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The weighted L^p Minkowski problem

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Abstract. The Minkowski problem in convex geometry concerns showing that a given Borel measure on the unit sphere is, up to perhaps a constant, some type of surface area measure of a convex body. Two types of Minkowski problems in particular are an active area of research: L^p Minkowski problems, introduced by Lutwak and Lutwak–Yang–Zhang, and weighted Minkowski problems, introduced by Livshyts. For the latter, the Gaussian Minkowski problem, whose primary investigators were Huang–Xi–Zhao, is the most prevalent. In this work, we consider weighted surface area in the L^p setting. We propose a framework going beyond the Gaussian setting by focusing on rotationally invariant measures, mirroring the recent development of the Gardner–Zvavitch inequality for rotationally invariant, log-concave measures. Our results include existence for all $p \in \mathbb{R}$ (with symmetry assumptions in certain instances). We also obtain uniqueness for $p \geq 1$ under a concavity assumption. Finally, we obtain results in the so-called *small mass regime* using degree theory, as instigated in the Gaussian case by Huang–Xi–Zhao.

1. Introduction

1.1. Motivation

Given a finite Borel measure μ on the unit sphere $\mathbb{S}^{n-1}$ in the n-dimensional Euclidean space $\mathbb{R}^n$, one may ask: does there exist a unique (up to translations) convex body K (compact, convex set with non-empty interior) such that $dS_K = d\mu$? Here, S_K is the surface area measure of K, which is obtained by pushing the Hausdorff measure on ∂K, the boundary of K, to $\mathbb{S}^{n-1}$, i.e., for every Borel $A \subset \mathbb{S}^{n-1}$,

$$S_K(A) = \mathcal{H}^{n-1}(n_K^{-1}(A)),$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure and $n_K \colon \partial K \to \mathbb{S}^{n-1}$ is the Gauss map, which associates an element y of ∂K with its outer unit normal.

Minkowski's existence theorem (see [99], p. 455) shows that if μ satisfies the following two conditions, then the answer is yes:

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(1) The measure μ is not concentrated on any great hemisphere, that is

$$\int_{\mathbb{S}^{n-1}} \langle \theta, \xi \rangle_+ \, d\mu(\xi) > 0 \quad \text{for all } \theta \in \mathbb{S}^{n-1}.$$

(2) The measure is centered, that is

$$\int_{\mathbb{S}^{n-1}} \xi \, d\mu(\xi) = 0.$$

The concept of the surface area measure of a convex body has seen many extensions over the past century, initiated by a series of papers by Lutwak [83, 84] in the 1990s establishing the modern L^p Brunn–Minkowski theory. It will be convenient to denote by $\mathcal{K}_o^n$ the set of all convex bodies containing the origin in their interior, and by $\mathcal{K}_e^n$ the set of all (origin)-symmetric convex bodies. A set K is origin-symmetric if $K = -K$. For brevity, we may simply say symmetric.

The question of showing that a given data measure on the sphere is the appropriate surface area measure of a convex body is a *Minkowski problem*. In many instances, one must introduce a constant into the Minkowski problem; in fact, this is more standard than not in the weighted setting (also in the Orlicz setting, see [54]). We say that a Minkowski problem is *constant-free* if there is no constant. In certain instances, only *even* data measures ν on the sphere are considered when solving a Minkowski problem, in which case the convex body whose relevant area measure fits the data will be symmetric.

The L^p Minkowski problem, concerning the L^p surface area measure $S_{K,p}$, is perhaps the most notable example of a Minkowski problem. Letting the support function of a convex body K be $h_K(x) = \sup_{y \in K} \langle x, y \rangle$, we have $dS_{K,p} = h_K^{1-p} dS_K$ for $K \in \mathcal{K}_o^n$. Existence in the even case was completely solved [19, 83, 87, 89] for $p \geq 0$ (note that, for $p = n$, one must introduce a constant). In fact, there is also uniqueness for $p \geq 1$, $p \neq n$ (uniqueness when $p = 0$ is still an open question, known as the famed log-Minkowski problem). For $p > 1$, $p \neq n$, in the non-even case, existence and uniqueness were settled in [29, 56]. For $p \in (0, 1)$, existence was settled in [26]. The regime $p < 1$, especially uniqueness and non-uniqueness (there are examples of non-uniqueness), is still an active area of research. See, e.g., [6, 7, 15–17, 20, 23, 25–27, 47, 48, 57, 60, 61, 74, 92, 102, 103, 109–111]. The reader is recommended to read the recent survey by Böröczky [14] for a detailed history of the L^p Minkowski problem.

In this work, we will focus on a particular type of Minkowski problem. Given a Borel measure μ on a metric measure space $(\mathcal{R}, d)$, the Minkowski content, or μ-weighted surface area, of a measurable set $A \subset \mathcal{R}$ is given by

$$\mu^+(\partial A) = \liminf_{\varepsilon \to 0} \frac{\mu(A_\varepsilon) - \mu(A)}{\varepsilon}, \tag{1.1}$$

where, as usual, $A_\varepsilon = \{x \in \mathcal{R} : d(x, A) \leq \varepsilon\}$. In this work, $\mathcal{R}$ will always be $\mathbb{R}^n$ for some n and the metric d will always be the Euclidean distance $|\cdot|$. In this case, $A_\varepsilon = A + \varepsilon B_2^n$. Here, $K + L = \{x + y : x \in K, y \in L\}$ is the so-called Minkowski sum of Borel sets K and L.

In our work, we will require that μ satisfies an *isoperimetric inequality*. Such inequalities control the Minkowski content $\mu^+(\partial A)$ of A by $\mu(A)$. Given a probability measure μ

on $\mathbb{R}^n$, its *isoperimetric constant* $h(\mu)$ is the largest constant h such that

$$\mu^+(\partial A) \geq h \min\{\mu(A), 1-\mu(A)\}. \tag{1.2}$$

The existence of $h(\mu)$, with bounds, when μ is log-concave was shown by Kannan, Lovász, and Simonovits [63], and improved by Bobkov [10]. Inequalities of the form (1.2) are called *Cheeger's inequalities*. Given a Borel measure μ on $\mathbb{R}^n$, its *isoperimetric function I_μ* is given by

$$I_\mu(t) = \inf\{\mu^+(\partial A) : \mu(A) = t,\ t \in (0, \mu(\mathbb{R}^n))\}. \tag{1.3}$$

Therefore, it is the largest function I such that

$$\mu^+(\partial A) \geq I(\mu(A)) \tag{1.4}$$

for every Borel set $A \subset \mathbb{R}^n$ with $\mu(A) \in (0, \mu(\mathbb{R}^n))$. However, the infimum in (1.3) may be zero in general; we say μ "has" an isoperimetric function if $I_\mu > 0$. Please, see the excellent textbook by Ledoux [73], in particular, p. 23. For a concrete example, (1.2) shows I_μ exists when μ is a log-concave measure with integrable density. If B is so that $\mu^+(\partial B) = I_\mu(\mu(B))$, then B is said to be an extremal set. In the setting of Borel measures on $\mathbb{R}^n$, the isoperimetric function is essentially only known for the Gaussian and Lebesgue measures. In practice, one usually establishes that a function I satisfies (1.4) (i.e., a *bound* for I_μ is shown).

A natural question is to obtain a formula for (1.1). We will follow the terminology by Livshyts [80]: for a convex body K and a Borel measure μ on the boundary of K of the form $d\mu(y) = \phi(y)\, d\mathcal{H}^{n-1}(y)$, the *weighted surface area of K with respect to μ* is defined by

$$S_K^\mu(E) = \int_{n_K^{-1}(E)} \phi(y)\, d\mathcal{H}^{n-1}(y)$$

for every Borel set $E \subset \mathbb{S}^{n-1}$.

We briefly mention how to define S_K^μ when μ is a Borel measure on $\mathbb{R}^n$. Essentially, the problem boils down to determining a canonical method to select how the measure behaves on the boundary of each convex body. We will always work with measures that have density. We recall that a measure μ on $\mathbb{R}^n$ is said to have density if it is absolutely continuous with respect to the Lebesgue measure, i.e., $d\mu(x) = \phi(x)\, dx$ for a nonnegative, locally integrable function ϕ. Under the minor technical assumption that μ has continuous density, there is no issue. For a given fixed convex body K, the minimal required assumption is that ∂K, up to a set of $(n-1)$-dimensional Hausdorff measure zero, is included in the Lebesgue set of the density of μ. We can make this precise.

Under the assumption that K is a convex body and that μ is a Borel measure on $\mathbb{R}^n$ with density ϕ whose Lebesgue set contains ∂K, one has that the lim inf in (1.1) is a limit and

$$\mu^+(\partial K) = \lim_{\varepsilon \to 0} \frac{\mu(K + \varepsilon B_2^n) - \mu(K)}{\varepsilon} = \int_{\mathbb{S}^{n-1}} dS_K^\mu(u).$$

If we additionally assume that $K \in \mathcal{K}_o^n$, its weighted L^p surface area is then, for $E \subset \mathbb{S}^{n-1}$ Borel,

$$S_{K,p}^{\mu}(E) = \int_{n_K^{-1}(E)} \langle y, n_K(y) \rangle^{1-p} \phi(y)\, d\mathcal{H}^{n-1}(y),$$

i.e., one has $dS_{K,p}^{\mu} = h_K^{1-p} dS_K^{\mu}$ and $S_{K,1}^{\mu} = S_K^{\mu}$. We outline in Section 2 formulas for more general versions of (1.1) that we will need throughout this work. Before listing our results, it is necessary to contextualize our results with respect to the broader literature.

1.2. Weighted Minkowski problems

In this paper, we establish a framework of weighted L^p Minkowski problems. Previously, the case of homogeneous measures was considered by Livshyts [80] and Wu [108]. The class of measures with radially decreasing densities was considered by Kryvonos and Langharst [70]. The Borel measure on $\mathbb{R}^n$ most pertinent to our investigations is the Gaussian measure

$$d\gamma_n(x) = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2}\, dx.$$

Huang, Xi, and Zhao [55] first established existence for the even weighted Minkowski problem for the Gaussian measure. Due to the lack of homogeneity, the constant in the problem cannot be removed. This is made precise by the result from K. Ball [4], who showed that $\gamma_n^+(\partial K) \leq 4n^{1/4}$, illustrating the necessity of the constant. Feng, Liu and Xu [39] later removed the even assumption. Liu [77] first considered the L^p Gaussian Minkowski problem, $p > 0$, without any symmetry assumptions. Recently, Feng, Hu and Xu [38] established existence for the even L^p Gaussian Minkowski problem when $p \leq 0$.

Like many fields of study, the Gaussian measure is a core measure in the theory, and the first to be characterized within the given framework. However, when a field matures, one must inevitably broaden one's horizons and consider other classes of measures. Liu and Tang [78] took a step in this direction; motivated by [85,88,90], they solved the weighted L^p Minkowski problem for $p \in \mathbb{R} \setminus \{0\}$ with weights that were s-concave approximations of a Gaussian, i.e., they considered probability measures μ whose densities were proportional to $(1 - s|x|^{\alpha})_+^{\beta}$ for various s, α and β. Hu [51] then considered the $p = 0$ case. In this work, we study the weighted L^p Minkowski problem for a large class of Borel measures on $\mathbb{R}^n$.

Definition 1.1. A Borel measure μ on $\mathbb{R}^n$ is said to be *rotationally invariant* if, for every Borel set $A \subset \mathbb{R}^n$ and $T \in O(n)$, one has $\mu(TA) = \mu(A)$.

When $p < 0$, we will require that the density of our measures satisfy the following growth regularity. For a set $E \subset \mathbb{R}^n$, we denote by $C^+(E)$ the set of nonnegative, continuous functions on E.

Definition 1.2. Fix $p \neq 0$. We say $\phi \in C^+(\mathbb{R}^n \setminus \{0\})$ has fast enough radial decay, or has property **(D)**$_p$, if $\phi \not\equiv 0$ and, for every $\theta \in \mathbb{S}^{n-1}$, one has

$$\lim_{t \to \infty} t^{n-p} \phi(t\theta) = 0.$$

We note that every finite, log-concave measure satisfies property **(D)**$_p$ for all p. Thus, there are a plethora of examples. Now that we have introduced the ambient measures we will consider, we list some technical definitions we need to state our results. Firstly, we set $\mathbb{R}_+ = [0,\infty)$ and

$$S_p^\mu(K) := S_{K,p}^\mu(\mathbb{S}^{n-1})$$

when $K \in \mathcal{K}_o^n$. We will use both notations interchangeably. When $p = 1$, this is precisely $\mu^+(\partial K)$.

We also have special terminology used for the $p = 0$ case. We recall that Böröczky, Lutwak, Yang, and Zhang [19] defined the subspace concentration condition in the context of Minkowski problems; the phenomenon of subspace concentration also appeared in the work of Klartag [64].

Definition 1.3. Let ν be a finite Borel measure on $\mathbb{S}^{n-1}$. The measure ν is said to satisfy *the subspace concentration inequality* if for every subspace ξ of $\mathbb{R}^n$ such that $0<\dim\xi<n$,

$$\nu(\xi\cap\mathbb{S}^{n-1}) \le \frac{1}{n}\,\nu(\mathbb{S}^{n-1})\dim\xi. \tag{1.5}$$

If the inequality (1.5) is always strict, then ν is said to satisfy the *strict subspace concentration inequality*.

With this in hand, we list our first result, which is when the measure on the sphere is even. We emphasize beforehand that, in the $p \ge 0$ cases, μ has only locally integrable density, while, in the $p < 0$ case, μ is a finite Borel measure (i.e., $\mu(\mathbb{R}^n) < \infty$). We recall that a function $\psi \in L^1_{\mathrm{loc}}(E)$ if and only if for every bounded Borel set $A \subset E$, one has $\int_A \psi(x)\,dx < \infty$. Notice that, if ψ is a function on $(0,\infty)$, the function $\psi(|\cdot|)$ is locally integrable on $\mathbb{R}^n$ if and only if $\int_0^R \psi(t)t^{n-1}\,dt < \infty$ for every $R > 0$ from polar coordinate integration.

Theorem 1.4. *Let $n\in\mathbb{N}$, let $p\in\mathbb{R}$ and let $\psi\in C^+(\mathbb{R}_+\setminus\{0\})$ satisfy*

(1) *if $p > 0$: $\psi\in L^1_{\mathrm{loc}}(\mathbb{R}_+\setminus\{0\})$ and $\int_0^R \psi(t)t^{n-1}\,dt < \infty$ for every $R > 0$;*
(2) *if $p = 0$: $\psi\in L^1_{\mathrm{loc}}(\mathbb{R}_+\setminus\{0\})$ and $\int_0^\infty \psi(t)^q t^{n-1}\,dt < \infty$ for some $q\in[1,\infty)$;*
(3) *if $p<0$: $\psi\in C^+(\mathbb{R}_+)\cap L^1(\mathbb{R}_+)$ is decreasing, has $\int_0^\infty \psi(t)t^{n-1}\,dt < \infty$, and satisfies property* **(D)**$_p$.

Let μ be the rotationally invariant measure on $\mathbb{R}^n$ with density $\psi(|\cdot|)$. Fix $a\in(0,\mu(\mathbb{R}^n))$. Suppose ν is an even, finite Borel measure on the sphere which,

(a) *if $p > 0$: is not concentrated on any great hemisphere;*
(b) *if $p = 0$: satisfies the strict subspace concentration inequality;*
(c) *if $p < 0$: is not concentrated on any great subsphere.*

Then, there exists a symmetric convex body $K = K(a)$ such that $\mu(K) = a$ and

$$\nu = \frac{\nu(\mathbb{S}^{n-1})}{S_p^\mu(K)}\,S_{K,p}^\mu.$$

Also, for $p\in\mathbb{R}\setminus\{0\}$, if the density of μ is positive on ∂K, then the converse direction holds as well.

Our next result removes the evenness assumption on ν. To achieve this, we require that μ be finite for all p. For ease of presentation, we shall henceforth normalize, without loss of generality, all finite Borel measures on $\mathbb{R}^n$ to probability, i.e., we shall assume that $\mu(\mathbb{R}^n) = 1$ when μ is finite.

Theorem 1.5. *Let $p \in \mathbb{R}$, let $n \in \mathbb{N}$, and let ψ be a function in $C^+(\mathbb{R}_+ \setminus \{0\})$ such that $\int_0^\infty \psi(t)t^{n-1}dt < \infty$. Furthermore, normalize ψ so that the rotationally invariant measure μ on $\mathbb{R}^n$ with density $\psi(|\cdot|)$ is probability.*

(1) *If $p \geq 0$, fix $a \in [1/2, 1)$.*

(2) *If $p < 0$, suppose additionally that $\psi \in C^+(\mathbb{R}_+)$ is decreasing and satisfies property* $(\mathbf{D})_p$. *Then, take $a \in (1/2, 1)$.*

Suppose ν is a finite Borel measure on the sphere which,

(a) *if $p > 0$: is not concentrated on any great hemisphere;*

(b) *if $p = 0$: satisfies the strict subspace concentration inequality;*

(c) *if $p < 0$: is not concentrated on any great subsphere.*

Then, there exists $K = K(a)$ in $\mathcal{K}_o^n$ such that $\mu(K) = a$ and

$$\nu = \frac{\nu(\mathbb{S}^{n-1})}{S_p^\mu(K)} S_{K,p}^\mu.$$

If ν is even, then K can be taken to be symmetric. For $p \in \mathbb{R} \setminus \{0\}$, if the density of μ is positive on ∂K, then the converse direction holds as well.

Please see Section 3 (for $p > 0$), Section 4 (for $p = 0$) and Section 5 (for $p < 0$) to find the proofs of each case of Theorems 1.4 and 1.5.

Concerning Theorem 1.5, the proof of each case uses in a critical way that μ is finite. Thus, that theorem says nothing about non-finite, rotationally invariant measures (e.g., volume). This is not surprising; we essentially used μ being finite to circumvent any requirement that ν has its barycenter at the origin. On the other hand, the $p = 0$ case of Theorem 1.4 holds for the Lebesgue measure as well; at a key step, we use Hölder's inequality to control μ-measures of convex bodies by their volumes, a step that is trivially unnecessary if μ is Lebesgue. However, for Lebesgue measure, assuming the *strict* subspace concentration inequality is more restrictive than necessary. It is well known that for volume, when ν is even, the weaker *subspace concentration condition* is necessary and sufficient [19]. Recall that ν is said to satisfy the subspace concentration condition precisely when equality occurring in (1.5) for a subspace ξ implies that there exists a subspace ξ' complementary to ξ so that equality occurs in (1.5) for ξ'.

The reader may wonder why we restrict our attention to the strict subspace concentration inequality for general measures μ, rather than pursuing this full condition. The reason is that the equality cases behave differently for non-Lebesgue measures. For example, the even Minkowski problem for the Gaussian cone volume measure, roughly corresponding to the $p = 0$ case of Theorem 1.4, was recently completed by Hu [50]. In that work, she constructed a family of bodies that yields asymptotic equality in the strict subspace concentration inequality when ν is the Gaussian cone volume measure. A similar phenomenon was observed in the so-called chord log-Minkowski problem [86]. For this reason, we did not pursue the full subspace concentration condition in this work, as it seems to be a volume-esque phenomenon.

1.3. Existence results in the small mass regime

Intuitively, the Gaussian measure behaves like the Lebesgue measure on convex bodies containing the origin with an asymptotically small measure. Huang, Xi and Zhao initiated a program on analyzing the constant-free, even Gaussian Minkowski problem by requiring K to have such a small measure. Feng, Liu and Xu [39] removed the symmetry assumptions. We will call this program the *small mass regime*. We remark here that the small mass regime in the planar case for the Gaussian measure was thoroughly investigated by Chen, Hu, Liu and Zhao in the beautiful work [24]. Liu [77] showed the existence of a solution to the constant-free, even, L^p Gaussian Minkowski problem, $p \geq 1$ in the small mass regime. Feng, Hu and Xu [38] later removed the assumption of symmetry. Tang [105] recently showed, when $1 \leq p < n$, the existence of a solution to the constant-free, even, L^p Gaussian Minkowski problem in the small mass regime, with a twist: the body K that solves the problem satisfies $\gamma_n(K) < 1/2$, opposite to the results mentioned above.

Our next set of results uses degree theory to work in the small mass regime. We will make use of the isoperimetric function I_μ from (1.4). However, when $p \geq n$, the property $\mathbf{(D)}_p$ is automatically implied by the other hypotheses on ψ, and we can actually avoid the use of the isoperimetric function. This is not surprising; the associated Monge–Ampère equation becomes much more pliable to classical PDE techniques.

Moreover, we will need to impose an additional assumption on μ. We recall that a function $\phi\colon \mathbb{R}^n \to \mathbb{R}_+$ is said to be radially decreasing if, for every $t \in (0,1)$ and $x \in \mathbb{R}^n$, $\phi(tx) \geq \phi(x)$.

Definition 1.6. We say a measure μ on $\mathbb{R}^n$ is *radially decreasing* if it has a locally integrable, radially decreasing density that is continuous on $\mathbb{R}^n \setminus \{0\}$.

We remark that a special case of Theorem 1.2 in [70] by Kryvonos and Langharst is the even, weighted L^p Minkowski problem (with a constant) for all finite, radially decreasing measures on $\mathbb{R}^n$. The results mentioned in Theorems 1.4 and 1.5 are more exhaustive. Even with radial decay, our results in the small mass regime for $p \in [1, n)$ require another technical property, which we call $\mathbf{(S)}_p$. We postpone its definition to Section 5 (see Definition 6.13). It would be unnatural to state our results at the level of this technical assumption.

Instead, we introduce the following natural class of Borel measures on $\mathbb{R}^n$ which satisfy this property. In particular, we will be mirroring the recent development of the Gardner–Zvavitch inequality: as conjectured by Gardner and Zvavitch in [45] (with the necessity of symmetry shown in [94]) and resolved by Eskenazis and Moschidis in [36] (using the tools established in [66]), γ_n is $(1/n)$-concave over the class $\mathcal{K}^n_e$ (see Section 1.4 to recall the definition of concavity for measures). Not too long afterwards, this was extended to a larger set of Borel measures. We define the following class of Borel measures on $\mathbb{R}^n$:

$$\mathcal{M}_n^\infty = \Big\{\mu : \frac{d\mu}{dx} = e^{-V(|x|)}, \text{ where } V\colon \mathbb{R}_+ \setminus \{0\} \to (-\infty,\infty] \text{ is increasing, satisfies for every } R>0,\ \int_0^R e^{-V(t)} t^{n-1}\,dt < \infty, \text{ and the function } t \mapsto V(e^t) \text{ is convex.}\Big\}$$

Cordero-Erausquin and Rotem [32] extended the result by Eskenazis and Moschidis to every measure $\mu \in \mathcal{M}_n^\infty$, i.e., every $\mu \in \mathcal{M}_n^\infty$ is $(1/n)$-concave over the class $\mathcal{K}_e^n$. The class $\mathcal{M}_n^\infty$ then contains every rotationally invariant, log-concave measure. But it also contains more. For example, it contains the Cauchy-type measures given by $d\mu_{q,b}(x) = (1+|x|^b)^{-q}\,dx$ for $q,b \geq 0$. Another group of examples includes: if $\mu_0 \in \mathcal{M}_n^\infty$, then so is also $d\mu_q(x) = |x|^{-q}\,d\mu_0(x)$ for $q \geq 0$. Here, μ_0 can even be taken as the Lebesgue measure. Clearly, every $\mu \in \mathcal{M}_n^\infty$ is radially decreasing. More pertinently, we show in Proposition 6.10 that property $\mathbf{(S)}_p$ holds for $\mu \in \mathcal{M}_n^\infty$.

Our previous results apply to the class $\mathcal{M}_n^\infty$. Indeed, the regime $p \geq 0$ of Theorem 1.4 applies to all of $\mathcal{M}_n^\infty$. As for the results requiring finite measures, we define the following subclass of probability measures:

$$\mathcal{M}_n = \{\mu \in \mathcal{M}_n^\infty : \mu(\mathbb{R}^n) = 1\}.$$

Then, the $p \geq 0$ regimes of Theorem 1.5 applies to $\mathcal{M}_n$. For those members of $\mathcal{M}_n$ whose densities satisfy property $\mathbf{(D)}_p$, we may apply the $p<0$ case of Theorems 1.4 and 1.5. Note that we only need to explicitly impose property $\mathbf{(D)}_p$ for measures in $\mathcal{M}_n^\infty$ that are not necessarily log-concave. Consider, for example, the measure μ with density $\phi(x) = |x|^{-q}$ for some $q>0$ to be determined. Clearly, $\mu \in \mathcal{M}_n^\infty$. To satisfy condition $\mathbf{(D)}_p$, we need $n-p<q$. But, to be integrable near the origin, we need $q<n$. Thus, as long as p and q satisfy $p > n-q > 0$, μ will be a finite measure that satisfies $\mathbf{(D)}_p$.

The following theorem is an application of the more general, but technical, Theorems 6.2, 6.3 and 6.22 and Lemma 2.8. Here, κ_n denotes the volume of the Euclidean unit ball in $\mathbb{R}^n$.

Theorem 1.7. *Fix $p \geq 1$. Let $\mu \in \mathcal{M}_n$ with density $\psi(|\cdot|)$ be such that $\psi \in C^1(\mathbb{R}_+)$. Let ν be a finite Borel measure on $\mathbb{S}^{n-1}$, not concentrated on any great hemisphere.*

(1) *If $p>n$, then, there exists $K \in \mathcal{K}_o^n$ such that $S_{K,p}^\mu = \nu$.*

(2) *If $p=n$, and if ν additionally satisfies $\nu(\mathbb{S}^{n-1}) \leq \psi(0) n\kappa_n$, then there exists $K \in \mathcal{K}_o^n$ such that $S_{K,n}^\mu = \nu$.*

(3) *Suppose ψ satisfies property $\mathbf{(D)}_p$ and μ has an isoperimetric function I_μ.*

   (a) *If $1 \leq p < n$, then there exists $a>0$ depending only on μ sufficiently close to 1, such that if ν is even and*

$$\nu(\mathbb{S}^{n-1}) < \big(n \min\{a I_\mu(a)^{p/(1-p)}, (1-a) I_\mu(1-a)^{p/(1-p)}\}\big)^{1-p},$$

then, there exist $K_1, K_2 \in \mathcal{K}_e^n$ such that $\mu(K_1) > a$, $\mu(K_2) < 1-a$, and

$$S_{K_1,p}^\mu = \nu = S_{K_2,p}^\mu.$$

(b) *If* $1 \le p < \infty$ *and if* ν *satisfies*

$$\nu(\mathbb{S}^{n-1}) < \left(\frac{n}{2}\right)^{1-p} I_\mu\left(\frac{1}{2}\right)^p,$$

then there exists $K \in \mathcal{K}_o^n$ *such that* $\mu(K) \ge 1/2$ *and* $\nu = S^\mu_{K,p}$.

1.4. Concavity of measures

For our uniqueness results to make sense, we must first discuss concavity of measures, in particular the so-called F-concave measures explored in [41, 42, 79, 80]. We say a Borel measure μ is F-concave, where F is an invertible, (strictly) monotonic, continuous function, if there exists a collection $\mathcal{C}$ of Borel sets with finite μ-measure such that, for every $K, L \in \mathcal{C}$ and every $0 < \lambda < 1$, one has

$$\mu((1-\lambda)K + \lambda L) \ge F^{-1}((1-\lambda)F(\mu(K)) + \lambda F(\mu(L))). \tag{1.6}$$

For a fixed $s \in [-\infty, \infty)$, a measure μ is s-concave over $\mathcal{C}$ if (1.6) holds for $F(x) = x^s$, i.e., for every $\lambda \in (0,1)$ and Borel sets $K, L \in \mathcal{C}$, one has

$$\mu((1-\lambda)K + \lambda L) \ge ((1-\lambda)\mu(K)^s + \lambda\mu(L)^s)^{1/s}.$$

The case $s = 0$ is log-concavity:

$$\mu((1-\lambda)K + \lambda L) \ge \mu(K)^{1-\lambda}\mu(L)^\lambda.$$

When $s = -\infty$, this should be read as $\mu((1-\lambda)K + \lambda L) \ge \min\{\mu(K), \mu(L)\}$. A function $f : \mathbb{R}^n \to [0, \infty]$ is said to be κ-concave, $\kappa \in \mathbb{R}$, if, for every $\lambda \in (0,1)$,

$$f((1-\lambda)x + \lambda y) \ge ((1-\lambda)f(x)^\kappa + \lambda f(y)^\kappa)^{1/\kappa}$$

for every x, y such that $0 < f(x)f(y) < \infty$. The case $\kappa = 0$ means f is log-concave, and $\kappa = \infty$ means f is constant.

In addition to the aforementioned result by Cordero-Erausquin and Rotem [32], showing that every $\mu \in \mathcal{M}_n^\infty$ is $(1/n)$-concave over $\mathcal{K}_e^n$, Aishwarya and Rotem [2] showed that, if V is a convex, q-homogeneous function and μ is a Borel measure such that $d\mu(x) = Ce^{-V(x)}dx$, for $C > 0$, then μ is $(\frac{q-1}{qn})$-concave over $\mathcal{K}_o^n$. When $V(x) = \frac{1}{2}|x|^2$, then one obtains that γ_n is $(\frac{1}{2n})$-concave over $\mathcal{K}_o^n$, which was previously shown by Kolesnikov and Livshyts [67].

Recall that a Borel measure is Radon if it is locally finite and (inner) regular. Borell's classification [12] states that a Radon measure μ is an s-concave measure (with $\mathcal{C}$ all Borel subsets of $\mathbb{R}^n$) if and only if it has density with respect to the Lebesgue measure that is $\kappa = s/(1-ns)$-concave (so, if $s = 1/n$, the measure is a multiple of the Lebesgue measure, and, if $s > 1/n$, the density is zero a.e.). Henceforth, we say such measures are s-concave in the sense of Borell. For brevity, we may simply say s-concave (without referencing some class $\mathcal{C}$).

Recall that a measure is said to be α-homogeneous if, for a Borel set A, one has $\mu(tA) = t^\alpha\mu(A)$, $\alpha, t > 0$, for all t such that tA is in the support of μ. The Lebesgue measure, which we denote as Vol_n, is an n-homogeneous, $(1/n)$-concave measure. To illustrate the use of s-concave measures in Minkowski problems, Livshyts [80] considered and solved the constant-free, even Minkowski problem for a measure that is α-homogeneous, $(1/\alpha)$-concave, in the sense of Borell, when $\alpha \geq n$; the result by Livshyts was extended to the L^p case, $p \geq n$, by Wu [108]. Kryvonos and Langharst [70] then did all $p \geq 1$.

For another example, the Gaussian measure γ_n is log-concave. In fact, γ_n is concave in the sense of (1.6) with F being the inverse function of $\gamma_1((-\infty, x))$, the so-called Ehrhard inequality [13, 34, 35, 72]; when $\mathscr{C}$ is taken to be $\mathcal{K}^n$, then there is equality in Ehrhard's inequality if and only if the two bodies under consideration are identical (the Ehrhard inequality still holds with $\mathscr{C}$ taken to be all Borel sets, but then equality conditions are still open). Using the Ehrhard inequality in the case of convex bodies, and the additional assumption that $\gamma_n(K) \geq 1/2$, Huang, Xi and Zhao [55] obtained uniqueness in the even Gaussian Minkowski problem (this is a special case of Proposition 1.8 below).

1.4.1. Uniqueness results. For our uniqueness results, we need to refer to Firey's L^p summation [40], denoted by $+_p$. Note that $p = 1$ is the usual Minkowski sum. We say μ is L^p F-concave if it satisfies (1.6) with Minkowski summation replaced by L^p summation. Note that L^p means of the form $(1-\lambda) \cdot K +_p \lambda \cdot L$, for $\lambda \in (0,1)$ and $K, L \in \mathcal{K}^n_o$, are increasing with respect to set-inclusion as p increases. Thus, if μ is F-concave, then it is also L^p F-concave (with the same F) for $p > 1$. If μ is shown to be L^p F-concave from this method (i.e., if one already knows μ is F-concave, and then uses monotonicity of L^p means to obtain μ is L^p F-concave), then equality in the L^p version of (1.6), $p > 1$, yields $K = L$, solely from equality forcing the L^p mean to be independent of p.

However, this does not use the L^p summation in a "smart" way. For example, Roysdon and Xing [97] showed, extending on the volume case by Lutwak [83], that, if μ is s-concave (in the sense of Borell), $s > 0$, then it is (sp)-concave with respect to L^p summation for $p \geq 1$. As for equality conditions, we recall that Dubuc [33] showed, as elaborated by Milman and Rotem [93], the following: let μ be s-concave (in the sense of Borell). If K and L are Borel sets such that, for some $\lambda \in (0,1)$, one has

$$\mu((1-\lambda)K + \lambda L)^s = (1-\lambda)\mu(K)^s + \lambda\mu(L)^s, \tag{1.7}$$

then, $K = aL + b$ for some $a > 0$ and $b \in \mathbb{R}^n$. In (1.7), if one replaces $+$ with $+_p$, and s with ps, then we lose the possibility of translation, i.e., equality holds in the (sp)-concavity of an s-concave measure with respect to L^p summation if and only if the two bodies are dilates.

As far as we are aware, the method of obtaining L^p F-concavity using monotonicity and the results by Roysdon–Xing and Lutwak for s-concave measures are the only examples of measures that are L^p F-concave for $p > 1$ (we save the discussion of $p < 1$ for the appendix). Thus, in our results below, the equality conditions for $p > 1$ should be read as $K = tL$. Using the L^p version of (1.6), we recall the following from Proposition 6.16 in [70].

Proposition 1.8. *Fix $p \geq 1$. Let μ, a Borel measure with density continuous on $\mathbb{R}^n \setminus \{0\}$, be L^p F-concave over $\mathscr{C} \subset \mathcal{K}^n_o$ such that F is differentiable. Suppose $K, L \in \mathscr{C}$ are such that $S^{\mu}_{K,p}$ and $S^{\mu}_{L,p}$ are finite Borel measures on $\mathbb{S}^{n-1}$ and*

$$S^{\mu}_{K,p} = S^{\mu}_{L,p}.$$

Then

$$\frac{F(\mu(L)) - F(\mu(K))}{F'(\mu(K))} \leq \frac{F(\mu(L)) - F(\mu(K))}{F'(\mu(L))}.$$

Furthermore, there is equality if and only if there is equality in the L^p version of (1.6).

As we have seen in our existence results, we were able to pin the measure of our convex bodies. Therefore, uniqueness follows as an immediate corollary of the above.

Theorem 1.9. *Fix $p \geq 1$. Let μ, a Borel measure with density continuous on $\mathbb{R}^n \setminus \{0\}$, be L^p F-concave over $\mathcal{C} \subset \mathcal{K}_o^n$ such that F is differentiable. Suppose $K, L \in \mathcal{C}$ are such that $\mu(K) = \mu(L)$, $S^\mu_{K,p}$ and $S^\mu_{L,p}$ are finite Borel measures on $\mathbb{S}^{n-1}$ and $S^\mu_{K,p} = S^\mu_{L,p}$. Then, there is equality in the L^p version of* (1.6).

Let us return to the case of s-concave measures from (1.7), and let $s > 0$. If μ is s-concave, then it is also L^p s-concave (due to monotonicity of L^p means) and L^p (sp)-concave (Roysdon–Xing). Suppose that we have equality in either of these L^p concavities. Then, since we also know that $\mu(K) = \mu(L)$, $K = L$. Consequently, "equality in (1.6)" in Theorem 1.9, in the case of s-concave measures for $p > 1$, should be read as $K = L$. For $p = 1$, one needs extra information to remove the translation from the equality conditions of (1.7) first, for instance, by assuming that both bodies are symmetric. We stated Proposition 1.8 and Theorem 1.9 in terms of a class $\mathcal{C} \subset \mathcal{K}_o^n$ to allow us to consider, for example, $\mathcal{C} = \mathcal{K}_e^n$. Thus, we can also apply Theorem 1.9 to the L^p $(1/n)$-concavity of $\mu \in \mathcal{M}_n^\infty$ over $\mathcal{K}_e^n$ for $p \geq 1$. If $p > 1$, then we must have $K = L$. However, for $p = 1$, outside the case of rotationally invariant measures that are also log-concave (where (1.7) and $\mu(K) = \mu(L)$ tell us that $K = L$), the equality conditions of (1.6) in this case are apparently still open.

This paper is organized as follows. In Section 2, we give many formulas for variations of surface area measure needed in this work (among other preliminary facts). In Section 3, we prove Theorem 3.7 and the $p > 0$ cases of Theorem 1.4 and Theorem 1.5, which are results concerning positive ranges for p. In Section 4, we prove the $p = 0$ cases of Theorems 1.4 and 1.5, which are the weighted log-Minkowski problems. In Section 5, we prove the $p < 0$ cases of Theorems 1.4 and 1.5, which concern negative p. For the constant-free results in the small mass regime, we prove Theorem 6.2 in Section 6.1, and Theorem 6.22 in Section 6.3. Finally, in Appendix A, we list examples of isoperimetric functions, concluding on a conjecture that would yield a sharp isoperimetric inequality for measures in $\mathcal{M}_n^\infty$.

2. Preliminaries

In this work, we will be using the radial function of a convex body $K \in \mathcal{K}_o^n$, which is the continuous function on $\mathbb{R}^n \setminus \{0\}$ given by $\rho_K(u) = \sup\{r > 0 : ru \in K\}$. Using radial functions, one has the volume formula

$$\mathrm{Vol}_n(K) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \rho_K(u)^n \, du,$$

where the integration is with respect to the spherical Lebesgue measure (i.e., $du = dS_{B_2^n}$).

We now briefly mention that in [43, 44, 75], the first two of which pre-date the Gaussian Minkowski problem, Ye and his various collaborators studied the so-called generalized volume setting. The framework introduces the general volume function G on $(0,\infty)\times\mathbb{S}^{n-1}$. From this, a measure on ∂K for $K\in\mathcal{K}^n_o$ is introduced as a type of push-forward of $G(\rho_K(u),u)$. This framework is very broad; indeed, set

$$G(t,u)=\int_0^t \phi(ru)r^{n-1}\,dr \tag{2.1}$$

for ϕ the density of a measure μ on $\mathbb{R}^n$ to obtain S^μ_K. They found necessary and sufficient conditions for existence in the associated Minkowski problem with a constant under various assumptions on $\partial_t G$. In particular, their results imply the L^p Gaussian Minkowski problem with a constant for all $p>0$, in the strict sense of the definition of Minkowski problem. Crucially, however, the approach, when translated to S^μ_K via (2.1), does not allow control of $\mu(K)$. The ability to do so, as first done by Huang, Xi and Zhao, is vital in Gaussian Minkowski problems, and is an important extra ingredient that we strove to keep. Consequently, the results for the Gaussian measure implied by [43, 44, 75] are very similar, but ultimately disjoint, from the majority of the mentioned results on Gaussian Minkowski problems with a constant, and, more pertinently, disjoint from the results we present herein. In addition to being able to control $\mu(K)$, the class of rotationally invariant measures we consider allows us to drop the assumptions on ϕ that one would obtain using the aforementioned generalized volume results.

Fix an arbitrary $K\in\mathcal{K}^n_o$. We say a convex body is strictly convex if its boundary does not contain a line-segment. The subgradient of h_K is precisely the set function given by

$$\partial h_K(u)=\{z\in\mathbb{R}^n : h_K(y)\geq h_K(u)+\langle z,y-u\rangle,\ \forall y\in K\}.$$

Denoting by ∇ the usual gradient operator, one has that $\partial h_K(u)$ is a singleton, namely $\nabla h_K(u)$, if and only if h_K is differentiable at u. In general, for $u\in\mathbb{S}^{n-1}$,

$$\partial h_K(u)=F(K,u)=\{y\in K : h_K(u)=\langle u,y\rangle\}, \tag{2.2}$$

where $F(K,u)$ is the face of K with outer-unit normal u. For $x\in\mathbb{R}^n\setminus\{0\}$, one has

$$\partial h_K(x)=F\Big(K,\frac{x}{|x|}\Big).$$

The Gauss map and the support function are related: n_K is invertible at $u\in\mathbb{S}^{n-1}$ if and only if h_K is differentiable at u, in which case $n_K^{-1}(u)=\nabla h_K(u)$, see Corollary 1.7.3 in [99]. Hence, K is strictly convex if and only if $h_K\in C^1$, see p. 115 of [99].

The surface area measure is closely related to the so-called Monge–Ampère measure: given a convex function h defined on an open, d-dimensional convex set Ω (equipped with the d-dimensional Hausdorff measure), its Monge–Ampère measure is precisely

$$\mu_h(E)=\mathcal{H}^d(N_h(E)),\quad\text{where}\quad N_h(E)=\bigcup_{x\in E}\partial h(x)$$

for a Borel subset $E\subset\Omega$. Here, ∂h is the subgradient of h; since we will only use this when h is the support function of a convex body, we do not define the subgradient of an arbitrary convex function. If h is C^2, then one obtains the following integral representation:

$$\mu_h(E) = \int_E \det \mathrm{Hess}(h(x))\, d\mathcal{H}^d(x), \tag{2.3}$$

where Hess denotes the Hessian on $\mathbb{R}^d$. As an example, setting $\Omega = \mathbb{S}^{n-1}$, the surface area measure is then the Monge–Ampère measure of the support function:

$$S_K(E) = \mathcal{H}^{n-1}\Big(\bigcup_{u\in E} F(K,u)\Big) = \mathcal{H}^{n-1}\Big(\bigcup_{u\in E} \partial h_K(u)\Big) = \mu_{h_K}(E). \tag{2.4}$$

If the boundary of K is a C^2 manifold with positive Gauss curvature everywhere, then we say that K is C^2_+. If K is C^2_+, then $h_K \in C^2(\mathbb{S}^{n-1})$ (see the bottom of page 115 in [99]) and S_K is absolutely continuous with respect to the spherical Lebesgue measure: $dS_K(u) = \det(\nabla^2 h_K + h_K I)\, du$. Here, I is the $(n-1)\times(n-1)$ identity matrix and ∇^2 is the spherical Hessian. We denote by ∇_s the spherical Gradient.

Since $K \in \mathcal{K}^n_o$, we have that $\rho_K(\theta)$ is continuous, and so there exists some $\theta_K \in \mathbb{S}^{n-1}$ such that $\rho_K(\theta_K)$ is maximal. One has that the line segment $[-\rho_K(\theta_K)\,\theta_K, \rho_K(\theta_K)\,\theta_K]$ is completely contained in K, and yet K is contained in the ball of radius $\rho_K(\theta_K)$. Let $g(t) = |t|$ if K is symmetric and $g(t) = t_+ = \max\{t, 0\}$ otherwise. From convexity, one has

$$h_K(u) \ge \rho_K(\theta_K)\, g(\langle \theta_K, u\rangle) \tag{2.5}$$

for all $u \in \mathbb{S}^{n-1}$. Note that, at the point θ_K, $\rho_K(\theta_K) = h(\theta_K)$. Recalling that h_K is 1-homogeneous, we have (by differentiating $h(tu)$ at $t = 1$) that, for every $u \in \mathbb{S}^{n-1}$ such that h_K is differentiable at u,

$$h_K(u) = \langle \nabla h_K(u), u\rangle. \tag{2.6}$$

Consequently, by the Cauchy–Schwarz inequality, $h_K(u) \le |\nabla h_K(u)|$. These two estimates yield

$$h_K(\theta_K)\, g(\langle u, \theta_K\rangle) \le |\nabla h_K(u)| \tag{2.7}$$

for $K \in \mathcal{K}^n_o$.

It is also well known that for every $u \in \mathbb{S}^{n-1}$, there exists a v such that

$$\rho_K(v)\, v = h_K(u)\, u + \nabla_s h_K(u).$$

This implies that

$$\rho_K^2(v) = h_K^2(u) + |\nabla_s h_K(u)|^2. \tag{2.8}$$

Recall also the fact that, when K is C^2_+,

$$\det(\nabla^2 h_K(u) + h_K(u) I) = \frac{(h_K(u)^2 + |\nabla_s h_K(u)|^2)^{n/2}}{h_K(u)} = \frac{\rho_K(v)^n}{h_K(u)}, \tag{2.9}$$

where v and u are related via (2.8). It follows from (2.9) that

$$h_K(u)\det(\nabla^2 h_K(u) + h_K(u)I) \leq \max_{v\in\mathbb{S}^{n-1}} \rho_K(v)^n = \max_{u\in\mathbb{S}^{n-1}} h_K(u)^n. \tag{2.10}$$

For every positive $f \in C(\mathbb{S}^{n-1})$, the *Wulff shape* of f is the convex body given by

$$[f] = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq f(u),\ \forall u \in \mathbb{S}^{n-1}\}.$$

One has that, for $K \in \mathcal{K}^n_o$, $[h_K] = K$. Since f is positive, $[f] \in \mathcal{K}^n_o$. Furthermore, if f is even, then $[f]$ is symmetric. Next, for $f \in C(\mathbb{S}^{n-1})$, Aleksandrov [1] defined a perturbation of $K \in \mathcal{K}^n$ to be the Wulff shape of the function

$$h_t(u) = h_K(u) + tf(u), \tag{2.11}$$

where $t \in (-\delta, \delta)$, with δ small enough so that h_t is positive for all u.

It was shown [55, 70] that, for almost all $u \in \mathbb{S}^{n-1}$ up to a set of spherical Lebesgue measure zero,

$$\left.\frac{\mathrm{d}\rho_{[h_t]}(u)}{\mathrm{d}t}\right|_{t=0} = \lim_{t\to 0}\frac{\rho_{[h_t]}(u) - \rho_K(u)}{t} = \frac{f(n_K(r_K(u)))}{h_K(n_K(r_K(u)))}\,\rho_K(u). \tag{2.12}$$

Here,

$$r_K(u) = \rho_K(u)u$$

is the radial map, which is defined almost everywhere. In fact, the identity (2.12) was proven using tools from [53]. Let us elaborate. In [53], they introduced the following concept: for $h, f \in C(\mathbb{S}^{n-1})$ and some small δ, define a function $h_t\colon \mathbb{S}^{n-1} \to (0,\infty)$ via

$$\log(h_t(u)) = \log(h(u)) + tf(u) + o(t,u), \tag{2.13}$$

where $o(t,u)/t \to 0$ as $t \to 0$ for all $u \in \mathbb{S}^{n-1}$. Then, the Wulff shapes $[h_t]$ are said to be the logarithmic family of Wulff shapes formed by the pair (h, f). One can readily verify that h_t defined by (2.11) is the logarithmic family of Wulff shapes formed by the pair $(h_K, f/h_K)$; then, (2.12) follows from Lemma 4.3 in [53]. However, there was actually no need to use this specific family. The same proof yields the following.

Proposition 2.1. *Let $K \in \mathcal{K}^n_o$ and $f \in C(\mathbb{S}^{n-1})$ be such that (h_K, f) is a logarithmic family. Then, defining h_t via (2.13), we have*

$$\left.\frac{\mathrm{d}\rho_{[h_t]}(u)}{\mathrm{d}t}\right|_{t=0} = \lim_{t\to 0}\frac{\rho_{[h_t]}(u) - \rho_K(u)}{t} = f(n_K(r_K(u)))\,\rho_K(u).$$

In fact, there exist constants $M, \delta > 0$ so that, for almost every $u \in \mathbb{S}^{n-1}$ and every $t \in (-\delta, \delta)$,

$$|\rho_{[h_t]}(u) - \rho_K(u)| \leq M|t|.$$

The following result was shown by Langharst and Kryvonos [70], extending on the partial cases by Hosle, Kolesnikov, and Livshyts [49, 80]; Gardner, Hug, Weil, Xing, and Ye [43]; Kolesnikov and Milman [68]; Bobkov [9]; and Aleksandrov [1].

Proposition 2.2. *Let μ be a Borel measure on $\mathbb{R}^n$ with locally integrable density ϕ. Let K be a convex body, such that ∂K, up to a set of $(n-1)$-dimensional Hausdorff measure zero, is in the Lebesgue set of ϕ. Then, for $f \in C(\mathbb{S}^{n-1})$, one has that*

$$\lim_{t\to 0} \frac{\mu([h_K + tf]) - \mu(K)}{t} = \int_{\mathbb{S}^{n-1}} f(u)\, dS_K^{\mu}(u). \tag{2.14}$$

The assumption that μ has locally integrable density can be dropped, i.e., μ can have singular components, as long as there exists a δ-neighborhood of ∂K not intersecting the region of $\mathbb{R}^n$ assigned singular mass. When $f = h_L$, where L is a compact, convex set, one has $[h_K + \varepsilon h_L] = K + \varepsilon L$ if $K \in \mathcal{K}_o^n$.

Recently, (2.14) was re-proven in [37] in the case when $K \in \mathcal{K}_o^n$ and μ has continuous density ω; we now elaborate. Recall that $S_{K,0}^{\mu}$ is the weighted cone measure, i.e., $dS_{K,0}^{\mu} = h_K dS_K^{\mu}$. Then, (2.14) can be written as, when $K \in \mathcal{K}_o^n$,

$$\lim_{t\to 0} \frac{\mu([h_K + tf]) - \mu(K)}{t} = \int_{\mathbb{S}^{n-1}} \frac{f(u)}{h_K(u)}\, dS_{K,0}^{\mu}(u),$$

which is what appears in Lemma 2.1 of [37] (note in that work, $S_{K,0}^{\mu}$ is called the dual ω-Orlicz moment of K). The choice to use $\frac{1}{h_K(u)} dS_{K,0}^{\mu}(u)$, and not simply S_K^{μ}, forces one to have the origin in the interior of K.

By using S_K^{μ} in (2.14), one can merely shift both the measure and the convex body K to remove any necessity for K to contain the origin; that is, the case for a general convex body K actually follows from the case containing the origin. Indeed, for a convex body K, let $K - \int_K x\,dx = K' \in \mathcal{K}_o^n$. For a Borel measure μ with density ϕ whose Lebesgue set contains ∂K, let μ' be the Borel measure with density $\phi'(y) = \phi(y + \int_K x\,dx)$. Then, notice that $S_{K'}^{\mu'} = S_K^{\mu}$, $\mu(K) = \mu'(K')$, and, for t small enough,

$$[h_{K'} + tf] = [h_K + tf] - \int_K x\,dx$$

implies

$$\mu'([h_{K'} + tf]) = \mu([h_K + tf]).$$

Schneider [100] recently gave a new, geometric proof of Proposition 2.2 for when μ has continuous density. This elegant proof is in contrast to the approach taken in [70], which used the radial variations (2.12) to prove (2.14). However, one advantage of using the condition (2.12) is that, if one uses Proposition 2.1, the exact same proof yields the following.

Theorem 2.3. *Let μ be a Borel measure on $\mathbb{R}^n$ with locally integrable density ϕ. Let $K \in \mathcal{K}_o^n$ be such that ∂K, up to a set of $(n-1)$-dimensional Hausdorff measure zero, is in the Lebesgue set of ϕ. Suppose $f \in C(\mathbb{S}^{n-1})$ is such that (h_K, f) is a logarithmic family, and define h_t via (2.13).*

Then, one has

$$\lim_{t\to 0} \frac{\mu([h_t]) - \mu(K)}{t} = \int_{\mathbb{S}^{n-1}} f(u)\, dS_{K,0}^{\mu}(u).$$

By taking the Taylor series expansion of e^x, a perturbation of h_K of the form $h_K e^{tf}$ satisfies the hypothesis of Theorem 2.3. Thus, we obtain the following corollary.

Corollary 2.4. *Let μ be a Borel measure on $\mathbb{R}^n$ with locally integrable density ϕ. Let $K \in \mathcal{K}_o^n$ be such that ∂K, up to a set of $(n-1)$-dimensional Hausdorff measure zero, is in the Lebesgue set of ϕ. Suppose $f \in C(\mathbb{S}^{n-1})$. Then,*

$$\lim_{t\to 0} \frac{\mu([h_K e^{tf}]) - \mu(K)}{t} = \int_{\mathbb{S}^{n-1}} f(u)\, dS^{\mu}_{K,0}(u).$$

This extends on the volume case from [18] and the Gaussian case from [38, 50].

The L^p surface area was also given a weighted analogue in [70]. The associated variational formula is the same as above, except $[h_K + th_L]$ is replaced with $[(h_K^p + th_L)^{1/p}]$ and dS^{μ}_K is replaced with $dS^{\mu}_{K,p}/p$; here, of course, K must contain the origin. This extends on partial cases by Lutwak [83], Wu [108], and Liu [77]. Note that in [70] it is assumed that $p \geq 1$, but this is not used in the proof.

Proposition 2.5. *Let $p \neq 0$, let K be a convex body in $\mathbb{R}^n$, and let μ be a Borel measure on $\mathbb{R}^n$ with a density whose Lebesgue set contains ∂K. Assume that $S^{\mu}_{K,p}$ is a finite Borel measure on $\mathbb{S}^{n-1}$. Then,*

$$\lim_{\varepsilon\to 0} \frac{\mu([(h_K^p + \varepsilon f)^{1/p}]) - \mu(K)}{\varepsilon} = \frac{1}{p}\int_{\mathbb{S}^{n-1}} f(u)\, dS^{\mu}_{K,p}(u).$$

Let μ be a Borel measure on $\mathbb{R}^n$. Then, Livshyts [80] introduced the *mixed measure* of Borel sets K and L as

$$\mu(K;L) := \liminf_{\varepsilon\to 0} \frac{\mu(K+\varepsilon L) - \mu(K)}{\varepsilon}.$$

In the same work, it was shown that, if μ has continuous density and K and L are convex sets, then the liminf is a limit and

$$\mu(K;L) = \int_{\mathbb{S}^{n-1}} h_L(u)\, dS^{\mu}_K(u). \tag{2.15}$$

In [70], the assumption that μ has continuous density was weakened to assuming that μ with a density whose Lebesgue set contains ∂K. When $\mu = \gamma_n$, the quantity $\gamma_n(K;L)$ would reappear in [55]. Notice that $\mu(K; B_2^n) = \mu^+(\partial K)$. Mixed measures were later systematically studied in [41, 42].

We mention now that Firey's L^p summation is precisely, for $a, b \geq 0$ and $K, L \in \mathcal{K}_o^n$,

$$a \cdot K +_p b \cdot L = [(ah_K^p + bh_L^p)^{1/p}].$$

The L^p version of mixed measures, where summation was replaced by L^p summation, was introduced in [70]: under the same assumptions on K, L and μ in (2.15), one has

$$\mu_p(K;L) := \lim_{\varepsilon\to 0} \frac{\mu(K +_p \varepsilon\cdot L) - \mu(K)}{\varepsilon} = \frac{1}{p}\int_{\mathbb{S}^{n-1}} h_L(u)^p\, dS^{\mu}_{K,p}(u), \tag{2.16}$$

whenever $S^{\mu}_{K,p}$ is a finite Borel measure on the sphere.

This extends on the Gaussian case from Liu [77] and the volume case by Lutwak [82], Lutwak, Yang, and Zhang [91], and Böröczky, Lutwak, Yang, and Zhang [19]. Notice that $\mu_p(K;K) = \frac{1}{p}\mu(K;K)$. While it is true that

$$\mathrm{Vol}_n(K) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_K(u)\, dS_K(u), \tag{2.17}$$

this does not hold for measures: $n\mu(K) \neq \mu(K;K)$. The following proposition, shown in Proposition 2.1 of [41], relates them.

Proposition 2.6. *Let μ be a radially decreasing measure on $\mathbb{R}^n$. Then, for every convex body $K \in \mathcal{K}_o^n$ such that the density of μ is defined on ∂K, one has*

$$n\mu(K) \geq \mu(K;K),$$

with equality if and only if for almost every $y \in \partial K$, the density of μ is a constant almost everywhere on $(0, y]$.

We conclude this section by obtaining an isoperimetric inequality for $S_p^\mu(K)$ when $K \in \mathcal{K}_o^n$ and μ is radially decreasing. To do this, we need another definition.

Definition 2.7. Fix $p > 0$, and let μ be a Borel measure on $\mathbb{R}^n$. We say μ has an L^p isoperimetric function over $\mathcal{K}_o^n$ (or $\mathcal{K}_e^n$) if there exists a nonnegative function I_p such that, for $K \in \mathcal{K}_o^n$ (or $\mathcal{K}_e^n$),

$$S_p^\mu(K) \geq I_p(\mu(K)).$$

We discuss in Appendix A existence of I_p. There is a plethora of examples, such as log-concave measures. The result of the following lemma establishes the existence of I_p when $p > 1$ for radially decreasing measures μ that have an isoperimetric function I_μ.

Lemma 2.8. *Let $K \in \mathcal{K}_o^n$ and let $p \geq 1$. Let μ be a radially decreasing measure on $\mathbb{R}^n$ and I a function satisfying* (1.4) *for μ and K. Then,*

$$S_p^\mu(K) \geq (n\mu(K))^{1-p} I(\mu(K))^p.$$

Proof. We apply Jensen's inequality to the probability measure $h_K(u)\, dS_K^\mu(u)/\mu(K;K)$ on the sphere to obtain

$$\begin{aligned}\Big(\frac{S_p^\mu(K)}{\mu(K;K)}\Big)^{1/p} &= \Big(\int_{\mathbb{S}^{n-1}} h_K(u)^{-p}\,\frac{h_K(u)\, dS_K^\mu(u)}{\mu(K;K)}\Big)^{1/p} \\ &\geq \int_{\mathbb{S}^{n-1}} h_K(u)^{-1}\,\frac{h_K(u)\, dS_K^\mu(u)}{\mu(K;K)} = \frac{\mu^+(\partial K)}{\mu(K;K)}.\end{aligned}$$

Upon re-arrangement, this becomes

$$S_p^\mu(K)^{1/p} \geq \mu(K;K)^{(1-p)/p}\mu^+(\partial K).$$

Applying Proposition 2.6, we obtain

$$S_p^\mu(K)^{1/p} \geq (n\mu(K))^{(1-p)/p}\mu^+(\partial K).$$

Using that $\mu^+(\partial K) \geq I(\mu(K))$ by hypothesis and raising both sides to the pth power yields the result. ∎

We note that the case $\mu = \gamma_n$ (and $I = I_{\gamma_n}$) of Lemma 2.8 was previously done in [38].

3. The case of positive p

In this section, we will prove the $p > 0$ cases of Theorems 1.4 and 1.5. At the end of the section, we also establish another result for when $p > n$, see Theorem 3.7 below. We will utilize variational techniques, which can be traced back to [29, 38, 39, 44, 46, 55, 70, 83, 87, 89, 107]. We denote by $C(\mathbb{S}^{n-1})$ the set of continuous functions on the sphere. If there is a "+" superscript, then the functions are additionally nonnegative; if there is a "e" subscript, then the functions are also even.

We will work with the following two functionals: the first,

$$\psi_\nu(f) := \mu([f]) - \frac{1}{p}\int_{\mathbb{S}^{n-1}} f^p(u)\,d\nu(u); \quad \psi_\nu(K) := \psi_\nu(h_K), \tag{3.1}$$

will be taken over $C_e^+(\mathbb{S}^{n-1})$, and the second,

$$\Omega_\nu(f) := -\frac{1}{p}\int_{\mathbb{S}^{n-1}} f^p(u)\,d\nu(u); \quad \Omega_\nu(K) := \Omega_\nu(h_K), \tag{3.2}$$

can be taken over $C^+(\mathbb{S}^{n-1})$ or $C_e^+(\mathbb{S}^{n-1})$. We break the proof into four steps:

(1) First, show that, for $p \neq 0$, any maximizer of (3.1) or (3.2) in $C_e^+(\mathbb{S}^{n-1})$ is the support function of a symmetric convex body, and any maximizer of (3.2) in $C^+(\mathbb{S}^{n-1})$ is the support function of a convex body containing the origin in its interior.

(2) Second, show that, when $p \neq 0$, a maximizer K of (3.1) satisfies $\nu = S^\mu_{K,p}$, and a maximizer K of (3.2) satisfies
$$\nu = \frac{\nu(\mathbb{S}^{n-1})}{S^\mu_p(K)}\, S^\mu_{K,p}.$$

(3) Third, establish the existence of a maximizer when the measure of the body is pinned and p is positive, that is, to prove the $p > 0$ cases of Theorems 1.4 and 1.5.

(4) Finally, establish the existence of a maximizer when $p > n$ and there are assumptions of symmetry and property $(\mathbf{D})_p$, that is, to prove Theorem 3.7.

Lemma 3.1. *Let μ be a Borel measure on $\mathbb{R}^n$, and fix a Borel measure ν on $\mathbb{S}^{n-1}$. Let $K \in \mathcal{K}^n_o$. Then, for a fixed $p \in \mathbb{R}$, $p \neq 0$:*

(1) *K solves*
$$\sup\{\psi_\nu(K) : K \in \mathcal{K}^n_e\} \tag{3.3}$$
if and only if h_K solves
$$\sup\{\psi_\nu(f) : f \in C_e^+(\mathbb{S}^{n-1})\}.$$

(2) *For a fixed $a \in (0, \mu(\mathbb{R}^n))$, K solves*
$$\sup\{\Omega_\nu(K) : \mu(K) = a, K \in \mathcal{K}^n_o\} \tag{3.4}$$
if and only if h_K solves
$$\sup\{\Omega_\nu(f) : \mu([f]) = a, f \in C^+(\mathbb{S}^{n-1})\}.$$

Additionally, in (3.4), if K is also assumed to be symmetric in the first optimization problem, then the set $C^+(\mathbb{S}^{n-1})$ is replaced by $C_e^+(\mathbb{S}^{n-1})$ in the second optimization problem.

Proof. Recall for $f \in C^+(\mathbb{S}^{n-1})$, $h_{[f]}(u) \leq f(u)$ point-wise and $[h_{[f]}] = [f]$. Suppose $p > 0$. Then,

$$h_{[f]}^p \leq f^p, \quad \text{but} \quad -\frac{1}{p} h_{[f]}^p \geq -\frac{1}{p} f^p.$$

If $p < 0$, then we instead have $h_{[f]}^p \geq f^p$, but still obtain

$$-\frac{1}{p} h_{[f]}^p \geq -\frac{1}{p} f^p.$$

Therefore, for all $p \neq 0$,

$$\Omega_\nu([f]) = -\frac{1}{p} \int_{\mathbb{S}^{n-1}} h_{[f]}^p(u)\, d\nu(u) \geq -\frac{1}{p} \int_{\mathbb{S}^{n-1}} f^p(u)\, d\nu(u) = \Omega_\nu(f).$$

Thus, a maximizer of Ω_ν is the support function of a convex body, which is symmetric when maximizing over $C_e^+(\mathbb{S}^{n-1})$. Since one always has $\mu([f]) = \mu([h_{[f]}])$, the above also shows $\psi_\nu([f]) \geq \psi_\nu(f)$. This establishes both equivalences. ∎

Lemma 3.2. *Let $K \in \mathcal{K}_o^n$, and let μ be a Borel measure on $\mathbb{R}^n$ with a density whose Lebesgue set contains ∂K. Then, for a fixed $p \in \mathbb{R}$, $p \neq 0$:*

(1) *If K solves (3.3), then $\nu = S_{K,p}^\mu$.*

(2) *If K solves (3.4), then $\nu = \frac{\nu(\mathbb{S}^{n-1})}{S_p^\mu(K)} S_{K,p}^\mu$.*

Proof. First suppose that K solves (3.3). Then, we perturb K by $f \in C_e^+(\mathbb{S}^{n-1})$: let

$$h_t = (h_K^p + tf)^{1/p},$$

where $t \in (-\delta, \delta)$, δ chosen so that h_t is strictly positive on $\mathbb{S}^{n-1}$. Then, by Proposition 2.5, we obtain

$$0 = \left.\frac{d\psi_\nu(h_t)}{dt}\right|_{t=0} = \frac{1}{p} \int_{\mathbb{S}^{n-1}} f(u)\, dS_{K,p}^\mu(u) - \frac{1}{p} \int_{\mathbb{S}^{n-1}} f(u)\, d\nu(u).$$

Since this is true for every $f \in C_e^+(\mathbb{S}^{n-1})$, we have by the Riesz representation theorem that $\nu = S_{K,p}^\mu$, thus establishing the first claim.

Next, suppose that K solves (3.4). Since perturbing K causes us to break the condition that $\mu(K) = a$, we must modify our direct variational approach with Lagrange multipliers. We define

$$\Phi(t, \varepsilon) = \Omega_\nu((h_K^p + tf + \varepsilon)^{1/p}) \quad \text{and} \quad \Psi(t, \varepsilon) = \mu([(h_K^p + tf + \varepsilon)^{1/p}]),$$

where $f \in C_e^+(\mathbb{S}^{n-1})$ if ν is even and one wants to take K symmetric, or $f \in C^+(\mathbb{S}^{n-1})$ otherwise. We again set $h_t = (h_K^p + tf)^{1/p}$. It can be noticed that

$$\Psi(0, 0) = \mu(K) = a,$$

and, from Proposition 2.5,

$$\Psi_\varepsilon(0, 0) = \frac{1}{p} S_p^\mu(K) \neq 0.$$

Here, Ψ_ε denotes partial differentiation of Ψ in the variable ε. Thus, when $t_0, \varepsilon_0 > 0$ are sufficiently small, we can apply the implicit function theorem on $R = (-t_0, t_0) \times (-\varepsilon_0, \varepsilon_0)$, to obtain that there exists a function $\xi(t)$ such that $(t, \xi(t))$ is the unique solution of $\Psi(t, \varepsilon) = a$ on R.

From $\Psi(t, \xi(t)) = a$, we get $\Psi_t(0,0) + \Psi_\xi(0,0)\xi'(0) = 0$, i.e.,

$$\xi'(0) = \frac{-p\Psi_t(0,0)}{S_p^\mu(K)}.$$

Recalling that, by hypothesis,

$$\Omega_\nu(h_K) = \sup\{\Omega_\nu(f) : \mu([f]) = a, f \in C^+(\mathbb{S}^{n-1})\},$$

(or $f \in C_e^+(\mathbb{S}^{n-1})$ if ν is even and we wish to take K symmetric), we find that the function $t \mapsto \Phi(t, \xi(t))$ is maximized at 0. Consequently, we have

$$\begin{aligned} 0 &= \frac{d}{dt}\Phi(t, \xi(t))\Big|_{t=0} = \Phi_t(0,0) + \Phi_\xi(0,0)\xi'(0) \\ &= \Phi_t(0,0) + \frac{\nu(\mathbb{S}^{n-1})\Psi_t(0,0)}{S_p^\mu(K)} = \frac{d}{dt}\Omega_\nu(h_t)\Big|_{t=0} + \frac{\nu(\mathbb{S}^{n-1})}{S_p^\mu(K)}\frac{d}{dt}\mu([h_t])\Big|_{t=0} \\ &= -\frac{1}{p}\int_{\mathbb{S}^{n-1}} f(u)\,d\nu(u) + \frac{\nu(\mathbb{S}^{n-1})}{pS_p^\mu(K)}\int_{\mathbb{S}^{n-1}} f(u)\,dS_{K,p}^\mu. \end{aligned}$$

The last step follows from Proposition 2.5. Then, by the arbitrariness of f, the second claim follows from the Riesz representation theorem. ■

There are a few steps that will be common to all proofs. Therefore, we list these as preparatory propositions. The first is fairly standard; we follow Lemma 6.3 in [54].

Proposition 3.3. *Let ν be a finite Borel measure on $\mathbb{S}^{n-1}$ not concentrated on any great hemisphere, let $g(t) = |t|$ or $g(t) = t_+$ and let $p > 0$. Then, there exists a constant $C_\nu(p) > 0$ such that*

$$\int_{\mathbb{S}^{n-1}} g(\langle\theta, u\rangle)^p\,d\nu(u) \geq C_\nu(p)^p\,\nu(\mathbb{S}^{n-1}).$$

Furthermore, for $p \geq 1$, $C_\nu(p) = C_\nu$, where $C_\nu \in (0,1]$ is a constant independent of p.

Proof. Observe that the map, for $\theta \in \mathbb{S}^{n-1}$,

$$\theta \to \int_{\mathbb{S}^{n-1}} \langle\theta, u\rangle_+\,d\nu(u)$$

is strictly positive, since ν is not concentrated on any hemisphere. Since ν is finite, we can find a constant C_ν such that

$$\frac{1}{\nu(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}} |\langle\theta, u\rangle|\,d\nu(u) \geq \frac{1}{\nu(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}} \langle\theta, u\rangle_+\,d\nu(u) \geq C_\nu > 0.$$

Furthermore, one has that $C_\nu \le 1$ since $|\langle \theta, u\rangle| \le 1$. Notice via Jensen's inequality that, when $p \ge 1$,

$$\frac{1}{\nu(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}} |\langle \theta, u\rangle|^p \, d\nu(u) \ge \frac{1}{\nu(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}} \langle \theta, u\rangle_+^p \, d\nu(u) \ge C_\nu^p,$$

and the claim follows. When $p \in (0,1)$, we do not have such nice control over $C_\nu(p)$. We simply use the fact that, since ν is not concentrated on any great hemisphere, the function

$$\theta \mapsto \int_{\mathbb{S}^{n-1}} \langle \theta, u\rangle_+^p \, d\nu(u)$$

is continuous and therefore obtains a strictly positive minimum on $\mathbb{S}^{n-1}$. ∎

Proposition 3.4. *Let $K \in \mathcal{K}_o^n$. Let μ be a Borel measure on $\mathbb{R}^n$ with density ϕ such that the Lebesgue set of ϕ contains ∂K and $\phi > C > 0$ on ∂K. Let $p \in \mathbb{R}$ be so that $S_{K,p}^\mu$ is a finite Borel measure. Then, $S_{K,p}^\mu$ is not concentrated on any great hemisphere.*

Proof. Let

$$c_1 = \begin{cases} \min_{u\in\mathbb{S}^{n-1}} h_K(u), & p < 1,\\ 1, & p = 1,\\ \max_{u\in\mathbb{S}^{n-1}} h_K(u), & p > 1.\end{cases}$$

Notice that $c_1 > 0$ by the assumptions on K. For all $p \in \mathbb{R}$, we have $h_K^{1-p} \ge c_1^{1-p}$ on $\mathbb{S}^{n-1}$. Observe that, for every $\theta \in \mathbb{S}^{n-1}$,

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} \langle \theta, u\rangle_+ \, dS_{K,p}^\mu(u) &= \int_{\partial K} h_K^{1-p}(n_K(u))\langle \theta, n_K(y)\rangle_+ \phi(y)\, d\mathcal{H}^{n-1}(y)\\ &\ge c_1^{1-p} C \int_{\partial K} \langle \theta, n_K(y)\rangle_+ \, d\mathcal{H}^{n-1}(y) = c_1^{1-p} C \int_{\mathbb{S}^{n-1}} \langle \theta, u\rangle_+ \, dS_K(u) > 0,\end{aligned}$$

since S_K is not concentrated on any great hemisphere. ∎

The next two propositions will be used to show non-degeneracy of the limit of the sequence of convex bodies, i.e., that the limiting body has a non-empty interior. The first proposition will be used in the proof of the $p \ge 0$ cases of Theorem 1.4. In those proofs, we are able to prove upper-and-lower bounds on sequences solving the optimization problems independently of each other, and so we may assume we have already shown boundedness when establishing the non-degeneracy.

Proposition 3.5. *Let $\psi \in C^+(\mathbb{R}_+ \setminus \{0\}) \cap L^1_{\mathrm{loc}}(\mathbb{R}_+ \setminus \{0\})$ be such that $\int_0^R \psi(t) t^{n-1}\, dt < \infty$ for every $R > 0$. Then, define the rotationally invariant Borel measure μ on $\mathbb{R}^n$ with density $\psi(|\cdot|)$ and fix $a \in (0, \mu(\mathbb{R}^n))$. Suppose $\{K_\ell\}$ is a bounded sequence of symmetric convex bodies such that $\mu(K_\ell) = a$ for all ℓ and one has K_ℓ converges to a symmetric, compact convex set K in the Hausdorff metric as $\ell \to \infty$. Then, $K \in \mathcal{K}_e^n$.*

Proof. We must show that the limiting set K is non-degenerate. By way of contradiction, suppose it is, i.e., that it has an empty interior. Then, we can find a sequence of directions θ_ℓ such that $\theta_\ell \in \operatorname{argmin}_{v\in\mathbb{S}^{n-1}} h_{K_\ell}(v)$ and, as $\ell \to \infty$, $h_{K_\ell}(\theta_\ell) \to 0$ and $\theta_\ell \to \theta$

for some $\theta \in \mathbb{S}^{n-1}$. Since each K_ℓ is symmetric, it is contained in the symmetric slab supporting K_ℓ with outer-unit normals $\pm\theta_\ell$, i.e., if we set

$$W(v) = \{x \in \mathbb{R}^n : |\langle v, x\rangle| \le 1\},$$

then $K_\ell \subset W(\theta_\ell / h_{K_\ell}(\theta_\ell)) = h_{K_\ell}(\theta_\ell) W(\theta_\ell)$. Without loss of generality, we may rotate K and each K_ℓ using the rotational invariance of μ and assume that $\theta_\ell = e_n$ for all ℓ. Then we have $K_\ell \subset h_{K_\ell}(e_n) W(e_n)$.

But also, since K is bounded and $K_\ell \to K$ by hypothesis, there exists $R = R(K) > 0$ such that $K, K_\ell \subset [-R, R]^n$ for ℓ large enough; also, $h_K(e_n) < R$ for ℓ large enough. Consequently, $K_\ell \subseteq [-R, R]^{n-1} \times [-h_{K_\ell}(e_n), h_{K_\ell}(e_n)]$.

Next, decompose $\mathbb{R}^n$ into $\mathbb{R}^{n-1} \times e_n\mathbb{R}$, and write $x = (\bar{x}, x_n)$, with $\bar{x} \in \mathbb{R}^{n-1}$. Then

$$\begin{aligned} a = \mu(K_\ell) &\le \mu\left([-R, R]^{n-1} \times [-h_{K_\ell}(e_n), h_{K_\ell}(e_n)]\right) \\ &= 2\int_{[-R,R]^{n-1}} \int_0^{h_{K_\ell}(e_n)} \psi(|(\bar{x}, x_n)|)\, dx_n\, d\bar{x} \\ &= 2h_{K_\ell}(e_n) \int_{[-R,R]^{n-1}} \Big(\frac{1}{h_{K_\ell}(e_n)} \int_0^{h_{K_\ell}(e_n)} \psi(|(\bar{x}, x_n)|)\, dx_n\Big)\, d\bar{x}. \end{aligned} \tag{3.5}$$

Abusing notation and keeping $|\cdot|$ for the Euclidean norm on $\mathbb{R}^{n-1}$, one then deduces that, for every $R > 0$,

$$\int_{RB_2^{n-1}} \psi(|\bar{x}|)\, d\bar{x} < \infty.$$

Notice that, by Lebesgue's differentiation theorem,

$$\lim_{\ell\to\infty} \frac{1}{h_{K_\ell}(e_n)} \int_0^{h_{K_\ell}(e_n)} \psi(|(\bar{x}, x_n)|)\, dx_n = \psi(|\bar{x}|).$$

Thus, by dominated convergence, we have

$$\begin{aligned} &\lim_{\ell\to\infty} \int_{[-R,R]^{n-1}} \Big(\frac{1}{h_{K_\ell}(e_n)} \int_0^{h_{K_\ell}(e_n)} \psi(|(\bar{x}, x_n)|)\, dx_n\Big)\, d\bar{x} \\ &= \int_{[-R,R]^{n-1}} \psi(|\bar{x}|)\, d\bar{x} < \int_{\sqrt{n-1}RB_2^{n-1}} \psi(|x|)\, d\bar{x} \\ &= \Big((n-1)\operatorname{Vol}_{n-1}(B_2^{n-1}) \int_0^{\sqrt{n-1}R} \psi(t)\, dt\Big) \cdot \Big(\int_0^{\sqrt{n-1}R} t^{n-2} \frac{\psi(t)dt}{\int_0^{\sqrt{n-1}R} \psi(t)\, dt}\Big), \end{aligned} \tag{3.6}$$

where the last equality follows from polar coordinates. Define the $(d-1)$th moment of ψ on $(0, r)$:

$$I_d(r) := \int_0^r t^{d-1} \frac{\psi(t)\, dt}{\int_0^r \psi(t)\, dt}.$$

We now show that $I_{n-1}(R)$ is finite for every $R < \infty$. We already have from polar coordinates that $\int_{RB_2^n} \psi(|x|)\, dx < \infty$ is totally equivalent to $I_n(R) < \infty$ for every $R < \infty$. We now show the rudimentary fact that if a function ψ has finite $(n-1)$th moment on $(0,R)$, then it has finite $(n-2)$nd moment on $(0,R)$ $(n \geq 2)$. Indeed, we obtain from Jensen's inequality,

$$\begin{aligned} I_{n-1}(R) &= \int_0^R (t^{n-1})^{(n-2)/(n-1)} \frac{\psi(t)dt}{\int_0^R \psi(t)\,dt} \leq \Big(\frac{\int_0^R \psi(t) t^{n-1} dt}{\int_0^R \psi(t)\,dt} \Big)^{(n-2)/(n-1)} \\ &= I_n(R)^{(n-2)/(n-1)} < \infty. \end{aligned}$$

We now obtain our contradiction. By (3.5), we have

$$0 < \frac{a}{2} \Big(\int_{[-R,R]^{n-1}} \Big(\frac{1}{h_{K_\ell}(e_n)} \int_0^{h_{K_\ell}(e_n)} \psi(|(\bar{x}, x_n)|)\,dx_n \Big)\,d\bar{x} \Big)^{-1} \leq h_{K_\ell}(e_n).$$

Sending $\ell \to \infty$, we have, by (3.6),

$$0 < \frac{a}{2} \Big(\Big((n-1)\,\mathrm{Vol}_{n-1}(B_2^{n-1}) \int_0^{\sqrt{n-1}R} \psi(t)\,dt \Big)\, I_{n-1}(\sqrt{n-1}\,R) \Big)^{-1} \leq 0,$$

which is a contradiction. We conclude the proof. ■

The next proposition will be used for non-degeneracy in the proofs of the $p \geq 0$ cases of Theorem 1.5.

Proposition 3.6. *Let μ be a rotationally invariant probability measure with a density continuous on $\mathbb{R}^n \setminus \{0\}$. Fix $a \in [1/2, 1)$. If K is a compact convex set such that $o \in K$ and $\mu(K) = a$, then $K \in \mathcal{K}_o^n$.*

Proof. By way of contradiction, suppose that the origin is on the boundary of K. Then, there exists $v \in \mathbb{S}^{n-1}$ such that $h_K(v) = 0$. Let $H^+ = \{x : \langle x, v\rangle \geq 0\}$ and define H^- similarly, but with $\geq$ replaced by $\leq$. Without loss of generality, $K \subset H^+$. Since μ is rotationally invariant, $\mu(H^+) = 1/2$. But, this means $1/2 \leq a = \mu(K) < \mu(H^+) = 1/2$, a contradiction. Thus, K contains the origin in its interior. ■

We are ready to prove our main results for rotationally invariant measures when $p > 0$.

Proof of the $p > 0$ case of Theorems 1.4 and 1.5. We will work with the functional Ω_ν. In light of Lemmas 3.1 and 3.2, it suffices to show that a solution exists to

(1) (3.4) for the $p > 0$ case of Theorem 1.5,

(2) and to the variant of (3.4) consisting of $C_e^+(\mathbb{S}^{n-1})$ under the assumption ν is even for the $p > 0$ case of Theorem 1.4 and the extra claim of the $p > 0$ case of Theorem 1.5, concerning the existence of $K \in \mathcal{K}_e^n$.

We note the converse direction in each case follows from Proposition 3.4. As for the forward directions, let K_ℓ be a maximizing sequence, that is,

$$\lim_{\ell \to \infty} \Omega_\nu(K_\ell) = \sup\{\Omega_\nu(K) : \mu(K) = a, K \in \mathcal{K}_o^n\}.$$

We claim that this sequence is bounded. Indeed, suppose not. If ν is even and we are aiming to show the existence of a symmetric maximizer, replace $\mathcal{K}_o^n$ with $\mathcal{K}_e^n$ in the maximization problem and let $g(t) = |t|$. Otherwise, let $g(t) = t_+$.

For each K_ℓ, there exists a $\theta_\ell \in \mathbb{S}^{n-1}$ such that $\rho_{K_\ell}(\theta_\ell)$ is maximal. Furthermore, from (2.5), one has $h_{K_\ell}(u) \geq \rho_{K_\ell}(\theta_\ell)\, g(\langle \theta_\ell, u\rangle)$. Notice then that, from the above estimate and Proposition 3.3,

$$\Omega_\nu(K_\ell) \leq -\frac{1}{p}\, \rho_{K_\ell}(\theta_\ell)^p \int_{\mathbb{S}^{n-1}} g(\langle \theta_\ell, u\rangle)^p \, d\nu(u) \leq -\frac{C_\nu(p)^p\, \nu(\mathbb{S}^{n-1})}{p}\, \rho_{K_\ell}(\theta_\ell)^p.$$

Sending $\ell \to \infty$, the above becomes $\lim_{\ell\to\infty} \Omega_\nu(K_\ell) = -\infty$, contradicting K_ℓ being a maximizing sequence. Therefore, the convex sets K_ℓ are uniformly bounded, and so, by Blaschke selection, then, by passing to a subsequence if need be, converge to a compact, convex set K containing the origin (which is symmetric if ν is even). For the $p > 0$ case of Theorem 1.5, the claim follows from Proposition 3.6. Similarly, for the $p > 0$ case of Theorem 1.4, the non-degeneracy follows from Proposition 3.5. ■

We are able to remove the constant in the even weighted L^p Minkowski problems when $p > n$, which generalizes the result by Wang [107] for the Gaussian measure, by using the property $(\mathbf{D})_p$.

Theorem 3.7. *Fix $p > n$. Let μ be an even Borel measure on $\mathbb{R}^n$ with locally integrable density ϕ satisfying property $(\mathbf{D})_p$ such that $\lim_{|x|\to 0} \phi(x) \in (0,\infty]$. Suppose ν is an even, finite Borel measure on the sphere. If ν is not concentrated on any great hemisphere, then there exists an origin symmetric convex body K such that*

$$\nu = S^{\mu}_{K,p}.$$

If ϕ is positive on ∂K, then the converse direction holds as well.

Proof. We will work with the functional ψ_ν. In light of Lemmas 3.1 and 3.2, it suffices to show that a solution to (3.3) exists under the assumption $p > n$. Let

$$\Psi(r,\nu) = \psi_\nu(h_{rB_2^n}) = \mu(rB_2^n) - \frac{r^p}{p}\, \nu(\mathbb{S}^{n-1}).$$

Let ϕ be the density of μ, and write

$$\mu(rB_2^n) = r^n\, \mathrm{Vol}_n(B_2^n)\, \frac{1}{\mathrm{Vol}_n(rB_2^n)} \int_{rB_2^n} \phi(x)\, dx.$$

Consequently,

$$\Psi(r,\nu) = r^n\, \mathrm{Vol}_n(B_2^n) \Big(\frac{1}{\mathrm{Vol}_n(rB_2^n)} \int_{rB_2^n} \phi(x)\, dx - r^{p-n}\, \frac{n}{p}\, \frac{\nu(\mathbb{S}^{n-1})}{\mathrm{Vol}_{n-1}(\mathbb{S}^{n-1})} \Big).$$

Therefore, as $r \to 0^+$,

$$r^{p-n}\, \frac{n}{p}\, \frac{\nu(\mathbb{S}^{n-1})}{\mathrm{Vol}_{n-1}(\mathbb{S}^{n-1})} \to 0,$$

but, from the Lebesgue differentiation theorem,

$$\frac{1}{\mathrm{Vol}_n(rB_2^n)}\int_{rB_2^n}\phi(x)\,dx \to \lim_{|x|\to 0}\phi(x) > 0$$

as $r\to 0$. This shows, $\Psi(r)>0$ for r small enough. On the other hand, write

$$\Psi(r,\nu) = r^p\Big(\frac{\mu(rB_2^n)}{r^p} - \frac{\nu(\mathbb{S}^{n-1})}{p}\Big).$$

Next, notice property $\mathbf{(D)}_p$ implies that, for every $x\in\mathbb{R}^n\setminus\{0\}$,

$$\lim_{r\to\infty} r^{n-p}\phi(rx) = 0.$$

Observe that

$$\lim_{r\to\infty}\frac{\mu(rB_2^n)}{r^p} = \lim_{r\to\infty}\frac{1}{r^p}\int_{rB_2^n}\phi(x)\,dx = \lim_{r\to\infty}\int_{B_2^n} r^{n-p}\phi(rx)\,dx.$$

For r large enough, $r^{n-p}\phi(rx)$ is dominated by, say, 1 for almost all $x\in B_2^n$. Hence, by dominated convergence, we obtain

$$\lim_{r\to\infty}\frac{\mu(rB_2^n)}{r^p} = 0.$$

Thus, $\Psi(r,\nu)<0$ for r large enough. We therefore let $R(\nu)=\sup\{r>0:\Psi(r,\nu)>0\}$; we have just shown that $R(\nu)\in(0,\infty)$. We emphasize here that we view R as a function from the set of Borel measures on the sphere to $(0,\infty)$.

Arguing as in the proofs of the $p>0$ case of Theorems 1.4 and 1.5, fix an arbitrary symmetric convex body K; then, from (2.5), $h_K(u)\geq \rho_K(\theta_K)|\langle\theta_K,u\rangle|$, where $\theta_K\in\mathbb{S}^{n-1}$ is so that $\rho_K(\theta_K)$ is maximal, and K is contained in the ball of radius $\rho_K(\theta_K)$.

We can now directly compute using the above estimate and Proposition 3.3:

$$\begin{aligned}\psi_\nu(h_K) &\leq \mu(K) - \frac{\rho_K(\theta_K)^p}{p}\int_{\mathbb{S}^{n-1}}|\langle\theta_K,u\rangle|^p\,d\nu(u) \leq \mu(K) - \frac{(\rho_K(\theta_K)C_\nu)^p}{p}\,\nu(\mathbb{S}^{n-1})\\ &\leq \mu(\rho_K(\theta_K)B_2^n) - \frac{(\rho_K(\theta_K)C_\nu)^p}{p}\,\nu(\mathbb{S}^{n-1}) = \Psi(\rho_K(\theta_K),C_\nu^p\nu).\end{aligned}$$

Therefore, $\psi_\nu(h_K)>0$ implies $\Psi(\rho_K(\theta_K),C_\nu^p\nu)>0$, which means

$$\rho_K(\theta_K)\leq R(C_\nu^p\nu)\leq \max\{R(C_\nu^p\nu),R(\nu)\} =: R,$$

and so $K\subseteq RB_2^n$. Formally, we can restrict our search for the maximizer to the set

$$\mathcal{F} = \{K\in\mathcal{K}_e^n : K\subset RB_2^n\}.$$

We have that $\sup\{\psi_\nu(h_K):K\in\mathcal{F}\}>0$ by the construction of R. Let $\{K_\ell\}\subset\mathcal{K}_e^n$ be a sequence of convex bodies so that $\lim_{\ell\to\infty}\psi_\nu(h_{K_\ell})=\sup\{\psi_\nu(h_K):K\in\mathcal{F}\}$. Via Blaschke selection, there exist some $K\in\mathcal{F}$ and a sub-sequence $\{K_{\ell_j}\}\subset\{K_\ell\}$ such that

$\lim_{j\to\infty}\psi_\nu(h_{K_{\ell_j}})=\psi_\nu(h_K)$, and so $\lim_{j\to\infty}K_{\ell_j}=K$ with respect to the Hausdorff metric. Finally, we see that, from the definition of $\mathcal{F}$,

$$\mu(K)=\lim_{j\to\infty}\mu(K_{\ell_j})\geq\lim_{j\to\infty}\psi_\nu(h_{K_{\ell_j}})>0,$$

and so K has non-empty interior, and is thus a convex body in the proper sense. The final claim follows from Proposition 3.4. ■

As we saw in the above proof, Theorem 3.7 actually only requires the slightly weaker assumption

$$\lim_{r\to\infty}\int_{B_2^n}\frac{\phi(rx)}{r^{p-n}}\,dx=0.$$

We note that any finite measure on $\mathbb{R}^n$ with continuous density, that is strictly positive at o, satisfies the hypotheses of Theorem 3.7.

4. The case when p is zero

This section is dedicated to proving the $p=0$ case of Theorems 1.4 and 1.5. Our approach follows ideas from [19, 53, 55, 71], especially [19]. We will be working with the entropy functional

$$\mathcal{E}_\nu(f)=-\frac{1}{\nu(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}}\log f(u)\,d\nu(u);\quad \mathcal{E}_\nu(K):=\mathcal{E}_\nu(h_K),$$

which will be taken over $C^+(\mathbb{S}^{n-1})$ or $C_e^+(\mathbb{S}^{n-1})$. We will consider the following optimization problem.

Lemma 4.1. *Let μ be a Borel measure on $\mathbb{R}^n$, and fix a Borel measure ν on $\mathbb{S}^{n-1}$. Let $K\in\mathcal{K}_o^n$. Then, for a fixed $a\in(0,\mu(\mathbb{R}^n))$, K solves*

$$\sup\{\mathcal{E}_\nu(K):\mu(K)=a,\,K\in\mathcal{K}_o^n\} \tag{4.1}$$

if and only if h_K solves

$$\sup\{\mathcal{E}_\nu(f):\mu([f])=a,\,f\in C^+(\mathbb{S}^{n-1})\}.$$

Additionally, if K is also assumed to be symmetric in the first optimization problem, then the set $C^+(\mathbb{S}^{n-1})$ is replaced by $C_e^+(\mathbb{S}^{n-1})$ in the second optimization problem.

Proof. Observe that the function $-\log x$ is monotonically decreasing to obtain, since we have $h_{[f]}(u)\leq f(u)$ point-wise,

$$\begin{aligned}\nu(\mathbb{S}^{n-1})\,\mathcal{E}_\nu([f])&=-\int_{\mathbb{S}^{n-1}}\log h_{[f]}(u)\,d\nu(u)\geq-\int_{\mathbb{S}^{n-1}}\log f(u)\,d\nu(u)\\&=\mathcal{E}_\nu(f)\,\nu(\mathbb{S}^{n-1}).\end{aligned}$$

The equivalence follows since $[h_{[f]}]=[f]$. ■

The next lemma shows that solving the optimization problem solves our weighted log-Minkowski problem.

Lemma 4.2. *Let $K \in \mathcal{K}_o^n$, and let μ be a measure on $\mathbb{R}^n$ with a density whose Lebesgue set contains ∂K. If K solves (4.1), then*

$$\nu = \frac{\nu(\mathbb{S}^{n-1})}{S_0^\mu(K)} S^\mu_{K,0}.$$

Proof. The proof is similar to the proof of the second claim in Lemma 3.2; define

$$\Phi^{(2)}(t,\varepsilon) = \mathcal{E}_\nu(h_K e^{tf+\varepsilon}) \quad \text{and} \quad \Psi^{(2)}(t,\varepsilon) = \mu([h_K e^{tf+\varepsilon}]).$$

It can again be noticed that

$$\Psi^{(2)}(0,0) = \mu(K) = a \quad \text{and} \quad \Psi^{(2)}_\varepsilon(0,0) = S_0^\mu(K) \neq 0.$$

Then, repeat the use of the implicit function theorem and the subsequent computations; note that one will have to use Corollary 2.4 in place of Proposition 2.5. ■

We now prove the main results of this section.

Proof of the $p = 0$ case of Theorems 1.4 and 1.5. In light of Lemmas 4.2 and 4.1, it suffices to show a solution exists to (4.1). Let K_ℓ be a maximizing sequence, that is,

$$\lim_{\ell\to\infty} \mathcal{E}_\nu(K_\ell) = \sup\{\mathcal{E}_\nu(K) : \mu(K) = a, K \in \mathcal{K}_o^n\}$$

for the $p = 0$ case of Theorem 1.5, the not necessarily even case, and

$$\lim_{\ell\to\infty} \mathcal{E}_\nu(K_\ell) = \sup\{\mathcal{E}_\nu(K) : \mu(K) = a, K \in \mathcal{K}_e^n\}$$

for the even case of (the $p = 0$ case of) Theorem 1.5 and (the $p = 0$ case of) Theorem 1.4.

Let $r(a)$ be such that $\mu(r(a)B_2^n) = a$, and set $B_a = r(a)B_2^n$. Then, observe that $\mathcal{E}_\nu(B_a) = -\log r(a)$. Consequently,

$$\lim_{\ell\to\infty} \mathcal{E}_\nu(K_\ell) \geq -\log r(a). \tag{4.2}$$

Following the approach from Theorem 6.3 in [19], since each K_ℓ is non-empty, there exist ellipsoids E_ℓ and vectors c_ℓ via John's theorem [62] such that $E_\ell \subset K_\ell \subset c_\ell + b(n)(E_\ell - c_\ell)$. Here, $b(n) = \sqrt{n}$ in the symmetric case (note $c_\ell = o$ in the symmetric case as well) and $b(n) = n$ in the general case. We then denote by $u_{1,\ell}, \dots, u_{n,\ell} \in \mathbb{S}^{n-1}$ the principal directions of E_ℓ indexed to satisfy $h_{1,\ell} \leq \dots \leq h_{n,\ell}$, where $h_{i,\ell} = h_{E_\ell}(u_{i,\ell})$, for $i = 1, \dots, n$. Defining the cross-polytopes

$$C_\ell = [\pm h_{1,\ell} u_{1,\ell}, \dots, \pm h_{n,\ell} u_{n,\ell}],$$

one obtains $C_\ell \subset E_\ell \subset \sqrt{n} C_\ell$. Consequently, we deduce

$$C_\ell \subset K_\ell \subset c_\ell + b(n)(\sqrt{n}\, C_\ell - c_\ell).$$

We must have that there exists a sequence of numbers A_ℓ such that $\mathrm{Vol}_n(C_\ell) \geq A_\ell > 0$. Indeed, if $\mathrm{Vol}_n(C_\ell) \to 0$, then $\mathrm{Vol}_n(K_\ell) \to 0$. In the case when μ is finite, i.e., when its density ϕ is in $L^1(\mathbb{R}^n)$, one obtains from dominated convergence that $\mu(K_\ell) \to 0$. Similarly, when $\phi \in L^q(\mathbb{R}^n)$ for some $q \in (1, \infty)$, we obtain, from Hölder's inequality,

$$\mu(K_\ell) = \int_{\mathbb{R}^n} \chi_{K_\ell}(x)\phi(x)\,dx \leq \|\phi\|_{L^q(\mathbb{R}^n)} \mathrm{Vol}_n(K_\ell)^{(q-1)/q},$$

which again goes to zero if $\mathrm{Vol}_n(K_\ell)$ does. In either case, this contradicts $\mu(K_\ell) = a$ for all ℓ.

Using the formula for the volume of a cross-polytope, we then obtain that

$$\prod_{i=1}^n h_{i,\ell} = \frac{n!\,\mathrm{Vol}_n(C_\ell)}{2^n} \geq \frac{n!\,A_\ell}{2^n}. \tag{4.3}$$

Notice that, with $\widetilde{C}_\ell = (n!\,A_\ell/2^n)^{-1/n} C_\ell$,

$$\mathcal{E}_\nu(\widetilde{C}_\ell) = -\frac{1}{\nu(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}} \log h_{\widetilde{C}_\ell}(u)\,d\nu(u) = -\frac{1}{n}\log\left(\frac{2^n}{n!\,A_\ell}\right) + \mathcal{E}_\nu(C_\ell). \tag{4.4}$$

By way of contradiction, suppose that the sequence $\{K_\ell\}$ is not bounded. Then, the sequence $\{C_\ell\}$ is not bounded. Therefore, by passing to a subsequence if need be, one has that $\lim_{\ell\to\infty} h_{n,\ell} = \infty$. One then obtains from (4.3) and from Lemma 6.2 in [19] that $\{\mathcal{E}_\nu(\widetilde{C}_\ell)\}$, and therefore $\{\mathcal{E}_\nu(C_\ell)\}$ via (4.4), are not bounded from below. Since $\mathcal{E}_\nu(C_\ell) \geq \mathcal{E}_\nu(K_\ell)$, this yields $\{\mathcal{E}_\nu(K_\ell)\}$ is not bounded from below, contradicting (4.2) for ℓ large enough. Thus, we must have that $\{K_\ell\}$ is bounded. Then, appealing to Propositions 3.6 and 3.5, for the $p = 0$ cases of Theorems 1.4 and 1.5, respectively, we have that the sequences are non-degenerate. Our claim then follows from Blaschke selection. ∎

5. The case of negative *p*

In this section, we consider $p < 0$. In contrast to the previous sections, we need an additional hypothesis on the measure. If μ is radially decreasing and rotationally invariant, then it has a density ϕ of the form $\phi(x) = \psi(|x|)$, where ψ is a continuous and decreasing function on $(0, \infty)$. We will need explicitly that $\lim_{t\to 0^+}\psi(t) \in (0, \infty)$; this forces continuity of ϕ on all of $\mathbb{R}^n$.

We will prove here the cases $p < 0$ of Theorems 1.4 and 1.5. The techniques from our approach can be traced through, e.g., [22, 25, 29, 38]. We start with the following two propositions for non-degeneracy. The first will be used in the proof of the $p < 0$ case of Theorem 1.4. However, we need non-degeneracy when establishing boundedness on the optimizing sequence. Thus, the content strays from that in Proposition 3.5.

Proposition 5.1. *Let $n \in \mathbb{N}$ and let $\psi \in C^+(\mathbb{R}_+ \setminus \{0\}) \cap L^1(\mathbb{R}_+ \setminus \{0\})$ be such that the integral $\int_0^\infty \psi(t)t^{n-1}\,dt < \infty$ and normalized so that the rotationally invariant measure μ on $\mathbb{R}^n$ with density $\psi(|\cdot|)$ is probability. Fix $a \in (0, 1)$ and suppose $\{K_\ell\}$ is a sequence of symmetric convex bodies such that $\mu(K_\ell) = a$ for all ℓ and one has K_ℓ converges to a symmetric closed convex set K in the Hausdorff metric as $\ell \to \infty$. Then, K is non-degenerate (i.e., has non-empty interior).*

Proof. By way of contradiction, suppose K is degenerate. Then, we can find a sequence of directions θ_ℓ such that $\theta_\ell \in \operatorname{argmin}_{v\in\mathbb{S}^{n-1}} h_{K_\ell}(v)$ and, as $\ell\to\infty$, $h_{K_\ell}(\theta_\ell)\to 0$ and $\theta_\ell\to\theta$ for some $\theta\in\mathbb{S}^{n-1}$. Since each K_ℓ is symmetric, it is contained in the symmetric slab supporting K_ℓ with outer-unit normals $\pm\theta_\ell$, i.e., if we set

$$W(v)=\{x\in\mathbb{R}^n : |\langle v,x\rangle|\le 1\},$$

then $K_\ell\subset W(\theta_\ell/h_{K_\ell}(\theta_\ell)) = h_{K_\ell}(\theta_\ell)W(\theta_\ell)$. Without loss of generality, we may rotate K and each K_ℓ using the rotational invariance of μ and assume that $\theta_\ell = e_n$ for all ℓ. Then we have $K_\ell\subset h_{K_\ell}(e_n)W(e_n)$. We again decompose $\mathbb{R}^n$ into $\mathbb{R}^{n-1}\times e_n\mathbb{R}$, and write $x=(\bar{x},x_n)$, with $\bar{x}\in\mathbb{R}^{n-1}$. Then,

$$\begin{aligned}
a=\mu(K_\ell) &< \mu(\mathbb{R}^{n-1}\times[-h_{K_\ell}(e_n),h_{K_\ell}(e_n)])\\
&= 2\int_{\mathbb{R}^{n-1}}\int_0^{h_{K_\ell}(e_n)} \psi(|(\bar{x},x_n)|)\,dx_n\,d\bar{x}\\
&= 2h_{K_\ell}(e_n)\int_{\mathbb{R}^{n-1}}\Big(\frac{1}{h_{K_\ell}(e_n)}\int_0^{h_{K_\ell}(e_n)} \psi(|(\bar{x},x_n)|)\,dx_n\Big)\,d\bar{x}.
\end{aligned}\tag{5.1}$$

Abusing notation and keeping $|\cdot|$ for the Euclidean norm on $\mathbb{R}^{n-1}$, one then deduces that $\int_{\mathbb{R}^{n-1}}\psi(|\bar{x}|)\,d\bar{x}<\infty$. Notice that, by Lebesgue's differentiation theorem,

$$\lim_{\ell\to\infty}\frac{1}{h_{K_\ell}(e_n)}\int_0^{h_{K_\ell}(e_n)}\psi(|(\bar{x},x_n)|)\,dx_n=\psi(|\bar{x}|).$$

Thus, we obtain

$$\lim_{\ell\to\infty}\int_{\mathbb{R}^{n-1}}\Big(\frac{1}{h_{K_\ell}(e_n)}\int_0^{h_{K_\ell}(e_n)}\psi(|(\bar{x},x_n)|)\,dx_n\Big)\,d\bar{x}=\int_{\mathbb{R}^{n-1}}\psi(|\bar{x}|)\,d\bar{x}.\tag{5.2}$$

To justify the limit: first, truncate $\mathbb{R}^{n-1}$ to RB_2^{n-1} for arbitrary R. Then, from the slightly stronger version of Lebesgue's differentiation theorem, which we recall asserts that

$$\lim_{\ell\to\infty}\frac{1}{h_{K_\ell}(e_n)}\int_0^{h_{K_\ell}(e_n)}|\psi(|(\bar{x},x_n)|)-\psi(|(\bar{x},0)|)|\,dx_n=0,$$

one easily finds that (5.2) holds (but with $\mathbb{R}^{n-1}$ replaced by RB_2^{n-1}). Then, send R to ∞ to establish (5.2) via monotone convergence. Next, observe from polar coordinates that

$$\int_{\mathbb{R}^n}\psi(|x|)\,dx=n\operatorname{Vol}_n(B_2^n)\int_0^\infty\psi(t)t^{n-1}\,dt,$$

i.e., given a function ψ on $(0,\infty)$, the function $\psi(|x|)$ is integrable on $\mathbb{R}^n$ if and only if ψ has finite $(n-1)$th moment on $\mathbb{R}_+$. We now show the rudimentary fact that if an integrable function ψ on $\mathbb{R}_+$ has finite $(n-1)$th moment, then it has finite $(n-2)$nd moment ($n\ge 2$). Indeed, we obtain from Jensen's inequality,

$$\begin{aligned}
\int_0^\infty\psi(t)t^{n-2}dt &= \Big(\int_0^\infty\psi(t)\,dt\Big)\cdot\Big(\int_0^\infty (t^{n-1})^{(n-2)/(n-1)}\frac{\psi(t)\,dt}{\int_0^\infty\psi(t)\,dt}\Big)\\
&\le\Big(\int_0^\infty\psi(t)dt\Big)\cdot\Big(\frac{\int_0^\infty\psi(t)t^{n-1}\,dt}{\int_0^\infty\psi(t)\,dt}\Big)^{(n-2)/(n-1)}<\infty.
\end{aligned}$$

Consequently, passing to polar coordinates on $\mathbb{R}^{n-1}$,

$$\int_{\mathbb{R}^{n-1}} \psi(|\bar{x}|)\, d\bar{x} = (n-1)\operatorname{Vol}_{n-1}(B_2^{n-1}) \int_0^\infty \psi(t) t^{n-2}\, dt < \infty. \tag{5.3}$$

With this in hand, we can obtain our contradiction. We have, from (5.1),

$$0 < \frac{a}{2}\left(\int_{\mathbb{R}^{n-1}} \left(\frac{1}{h_{K_\ell}(e_n)} \int_0^{h_{K_\ell}(e_n)} \psi(|(\bar{x}, x_n)|)\, dx_n\right) d\bar{x}\right)^{-1} < h_{K_\ell}(e_n).$$

Taking the limit as $\ell \to \infty$, we obtain, from (5.2),

$$0 < \frac{a}{2}\left(\int_{\mathbb{R}^{n-1}} \psi(|\bar{x}|)\, d\bar{x}\right)^{-1} \le 0,$$

where we used (5.3) to deduce the strictness away from zero. We have reached the desired contradiction, and we conclude the proof. ■

The next proposition will be used to prove the $p < 0$ case of Theorem 1.5.

Proposition 5.2. *Let μ be a rotationally invariant probability measure with a density continuous on $\mathbb{R}^n \setminus \{0\}$. Fix $a \in (1/2, 1)$. If K is a closed convex set so that $o \in K$ and $\mu(K) = a$, then $o \in \operatorname{int}(K)$.*

Proof. By way of contradiction, suppose that the origin is on the boundary of K. Then, there exists $v \in \mathbb{S}^{n-1}$ such that $h_K(v) = 0$. Let $H^+ = \{x : \langle x, v\rangle \ge 0\}$ and define H^- similarly, but with $\ge$ replaced by $\le$. Without loss of generality, $K \subseteq H^+$. Since μ is rotationally invariant, $\mu(H^+) = 1/2$. But this means $1/2 < a = \mu(K) \le \mu(H^+) = 1/2$, a contradiction. Thus, K contains the origin in its interior. ■

Unlike Proposition 3.6, we do not have boundedness in Proposition 5.2. Thus, we had to assume $a > 1/2$ to prevent $K = H^+$. We next prove the $p < 0$ cases of Theorems 1.4 and 1.5 when the data measure ν is zero on every great subsphere. Then, we prove the general case via approximation.

Lemma 5.3. *Let μ be a rotationally invariant, radially decreasing probability measure on $\mathbb{R}^n$ with continuous density. Let $p < 0$. We consider two cases.*

(1) *Fix $a \in (0, 1)$. Let ν be an even, finite Borel measure on $\mathbb{S}^{n-1}$ that vanishes on all great subspheres. Then, there exists a symmetric convex body K with $\mu(K) = a$ solving (the variant of) (3.4) (with $\mathcal{K}_e^n$) for the given μ and ν.*

(2) *Suppose also that, if $d\mu(x) = \psi(|\cdot|)\, dx$, then $\psi \in L^1(\mathbb{R}_+)$. Fix $a \in (1/2, 1)$. Let ν be a finite Borel measure on $\mathbb{S}^{n-1}$ that vanishes on all great subspheres. Then, there exists a convex body K with $\mu(K) = a$ solving (3.4) for the given μ and ν.*

Proof. Let K_ℓ be a sequence tending to the supremum, that is,

$$\begin{cases} \lim_{\ell\to\infty} \Omega_\nu(K_\ell) = \sup\{\Omega_\nu(K) : \mu(K) = a, K \in \mathcal{K}_e^n\} > 0 & \text{(case 1)}, \\ \lim_{\ell\to\infty} \Omega_\nu(K_\ell) = \sup\{\Omega_\nu(K) : \mu(K) = a, K \in \mathcal{K}_o^n\} > 0 & \text{(case 2)}. \end{cases} \tag{5.4}$$

It suffices to show that the sequence $\{K_\ell\}$ has a non-degenerate limit and is contained in a large ball; by Blaschke selection, this would then yield the result that there exists a convex body K such that up to a subsequence, $K_\ell \to K$ and, by construction, K is a convex body with $\mu(K) = a > 0$ (which is symmetric for case 1).

The fact that the limit is non-degenerate follows from Proposition 5.1 for case 1 and Proposition 5.2 for case 2; we now show that K_ℓ is bounded. Again, by way of contradiction, suppose otherwise. From polarity, the sequence K_ℓ° is degenerate. That is, by passing to a subsequence if needed, K_ℓ° converges to a compact convex set K° and there exists $v_0 \in \mathbb{S}^{n-1}$ such that $h_{K^\circ}(v_0) = 0$. Define the set, for every $r \in \mathbb{S}^{n-1}$,

$$\omega_r(v_0) = \{u \in \mathbb{S}^{n-1} : |\langle u, v_0\rangle| > r\}.$$

Then, by Lemma 6.2 in [54], $\rho_{K_\ell^\circ} \to 0$ on $\omega_r(v_0)$. Since we have already shown that h_{K_ℓ} is bounded from below, it follows that $\rho_{K_\ell^\circ}$ is bounded from above. Consequently, $K_\ell^\circ \subset RB_2^n$ for some $R > 0$. Then,

$$\begin{aligned} -p\Omega_\nu(K_\ell) &= \int_{\mathbb{S}^{n-1}} h_{K_\ell}(u)^p \, d\nu(u) \\ &= \int_{\omega_r(v_0)} \rho_{K_\ell^\circ}(u)^{-p} \, d\nu(u) + \int_{\mathbb{S}^{n-1}\setminus\omega_r(v_0)} \rho_{K_\ell^\circ}(u)^{-p} \, d\nu(u) \\ &\le \int_{\omega_r(v_0)} \rho_{K_\ell^\circ}(u)^{-p} \, d\nu(u) + R^{-p}\,\nu(\mathbb{S}^{n-1} \setminus \omega_r(v_0)). \end{aligned}$$

Since ν vanishes on all great subspheres, we obtain

$$\lim_{r\to 0^+} \nu(\mathbb{S}^{n-1} \setminus \omega_r(v_0)) = \nu(\mathbb{S}^{n-1} \cap v_0^\perp) = 0.$$

We deduce that $\lim_{\ell\to\infty} -p\,\Omega_\nu(K_\ell) = 0$, contradicting (5.4). ∎

To establish our next set of results, we will use the theory of partial differential equations (and, more specifically, Monge–Ampère equations) in conjunction with an approximation argument. Suppose that we have $K \in \mathcal{K}_o^n$ and a Borel measure ν on $\mathbb{S}^{n-1}$ such that

$$\nu = S_{K,p}^\mu$$

where μ is a Borel measure on $\mathbb{R}^n$ with density ϕ whose Lebesgue set contains ∂K. If ν has strictly positive density f on $\mathbb{S}^{n-1}$ and K has C^2 smooth boundary, then this can be rewritten as the following Monge–Ampère equation:

$$h_K^{1-p}\phi(\nabla h_K)\det(\nabla^2 h_K + h_K I) = f. \tag{5.5}$$

Our first lemma transforms the Minkowski problem on $\mathbb{S}^{n-1}$ into one on $\mathbb{R}^{n-1}$. We will use radial projections. This technique can be traced through pages 13–17 of [95] and [28,52,101,106]. We use the usual notation that, for $e \in \mathbb{S}^{n-1}$, $e^\perp = \{x \in \mathbb{R}^n : \langle x, e\rangle = 0\}$ is the hyperplane through the origin orthogonal to e. Then, $H_e := e^\perp + e$ is the hyperplane tangential to $\mathbb{S}^{n-1}$ at e. We denote by $\pi_e : e^\perp \to \mathbb{S}^{n-1}$ the radial projection of H_e to $\mathbb{S}^{n-1}$, i.e.,

$$\pi_e(y) = \frac{y + e}{\sqrt{1 + |y|^2}}, \quad \text{which yields} \quad \langle \pi_e(y), e\rangle = \frac{1}{\sqrt{1 + |y|^2}}. \tag{5.6}$$

Proposition 5.4. *Fix $K \in \mathcal{K}_o^n$ and let $h = h_K$ solve (5.5) for a given nonnegative function f on $\mathbb{S}^{n-1}$ and a Borel measure μ on $\mathbb{R}^n$ with continuous density ϕ. Fix $e \in \mathbb{S}^{n-1}$ and define $H_K, F : e^\perp \to \mathbb{R}_+$ as*

$$H_K(y) = h_K(e+y) \quad \text{and} \quad F(y) = (1+|y|^2)^{-(n+p)/2} f\Big(\frac{e+y}{\sqrt{1+|y|^2}}\Big).$$

Then, H_K solves the following Monge–Ampère equation weakly on $e^\perp \approx \mathbb{R}^{n-1}$:

$$H_K(y)^{1-p}\phi(\nabla H_K(y) - (H_K(y) - \langle \nabla H_K(y), e\rangle)\cdot e) \det \mathrm{Hess}(H_K(y)) = G(y).$$

Proof. Observe that we can write, from (5.6),

$$H_K(y) = \sqrt{1+|y|^2}\, h_K(\pi_e(y)). \tag{5.7}$$

From this representation, we obtain via (2.2) that

$$\partial H_K(y) = \partial h_K(y+e)|_{y\in e^\perp} = F(K, y+e)|_{y\in e^\perp} = F(K, \pi_e(y))|_{y\in e^\perp}.$$

Using the 1-homogeneity of the support function, we differentiate (5.7) to obtain

$$\nabla H_K(y) = \nabla h_K(v)|_{v=\pi_e(y)}.$$

Therefore, there exists $t \in \mathbb{R}$ such that

$$\nabla h_K(v) = \nabla H_K(y) - te, \tag{5.8}$$

with $v = \pi_e(y)$. Then, using (2.6),

$$h_K(v) = \langle \nabla h_K(v), v\rangle = \langle \nabla H_K(y), v\rangle - t\langle e, v\rangle.$$

Multiplying by $\sqrt{1+|y|^2}$ and using (5.6) and (5.7), we obtain

$$t = \langle \nabla H_K(y), y\rangle - H_K(y).$$

Consequently, (5.8) becomes

$$\nabla h_K(v) = \nabla H_K(y) + (H_K(y) - \langle \nabla H_K(y), y\rangle)\cdot e,$$

From (2.4), one has

$$\mathcal{H}^{n-1}\Big(\bigcup_{u\in\pi_e(E)} (F(K,u)|_{u\in e^\perp})\Big) = \int_{\pi_e(E)} \langle u, e\rangle\, dS_K(u).$$

We may re-formulate (5.5) as equality in the following:

$$dS_K(u) = \frac{1}{\phi(\nabla h_K(u))} h_K(u)^{p-1} f(u)\, du$$

(note the fact that $S_K(E) < S_K(\mathbb{S}^{n-1}) < \infty$ manifestly yields the integrability of the right-hand side over $\mathbb{S}^{n-1}$). Consequently, using (2.3) and (2.4), we obtain, for an arbitrary Borel $E \subset e^\perp$,

$$\begin{aligned}\int_E \det \mathrm{Hess}(H_K(y))\, d\mathcal{H}^{n-1}(y) &= \int_{\pi_e(E)} \langle u, e\rangle\, dS_K(u)\\ &= \int_{\pi_e(E)} \frac{\langle u, e\rangle}{\phi(\nabla h_K(u))} h_K(u)^{p-1} f(u)\, du\\ &= \int_E \frac{1}{\phi(\nabla H_K(y) + (H_K(y) - \langle \nabla H_K(y), y\rangle)\cdot e)} H_K(y)^{p-1} F(y)\, d\mathcal{H}^{n-1}(y),\end{aligned}$$

where, in the last line, we used that the determinant of the Jacobian of the coordinate transformation $v = \pi_e(y)$ is $(1 + |y|^2)^{-n/2}$. We note that at this point, we have shown weak equality in

$$\det \mathrm{Hess}(H_K(y)) = \frac{1}{\phi(\nabla H_K(y) + (H_K(y) - \langle \nabla H_K(y), y\rangle)\cdot e)} H_K(y)^{p-1} F(y).$$

The claim follows from this; if we define a measure M_1 on $e^\perp$ as the measure with density $\det \mathrm{Hess}(H_K(y))$, and

$$dM_2(y) = \frac{1}{\phi(\nabla H_K(y) + (H_K(y) - \langle \nabla H_K(y), y\rangle)\cdot e)} H_K(y)^{p-1} F(y)\, dy,$$

then, by the definition of weak equality,

$$\int_E g(y)\, dM_1(y) = \int_E g(y)\, dM_2(y)$$

for every continuous function g and compact E. Setting

$$g(y) = \phi(\nabla H_K(y) + (H_K(y) - \langle \nabla H_K(y), y\rangle)\cdot e) H_K(y)^{1-p}$$

yields the claimed equality. ■

Having established Proposition 5.4, the proof of the following lemma is line-by-line the same as that of Theorem 3.1 in [38].

Lemma 5.5. *Let μ be a Borel measure on $\mathbb{R}^n$ with continuous density ϕ. Fix $p \in \mathbb{R}$, $p \neq 0$. Suppose a function f on $\mathbb{S}^{n-1}$ and $K \in \mathcal{K}_o^n$ satisfy $0 < c_1 \le f \le c_2$ on $\mathbb{S}^{n-1}$ and $dS^\mu_{K,p}(u) = f(u)\, du$. Then:*

(1) *∂K is C^1 and strictly convex, and so h_K is C^1 smooth on $\mathbb{R}^n \setminus \{0\}$;*
(2) *if f is continuous, then the restriction of h_K to $\mathbb{S}^{n-1}$ is in $C^{1,\alpha}$ for any $\alpha \in (0, 1)$;*
(3) *if $f \in C^\alpha(\mathbb{S}^{n-1})$ for any $\alpha \in (0, 1)$, then h_K is $C^{2,\alpha}$ on $\mathbb{S}^{n-1}$.*

With the regularity from Lemma 5.5 in hand, we are ready to prove the main results of this section.

Proof of the $p < 0$ case of Theorems 1.4 and 1.5. We first approximate ν with a sequence of Borel measures ν_ℓ such that $d\nu_\ell = f_\ell\, du$, with $f_\ell \in C^+(\mathbb{S}^{n-1})$ (or $f_\ell \in C_e^+(\mathbb{S}^{n-1})$ in the case of even ν). We note that ν_ℓ vanishes on all great subspheres. Then, by Lemma 5.3 and Lemma 3.2, there exists a sequence $\{K_\ell\} \subset \mathcal{K}_o^n(\mathcal{K}_e^n)$ such that

$$\nu_\ell = \lambda_\ell S^\mu_{K_\ell,p}, \quad \text{with} \quad \lambda_\ell = \frac{\nu(\mathbb{S}^{n-1})}{S_p^\mu(K_\ell)}.$$

But actually, we may assume that $0 < f_\ell \in C_e^\alpha(\mathbb{S}^{n-1})$ for some $\alpha \in (0,1)$. Then, the following Monge–Ampère equation holds weakly:

$$h_{K_\ell}^{1-p}\, \psi(|\nabla h_{K_\ell}|) \det(\nabla^2 h_{K_\ell} + h_{K_\ell} I) = \frac{f_\ell}{\lambda_\ell},$$

where we used that μ has a density of the form $\psi(|x|)$, with ψ decreasing and continuous on $\mathbb{R}_+$.

From item (3) of Lemma 5.5, h_K is then $C^{2,\alpha}$ smooth, and consequently each K_ℓ is smooth and uniformly convex. We will again show that the sequence is uniformly bounded; via Blaschke selection, the existence of a $K \in \mathcal{K}_o^n$ (or $\mathcal{K}_e^n$) such that $\mu(K) = a$ and $\nu = \lambda S^\mu_{K,p}$ follows.

The proof of the fact that the sequence is non-degenerate follows from Proposition 5.1 for the $p < 0$ case of Theorem 1.4 and Proposition 5.2 for the $p < 0$ case of Theorem 1.5. All that remains is the upper bound. Let h_{K_ℓ} attain its maximum on $\mathbb{S}^{n-1}$ at θ_ℓ. We again let $g(t) = |t|$ if ν is even and the K_ℓ are symmetric and $g(t) = t_+$ otherwise.

Since ν is not concentrated on any great subsphere, there exist $\varepsilon, \delta > 0$ such that

$$\int_{\{u\in\mathbb{S}^{n-1}:g(\langle u,\theta_\ell\rangle)>\delta\}} f_\ell(u)\, du > \varepsilon > 0 \tag{5.9}$$

for large enough ℓ. Combining (5.9), (5.5), and (2.10) yields

$$\begin{aligned}
0 < \frac{\varepsilon}{\lambda_\ell} &< \int_{\{u\in\mathbb{S}^{n-1}:g(\langle u,\theta_\ell\rangle)>\delta\}} \frac{f_\ell(u)}{\lambda_\ell}\, du \\
&= \int_{\{u\in\mathbb{S}^{n-1}:g(\langle u,\theta_\ell\rangle)>\delta\}} h_{K_\ell}^{1-p}(u)\, \psi(|\nabla h_{K_\ell}(u)|) \det(\nabla^2 h_{K_\ell} + h_{K_\ell} I)\, du \\
&\le h_{K_\ell}(\theta_\ell)^{-p} \int_{\{u\in\mathbb{S}^{n-1}:g(\langle u,\theta_\ell\rangle)>\delta\}} \psi(|\nabla h_{K_\ell}(u)|)\, h_{K_\ell}(u) \det(\nabla^2 h_{K_\ell} + h_{K_\ell} I)\, du \\
&\le h_{K_\ell}(\theta_\ell)^{n-p} \int_{\{u\in\mathbb{S}^{n-1}:g(\langle u,\theta_\ell\rangle)>\delta\}} \psi(|\nabla h_{K_\ell}(u)|)\, du.
\end{aligned}$$

If $g(\langle u, \theta_\ell\rangle) > \delta$, then (2.7) yields

$$|\nabla h_{K_\ell}(u)| > \delta h_{K_\ell}(\theta_\ell). \tag{5.10}$$

However, since μ is radially decreasing, the function ψ is decreasing. Hence, (5.10) provides the bound

$$\psi(|\nabla h_{K_\ell}(u)|) \le \psi(\delta h_{K_\ell}(\theta_\ell)) \tag{5.11}$$

whenever $u \in \mathbb{S}^{n-1}$ is so that $g(\langle u, \theta_\ell \rangle) > \delta$. Therefore, we can continue using (5.11) to obtain

$$\begin{aligned} 0 < \varepsilon &< \lambda_\ell \, h_{K_\ell}(\theta_\ell)^{n-p} \, \psi(\delta h_{K_\ell}(\theta_\ell)) \int_{\{u \in \mathbb{S}^{n-1} : g(\langle u, \theta_\ell \rangle) > \delta\}} du \\ &\leq \lambda_\ell \, h_{K_\ell}(\theta_\ell)^{n-p} \, \psi(\delta h_{K_\ell}(\theta_\ell)) \, \mathrm{Vol}_{n-1}(\mathbb{S}^{n-1}). \end{aligned}$$

But the final line of the above goes to zero as $\ell \to \infty$ (since μ satisfies property $\mathbf{(D)}_p$), a contradiction. The final claim follows from Proposition 3.4. ■

6. The small mass regime

In this section, we analyze smooth solutions to (5.5) when $p > -n-1$ by utilizing degree theory. Our technique follows largely [55], and can be traced through [38, 39, 77] as well. We will impose that our function ϕ is radially decreasing and rotationally invariant. Therefore, for $x \in \mathbb{R}^n \setminus \{0\}$, $\phi(x) = \psi(|x|)$ for some continuous, decreasing function ψ on $(0, \infty)$. In this case, (5.5) becomes, after a change of notation that will be exclusive to this section,

$$h^{1-p} \, \psi(|\nabla h|) \, \det(h_{ij} + h\delta_{ij}) = f. \tag{6.1}$$

We follow the usual notation that $h_i = \partial_i h$ and $h_{ij} = \partial^2_{ij} h$.

Definition 6.1. For a given nonnegative function $f \in C(\mathbb{S}^{n-1})$, nonnegative, decreasing $\psi \in C^1(\mathbb{R}_+ \setminus \{0\})$ and $p \in \mathbb{R}$, we say h solves (6.1) for the triple (f, p, ψ) if

(1) $h \in C^2(\mathbb{S}^{n-1})$,

(2) h is nonnegative, convex and 1-homogeneously extended to $\mathbb{R}^n$, and

(3) (6.1) holds true.

Our main theorems for this section are as follows.

Theorem 6.2. *Fix $n \in \mathbb{N}$ and $0 < p < n$. Suppose $\psi \in C^1(\mathbb{R}_+)$ is strictly decreasing in its support, satisfies properties $\mathbf{(D)}_p$ and $\mathbf{(S)}_p$, is such that $\int_0^\infty \psi(t) t^{n-1} \, dt < \infty$, and is normalized so that the rotationally invariant measure μ on $\mathbb{R}^n$ with density $\psi(|\cdot|)$ is probability. Suppose additionally that μ has L^p isoperimetric function I_p over $\mathcal{K}^n_e$.*

Let ν be an even, finite Borel measure on $\mathbb{S}^{n-1}$, not concentrated on any great hemisphere, such that

$$\nu(\mathbb{S}^{n-1}) < \min\{I_p(a), I_p(1-a)\},$$

where a is a constant depending only on μ sufficiently close to 1. In fact, one can take $a = 1/2$. Then, there exist $K_1, K_2 \in \mathcal{K}^n_e$ such that $\mu(K_1) > a$, $\mu(K_2) < 1 - a$, and

$$S^\mu_{K_1,p} = \nu = S^\mu_{K_2,p}.$$

To explain the necessity of our assumptions, we will first establish Theorem 6.2 in the case when ν is a multiple of the spherical Lebesgue measure (the so-called isotropic curvature flow problem), in Lemma 6.5 and Proposition 6.8 below. We require that the density of μ is radially decreasing and continuous at zero at this moment. We establish that there are at least two solutions to the isotropic curvature problem (each of which is a centered Euclidean ball). However, we need that there are exactly two solutions; this is precisely property $(\mathbf{S})_p$, given in Definition 6.13. Pre-supposing such a fact may seem extreme. However, write $d\mu(x) = \psi(|x|)\,dx$. Then, in Proposition 6.9, we show that if ψ is concave far enough down its support, then property $(\mathbf{S})_p$ holds.

Finally, in Lemma 6.12 and Theorem 6.15, we use degree theory to solve Theorem 6.2 when ν has smooth density f. The presence of property $(\mathbf{D})_p$ is present at all steps. If we only care about a single solution, then we can drop the assumption of symmetry and expand the range of p.

Theorem 6.3. *Fix $p \geq 1$. Suppose $\psi \in C^1(\mathbb{R}_+)$ is strictly decreasing on its support, satisfies properties $(\mathbf{D})_p$ and $(\mathbf{S})_p$, is such that $\int_0^\infty \psi(t)t^{n-1}\,dt < \infty$, and is normalized so that the rotationally invariant measure μ on $\mathbb{R}^n$ with density $\psi(|\cdot|)$ is probability. Suppose additionally that μ has L^p isoperimetric function I_p over $\mathcal{K}_o^n$.*

Let ν be a finite Borel measure on $\mathbb{S}^{n-1}$, not concentrated on any great hemisphere, such that

$$\nu(\mathbb{S}^{n-1}) < I_p\left(\frac{1}{2}\right).$$

Then, there exists $K \in \mathcal{K}_o^n$ such that $\mu(K) \geq 1/2$ and $\nu = S^\mu_{K,p}$. Furthermore, if ν is even, then $K \in \mathcal{K}_e^n$.

In the next lemma, we first consider the case when f is a constant; this is the so-called isotropic curvature problem. Recently, Ivaki and Milman [59], in a major breakthrough, established the following concerning solutions to isotropic curvature problems.

Proposition 6.4 (Solution to isotropic curvature problems, Theorem 1.3 in [59]). *Suppose $K \in \mathcal{K}_e^n$ and that ∂K is smooth, strictly convex, and has strictly positive Gauss curvature κ.*

Let $\varphi\colon (0,\infty) \times (0,\infty) \to (0,\infty)$ be C^1 with $\partial_1\varphi \geq 0, \partial_2\varphi \geq 0$ such that at least one of these inequalities is strict. If K satisfies

$$\varphi(h, |\nabla h|)\kappa = h^{n+2},$$

then K is a centered Euclidean ball.

Recall that ∇, the gradient on $\mathbb{R}^n$, and ∇_s, the gradient on $\mathbb{S}^{n-1}$, are related via

$$\nabla h(u) = \nabla_s h(u) + h(u)u. \tag{6.2}$$

We note that if h is a centered Euclidean ball, then its support function h is constant on $\mathbb{S}^{n-1}$. Therefore, we refer to support functions of Euclidean balls as *constant solutions* to (6.1). If h is a (nonnegative) constant function on $\mathbb{S}^{n-1}$, then the above becomes $\nabla h(u) = h \cdot u$; in particular, we have $|\nabla h(u)| = h$. Also, we recall that $\det(\nabla^2 h_K + hI)$ is precisely the reciprocal of the Gauss curvature of K (as a function of its outer-unit normals).

Lemma 6.5. *Fix $p > -n-1$ and a nonnegative, decreasing $\psi \in C^1(\mathbb{R}_+ \setminus \{0\})$. Let $c > 0$ and suppose that h solves (6.1) for the triple (c, p, ψ). If $[h] \in \mathcal{K}_e^n$, then h has to be constant.*

Proof. Set
$$\varphi(x,t) = cx^{n+p+1}\psi(t)^{-1}.$$
Observe that (6.1) can be re-written as $\varphi(h, |\nabla h|)\kappa = h^{n+2}$, where κ denotes the Gauss curvature of $[h]$. Observe also that
$$\partial_t\varphi(x,t) = cx^{n+p+1}\frac{d}{dt}\psi(t)^{-1} \geq 0 \quad \text{and} \quad \partial_x\varphi(x,t) = c(n+p+1)x^{n+p}\psi(t)^{-1} > 0.$$
Then, according to Proposition 6.4, h is constant. ∎

Recently, Ivaki [58] showed the following variant of Theorem 1.3 in [59].

Proposition 6.6 (Solution to isotropic curvature problems, Theorem 1.2 in [58]). *Suppose $K \in \mathcal{K}^n_o$ has support function h. Suppose also that ∂K is smooth, strictly convex and has strictly positive Gauss curvature κ.*

Let $\varphi\colon (0,\infty)\times(0,\infty) \to (0,\infty)$ be C^1 with $\partial_1\varphi \geq 0, \partial_2\varphi \geq 0$ such that at least one of these inequalities is strict. If K satisfies
$$\varphi(h, |\nabla h|)\kappa = 1,$$
then K is a centered Euclidean ball. Furthermore, if $\partial_1\varphi = 0$, then the same assumption holds with $K \in \mathcal{K}^n$.

We can then obtain the following variant of Lemma 6.5.

Lemma 6.7. *Fix $p \geq 1$ and a nonnegative, decreasing $\psi \in C^1(\mathbb{R}_+ \setminus \{0\})$. Let $c > 0$ and suppose that h solves (6.1) for the triple (c, p, ψ). If $p > 1$ and $[h] \in \mathcal{K}^n_o$, or $p = 1$ and $[h] \in \mathcal{K}^n$, then h has to be constant.*

Proof. Set $\varphi(x,t) = cx^{p-1}\psi(t)^{-1}$, where $p \geq 1$. Then, the equation (6.1) can be re-written as $\varphi(h, |\nabla h|)\kappa = 1$, where κ denotes the Gauss curvature of $[h]$. Also, observe that
$$\partial_t\varphi(x,t) = cx^{p-1}\frac{d}{dt}\psi(t)^{-1} \geq 0,$$
and, if $p > 1$,
$$\partial_x\varphi(x,t) = c(p-1)x^{p-2}\psi(t)^{-1} > 0.$$
If $p = 1$, then we see $\partial_x\varphi(x,t) = 0$. Then, according to Proposition 6.6, h is constant. ∎

6.1. Small *p*

Now we turn our attention to $p \in (-n-1, n)$. We first show that, under property $\mathbf{(D)}_p$, there exist two constant solutions by making c small enough when $p \in (-n-1, n)$. We recall that the support of a nonnegative function is given by
$$\operatorname{supp}(f) = \{x : f(x) > 0\}.$$
We define the set
$$L_p(\psi) = \left\{t \in \operatorname{supp}(\psi) : \psi(t) = -\frac{t\psi'(t)}{(n-p)}\right\}. \tag{6.3}$$

Recall that ψ is said to have property $(\mathbf{D})_p$ if

$$\lim_{t\to\infty} t^{n-p}\,\psi(t) = 0.$$

Proposition 6.8. *Fix $-n-1 < p < n$. Let $\psi \in C^1(\mathbb{R}_+)$ be a nonnegative, strictly decreasing function satisfying property $(\mathbf{D})_p$ such that $\psi \not\equiv 0$.*

For $c < \max_{t\in \operatorname{supp}(\psi)} t^{n-p}\psi(t)$, there exist two constant solutions h_1 and h_2 solving (6.1) with $f = c$. If $\sup(L_p(\psi)) < \sup(\operatorname{supp}(\psi))$, then c can be made small enough so that there are exactly two constant solutions.

Proof. First, notice that if h is a constant, then (6.1) becomes $h^{n-p}\psi(h) = c$. Therefore, if we set $g(t) = t^{n-p}\psi(t)$, we are looking at the level curve $g(t) = c$. We will analyze the critical points of g. By the hypotheses on ψ, g is differentiable on $(0,\infty)$. Observe that

$$g'(t) = t^{n-p-1}((n-p)\psi(t) + t\psi'(t)) \quad \text{for } t \in (0,\infty).$$

Therefore, $t^\star$ is a critical point of g if and only if $t^\star \in L_p(\psi)$.

First, we show that 0 is not an accumulation point of $L_p(\psi)$. By way of contradiction, suppose it is. Then, there exists a sequence $t_k \in L_p(\psi)$ tending to zero. Observe then that

$$\liminf_{t\to 0}\psi(t) \le \lim_{k\to\infty}\psi(t_k) = -\frac{\psi'(t_k)}{(n-p)}\,t_k = 0. \tag{6.4}$$

Indeed, fix a small $\varepsilon > 0$. Then, since ψ is C^1 smooth on $[0,\varepsilon]$, its derivative is bounded on that interval. Thus, there exists $M > 0$ such that $|\psi'(t)| \le M$ for every $t \in [0,\varepsilon]$. Recalling that $t_k \to 0$, we deduce that, for $\delta \in (0,\varepsilon)$, there exists $N \in \mathbb{N}$ such that $t_k < \delta$ for $k > N$. Combining these facts, we have

$$\left|\frac{\psi'(t_k)}{(n-p)}\,t_k\right| \le \frac{M}{(n-p)}\,\delta \quad \text{for } k > N.$$

We deduce that the limit is zero by sending $\delta \to 0$.

However, by the definition of the set $L_p(\psi)$,

$$-\frac{\psi'(t_k)}{(n-p)}\,t_k = \psi(t_k) \quad \text{for all } k.$$

Consequently, from (6.4) we deduce that $\psi(0) = 0$. But ψ is decreasing and nonnegative; this implies $\psi \equiv 0$, a contradiction.

Let $t_1 = \inf(L_p(\psi))$. Since $\lim_{t\to 0^+} g(t) = 0$, we can pick any $c \in (0, g(t_1))$ and define $T_1 \in (0,t_1)$ to be so that $c = g(T_1)$. Notice that g is increasing on $(0,t_1)$. Next, we consider two cases. The first case is $\sup(L_p(\psi)) < \sup(\operatorname{supp}(\psi))$. Then, since

$$\lim_{t\to \sup(\operatorname{supp}(\psi))} g(t) = 0,$$

we find that g is decreasing on $(\sup(L_p(\psi)), \sup(\operatorname{supp}(\psi)))$. Let $b = \inf_{t\in L_p(\psi)} g(t)$. Then, we lower c if necessary (and thus pick a new T_1) so that $c < b$, to obtain $g(T_1) = c < b$ and there exists a unique $T_2 \in (\sup(L_p(\psi)), \sup(\operatorname{supp}(\psi)))$ such that $g(T_2) = c$; additionally, $g(t) \ge b > c$ on $(t_1, \sup(L_p(\psi)))$. This shows that T_1 and T_2 are the only $t \in \operatorname{supp}(\psi)$ such that $g(t) = c$.

Finally, we consider the case where $\sup(L_p(\psi)) = \sup(\mathrm{supp}(\psi))$. Then, there exists a sequence $\{t_k\} \subset L_p(\psi)$ that tends to $\sup(\mathrm{supp}(\psi))$. Consequently, we have

$$0 = (n-p) \lim_{k\to\infty} t_k^{n-p} \psi(t_k) = \lim_{k\to\infty} t_k^{n-p+1}(-\psi'(t_k)).$$

Therefore, there exists N such that, for $k \geq N$,

$$t_k^{n-p+1}(-\psi'(t_k)) < c(n-p);$$

note that we take $N = N(c)$ to be the smallest possible for our given c. This can be rewritten as

$$t_k^{n-p} \psi(t_k) < c.$$

Therefore, $g(t_N) < c$, but $g(t_{N-1}) > c$. By the intermediate value theorem, we pick T_2 to be the point such that $g(T_2) = c$ and $t_{N-1} < T_2 < t_N$. ∎

The condition $\sup(L_p(\psi)) < \sup(\mathrm{supp}(\psi))$ seems somewhat mysterious. We give a few sufficient conditions so that this is the case.

Proposition 6.9. *Fix $-n-1 < p < n$. Let $\psi \in C^1(\mathbb{R}_+)$ be a nonnegative, strictly decreasing function satisfying property* $(\mathbf{D})_p$ *such that there exists $T > 0$ such that ψ is concave on $[T, \sup(\mathrm{supp}(\psi)))$. Then, there exists a small enough c so that there are exactly two constant solutions h_1 and h_2 that solve* (6.1)*.*

Proof. Clearly, if the cardinality of $L_p(\psi)$ is finite, then $\sup(L_p(\psi)) < \sup(\mathrm{supp}(\psi))$. Suppose that the cardinality of $L_p(\psi)$ is infinite. Then, if $t_1, t_2 \in L_p(\psi)$, with $t_1 < t_2$, one has

$$-\frac{\psi'(t_2)}{(n-p)} t_2 = \psi(t_2) < \psi(t_1) = -\frac{\psi'(t_1)}{(n-p)} t_1.$$

Since ψ is decreasing, $-\psi'$ is a positive function. Then, the above re-writes as

$$-\psi'(t_2) < -\psi'(t_1)\frac{t_1}{t_2} < -\psi'(t_1).$$

Thus, the fact that ψ is decreasing implies that ψ' is strictly increasing in $L_p(\psi)$. However, by hypothesis, ψ' decreases in $[T, \sup(\mathrm{supp}(\psi)))$. For both facts to be true, we must have $\sup(L_p(\psi)) \leq T$. The claim then follows from Proposition 6.8. ∎

Proposition 6.9 is a bit weak in the sense that many of the measures that we care about are eventually convex. For example, e^{-t} is convex on $\mathbb{R}_+$, and $e^{-t^2/2}$ is concave on $[0,1)$ and convex on $(1,\infty)$. We now show that, if $\psi = e^{-V}$ where $t \mapsto V(e^t)$ is convex (e.g., V convex), $L_p(\psi)$ is a singleton.

Proposition 6.10. *Fix $-n-1 < p < n$. Let $\psi \in C^1(\mathbb{R}_+)$ be a nonnegative, strictly decreasing function satisfying property* $(\mathbf{D})_p$ *such that $\psi = e^{-V}$ with $t \mapsto V(e^t)$ convex. Then, there exists a small enough c so that there are exactly two constant solutions h_1 and h_2 that solve* (6.1)*.*

Proof. As shown in the proof of Proposition 6.8, if we set $g(t) = t^{n-p}\psi(t)$, then we have $\lim_{t\to 0} g(t) = \lim_{t\to\infty} g(t) = 0$. Thus, $L_p(\psi)$ is non-empty, being the critical points of g. We now claim that $L_p(\psi)$ is a singleton. Inserting that $\psi = e^{-V}$, the condition $t \in L_p(\psi)$ can be written as

$$\frac{(n-p)}{t} = V'(t). \tag{6.5}$$

We take a moment to consider the illuminating case where ψ is log-concave, and so V is convex. Then, V' is increasing. However, the function $(n-p)/t$ is decreasing; these two functions can agree at only a point.

In our more general case, with $V(e^t)$ convex, we have that $(V(e^t))'$ is increasing. Also, from the chain rule, $(V(e^t))' = V'(e^t)e^t$. Therefore, (6.5) becomes, from the chain rule and a change of variable,

$$(n-p) = (V(e^t))',$$

which can only hold at a single point. The claim then follows from Proposition 6.8. ■

We note that Proposition 6.8 yields a procedure to determine the number of solutions given the value of c given ψ and knowing $p \in (-n-1, n)$. For example, the Gaussian case. We illustrate this in the following proposition.

Proposition 6.11. *Suppose $h \in C^2$ solves the equation*

$$e^{-|\nabla h|^2/2} h^{1-p} \det(h_{ij} + h\delta_{ij}) = c(2\pi)^{n/2}.$$

If $p = 1$, or $p > -n-1$ and h is even and positive, or $p > 1$ and h is positive, then h is constant, i.e., $[h]$ must be a centered Euclidean ball. Furthermore, we can characterize the number of solutions. If $p > n$, then for every $c > 0$, there is exactly one solution. If $p = n$, there is exactly one solution for $c \in (0, (2\pi)^{-n/2})$ and no solution for $c > (2\pi)^{-n/2}$. Finally, suppose $p \in (-n-1, n)$. Let

$$C(n,p) = (2\pi)^{-n/2}(n-p)^{(n-p)/2} e^{-(n-p)/2}.$$

1. *If $c < C(n,p)$, there are exactly two solutions.*
2. *If $c = C(n,p)$, there is exactly one solution.*
3. *If $c > C(n,p)$, there are no solutions.*

Proof. According to the proof of Proposition 6.8, the number of solutions is determined by the number of solutions to $g(t) = c$, with $g(t) = (2\pi)^{-n/2} t^{n-p} e^{-t^2/2}$. Observe that, if $p > n$, then $\lim_{t\to 0^+} g(t) = \infty$ and $g(t)$ is strictly decreasing to 0 on $(0,\infty)$ (with g having no critical points). Thus, g is a bijection from $(0,\infty)$ to $(0,\infty)$, and so $g(t) = c$ has a unique solution for every $c > 0$. If $p = n$, then g is a bijection from $\mathbb{R}_+$ to $(0, (2\pi)^{-n/2}]$, and the claim follows.

For $p \in (-n-1, n)$, we examine the number of critical points of g, which is determined by solving

$$\psi(t) = -\frac{t\psi'(t)}{(n-p)}$$

with $\psi(t) = (2\pi)^{-n/2} e^{-t^2/2}$. Then, the formula reduces to $t^2 = (n-p)$. Thus, the critical point of g is at $t = \sqrt{n-p}$. Therefore, we see that g increases on $(0, \sqrt{n-p})$ from 0 to its maximum $g = C(n, p)$, and then decreases on $(\sqrt{n-p}, \infty)$ to 0.

Thus, for every $T \in (0, \sqrt{n-p})$, $g(t) = g(T)$ has exactly two solutions T_1 and T_2 which satisfy $T_1 < \sqrt{n-p} < T_2$. Notice that this also shows that if $c > g(\sqrt{n-p})$ $(= C(n, p))$, then there are no solutions. The fact that these are the only possible solutions is from Lemmas 6.5 and 6.7. ■

This extends the result by Chen, Hu, Liu and Zhao [24] in the case $p = 1, n = 2$.

In the next lemma, we analyze the case when f is α-Hölder. We recall the Hölder norm of a function u over a set U is given by

$$\|u\|_{C^{2,\alpha}(U)} = \sum_{|\beta| \le 2} \sup_{x \in U} |D^\beta u(x)| + \sum_{|\beta| = k} [D^\beta u]_{C^\alpha(U)}$$

and

$$[D^\beta u]_{C^\alpha(U)} := \sup_{x, y \in U, x \neq y} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\alpha} \right\}.$$

Lemma 6.12. *Let $f \in C_e^\alpha(\mathbb{S}^{n-1})$ be such that $1/\tau < f < \tau$ for some positive constant τ. Let $0 < p < n$ and let $\psi \in C^1(\mathbb{R}_+ \setminus \{0\})$ be a function that is strictly decreasing and satisfies property* $(\mathbf{D})_p$. *Suppose h is an even function that solves (6.1) for the triple (f, p, ψ).*

Then, there exists a constant $\tau' > 0$, depending only on τ and ψ, such that

$$\frac{1}{\tau'} < h < \tau' \quad \textit{and} \quad \|h\|_{C^{2,\alpha}} < \tau'.$$

Proof. We first show that h is bounded from above. Let $\{u_k\}$ be a sequence in $\mathbb{S}^{n-1}$ such that $\lim_{k \to \infty} h(u_k) = \sup_{u \in \mathbb{S}^{n-1}} h(u)$. Then from equation (6.1),

$$\begin{aligned} \frac{1}{\tau} < f(u_k) &= h(u_k)^{1-p} \psi(|\nabla h(u_k)|) \det(h_{ij} + h\delta_{ij})|_{u_k} \\ &= h(u_k)^{n-p} \psi(|\nabla h(u_k)|) \frac{\det(h_{ij} + h\delta_{ij})|_{u_k}}{h(u_k)^{n-1}} \\ &\le \frac{\det(h_{ij} + h\delta_{ij})|_{u_k}}{h(u_k)^{n-1}} h(u_k)^{n-p} \psi(h(u_k)), \end{aligned} \tag{6.6}$$

where, in the last step, we used ψ was decreasing and (2.6). If h is not bounded from above, then $h(u_k) \to \infty$, but, from (2.9),

$$\frac{\det(h_{ij} + h\delta_{ij})|_{u_k}}{h(u_k)^{n-1}} \to 1 \quad \text{and} \quad \lim_{k \to \infty} h(u_k)^{n-p} \psi(h(u_k)) = 0;$$

this contradicts (6.6). Thus $h_{\max} < C_0$ for some positive constant C_0 that depends only on τ and ψ.

Next, we will show that h also has a positive lower bound. We first establish that

$$h_{\max} \ge C_1 \tag{6.7}$$

for some constant $C_1 > 0$ depending only on τ and ψ. Indeed, let u_0 be a direction so that $h(u_0) = \sup_{u \in \mathbb{S}^{n-1}} h(u) = h_{\max}$. Then, from (2.10) and (6.1), the following holds:

$$\begin{aligned} h(u_0)^{n-1} &\geq \det(h_{ij} + h\delta_{ij})|_{u_0} = f(u_0)h(u_0)^{p-1} \frac{1}{\psi(|\nabla h(u_0)|)} \\ &\geq f(u_0)h(u_0)^{p-1} \frac{1}{\psi(h(u_0))} \geq \frac{1}{\tau} h(u_0)^{p-1} \cdot \min_{u \in \mathbb{S}^{n-1}} \psi(h(u))^{-1}. \end{aligned}$$

Since $p < n$, the claim follows with

$$C_1 := \big(\min_{u \in \mathbb{S}^{n-1}} (\tau\psi(h(u)))^{-1} \big)^{1/(n-p)}.$$

Next, using (6.1), we have

$$\frac{1}{n} h \det(h_{ij} + h\delta_{ij}) = \frac{1}{n\psi(|\nabla h|)} h^p f \geq \frac{1}{n\psi(h)} h^p f \geq \min_{u \in \mathbb{S}^{n-1}} (n\tau\psi(h(u)))^{-1} h^p.$$

We obtain from (2.17) that the total integral of the left-hand side on $\mathbb{S}^{n-1}$ is the volume of $[h]$. Therefore, by setting $C_2 = \min_{u \in \mathbb{S}^{n-1}} (n\tau\psi(h(u)))^{-1}$, we have

$$\text{Vol}_n([h]) \geq C_2 \int_{\mathbb{S}^{n-1}} h^p(u)\, du. \tag{6.8}$$

Combining (6.7), (6.8), and (2.5), we have

$$\text{Vol}_n([h]) \geq C_2 h_{\max}^p \int_{\mathbb{S}^{n-1}} |\langle u, u_0 \rangle|^p \, du \geq C_1^p C_2 \int_{\mathbb{S}^{n-1}} |\langle u, u_0 \rangle|^p \, du =: C_3. \tag{6.9}$$

Also note the that

$$[h] \subset (h_{\max} B_2^n) \cap \{x \in \mathbb{R}^n : |\langle x, u_0 \rangle| \leq h_{\min}\}.$$

This implies

$$\text{Vol}_n([h]) \leq 2^n h_{\max}^{n-1} h_{\min} < 2^n C_0^{n-1} h_{\min}. \tag{6.10}$$

Combining (6.9) and (6.10), $h_{\min}$ is such that

$$h_{\min} > \frac{1}{2^n C_0^{n-1} C_3}.$$

We then define τ' so that $\tau' > C_0$ and $0 < 1/\tau' < 1/(2^n C_0^{n-1} C_3)$. Finally, by Lemma 5.5, the $C^{2,\alpha}$ a priori estimate is also established. ■

We will recall from Proposition 6.8 that, for c small enough, we can find two constant solutions to the isotropic problem (6.1) with $f = c$. We gave examples in Propositions 6.9 and 6.10 when there are exactly two solutions; including when the measure μ with density $\psi(|\cdot|)$ is in $\mathcal{M}_n^\infty$. We will need the existence of exactly two solutions for the following theorem. Therefore, we make it an explicit property.

Definition 6.13. Fix $0 < p < n$. Let $\psi \in C^1(\mathbb{R}_+)$ be nonnegative and strictly decreasing on its support. We say ψ has property $\mathbf{(S)}_p$ if $\sup(L_p(\psi)) < \sup(\operatorname{supp}(\psi))$, where $L_p(\psi)$ is given by (6.3).

Lemma 6.14. *Fix $0 < p < n$. Suppose $\psi \in C^1(\mathbb{R}_+)$ is strictly decreasing on its support, satisfies properties $\mathbf{(D)}_p$ and $\mathbf{(S)}_p$, is such that $\int_0^\infty \psi(t)t^{n-1}dt < \infty$ and normalized so that the rotationally invariant measure μ on $\mathbb{R}^n$ with density $\psi(|\cdot|)$ is probability. Then, there exist $C, a, h_1, h_2 > 0$ such that, if $c < C$, h_1 and h_2 are the unique solutions to (6.1) with $f = c$, and the centered Euclidean balls $[h_1]$ and $[h_2]$ satisfy*

$$\mu([h_1]) > a \quad \text{and} \quad \mu([h_2]) < 1 - a.$$

Furthermore, a can be made independent of c, h_1 and h_2. Additionally, C can be small enough so that

$$\mu([h_2]) > \frac{1}{2} > \mu([h_1]).$$

Proof. Recall the isotropic equation with respect to (6.1), i.e.,

$$h^{1-p}\,\psi(|\nabla h|)\,\det(h_{ij} + h\delta_{ij}) = c. \tag{6.11}$$

From Proposition 6.8, there exist two constant solutions to (6.11), say h_1 and h_2, for

$$0 < c < C := \max_{t \in \operatorname{supp}(\psi)} t^{n-p}\,\psi(t);$$

by property $\mathbf{(S)}_p$, c can be made small enough so that these are the only two. Recall that c is related to h_1 and h_2 via $h_1^{n-p}\,\psi(h_1) = c = h_2^{n-p}\,\psi(h_2)$. From Lemma 6.5, these solutions, which are the support functions of centered Euclidean balls, are the only solutions among positive, even $C^2(\mathbb{S}^{n-1})$ convex functions. Set $h_3 = \inf L_p(\psi) > 0$. We then have

$$0 < h_1 < h_3 < h_2.$$

Observe that $[h_1] \subset [h_3] \subset [h_2]$. Thus,

$$\mu([h_1]) < \mu([h_3]) < \mu([h_2]) < 1.$$

But also, $\mu([h_2]) = 1 - \mu(\mathbb{R}^n \setminus [h_2])$. Naively, one can set, say,

$$a = \frac{99}{100}\,\min\{\mu([h_1]), \mu(\mathbb{R}^n \setminus [h_2])\}$$

to obtain $\mu([h_1]) > a$ and $\mu([h_2]) < 1 - a$. However, this bound depends on h_1, h_2, and c. Instead, set

$$a = \min\{\mu([h_3]), \mu(\mathbb{R}^n \setminus [h_3])\}.$$

Then,

$$\mu([h_2]) > \mu([h_3]) \geq a$$

and

$$\mu([h_1]) < \mu([h_3]) = 1 - \mu(\mathbb{R}^n \setminus [h_3]) \leq 1 - a.$$

For the final claim, we recall that it was shown $h_1 \in (0, h_3)$ and $h_2 \in (h_4, h_5)$, where $h_4 = \sup L_p(\psi)$ and $h_5 = \sup(\operatorname{supp}(\psi))$. On $(0, h_5)$, the function $h \mapsto \mu([h])$ is increasing from 0 to 1. Thus, there exist h_0 and h_6 such that $\mu([h]) < 1/2$ if we pick $h \in (0, h_0)$ and $\mu([h]) > 1/2$ if we pick $h \in (h_6, h_5)$. Consequently, we first set

$$C = \min\{h_0^{n-p}\psi(h_0), h_3^{n-p}\psi(h_3), h_4^{n-p}\psi(h_4), h_6^{n-p}\psi(h_6)\}$$

and pick any $c \in (0, C)$ to obtain the corresponding, unique, h_1 and h_2, that, by construction, satisfy $\mu([h_1]) < 1/2 < \mu([h_2])$. ■

For the reader unfamiliar with degree theory, we recommend the work by Li [76] for a thorough review. Recall I_p isoperimetric functions, which we introduced in Definition 2.7.

Theorem 6.15. *Fix $0 < p < n$. Suppose $\psi \in C^1(\mathbb{R}_+)$ is strictly decreasing on its support, satisfies properties $(\mathbf{D})_p$ and $(\mathbf{S})_p$, is such that $\int_0^\infty \psi(t)t^{n-1}dt < \infty$ and normalized so that the rotationally invariant measure μ on $\mathbb{R}^n$ with density $\psi(|\cdot|)$ is probability. Suppose additionally that μ has L^p isoperimetric function I_p over $\mathcal{K}_e^n$.*

Let $f \in C_e^\alpha(\mathbb{S}^{n-1})$ be such that $1/\tau < f < \tau$ for some positive constant τ and

$$\|f\|_{L^1} < f_a := \min\{I_p(a), I_p(1-a)\},$$

where a is a constant depending only on μ sufficiently close to 1. Then, there exist $K_1, K_2 \in \mathcal{K}_e^n$ such that $\mu(K_1) > a$ and $\mu(K_2) < 1-a$ and both h_{K_1} and h_{K_2} solve (6.1) for the triple (f, p, ψ). Furthermore, one can pick $a = 1/2$.

Proof. We will work by using a degree theoretic argument. Firstly, we consider the isotropic equation (6.11), which is (6.1) when $f = c$, a positive constant. By Lemma 6.14, if we take $0 < c < C := \max_{t \in \operatorname{supp}(\psi)} t^{n-p}\psi(t)$, then we guarantee the existence of two solutions to the isotropic equation (6.11). We also obtain a constant $a \in (0, 1)$, independent of c, bounding the measures of the Wulff shapes of these solutions. Since a is independent of c, we can lower c if need be so that $\|c\|_{L^1} < f_a$ and the two solutions to (6.11) with this c, denoted say h_1 and h_2, are the only two. Observe that $[h_1]$ and $[h_2]$ are centered Euclidean balls satisfying $\mu([h_1]) > a$ and $\mu([h_2]) < 1-a$. In fact, we saw we can lower c even more, if it is necessary, and change a to the more concrete value $a = 1/2$. If one decides to do so, continue to denote by h_1 and h_2 the two unique solutions to (6.11).

We denote by $\Delta_{\mathbb{S}^{n-1}}$ the spherical Laplacian. We recall that $\Delta_{\mathbb{S}^{n-1}}$ has a discrete spectrum. Therefore, by making c smaller if necessary, we have that the linearized operator of the equation (6.11), i.e.,

$$L_{h_i}(g) = h_i^{\,n-2}\Big(\Delta_{\mathbb{S}^{n-1}} g + \Big((n-p) + h_i \cdot \frac{\psi'(h_i)}{\psi(h_i)}\Big)g\Big) \tag{6.12}$$

is invertible for $i = 1, 2$.

Define a family of operators $F_t : C^{2,\alpha}(\mathbb{S}^{n-1}) \to C^\alpha(\mathbb{S}^{n-1})$ by

$$F_t(h) = \det(\nabla^2 h + hI) - h^{p-1}\frac{1}{\psi(|\nabla h|)} f_t,$$

where $f_t = (1-t)c + tf$ for some $t \in [0, 1]$. Since $f \in C_e^\alpha(\mathbb{S}^{n-1})$, there exists a constant $\tau > 0$ such that $1/\tau < f, c < \tau$ and $\|f\|_{C^\alpha} < \tau$. Then for each $t \in [0, 1]$, f_t has the same bound as f, i.e., $1/\tau < f_t < \tau$, $\|f_t\|_{L^1} < f_a$, and $\|f_t\|_{C^\alpha} < \tau$. We then let $\tau' > 0$ be the constant in Lemma 6.12.

For ease, set

$$\mathcal{S}_e = C^{2,\alpha}(\mathbb{S}^{n-1}) \cap \{h \in C_e^+(\mathbb{S}^{n-1}) : h \text{ is convex}\},$$

and define the sets $O_1, O_2 \subset \mathcal{S}_e$ by

$$O_1 = \Big\{h \in \mathcal{S}_e : \frac{1}{\tau'} < h < \tau', \|h\|_{C^{2,\alpha}} < \tau', \mu([h]) > a\Big\}$$

and

$$O_2 = \Big\{h \in \mathcal{S}_e : \frac{1}{\tau'} < h < \tau', \|h\|_{C^{2,\alpha}} < \tau', \mu([h]) < 1 - a\Big\}.$$

Under the $\|\cdot\|_{C^{2,\alpha}}$ norm, it is easy to see that each O_k is open and bounded. We claim that

$$\partial O_k \cap F_t^{-1}(0) = \emptyset \tag{6.13}$$

for $k = 1, 2$ and $t \in [0, 1]$. Indeed, if $h^{(k)} \in \partial O_k \cap F_t^{-1}(0)$, i.e., if $F_t(h^{(k)}) = 0$, then $h^{(k)}$ solves

$$(h^{(k)})^{1-p}\, \psi(|\nabla h^{(k)}|)\, \det(h_{ij}^{(k)} + h^{(k)}\delta_{ij}) = f_t,$$

$\mu([h^{(1)}]) \geq a$ and $\mu([h^{(2)}]) \leq 1 - a$. Since $h^{(k)} \in \partial O_k$, it must achieve equality in at least one of the conditions defining O_k. However, Lemma 6.12 guarantees that

$$\frac{1}{\tau'} < h^{(k)} < \tau' \quad \text{and} \quad \|h^{(k)}\|_{C^{2,\alpha}} < \tau'$$

are strictly satisfied. Therefore, the only way $h^{(k)}$ can lie on the boundary ∂O_k is if it satisfies the measure condition with equality, i.e., $\mu([h^{(1)}]) = a$ (or $\mu([h^{(2)}]) = 1 - a$). Next, by hypothesis, we have

$$\|f_t\|_{L^1} = S_p^{\mu}([h^{(1)}]) \geq I_p(a) > 0 \text{ and } \|f_t\|_{L^1} = S_p^{\mu}([h^{(2)}]) \geq I_p(1-a) > 0,$$

which contradicts the condition $\|f\|_{L^1} < f_a$. This proves (6.13). Then, by Proposition 2.2 in [76], we obtain

$$\deg(F_0, O_k, 0) = \deg(F_1, O_k, 0) \neq 0,$$

i.e., the degree $\deg(F_t, O_k, 0)$ is well-defined for $t \in [0, 1]$ and does not depend on t. We now show that $\deg(F_1, O_k, 0) \neq 0$ by showing that $\deg(F_0, O_k, 0) \neq 0$.

For this, we will use our linear operator from (6.12). Recalling that it is invertible, we can use Proposition 2.3 in [76] and property $(\mathbf{S})_p$ to obtain

$$\deg(F_0, O_1, 0) = \deg(L_{h_1}, O_1, 0) \neq 0 \quad \text{and} \quad \deg(F_0, O_2, 0) = \deg(L_{h_2}, O_2, 0) \neq 0,$$

where the last inequalities in each follow from Proposition 2.4 in [76]. This implies the existence of $h^{(k)} \in O_k$ such that $F_1(h^{(k)}) = 0$. The claim follows. ∎

Theorem 6.2 then follows from Theorem 6.15 via approximation arguments, like in the proof of the $p < 0$ case of Theorems 1.4 and 1.5.

6.2. Non-symmetric case

In this section, we restrict ourselves to a single solution in order to drop the assumption of symmetry. Recalling that Proposition 6.8 yields two solutions to the isotropic equation (6.11), and **(S)**$_p$ yields exactly two solutions via Lemma 6.14, we will take only the "larger".

Lemma 6.16. *Let $n \in \mathbb{N}$, $p \in \mathbb{R}$ and let $\psi \in C^1(\mathbb{R}_+ \setminus \{0\})$ be a function that is strictly decreasing and satisfies property* **(D)**$_p$*, such that $\int_0^\infty \psi(t)t^{n-1}\,dt < \infty$. Normalize ψ so that the rotationally invariant measure μ on $\mathbb{R}^n$ with density $\psi(|\cdot|)$ is probability.*

Let $f \in C^\alpha(\mathbb{S}^{n-1})$ be such that $1/\tau < f < \tau$ for some positive constant τ. Suppose h is a function with $\mu([h]) \geq 1/2$ that solves (6.1) for the triple (f, p, ψ). Then, there exists a constant $\tau' > 0$, depending only on τ and ψ, such that

$$\frac{1}{\tau'} < h < \tau' \quad \text{and} \quad \|h\|_{C^{2,\alpha}} < \tau'.$$

Proof. We first note that we have $h_{\max} < C_0$ for some positive constant C_0, which depends only on τ and ψ, as the proof is the same as the first part of Lemma 6.12. Next, from Proposition 3.6, we then obtain $[h] \in \mathcal{K}_o^n$. This yields a positive lower bound on h, i.e., $h \geq \min_{u \in \mathbb{S}^{n-1}} h(u) > 0$. However, we claim we can find a uniform bound away from zero depending only on ψ and τ. Suppose not, i.e.,

$$\inf_{K \in \mathcal{Q}} \Big\{ \min_{u \in \mathbb{S}^{n-1}} h_K(u) \Big\} = 0,$$

where

$$\mathcal{Q} := \{K \in \mathcal{K}_o^n : \mu(K) \geq 1/2 \text{ and } h_K \text{ solves (6.1) for the triple } (f, p, \psi)\}.$$

Then, for any $\varepsilon > 0$, there exist $K_\varepsilon \in \mathcal{K}_o^n$ and $u_\varepsilon \in \mathbb{S}^{n-1}$ such that

$$h_{K_\varepsilon}(u_\varepsilon) := \min_{u \in \mathbb{S}^{n-1}} h_{K_\varepsilon}(u) < \varepsilon.$$

Thus,

$$K_\varepsilon \subset (H_\varepsilon + \varepsilon u_\varepsilon) \cap RB_2^n,$$

where R is a finite, positive constant, since $\max_{u \in \mathbb{S}^{n-1}} h_{K_\varepsilon}(u) < C_0$ and

$$H_\varepsilon = \{x \in \mathbb{R}^n : \langle x, u_\varepsilon \rangle \leq 0\}.$$

Next, observe that

$$\frac{1}{2} = \mu(H_\varepsilon) = \mu(H_\varepsilon \cap RB_2^n) + \mu(H_\varepsilon \setminus RB_2^n),$$

where $\mu(H_\varepsilon \setminus RB_2^n) = c > 0$ for some constant c that is strictly positive and independent of ε due to the rotational invariance of μ and the fact that u_ε is a unit vector.

Observe that

$$\begin{aligned}\mu(K_\varepsilon) &\leq \mu(\{x \in \mathbb{R}^n : \langle x, u_\varepsilon \rangle \leq \varepsilon\} \cap RB_2^n) \\ &= \mu(H_\varepsilon \cap RB_2^n) + \mu(\{x \in \mathbb{R}^n : 0 < \langle x, u_\varepsilon \rangle \leq \varepsilon\} \cap RB_2^n).\end{aligned}$$

For sufficiently small ε, the measure of this thin slice is strictly less than c. Thus,

$$\mu(K_\varepsilon) < \mu(H_\varepsilon \cap RB_2^n) + c = \frac{1}{2}\cdot$$

This contradicts the assumption that $\mu(K_\varepsilon) \geq 1/2$. Consequently, there exists a constant that depends only on τ and ψ bounding $\inf_{K\in\mathcal{Q}}\{\min_{u\in\mathbb{S}^{n-1}} h_K(u)\}$ away from zero. ∎

Having established Lemma 6.16, the proof of the following theorem is exactly the same as Theorem 6.15, and is omitted. We simply note that one uses Lemma 6.7 for the uniqueness of the isotropic solutions, and we also have the necessary analogue of Lemma 6.14 when $p \geq n$, which means specifically we can find a unique solution to the isotropic problem (6.11) satisfying $\mu([h]) \geq 1/2$ when c is small enough. In fact, the properties **(D)**$_p$ and **(S)**$_p$ are actually unnecessary. This follows from the function $g(t) = t^{n-p}\psi(t)$ being always decreasing when $p \geq n$ and ψ is decreasing, and the fact that $g(t)$ is a bijection from $(0,\infty)$ to $(0,\infty)$ for $p > n$ and from $[0,\infty)$ to $(0,\psi(0)]$ when $p = n$.

Theorem 6.17. *Fix $p \geq 1$. Suppose $\psi \in C^1(\mathbb{R}_+)$ is strictly decreasing on its support, satisfies properties* **(D)**$_p$ *and* **(S)**$_p$*, is such that $\int_0^\infty \psi(t)t^{n-1}\,dt < \infty$ and normalized so that the rotationally invariant measure μ on $\mathbb{R}^n$ with density $\psi(|\cdot|)$ is probability. Suppose additionally that μ has L^p isoperimetric function I_p over $\mathcal{K}_o^n$.*

Let $f \in C^\alpha(\mathbb{S}^{n-1})$ be such that $1/\tau < f < \tau$ for some positive constant τ (if $p = n$, additionally assume that $\tau < \psi(0)$). Suppose also that

$$\|f\|_{L^1} < f_a := I_p\Big(\frac{1}{2}\Big).$$

Then, there exists $K \in \mathcal{K}_o^n$ such that $\mu(K) \geq 1/2$ and h_K solves (6.1) for the triple (f, p, ψ). Furthermore, if $f \in C_e^\alpha(\mathbb{S}^{n-1})$, then $K \in \mathcal{K}_e^n$.

Theorem 6.3 then follows from Theorem 6.17 by approximation. As we show in the next section, we can use a completely different method to study the range $p \geq n$ and remove the norm assumption on f.

6.3. Large *p*

In this subsection, we consider when $p \geq n$. We are able to drop the symmetry assumption in this case. We no longer need to assume property **(D)**$_p$, as, for $p \geq n$, the fact that ψ is strictly decreasing implies that this property is true. We first establish the uniqueness.

Lemma 6.18. *Fix $p \geq n$ and a nonnegative function $f \in C(\mathbb{S}^{n-1})$. Let $\psi \in C^1(\mathbb{R}_+)$ be a nonnegative, strictly decreasing function. Suppose the equation*

$$h^{1-p}\,\psi(|\nabla h|)\det(h_{ij} + h\delta_{ij}) = f \tag{6.14}$$

is solved by a positive $h \in C^{2,\alpha}(\mathbb{S}^{n-1})$. Then, h is unique.

Proof. Let $h_1, h_2 \in C^{2,\alpha}(\mathbb{S}^{n-1})$ be two positive solutions of the equation (6.14) and set $G(x) = h_1(x)/h_2(x)$. By hypothesis, G is positive and $G \in C^2(\mathbb{S}^{n-1})$. Then, there exists $u_0 \in \mathbb{S}^{n-1}$ such that G reaches its maximum at u_0. Thus,

$$0 = \nabla G(u_0) = \frac{(\nabla h_1(u_0))h_2(u_0) - h_1(u_0)\nabla h_2(u_0)}{h_2(u_0)^2}$$

and

$$0 \geq \nabla^2 G(u_0) = \frac{h_2(u_0)\nabla^2 h_1(u_0) - h_1(u_0)\nabla^2 h_2(u_0)}{h_2^2(u_0)},$$

i.e.,

$$\frac{\nabla^2 h_1(u_0)}{h_1(u_0)} \leq \frac{\nabla^2 h_2(u_0)}{h_2(u_0)}.$$

Hence,

$$\begin{aligned} f(u_0)&h_1(u_0)^{p-1}\psi(|\nabla h_1(u_0)|)^{-1} \\ &= h_1(u_0)^{n-1}\det\left(\frac{\nabla^2 h_1(u_0)}{h_1(u_0)} + I\right) \leq h_1(u_0)^{n-1}\det\left(\frac{\nabla^2 h_2(u_0)}{h_2(u_0)} + I\right) \\ &= \frac{h_1(u_0)^{n-1}}{h_2(u_0)^{n-1}} f(u_0)h_2(u_0)^{p-1}\psi(|\nabla h_2(u_0)|)^{-1}, \end{aligned}$$

i.e.,

$$h_1^{p-n}(u_0)\psi(|\nabla h_1(u_0)|)^{-1} \leq h_2(u_0)^{p-n}\psi(|\nabla h_2(u_0)|)^{-1}.$$

Set

$$\frac{|\nabla h_1(u_0)|}{h_1(u_0)} = \frac{|\nabla h_2(u_0)|}{h_2(u_0)} = c,$$

and then

$$h_1^{p-n}(u_0)\psi(ch_1(u_0))^{-1} \leq h_2^{p-n}(u_0)\psi(ch_2(u_0))^{-1},$$

Observe that the function $t \mapsto t^{p-n}\psi(ct)^{-1}$ is increasing, since $p \geq n$ and ψ is decreasing. Hence

$$h_1(u_0) \leq h_2(u_0),$$

which implies $h_1 \leq h_2$ on $\mathbb{S}^{n-1}$. Similarly, we get $h_1 \geq h_2$ on $\mathbb{S}^{n-1}$. Thus $h_1 = h_2$. ■

Then, we turn to the existence. Our first step is again a $C^{2,\alpha}$-estimate.

Lemma 6.19. *For $p \geq n$, let $f \in C^\alpha(\mathbb{S}^{n-1})$ be such that $1/\tau < f < \tau$ for some positive constant τ when $p > n$, and $f < \psi(0)$ when $p = n$. Let $\psi \in C^1(\mathbb{R}_+ \setminus \{0\})$ be a function that is strictly decreasing. Suppose h solves (6.1) for the triple (f, p, ψ). Then, there exists a constant $\tau' > 0$, depending only on τ and ψ, such that*

$$\frac{1}{\tau'} < h < \tau' \quad \textit{and} \quad \|h\|_{C^{2,\alpha}} < \tau'.$$

Proof. The upper bound is the same as Lemma 6.12. We only need to prove the positive lower bound. Let $\{v_k\}$ be a sequence in $\mathbb{S}^{n-1}$ such that $\lim_{k\to\infty} h(v_k) = \inf_{v\in\mathbb{S}^{n-1}} h(v)$. Then from (6.1) and (2.9),

$$\begin{aligned} \tau > f(v_k) &= h^{1-p}(v_k)\psi(|\nabla h(v_k)|)\det(h_{ij} + h\delta_{ij})|_{v_k} \\ &\geq h^{n-p}(v_k)\psi(|\nabla h(v_k)|) = h^{n-p}(v_k)\psi(h(v_k))\frac{\psi(|\nabla h(v_k)|)}{\psi(h(v_k))}. \end{aligned} \tag{6.15}$$

By way of contradiction, suppose $h(v_k) \to 0$. From (6.2),

$$\frac{\psi(|\nabla h(v_k)|)}{\psi(h(v_k))} \to 1 \quad \text{as } k \to \infty.$$

On the other hand,

$$\begin{aligned} h(v_k)^{n-p}\,\psi(h(v_k)) &\to \infty && \text{as } k \to \infty \text{ when } p > n, \text{ and} \\ h(v_k)^{n-p}\,\psi(h(v_k)) &\to \psi(0) && \text{as } k \to \infty \text{ when } p = n; \end{aligned}$$

this contradicts (6.15). Thus $h > 1/\tau'$ for some positive constant τ' that depends only on τ and ψ.

Moreover, by Lemma 5.5, the $C^{2,\alpha}$ a priori estimate is also established. ■

In the next lemma, we introduce the linear operator of (6.14) and show it is invertible.

Lemma 6.20. *Fix $p \geq n$. Let $\psi \in C^1(\mathbb{R}_+)$ be a strictly decreasing nonnegative function. Fix a positive function $f \in C^\alpha(\mathbb{S}^{n-1})$. Suppose that $h \in C^+(\mathbb{S}^{n-1})$ is a convex function solving (6.14). Then the linearized operator of (6.14) at h, defined in (6.16) below, is invertible.*

Proof. Let $\Gamma \in C^2(\mathbb{S}^{n-1})$ and set $h_\varepsilon = he^{\varepsilon\Gamma}$. Then, the linear operator L_h of (6.14) at h acting on Γ, $L_h(\Gamma) \in C(\mathbb{S}^{n-1})$, is given by

$$\begin{aligned} L_h(\Gamma) &:= \frac{d}{d\varepsilon}\det((h_\varepsilon)_{ij} + h_\varepsilon\delta_{ij})|_{\varepsilon=0} - f\frac{d}{d\varepsilon}[h_\varepsilon^{p-1}\psi(|\nabla h_\varepsilon|)^{-1}]|_{\varepsilon=0} \\ &= \sum_{i,j}(\omega^{ij}(h_{ij}\Gamma + h_j\Gamma_i + h_i\Gamma_j + h\Gamma_{ij} + h\Gamma\delta_{ij})) \\ &\quad - f\Big[(p-1)h^{p-1}\Gamma\psi(|\nabla h|)^{-1} + h^{p-1}(\psi(|\nabla h|)^{-1})'\frac{1}{|\nabla h|} \\ &\quad \cdot\Big(\Big(h^2 + \sum_i h_i^2\Big)\Gamma + \sum_i h(h_i\Gamma_i)\Big)\Big], \end{aligned} \tag{6.16}$$

where ω^{ij} is the cofactor matrix of $(h_{ij} + h\delta_{ij})_{i,j}$. Since $p \geq n$ and ψ is strictly decreasing, we deduce

$$\begin{aligned} L_h(1) &= \sum_{i,j}\omega^{ij}(h_{ij} + h\delta_{ij}) \\ &\quad - f\Big[(p-1)h^{p-1}\psi(|\nabla h|)^{-1} + h^{p-1}(\psi(|\nabla h|)^{-1})'\frac{1}{|\nabla h|}\Big(h^2 + \sum_i h_i^2\Big)\Big] \\ &= f\big[(n-p)h^{p-1}\psi(|\nabla h|)^{-1} - h^{p-1}(\psi(|\nabla h|)^{-1})'|\nabla h|\big] < 0. \end{aligned}$$

Observe that we can write

$$L_h(\Gamma) = \Gamma L_h(1) + \sum_{i,j}\omega^{ij}(h_j\Gamma_i + h_i\Gamma_j + h\Gamma_{ij}) - \sum_i fh^{p-1}hh_i\Gamma_i. \tag{6.17}$$

Suppose Γ is a function so that $L_h(\Gamma) = 0$. Let $u_0 \in \mathbb{S}^{n-1}$ be a point where Γ obtains its maximum, which implies $\nabla_s \Gamma(u_0) = 0$ (so $\Gamma_i(u_0) = 0$). Then, (6.17) becomes

$$\Gamma(u_0)\, L_h(1)[u_0] + \sum_{i,j} \omega^{ij}(u_0) h(u_0) \Gamma_{ij}(u_0) = 0.$$

Combining $L_h(1) < 0$ and the fact that ω^{ij} is positive definite (since h is convex), with Γ_{ij} being semi-negative definite at u_0, we obtain

$$\Gamma(u_0) \leq 0,$$

and therefore $\Gamma \leq 0$. Similarly, by repeating the above procedure at a point where Γ obtains its minimum, we obtain $\Gamma \geq 0$. Thus,

$$\Gamma \equiv 0.$$

This shows that (6.16) is invertible. ∎

Finally, we use convex combinations and the continuity of the Monge–Ampère equation to establish existence.

Theorem 6.21. *Fix $\alpha \in (0, 1)$ and $p \geq n$. Let $\psi \in C^1(\mathbb{R}_+)$ be a strictly decreasing nonnegative function. Let $f \in C^\alpha(\mathbb{S}^{n-1})$ be such that*

(1) *when $p > n$: $1/\tau < f < \tau$ for some positive constant τ.*

(2) *when $p = n$: $f < \psi(0)$.*

Then, there exists a unique, positive, convex solution $h \in C^{2,\alpha}(\mathbb{S}^{n-1})$ to (6.14) for the triple (f, p, ψ).

Proof. We first show that there exists a constant solution to (6.14). From Proposition 6.8, we see that the condition $\lim_{t\to\infty} t^{n-p}\psi(t) = 0$ yields such a solution when $f = c_0$ for c_0 small enough. Since c_0 is strictly positive, we can find a constant $0 < \tau$ such that $1/\tau < c_0 < \tau$. Notice that, when $p = n$, the condition reduces to ψ being decreasing. Since ψ has its maximum at the origin, we are free to take $\tau = \psi(0)$ when $p = n$.

Now, consider a family of functions:

$$h^{1-p}\psi(|\nabla h|)\det(h_{ij} + h\delta_{ij}) = f_t, \tag{6.18}$$

where $f_t = (1-t)c_0 + tf$, $t \in [0, 1]$ and c_0, f such that $1/\tau < c_0, f < \tau$. Thus $1/\tau < f_t < \tau$. For ease, set $\mathcal{S} = C^{2,\alpha}(\mathbb{S}^{n-1}) \cap \{h \in C^+(\mathbb{S}^{n-1}) : h \text{ is convex}\}$. Next, define

$$O = \{t \in [0, 1] : (6.18) \text{ admits a solution } h_t \in \mathcal{S}\}.$$

We have shown that $0 \in O$, and thus O is not empty. Next, we prove that O is an open set. Set $F(t, h) := h^{1-p}\psi(|\nabla h|)\det(h_{ij} + h\delta_{ij}) - f_t$. For any $t^\star \in O$, by the definition of O, there exists a solution $h_{t^\star} \in \mathcal{S}$ that satisfies equation (6.18), i.e., $F(t^\star, h_{t^\star}) = 0$. From Lemma 6.20, we may use the Implicit Function Theorem to obtain $r, r_1 > 0$ such that, if we define the following metric balls, $B_r(t^\star) \subset [0, 1]$ and $B_{r_1}(h_{t^\star}) \subset \mathcal{S}$, then there exists a unique $u \in C^1(B_r(t^\star); B_{r_1}(h_{t^\star}))$ with the following property: $u(t^\star) = h_{t^\star}$ and $F(t, u(t)) = 0$ for any $t \in B_r(t^\star)$. Thus, O is an open set.

Next, we prove that O is a closed set. Let $\{t_k\}_{k=1}^{\infty}$ be a sequence in O and suppose $t_k \to t_0$ as $k \to \infty$. We will show that $t_0 \in O$. From the definition of O, for every t_k, there exists a solution $h^{(k)} := h_{t_k} \in \mathcal{S}$ that satisfies (6.18). Notice that $1/\tau < f_{t_k} < \tau$. Thus, by Lemma 6.19, there exists a constant τ', which only depends on τ and ψ, such that $\|h^{(k)}\|_{C^{2,\alpha}} < \tau'$. Then, by the Arzelà–Ascoli theorem, there exists a subsequence $\{h^{(k_\ell)}\} \subset \{h^{(k)}\}$ and $h^{(0)} \in \mathcal{S}$ satisfying $h^{(k_\ell)} \to h^{(0)}$ as $\ell \to \infty$. Moreover,

$$(h^{(k_\ell)})^{1-p}\,\psi(|\nabla h^{(k_\ell)}|)\,\det(h_{ij}^{(k_\ell)} + h^{(k_\ell)}\delta_{ij}) = f_{t_{k_\ell}}.$$

Taking the limit as $\ell \to \infty$ in the above equation, we deduce

$$(h^{(0)})^{1-p}\,\psi(|\nabla h^{(0)}|)\,\det(h_{ij}^{(0)} + h^{(0)}\delta_{ij}) = f_{t_0},$$

which means $h_{t_0} := h^{(0)}$ solves (6.18) when $t = t_0$. By definition, $t_0 \in O$. Thus, O is a closed set.

Consequently, O is both open and closed in $[0, 1]$; since $[0, 1]$ is connected, we must have $O = [0, 1]$. In particular, when $t = 1$, there exists a solution $h := h_1 \in \mathcal{S}$ to equation (6.14). From Lemma 6.19, this solution is positive. The uniqueness of the solution then follows from Lemma 6.18. ∎

We deduce the following from Theorem 6.21 via a standard approximation argument.

Theorem 6.22. *Fix $n \in \mathbb{N}$ and $p \geq n$. Suppose $\psi \in C^1(\mathbb{R}_+)$ is strictly decreasing on its support. Let μ be the rotationally invariant measure on $\mathbb{R}^n$ with density $\psi(|\cdot|)$. Let ν be a finite Borel measure on $\mathbb{S}^{n-1}$, not concentrated on any great hemisphere.*

If $p > n$, then, there exists $K \in \mathcal{K}_o^n$ such that $S_{K,p}^{\mu} = \nu$. If $p = n$, and we additionally assume that $\nu(\mathbb{S}^{n-1}) \leq \psi(0) n\kappa_n$, then, there exists $K \in \mathcal{K}_o^n$ such that $S_{K,n}^{\mu} = \nu$.

A. Isoperimetric inequalities

This section is dedicated to discussing the isoperimetric function of a measure on $\mathbb{R}^n$, which we recall is the largest function I_μ such that $\mu^+(\partial A) \geq I_\mu(\mu(A))$ for all Borel sets A. Let us discuss a few examples. The first three can be found in the work of Bakry and Ledoux [3], and the last is from Barthe and Maurey [5]:

(1) The classic isoperimetric inequality on $\mathbb{R}^n$ is precisely

$$I_{\mathrm{Vol}_n}(t) = n\,\mathrm{Vol}_n(B_2^n)^{1/n}\, t^{(n-1)/n},$$

and the extremal sets are the Euclidean balls.

(2) Set $\Phi(t) = \gamma_1((-\infty, t))$. Then, $I_{\gamma_n} = \Phi' \circ \Phi^{-1}$. Here, Φ' denotes the derivative of Φ in t, Φ^{-1} denotes the inverse of Φ, and $\circ$ denotes function composition. The extremal sets are half-spaces.

(3) Let μ be a probability measure on $\mathbb{R}^n$ with smooth, strictly positive density e^{-V} such that $\mathrm{Hess}(V(x)) \geq \rho\,\mathrm{Id}_n$. Here, $\rho > 0$, Id_n is the $n \times n$ identity matrix, and the inequality is with respect to symmetric matrices uniformly for every $x \in \mathbb{R}^n$. Then, $I_\mu \geq \sqrt{\rho}\, I_{\gamma_n}$.

(4) Let μ be the uniform measure on the cube $[-1, 1]^n$. Then, $I_\mu \geq \sqrt{2\pi}\, I_{\gamma_n}$.

All of the above examples are log-concave measures. Bobkov [8] completed the picture for this class of measures: if μ is a log-concave probability measure on $\mathbb{R}^n$, then, for every Borel set A, every point $x_0 \in \mathbb{R}^n$ and every $r > 0$, one has

$$2r\mu^+(\partial A) \geq -\mu(A)\log\mu(A) - (1-\mu(A))\log(1-\mu(A)) + \log(\mu(rB_2^n + x_0)).$$

Since x_0 is arbitrary, we can set $x_0 = \int_{\mathbb{R}^n} x\, d\mu(x)$. Then, we can find an $r = r(\mu(A))$, sufficiently large, so that the sum of the last two terms is zero, i.e., the function $r(t)$ for $t \in [0,1]$ is implicitly defined by

$$\mu\Big(rB_2^n + \int_{\mathbb{R}^n} x d\mu(x)\Big) = (1-\mu(A))^{(1-\mu(A))}.$$

Then, one obtains $\mu^+(\partial A) \geq \ell(\mu(A))$, i.e., $I_\mu \geq \ell$. Here, $\ell(t) = -\frac{t}{2r(t)}\log t$ is a function on $[0,1]$.

Bobkov [9] also studied s-concave measures (in the sense of Borell). If μ be s-concave, then

$$I_\mu \geq \frac{c(s)}{m(\mu)}(\min\{\mu(A), 1-\mu(A)\})^{1-s},$$

where $m(\mu)$ is defined so that $\mu(\{x : |x| \leq m(\mu)\}) = 1/2$ and

$$c(s) = \begin{cases} \frac{1}{40}(2^{-s}-1)/(2^{s^2-2s}-1), & s < 0, \\ 1/80, & s = 0, \\ 1/16, & s \in (0,1]. \end{cases}$$

From these results, we deduce that any finite, rotationally invariant s-concave measure μ whose density is C^1, has its maximum at the origin, and satisfies Property $(\mathbf{D})_\mathbf{p}$ (for a fixed $p \in (-n-1, n)$) is valid for Theorem 1.7. In fact, we see that, for log-concavity ($s = 0$), we have two different choices of isoperimetric function. But we also have, from the groundbreaking work by Milman and Rotem [93], two different choices of isoperimetric functions for $s > 0$ under an additional homogeneity assumption: Let μ be s-concave and $1/s$-homogeneous with $s \in (0, 1/n]$. Then,

$$I_\mu(t) = \frac{1}{q}\,\mu(B_2^n)^q\, t^{1-q}, \quad \text{where} \quad \frac{1}{q} = \frac{1}{s} + n,$$

and the extremal sets are precisely dilates of B_2^n.

These examples illustrate Theorem 1.7. We conclude by discussing Cheeger constants. We say that a probability measure μ satisfies an isoperimetric inequality of Cheeger-type if there exists a constant c such that

$$I_\mu(t) \geq c\min\{t, 1-t\}, \tag{A.1}$$

i.e., if c satisfies

$$\mu^+(\partial A) \geq c\min\{\mu(A), 1-\mu(A)\}.$$

The largest constant c for which (A.1) holds is called the isoperimetric, or Cheeger, constant of μ, and this will be denoted as

$$\mathrm{Is}(\mu) = \inf \frac{\mu^+(\partial A)}{\min\{\mu(A), 1-\mu(A)\}},$$

where the infimum runs over all open sets with smooth boundary such that $\mu(A) \in (0,1)$.

As in the case of the isoperimetric function, one usually shows a constant c satisfies (A.1), thus bounding $Is(\mu)$ from below. The isoperimetric constant was introduced in the context of spectral gaps and isoperimetric inequalities on compact manifolds by Cheeger [21].

Let μ be the Borel measure on $\mathbb{R}$ with density $\chi_{[0,\infty)}(t)e^{-t}$. Talagrand [104] showed that $I_\mu(t) = \min\{t, 1-t\}$, which also yields that $\mathrm{Is}(\mu) = 1$. This is a rare instance where the optimal isoperimetric inequality of the form (1.4) coincides with the optimal Cheeger-type isoperimetric inequality (A.1). It is natural to try and extend this result to higher dimensions. Bobkov and Houdré [11] showed that, if μ is a probability measure on $\mathbb{R}$ and μ^n is the product measure on $\mathbb{R}^n$ given by the n-fold product of μ, then

$$\mathrm{Is}(\mu^n) \geq \frac{1}{2\sqrt{6}}\,\mathrm{Is}(\mu).$$

It behooves us to mention the following concerning the isoperimetric constant. A natural question is to bound $\mathrm{Is}(\mu)$ from below independently of μ and the dimension. Firstly, set $\mathrm{Is}(n) = \inf \mathrm{Is}(\mu)$, where the infimum runs over all isotropic, log-concave measures μ on $\mathbb{R}^n$ (recall a measure is isotropic if $\int_{\mathbb{R}^n} x_i \, d\mu(x) = 0$ and $\int_{\mathbb{R}^n} x_i x_j \, d\mu(x) = \delta_{i,j}$ for $i, j = 1, \dots, n$). The KLS conjecture [63] suggests that there exists a constant C independent of the dimension such that $\mathrm{Is}(n) \geq C$. This conjecture is still open. As of writing, the most recent progress is due to Klartag [65]: $\mathrm{Is}(n) \geq C(\log n)^{-1/2}$ for some absolute constant C.

We turn to the L^p isoperimetric function from Definition 2.7. We first recall the following from Theorem 6.1 in [70].

Proposition A.1 (L^p Minkowski's first inequality for F-concave measures). *Fix $p \in \mathbb{R}$, $p \neq 0$. Let μ be a Borel measure on $\mathbb{R}^n$ such that μ is L^p F-concave and F is differentiable, with respect to a class of Borel sets $\mathcal{C}$. Then, for every $K, L \in \mathcal{C}$, one has that*

$$\mu_p(K; L) \geq \mu_p(K; K) + \frac{F(\mu(L)) - F(\mu(K))}{F'(\mu(K))}.$$

Next, notice that from (2.16),

$$\begin{aligned}\frac{r^{1/p}}{p} S_p^\mu(K) &= r^{1/p} \lim_{\varepsilon \to 0} \frac{\mu(K +_p \varepsilon \cdot B_2^n) - \mu(K)}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \frac{\mu(K +_p \varepsilon \cdot rB_2^n) - \mu(K)}{\varepsilon} = \mu_p(K; rB_2^n).\end{aligned}$$

Therefore, for $p > 0$ and $r > 0$, we obtain from Proposition A.1 that

$$S_p^\mu(K) \geq r^{-1/p}\Big[\mu(K;K) + p\frac{F(\mu(rB_2^n)) - F(\mu(K))}{F'(\mu(K))}\Big], \tag{A.2}$$

where we used that $p\mu_p(K;K) = \mu(K;K)$. Since $\mu(K;K) \geq 0$, we see that (A.2) can always be made positive. Indeed, define $r = r(\mu(K))$ as follows: pick r so that

$$\mathrm{sgn}(F'(\mu(K)))\, F(\mu(rB_2^n)) > \mathrm{sgn}(F'(\mu(K)))\, F(\mu(K)).$$

Therefore, for $p > 0$, (A.2) reduces the question of establishing an L^p isoperimetric inequality to establishing a measure is L^p F-concave. Following Hosle, Kolesnikov, and Livshyts [49], let us consider the case where $F(x) = x^{q/n}$ for some $q > 0$, i.e., μ satisfies the following (p, q)-Brunn–Minkowski-type inequality:

$$\mu((1-\lambda) \cdot K +_p \lambda \cdot L) \geq \big((1-\lambda)\,\mu(K)^{q/n} + \lambda\mu(L)^{q/n}\big)^{n/q}. \tag{A.3}$$

Then, (A.2) becomes

$$S_p^{\mu}(K) \geq r^{-1/p} \left(\mu(K;K) + \frac{np}{q}\left(\mu(rB_2^n)^{q/n} - \mu(K)^{q/n}\right)\mu(K)^{(n-q)/q}\right). \tag{A.4}$$

We have essentially covered the known examples of (A.3) when $p \geq 1$ in Section 1.4. As for $p < 1$, this is still an active area of research (even in the case of the Lebesgue measure). For example, in [18] it was shown that the case $(0,0)$ for volume and symmetric convex bodies implies the case (p, p) for all $p > 0$; the case $(0,0)$ was then verified for $n = 2$ in the same work, when both bodies are balls [31], and also when K is a ball and L is close to K [30]. Saroglou [98] established the $(0,0)$ case for volume when all bodies are symmetric under orthogonal coordinate hyperplanes, and showed that the case $(0,0)$ for symmetric convex bodies and volume implies the same for all even log-concave measures. As for other positive results, the (p, p)-inequality for volume has been shown to hold for p close enough to 1 (see, e.g., Kolesnikov and Milman, [69], Putterman [96], and Chen, Huang, Li and Liu [25]). For measures besides volume, Livshyts [81] recently established that every even log-concave measure satisfies a $(1, n^{-3-o(1)})$ inequality over symmetric convex bodies. The reader is referred to the work by Hosle, Kolesnikov and Livshyts [49] for a thorough overview of other known results and the establishment of more partial cases, including results for measures besides Lebesgue.

For $\mu \in \mathcal{M}_n^{\infty}$, when $\mathcal{C} = \mathcal{K}_e^n$, (A.4) is valid when $q = 1$ and $p \geq 1$. The quantity $\mu(K;K)$ seems somewhat mysterious. We conclude with a conjecture that would supply a more quantifiable isoperimetric inequality for measures in $\mathcal{M}_n^{\infty}$ using the $(1/n)$-concavity. We follow a schema by Bobkov [9]. Firstly, one can verify, if μ is a probability measure on $\mathbb{R}^n$, K is a convex set and $r > 0$, that

$$\mu^+(\partial K) = \lim_{\varepsilon \to 0^+} \frac{\mu\big((1-\varepsilon)K + \varepsilon r B_2^n\big) + \mu\big((1-\varepsilon)(\mathbb{R}^n \setminus \operatorname{int}(K)) + \varepsilon r B_2^n\big) - 1}{2r\varepsilon}. \tag{A.5}$$

Indeed, this follows from the fact that $\mu(K; rB_2^n) = r\mu(K; B_2^n)$ and (2.15), which actually holds in this instance; $\partial\,(\mathbb{R}^n \setminus \operatorname{int}(K))$ is the same as ∂K, but with opposite outer-unit normal.

Suppose also μ is an F-concave probability measure, in the sense of (1.6), over a collection of sets $\mathcal{C}$, such that F is C^1 smooth, $\{rB_2^n\}_{r>0} \subset \mathcal{C}$, and, if $K \in \mathcal{C}$, then so too is $(\mathbb{R}^n \setminus \operatorname{int}(K))$. Then, setting $I = (F^{-1})' \circ F$, from (A.5) one has that, for every $r > 0$ and $K \in \mathcal{C}$,

$$\begin{aligned} 2r\mu^+(\partial K) \geq{}& I(\mu(K))(F(\mu(rB_2^n)) - F(\mu(K))) \\ &+ I(1-\mu(K))(F(\mu(rB_2^n)) - F(1-\mu(K))). \end{aligned} \tag{A.6}$$

The result on s-concave probability measures by Bobkov followed by setting $F(t) = t^s$ in (A.6), and some further analysis.

If F satisfies $F(t) = -F(1-t)$, i.e., F is an odd function about $t = 1/2$, then (A.6) reduces to

$$\mu^+(\partial A) \geq \frac{I(\mu(A))(F(\mu(rB_2^n)))}{r}. \tag{A.7}$$

The Ehrhard inequality is precisely the statement that γ_n is Φ^{-1}-concave. Furthermore, Φ^{-1} is odd about $1/2$, and so (A.7) holds with $\mu = \gamma_n$ and $F = \Phi^{-1}$. Sending $r \to \infty$, and noting that $\lim_{r\to\infty} \Phi^{-1}(\mu(rB_2^n))/r = 1$, recovers $I_{\gamma_n} = \Phi' \circ \Phi^{-1}$.

Conjecture. *Let μ be a probability measure over $\mathbb{R}^n$ that is $(1/n)$-concave over $\mathcal{K}_e^n$. Then, for every $K, L \in \mathcal{K}_e^n$ and $t \in (0,1)$,*

$$\mu((1-t)(\mathbb{R}^n \setminus \mathrm{int}(K)) + tL) \geq \big((1-t)\mu((\mathbb{R}^n \setminus \mathrm{int}(K)))^{1/n} + t\mu(L)^{1/n}\big)^n.$$

Suppose that the conjecture is true. Then, we could consider the case where $\mathcal{C} \supset \mathcal{K}_e^n$ and $F(x) = x^{1/n}$ in (A.6). Then $(F^{-1})'(x) = nx^{n-1}$ and $I(x) = nx^{(n-1)/n}$. Consequently, (A.6) becomes, for every $r > 0$ and $K \in \mathcal{K}_e^n$,

$$\mu^+(\partial K) \geq \frac{n}{2r}\big[\mu(rB_2^n)^{1/n}\big(\mu(K)^{(n-1)/n} + (1-\mu(K))^{(n-1)/n}\big) - 1\big]. \tag{A.8}$$

Suppose μ has continuous density that contains the origin in the interior of its connected support. We then claim that we can define a function $r = r(\mu(K))$ so that the right-hand side of (A.8) is strictly positive. Indeed, we would need

$$\mu(rB_2^n) > (\mu(K)^{(n-1)/n} + (1-\mu(K))^{(n-1)/n})^{-n} = z(\mu(K)),$$

where

$$z(t) := (t^{(n-1)/n} + (1-t)^{(n-1)/n})^{-n}.$$

Observe that $z(t)$ maps $[0,1]$ onto $[1/2, 1]$, with maximum $z(0) = z(1) = 1$ and minimum $z(1/2) = 1/2$. The function $r \to \mu(rB_2^n)$ is strictly increasing on its support. Thus, it has an inverse function, say $\mathcal{R}_\mu$. Therefore, the function we need is, for $\varepsilon > 0$,

$$r_\mu(t,\varepsilon) = \mathcal{R}_\mu(z(t)) + \varepsilon, \tag{A.9}$$

as this yields

$$\mu(r_\mu(\mu(K),\varepsilon)B_2^n) > \mu(\mathcal{R}_\mu(z(\mu(K)))B_2^n) = z(\mu(K)).$$

We make this formal: let $\mu \in \mathcal{M}_n^\infty$. Then, under the condition that the above conjecture is true, one has, for every $K \in \mathcal{K}_e^n$ and $\varepsilon > 0$,

$$\mu^+(\partial K) \geq \mathcal{I}_\mu(\mu(K),\varepsilon),$$

where

$$\mathcal{I}_\mu(t,\varepsilon) = \frac{n}{2r_\mu(t,\varepsilon)}\big(\mu(r_\mu(t,\varepsilon)B_2^n)^{1/n}(t^{(n-1)/n} + (1-t)^{(n-1)/n}) - 1\big),$$

and $r_\mu(t,\varepsilon)$ is the function defined in (A.9).

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