb{N}}$ is the system of Rademacher functions. Then there exists a positive constant C such that

$$\|f_t\|_{B^{1/p-1,\tau}_{p,q}} \le C N^{1/q}$$

holds uniformly in $t \in [0,1]$.

Proof. For any $\sigma \in \{1,\dots,N\}$, let $\mathcal{L}(\sigma) := \{b_\kappa + N - \sigma : \kappa = 1,\dots,2^{2N}\}$. Clearly, the sets $\mathcal{L}(\sigma)$ are disjoint and $\sharp(\mathcal{L}(\sigma)) = 2^{2N}$ for all $\sigma \in \{1,\dots,N\}$. For any $t \in [0,1]$, we rewrite f_t as in (7.9):

$$f_t = \sum_{\sigma=1}^{N} 2^{(2N-\sigma)/p-2N/q}\, g_{\sigma,t},$$

where

$$g_{\sigma,t}(\cdot) := \sum_{\ell\in\mathcal{L}(\sigma)} 2^{\ell(1/p'-1/2)} \sum_{\{\rho\in\mathbb{N}:\,0<2^{2N+2-\ell}\rho<1\}} r_{\ell+\sigma}(t)\, 2^{\ell/2}\, \eta(2^\ell(\cdot - 2^{2N+2-\ell-\sigma}\rho)).$$

We aim to use Proposition 7.5 (ii) with $m := 2N-\sigma$. To this end, observe that the points

$$\{2^{m+2-\ell}\rho\}_{\{\rho\in\mathbb{N}:\,0<2^{2N+2-\ell}\rho<1\}}$$

are $2^{m-\ell}$-separated with $m := 2N-\sigma$. Using this and Proposition 7.5 (ii) with

$$\beta_{2N-\sigma} := 2^{(2N-\sigma)/p-2N/q},$$

we conclude that, for any $t \in [0,1]$,

$$\|f_t\|_{B^{1/p-1,\tau}_{p,q}} \lesssim \Big[\sum_{\sigma=1}^{N} 2^{(2N-\sigma/p-2N/q)q}\, 2^{-(2N-\sigma)q/p}\, \sharp(\mathcal{L}(\sigma))\Big]^{1/q} = N^{1/q},$$

where the implicit positive constant is independent of t. This finishes the proof. ∎

Let

$$E := E^N := \bigcup_{\kappa=1}^{4^N} E_\kappa,$$

where E_κ is as in (7.23). For any $t \in [0,1]$ and $f \in L^1_{\rm loc}$, define

$$T_t f := \sum_{(j,\mu)\in E} r_j(t)\langle f, h_{j,\mu}\rangle h_{j,\mu}.$$

Then we have the following lower bound.

Proposition 7.13. *Let $0 < q \le p < \infty$ and $\tau \in [0,\infty)$. Then there exists a positive constant $C := C_{p,q,\tau}$, depending on p, q, and τ, such that, for sufficiently large N,*

$$\left[\int_0^1 \int_0^1 \|T_{t_1} f_{t_2}\|^q_{B^{1/p-1,\tau}_{p,q}}\, dt_1\, dt_2\right]^{1/q} \ge CNZ^{1/q},$$

where Z is as in (7.22).

Proof. Similarly to (7.15), under the assumption of $q \le p$, we only need to estimate the lower bound of

$$\left\{\sum_{\kappa=1}^{2^{2N}} \sum_{k\in U_\kappa} \left(\int_0^1 \left[\int_0^1 \int_0^1 |2^{k(1/p-1)}\psi_k * T_{t_1} f_{t_2}(x)|^q\, dt_1\, dt_2\right]^{p/q} dx\right)^{q/p}\right\}^{1/q}. \tag{7.25}$$

Using Khinchine's inequality (see Lemma 7.8), we find that (7.25) is equivalent to

$$\begin{aligned}\Bigg\{\sum_{\kappa=1}^{2^{2N}} \sum_{k\in U_\kappa} \Bigg(\int_0^1 \Bigg[\sum_{j\in U_\kappa}\sum_{\ell\in\mathcal{L}}\Bigg| \sum_{\{\mu:\,(j,\mu)\in E_\kappa\}} &\langle 2^{\ell/p'} H_\ell, h_{j,\mu}\rangle \\ &\times 2^{k(1/p-1)}\, 2^{2N(1/p-1/q)}\, \psi_k * h_{j,\mu}(x)\Bigg|^2\Bigg]^{p/2} dx\Bigg)^{q/p}\Bigg\}^{1/q},\end{aligned} \tag{7.26}$$

where H_ℓ is as in (7.24). We drop all the terms with $(j,\ell) \ne (k, b_\kappa + N)$, and hence (7.26) is bounded from below by

$$\begin{aligned}\Bigg\{\sum_{\kappa=1}^{2^{2N}} \sum_{k\in U_\kappa} \Bigg[\int_0^1 \Bigg| \sum_{\{\mu:\,(k,\mu)\in E_\kappa\}} &\langle 2^{(b_\kappa+N)/p'} H_{b_\kappa+N}, h_{k,\mu}\rangle \\ &\times 2^{k(1/p-1)}\, 2^{2N(1/p-1/q)}\, \psi_k * h_{k,\mu}(x)\Bigg|^p dx\Bigg]^{q/p}\Bigg\}^{1/q}.\end{aligned} \tag{7.27}$$

Repeating an argument similar to that used in the proof of the identity (52) in [52], we obtain, for $\mu = \mu_n := 2^{k-b_\kappa+N+2}n$ with $n \in \mathbb{N}$,

$$\langle H_{b_\kappa+N}, h_{k,\mu_n}\rangle = N2^{k/2-b_\kappa-N}\int_0^{1/2} \eta(x)\, dx.$$

Recall that, for any $k \in \mathbb{N}$ and $n \in \mathbb{N}$ with $\mu_n < 2^k$, the interval J_{k,μ_n} is as in (7.18). Applying this, the fact that the intervals $\{J_{k,\mu_n}\}_{n\in\mathbb{N}}$ are disjoint, and (7.5), we conclude that (7.27) is bounded from below by

$$\Big\{\sum_{\kappa=1}^{2^{2N}}\sum_{k\in U_\kappa}\Big[\sum_{\{n:\,0<2^{k-b_\kappa+N+2}n<2^k\}}\int_{J_{k,\mu_n}}\Big|2^{(b_\kappa+N)/p'}N\,2^{k-b_\kappa-N} \times 2^{k(1/p-1)}\,2^{2N(1/p-1/q)}\int_0^{1/2}\eta(u)\,du\Big|^p\,dx\Big]^{q/p}\Big\}^{1/q}.$$

Combining this, the fact that

$$\sum_{\{n:\,0<2^{k-b_\kappa+N+2}n<2^k\}}|J_{k,\mu_n}|\sim 2^{b_\kappa-N-k},$$

and (7.22), we further find that the last expression is bounded from below by

$$\begin{aligned}&\Big\{\sum_{\kappa=1}^{2^{2N}}\sum_{k\in U_\kappa}2^{(b_\kappa-N-k)q/p}\,\big|2^{(b_\kappa+N)/p'}N\,2^{k-b_\kappa-N}\,2^{k(1/p-1)}\,2^{2N(1/p-1/q)}\big|^q\Big\}^{1/q}\\&\quad=\Big(\sum_{\kappa=1}^{2^{2N}}\sum_{k\in U_\kappa}N^q2^{-2N}\Big)^{1/q}=NZ^{1/q}.\end{aligned}$$

This finishes the proof of Proposition 7.13. ∎

Now, we are in the position to prove the endpoint case of Theorem 7.4.

Proof of Theorem 7.4: Case $s=1/p-1$. We consider the following two cases on p,q.

Case (1): $q\le p$.

In this case, applying Lemma 7.12, Proposition 7.13, and an argument similar to that used in the proof of the case $s<1/p-1$ and $q\le p$ of Theorem 7.4, we conclude that

$$\max_{\pm}\|P_{E^\pm}\|_{B^{1/p-1,\tau}_{p,q}}\gtrsim N^{1-1/q}Z^{1/q}\ge N,$$

where $E^\pm$ is as in (7.19). In the last step we have used $Z\sim N$. This means that T is not an isomorphism in this case.

Case (2): $q>p$.

In this case, let $0<p_0<q_0\le\infty$, $s_0:=1/p_0-1$, and $\tau_0\in[0,1/p_0)$. Suppose that $T\colon B^{1/p_0-1,\tau_0}_{p_0,q_0}\to b^{1/p_0-1,\tau_0}_{p_0,q_0}$ is an isomorphism. Now, we choose $p_1\in(1,\infty)$, $s_1=1/p_1-1$, and $\tau_1\in(1/p_1,1)$. Let $\theta\in(0,1)$ be such that

$$(1-\theta)\tau_0+\theta\tau_1\le\frac{1-\theta}{p_0}+\frac{\theta}{p_1}. \tag{7.28}$$

Such $\theta \in (0,1)$ exists due to $\tau_0 \in [0, 1/p_0)$. Then fix $q_1 \in (0, p_1)$ sufficiently small such that

$$\frac{1-\theta}{q_0} + \frac{\theta}{q_1} \geq \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \tag{7.29}$$

Let

$$s := (1-\theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \tau := (1-\theta)\tau_0 + \theta\tau_1.$$

Using (7.28) and (7.29), we obtain $0 < q \leq p < \infty$, $s = 1/p - 1$, and $\tau \in (0, 1/p]$. This, together with Proposition 7.13, further implies that, for any given sufficient large $N \in \mathbb{N}$, there exist $t_1, t_2 \in [0,1]$ such that

$$\|T_{t_1} f_{t_2}\|_{B^{1/p-1,\tau}_{p,q}} \gtrsim N^{1+1/q}. \tag{7.30}$$

On the other hand, by Remark 3.2 (iii), we find that

$$B^{s_1,\tau_1}_{p_1,q_1} = B^{\tau_1-1}_{\infty,\infty}.$$

Moreover, since $\tau_1 \in (1/p_1, 1)$, from Theorem 3.41 in [65], we deduce that $T\colon B^{\tau_1-1}_{\infty,\infty} \to b^{\tau_1-1}_{\infty,\infty}$ is an isomorphism. Applying an argument similar to that used in the proof of (7.21), we find that

$$\|T_{t_1} f_{t_2}\|_{B^{1/p-1,\tau}_{p,q}} \lesssim N^{1/q},$$

which, combined with (7.30), further implies an obvious contradiction. Thus,

$$T : B^{1/p_0-1,\tau_0}_{p_0,q_0} \to b^{1/p_0-1,\tau_0}_{p_0,q_0}$$

is not an isomorphism in this case. This finishes the proof of Theorem 7.4. ∎

7.2. Necessity of the upper bound $s < 1/p - \tau$

The following conclusion shows that, when $s \in (1/p - \tau, \infty)$, $T\colon B^{s,\tau}_{p,q} \to b^{s,\tau}_{p,q}$ is not an isomorphism, which is a part of Theorem 2.1 in [77].

Lemma 7.14. *Let $p, q \in (0, \infty]$, $s \in \mathbb{R}$, and $\tau \in [0, 1/p]$. Then, for any $h \in \mathcal{H}$, $h \in B^{s,\tau}_{p,q}$ if and only if either $s = 1/p$, $q = \infty$, and $s \leq 1/p - \tau$ or $s < 1/p$, $q \in (0, \infty]$, and $s \leq 1/p - \tau$.*

We are forced to consider the endpoint case $s = 1/p - \tau$. Let $p, q \in (0, \infty]$ and $\tau \in [0, 1/p]$. Observe that Lemma 7.14 implies that, when $\tau = 0$ and $q \in (0, \infty)$, $h \notin B^{1/p,0}_{p,q}$. This immediately shows that $S\colon b^{s,\tau}_{p,q} \to B^{s,\tau}_{p,q}$ is not an isomorphism. Thus, in what follows, when $\tau = 0$, we always consider $q = \infty$. For convenience, we say the Besov-type space $B^{1/p-\tau,\tau}_{p,q}$ is a bmo-*type space* if either $p \in (0, \infty)$, $q \in (0, \infty]$, and $\tau \in (0, 1/p]$, or $p \in (0, \infty]$, $q = \infty$, and $\tau = 0$. The following function plays a major role in this situation. Let

$$f := \sum_{j=0}^{\infty} 2^{-j/2} h_{j,0} \tag{7.31}$$

and, for any $N \in \mathbb{N}$, the partial sum f_N of f is defined by setting

$$f_N := \sum_{j=0}^{N-1} 2^{-j/2} h_{j,0}. \tag{7.32}$$

Obviously, f is an unbounded compactly supported function in L^2.

To study the properties of f_N, we need to recall the characterization of $B^0_{\infty,\infty}$ in terms of local means; see, for instance, Subsections 2.4.6 and 2.5.3 in [58]. Let $\psi_0, \psi \in C_c^\infty$ be two functions supported in $(-1/2, 1/2)$ satisfying (7.1), (7.2), and (7.3) with some $M \in \mathbb{N}$. Then, for any $f \in B^0_{\infty,\infty}$,

$$\|f\|_{B^0_{\infty,\infty}} \sim \sup_{x\in\mathbb{R}} \sup_{k\in\mathbb{Z}_+} |\psi_k * f(x)|. \tag{7.33}$$

Now, we present some properties related to f_N as follows.

Lemma 7.15. *Let f_N for any $N \in \mathbb{N}$ be the same as in (7.32). Then the following assertions hold.*

(i) *The sequence $\{f_N\}_{N\in\mathbb{N}}$ is unbounded in all the* bmo-*type spaces.*

(ii) *The sequence $\{\delta(f_N)\}_{N\in\mathbb{N}}$ is bounded in all the sequence spaces of* bmo-*type spaces.*

Proof. Fix a bmo-type space $B^{1/p-\tau,\tau}_{p,q}$ for some p, q, and τ, and consider its sequence space $b^{1/p-\tau,\tau}_{p,q}$.

We first prove (i). Let ψ_0 and ψ be the same as in (7.33). Without loss of generality, we may assume that

$$\int_{-1/2}^{0} \psi(z)\, dz \neq 0.$$

Fix $N \in \mathbb{N}$. By Lemma 7.14, we conclude

$$f_N \in B^{1/p-\tau,\tau}_{p,q}.$$

Note that, for any $j \in \{0, \dots, N-1\}$ and $k \geq N-1$, $h_{j,0} \equiv 2^{j/2}$ on $(0, 2^{-k-1})$. Therefore, when $k \geq N-1$,

$$\begin{aligned}\psi_k * f_N(0) &= 2^k \sum_{j=0}^{N-1} \int_0^{2^{-j}} \psi(-2^k y)\, 2^{-j/2} h_{j,0}(y)\, dy \\ &= 2^k \sum_{j=0}^{N-1} \int_0^{2^{-k-1}} \psi(-2^k y)\, dy = N \int_0^{1/2} \psi(-z)\, dz.\end{aligned}$$

This, together with (7.33), further implies that

$$\|f_N\|_{B^0_{\infty,\infty}} \gtrsim N. \tag{7.34}$$

Moreover, by Lemma 3.6, we find that

$$B^{1/p-\tau,\tau}_{p,q} \hookrightarrow B^0_{\infty,\infty}. \tag{7.35}$$

Using this and (7.34), we conclude that

$$\|f_N\|_{B^{1/p-\tau,\tau}_{p,q}} \gtrsim \|f_N\|_{B^0_{\infty,\infty}} \gtrsim N.$$

Thus, (i) holds.

Next, we show (ii). Notice that

$$\sup_{N\in\mathbb{N}} \|\delta(f_N)\|_{b^{1/p-\tau,\tau}_{p,q}} \le \sup_{L\in\mathbb{Z}} 2^{L\tau}\Big[\sum_{j=\max\{L,0\}}^{\infty} 2^{j(1/p-\tau+1/2-1/p)q}\, 2^{-jq/2}\Big]^{1/q} \sim \sup_{L\in\mathbb{Z}} 2^{L\tau}\, 2^{-\max\{L,0\}\tau} = 1 < \infty, \tag{7.36}$$

which still holds with a slight modification made when $q = \infty$. This finishes the proof of (ii) and hence Lemma 7.15. ∎

Concerning f in (7.31), we have the following lemma.

Lemma 7.16. *Let f be the same as in (7.31). Then the following assertions hold.*

(i) *The function f does not belong to any* bmo*-type space.*

(ii) *The sequence $\delta(f)$ belongs to all the sequence spaces of* bmo*-type spaces.*

Proof. Fix a bmo-type space $B^{1/p-\tau,\tau}_{p,q}$ for some p, q, and τ.

We first show (i). To this end, we suppose that $f \in B^{1/p-\tau,\tau}_{p,q}$. By (6.5), we conclude that, for any $N \in \mathbb{N}$,

$$E_N f = \sum_{j=0}^{N-1} 2^{-j/2}\, h_{j,0} = f_N,$$

where E_N is the same as in (6.3). From this, Theorem 1.9(ii) in [17], and the embedding (7.35), we deduce that, for any $N \in \mathbb{N}$,

$$\|f_N\|_{B^0_{\infty,\infty}} = \|E_N f\|_{B^0_{\infty,\infty}} \le \sup_{N\in\mathbb{Z}_+} \|E_N\|_{B^0_{\infty,\infty}\to B^0_{\infty,\infty}} \|f\|_{B^0_{\infty,\infty}} \lesssim \|f\|_{B^{1/p-\tau,\tau}_{p,q}} < \infty.$$

This contradicts (7.34) and hence $f \notin B^{1/p-\tau,\tau}_{p,q}$. This finishes the proof of (i).

On the other hand, (ii) follows directly from (7.36), which then completes the proof of Lemma 7.16. ∎

As an immediate consequence of Lemma 7.16, we obtain the following interesting result.

Corollary 7.17. *Let $p, q \in (0, \infty]$ and $\tau \in [0, 1/p]$. If the space $B^{1/p-\tau,\tau}_{p,q}$ is a* bmo*-type space, then*

$$f \in S(b^{1/p-\tau,\tau}_{p,q}) \setminus B^{1/p-\tau,\tau}_{p,q},$$

where f is as in (7.31) and S is as in (5.1).

Remark 7.18. In Theorem 1.10 of [19], Garrigós, Seeger, and Ullrich have shown that, for any $p \in [1,\infty)$,

$$f \in S(b^{1/p,0}_{p,\infty}) \setminus B^{1/p,0}_{p,\infty}, \tag{7.37}$$

where f is the same as in (7.31). Let us mention that Lemma 7.16(i) implies that (7.37) holds for any $p \in (0,\infty]$ because, from the Sobolev-type embeddings (see, for instance, Corollary 2.2 in [76]), it follows that $B^{1/p,0}_{p,\infty} \hookrightarrow B^{0}_{\infty,\infty}$ for any $p \in (0,\infty]$.

7.3. Necessity of the upper bound $s < 1$

In this subsection, we work with a standard extremal function for the Haar system. Let ρ be a smooth function such that $\rho \in C_{\mathrm{c}}^{\infty}$, $\operatorname{supp}\rho \subset [-1,2]$, and $\rho(x) = 1$ if $x \in [0,1]$. For any $x \in \mathbb{R}$, let

$$f(x) := \rho(x)x. \tag{7.38}$$

Clearly, $f \in C_{\mathrm{c}}^{\infty}$.

Proposition 7.19. *Let $p, q \in (0,\infty]$ and $\tau \in [0, 1/p]$. Then T is not an isomorphism from $B^{1,\tau}_{p,q}$ onto $b^{1,\tau}_{p,q}$.*

Proof. We consider the following two cases on $q \in (0,\infty]$.

Case (1): $q \in (0,\infty)$.

Let f be the same as in (7.38). Since $f \in C_{\mathrm{c}}^{\infty}$, it easily follows $f \in B^{1,\tau}_{p,q}$. Now, we turn to the Fourier–Haar series of f. Indeed, we only need to consider the Fourier–Haar series of f on $[0,1)$. Elementary calculations yield that, for any $x \in [0,1)$,

$$f(x) = \frac{1}{2} - \frac{1}{4}\sum_{j=0}^{\infty} 2^{-3j/2} \sum_{k=0}^{2^j-1} h_{j,k}(x).$$

By this and $q \in (0,\infty)$, we find that

$$\|T(f)\|_{b^{1,\tau}_{p,q}} \gtrsim \left[\sum_{j=0}^{\infty} 2^{j(1+1/2-1/p)q}\Bigg(\sum_{\{m:\, Q_{j,m}\subset[0,1)\}} 2^{-3jp/2}\Bigg)^{q/p}\right]^{1/q} = \left(\sum_{j=0}^{\infty} 1\right)^{1/q} = \infty,$$

which still holds with a slight modification made when $p = \infty$. This shows that

$$T(f) \notin b^{1,\tau}_{p,q},$$

and hence the desired conclusion holds in this case.

Case (2). $q = \infty$.

Let $\phi \in C_{\mathrm{c}}^{\infty}(0,1)$ satisfy $\phi \equiv 1$ on $[1/4, 3/4]$. For any large $N \in (2,\infty) \cap \mathbb{N}$, let

$$\mathcal{E}_N := \left\{ j \in \mathbb{N} : \frac{N}{4} < j < \frac{N}{2} \right\}$$

and, for any $j \in \mathcal{E}_N$ and $x \in \mathbb{R}$, let

$$\phi_j(x) := e^{2\pi i 2^j x}\phi(x).$$

Next, we consider the following two cases on $p \in (0, \infty]$.

If $p \in (0, \infty)$, then, for any $t \in [0, 1]$ and $x \in \mathbb{R}$, define

$$f_{N,t}(x) := \sum_{j\in\mathcal{E}_N} \frac{r_j(t)}{2^j}\phi_j(x),$$

where $\{r_j\}_{j\in\mathcal{E}_N}$ are the Rademacher functions.

Using Lemma 7.3 in [18] and Lemma 7.1, we find that, for any $t \in [0, 1]$,

$$\begin{aligned}
\|f_{N,t}\|_{B^{1,\tau}_{p,\infty}} &\lesssim \sup_{P\in\mathfrak{D}} |P|^{-\tau} \sup_{k\in\mathbb{Z}_+\cap[j_P,\infty)} 2^{k(3/2-1/p)} \left\| \sum_{j\in\mathcal{E}_N} 2^{-j}|\psi_k * \phi_j| \right\|_{L^p(P)} \\
&\lesssim \sup_{P\in\mathfrak{D}} |P|^{-\tau} \sup_{k\in\mathbb{Z}_+\cap[j_P,\infty)} 2^{k(3/2-1/p)} \sum_{j\in\mathcal{E}_N} 2^{-j}\, 2^{-|j-k|M}\, 2^{-k/2}\, |P\cap[0,1]|^{1/p} \\
&\lesssim \sup_{P\in\mathfrak{D}} |P|^{-\tau}|P\cap[0,1]|^{1/p} \sup_{k\in\mathbb{Z}_+} 2^{k(1-1/p)} \sum_{j\in\mathbb{Z}_+} 2^{-j}\, 2^{-|j-k|M} \\
&\sim \sup_{P\in\mathfrak{D}} |P|^{-\tau}|P\cap[0,1]|^{1/p} \sup_{k\in\mathbb{Z}_+} 2^{-k/p} \sim 1,
\end{aligned}$$

where the second step used the fact that, for any $k, j \in \mathbb{Z}_+$,

$$|\psi_k * \phi_j| \lesssim 2^{-|j-k|M}\, 2^{-k/2}$$

with $M \in \mathbb{N} \cap (1, \infty)$ (see Lemma 7.3 in [18]) and the implicit positive constants are independent of t.

On the other hand, we have

$$\|T(f_{N,t})\|_{b^{1,\tau}_{p,\infty}} \ge 2^{N(3/2-1/p)}\left(\sum_{\{m\in\mathbb{Z}:\, Q_{k,m}\subset(1/4,3/4)\}} |\langle f_{N,t}, h_{N,m}\rangle|^p\right)^{1/p}.$$

Using this and the estimation of (8.10) in [19], we find that there exists $t_1 \in [0, 1]$ such that

$$\|T(f_{N,t_1})\|_{b^{1,\tau}_{p,\infty}} \gtrsim \sqrt{N}.$$

Therefore, T is not an isomorphism from $B^{1,\tau}_{p,\infty}$ onto $b^{1,\tau}_{p,\infty}$ in the case where $p \in (0, \infty)$.

If $p = \infty$, then, for any $x \in \mathbb{R}$, define

$$f_N(x) := \sum_{j\in\mathcal{E}_N} 2^{-j}\phi_j(x).$$

Then, using the estimate in (8.12) of [19] with $q := \infty$, we obtain $\|f_N\|_{B^1_{\infty,\infty}} \lesssim 1$. However, from inequality (8.14) in [19], it follows that

$$\|T(f_N)\|_{b^1_{\infty,\infty}} \gtrsim N.$$

This further implies the desired conclusion in this case, which completes the proof of Lemma 7.19. ∎

Remark 7.20. By an interpolation argument similar to (7.21) and Proposition 7.19, one can easily find that, when $s \in (1, \infty)$, $p, q \in (0, \infty]$, and $\tau \in [0, 1/p]$, T is not an isomorphism from $B^{s,\tau}_{p,q}$ onto $b^{s,\tau}_{p,q}$. We omit further details.

We summarize the results of this subsection as follows.

Corollary 7.21. *Let $p, q \in (0, \infty]$, $s \in \mathbb{R}$, and $\tau \in [0, 1/p]$. If $T\colon B^{s,\tau}_{p,q} \to b^{s,\tau}_{p,q}$ is an isomorphism, then*

$$s \in \Big(\frac{1}{p} - 1, \min\Big\{1, \frac{1}{p} - \tau\Big\}\Big).$$

Proof. The necessity of the upper bound has been shown in Subsections 7.2 and 7.3. The necessity of the lower bound is treated in Sections 7.1. ∎

7.4. Lifted local bounded mean oscillation spaces

In this subsection, we consider the Haar characterization of some spacial cases of Besov-type spaces, namely the local bounded mean oscillation spaces, which have a close connection with Navier–Stokes equations (see, for instance, [29] and Chapter 6 of [65]). For more details of bounded mean oscillation spaces, we refer to [8, 26] and the references therein.

Recall that the *space* bmo is the collection of all $f \in L^1_{\mathrm{loc}}$ such that

$$\|f\|_{\mathrm{bmo}} := \sup_{|P|=1} \int_P |f(x)|\,dx + \sup_{|P|\le 1} \frac{1}{|P|} \int_P |f(x) - f_P|\,dx < \infty,$$

where, for any interval P,

$$f_P := \frac{1}{|P|} \int_P f(x)\,dx$$

and the suprema are taken over all finite intervals P, either with length equal to 1 or with length not more than 1. Let $s \in \mathbb{R}$. The *lift operator* J^s is defined by setting, for any $f \in \mathcal{S}'$,

$$J^s f := \big[(1 + |\cdot|^2)^{-s/2} \widehat{f}\,\big]^\vee,$$

which is a bijection on both $\mathcal{S}$ and $\mathcal{S}'$ onto themselves. Then, for any $s \in \mathbb{R}$, the *lifted local bounded mean oscillation spaces* bmo^s is defined by setting

$$\mathrm{bmo}^s := J^s\mathrm{bmo} := \{J^s f : f \in \mathrm{bmo}\}.$$

It is known that

$$\mathrm{bmo}^s = B^{s,1/2}_{2,2} = F^s_{\infty,2} \tag{7.39}$$

in the sense of equivalent norms; see, for instance, equation (3.193) in [65]. By this, we can immediately formulate the consequences of our general results with respect to this special case. We first consider $s = 0$. Let us start with the Daubechies wavelet characterization of bmo. This yields

$$\|f\|_{\mathrm{bmo}} \sim \|\{\langle f, \psi_{j,k}\rangle\}_{j\in\mathbb{N}_{-1},k\in\mathbb{Z}}\|_{b^{0,1/2}_{2,2}} \tag{7.40}$$

for any pair (ϕ, ψ) of sufficiently smooth compactly supported generators of a wavelet system. Wojtaszczyk (see Exersize 8.8 in [69] with BMO instead of bmo) and Triebel (see Section 3.3.1 in [65]) have shown that $\phi, \psi \in C^1$ is sufficient. More than 40 years ago, Wojtaszczyk [68] has shown a similar characterization as in (7.40) of BMO(0, 1) by using the Franklin system.

Applying (7.39) and Theorem 2.2, we obtain the following conclusion.

Corollary 7.22. *Let $s \in \mathbb{R}$. Then the mapping $T\colon f \mapsto \delta(f)$ is an isomorphism from* bmo^s *onto $b^{s,1/2}_{2,2}$ if and only if $s \in (-1/2, 0)$.*

Remark 7.23. (i) As pointed out in Remark 3.45 of [65] by Triebel, it is not clear whether the space bmo^s admits the Haar expansion if $s \in (-1, -1/2]$. From Corollary 7.22, we infer that, when $s \le -1/2$, the Haar characterization for the space bmo^s is no longer true. Notably, this failure can be also deduced from the argument in Remark 12.1 of [18], due to the counterexamples developed in [52]. This fact gives a negative answer to aforementioned question.

(ii) The negative statement in (i) concerning bmo can be supplemented by a positive result in the homogeneous situation due to Chang and Fefferman [8]. Let $\mathrm{BMO_d}$ denote the *dyadic* BMO, i.e., the class of all locally integrable functions f on $\mathbb{R}$ such that

$$\sup_{P \in \mathcal{D}} \frac{1}{|P|} \int_P |f(x) - f_P|\, dx < \infty.$$

Obviously, $\mathrm{BMO} \hookrightarrow \mathrm{BMO_d}$. In Proposition on p. 26 of [8], Chang and Fefferman proved that a function $f \in L^1_{\mathrm{loc}}$ belongs to $\mathrm{BMO_d}$ if and only if there exists a positive constant C such that, for any dyadic interval $P \in \mathcal{D}$,

$$\sum_{j=\max\{j_P,0\}}^{\infty} \sum_{\{m\colon Q_{j,m} \subset P\}} |\langle f, h_{j,m}\rangle|^2 \le C|P|.$$

8. Remaining proofs

It remains to prove Corollary 2.6. To this end, we need to give an interpretation about the pointwise multiplication; see, for instance, Section 4.2 in [49]. For any $k \in \mathbb{Z}_+$ and $f \in \mathcal{S}'$, let

$$S^k f := \sum_{j=0}^{k} S_j f,$$

where, for any $j \in \mathbb{Z}_+$, S_j is the same as in (3.1). It is well known that, for any $k \in \mathbb{Z}_+$ and $f \in \mathcal{S}'$, $S^k f \in C^\infty$ with all of its derivatives having polynomial growth at infinity; see, for instance, Theorem 2.3.20 in [21].

Definition 8.1. Let $f, g \in \mathcal{S}'$. The *product* of f and g is defined by setting

$$fg := \lim_{k\to\infty} S^k f\, S^k g$$

if the limit on the right-hand side exists in $\mathcal{S}'$.

We need the following useful conclusions on the properties of S^k, which is a part of equation (5) in Subsection 4.2.1 of [49].

Lemma 8.2. *If $p \in (0,\infty]$, $q \in (0,\infty)$, and $s \in \mathbb{R}$, then, for any $f \in B^s_{p,q}$,*

$$\lim_{k\to\infty} \|S^k f - f\|_{B^s_{p,q}} = 0.$$

Definition 8.3. Let $X \hookrightarrow \mathcal{S}'$ be a quasi-Banach space. Then the *pointwise multiplier space* $M(X)$ of X is defined to be the set of all $m \in \mathcal{S}'$ such that, for any $f \in X$, $mf \in X$ and

$$\|m\|_{M(X)} := \sup_{\{f\in X,\ f\neq \mathbf{0}\}} \frac{\|mf\|_X}{\|f\|_X} < \infty,$$

where $\mathbf{0}$ denotes the *null element* of X.

The following lemma is useful for the proof of Corollary 2.6.

Lemma 8.4. *Let $p \in [1,\infty)$, $q \in (1,\infty)$, and $s \in \mathbb{R}$. Then, for any $f \in B^{-s}_{p',q'}$, $g \in B^s_{p,q}$, and $m \in M(B^s_{p,q}) \cap L^1_{\rm loc}$,*

$$\langle mf, g\rangle = \langle f, mg\rangle.$$

Proof. Fix $f \in B^{-s}_{p',q'}$, $g \in B^s_{p,q}$, and $m \in M(B^s_{p,q}) \cap L^1_{\rm loc}$. Using Lemma 2.12 in [35] and Lemma 4.1, we find that

$$m \in M(B^s_{p,q}) = M(B^{-s}_{p',q'}).$$

Note that, for any $\varphi \in \mathcal{S}$ and $k, j \in \mathbb{Z}_+$,

$$\begin{aligned}
&|\langle mf,\varphi\rangle - \langle f, m\varphi\rangle| \\
&\quad\le |\langle mf,\varphi\rangle - \langle mS^k f,\varphi\rangle| + |\langle mS^k f,\varphi\rangle - \langle S^j mS^k f,\varphi\rangle| \\
&\qquad + |\langle S^j mS^k f,\varphi\rangle - \langle f, m\varphi\rangle| \\
&\quad =: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3.
\end{aligned}$$

We first estimate I_1. Since $m \in M(B^{-s}_{p',q'})$, from Lemma 4.1, we infer that, for any $\varphi \in \mathcal{S}$ and $k \in \mathbb{Z}_+$,

$$\mathrm{I}_1 \le \|m(f - S^k f)\|_{B^{-s}_{p',q'}} \|\varphi\|_{B^s_{p,q}} \lesssim \|f - S^k f\|_{B^{-s}_{p',q'}} \|\varphi\|_{B^s_{p,q}}.$$

Now, we estimate I_2. It is known that, for any $h \in \mathcal{S}'$,

$$S^j h \to h \tag{8.1}$$

in $\mathcal{S}'$ as $j \to \infty$ (see, for instance, equation (2) in Subsection 4.2.1 of [49]). For any given $k \in \mathbb{Z}_+$, by (8.1) and the fact that $S^k f \in C^\infty$ with all of its derivatives having polynomial growth at infinity, we conclude that $S^j mS^k f \to mS^k f$ in $\mathcal{S}'$ as $j \to \infty$. This further implies that, for any given $k \in \mathbb{Z}_+$,

$$\lim_{j\to\infty} \mathrm{I}_2 = 0.$$

Next, we estimate I_3. Using (8.1) again and Lemma 8.2(ii), we find that, for any $k \in \mathbb{Z}_+$,

$$\lim_{j\to\infty} \mathrm{I}_3 = |\langle mS^k f, \varphi\rangle - \langle f, m\varphi\rangle|.$$

This, together with the assumption that $m \in M(B^s_{p,q}) \cap L^1_{\mathrm{loc}}$ and Lemma 4.1, further implies that, for any $k \in \mathbb{Z}_+$,

$$\lim_{j\to\infty} \mathrm{I}_3 = |\langle S^k f - f, m\varphi\rangle| \lesssim \|f - S^k f\|_{B^{-s}_{p',q'}} \|\varphi\|_{B^s_{p,q}}.$$

Combining above estimates and using Lemma 8.2, we conclude that, for any $\varphi \in \mathcal{S}$,

$$|\langle mf, \varphi\rangle - \langle f, m\varphi\rangle| \le \lim_{k\to\infty} \lim_{j\to\infty} (\mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3) = 0. \tag{8.2}$$

Since $\mathcal{S}$ is dense in $B^s_{p,q}$, it follows that there exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ in $\mathcal{S}$ such that $\varphi_n \to g$ in $B^s_{p,q}$. From this, (8.2), Lemma 4.1, and the assumption that $m \in M(B^s_{p,q}) \cap L^1_{\mathrm{loc}}$, we deduce that, for any $n \in \mathbb{N}$,

$$\begin{aligned} |\langle mf, g\rangle - \langle f, mg\rangle| &\le |\langle mf, g\rangle - \langle mf, \varphi_n\rangle| + |\langle f, m\varphi_n\rangle - \langle f, mg\rangle| \\ &\lesssim \|f\|_{B^{-s}_{p',q'}} \|g - \varphi_n\|_{B^s_{p,q}} \to 0 \end{aligned}$$

as $n \to \infty$. This finishes the proof of Lemma 8.4. ∎

Now, we are ready to prove Corollary 2.6.

Proof of Corollary 2.6. Let $\varepsilon \in (0, \infty)$ be such that $s > 1/p - \tau - 1 + \varepsilon$. Applying Theorem 1 in Section 4.6.3 of [49] and Lemma 2.12 in [35], we obtain

$$\mathbf{1}_{(0,\infty)} \in M(B^{-1+\varepsilon}_{\infty,2}) = M(B^{1-\varepsilon}_{1,2}).$$

Clearly, $\mathbf{1}_{\{0\}} = \mathbf{0}$ in $\mathcal{S}'$ and $\mathbf{0} \in M(B^{1-\varepsilon}_{1,2})$. Thus, $\mathbf{1}_{[0,\infty)} \in M(B^{1-\varepsilon}_{1,2})$. From these and from the translation invariance and the reflection invariance of Besov spaces, it follows that, for any $a \in \mathbb{R}$,

$$\mathbf{1}_{(a,\infty)}, \mathbf{1}_{[a,\infty)}, \mathbf{1}_{(-\infty,a)}, \mathbf{1}_{(-\infty,a]} \in M(B^{1-\varepsilon}_{1,2}).$$

This further implies that, for any interval $I \subset \mathbb{R}$,

$$\mathbf{1}_I \in M(B^{1-\varepsilon}_{1,2}) \cap L^1_{\mathrm{loc}}. \tag{8.3}$$

Now, we assume that I is either a dyadic interval or $I := [0, \infty)$. Then, by Lemma 8.4, (8.3), (4.1), and (4.2), we find that, for any $f \in B^{s,\tau}_{p,q}$ and $h \in \mathcal{H}$,

$$\langle \mathbf{1}_I f, h\rangle = \langle f, h\mathbf{1}_I\rangle,$$

which further implies that

$$\|\delta(\mathbf{1}_I f)\|_{b^{s,\tau}_{p,q}} \le \|\delta(f)\|_{b^{s,\tau}_{p,q}}. \tag{8.4}$$

Applying this and Theorem 2.2, we conclude the desired conclusion in this case. By an argument similar to that used in the proof of (8.4), we obtain the desired conclusion in the case where $I := [0, a)$ for any $a \in \mathbb{N}$.

Since Besov-type spaces are translation invariant, it remains to show the desired result in the case where $I := [0, b)$ for some $b \in (0, \infty) \setminus \mathbb{N}$. Without loss of generality, we may assume that $[0, c) \subset I \subset [0, a)$, where either $a, c \in \mathbb{N}$ or $a, c \in \{2^{-j} : j \in \mathbb{N}\}$ satisfy $0 < c < b < a$ and $a \leq 2c$. From a standard argument that used in the proof that the definition of Besov-type spaces is independent of the choice of φ_0 (see, for instance, the proofs of Lemma 4.1 in [76] and Theorem 1.1 in [71]), we infer that the mapping $f \mapsto f(\lambda\cdot)$ is uniformly bounded from $B^{s,\tau}_{p,q}$ into itself with respect to $\lambda \in [1/2, 2]$. This further implies that the desired result holds in the case where $I := [0, b)$ for some $b \in (0, \infty) \setminus \mathbb{N}$, which completes the proof of Corollary 2.6. ∎

Acknowledgements. Yirui Zhao would like to thank Yinqin Li for many helpful discussions on pointwise multipliers. The authors would also like to express their deep gratitude to Gustavo Garrigós (Murcia) for pointing out a serious flaw in the previous version of our article and, particularly, for his several valuable suggestions concerning the necessity parts presented in Subsection 7.1. The authors would also like to thank the referee for his/her carefully reading and stimulating remarks which indeed improve the quality of this article.

Funding. This project is partially supported by the National Natural Science Foundation of China (Grants no. 12431006 and 12371093), the Beijing Natural Science Foundation (Grant no. 1262011), and the Fundamental Research Funds for the Central Universities (Grant no. 2253200028).

References

[1] Amann, H.: *Linear and quasilinear parabolic problems. Vol. II. Function spaces*. Monogr. Math. 106, Birkhäuser/Springer, Cham, 2019. Zbl 1448.46004 MR 3930629

[2] Bergh, J. and Löfström, J.: *Interpolation spaces. An introduction*. Grundlehren math. Wiss. 223, Springer, Berlin-New York, 1976. Zbl 0344.46071 MR 0482275

[3] Bočkarev, S. V.: The coefficients of Fourier series in Haar's system. *Mat. Sb. (N.S.)* **80(122)** (1969), 97–116. Zbl 0181.07102 MR 0249914

[4] Bonami, A., Cao, J., Ky, L. D., Liu, L., Yang, D. and Yuan, W.: Multiplication between Hardy spaces and their dual spaces. *J. Math. Pures Appl. (9)* **131** (2019), 130–170. Zbl 1431.42038 MR 4021172

[5] Bonami, A., Iwaniec, T., Jones, P. and Zinsmeister, M.: On the product of functions in BMO and H^1. *Ann. Inst. Fourier (Grenoble)* **57** (2007), no. 5, 1405–1439. Zbl 1132.42010 MR 2364134

[6] Bonami, A., Liu, L., Yang, D. and Yuan, W.: Pointwise multipliers of Zygmund classes on $\mathbb{R}^n$. *J. Geom. Anal.* **31** (2021), no. 9, 8879–8902. Zbl 1471.42025 MR 4302202

[7] Cao, J., Ky, L. D. and Yang, D.: Bilinear decompositions of products of local Hardy and Lipschitz or BMO spaces through wavelets. *Commun. Contemp. Math.* **20** (2018), no. 3, article no. 1750025, 30 pp. Zbl 1385.42019 MR 3766731

[8] Chang, S.-Y. A. and Fefferman, R.: Some recent developments in Fourier analysis and H^p-theory on product domains. *Bull. Amer. Math. Soc. (N.S.)* **12** (1985), no. 1, 1–43. Zbl 0557.42007 MR 0766959

[9] Coifman, R. R., Jones, P. W. and Semmes, S.: Two elementary proofs of the L^2 boundedness of Cauchy integrals on Lipschitz curves. *J. Amer. Math. Soc.* **2** (1989), no. 3, 553–564. Zbl 0713.42010 MR 0986825

[10] Danchin, R. and Mucha, P. B.: A Lagrangian approach for the incompressible Navier–Stokes equations with variable density. *Comm. Pure Appl. Math.* **65** (2012), no. 10, 1458–1480. Zbl 1247.35088 MR 2957705

[11] Drihem, D.: Some embeddings and equivalent norms of the $\mathcal{L}^{\lambda,s}_{p,q}$ spaces. *Funct. Approx. Comment. Math.* **41** (2009), no. part 1, 15–40. Zbl 1188.46020 MR 2568794

[12] El Baraka, A.: An embedding theorem for Campanato spaces. *Electron. J. Differ. Equ.* (2002), article no. 66, 17 pp. Zbl 1002.46024 MR 1921139

[13] El Baraka, A.: Function spaces of BMO and Campanato type. In *Proceedings of the 2002 Fez Conference on partial differential equations*, pp. 109–115. Electron. J. Differ. Equ. Conf. 9, Southwest Texas State Univ., San Marcos, TX, 2002. Zbl 1031.46036 MR 1976688

[14] El Baraka, A.: Littlewood–Paley characterization for Campanato spaces. *J. Funct. Spaces Appl.* **4** (2006), no. 2, 193–220. Zbl 1133.42031 MR 2227045

[15] Frazier, M. and Jawerth, B.: A discrete transform and decompositions of distribution spaces. *J. Funct. Anal.* **93** (1990), no. 1, 34–170. Zbl 0716.46031 MR 1070037

[16] Garrigós, G., Seeger, A. and Ullrich, T.: The Haar system as a Schauder basis in spaces of Hardy–Sobolev type. *J. Fourier Anal. Appl.* **24** (2018), no. 5, 1319–1339. Zbl 1405.46025 MR 3856231

[17] Garrigós, G., Seeger, A. and Ullrich, T.: Basis properties of the Haar system in limiting Besov spaces. In *Geometric aspects of harmonic analysis*, pp. 361–424. Springer INdAM Ser. 45, Springer, Cham, 2021. Zbl 1487.42066 MR 4390231

[18] Garrigós, G., Seeger, A. and Ullrich, T.: The Haar system in Triebel–Lizorkin spaces: endpoint results. *J. Geom. Anal.* **31** (2021), no. 9, 9045–9089. Zbl 1478.46032 MR 4302211

[19] Garrigós, G., Seeger, A. and Ullrich, T.: Haar frame characterizations of Besov–Sobolev spaces and optimal embeddings into their dyadic counterparts. *J. Fourier Anal. Appl.* **29** (2023), no. 3, article no. 39, 51 pp. Zbl 1526.46023 MR 4603296

[20] Goldberg, D.: A local version of real Hardy spaces. *Duke Math. J.* **46** (1979), no. 1, 27–42. Zbl 0409.46060 MR 0523600

[21] Grafakos, L.: *Classical Fourier analysis*. Third edition. Grad. Texts in Math. 249, Springer, New York, 2014. Zbl 1304.42001 MR 3243734

[22] Gulisashvili, A. B.: On multipliers in Besov spaces. (Russian) *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **135** (1984), 36–50; English traslation *J. Math. Sci.* **31** (1985), 2662–2672. Zbl 0564.46026 MR 0741693

[23] Haroske, D. D. and Triebel, H.: Morrey smoothness spaces: a new approach. *Sci. China Math.* **66** (2023), no. 6, 1301–1358. Zbl 1529.46025 MR 4596050

[24] Hytönen, T. P.: The sharp weighted bound for general Calderón–Zygmund operators. *Ann. of Math. (2)* **175** (2012), no. 3, 1473–1506. Zbl 1250.42036 MR 2912709

[25] Janson, S.: On functions with conditions on the mean oscillation. *Ark. Mat.* **14** (1976), no. 2, 189–196. Zbl 0341.43005 MR 0438030

[26] John, F. and Nirenberg, L.: On functions of bounded mean oscillation. *Comm. Pure Appl. Math.* **14** (1961), 415–426. Zbl 0102.04302 MR 0131498

[27] Kahane, J.-P. and Lemarié-Rieusset, P. G.: *Fourier series and wavelets*. Stud. Develop. Modern Math. 3, Gordon and Breach, London, 1996. Zbl 0966.42002

[28] Kashin, B. S. and Saakyan, A. A.: *Orthogonal series*. Transl. Math. Monogr. 75, American Mathematical Society, Providence, RI, 1989. Zbl 0668.42011 MR 1007141

[29] Koch, H. and Tataru, D.: Well-posedness for the Navier–Stokes equations. *Adv. Math.* **157** (2001), no. 1, 22–35. Zbl 0972.35084 MR 1808843

[30] Lemarié-Rieusset, P. G.: *The Navier–Stokes problem in the 21st century*. CRC Press, Boca Raton, FL, 2016. Zbl 1342.76029 MR 3469428

[31] Lemarié-Rieusset, P. G.: Sobolev multipliers, maximal functions and parabolic equations with a quadratic nonlinearity. *J. Funct. Anal.* **274** (2018), no. 3, 659–694. Zbl 1386.35206 MR 3734981

[32] Li, P., Xiao, J. and Yang, Q.: Global mild solutions to modified Navier–Stokes equations with small initial data in critical Besov-Q spaces. *Electron. J. Differential Equations* (2014), article no. 185, 37 pp. Zbl 1301.35091 MR 3262056

[33] Li, P. and Yang, Q.: Wavelets and the well-posedness of incompressible magneto-hydrodynamic equations in Besov type Q-space. *J. Math. Anal. Appl.* **405** (2013), no. 2, 661–686. Zbl 1309.35086 MR 3061042

[34] Li, P. and Zhai, Z.: Riesz transforms on Q-type spaces with application to quasi-geostrophic equation. *Taiwanese J. Math.* **16** (2012), no. 6, 2107–2132. Zbl 1259.35160 MR 3001839

[35] Li, Y., Sickel, W., Yang, D. and Yuan, W.: Characterizations of pointwise multipliers of Besov spaces in endpoint cases with an application to the duality principle. *J. Funct. Anal.* **286** (2024), no. 1, article no. 110198, 38 pp. Zbl 1539.46023 MR 4658502

[36] Li, Y., Sickel, W., Yang, D. and Yuan, W.: Wavelet and Fourier analytic characterizations of pointwise multipliers of Besov spaces $B^s_{p,p}(\mathbb{R}^n)$ with $0 < p \leq 1$. *J. Funct. Anal.* **287** (2024), no. 12, article no. 110654, 50 pp. Zbl 1569.46032 MR 4796722

[37] Liang, Y., Yang, D., Yuan, W., Sawano, Y. and Ullrich, T.: A new framework for generalized Besov-type and Triebel–Lizorkin-type spaces. *Dissertationes Math.* **489** (2013), 114 pp. Zbl 1283.46027 MR 3099066

[38] Luecking, D. H.: Embedding theorems for spaces of analytic functions via Khinchine's inequality. *Michigan Math. J.* **40** (1993), no. 2, 333–358. Zbl 0801.46019 MR 1226835

[39] Maz'ya, V. G. and Shaposhnikova, T. O.: *Theory of multipliers in spaces of differentiable functions*. Monogr. Stud. Math. 23, Pitman, Boston, MA, 1985. Zbl 0645.46031 MR 0785568

[40] Maz'ya, V. G. and Shaposhnikova, T. O.: *Theory of Sobolev multipliers. With applications to differential and integral operators*. Grundlehren math. Wiss. 337, Springer, Berlin, 2009. Zbl 1157.46001 MR 2457601

[41] Meyer, Y.: *Wavelets and operators*. Cambridge Stud. Adv. Math. 37, Cambridge University Press, Cambridge, 1992. Zbl 0776.42019 MR 1228209

[42] Nakai, E.: Pointwise multipliers on several functions spaces –a survey. *Linear Nonlinear Anal.* **3** (2017), no. 1, 27–59. Zbl 1375.42007 MR 3638666

[43] Nakai, E. and Yabuta, K.: Pointwise multipliers for functions of bounded mean oscillation. *J. Math. Soc. Japan* **37** (1985), no. 2, 207–218. Zbl 0546.42019 MR 0780660

[44] Nazarov, F., Treil, S. and Volberg, A.: The Tb-theorem on non-homogeneous spaces. *Acta Math.* **190** (2003), no. 2, 151–239. Zbl 1065.42014 MR 1998349

[45] Oswald, P.: L^p-Approximation durch Reihen nach dem Haar-Orthogonalsystem und dem Faber–Schauder-System. *J. Approx. Theory* **33** (1981), no. 1, 1–27. Zbl 0491.41021 MR 0639218

[46] Oswald, P.: On inequalities for spline approximation and spline systems in the space L^p $(0 < p < 1)$. In *Approximation and function spaces (Gdańsk, 1979)*, pp. 531–552. North-Holland, Amsterdam-New York, 1981. Zbl 0485.41014 MR 0649459

[47] Oswald, P.: Multivariate Haar systems in Besov function spaces. *Mat. Sb.* **212** (2021), no. 6, 73–108; English translation: *Sb. Math.* **212** (2021), no. 6, 810–842. Zbl 1479.42095 MR 4265429

[48] Ropela, S.: Spline bases in Besov spaces. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **24** (1976), no. 5, 319–325. Zbl 0328.41008 MR 0417660

[49] Runst, T. and Sickel, W.: *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*. De Gruyter Ser. Nonlinear Anal. Appl. 3, de Gruyter, Berlin, 1996. Zbl 0873.35001 MR 1419319

[50] Sawano, Y., Yang, D. and Yuan, W.: New applications of Besov-type and Triebel–Lizorkin-type spaces. *J. Math. Anal. Appl.* **363** (2010), no. 1, 73–85. Zbl 1185.42022 MR 2559042

[51] Schneider, C. and Vybíral, J.: Non-smooth atomic decompositions, traces on Lipschitz domains, and pointwise multipliers in function spaces. *J. Funct. Anal.* **264** (2013), no. 5, 1197–1237. Zbl 1267.46053 MR 3010019

[52] Seeger, A. and Ullrich, T.: Haar projection numbers and failure of unconditional convergence in Sobolev spaces. *Math. Z.* **285** (2017), no. 1-2, 91–119. Zbl 1369.46032 MR 3598805

[53] Stegenga, D. A.: Bounded Toeplitz operators on H^1 and applications of the duality between H^1 and the functions of bounded mean oscillation. *Amer. J. Math.* **98** (1976), no. 3, 573–589. Zbl 0335.47018 MR 0420326

[54] Strichartz, R. S.: Multipliers on fractional Sobolev spaces. *J. Math. Mech.* **16** (1967), 1031–1060. Zbl 0145.38301 MR 0215084

[55] Triebel, H.: On Besov–Hardy–Sobolev spaces in domains and regular elliptic boundary value problems. The case $0 < p \leq \infty$. *Comm. Partial Differential Equations* **3** (1978), no. 12, 1083–1164. Zbl 0403.35034 MR 0512083

[56] Triebel, H.: On Haar bases in Besov spaces. *Serdica* **4** (1978), no. 4, 330–343. Zbl 0439.46025 MR 0543155

[57] Triebel, H.: *Theory of function spaces*. Monogr. Math. 78, Birkhäuser, Basel, 1983. Zbl 0546.46027 MR 0781540

[58] Triebel, H.: *Theory of function spaces. II*. Monogr. Math. 84, Birkhäuser, Basel, 1992. Zbl 0763.46025 MR 1163193

[59] Triebel, H.: Function spaces in Lipschitz domains and on Lipschitz manifolds. Characteristic functions as pointwise multipliers. *Rev. Mat. Complut.* **15** (2002), no. 2, 475–524. Zbl 1034.46033 MR 1951822

[60] Triebel, H.: *Theory of function spaces. III*. Monogr. Math. 100, Birkhäuser, Basel, 2006. Zbl 1104.46001 MR 2250142

[61] Triebel, H.: *Function spaces and wavelets on domains*. EMS Tracts Math. 7, European Mathematical Society (EMS), Zürich, 2008. Zbl 1158.46002 MR 2455724

[62] Triebel, H.: *Bases in function spaces, sampling, discrepancy, numerical integration*. EMS Tracts Math. 11, European Mathematical Society (EMS), Zürich, 2010. Zbl 1202.46002 MR 2667814

[63] Triebel, H.: Characterizations of some function spaces in terms of Haar wavelets. *Comment. Math.* **53** (2013), no. 2, 135–153. Zbl 1298.46037 MR 3155043

[64] Triebel, H.: *Local function spaces, heat and Navier–Stokes equations*. EMS Tracts Math. 20, European Mathematical Society (EMS), Zürich, 2013. Zbl 1280.46002 MR 3086433

[65] Triebel, H.: *Hybrid function spaces, heat and Navier–Stokes equations*. EMS Tracts Math. 24, European Mathematical Society (EMS), Zürich, 2014. Zbl 1330.46003 MR 3308920

[66] Triebel, H.: *Tempered homogeneous function spaces*. EMS Ser. Lect. Math., European Mathematical Society (EMS), Zürich, 2015. Zbl 1336.46004 MR 3409094

[67] Triebel, H.: *Theory of function spaces IV*. Monogr. Math. 107, Birkhäuser/Springer, Cham, 2020. Zbl 1445.46002 MR 4298338

[68] Wojtaszczyk, P.: The Franklin system is an unconditional basis in H_1. *Ark. Mat.* **20** (1982), no. 2, 293–300. Zbl 0534.46038 MR 0686177

[69] Wojtaszczyk, P.: *A mathematical introduction to wavelets*. London Math. Soc. Stud. Texts 37, Cambridge University Press, Cambridge, 1997. Zbl 0865.42026 MR 1436437

[70] Yang, D. and Yuan, W.: A new class of function spaces connecting Triebel–Lizorkin spaces and Q spaces. *J. Funct. Anal.* **255** (2008), no. 10, 2760–2809. Zbl 1169.46016 MR 2464191

[71] Yang, D. and Yuan, W.: Characterizations of Besov-type and Triebel–Lizorkin-type spaces via maximal functions and local means. *Nonlinear Anal.* **73** (2010), no. 12, 3805–3820. Zbl 1225.46033 MR 2728556

[72] Yang, D. and Yuan, W.: New Besov-type spaces and Triebel–Lizorkin-type spaces including Q spaces. *Math. Z.* **265** (2010), no. 2, 451–480. Zbl 1191.42011 MR 2609320

[73] Yang, D. and Yuan, W.: Relations among Besov-type spaces, Triebel–Lizorkin-type spaces and generalized Carleson measure spaces. *Appl. Anal.* **92** (2013), no. 3, 549–561. Zbl 1278.42032 MR 3021276

[74] Yang, D., Yuan, W. and Zhang, Y.: Bilinear decomposition and divergence-curl estimates on products related to local Hardy spaces and their dual spaces. *J. Funct. Anal.* **280** (2021), no. 2, article no. 108796, 74 pp. Zbl 1457.42036 MR 4159273

[75] Yang, D., Yuan, W. and Zhuo, C.: Complex interpolation on Besov-type and Triebel–Lizorkin-type spaces. *Anal. Appl. (Singap.)* **11** (2013), no. 5, article no. 1350021, 45 pp. Zbl 1294.46020 MR 3104106

[76] Yuan, W., Sickel, W. and Yang, D.: *Morrey and Campanato meet Besov, Lizorkin and Triebel*. Lecture Notes in Math. 2005, Springer, Berlin, 2010. MR 2683024

[77] Yuan, W., Sickel, W. and Yang, D.: The Haar system in Besov-type spaces. *Studia Math.* **253** (2020), no. 2, 129–162. Zbl 1443.42020 MR 4078220

[78] Yuan, W., Sickel, W. and Yang, D.: Regularity of characteristic functions in Besov-type and Triebel–Lizorkin-type spaces. *J. Fourier Anal. Appl.* **31** (2025), no. 3, article no. 25, 64 pp. Zbl 08051203 MR 4894931

[79] Zhang, Y., Yang, D. and Yuan, W.: Real-variable characterizations of local Orlicz-slice Hardy spaces with application to bilinear decompositions. *Commun. Contemp. Math.* **24** (2022), no. 6, article no. 2150004, 35 pp. Zbl 1494.42027 MR 4466160

Received May 16, 2025; revised February 17, 2026.

Winfried Sickel
Mathematisches Institut, Friedrich-Schiller-Universität Jena
07743 Jena, Germany;
winfried.sickel@uni-jena.de
Author IDs: zbMATH sickel.winfried MR 161535 ORCID 0000-0001-8609-077X

Dachun Yang
Laboratory of Mathematics and Complex Systems (Ministry of Education of China), School of Mathematical Sciences, Institute for Advanced Study, Beijing Normal University
100875 Beijing, P. R. China;
dcyang@bnu.edu.cn
Author IDs: zbMATH yang.dachun MR 317762 ORCID 0000-0001-9024-3345

Wen Yuan
Laboratory of Mathematics and Complex Systems (Ministry of Education of China), School of Mathematical Sciences, Beijing Normal University
100875 Beijing, P. R. China;
wenyuan@bnu.edu.cn
Author IDs: zbMATH yuan.wen MR 743517 ORCID 0000-0002-4072-245X

Yirui Zhao
Laboratory of Mathematics and Complex Systems (Ministry of Education of China), School of Mathematical Sciences, Beijing Normal University
100875 Beijing, P. R. China;
yiruizhao@mail.bnu.edu.cn
Author IDs: zbMATH zhao.yirui MR 1577643 ORCID 0009-0001-7875-2574