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Grand orbit relations in wandering domains

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Abstract. One of the fundamental distinctions in McMullen and Sullivan's description of the Teichmüller space of a complex dynamical system is between discrete and indiscrete grand orbit relations. We investigate these on the Fatou set of transcendental entire maps and provide criteria to distinguish between the two types. Furthermore, we show that discrete and indiscrete grand orbit relations may coexist non-trivially in a wandering domain, a phenomenon which does not occur for any other type of Fatou component. One of the tools used is a novel quasiconformal surgery construction of independent interest.

1. Introduction

Given a holomorphic endomorphism $f: \mathcal{S} \to \mathcal{S}$ of a Riemann surface $\mathcal{S}$, we consider the dynamical system generated by the iterates of f, denoted $f^n := f \circ \cdots \circ f$, n times. These classes of dynamical systems appear naturally as complexifications of real analytic iterative processes, but they are also of independent interest because of their deep connections to other areas of mathematics like analysis, topology, or number theory. We consider two different settings: if $\mathcal{S} = \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, then f is a rational map; if $\mathcal{S} = \mathbb{C}$ and f does not extend to $\widehat{\mathbb{C}}$ (i.e., infinity is an essential singularity), then f is a transcendental entire map.

There is a dynamically natural partition of $\mathcal{S}$ into the *Fatou set*, $F(f)$, defined as the set of points $z \in \mathcal{S}$ where the iterates $\{f^n\}_{n\in\mathbb{N}}$ form a normal family in some neighborhood of z, and its complement, the *Julia set* $J(f) := \mathcal{S} \setminus F(f)$, which is the locus of chaotic dynamics. A connected component of $F(f)$ is called a *Fatou component* and it is either periodic, preperiodic, or wandering, the latter being possible only when f is transcendental. For background on the dynamics of rational and transcendental maps, see, e.g., [5, 24].

Given a point $z \in \mathcal{S}$, its *orbit* is the set of its forward iterates $(f^n(z))$ for $n \in \mathbb{N}$, while its *grand orbit* is defined as

$$\mathrm{GO}(z, f) = \mathrm{GO}(z) := \{w \in \mathcal{S} \mid f^n(z) = f^m(w) \text{ for some } n, m \in \mathbb{N}\}.$$

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The grand orbit of a set $U \subset \mathcal{S}$, denoted by $\mathrm{GO}(U)$, is the union of all grand orbits of the points in U. The equivalence relation on $V = \mathrm{GO}(U)$ whose equivalence classes consist of grand orbits is known as the *grand orbit relation* (or *GO relation*). We say it is *discrete* if $\mathrm{GO}(z)$ is a discrete set for all $z \in V$; otherwise, we call it *indiscrete*.

It is not hard to show that, in the case of (pre)periodic Fatou components, after removing the grand orbits of periodic points and singular values, the nature of the grand orbit relation depends only on the type of component considered: it is discrete in attracting basins, parabolic basins, and Baker domains, and it is indiscrete in superattracting basins and rotation domains [14, 23].

However, in the case of wandering domains, there are examples of both discrete and indiscrete grand orbit relations [16]. Our goal in this paper is to further explore orbit relations in wandering domains. More precisely, we will provide necessary and sufficient criteria for discreteness of the grand orbit equivalence relation and, rather strikingly, prove the existence of wandering domains in which discrete and indiscrete grand orbit relations coexist non-trivially.

For our examples, we first construct an abstract model of the desired internal dynamics as an infinite sequence of Blaschke products. Using quasiconformal surgery, we "paste" this model into an actual wandering domain of an entire function. Since our quasiconformal surgery technique employs some novel techniques and is of independent interest, we state it as a separate result.

Motivation and statement of results

It is frequently of interest to understand the conjugacy classes of dynamical systems, where the type of conjugacies (conformal, smooth, quasiconformal, topological, measurable, etc.) may depend on the particular setting. In complex dynamics, quasiconformal conjugacies have turned out to be particularly important. We are interested in the deformation space (also known as *moduli space*) of a given holomorphic map f, rational or transcendental, which consists of the set of maps that are quasiconformally conjugate to f, modulo conformal conjugacy. As in the classical case of Riemann surfaces, moduli spaces often have orbifold singularities, and it is frequently more convenient to work with *Teichmüller spaces*, constructed as marked versions of moduli spaces.

In [23], McMullen and Sullivan developed a very general framework of moduli and Teichmüller spaces for complex dynamical systems. In addition, they showed that the Teichmüller space of a rational map is a finite-dimensional complex manifold, naturally factoring as the product of Teichmüller spaces associated to periodic Fatou components and one factor corresponding to invariant line fields on the Julia set.

More precisely, given a rational or entire map f, one considers the set of (*dynamically*) *marked points* of f, defined as

$$\widehat{J}(f) := \overline{\mathrm{GO}(S(f) \cup \mathrm{Per}(f))},$$

where $S(f)$ denotes the set of singular values of f (critical and asymptotic values, and accumulations thereof) and $\mathrm{Per}(f)$ the set of periodic points. This set is a completely invariant closed set containing the Julia set $J(f)$, since repelling periodic points are dense in $J(f)$. Its complement $\widehat{F}(f) := \mathbb{C} \setminus \widehat{J}(f)$ is thus an open completely invariant subset of the Fatou set, and the restriction $f|_{\widehat{F}(f)} \colon \widehat{F}(f) \to \widehat{F}(f)$ is a covering map.

Each grand orbit V of a connected component of $\widehat{F}(f)$ contributes a factor $\mathcal{T}(V, f)$ to the Teichmüller space $\mathcal{T}(f)$ of f in the following way: if the grand orbit relation on V is discrete, then its grand orbit space V/f is a (connected) Riemann surface, and $\mathcal{T}(V, f)$ is the Teichmüller space of V/f. If the grand orbit relation is indiscrete, then either all components of V are punctured disks or they are all annuli of finite modulus, and the closures of grand orbit equivalence classes consist of disjoint unions of analytic Jordan curves. If V_0 is any component of V, these curves foliate V_0, and in this case $\mathcal{T}(V, f)$ is the Teichmüller space of the foliated disk or annulus V_0. (This construction does not depend on the choice of the component V_0.)

Lastly, there is a factor $\mathcal{T}(\widehat{J}(f), f)$ supported on $\widehat{J}(f)$. In the case of rational maps, $\mathcal{T}(\widehat{J}(f), f)$ corresponds to the space of measurable invariant line fields on the Julia set, and it is conjectured that it is always trivial except in the case of flexible Lattès maps.

In order to use this description of $\mathcal{T}(f)$ for a transcendental entire map f, it is essential to understand the nature of the grand orbit relation in the components of $\widehat{F}(f)$. For the components contained in basins of attraction, parabolic basins, and rotation domains, the situation is exactly as it is for rational maps, see [16]. (The only difference is that there may be infinitely many such grand orbits, and each one may contain infinitely many grand orbits of singular values.)

The situation becomes more interesting if V is a subset of the grand orbit of a wandering domain or of a cycle of Baker domains. In the case of a Baker domain, the grand orbit relation is discrete, but the type of the Riemann surface V/f depends on the type of Baker domain (hyperbolic, simply or doubly parabolic, see [14, 20]). As pointed out above, in the case of a wandering domain, examples in [16] show that the grand orbit relation can be either discrete or indiscrete. Given the variety of possible dynamics in wandering domains, two natural questions arise.

Question 1. *Can points with discrete and indiscrete grand orbits coexist in a single component of $\widehat{F}(f)$?*

Question 2. *Can components of $\widehat{F}(f)$ with discrete and indiscrete grand orbit relations coexist in a single wandering domain?*

We will see that the answer to the first question is "no", and the answer to the second one is "yes". Note that from our definition, a set U with an indiscrete GO relation could still contain points with discrete grand orbits. The negative answer to Question 1 shows that this does not happen in the case where U is a component of $\widehat{F}(f)$. (It does happen for components of the Fatou set $F(f)$, which is the main reason to introduce the set $\widehat{F}(f)$. E.g., in the immediate basin of an attracting fixed point, almost all grand orbits are discrete, but the grand orbit of the fixed point is indiscrete.)

The following theorem, whose proof can be found in Section 6, answers the first question.

Theorem A (GO relations in $\widehat{F}(f)$). *Let f be an entire map and let V be the grand orbit of a component of $\widehat{F}(f)$. Then the following are equivalent*:

(a) *The grand orbit relation in V is discrete.*

(b) *There exists $z \in V$ such that* $\mathrm{GO}(z)$ *is discrete.*

(c) *For each $z \in V$, there exists a neighborhood U of z such that* $\mathrm{GO}(z') \cap U = \{z'\}$ *for all $z' \in U$.*

If, in addition, V is contained in the grand orbit of a wandering domain, then (a)–(c) *are equivalent to*

(d) *there exists a simply connected domain $W \subset V$ such that $f|_{f^n(W)}$ is injective for all $n \geq 0$.*

It follows from this theorem that for any component U of $\widehat{F}(f)$, the GO relation is discrete on $\mathrm{GO}(U)$ if and only if $\mathrm{GO}(z) \cap U$ is discrete for any $z \in U$. Thus, from now on we will interchangeably talk about discreteness of the GO relation in components of $\widehat{F}(f)$ or in their grand orbits.

Before addressing the second question, let us first define the *small orbit* of a point z under f as

$$\mathrm{SO}(z, f) = \mathrm{SO}(z) := \{w \in \mathcal{S} \mid f^n(z) = f^n(w) \text{ for some } n \in \mathbb{N}\} \subset \mathrm{GO}(z).$$

As in the case of grand orbits, the small orbit $\mathrm{SO}(U)$ of a set $U \subset \mathcal{S}$ is the union of all small orbits of the points in U, and the *small orbit relation* (or *SO relation*) on U is the equivalence relation whose equivalence classes are the small orbits $\mathrm{SO}(z)$ of points $z \in U$. It is a simple yet important fact that in the case of a wandering domain U, small orbits and grand orbits coincide, i.e., $\mathrm{SO}(z) \cap U = \mathrm{GO}(z) \cap U$ for all $z \in U$.

In order to answer the second question affirmatively, we construct transcendental entire functions with a sequence of wandering domains on which the dynamics mimics the dynamics of a polynomial P on an increasing sequence of bounded domains containing the Julia set $J(P)$. These examples are constructed in such a way that the small orbit relations on the different Fatou components of the polynomial "transfer" to the grand orbit relation of the transcendental map on the sequence of wandering domains[1].

Theorem B (Wandering domains arising from polynomial dynamics). *Let P be an arbitrary polynomial of degree $d \geq 2$. Let (D_n), $n \geq 0$, be a nested sequence of topological disks bounded by simple closed curves containing $J(P)$, and such that $P\colon D_n \to D_{n+1}$ is proper of degree d. Then there exist a transcendental entire function g with an orbit of wandering domains (U_n), $n \geq 0$, and a sequence of conformal maps $h_n\colon D_n \to U_n$ such that*

$$g|_{U_n} = h_{n+1} \circ P \circ h_n^{-1}.$$

Furthermore,

(a) *for any $z \in D_0$,*

$$h_0(\mathrm{SO}(z, P) \cap D_0) = \mathrm{GO}(h_0(z), g) \cap U_0;$$

(b) *if P is not conjugate to $z \mapsto z^d$, then $h_0(J(P)) \subset \widehat{J}(g) \cap U_0$, and the image under h_0 of every Fatou component of P contains at least one component of $\widehat{F}(g)$.*

Observe that such a sequence of topological disks (D_n) always exists, given by domains bounded by equipotentials in the basin of infinity. Actually, it is easy to see that every such sequence is of this form.

[1] We thank Curtis McMullen for generously suggesting this beautiful idea to us.

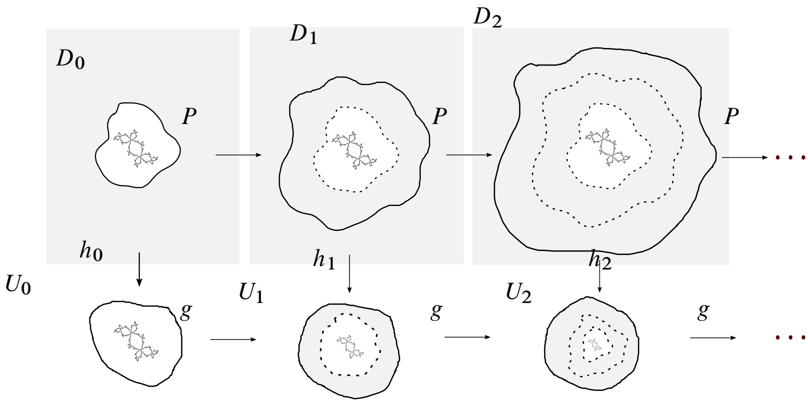

Figure 1. Sketch of Theorem B in the case where P is a quadratic polynomial with a period 3 attracting orbit (whose Julia set is commonly known as "Douady's rabbit").

See Figure 1 for an illustration of the statement of the theorem. Part (a) says that h_0 sends the small orbit relations for P on D_0 to the grand orbit relations for g on U_0. Note that the *traces* of $J(P)$ in the wandering domains (i.e., the images of $J(P)$ under h_n) map to each other under g. Hence, from Theorem A and noting that the SO relation for P is always indiscrete on the basin of infinity while being discrete on any basin of attraction of a finite cycle, we obtain the following corollary of Theorem B, which answers the second question (Question 2) posed above.

Corollary C (Coexistence of different GO relations in a wandering domain). *There exists a transcendental entire function g with a simply connected wandering domain U and two connected components W_1 and W_2 of $U \cap \widehat{F}(g)$ such that the grand orbit relation is discrete on W_1 and indiscrete on W_2.*

The corollary only states the simplest version of coexistence of discrete and indiscrete grand orbit relations in wandering domains. However, the construction in Theorem B is very flexible, and by choosing an appropriate polynomial P, one can obtain infinitely many components of $U \cap \widehat{F}(g)$ in a wandering domain U, and many combinations in terms of orbit relations. For details, see Theorem 7.3.

Observe that Theorem B provides only examples of *simply connected* domains. This is not a consequence of the limitations of our technique, but rather of the non-existence of such examples for multiply connected wandering domains. It follows from Theorem A and the dynamics of entire functions in multiply connected wandering domains, as described in [7], that grand orbit relations in multiply connected wandering domains are always indiscrete; see Corollary 6.9.

The proof of Theorem B proceeds in two steps. We first build a sequence of Blaschke products ($b_n : \mathbb{D} \to \mathbb{D}$) which serves as an abstract model of the desired internal dynamics in the orbit of the wandering domain (see Proposition 7.1). These maps b_n reflect the dynamics of P inside the domains D_n. (For an illustration of the construction, see Figure 3 in Section 7.) Secondly, using quasiconformal surgery, we show that the sequence (b_n) can be realized by a sequence of wandering domains of a transcendental entire map.

Informally speaking, we start with a given map f possessing an orbit of wandering domains $(\widetilde{U}_n)$ such that $f|_{\widetilde{U}_n}: \widetilde{U}_n \to \widetilde{U}_{n+1}$ is proper of degree d, and we "replace" f inside each $\widetilde{U}_n$ by the Blaschke product b_n. In our construction, we establish uniform estimates on quasiconformal distortion to ensure that the iterates of the modified map are uniformly K-quasiregular for some constant K. It is then a standard result from quasiconformal surgery that the new function is quasiconformally conjugate to a transcendental entire map f with the desired properties.

This "replacement" surgery procedure in the whole wandering orbit is new and works in a slightly more general setting, without assuming a constant degree for all Blaschke products. However, to establish our uniform estimates, we require both sequences (b_n) and $(f|_{\widetilde{U}_n})$ to be *uniformly hyperbolic*, which means that the degrees in the sequences are uniformly bounded and that the critical points stay within bounded hyperbolic distance from an orbit (see Sections 4 and 5). The technique is of independent interest and summarized in the following theorem.

Theorem D (A replacement surgery procedure). *Let f be an entire function with an orbit of simply connected wandering domains $(\widetilde{U}_n)$ and let $(b_n: \mathbb{D} \to \mathbb{D})$ be a sequence of Blaschke products such that $2 \le \deg(f|_{U_n}) = \deg(b_n)$ for all $n \ge 0$. Suppose both sequences $(f|_{\widetilde{U}_n})$ and (b_n) are uniformly hyperbolic. Then there exists an entire function g with an orbit of simply connected wandering domains (U_n) such that the following hold:*

(a) *There is a sequence of conformal maps $\theta_n: U_n \to \mathbb{D}$ such that*

$$g|_{U_n} = \theta_{n+1}^{-1} \circ b_n \circ \theta_n.$$

(b) *There are neighborhoods V and $\widetilde{V}$ of $\mathbb{C} \setminus \bigcup_{n\ge 0} U_n$ and $\mathbb{C} \setminus \bigcup_{n \ge 0} \widetilde{U}_n$, respectively, and a quasiconformal map $\Phi: \mathbb{C} \to \mathbb{C}$ such that*

$$g|_V = \Phi \circ f|_{\widetilde{V}} \circ \Phi^{-1}.$$

Furthermore, $\Phi(\partial \widetilde{U}_n) = \partial U_n$ for all n.

We remark that this surgery operation only changes the dynamics of the original map f inside the wandering orbit $(\widetilde{U}_n)$, so the dynamics of g outside its wandering orbit (U_n) are quasiconformally conjugate to the dynamics of f outside $(\widetilde{U}_n)$. In contrast, for examples constructed using approximation theory, there is almost no control on the resulting entire function outside the prescribed domains. For more information on the use of approximation theory to construct functions with wandering domains, we refer to the seminal paper by Eremenko and Lyubich [10], as well as to several recent interesting constructions; see, e.g., [4, 8, 11, 22].

Combining Theorem D with results in [4], in Section 5 we prove the following realization statement, generalizing Corollary 1.10 in [13].

Theorem E (Realization of abstract wandering sequences). *Let (b_n) be a sequence of uniformly hyperbolic Blaschke products. Then there exist a transcendental entire function g with an orbit of simply connected wandering domains (U_n) and a sequence of conformal maps $(\theta_n: U_n \to \mathbb{D})$ such that*

$$g|_{U_n} = \theta_{n+1}^{-1} \circ b_n \circ \theta_n.$$

The existence of wandering domains associated to sequences of finite Blaschke products can also be proved using approximation theory, see Theorem 1.5 in [12].

Structure of the article. Section 2 has definitions, notation, and some preliminary results on holomorphic self-maps of the unit disk, while Section 3 contains results on quasiconformal and quasisymmetric maps needed for our quasiconformal surgery. In Section 4, we define the notion of uniformly hyperbolic Blaschke products and prove the main properties we will need. Section 5 deals with uniformly hyperbolic wandering domains and contains the proofs of Theorem D and Theorem E. In Section 6, we discuss grand orbit relations and prove Theorem A. Finally, in Section 7 we study polynomials in relation to uniformly hyperbolic Blaschke products and give the proofs of Theorem B and Corollary C.

2. Preliminaries

2.1. Notation and basic definitions

We denote by $\mathbb{C}$ the complex plane and by $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the extended complex plane. We denote the closure of a set A (either in $\mathbb{C}$ or in $\widehat{\mathbb{C}}$) by $\overline{A}$. We write $A \subset B$ to indicate that A is a subset of B, including the possibility that $A = B$. In cases where we want to stress that A is a proper subset of B, we use the notation $A \subsetneq B$ instead. For sets $A, B \subset \mathbb{C}$, we write $A \Subset B$ to indicate that A is *compactly contained* in B, i.e., $\overline{A}$ is compact and $\overline{A} \subset B$. Given a set $A \subset \mathbb{C}$ and a constant $c \in \mathbb{C}$, we write $A + c := \{z + c : z \in A\}$ for the translate of the set A by c.

For $r > 0$, we denote by $\mathbb{D}_r$ and $\mathbb{S}_r$ the open disk and the circle of radius r centered at 0, respectively. In the case $r = 1$, we write $\mathbb{D} := \mathbb{D}_1$ and $\mathbb{S} := \mathbb{S}_1$. For $0 < r < R$, $\mathbb{A}_{r,R} := \{r < |z| < R\}$ is the round annulus with inner radius r and outer radius R. When $R = 1$, we write $\mathbb{A}_r := \mathbb{A}_{r,1}$. We refer to the conformal modulus of a topological annulus A as $\operatorname{mod}(A)$. Sequences of points, sets, and functions such as (x_n) are indexed by $n \geq 0$, unless otherwise specified. For a simply connected domain $U \subset \mathbb{C}$, $d_U(\cdot,\cdot)$ denotes the hyperbolic distance in U, while $\rho_U(\cdot)$ denotes the density of the hyperbolic metric.

Given domains $U, V \subset \mathbb{C}$ and a holomorphic map $f: U \to V$, we denote by $\operatorname{Crit}(f)$ the set of its critical points and by $S(f)$ the set of its singular (critical and asymptotic) values. Moreover, if f is a proper map, we denote by $\deg(f)$ the topological degree of f.

Definition 2.1. Let $U_n \subset \mathbb{C}$ and let $f_n: U_n \to U_{n+1}$ be holomorphic proper maps. The sequence (f_n) has *uniformly bounded degree* if there exists $d \in \mathbb{N}$ such that $\deg(f_n) \leq d < \infty$ for all $n \geq 0$. Moreover, an (f_n)-*orbit* (or simply an *orbit* when the context is clear) is a sequence of points (z_n), $z_n \in U_n$, such that $f_n(z_n) = z_{n+1}$.

The notions of topological and (quasi-)conformal equivalence are usually defined for pairs of maps. In this paper, we extend this definition to sequences of maps.

Definition 2.2. Let $U_n, V_n \subset \mathbb{C}$ and let $f_n: U_n \to U_{n+1}$ and $g_n: V_n \to V_{n+1}$ be holomorphic functions. We say that the sequences (f_n) and (g_n) are *topologically equivalent* if there exists a sequence of homeomorphisms $h_n: U_n \to V_n$ such that

$$g_n = h_{n+1} \circ f_n \circ h_n^{-1} \quad \text{for all } n \geq 0. \tag{2.1}$$

If, in addition, there exists $K \geq 1$ such that all maps h_n are K-quasiconformal, then we say that the sequences are (K-)*quasiconformally equivalent*. If all maps h_n are conformal, the sequences are said to be *conformally equivalent*.

Remark. Orbits and singular values are preserved under topological equivalence, that is, if (f_n) and (g_n) are sequences of maps as in Definition 2.2 that are topologically equivalent under a sequence of homeomorphisms (h_n), then (z_n) is an (f_n)-orbit if and only if $(h_n(z_n))$ is a (g_n)-orbit. Moreover, $h_n(\mathrm{Crit}(f_n)) = \mathrm{Crit}(g_n)$ and $h_{n+1}(S(f_n)) = S(g_n)$ for all $n \geq 0$.

If f is an entire map with a wandering domain U_0, we will denote by U_{n+1} the Fatou component that contains $f(U_n)$, $n \geq 0$, and say that (U_n) is an *orbit* of wandering domains. Note that if $\deg(f|_{U_n})$ is finite, then $U_{n+1} = f(U_n)$, and $(f_n := f|_{U_n})$ is a sequence as above.

Finite Blaschke products are proper holomorphic self-maps of the unit disk. It is well known that they have a finite degree $d \geq 1$ and can be written as

$$b(z) = e^{i\theta} \prod_{k=1}^{d} \frac{z - a_k}{1 - \overline{a_k}\, z},$$

where $\theta \in \mathbb{R}$ and $|a_k| < 1$ for $k = 1, \ldots, d$. We will frequently work with infinite sequences of Blaschke products (b_n), $n \geq 0$.

2.2. Estimates for holomorphic self-maps of the unit disc

For $\rho \in (0, 1)$, let

$$\mathcal{F}_\rho = \{g : \mathbb{D} \to \mathbb{D} \text{ analytic}, g(0) = 0, |g'(0)| = \rho\}.$$

The following is a variation of the Schwarz lemma. It follows from Corollary 2.4 in [4], but we include a direct proof for completeness.

Lemma 2.3 (Uniform Schwarz lemma). *Let $\rho, r \in (0, 1)$. Then there exists $c \in (0, 1)$, explicitly given by $c = (\rho + r)/(1 + r\rho)$, such that $|g(z)| \leq c|z|$ for all $g \in \mathcal{F}_\rho$ and all $|z| \leq r$.*

In particular, if $\rho \leq r$, then

$$|g(z)| \leq \frac{2r^2}{1 + r^2} < r.$$

Remark. The general statement without the explicit formula for c can be proved using the Schwarz lemma and a normality argument.

Proof of Lemma 2.3. Suppose $g \in \mathcal{F}_\rho$. By composing g with a rotation, we may assume $g'(0) = \rho$. Define $h(z) = g(z)/z$, with $h(0) = g'(0) = \rho$. We have to show that

$$|h(z)| \leq c = \frac{\rho + r}{1 + r\rho} \quad \text{for } |z| \leq r.$$

The function h is an analytic map from $\mathbb{D}$ into itself, so by the Schwarz lemma it satisfies

$$d_{\mathbb{D}}(h(z), h(w)) \leq d_{\mathbb{D}}(z, w) \quad \text{for all } z, w \in \mathbb{D}.$$

Fixing $|z| \leq r$, we get that

$$d_{\mathbb{D}}(\rho, h(z)) = d_{\mathbb{D}}(h(0), h(z)) \leq d_{\mathbb{D}}(0, z) \leq d_{\mathbb{D}}(0, r),$$

and thus

$$d_{\mathbb{D}}(0, h(z)) \leq d_{\mathbb{D}}(0, \rho) + d_{\mathbb{D}}(\rho, h(z)) \leq d_{\mathbb{D}}(0, \rho) + d_{\mathbb{D}}(0, r).$$

Since $d_{\mathbb{D}}(0, r) = d_{\mathbb{D}}(0, -r)$, and since the interval $(-1, 1)$ is a hyperbolic geodesic containing $-r$, 0, and ρ, we get that

$$d_{\mathbb{D}}(0, \rho) + d_{\mathbb{D}}(0, r) = d_{\mathbb{D}}(-r, \rho).$$

The fact that

$$T(w) := \frac{w + r}{1 + rw}$$

is a hyperbolic isometry gives

$$d_{\mathbb{D}}(-r, \rho) = d_{\mathbb{D}}(T(-r), T(\rho)) = d_{\mathbb{D}}\Big(0, \frac{\rho + r}{1 + r\rho}\Big).$$

Combining all the inequalities and equalities above, we get

$$d_{\mathbb{D}}(0, h(z)) \leq d_{\mathbb{D}}\Big(0, \frac{\rho + r}{1 + r\rho}\Big),$$

so that

$$|h(z)| \leq \frac{\rho + r}{1 + r\rho}.$$

Finally, suppose that $0 < \rho \leq r < 1$. The function

$$\rho \mapsto c(\rho) = \frac{\rho + r}{1 + \rho r}$$

is increasing on $[0, 1]$, so

$$c(\rho) \leq c(r) = \frac{2r}{1 + r^2}$$

and the result follows directly. ∎

Corollary 2.4. *Assume that $0 < \rho \leq r < 1$, and $g \in \mathcal{F}_\rho$. Then $g^{-1}(\overline{\mathbb{D}}_r)$ contains the disk $\overline{\mathbb{D}}_R$ with $R = \sqrt{r/(2 - r)} \in (r, 1)$.*

Proof. Let $R = \sqrt{r/(2 - r)}$. Then $0 < R < 1$, and $2R^2/(1 + R^2) = r$, so applying Lemma 2.3 with R instead of r gives that $|g(z)| \leq r < R$ whenever $|z| \leq R$ and $\rho \leq R$, which is satisfied because $\rho \leq r$ and $r < R$. ∎

3. Quasiconformal and quasisymmetric maps

In the proof of Theorem 4.6, which is the basis for Theorem D, we will require some results on quasiconformal and quasisymmetric maps. For a general reference, see [1, 21], and for results in the context of holomorphic dynamics, see [9].

Let X and Y be connected subsets of $\mathbb{C}$, and let $f: X \to Y$ be a homeomorphism. We say that f is H*-quasisymmetric* if $H \geq 1$ is a constant such that for $x_1, x_2, x_3 \in X$,

$$|f(x_1) - f(x_2)| \leq H|f(x_2) - f(x_3)| \quad \text{whenever } |x_1 - x_2| \leq |x_2 - x_3|.$$

In the general theory of quasisymmetric maps on metric spaces, homeomorphisms satisfying this condition are usually called "weakly H-quasisymmetric". However, by Corollary 10.22 in [18], for every weakly H-quasisymmetric homeomorphism between connected subsets of $\mathbb{C}$, there exists a homeomorphism $\eta: [0, \infty) \to [0, \infty)$, depending only on H, such that f is strongly η-quasisymmetric, i.e.,

$$|f(x_1) - f(x_2)| \leq \eta(t)|f(x_2) - f(x_3)| \quad \text{whenever } |x_1 - x_2| \leq t|x_2 - x_3|.$$

With this stronger condition, it is immediate that the composition $f_2 \circ f_1$ of strongly η_k-quasisymmetric homeomorphisms f_k, $k = 1, 2$, is strongly $(\eta_2 \circ \eta_1)$-quasisymmetric, which in turn implies that the composition $f_2 \circ f_1$ of H_k-quasisymmetric homeomorphisms f_k is H-quasisymmetric, with H only depending on H_1 and H_2. We say that $f: X \to \mathbb{C}$ is an H*-quasisymmetric embedding* if it is an H-quasisymmetric homeomorphism onto its image $f(X)$.

By Theorem 11.14 in [18], every K-quasiconformal map $f: \mathbb{C} \to \mathbb{C}$ is H-quasisymmetric with H only depending on K, and every H-quasisymmetric $f: \mathbb{C} \to \mathbb{C}$ is K-quasiconformal, with K depending only on H. (Note that conformal maps from a bounded onto an unbounded domain are never quasisymmetric, so the assumption that f is defined in the whole plane is important here.)

Let γ be a Jordan curve in $\mathbb{C}$. We say that γ is a K*-quasicircle* if it is the image of the unit circle under a K-quasiconformal self-map of $\mathbb{C}$. Ahlfors showed that there is a simple geometric characterization of quasicircles, see Chapter IV of [1]. For distinct points $z, w \in \gamma$, let $\operatorname{diam}_\gamma(z, w)$ denote the minimum of the diameters of the two components of $\gamma \setminus \{z, w\}$. We say that γ is of M*-bounded turning* if for all distinct $z, w \in \gamma$, we have $\operatorname{diam}_\gamma(z, w) \leq M|z - w|$. The result of Ahlfors is that every K-quasicircle is of M-bounded turning, with $M = M(K)$, and that every Jordan curve of M-bounded turning is a K-quasicircle, with $K = K(M)$.

Ahlfors and Beurling gave an explicit formula showing that every H-quasisymmetric mapping $f: \mathbb{R} \to \mathbb{R}$ extends to a K-quasiconformal map $F: \mathbb{C} \to \mathbb{C}$, with K depending only on H, see Chapter IV of [1]. Combining this with the characterization of quasicircles given above, Tukia generalized this result to show the following.

Theorem 3.1 (Quasiconformal extension of quasisymmetric embeddings of circles [25]). *Let $f: S \to \mathbb{C}$ be an H-quasisymmetric embedding of a circle or a line $S \subset \mathbb{C}$. Then f extends to a K-quasiconformal map $F: \mathbb{C} \to \mathbb{C}$, with K depending only on H. In particular, the image $f(S)$ is a K-quasicircle.*

Remark. In Tukia's paper, the result is stated for quasisymmetric embeddings of the real line. However, it is not hard to show that this result is true for arbitrary circles and lines (and Tukia's arguments in the proof show it as well.)

We say that two domains U and V are K*-quasiconformally equivalent* if there exists a K-quasiconformal isomorphism $\varphi\colon U \to V$.

Theorem 3.2 (Extension of quasiconformal maps on compact sets, Theorem 8.1 in [21]). *Let $\psi\colon D \to D'$ be a K-quasiconformal homeomorphism between domains $D, D' \subset \mathbb{C}$, and let $E \subset D$ be compact. Then, there exist $K' = K'(K, D, E)$ and a K'-quasiconformal map $\Psi\colon \mathbb{C} \to \mathbb{C}$ such that $\psi = \Psi$ on E.*

One immediate consequence of this theorem is that an embedding of a circle which extends to a conformal map of an annular neighborhood is a quasisymmetric embedding, in a quantitative way. The following corollary is the precise version of this statement.

Corollary 3.3 (Quasisymmetric image of a circle). *Let $r > 0$ and $0 < t < 1$, and let $\psi\colon \mathbb{A}_{rt,r/t} \to \mathbb{C}$ be a conformal embedding. Then $\psi|_{\mathbb{S}_r}\colon \mathbb{S}_r \to \mathbb{C}$ is an H-quasisymmetric embedding, where H depends only on t.*

Proof. By rescaling the annulus, we may assume that $r = 1$. Theorem 3.2 implies that $\psi|_{\mathbb{S}}$ has a K-quasiconformal extension to the plane, where K depends only on t. This extension is then H-quasisymmetric, with H again depending only on t, establishing the claim. ∎

We will also need the following interpolation result, which is a quantitative version of Proposition 2.30(b) in [9].

Theorem 3.4 (Interpolation between boundary maps of annuli). *Let $K_0 \geq 1$ be a constant, let $\mathbb{A}_{r,R}$ be a round annulus of modulus $M = \frac{1}{2\pi}\log\frac{R}{r}$, and let $A \subset \mathbb{C}$ be a topological annulus whose inner and outer boundary are Jordan curves γ_i and γ_o, respectively, with $K_0^{-1}M \leq \operatorname{mod} A \leq K_0 M$. Furthermore, assume that φ_i and φ_o are orientation-preserving H-quasisymmetric homeomorphisms from the inner and outer boundary circles of $\mathbb{A}_{r,R}$ to γ_i and γ_o, respectively. Then there exists a homeomorphism $\varphi\colon \overline{\mathbb{A}}_{r,R} \longrightarrow \overline{A}$ which is K-quasiconformal in $\mathbb{A}_{r,R}$ with $K = K(H, K_0, M)$, and coincides with φ_i and φ_o on the respective inner and outer boundaries of $\mathbb{A}_{r,R}$.*

Before proving this theorem, we will first need some auxiliary results. In the statement of the following theorem, we say that a map $f\colon X \to \mathbb{C}$ on an arbitrary subset $X \subset \mathbb{C}$ is K-quasiconformal if it extends to a K-quasiconformal embedding of a neighborhood of X. Furthermore, for disjoint Jordan curves $S, T \subset \mathbb{C}$, we use the notation $S \prec T$ to indicate that S is contained in the bounded component of $\mathbb{C} \setminus T$, i.e., that S is "inside" T.

Theorem 3.5 (Quantitative quasiconformal extension theorem, see 5.8 in [26]). *Suppose $0 < a < b < c < d$ and $K_0, M \geq 1$ are constants. Let $D = \mathbb{A}_{a,b} \cup \mathbb{A}_{c,d}$, and let $f_0\colon \overline{D} \to \mathbb{C}$ be a K_0-quasiconformal embedding. Assume that the images of the four boundary circles are nested in the same way that the circles themselves are, i.e., that $f_0(\mathbb{S}_a) \prec f_0(\mathbb{S}_b) \prec f_0(\mathbb{S}_c) \prec f_0(\mathbb{S}_d)$, and that* $\operatorname{diam} f_0(\mathbb{S}_d) \leq M \operatorname{diam} f_0(\mathbb{S}_a)$. *Then there exists a K-quasiconformal map $f\colon \overline{\mathbb{A}}_{a,d} \to \mathbb{C}$ such that $f = f_0$ on $\mathbb{S}_a \cup \mathbb{S}_d$, where $K = K(a, b, c, d, K_0, M)$.*

Remark 3.6. The cited theorem actually applies to quasiconformal maps in arbitrary dimension, in which case the constant K additionally depends on the dimension.

Corollary 3.7 (Interpolation between circle maps). *Suppose* $0 < r_1 < r_2$ *and* $H \geq 1$ *are constants. Let* $f_j : \mathbb{S}_{r_j} \to \mathbb{S}_{r_j}$ *be* H*-quasisymmetric for* $j = 1, 2$*. Then there exist* $K = K(r_1, r_2, H)$*, and a* K*-quasiconformal map* $f : \mathbb{C} \to \mathbb{C}$*, such that* $f = f_j$ *on* $\mathbb{S}_{r_j}$ *for* $j = 1, 2$*.*

Proof. By the Beurling–Ahlfors extension theorem, see Chapter IV of [1], there exists $K_0 = K_0(H)$ such that each f_j extends continuously to a K_0-quasiconformal self-map of the disk $\mathbb{D}_{r_j}$, normalized by $f_j(0) = 0$, which can then be extended by reflection to a K_0-quasiconformal map of the plane. By Mori's theorem, see Chapter III of [1], we have the uniform Hölder estimates

$$|f_j(z) - f_j(w)| \leq M_j |z - w|^{1/K_0}$$

whenever $z, w \in \overline{\mathbb{D}}_{r_j}$, with explicit constants $M_j = 16 r_j^{1-1/K_0}$. In particular, this shows that for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, r_j, K_0) > 0$ such that $||f_j(z)| - r_j| \leq \varepsilon$ whenever $||z| - r_j| \leq \delta$. (This holds without the assumption that $|z| \leq r_j$, using the fact that f_j is defined by reflection for $|z| > r_j$.) Picking $\varepsilon > 0$ small enough such that $r_1 + \varepsilon < r_2 - \varepsilon$, we use the corresponding δ (reducing it to be smaller than ε, if necessary) to define the radii to use in applying Theorem 3.5 as $a = r_1$, $b = r_1 + \delta$, $c = r_2 - \delta$, and $d = r_2$. Defining

$$D = \mathbb{A}_{r_1, r_1+\delta} \cup \mathbb{A}_{r_2-\delta, r_2}$$

and $f_0 : D \to \mathbb{C}$ by

$$f_0(z) = \begin{cases} f_1(z) & \text{for } r_1 \leq |z| < r_1 + \delta, \\ f_2(z) & \text{for } r_2 - \delta < |z| \leq r_2, \end{cases}$$

we have that f_0 is a K_0-quasiconformal embedding respecting the nesting of the boundary circles, and that $\operatorname{diam} f_0(\mathbb{S}_{r_2}) = M \operatorname{diam} f_0(\mathbb{S}_{r_1})$ with $M = r_2/r_1$. This shows that the assumptions of Theorem 3.5 are satisfied, so that there exists a K-quasiconformal map $f : \overline{\mathbb{A}}_{r_1, r_2} \to \mathbb{C}$ such that $f = f_0 = f_1$ on $\mathbb{S}_{r_1}$ and $f = f_0 = f_2$ on $\mathbb{S}_{r_2}$, where $K = K(r_1, r_2, H)$. Extending f by f_1 on $\mathbb{D}_{r_1}$, and by f_2 on $\mathbb{C} \setminus \overline{\mathbb{D}}_{r_2}$, we end up with the claimed extension to the plane. ∎

Lemma 3.8 (Extension of qc maps between annuli). *Let* $0 < r < 1$*, and let* $\psi : \mathbb{A}_r \to A$ *be a* K*-quasiconformal map onto an annulus* A *bounded by two* K*-quasicircles. Then* ψ *has a* K'*-quasiconformal extension to* $\mathbb{C}$*, with* K' *only depending on* K *and* r*.*

Proof. Let Γ^i and Γ^o denote the inner and outer boundaries of A respectively. Every K-quasicircle Γ admits a K^2-quasiconformal reflection (see, e.g., Section D in Chapter IV of [1] or Proposition 4.9.12 in [19]), i.e., a K^2-quasiconformal involution φ_Γ fixing every point of Γ and interchanging its complementary components.[2] Using the reflection with

[2]In fact, something slightly stronger is true: K-quasicircles always admit K-quasiconformal reflections. However, a K^2-quasiconformal reflection suffices for our purposes, and the proof of this result is much easier.

respect to the circle of radius $s > 0$, $\tau_s(z) = s^2/\bar{z}$, for $s = 1$ and $s = r$, and the two reflections φ_{Γ^i} and φ_{Γ^o}, we can extend ψ to $\mathbb{A}_{r^2,1/r}$ by

$$\widehat{\psi}(z) = \begin{cases} \psi(z) & \text{if } z \in \mathbb{A}_r, \\ \varphi_{\Gamma^o} \circ \psi \circ \tau_1(z) & \text{if } 1 \leq |z| < 1/r, \\ \varphi_{\Gamma_i} \circ \psi \circ \tau_r(z) & \text{if } r^2 < |z| \leq r. \end{cases}$$

Then, $\widehat{\psi}\colon \mathbb{A}_{r^2,1/r} \to \widehat{A}$ is a K^3-quasiconformal map, where $\widehat{A}$ is a topological annulus containing the closure of A. Applying Theorem 3.2 with $D = \mathbb{A}_{r^2,1/r}$ and $E = \overline{\mathbb{A}}_r$, we get the desired K'-quasiconformal extension $\Psi\colon \mathbb{C} \to \mathbb{C}$, with K' only depending on K and r. ∎

We are finally ready for the proof of Theorem 3.4.

Proof of Theorem 3.4. By scaling the round annulus, we can assume that $R = 1$, so that $\mathbb{A}_{r,R} = \mathbb{A}_r$ and $M = \frac{1}{2\pi} \log \frac{1}{r}$. From the condition on the moduli of $\mathbb{A}_r$ and A, we know that there exists a K_0-quasiconformal isomorphism $\psi\colon \mathbb{A}_r \to A$. Note that by Theorem 3.1, the curves bounding A are K_1-quasicircles, where K_1 only depends on H. By Lemma 3.8, ψ extends to a K_2-quasiconformal map of the plane, with K_2 only depending on K_0, H and M. Then $\widetilde{\varphi}_i = \psi^{-1} \circ \varphi_i$ and $\widetilde{\varphi}_o = \psi^{-1} \circ \varphi_o$ are orientation-preserving H_1-quasisymmetric self-maps of the inner and outer boundary circles of $\mathbb{A}_r$, with H_1 only depending on K_2 and H.

By Corollary 3.7, there exists a K_3-quasiconformal homeomorphism $\widetilde{\varphi}\colon \mathbb{A}_r \to \mathbb{A}_r$ with boundary maps $\widetilde{\varphi}_i$ and $\widetilde{\varphi}_o$, where K_3 only depends on M and H_1. Defining $\varphi := \psi \circ \widetilde{\varphi}$, we obtain the desired K-quasiconformal map $\varphi\colon \mathbb{A}_r \to \overline{A}$, with $K = K_2 \cdot K_3$ only depending on K_0, H, and M. ∎

4. Uniformly hyperbolic Blaschke products

A Blaschke product $b\colon \mathbb{D} \to \mathbb{D}$ of degree $2 \leq d < \infty$ is *hyperbolic* if it has a fixed point z_0 inside $\mathbb{D}$, which by the Schwarz lemma is unique. Moreover, $|b'(z_0)| < 1$ and z_0 is a global attractor under b. Note that we always consider Blaschke products restricted to the unit disk. In particular, when speaking about the critical points and critical values of a Blaschke product, we only consider those inside the unit disk.

In the following, we will normalize our Blaschke products to fix zero, so that

$$b(z) = z \prod_{k=1}^{d-1} \frac{z - a_k}{1 - \overline{a_k}\, z}, \quad a_k \in \mathbb{D}. \tag{4.1}$$

Then, the *multiplier* of the fixed point at $a_0 = 0$ is

$$b'(0) = (-1)^{d-1} \prod_{k=1}^{d-1} a_k, \tag{4.2}$$

and therefore in particular $|b'(0)| \leq \min_k |a_k|$.

Hyperbolic Blaschke products of the form (4.1) satisfy the following properties.

Lemma 4.1 (Properties of hyperbolic Blaschke products). *Let b be a Blaschke product of the form* (4.1). *Then the critical points of b in the unit disk belong to the hyperbolic convex hull of its zeros $\{a_0, a_1, \ldots, a_{d-1}\}$. Furthermore, given a radius r satisfying $\max_k |a_k| < r < 1$, then $D_r := b^{-1}(\mathbb{D}_r)$ satisfies the following*:

(a) *D_r is an analytic Jordan domain and $\overline{\mathbb{D}}_r \subset D_r$;*

(b) *$A^r := D_r \setminus \overline{\mathbb{D}}_r$ is an annulus whose modulus satisfies*

$$\frac{1}{4\pi} \log \frac{1}{r(2-r)} \leq \operatorname{mod}(A^r) \leq \frac{d-1}{2\pi d} \log \frac{1}{r},$$

where the right equality is satisfied if $b(z) = z^2$.

(c) $|b'(0)| < r^{d-1}$.

Proof. The fact that the critical points of b belong to the hyperbolic convex hull of the zeros of b is a classical result by Walsh, see Theorem 6.1.6 in [17].

Corollary 2.4 shows that we have $\overline{\mathbb{D}}_r \subset D_r$, and the assumption $r > \max_k |a_k|$ implies that all zeros of b are contained in $\mathbb{D}_r$, hence in the connected component of D_r containing $\mathbb{D}_r$. Since every connected component of D_r must contain at least one zero, this shows that D_r is connected, and by the maximum principle, it is also simply connected. The assumption further implies that all the critical values of b are contained in $\mathbb{D}_r$, thus $\partial D_r = b^{-1}(\mathbb{S}_r)$ is an analytic Jordan curve.

For (b), note that b maps $\mathbb{D} \setminus \overline{D}_r$ to $\mathbb{D} \setminus \overline{\mathbb{D}}_r$ with degree d. Since the latter has modulus $-\frac{1}{2\pi} \log r$, it follows that

$$\operatorname{mod}(\mathbb{D} \setminus \overline{D}_r) = -\frac{1}{2\pi d} \log r.$$

Now, by the Grötzsch principle, we have

$$\operatorname{mod}(A^r) + \operatorname{mod}(\mathbb{D} \setminus \overline{D}_r) \leq \operatorname{mod}(\mathbb{D} \setminus \overline{\mathbb{D}}_r),$$

which gives the right inequality. Observe that if $b(z) = z^2$, $D_r = \{|z| < r^{1/d}\}$ and the modulus is directly computable. To see the left inequality, let $\rho := |b'(0)|$ and note that by our assumption on r, $|\rho| < r$. Corollary 2.4 implies that A^r contains the round annulus $\{r < |z| < R\}$, where $R := \sqrt{r/(2-r)}$, and the estimate follows.

Lastly, (c) follows directly from (4.2). ∎

Next we consider sequences (b_n) of Blaschke products. For any such sequence, we denote by $a_0^n = 0, a_1^n, \ldots, a_{d_n-1}^n$ the zeros of b_n, where $d_n := \deg(b_n)$. Sometimes, it will be convenient to think of the domains and ranges of the maps b_n as distinct copies of the unit disk $\mathbb{D}^{(n)}$, where $b_n : \mathbb{D}^{(n)} \to \mathbb{D}^{(n+1)}$ for any $n \geq 0$.

Definition 4.2 (Uniformly hyperbolic sequence of Blaschke products). We say that a sequence (b_n) of hyperbolic Blaschke products of the form (4.1) is *uniformly hyperbolic* if (b_n) has uniformly bounded degree and there exists $0 < r < 1$ such that $|a_k^n| < r$ for every $0 \leq k < \deg(b_n)$ and every $n \in \mathbb{N}$.

Remark. Recall that, by definition, hyperbolic Blaschke products are of degree at least 2, and so if (b_n) is uniformly hyperbolic, then $2 \leq \min_n \deg(b_n) \leq \max_n \deg(b_n) < \infty$.

As a consequence of Lemma 4.1, uniformly hyperbolic sequences of Blaschke products have the following properties. Recall that for a sequence (b_n) of Blaschke products, we denote by $B_n\colon \mathbb{D}^{(0)} \to \mathbb{D}^{(n)}$ the composition of the n first maps; that is, $B_n := b_{n-1} \circ \cdots \circ b_0$, $n \geq 1$.

Lemma 4.3 (Properties of uniformly hyperbolic sequences of Blaschke products). *Let (b_n) be a sequence of uniformly hyperbolic Blaschke products and let $0 < r < 1$ be a radius provided by Definition 4.2. For $n \geq 0$, let*

$$A_n := A_n^r := b_n^{-1}(\mathbb{D}_r) \setminus \mathbb{D}_r \ \subset \mathbb{D}^{(n)}. \tag{4.3}$$

Then, the following hold:

(1) *$B_n(z) \to 0$ as $n \to \infty$ for all $z \in \mathbb{D}$.*

(2) *For each $n \geq 0$, A_n is an annulus, and there exist constants $m, M \in (0, \infty)$, independent of n, such that $m \leq \operatorname{mod} A_n \leq M$.*

(3) *For each $z \in \mathbb{D}$, there exists at most one $n \in \mathbb{N}$ such that $B_n(z) \in A_n$.*

Proof. Note that, for each n, $\rho_n := |b_n'(0)| \leq r^{d_n - 1}$ by Lemma 4.1. Hence, we have that $\sum_n (1 - \rho_n) = \infty$ and so (1) holds, see Theorem 2.1 in [4]. Item (2) is a direct consequence of Lemma 4.1. To see (3), fix $z \in \mathbb{D}$ and assume that $B_n(z) \in A_n$. Then $B_{n+1}(z) \in b_n(A_n) \subset \mathbb{D}_r$. By the Schwarz lemma, $b_j(\mathbb{D}_r) \subset \mathbb{D}_r$ for every $j \geq 0$, hence $B_{n+k}(z) \in \mathbb{D}_r$ for all $k \geq 1$. Since $\mathbb{D}_r \cap A_n = \emptyset$, this establishes (3). ∎

A priori, it is not obvious that uniform hyperbolicity is preserved under conformal equivalence, but the following theorem provides alternative characterizations, which are more convenient in this respect.

Theorem 4.4 (Characterization of uniformly hyperbolic sequences of Blaschke products). *Let (b_n) be a sequence of Blaschke products of uniformly bounded degree fixing 0. Then the following are equivalent*:

(a) *(b_n) is uniformly hyperbolic.*

(b) *There exists $s \in (0, 1)$ such that $\operatorname{Crit}(b_n) \subset \mathbb{D}_s$ for all n.*

(c) *Given any orbit (z_n), there exists a constant C such that $d_{\mathbb{D}^{(n)}}(z_n, c) \leq C$ for all $c \in \operatorname{Crit}(b_n)$.*

(d) *There exist an orbit (z_n) and a constant C such that $d_{\mathbb{D}^{(n)}}(z_n, c) \leq C$ for all $c \in \operatorname{Crit}(b_n)$.*

Proof. Equivalence of (b) and (c) is immediate from the fact that $|z_n| \leq |z_0|$ for all n by the Schwarz lemma. The equivalence between (c) and (d) follows from the fact that the hyperbolic distance between orbits is a decreasing function of n.

Condition (a) implies (b) since the critical points of b_n are contained in the hyperbolic convex hull of its zeros, see Lemma 4.1. It remains to prove that (b) implies (a). We will prove this claim by contradiction. Assume that there exists a constant $s \in (0, 1)$ such for all n, $\operatorname{Crit}(b_n) \subset \mathbb{D}_s$, but that the sequence (b_n) is not uniformly hyperbolic. Let $c_1^n, \ldots, c_{d_n-1}^n$ denote the critical points of b_n (not necessarily distinct) while, as usual, we denote the zeros of b_n by $a_0^n, a_1^n, \ldots, a_{d_n-1}^n$. By passing to a subsequence, we may assume that

- the sequence (b_n) has constant degree $d \geq 2$,
- the limits $\lim_{n\to\infty} c_k^n = c_k$ exist for all $k = 1, \dots, d-1$, with $|c_k| \leq s$,
- the limits $\lim_{n\to\infty} a_k^n = a_k$ exists for all $k = 1, \dots, d-1$, with $|a_k| \leq 1$, and
- $|a_k| = 1$ for at least one index k.

Then

$$b_n(z) = z \prod_{k=1}^{d-1} \frac{z - a_k^n}{1 - \overline{a_k^n}\, z} \quad \longrightarrow \quad b_\infty(z) = z \prod_{k=1}^{d-1} \frac{z - a_k}{1 - \overline{a_k}\, z} \quad \text{for } n \to \infty,$$

locally uniformly in $\mathbb{D}$. For any k for which $|a_k| = 1$, the factor $(z - a_k)/(1 - \overline{a_k}\, z)$ is a constant equal to $-a_k$, so b_∞ is a Blaschke product of degree $1 \leq d_\infty \leq d-1$. However, by Hurwitz's theorem, b_∞ has $d-1$ critical points $c_1, \dots, c_{d-1}$ in the unit disk, which implies that the degree of b_∞ is d, a contradiction. ∎

Using the characterization of uniformly hyperbolic Blaschke products provided by Theorem 4.4 (c), we deduce that uniform hyperbolicity is preserved under quasiconformal equivalence.

Corollary 4.5 (Uniform hyperbolicity is a quasiconformal invariant). *Let (b_n) and $(\tilde{b}_n)$ be two sequences of Blaschke products fixing zero that are quasiconformally equivalent. Then (b_n) is uniformly hyperbolic if and only if $(\tilde{b}_n)$ is uniformly hyperbolic.*

Proof. Note that it suffices to prove one direction, since the inverse of a K-quasiconformal map is K-quasiconformal. Suppose (b_n) is uniformly hyperbolic. By assumption, there exists $K \geq 1$ and a sequence of K-quasiconformal maps (h_n) such that

$$b_n = h_{n+1} \circ \tilde{b}_n \circ h_n^{-1} \quad \text{for all } n.$$

Since $\mathrm{Crit}(\tilde{b}_n) = h_n(\mathrm{Crit}(b_n))$ for all n and since the family of K-quasiconformal maps is equicontinuous on $\mathbb{D}$ with respect to the hyperbolic metric (see, for example, Theorem 4.4.1 in [19]), it follows from Theorem 4.4 (c) that $(\tilde{b}_n)$ is uniformly hyperbolic. ∎

We conclude the section with the following theorem, which plays a crucial role in the proof of Theorem D. Roughly speaking, it tells us that any two sequences of uniformly hyperbolic Blaschke products are K-quasiconformally equivalent near the boundary of the unit disk, or more precisely, in the sequence of annuli $\mathbb{D}^{(n)} \setminus (\mathbb{D}_r \cup A_n)$, where the constant K is independent of n. In the statement and proof of the following theorem, we again use the notations $\mathbb{D}^{(n)}$ and $\widetilde{\mathbb{D}}^{(n)}$ to denote different copies of $\mathbb{D}$.

Theorem 4.6 (K-quasiconformal equivalence of uniformly hyperbolic Blaschke products). *Let $(b_n\colon \mathbb{D}^{(n)} \to \mathbb{D}^{(n+1)})$ and $(\tilde{b}_n\colon \widetilde{\mathbb{D}}^{(n)} \to \widetilde{\mathbb{D}}^{(n+1)})$ be two sequences of uniformly hyperbolic Blaschke products with* $\deg(b_n) = \deg(\tilde{b}_n) =: d_n$. *Then there exist $r \in (0,1)$, $K \geq 1$ and a sequence of K-quasiconformal maps $h_n\colon \mathbb{D}^{(n)} \to \widetilde{\mathbb{D}}^{(n)}$ such that, for each $n \in \mathbb{N}$,*

(a) *$h_n(z) = z$ for $|z| \leq r$;*

(b) *$b_n = h_{n+1}^{-1} \circ \tilde{b}_n \circ h_n$ on the annulus $C_n := \mathbb{D}^{(n)} \setminus b_n^{-1}(\mathbb{D}_r)$.*

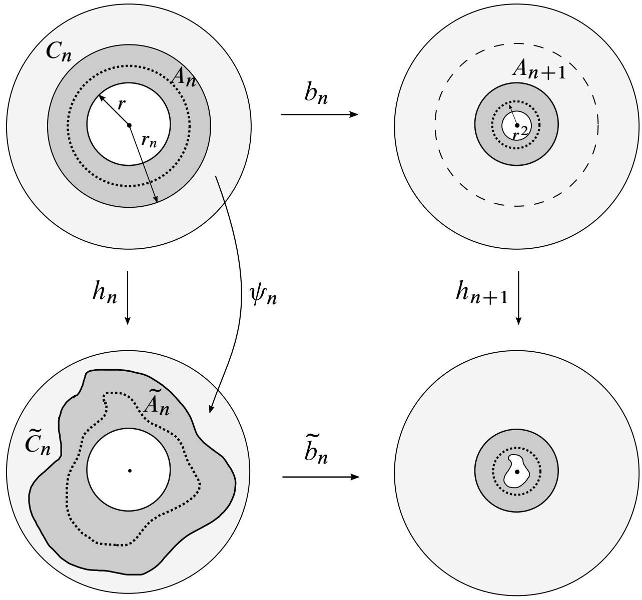

Figure 2. Illustration of the sets and maps involved in the proof of Theorem 4.6. The annulus A_n and its images under h_n and $\tilde{b}_n$ are depicted in dark gray, while C_n and its images are shown in light gray. The map ψ_n sends $\mathbb{A}_{r^{2/d_n}}$, whose inner boundary is drawn with dotted lines, conformally to $\tilde{b}_n^{-1}(\mathbb{A}_{r^2})$.

Proof. It suffices to show that the theorem holds when the maps b_n are of the form $b_n(z) = z^{d_n}$, with $2 \leq d_n \leq \max_n d_n =: d < \infty$. Let us assume (b_n) is of such form, and let $(\tilde{b}_n)$ be any other sequence of uniformly hyperbolic Blaschke products as in the statement. For an illustration of the construction in the following proof, see Figure 2.

By Lemma 4.1, there exists $r \in (0, 1)$ such that all zeros, critical points, and critical values of all b_n and $\tilde{b}_n$ are contained in $\mathbb{D}_{r^2}$. Note that the preimage of the circle $\mathbb{S}_r$ under b_n is the circle $\mathbb{S}_{r_n}$ with radius $r_n := r^{1/d_n}$, and similarly, the preimage of the circle $\mathbb{S}_{r^2}$ under b_n is the circle $\mathbb{S}_{r_n^2}$. We then have the uniform bounds

$$0 < r \leq r^{1/2} \leq r_n \leq r^{1/d} < 1. \tag{4.4}$$

With our assumption on the form of b_n, we have that for all $n \geq 0$,

$$A_n := b_n^{-1}(\mathbb{D}_r) \setminus \mathbb{D}_r = \mathbb{S}_r \cup \mathbb{A}_{r,r_n} \quad \text{and} \quad C_n = \mathbb{A}_{r_n} \cup \mathbb{S}_{r_n}.$$

Likewise, let

$$\widetilde{A}_n := \tilde{b}_n^{-1}(\mathbb{D}_r) \setminus \mathbb{D}_r \quad \text{and} \quad \widetilde{C}_n := \widetilde{\mathbb{D}}^{(n)} \setminus \tilde{b}_n^{-1}(\mathbb{D}_r),$$

and observe that with this notation, the domain and codomain of h_n can be written as the unions

$$\mathbb{D}^{(n)} = \overline{\mathbb{D}}_r \cup \operatorname{int}(A_n) \cup \mathbb{S}_{r_n} \cup \operatorname{int}(C_n) \quad \text{and} \quad \widetilde{\mathbb{D}}^{(n)} = \overline{\mathbb{D}}_r \cup \operatorname{int}(\widetilde{A}_n) \cup \tilde{b}_n^{-1}(\mathbb{S}_r) \cup \operatorname{int}(\widetilde{C}_n).$$

Then, for all $n \geq 0$, we define the restriction of h_n to each of those subsets of $\mathbb{D}^{(n)}$ as follows:

(1) $h_n : \overline{\mathbb{D}}_r \to \overline{\mathbb{D}}_r$ is defined as the identity.

(2) $h_n : \mathbb{S}_{r_n} \to \tilde{b}_n^{-1}(\mathbb{S}_r)$ is defined as one of the d_n maps satisfying the functional equation

$$\tilde{b}_n \circ h_n(z) = h_{n+1} \circ b_n(z) \, (= b_n(z)) \,. \tag{4.5}$$

The procedure to define such a map is standard: if we choose the image of one point, say $h_n(r_n)$, to be one of the d points in $\tilde{b}_n^{-1}(b_n(r_n))$, the map h_n is defined in the whole of $\mathbb{S}_{r_n}$ by lifting (4.5).

(3) $h_n : \mathrm{int}(A_n) \to \mathrm{int}(\widetilde{A}_n)$ is a K-quasiconformal map defined by interpolation between its boundary maps defined in (1) and (2), $h_n|_{\mathbb{S}_{r_n}}$ and $h_n|_{\mathbb{S}_r}$, using Theorem 3.4.

Claim. *The constant K can be chosen independent of n.*

Proof of Claim. To see this, we must check that the constants H, K_0, and M appearing in the assumptions of Theorem 3.4 can be chosen uniformly, i.e., independent of n. From the uniform bounds (4.4), the annuli $\mathbb{A}_{r,r_n}$ have moduli uniformly bounded above and below, and by Lemma 4.1, the same is true for the annuli $\widetilde{A}_n$. It remains to show that the inner and outer boundary maps are H-quasisymmetric, with a constant H independent of n. For the inner boundary maps, the identity, this is trivially true, so we only need to establish it for the outer boundary maps. To do so, we will show that $h_n|_{\mathbb{S}_{r_n}}$ is the restriction of a conformal map defined on an annular neighborhood of $\mathbb{S}_{r_n}$, and apply Corollary 3.3.

Observe that by our choice of r,

$$b_n : b_n^{-1}(\mathbb{A}_{r^2}) \to \mathbb{A}_{r^2} \quad \text{and} \quad \tilde{b}_n : \tilde{b}_n^{-1}(\mathbb{A}_{r^2}) \to \mathbb{A}_{r^2}$$

are analytic d_n-to-1 covering maps. Hence, the branch of $\psi_n := \tilde{b}_n^{-1} \circ b_n$ which agrees with the map h_n on $\mathbb{S}_{r_n}$, as chosen in (2), is a conformal map from $\mathbb{A}_{r_n^2}$ to $\tilde{b}_n^{-1}(\mathbb{A}_{r^2})$, with $\psi_n|_{\mathbb{S}_{r_n}} = h_n$. Again using the uniform bounds (4.4), we see that choosing $t = r^{1/d} \in (0,1)$, we have

$$r_n^2 \leq t \cdot r_n \quad \text{and} \quad \frac{r_n}{t} \leq 1.$$

Hence, $\mathbb{A}_{t r_n, r_n/t} \subset \mathbb{A}_{r_n^2}$, which is the domain of ψ_n. From Corollary 3.3 we conclude that $h_n = \psi_n|_{\mathbb{S}_{r_n}}$ is H-quasisymmetric, with H depending only on $t = r^{1/d}$, which in turn is independent of n. △

(4) $h_n : \mathrm{int}(C_n) \to \mathrm{int}(\widetilde{C}_n)$ is defined iteratively, using that h_n has been defined in $\overline{\mathbb{D}}_{r_n}$ for all n: notice that for every $n \geq 0$, the maps

$$b_n : b_n^{-1}(A_{n+1}) \to A_{n+1} \quad \text{and} \quad \tilde{b}_n : \tilde{b}_n^{-1}(\widetilde{A}_{n+1}) \to \widetilde{A}_{n+1}$$

are covering maps of degree d_n between annuli (belonging to $\mathbb{D}^{(n)}$ and $\mathbb{D}^{(n+1)}$, $\widetilde{\mathbb{D}}^{(n)}$ and $\widetilde{\mathbb{D}}^{(n+1)}$, respectively). Hence, the K-quasiconformal map $h_{n+1} : A_{n+1} \to \widetilde{A}_{n+1}$ can be *lifted* to a K-quasiconformal map $h_n : b_n^{-1}(A_{n+1}) \to \tilde{b}_n^{-1}(\widetilde{A}_{n+1})$, so that the functional equation

$$\tilde{b}_n \circ h_n(z) = h_{n+1} \circ b_n(z), \tag{4.6}$$

is satisfied for every $n \geq 0$.

Once this is done, we proceed in the same way to extend h_n to the next sequence of annuli $b_n^{-1}(b_{n+1}^{-1}(A_{n+2})) \subset \mathbb{D}^{(n)}$. It follows from Lemma 4.3 (3) that in $\mathbb{D}^{(n)}$, these annuli are disjoint from the previous ones. Iterating this process infinitely many times, and since all points in C_n are eventually mapped to A_j for some $j \geq n$, we extend h_n to the whole of C_n for every n.

We have now defined maps $h_n : \mathbb{D}^{(n)} \to \widetilde{\mathbb{D}}^{(n)}$, continuous by construction, K-quasiconformal for some uniform K by (3), and so that (a) and (b) in the statement of the theorem hold by (1) and (4), respectively, which concludes the proof. ∎

5. Uniform hyperbolicity in wandering domains

In Section 4, we introduced the notion of uniform hyperbolicity for sequences of Blaschke products. In this section, we define this concept for the restriction of an entire map to an orbit of wandering domains, discuss the relation between the two definitions, as well as with the notion of strong contraction, and prove Theorem D and Theorem E.

Definition 5.1 (Admissible orbit of wandering domains). Let f be an entire function with an orbit of wandering domains (U_n). We say that this orbit is *admissible* if each U_n is simply connected, and the degrees $d_n = \deg f|_{U_n}$ satisfy $2 \leq d_n \leq d < \infty$ for some $d \in \mathbb{N}$ and all n.

Note that any entire function with an admissible orbit of wandering domains (U_n) has an *associated sequence* of finite Blaschke products $(b_n := \varphi_{n+1} \circ f \circ \varphi_n^{-1})$, which can be chosen to fix 0 by considering suitable Riemann maps (φ_n).

Definition 5.2 (Uniform hyperbolicity in wandering domains). Let f be an entire function with an admissible orbit of wandering domains (U_n). We say that $(f|_{U_n})$ is *uniformly hyperbolic* if there exist an orbit $(z_n = f^n(z_0))$ and a constant C such that $d_{U_n}(z_n, c) \leq C$ for all $c \in \operatorname{Crit}(f|_{U_n})$ and all $n \geq 0$.

As a direct consequence of Theorem 4.4 and the invariance of the hyperbolic metric under conformal maps, we have the following.

Theorem 5.3 (Equivalence of definitions). *Let f be an entire function with an admissible orbit of wandering domains (U_n). Let (b_n) be a sequence of Blaschke products fixing zero associated with $(f|_{U_n})$. Then $(f|_{U_n})$ is uniformly hyperbolic if and only if (b_n) is uniformly hyperbolic. Furthermore, if $(f|_{U_n})$ is uniformly hyperbolic, and if (z_n) is any orbit in (U_n), then there exists a constant C such that $d_{U_n}(z_n, c) \leq C$ for all $c \in \operatorname{Crit}(f|_{U_n})$ and all $n \geq 0$.*

One possible classification of orbits of wandering domains, (U_n), is proven in [4], in terms of the sequence of hyperbolic distances $d_{U_n}(z_n, z_n')$ between points in different orbits (z_n) (z_n'), giving rise to three types: *contracting*, *semi-contracting* and *eventually isometric*. One can give an even finer classification in the contracting case according to the rate of contraction with which iterates cluster together. More precisely, we say that $(f|_{U_n})$ is *strongly contracting* if there exists $c \in (0,1)$ such that

$$(5.1)\quad d_{U_n}(f^n(z), f^n(z')) = O(c^n) \quad \text{for one pair (and hence for all pairs) } z, z' \in U_0,$$

see Definition 1.2 in [4]. The following proposition shows that uniform hyperbolicity is a strictly stronger condition than strong contraction.

Proposition 5.4 (Uniform hyperbolicity implies strong contraction). *Let f be an entire function with an admissible orbit of wandering domains (U_n). If $(f|_{U_n})$ is uniformly hyperbolic, then $(f|_{U_n})$ is strongly contracting. The converse is in general not true, i.e., there exist entire functions f such that $(f|_{U_n})$ is strongly contracting, but not uniformly hyperbolic.*

Proof. It is proven in Theorem B and Lemma 3.1 (b) of [4] that strong contraction is equivalent to requiring that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |b_k'(0)| < 1, \tag{5.2}$$

for any (and hence all) Blaschke sequences fixing 0 associated to (U_n). If we assume that $(f|_{U_n})$ is uniformly hyperbolic, then it follows from Lemma 4.3 and Theorem 5.3 that for any associated sequence of Blaschke products (b_n) that fix 0, there exists $r < 1$ such that $|b_n'(0)| \le r$ for all n, which implies that the average in (5.2) is smaller than $r < 1$.

In order to construct an example for which the reverse implication fails, consider first the sequence of Blaschke products

$$b_n(z) := \begin{cases} z^2, & \text{for } n = 2k-1, k \ge 1, \\ z\,\frac{z+a_n}{1+a_n z}, & \text{for } n = 2k, k \ge 1, \end{cases}$$

where $a_n \in (0,1)$, $a_2 = 1/2$, and $a_n \to 1$ for $n \to \infty$. We have that

$$b_n(0) = 0, \quad b_{2k-1}'(0) = 0 \quad \text{and} \quad b_{2k}'(0) = a_{2k} \to 1.$$

Let c_n be the critical point of b_n, and note that $c_{2k} \to -1$ as $k \to \infty$. Now let $D_n := \{z : |z-4n| < 1\}$ and let $f_n\colon D_n \to D_{n+1}$ be defined by $f_n = T_{n+1}\circ b_n \circ T_n^{-1}$, where $T_n(z) = z + 4n$. Finally, let $B_n = b_n \circ\cdots\circ b_1$ and $F_n = f_n\circ\cdots\circ f_1$. We apply Theorem 5.3 in [4] to obtain a transcendental entire function f with an orbit of simply connected wandering domains (U_n) such that the following hold:

(1) $f(4n) = 4(n+1)$, $f^n(1/2) = F_n(1/2)$, $f(C_n) = f_n(C_n)$, and $f'(C_n) = f_n'(C_n) = 0$ for $n \ge 1$, where $C_n = c_n + 4n$.

(2) $d_{U_n}(f^n(0), f^n(1/2)) < K_n \cdot d_{D_n}(F_n(0), F_n(1/2))$ for some $K_n > 1$ with $K_n \to 1$ as $n \to \infty$.

(3) U_n is asymptotically equal to D_n (as $n \to \infty$).

Since $b_n(0) = 0$ and $b_{2k} = z^2$, it follows by the Schwarz lemma that

$$\operatorname{dist}(F_n(0), F_n(1/2)) = \operatorname{dist}(0, F_n(1/2)) < \frac{1}{2^n}.$$

Since $d_{\mathbb{D}}(0,z) = \log(1+|z|)/(1-|z|)$ we deduce that there exists $1 < c < 2$ such that

$$d_{D_n}(0, F_n(1/2)) = d_{\mathbb{D}}(0, B_n(1/2)) < \frac{1}{c^n},$$

for n sufficiently large. It now follows by (3), together with (5.1), that $(f|_{U_n})$ is strongly contracting. Moreover, by (1), $f(C_n) = f_n(C_n)$. Since $c_{2k} \to -1$ as $k \to \infty$, we deduce by (3) that $C_{2k} \to \partial U_{2k}$. Also, since $f(4n) = 4(n+1)$ (again by (1)), we have that $d_{U_{2k}}(8k, C_{2k}) \to \infty$. Since C_{2k} is the critical point of f in U_{2k}, we deduce by Theorem 5.3 that $(f|_{U_n})$ is not uniformly hyperbolic. ∎

Observation 5.5 (Wandering domains arising as lifts). A well-known method for obtaining wandering domains is by *lifting* invariant components of a self-map F of $\mathbb{C} \setminus \{0\}$ by the exponential map. Namely, for any such F, there is a (non-unique) transcendental entire function f such that $\exp \circ f = F \circ \exp$; see p. 3 of [4] for more details. Suppose that F has an invariant Fatou component Δ and f has an orbit of wandering domains (U_n), so that $\exp(U_n) = \Delta$ for all n. If Δ is a (super)attracting basin, then $(f|_{U_n})$ is uniformly hyperbolic, whereas if Δ is a parabolic basin, $(f|_{U_n})$ is not uniformly hyperbolic, as it is not strongly contracting; see Corollary 3.5 in [4]. If Δ is a Siegel disk or a univalent Baker domain, $(f|_{U_n})$ is not uniformly hyperbolic either, since $\deg f|_{U_n} = 1$ for all n. Finally, if Δ is a Baker domain where F has finite degree greater than one, then Δ has finitely many critical points c_j, $j = 1, \ldots, n$, and since for any $z \in \Delta$, $F^n(z) \to \infty$, we have $d_\Delta(F^n(z), c_j) \to \infty$ for all j as $n \to \infty$. Since the hyperbolic metric is conformally invariant, in particular under the inverse branches of exp, it follows by Theorem 5.3 that $(f|_{U_n})$ is not uniformly hyperbolic.

We are now ready to prove Theorem D, using the tools developed in the previous sections. Given an orbit of wandering domains $(\widetilde{U}_n)$ such that $(f|_{\widetilde{U}_n})$ is uniformly hyperbolic and a sequence of prescribed uniformly hyperbolic Blaschke products (b_n), we will perform surgery to "change" the map f in $(\widetilde{U}_n)$, so that (b_n) becomes an associated Blaschke sequence of the new wandering orbit.

Proof of Theorem D. Let $(\tilde{b}_n)$ be a sequence of Blaschke products fixing zero associated with $(f|_{\widetilde{U}_n})$. In particular, each $\tilde{b}_n$ is of degree $d_n := \deg(f|_{\widetilde{U}_n}) = \deg(b_n)$ and by Theorem 5.3, $(\tilde{b}_n)$ is uniformly hyperbolic. Denote by $\varphi_n \colon \widetilde{U}_n \to \mathbb{D}$ the Riemann maps such that

$$\tilde{b}_n \circ \varphi_n = \varphi_{n+1} \circ f, \qquad \text{for } n \geq 0.$$

By Theorem 4.6, applied with the roles of $(\tilde{b}_n)$ and (b_n) reversed, there exist $K \geq 1$, $0 < r < 1$ and a sequence of K-quasiconformal maps $h_n \colon \mathbb{D} \to \mathbb{D}$ such that h_n is the identity on $\overline{\mathbb{D}}_r$ and

$$\tilde{b}_n = h_{n+1}^{-1} \circ b_n \circ h_n \quad \text{on} \quad C_n := \mathbb{D} \setminus \tilde{b}_n^{-1}(\mathbb{D}_r).$$

Note that if $A_n := \tilde{b}_n^{-1}(\mathbb{D}_r) \setminus \mathbb{D}_r$, then $\mathbb{D} = \mathbb{D}_r \cup A_n \cup C_n$ for each n. Define a new sequence of maps

$$\begin{aligned} g_n : \quad \mathbb{D} &\longrightarrow \mathbb{D}, \\ z &\longmapsto h_{n+1}^{-1} \circ b_n \circ h_n(z). \end{aligned}$$

By construction, g_n is K^2-quasiregular on $\mathbb{D}$ but furthermore, for each $n \geq 0$,

$$g_n = b_n \quad \text{on } \mathbb{D}_r \quad \text{and} \quad g_n = \tilde{b}_n \quad \text{on } C_n.$$

Moreover, the sequences (g_n) and (b_n) are quasiconformally equivalent on $\mathbb{D}$.

Using the sequence (g_n), we modify the initial map f by defining

$$F(z) := \begin{cases} f(z), & \text{if } z \notin \bigcup_{n\geq 0} \widetilde{U}_n, \\ \varphi_{n+1}^{-1} \circ g_n \circ \varphi_n(z), & \text{if } z \in \widetilde{U}_n. \end{cases}$$

Notice that continuity of F is ensured by construction, since $g_n(z) = \tilde{b}_n(z)$ for all $z \in C_n$, which implies that $f(z) = F(z)$ on a neighborhood of $\partial\widetilde{U}_n$.

Letting $\psi_n := h_n \circ \varphi_n$ for $n \geq 0$, the following diagram commutes:

$$\begin{tikzcd} \cdots \arrow[r, "F"] & \widetilde{U}_n \arrow[r, "F"] \arrow[d, "\psi_n"] & \widetilde{U}_{n+1} \arrow[r, "F"] \arrow[d, "\psi_{n+1}"] & \cdots \\ \cdots \arrow[r, "b_{n-1}"] & \mathbb{D} \arrow[r, "b_n"] & \mathbb{D} \arrow[r, "b_{n+1}"] & \cdots \end{tikzcd}$$

Note that F is K^2-quasiregular, and holomorphic outside the topological annuli

$$\mathcal{A}_n := \varphi_n^{-1}(A_n) = \varphi_n^{-1}(g_n^{-1}(\mathbb{D}_r) \setminus \mathbb{D}_r), \quad n \geq 0.$$

Moreover, note that for every $k \geq 0$ and $z \in U_k$, the iterates of F satisfy

$$F^n(z) = \psi_{n+k}^{-1} \circ b_{n-1+k} \circ \cdots \circ b_k \circ \psi_k(z), \quad n \in \mathbb{N},$$

and hence they are uniformly K^2-quasiregular.

This is enough to conclude, by Sullivan's straightening theorem, see Theorem 5.6 in [9], that there exists an entire function g quasiconformally conjugate to F under a global quasiconformal homeomorphism $\Phi: \mathbb{C} \to \mathbb{C}$, exhibiting an orbit of wandering domains (U_n) with $U_n = \Phi(\widetilde{U}_n)$. Nevertheless, in order to show that (b_n) is an associated Blaschke sequence of g in (U_n), we shall look more closely into the construction of Φ.

We can define a complex structure μ, which has bounded dilatation and is F^*-invariant. For details on the following kind of construction, see Section 1.5 of [9]. Let $U := \bigcup_{n\geq 0} \widetilde{U}_n$ and define

$$\mu := \begin{cases} \psi_n^*(\mu_0) & \text{on } \widetilde{U}_n \text{ for every } n \geq 0, \\ (F^m)^*\mu & \text{on } F^{-m}(U) \setminus F^{-m+1}(U) \text{ for every } m \geq 1, \\ \mu_0 & \text{on } \mathbb{C} \setminus \bigcup_{m\geq 0} F^{-m}(U), \end{cases}$$

where $\mu_0 \equiv 0$. It is easy to check that

(1) μ is F-invariant on U by definition of F, and hence $F^*\mu = \mu$ by construction;

(2) $\mu = \mu_0$ on $\varphi_n^{-1}(\mathbb{D}_r)$ for all n, since $\psi_n = \varphi_n$ is conformal on this domain;

(3) μ has dilatation at most K in $\widetilde{U}_n$ for any $n \geq 0$, since the maps ψ_n are uniformly K-quasiconformal. Therefore μ has dilatation at most K in the whole complex plane, since all the pullbacks under F are made by holomorphic maps.

We can then apply the measurable Riemann mapping theorem (see, e.g., Theorem 5.3.2 in [2] or Theorem 1.28 in [9]) to obtain a quasiconformal homeomorphism $\Phi: \mathbb{C} \to \mathbb{C}$ such that $\Phi^*\mu_0 = \mu$ and an entire map $g(z) := \Phi \circ F \circ \Phi^{-1}(z)$ such that $U_n := \Phi(\widetilde{U}_n)$, $n \geq 0$, is an orbit of wandering domains.

Now consider the maps

$$\theta_n := \psi_n \circ \Phi^{-1} : U_n \to \mathbb{D},$$

which are K-quasiconformal. Then, the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "g"] & U_n \arrow[r, "g"] & U_{n+1} \arrow[r, "g"] & \cdots \\
\cdots \arrow[r, "F"] & \widetilde{U}_n \arrow[r, "F"] \arrow[u, "\Phi"] \arrow[d, "\psi_n"] & \widetilde{U}_{n+1} \arrow[r, "F"] \arrow[u, "\Phi"] \arrow[d, "\psi_{n+1}"] & \cdots \\
\cdots \arrow[r, "b_{n-1}"] & \mathbb{D} \arrow[r, "b_n"] & \mathbb{D} \arrow[r, "b_{n+1}"] & \cdots
\end{tikzcd}$$

Moreover,

$$\theta_n^*(\mu_0) = (\Phi^{-1})^*((\psi_n)^*(\mu_0)) = (\Phi^{-1})^*\mu = \mu_0.$$

Hence, by Weyl's lemma, θ_n is conformal for all n and the proof is complete. ∎

We would like to use Theorem D to prove Theorem E, which claims the existence of entire maps with an orbit of wandering domains with a prescribed uniformly hyperbolic Blaschke sequence. In view of Theorem D, given such a Blaschke sequence (b_n) with uniformly bounded degrees (d_n), we only need to construct an entire function f such that $(f|_{\widetilde{U}_n})$ is uniformly hyperbolic in an orbit $(\widetilde{U}_n)$ of wandering domains with matching degrees, i.e. such that, $\deg(f|_{\widetilde{U}_n}) = d_n$ for all n. The function f is provided by Theorem 5.3 in [4], as shown in the following lemma.

Lemma 5.6 (Existence of uniformly hyperbolic $(f|_{\widetilde{U}_n})$ with prescribed degrees). *Let (d_n) be a sequence of natural numbers with $2 \le d_n \le d$ for some $d \in \mathbb{N}$ and all $n \in \mathbb{N}$. Then there exists an entire function f with an orbit of simply connected wandering domains $(\widetilde{U}_n)$ such that $(f|_{\widetilde{U}_n})$ is uniformly hyperbolic and* $\deg(f|_{\widetilde{U}_n}) = d_n$ *for all n.*

Proof. Let (b_n) be a sequence of uniformly hyperbolic Blaschke products of the form (4.1) with $\deg(b_n) = d_n$, such that b_n has $d_n - 1$ critical points, say c_n^k, $1 \le k \le d_n - 1$, of multiplicity 1 in $\mathbb{D}$. Note that such a sequence always exists, see, for example, Theorem 1 in [27]. It follows from Theorem 4.4(b) that there exists $s \in (0, 1)$ such that $c_n^k \in \mathbb{D}_s$ for all n and k.

Now let $D_n := \{z : |z - 4n| < 1\}$ and let $f_n \colon D_n \to D_{n+1}$ be defined by $f_n = T_{n+1} \circ b_n \circ T_n^{-1}$, where $T_n(z) = z + 4n$. We apply Theorem 5.3 in [4] to obtain a transcendental entire function f with an orbit of simply connected wandering domains $(\widetilde{U}_n)$ such that the following hold:

(1) $f(4n) = 4(n+1)$, and $f'(C_n^k) = 0$ for $n \in \mathbb{N}$, $1 \le k \le d_n - 1$, where $C_n^k = c_n^k + 4n$.

(2) $\deg(f|_{\widetilde{U}_n}) = d_n$ for all $n \in \mathbb{N}$.

(3) $\widetilde{U}_n$ is asymptotically equal to D_n (as $n \to \infty$).

Note that by (1) and (2), $\operatorname{Crit}(f|_{\widetilde{U}_n}) = \bigcup_k C_n^k$ for every n. Since $c_n^k \in \mathbb{D}_s$ for all n, k we deduce that $C_n^k \in D_{n,s} := \{z : |z - 4n| < s\}$. It now follows by (3) that there exists $C > 0$ such that $d_{\widetilde{U}_n}(4n, C_n^k) \le C$. We conclude that $(f|_{\widetilde{U}_n})$ is uniformly hyperbolic. ∎

Proof of Theorem E. Let (b_n) be a given sequence of uniformly hyperbolic Blaschke products, and let $d_n := \deg(b_n)$ for each n. Finally, let f be the entire map with wandering domains $(\widetilde{U}_n)$ provided by Lemma 5.6, such that $(f|_{\widetilde{U}_n})$ is uniformly hyperbolic and $\deg(f|_{\widetilde{U}_n}) = d_n$. Applying Theorem D to f and (b_n), we obtain an entire function with the desired properties. ∎

6. Grand orbit relations

In this section, we discuss properties of grand orbit relations and prove Theorem A. We start by giving several characterizations of $\widehat{J}(f)$. Let f be an entire map, and define

$$\widehat{S}(f) := \overline{\mathrm{GO}(S(f))}.$$

Recall from the introduction that $\widehat{J}(f)$ denotes the closure of the grand orbits of its singular values and periodic points. Note that $J(f) \subset \widehat{S}(f)$, and that any (super)attracting cycle also belongs to $\widehat{S}(f)$, since it must be contained in the accumulation set of the orbit of some singular value. Thus, we have that

$$\widehat{J}(f) = \widehat{S}(f) \cup \overline{\mathrm{GO}(\mathrm{Per}(f))} = \overline{\mathrm{GO}(S(f) \cup \{\text{Siegel periodic points}\})}. \tag{6.1}$$

For the rest of the section, we fix an entire function f and write

$$\widehat{J} := \widehat{J}(f), \quad \text{and} \quad \widehat{S} := \widehat{S}(f).$$

Before proceeding, we need some basic facts on holomorphic motions.

Definition 6.1. Let $U \subsetneq \mathbb{C}$ be a simply connected domain, $\lambda_0 \in U$, and $E \subset \widehat{\mathbb{C}}$ be an arbitrary set. A *holomorphic motion* of E over (U, λ_0) is a function $H : U \times E \to \widehat{\mathbb{C}}$ such that

(a) for any $z \in E$, the map $\lambda \mapsto H(\lambda, z)$ is holomorphic in U;

(b) for any $\lambda \in U$, the map $z \mapsto H(\lambda, z)$ is injective in E;

(c) $H(\lambda_0, z) = z$ for all $z \in E$.

Remark. Typically, holomorphic motions are defined over $\mathbb{D}$ with basepoint $\lambda_0 = 0$. However, composition (in the λ-variable) with a Riemann map of U onto the unit disk shows that the theory with our slightly more general definition is completely equivalent to the one in the literature.

Mañé, Sad, and Sullivan introduced the concept of holomorphic motions and proved the following remarkable λ-*lemma*:

Theorem 6.2 (Theorem 12.1.2 in [2]). *Let H be a holomorphic motion of a set E over (U, λ_0). Then H has a unique extension to a holomorphic motion of the closure $\overline{E}$ (taken in $\widehat{\mathbb{C}}$) such that*

(a) *$(\lambda, z) \to H(\lambda, z)$ is jointly continuous in $U \times \overline{E}$, and*

(b) *for each $\lambda \in U$, the map $z \mapsto H(\lambda, z)$ is a quasisymmetric embedding of $\overline{E}$ into $\widehat{\mathbb{C}}$.*

Remark. Złodkowski later proved a stronger version of the λ-lemma, which says that H actually extends (non-uniquely) to a holomorphic motion of $\widehat{\mathbb{C}}$. For more background and proofs, see Chapter 12 of [2] and Section 5.2 of [19].

The key fact to proving Theorem A is the following lemma.

Lemma 6.3 (Holomorphic motions of grand orbits). *Let $U \subset \mathbb{C} \setminus \widehat{J}$ be a simply connected open set and let $\lambda_0 \in U$. Then there exists a unique holomorphic motion*

$$\begin{array}{rccl} H: & U \times \mathrm{GO}(\lambda_0) & \longrightarrow & \widehat{\mathbb{C}}, \\ & (\lambda, z) & \longmapsto & H(\lambda, z) =: h_z(\lambda), \end{array}$$

such that $H(\lambda, \cdot)$ maps $\mathrm{GO}(\lambda_0)$ to $\mathrm{GO}(\lambda)$, preserving orbit relations, i.e., if $z \in \mathrm{GO}(\lambda_0)$ with $f^n(z) = f^m(\lambda_0)$, then $f^n(H(\lambda, z)) = f^m(\lambda)$.

Proof. If $z \in \mathrm{GO}(\lambda_0)$, we have that $f^n(z) = f^m(\lambda_0)$ for some $n, m \in \mathbb{N}$. Since $z, \lambda_0 \notin \widehat{S}$, we know that $(f^n)'(z) \neq 0$, so there exists a holomorphic branch g of f^{-n} in a neighborhood of $\lambda_m = f^m(\lambda_0)$ such that $g(\lambda_m) = z$. With this branch, we define $h_z = g \circ f^m$, so that h_z is a holomorphic branch of $f^{-n} \circ f^m$, defined in a neighborhood of λ_0, with $h_z(\lambda_0) = z$. Now let γ be any continuous path in U starting at λ_0. Since $U_m = f^m(U)$ is disjoint from $\widehat{S}$, every point in U_m is a regular value for f^n. In particular, this implies that every local holomorphic branch of f^{-n} can be analytically continued along any continuous path in U_m. In particular, the branch g of f^{-n} has an analytic continuation along $\gamma_m = f^m \circ \gamma$, so $h_z = g \circ f^m$ has an analytic continuation along γ. Since U is simply connected, the monodromy theorem then shows that h_z extends to a holomorphic function in U. By the uniqueness theorem, this analytic continuation is the unique holomorphic branch of $f^{-n} \circ f^m$ in U satisfying $h_z(\lambda_0) = z$. It satisfies the functional equation $f^n(h_z(\lambda)) = f^m(\lambda)$ for all $\lambda \in U$, so this construction uniquely defines a function $H(\lambda, z) = h_z(\lambda)$ for $(\lambda, z) \in U \times \mathrm{GO}(\lambda_0)$ with the following properties:

- for each $z \in \mathrm{GO}(\lambda_0)$, the function $\lambda \mapsto H(\lambda, z)$ is holomorphic in U,
- if $f^n(z) = f^m(\lambda_0)$, then $f^n(H(\lambda, z)) = f^m(\lambda)$.

The second assertion implies in particular that for any $\lambda_1 \in U$ we have $H(\lambda_1, \mathrm{GO}(\lambda_0)) \subset \mathrm{GO}(\lambda_1)$. We claim that, in fact, equality holds. In order to show this, let $z \in \mathrm{GO}(\lambda_1)$. Then there exist $m, n \in \mathbb{N}$ with $f^n(z) = f^m(\lambda_1)$. Repeating the argument above with λ_1 instead of λ_0, we obtain a unique holomorphic branch h_z^1 of $f^{-n} \circ f^m$ in U, satisfying $h_z^1(\lambda_1) = z$. Defining $z_0 = h_z^1(\lambda_0) \in \mathrm{GO}(\lambda_0)$, the function h_z^1 is also the unique holomorphic branch of $f^{-n} \circ f^m$ with $h_z^1(\lambda_0) = z_0$, so by construction we have that $z = h_{z_0}(\lambda_1) \in H(\lambda_1, \mathrm{GO}(\lambda_0))$, establishing the opposite inclusion $\mathrm{GO}(\lambda_1) \subset H(\lambda_1, \mathrm{GO}(\lambda_0))$ and hence the equality $H(\lambda_1, \mathrm{GO}(\lambda_0)) = \mathrm{GO}(\lambda_1)$.

We still need to show that the map $H(\lambda, \cdot)$ is injective for every $\lambda \in U$. Suppose to the contrary that $H(\lambda^*, \cdot)$ is not injective for some $\lambda^* \in U$. Then there exist distinct $z_1, z_2 \in \mathrm{GO}(\lambda_0)$ with $z^* = h_{z_1}(\lambda^*) = h_{z_2}(\lambda^*)$. By construction, h_{z_1} is a holomorphic branch of $f^{-n_1} \circ f^{m_1}$ for some $m_1, n_1 \in \mathbb{N}$, so that $f^{n_1} \circ h_{z_1} = f^{m_1}$. Similarly, there exist $m_2, n_2 \in \mathbb{N}$ with $f^{n_2} \circ h_{z_2} = f^{m_2}$. Composing either of these functional equations with a suitable iterate of f, we may assume that $m_1 = m_2$, so that

$$f^{n_1}(h_{z_1}(\lambda)) = f^{n_2}(h_{z_2}(\lambda)) = f^m(\lambda) \tag{6.2}$$

for all $\lambda \in U$. Since $h_{z_1}(\lambda_0) = z_1 \neq z_2 = h_{z_2}(\lambda_0)$, we have that λ^* is an isolated solution to the equation $h_{z_1}(\lambda) = h_{z_2}(\lambda)$. We now distinguish two cases.

Suppose $n_1 = n_2 = n$. Then, since $h_{z_1}(\lambda) \neq h_{z_2}(\lambda)$ in a punctured neighborhood of λ^*, it follows that z^* is a solution of multiplicity ≥ 2 of $f^n(z) = f^m(\lambda^*)$ and hence $(f^n)'(z^*) = 0$. Thus $z^* \in \widehat{S}$, which implies that $\lambda^* \in U \cap \widehat{S}$ (because $f^m(\lambda^*) = f^n(z^*)$), a contradiction.

It is enough then to deal with the case $n_1 > n_2$, since the reverse case is symmetric. In this situation, (6.2) with $\lambda = \lambda^*$ implies that $f^{n_2}(z^*)$ is periodic of period $n_1 - n_2$, and hence z^* is preperiodic (maybe periodic, if $n_2 = 0$). Thus, both z^* and λ^* are in $\widehat{J}$, leading to a contradiction. ∎

As a first consequence, we have that either all or none of the grand orbits of points in a component of $\mathbb{C} \setminus \widehat{J}$ are discrete.

Corollary 6.4 (No coexistence of discrete and indiscrete grand orbits). *Let U be a component of $\mathbb{C} \setminus \widehat{J}$. Given any $\lambda_0, \lambda_1 \in U$,* $\mathrm{GO}(\lambda_0)$ *is discrete if and only if* $\mathrm{GO}(\lambda_1)$ *is discrete.*

Proof. Let $\lambda_0, \lambda_1 \in U$ and let $V \subset U$ be an open simply connected set containing λ_0 and λ_1. By Lemma 6.3, there exists a holomorphic motion $H\colon V \times \mathrm{GO}(\lambda_0) \to \mathbb{C}$, and $H(\lambda_1, \mathrm{GO}(\lambda_0)) = \mathrm{GO}(\lambda_1)$. By the λ-lemma, $H(\lambda_1, \cdot)$ is a homeomorphism and hence the image of a discrete set must be a discrete set, which concludes the proof. ∎

In addition, we are able to prove that discreteness of orbits is equivalent to this seemingly stronger notion.

Definition 6.5 (Strongly discrete GO). We say that the grand orbit of a point $z \in \mathbb{C} \setminus \widehat{J}$ is *strongly discrete* if there exists a neighborhood U of z such that $\mathrm{GO}(z') \cap U = \{z'\}$ for every $z' \in U$.

Observation 6.6. The definition of $\mathrm{GO}(z)$ being strongly discrete is independent of the representative of $\mathrm{GO}(z)$ for which such a neighborhood U exists. In other words, the grand orbit of z is strongly discrete if and only if for all $w \in \mathrm{GO}(z)$, there exists a neighborhood U_w of w such that $\mathrm{GO}(w') \cap U_w = \{w'\}$ for every $w' \in U_w$. To see this, suppose that $\mathrm{GO}(z)$ is strongly discrete and let U_z be a neighborhood of z such that

$$\mathrm{GO}(z') \cap U_z = \{z'\} \quad \text{for every } z' \in U_z. \tag{6.3}$$

Assume for the sake of contradiction that there is w such that $f^n(w) = f^m(z)$ for some $n, m \geq 0$, and so that in any neighborhood W of w, there are at least two elements of the same grand orbit. But for W of sufficiently small diameter, we have that $f^m(W) \subset f^n(U_z)$, and by (6.3), $f^n(U_z)$ contains a unique representative of each grand orbit of the points it comprises. This means that $\deg(f^m|_W) \geq 2$ for any neighborhood W of w, which contradicts the fact that since $w \notin \widehat{J}$, f^m is a local homeomorphism in some neighborhood of w.

Proposition 6.7 (Discrete grand orbits are strongly discrete). *Let $z \in \mathbb{C} \setminus \widehat{J}$. The grand orbit of z is discrete if and only if it is strongly discrete.*

Proof. By Observation 6.6, it is immediate that strongly discrete implies discrete.

Now suppose that $\lambda_0 \in \mathbb{C} \setminus \hat{J}$ and $\mathrm{GO}(\lambda_0)$ is discrete but not strongly discrete. Then, since $\mathrm{GO}(\lambda_0)$ is discrete, there exists a simply connected neighborhood U_0 of λ_0 such that $\mathrm{GO}(\lambda_0) \cap U_0 = \{\lambda_0\}$, and two sequences (x_n) and (x'_n) in U_0 such that $x'_n \in \mathrm{GO}(x_n)$ for all $n \geq 0$ and $x_n, x'_n \to \lambda_0$ when $n \to \infty$.

By Lemma 6.3, there exists a holomorphic motion $H(\lambda, z)$, where $H: U_0 \times \mathrm{GO}(\lambda_0) \to \mathbb{C}$. In particular, $H(x_n, \mathrm{GO}(\lambda_0)) = \mathrm{GO}(x_n)$. By the λ-lemma (Theorem 6.2), H extends to a holomorphic motion of the closure.

Hence, for every $n \geq 0$ we may choose $z_n \in \mathrm{GO}(\lambda_0)$ such that

$$H(x_n, z_n) = x'_n.$$

Observe that $z_n \notin U_0$ for any $n \geq 0$, since $\mathrm{GO}(\lambda_0) \cap U_0 = \{\lambda_0\}$. Since $\hat{\mathbb{C}} \setminus U_0$ is compact, there exists a convergent subsequence

$$z_{n_k} \to z_\infty \in \overline{\mathrm{GO}(\lambda_0)} \setminus U_0 \quad \text{as } k \to \infty.$$

But we know that $x_n \to \lambda_0$. Hence, as $k \to \infty$

$$\lambda_0 \longleftarrow x'_{n_k} = H(x_{n_k}, z_{n_k}) \longrightarrow H(\lambda_0, z_\infty) = z_\infty \notin U_0,$$

where the right-hand side follows from H being jointly continuous and $H(\lambda_0, \cdot)$ being the identity. This is a contradiction and hence the Proposition is proven. ■

As a consequence, we obtain the following criterion for *discreteness* of grand orbits.

Corollary 6.8 (Discreteness criterion). *Let $z \in \mathbb{C} \setminus \hat{J}$. The grand orbit of z is discrete if and only if there exists a neighborhood U of z such that $f^n(U) \cap f^m(U) = \emptyset$ for all $n \neq m$ and $f|_{f^n(U)}$ is injective for all $n \geq 0$.*

Proof. If the grand orbit of z is discrete, then it is strongly discrete, and so there exists a neighborhood $U \subset \mathbb{C} \setminus \hat{J}$ of z such that every point is the unique representative of its grand orbit in U. By contradiction, suppose $w \in f^n(U) \cap f^m(U)$, for some $m > n$. This implies there exists $w' \in f^n(U)$ such that $f^{m-n}(w') = w$. Note that $w \neq w'$, since periodic points belong to $\hat{J}$. Hence, w and w' are in the same grand orbit, and so are their (distinct) preimages in U under f^n, a contradiction. Likewise, if $f|_{f^n(U)}$ was not injective for some $n \geq 0$, there would exist two distinct points $w, w' \in f^n(U)$ mapping to the same point in $f^{n+1}(U)$, and hence belonging to the same grand orbit. Taking their preimages under f^n in U leads again to a contradiction.

Now suppose that the disjointness and injectivity conditions are satisfied for a certain neighborhood U, and assume by contradiction that the grand orbit is not discrete. It follows by Proposition 6.7 that the grand orbit is not strongly discrete, i.e., that U contains two points $w, w' \in U$ such that $f^n(w) = f^m(w') =: w^*$ for some $m, n \in \mathbb{N}$. If $n = m$, this violates the injectivity of $f|_{f^k(U)}$ for some $k \leq n - 1$, so it is not possible. If, say, $m > n$, then $w^* \in f^n(U) \cap f^m(U)$, a contradiction. ■

We can now easily deduce Theorem A.

Proof of Theorem A. Let V be the grand orbit of a component of $\widehat{F}(f)$ for an entire map f. Recall that we defined the orbit relation in V to be indiscrete if $\mathrm{GO}(z)$ is an indiscrete set for some $z \in V$. Since, by Corollary 6.4 and by Proposition 6.7, either the grand orbit of all or of none of the points in V is discrete, and being discrete is equivalent to being strongly discrete, the equivalence between (a), (b) and (c) follows. Suppose that, additionally, V is in the grand orbit of a wandering domain U, and let $z \in V$. By passing to an iterate, we may assume that $z \in U$. Let $W := U \cap V$, which contains the point z. Then, since $f^n(U) \cap f^m(U) = \emptyset$ for all $n \neq m$, $f^n(W) \cap f^m(W) = \emptyset$ for all $n \neq m$, and (b) and (d) are equivalent by Corollary 6.8. ∎

In [7], a thorough description of the dynamics of entire maps in multiply connected wandering domains is provided. The combination of the findings in [7] with Theorem A results in the following concise proof of the fact that the orbit relation is always indiscrete in such domains.

Corollary 6.9 (GO relations in multiply connected wandering domains). *Let f be an entire map with a multiply connected wandering domain U. Then, the grand orbit relation in every component of* $\mathrm{GO}(U) \setminus \widehat{S}(f)$ *is indiscrete.*

Proof. Let $\widetilde{U}$ be a connected component of $U \setminus \widehat{S}$, and, for each $n \geq 1$, denote $U_{n+1} := f(U_n)$, with $U_0 = U$. Let $V \subset \widetilde{U}$ be any simply connected domain. Then, by Theorem 1.3 in [7], there exists $N \in \mathbb{N}$ and a sequence of round annuli (C_n) such that $f^n(V) \subset C_n \subset U_n$ for all $n \geq N$. Let $D := f^N(V)$. It follows by Theorem 5.2 in [7] that for every $m \geq 0$, there exists a round annulus $A_m \supset C_N \supset D$ in U_N such that for every $z \in A_m$ and $m \in \mathbb{N}$, $f^m(z) = q_m \cdot (\varphi_m(z))^{d_m}$, where $q_m > 0$, φ_m is a conformal map on A_m such that $|\varphi_m(z)| \to |z|$ as $m \to \infty$ and $d_m \in \mathbb{N}$ is such that $d_m \to \infty$ as $m \to \infty$.

Let us fix any arc, α, of points of the same modulus contained in D. Then, for each m sufficiently large, there exists a curve $\gamma_m \subset D$ such that $\varphi_m(\gamma_m) = \alpha$. In addition, by increasing m if necessary, since $d_m \to \infty$, $f^m(\gamma_m) = q_m(\varphi_m(\gamma_m))^{d_m}$ is a circle traversed at least twice. This implies that $f|_{f^{N+j}(V)}$ is not injective for some $1 \leq j \leq m$, and so by Theorem A, the orbit relation in $\widetilde{U}$ is indiscrete. ∎

To conclude this section and our general discussion on discreteness of grand orbit relations, we show in the following lemma how (grand and small) orbit relations may vary (or be preserved) under conjugation via sequences of homeomorphisms. This is actually all we will need for the proof of the second part of Theorem B.

Lemma 6.10 (Orbit relations under topological equivalence). *Let f and g be entire or rational maps and suppose that $(f_n := f|_{f^n(U)})$ and $(g_n := g|_{g^n(V)})$ are topologically equivalent for some connected domains $U \subset \widehat{\mathbb{C}} \setminus \widehat{J}(f)$ and $V \subset \widehat{\mathbb{C}} \setminus \widehat{J}(g)$. I.e., assume that there exists a sequence $(h_n\colon f^n(U) \to g^n(V))$ of homeomorphisms such that*

$$g_n = h_{n+1} \circ f_n \circ h_n^{-1} \quad \textit{for all } n \geq 0. \tag{6.4}$$

Assume further that the domains $g^n(V)$ are pairwise disjoint. Then, for any $z \in U$,

$$h_0(\mathrm{SO}(z, f) \cap U) = \mathrm{GO}(h_0(z), g) \cap V = \mathrm{SO}(h_0(z), g) \cap V; \tag{6.5}$$

Consequently,

(a) *the* SO *relation for* f *on* U *is discrete if and only if the* GO *relation for* g *on* V *is discrete.*

(b) *the type of* GO *relation for* f *on* U *is the same as that for* g *on* V*, except when* U *is contained in the grand orbit of a rotation domain, in which case the* GO *relation is indiscrete for* f *on* U *and discrete for* g *on* V*.*

Proof. The equalities in (6.5) are immediate from the commutative diagram and the fact that on an orbit of disjoint domains grand orbits and small orbits coincide. Part (a) follows from (6.5) and the fact that the maps h_n are homeomorphisms, thus preserving discreteness and indiscreteness of sets. Part (b) is immediate in the case that the sets $(f^n(U))$ are disjoint (and hence GO and SO coincide). Otherwise, U belongs to a preperiodic Fatou component of f, and we may assume without loss of generality that it is periodic. If U is part of an attracting or parabolic basin, or a Baker domain, then the GO relation on U for f is discrete and hence so is the SO relation. If U is part of a superattracting basin, it is easy to see that both are indiscrete; see Section 6 of [23] or Section 4 of [16] for details. Lastly, if U is part of a rotation domain, the GO relation for f on U is indiscrete while the SO relation is discrete, since f is univalent on U. ∎

7. Coexistence of different orbit relations in wandering domains

The goal of this section is to prove Theorem B and Corollary C. That is, given a polynomial P of degree $d \geq 2$, we wish to construct an entire function g with an orbit of wandering domains (U_n), so that $(g|_{U_n})$ is conformally equivalent to $(P|_{D_n})$ for all n, where (D_n) is an increasing sequence of topological disks bounded by equipotentials. We will then use Lemma 6.10 to transfer the grand orbit relations for P on the domains D_n to those for g on the wandering domains.

We will construct g in two steps. First we define a sequence of uniformly hyperbolic Blaschke products (b_n) reflecting the dynamics of P on D_n (Proposition 7.1). Then we use Theorem D to insert the Blaschke sequence inside a wandering orbit of an appropriately chosen function f (Propositions 7.2).

Given a polynomial P of degree $d \geq 2$, by Böttcher's theorem (see, e.g., Section 9 in [24]), there exists a conformal map φ in a neighborhood of ∞, with $\varphi(\infty) = \infty$, conjugating P to $w \mapsto w^d$, i.e., satisfying the functional equation $\varphi(w^d) = P(\varphi(w))$. Fix $R > 1$ such that the domain of φ contains $\widehat{\mathbb{C}} \setminus \mathbb{D}_R$ and, for each $n \in \mathbb{N}$, define

$$R_n := R^{d^n}, \quad \gamma_n := \varphi(\mathbb{S}_{R_n}) \quad \text{and} \quad D_n := \operatorname{int}(\gamma_n), \tag{7.1}$$

where $\operatorname{int}(\gamma_n)$ denotes the bounded connected component of $\mathbb{C} \setminus \gamma_n$. Note that for all $n \geq 0$ we have that $\overline{D}_n \subset D_{n+1}$, $J(P)$ and all critical points of P are contained in D_n, and the restriction $P|_{D_n} : D_n \to D_{n+1}$ is a proper map of degree d. Note that we can view $(P|_{D_n})$ as a sequence of proper holomorphic maps in the sense of Definition 2.1.

Proposition 7.1 (The prescribed Blaschke sequence). *Let* P *be a polynomial of degree* $d \geq 2$ *and let* (D_n) *be defined as in* (7.1)*. Then there exists a sequence* (b_n) *of uniformly hyperbolic Blaschke products of degree* d *conformally equivalent to the sequence* $(P|_{D_n})$ *under a sequence of conformal maps* $(\psi_n : D_n \to \mathbb{D})$*.*

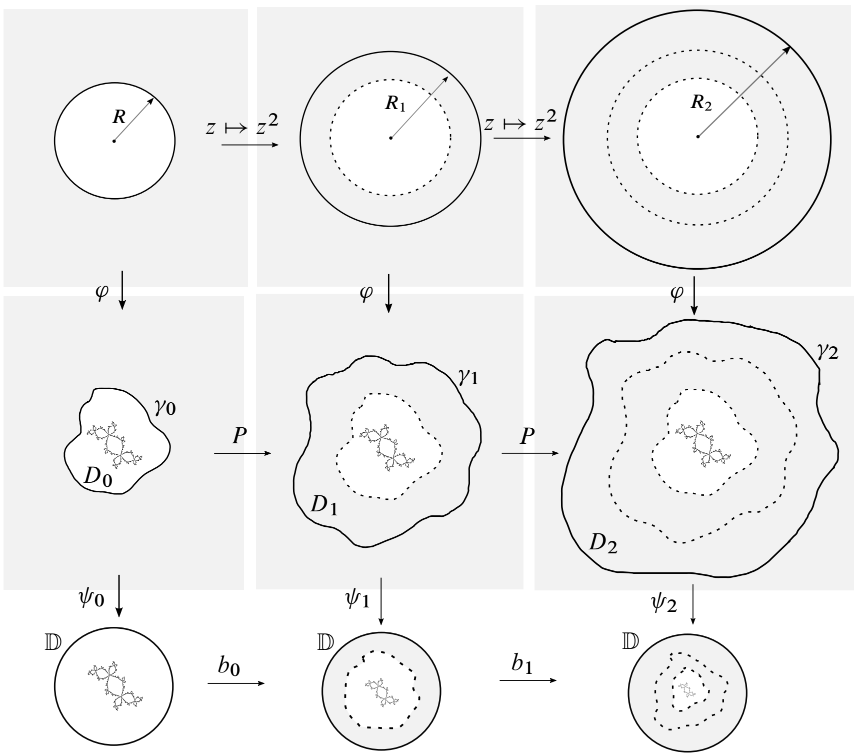

Figure 3. Illustration of the construction of the sets (D_n) and the associated sequence of Blaschke products (b_n) for the polynomial $P(z) := z^2 + c$, with $c = -0.123 + 0.745i$ (Douady's rabbit).

See Figure 3 for an illustration of the construction of the sets D_n and the associated Blaschke sequence (b_n).

Proof. By conjugating with an affine transformation, we may assume that $P(0) = 0$. For each $n \geq 0$, choose a conformal map $\psi_n \colon D_n \to \mathbb{D}$, normalized so that $\psi_n(0) = 0$, and define the degree d Blaschke product

$$b_n(z) := \psi_{n+1} \circ P \circ \psi_n^{-1}(z), \quad \text{for } z \in \mathbb{D}. \tag{7.2}$$

We claim that the sequence (b_n) is uniformly hyperbolic. Indeed, by Theorem 4.4(d), it suffices to show that there exists a constant C such that $d_{\mathbb{D}}(0, c) \leq C$ for all $c \in \operatorname{Crit}(b_n)$ and all n. Recall that $\operatorname{Crit}(P) \subset D_0$ and that $\psi_n(\operatorname{Crit}(P)) = \operatorname{Crit}(b_n)$ for all n. Since $\overline{D}_n \subset D_{n+1}$, we have $\rho_{D_n}(z) > \rho_{D_{n+1}}(z)$ for all $z \in D_n$ by the Schwarz–Pick lemma (see Theorem 10.5 in [3]), while the maps ψ_n preserve hyperbolic distances. In particular, for any $c \in \operatorname{Crit}(P)$,

$$d_{\mathbb{D}}(0, \psi_n(c)) = d_{D_n}(0, c) \geq d_{D_{n+1}}(0, c) = d_{\mathbb{D}}(0, \psi_{n+1}(c)).$$

The claim now follows with $C := \max_{z \in \operatorname{Crit}(b_0)} d_{\mathbb{D}}(0, z)$. ∎

We aim to use Theorem D to insert the sequence of Blaschke products (b_n) obtained in Proposition 7.1 inside an orbit of wandering domains $(\widetilde{U}_n)$ of a function f (the *template* function) such that $\deg(f|_{\widetilde{U}_n}) = d$. Since in this case the sequence $\deg(f|_{\widetilde{U}_n})$ is constant, f may be chosen to be a lift of a self-map of $\mathbb{C} \setminus \{0\}$, as explained in Observation 5.5. Using this procedure and unlike in Lemma 5.6, where f is obtained via approximation techniques, the grand orbit of the singular set $S(f)$ of f is known. The following is an appropriate choice of f, a reformulation of Proposition 4.1 in [13], in a version that fits our purposes.

Proposition 7.2 (The template function). *Let $d \geq 2$. Then there exists an entire function f with no finite asymptotic values and an orbit of simply connected wandering domains $\widehat{U} = (\widetilde{U}_n)_{n\geq 0}$ such that*

(a) *$(f|_{\widetilde{U}_n})$ is uniformly hyperbolic, with $\deg f|_{\widetilde{U}_n} = d$ for all $n \geq 0$;*

(b) *each $\widetilde{U}_n$ contains exactly one critical point c_n of multiplicity $d-1$, and $f(c_n) = c_{n+1}$;*

(c) *there are no critical points in $\mathrm{GO}(\widehat{U}) \setminus \widehat{U}$;*

Proof. Following the proof of Proposition 4.1 in [13], consider the functions

$$q(z) := 2 \cdot \sum_{j=1}^{d-1} \binom{d-1}{j} \cdot \frac{z^j}{j} \quad \text{and} \quad F(z) := -z^2 \cdot \exp(q(z) - c),$$

for $c := q(-1)$. The function F has two superattracting fixed points at -1 and 0, no further critical points, and a single finite asymptotic value at 0. Moreover, the Fatou component W containing -1 is simply connected, and the restriction $F\colon W \to W$ is conjugate to $z \mapsto z^d$ on the unit disk. Since $F^{-1}(0) = \{0\}$, we can lift F under the exponential map to obtain a map f, defined by

$$f(w) := 2w + q(\exp(w)) - c + \pi i.$$

Since the only finite asymptotic value of F is the fixed point at zero, which is an omitted value for the exponential, the lift f does not have any finite asymptotic values (see, e.g., Proposition 3.10 in [15]). For each $k \in \mathbb{Z}$, let W_k be the connected component of $\exp^{-1}(W)$ containing $z_k = (2k-1)\pi i$. Each W_k is a Fatou component of f (see, e.g., [6]), and since W is simply connected with $0 \notin W$, each W_k is simply connected, and is mapped conformally onto W by the exponential function. In particular, this implies that the W_k are all distinct, i.e., $W_k \neq W_l$ for $k \neq l$. Furthermore, since $f(z_k) = z_{2k}$, we also know that $f(W_k) = W_{2k}$, and that $\deg f|_{W_{2k}} = d$, with exactly one critical point z_k of multiplicity $d-1$, mapping to the critical point z_{2k}. This shows that W_0 is a fixed superattracting basin for f, whereas for every odd $m \in \mathbb{Z}$, the sequence $\widehat{W}_m = (W_{2^n m})_{n\geq 0}$ is an orbit of wandering domains, with $\widehat{W}_m \cap \widehat{W}_l = \emptyset$ for $m \neq l$. Since all the W_k are simply connected, and $\deg f|_{W_k} = d \geq 2$, all of these orbits of wandering domains are admissible. Furthermore, since the critical points in each $\widehat{W}_m$ form a single orbit, f is uniformly hyperbolic on each of them. Finally, choosing $\widehat{U} = \widehat{W}_1$, i.e., $U_n = W_{2^n}$ for $n \geq 0$, assertion (c) follows from the fact that all the critical points of f are contained in the disjoint union $\bigcup_{m \text{ odd}} \widehat{W}_m$ of orbits of wandering domains. ∎

We are now ready to prove Theorem B.

Proof of Theorem B. By Proposition 7.1, we know that there exists a sequence of Blaschke products $(b_n)_{n\geq 0}$ conformally equivalent to $(P|_{D_n})_{n\geq 0}$, under a sequence of conformal maps $\psi_n\colon D_n \to \mathbb{D}^{(n)}$. Let f be the entire function from Proposition 7.2 and let $\widetilde{U} = (\widetilde{U}_n)_{n\geq 0}$ be its orbit of wandering domains. By Theorem D, we can replace the dynamics of $(f|_{\widetilde{U}_n})$ by the Blaschke sequence (b_n), obtaining an entire map g with an orbit of wandering domains $\widehat{U} = (U_n)$ such that $(g|_{U_n})$ is conformally equivalent to (b_n) under a sequence of conformal maps (Θ_n) and hence also to $(P|_{D_n})$ under $(h_n := \Theta_n^{-1} \circ \psi_n)$. For an illustration of the construction, see the following commutative diagram:

$$\begin{tikzcd}
U_0 \arrow[r, "g"] \arrow[d, "\Theta_0"] & \cdots \arrow[r, "g"] & U_n \arrow[r, "g"] \arrow[d, "\Theta_n"] & U_{n+1} \arrow[r, "g"] \arrow[d, "\Theta_{n+1}"] & \cdots \\
\mathbb{D}^{(0)} \arrow[r, "b_0"] & \cdots \arrow[r, "b_{n-1}"] & \mathbb{D}^{(n)} \arrow[r, "b_n"] & \mathbb{D}^{(n+1)} \arrow[r, "b_{n+1}"] & \cdots \\
D_0 \arrow[r, "P"] \arrow[u, "\psi_0"'] \arrow[uu, bend left, "h_0"] & \cdots \arrow[r, "P"] & D_n \arrow[r, "P"] \arrow[u, "\psi_n"'] & D_{n+1} \arrow[r, "P"] \arrow[u, "\psi_{n+1}"'] & \cdots
\end{tikzcd}$$

Since f does not have any finite asymptotic values, by Theorem D the map g does not have any finite asymptotic values either. The only critical points of f in $\operatorname{GO}(\widetilde{U})$ are the critical points of multiplicity $d-1$ in each $\widetilde{U}_n$. After the surgery, our new map g still has $d-1$ critical points in each U_n (counted with multiplicity), given by $h_n(\operatorname{Crit}(P))$, and no critical points in $\operatorname{GO}(\widehat{U}) \setminus \widehat{U}$. It follows that

$$\widehat{S}(g) = \overline{\operatorname{GO}(h_n(\operatorname{Crit}(P))}.$$

We can now apply Lemma 6.10 with $U = D_0$ and $V = U_0$ to obtain statement (a) in Theorem B directly from (6.5).

To prove (b), note that our assumption that P is not conjugate to $z \mapsto z^d$ implies that P does not have any finite exceptional points (i.e., points with finite grand orbit), so that the Julia set is the set of accumulation points of the preimages of any point in $\mathbb{C}$. So if $c \in \mathbb{C}$ is a critical point of P, and $z \in J(P)$ is arbitrary, then there exist sequences $n_k \to \infty$ and (z_k) in D_0 with $z_k \to z$, such that $P^{n_k}(z_k) = c$. Using the conformal equivalence, we have that

$$g^{n_k}(h_0(z_k)) = h_{n_k}(P^{n_k}(z_k)) = h_{n_k}(c).$$

Since $h_n(c)$ is a critical point of g for every n, it follows that

$$h_0(z_k) \in \operatorname{GO}(\operatorname{Crit}(g)) \subset \widehat{S}(g).$$

Thus

$$h_0(z) = \lim_{k\to\infty} h_0(z_k) \in \widehat{S}(g) \subset \widehat{J}(g).$$

This proves that $h_0(J(P)) \subset \widehat{J}(g)$.

It remains to show that if V is a Fatou component of P, then $h_0(V)$ contains at least one component of $\widehat{F}(g)$. Define

$$A = \{w \in \mathbb{C} : P^{m+k}(w) = P^k(c) \text{ for some } m,k \geq 0,\ c \in \operatorname{Crit}(P)\} \subset \operatorname{GO}(\operatorname{Crit}(P)) \cap D_0.$$

Then

$$\widehat{J}(g) \cap U_0 = h_0(A)$$

(see (7.3) in the proof of Theorem 7.3). Since $\mathrm{Crit}(P)$ is finite, $\mathrm{GO}(\mathrm{Crit}(P))$ is countable, hence so is A, and therefore $h_0(A)$ is countable. In particular, the open set $h_0(V) \subset U_0$ cannot be contained in $h_0(A)$, so we may choose $w \in h_0(V) \setminus \widehat{J}(g)$. Let W be the connected component of $\widehat{F}(g) = \mathbb{C} \setminus \widehat{J}(g)$ containing w. Moreover, $\partial V \subset J(P)$ and we have already shown that $h_0(J(P)) \subset \widehat{J}(g)$, so

$$\partial(h_0(V)) = h_0(\partial V) \subset \widehat{J}(g).$$

Hence W cannot meet $\partial(h_0(V))$, and therefore $W \subset h_0(V)$. Consequently, $W \subset h_0(V)$, i.e., $h_0(V)$ contains a connected component of $\widehat{F}(g)$. ■

Corollary C now follows readily, from the simplest possible instance of Theorem B.

Proof of Corollary C. Let P be a quadratic polynomial given by $P(z) := \lambda z + z^2$ with $0 < |\lambda| < 1$, so that P has an attracting fixed point at 0 and its Julia set is a quasicircle. Applying Theorem B for this choice of P and a sequence of domains (D_n) as in (7.1), we obtain a corresponding transcendental entire function g with an orbit of simply connected wandering domains (U_n). The set $\widehat{F}(P) \cap D_0$ has two connected components V_1 and V_2, where V_1 is the attracting basin of zero with infinitely many points removed, while V_2 is the intersection of the superattracting basin of infinity with D_0. The grand orbit relation of P is discrete in $\mathrm{GO}(V_1)$ and indiscrete in $\mathrm{GO}(V_2)$, see Section 6 of [23]. By Theorem B, h_0 maps V_1 and V_2 into different components of $\widehat{F}(g)$ say W_1 and W_2 with the same type of GO relations as V_1 and V_2. ■

For the sake of simplicity, we chose to state and prove Corollary C for a quadratic polynomial with an attracting fixed point. The choice of other polynomials, possibly of higher degree, with different combinations of types of Fatou components would lead to the existence of wandering domains that decompose into different collections of components with discrete and indiscrete grand orbit relations, once the closure of the singular set is removed. As a final result in our paper, we show in more detail how the different components of $\widehat{F}(P)$ and $\widehat{F}(g)$ correspond to each other.

Theorem 7.3 (Correspondence between components). *Under the setup in Theorem* B, *the conformal map h_0 induces a map $V \mapsto V^*$ from components of $\widehat{F}(P) \cap D_0$ to components of $\widehat{F}(g) \cap U_0$ such that*

(a) *$V \subset h_0^{-1}(V^*)$ for every component V of $\widehat{F}(P) \cap D_0$,*

(b) *for every component W of $\widehat{F}(g) \cap U_0$, there exists a component V of $\widehat{F}(P) \cap D_0$ with $V^* = W$, and*

(c) *if V_1 and V_2 are different components of $\widehat{F}(P) \cap D_0$ with $V_1^* = V_2^*$, then V_1 and V_2 are contained in the same superattracting basin of P.*

Remark. Item (a) is equivalent to the simple assertion that

$$\widehat{F}(P) \cap D_0 \subset h_0^{-1}(\widehat{F}(g) \cap U_0),$$

and assertions (b) and (c) say in a somewhat more precise way that the map $V \mapsto V^*$ from connected components of $\widehat{F}(P) \cap D_0$ to $\widehat{F}(g) \cap U_0$ induced by inclusion via h_0 is onto, and that it is a one-to-one correspondence away from superattracting basins of P. Finally, observe that the case $P(z) = z^d$ is special because $\widehat{S}(P) = \{0\}$ and does not contain $J(P)$.

In the following proof, we again adopt the notation $\mathbb{D}^{(n)}$ to denote different copies of $\mathbb{D}$.

Proof. Let $S_0 = \mathrm{GO}(S(g)) \cap U_0$. Since f does not have any asymptotic values, and all the critical points in the grand orbit of U_0 are contained in the sequence (U_n), we have that $z_0 \in S_0$ if and only if there exist $k, m, n \geq 0$ and a critical point $c_m \in U_m$ such that

$$g^n(z_0) = g^k(c_m).$$

Since $g^n(z_0) \in U_n$ and $g^k(c_m) \in U_{m+k}$, this is only possible if $n = m + k$ and thus

$$g^{m+k}(z_0) = g^k(c_m).$$

Writing $c = h_m(c_m)$ and $w = h_0^{-1}(z_0)$, this shows that $z_0 = h_0(w) \in S_0$ if and only if there exist $k, m \geq 0$ and $c \in \mathrm{Crit}(P)$ such that $P^{m+k}(w) = P^k(c)$. Since $c \in D_0 \subset D_m$, any solution $w \in \mathbb{C}$ of this equation (not assuming $w \in D_0$ a priori) satisfies

$$w \in P^{-(m+k)}(D_k) = P^{-m}(D_0) \subset D_0,$$

since

$$P^{-1}(D_0) \subset D_0 \quad \text{and} \quad P^{-1}(D_{n+1}) = D_n \quad \text{for all } n \geq 0.$$

This shows that $S_0 = h_0(A)$ for the set

$$\begin{aligned} A &= \{w \in \mathbb{C} : P^{m+k}(w) = P^k(c) \text{ for some } m, k \geq 0 \text{ and } c \in \mathrm{Crit}(P)\} \\ &\subset \mathrm{GO}(\mathrm{Crit}(P)) \cap D_0, \end{aligned}$$

and thus

$$\widehat{J}(g) \cap U_0 = \widehat{S}(g) \cap U_0 = \overline{S_0} = h_0(\overline{A}) \subset h_0(\widehat{J}(P) \cap D_0), \tag{7.3}$$

where the first equality follows from the fact that U_0 does not contain any periodic points of f and the last inclusion follows from the fact that

$$\overline{A} \subset \widehat{S}(P) \subset \widehat{J}(P).$$

Taking complements and applying h_0^{-1} to both sides, we conclude that

$$\widehat{F}(P) \cap D_0 \subset h_0^{-1}(\widehat{F}(g) \cap U_0),$$

establishing (a).

Since A contains the backward orbit of all critical points, we have that $J(P) \subset \overline{A}$, so

$$\widehat{J}(g) \cap U_0 = h_0(\overline{A}) \supset h_0(J(P)).$$

Taking complements and applying h_0^{-1} yields

$$h_0^{-1}(\widehat{F}(g) \cap U_0) \subset F(P) \cap D_0 \subset F(P),$$

which shows that every component of $h_0^{-1}(\widehat{F}(g) \cap U_0)$ is contained in some Fatou component of P.

We now treat the different types of Fatou components separately. If G is a Fatou component of P associated with an attracting or parabolic periodic cycle, then $G \cap \widehat{S}(P)$ is a countable set, so $V = G \setminus \widehat{S}(P)$ is connected. Let V^* be the connected component of $\widehat{F}(g) \cap U_0$ such that $V \subset h_0^{-1}(V^*)$. Then $h_0^{-1}(V^*) \subset G$, with $G \setminus h_0^{-1}(V^*)$ countable. This shows that G contains exactly one component V of $\widehat{F}(P) \cap D_0$, and $h_0(G)$ contains exactly one component V^* of $\widehat{F}(g) \cap U_0$, establishing claims (b) and (c) for the case of components contained in attracting or parabolic basins.

If G is a Fatou component of P associated with a cycle of Siegel disks, we claim that

$$\overline{A} \cap G = \widehat{S}(P) \cap G.$$

We know that $\overline{A} \subset \widehat{S}(P)$, so we only need to establish the reverse inclusion. Since

$$\widehat{S}(P) = \overline{\mathrm{GO}(S(P))},$$

it is enough to show that

$$\mathrm{GO}(S(P)) \cap G \subset \overline{A}.$$

Let $z \in \mathrm{GO}(S(P)) \cap G$. Then there exist $m, n \geq 0$ and $c \in \mathrm{Crit}(P)$ with

$$w^* = P^m(z) = P^n(c).$$

By adding an integer constant to both m and n, we may assume that w^* is in the cycle of Siegel disks. Then there exists a sequence of integers $k_j \to \infty$ such that

$$P^{k_j}(w) \to w$$

uniformly in a neighborhood of w^*. In particular, there exists a sequence $w_j \to w^*$ such that

$$P^{k_j}(w_j) = w^*.$$

Since P^m is an open mapping, there exists a sequence $z_j \to z$ such that

$$P^m(z_j) = w_j,$$

so that

$$P^{m+k_j}(z_j) = P^{k_j}(w_j) = w^* = P^n(c).$$

We may assume that $m + k_j \geq n$ for all j, so that $z_j \in A$ for all j, and hence $z \in \overline{A}$, as claimed.

The claim directly implies that

$$h_0^{-1}(\widehat{S}(g) \cap U_0) \cap G = \widehat{S}(P) \cap G.$$

We know that
$$\widehat{J}(g) \cap U_0 = \widehat{S}(g) \cap U_0$$
and that $(\widehat{J}(P) \cap G) \setminus (\widehat{S}(P) \cap G)$ contains at most one point (if G contains a periodic point which is not in $\widehat{S}(P)$), so for every component V of $\widehat{F}(P) \cap G$, there exists a unique component V^* of $\widehat{F}(g) \cap U_0$ such that either $V = h_0^{-1}(V^*)$, or $V = h_0^{-1}(V^*) \setminus \{z_0\}$ for some $z_0 \in G$. This establishes (b) and (c) in the case of Fatou components associated with Siegel disks.

Lastly, if G is a Fatou component of P associated with a superattracting basin (including the case where G is the basin of infinity), (c) is vacuously true, and (b) follows easily from the fact that $\widehat{F}(P)$ is open and dense in $F(P)$. ■

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