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Schatten class properties of weighted commutators

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Abstract. In this paper, we establish the Schatten class properties of the commutator $[M_f, (-\Delta)^{\frac{\varepsilon}{2}}]$ of a power $0 < \varepsilon < 1$ of the Laplacian with the multiplication operator on weighted spaces. For instance, we show that this commutator belongs to the weak Schatten class $\mathcal{L}_{\frac{d}{1-\varepsilon},\infty}$ associated with the weighted Hilbert space $L_2(\mathbb{R}^d, w(x)\,dx)$ if and only if f is from the homogeneous Sobolev space $\dot{W}^1_{\frac{d}{1-\varepsilon}}(\mathbb{R}^d)$. Our results generalize the recent result due to Frank, Sukochev and Zanin.

1. Introduction

For each $1 \le j \le d$, denote by R_j the j-th Riesz transform on $\mathbb{R}^d$, $d \in \mathbb{N}$. For a locally integrable function f, the commutator $[M_f, R_j]$ is defined by

$$[M_f, R_j] = M_f R_j - R_j M_f.$$

Here, M_f is the operator of multiplication by f, that is, $M_f g = fg$. In the remarkable paper [9], Coifman, Rochberg, and Weiss proved that the commutator $[M_f, R_j]$ is bounded on $L_p(\mathbb{R}^d)$, $1 < p < \infty$, if and only if $f \in \mathrm{BMO}(\mathbb{R}^d)$, the class of functions of bounded mean oscillations. Moreover, the commutator is compact on $L_2(\mathbb{R}^d)$ if and only if $f \in \mathrm{VMO}(\mathbb{R}^d)$. These theorems were extended into a weighted setting in [4, 22, 28].

In the present paper, we mainly concern the Schatten properties of commutators. In [26], Janson and Wolff established that, for $d \ge 2$ and $d < p < \infty$, the commutator $[M_f, R_j]$ belongs to the Schatten class $\mathcal{L}_p(L_2(\mathbb{R}^d))$ if and only if f belongs to a homogeneous fractional Sobolev space $\dot{W}^{d/p}_d(\mathbb{R}^d)$; for $0 < p \le d$, they showed that $[M_f, R_j] \in \mathcal{L}_p(L_2(\mathbb{R}^d))$ if and only if f is constant. We refer the reader to [25, 42] for alternative proofs. At the critical index $p = d$, there exists a *cut-off phenomenon*: Connes, Sullivan, and Teleman [10] established that the commutator $[M_f, R_j]$ belongs to the weak Schatten class $\mathcal{L}_{d,\infty}(L_2(\mathbb{R}^d))$ if and only if f is in the homogeneous first-order Sobolev space $\dot{W}^1_d(\mathbb{R}^d)$ (see [17, 35] for alternative proofs). While for the case $d = 1$ and $0 < p < \infty$, the commutator of the Hilbert transform $[M_f, H] \in \mathcal{L}_p(L_2(\mathbb{R}))$ if and only if f is in a Besov space, see, e.g., [40] and Theorem 5.7 in [41]. Recently, the above mentioned results were also extended into a weighted setting, see, e.g., [19, 29, 30].

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We continue this line of research to study the Schatten properties of the commutator

$$[M_f, (-\Delta)^{\frac{\varepsilon}{2}}], \quad \varepsilon \in (0,1),$$

defined on $L_2(\mathbb{R}^d)$. Let us formulate precisely the previous results on such commutators. The boundedness was first studied by Murray [38] for dimension $d = 1$, and by Janson and Peetre [25] for general dimension $d \geq 1$. Its weighted extension is referred to [7]. Moreover, in [25], the authors showed that, for each $\varepsilon \in (0,1)$ and $d/(1-\varepsilon) < p < \infty$,

$$[M_f, (-\Delta)^{\frac{\varepsilon}{2}}] \in \mathcal{L}_p(L_2(\mathbb{R}^d)) \iff f \in \dot{W}_p^{\varepsilon+\frac{d}{p}}(\mathbb{R}^d) \tag{1.1}$$

and

$$[M_f, (-\Delta)^{\frac{\varepsilon}{2}}] \in \mathcal{L}_{\frac{d}{1-\varepsilon}}(L_2(\mathbb{R}^d)) \iff f \text{ is constant.}$$

Only very recently, the "cut-off phenomenon" of the commutator $[M_f, (-\Delta)^{\frac{\varepsilon}{2}}]$ was discovered by Frank, Sukochev and Zanin [18] via the powerful technique of double operator integrals. Precisely, it was proved in Theorem 1.1 of [18] that for each $d \geq 1$ and $\varepsilon \in (0,1)$, one has

$$[M_f, (-\Delta)^{\frac{\varepsilon}{2}}] \in \mathcal{L}_{\frac{d}{1-\varepsilon},\infty}(L_2(\mathbb{R}^d)) \iff f \in \dot{W}^1_{\frac{d}{1-\varepsilon}}(\mathbb{R}^d), \tag{1.2}$$

as well as

$$\lim_{t\to\infty} t^{\frac{1-\varepsilon}{d}} \mu_{\mathcal{L}_\infty(L_2(\mathbb{R}^d))}(t, [M_f, (-\Delta)^{\frac{\varepsilon}{2}}]) = c_{d,\varepsilon} \|f\|_{\dot{W}^1_{\frac{d}{1-\varepsilon}}(\mathbb{R}^d)}. \tag{1.3}$$

Here, $t \mapsto \mu_{\mathcal{L}_\infty(L_2(\mathbb{R}^d))}(t,x)$ denotes the singular value function of $x \in \mathcal{L}_\infty(L_2(\mathbb{R}^d))$.

Note that the Schatten classes mentioned above are defined based on the Hilbert space $L_2(\mathbb{R}^d)$ associated with the usual Lebesgue measure. In this paper, our aim is to study the sufficient and necessary condition on f such that the commutator $[M_f, (-\Delta)^{\frac{\varepsilon}{2}}]$ belongs to the (weak) Schatten classes $\mathcal{L}_p(L_2(\mathbb{R}^d, w(x)\,dx))$ or $\mathcal{L}_{\frac{d}{1-\varepsilon},\infty}(L_2(\mathbb{R}^d, w(x)\,dx))$ with respect to the weighted Lebesgue space $L_2(\mathbb{R}^d, w(x)\,dx)$. Our main results concern weighted generalizations of (1.1), (1.2) and (1.3). The notation $A_2(\mathbb{R}^d)$ denotes the Muckenhoupt A_2 weight class defined on $\mathbb{R}^d$.

Theorem 1.1. *Let $d \geq 1$, $w \in A_2(\mathbb{R}^d)$, $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$, $\varepsilon \in (0,1)$ and $p > \frac{d}{1-\varepsilon}$.*

(i) *If $f \in \dot{W}_p^{\varepsilon+\frac{d}{p}}(\mathbb{R}^d)$, then $[M_f, (-\Delta)^{\frac{\varepsilon}{2}}] \in \mathcal{L}_p(L_2(\mathbb{R}^d, w(x)\,dx))$ and*

$$\|[M_f, (-\Delta)^{\frac{\varepsilon}{2}}]\|_{\mathcal{L}_p(L_2(\mathbb{R}^d, w(x)\,dx))} \leq c_{w,p,d,\varepsilon} \|f\|_{\dot{W}_p^{\varepsilon+\frac{d}{p}}(\mathbb{R}^d)}.$$

(ii) *If $[M_f, (-\Delta)^{\frac{\varepsilon}{2}}] \in \mathcal{L}_p(L_2(\mathbb{R}^d, w(x)\,dx))$, then $f \in \dot{W}_p^{\varepsilon+\frac{d}{p}}(\mathbb{R}^d)$ and*

$$\|f\|_{\dot{W}_p^{\varepsilon+\frac{d}{p}}(\mathbb{R}^d)} \leq c_{w,p,d,\varepsilon} \|[M_f, (-\Delta)^{\frac{\varepsilon}{2}}]\|_{\mathcal{L}_p(L_2(\mathbb{R}^d, w(x)\,dx))}.$$

Theorem 1.2. *Let $d \geq 1$, $w \in A_2(\mathbb{R}^d)$, $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$, and $\varepsilon \in (0,1)$.*

(i) *If $f \in \dot{W}^1_{\frac{d}{1-\varepsilon}}(\mathbb{R}^d)$, then $[M_f, (-\Delta)^{\frac{\varepsilon}{2}}] \in \mathcal{L}_{\frac{d}{1-\varepsilon},\infty}(L_2(\mathbb{R}^d, w(x)\,dx))$ and*

$$\|[M_f, (-\Delta)^{\frac{\varepsilon}{2}}]\|_{\mathcal{L}_{\frac{d}{1-\varepsilon},\infty}(L_2(\mathbb{R}^d, w(x)\,dx))} \leq c_{w,d,\varepsilon} \|f\|_{\dot{W}^1_{\frac{d}{1-\varepsilon}}(\mathbb{R}^d)}.$$

(ii) *If* $[M_f,(-\Delta)^{\frac{\varepsilon}{2}}]\in\mathcal{L}_{\frac{d}{1-\varepsilon},\infty}(L_2(\mathbb{R}^d,w(x)\,dx))$, *then* $f\in\dot{W}^1_{\frac{d}{1-\varepsilon}}(\mathbb{R}^d)$ *and*

$$\|f\|_{\dot{W}^1_{\frac{d}{1-\varepsilon}}(\mathbb{R}^d)}\le c_{w,d,\varepsilon}\|[M_f,(-\Delta)^{\frac{\varepsilon}{2}}]\|_{\mathcal{L}_{\frac{d}{1-\varepsilon},\infty}(L_2(\mathbb{R}^d,w(x)\,dx))}.$$

Theorem 1.3. *Let* $d\ge 1$, $w\in A_2(\mathbb{R}^d)$ *and* $\varepsilon\in(0,1)$. *For every* $f\in\dot{W}^1_{\frac{d}{1-\varepsilon}}(\mathbb{R}^d)$, *there exists the limit*

$$\lim_{t\to\infty}t^{\frac{1-\varepsilon}{d}}\mu_{\mathcal{L}_\infty(L_2(\mathbb{R}^d,w(x)\,dx))}(t,[M_f,(-\Delta)^{\frac{\varepsilon}{2}}])=c_{d,\varepsilon}\|f\|_{\dot{W}^1_{\frac{d}{1-\varepsilon}}(\mathbb{R}^d)}.$$

Note that Theorem 1.2 and Theorem 1.3 imply the "constant phenomenon". That is, for $\varepsilon\in(0,1)$ and $0<p\le\frac{d}{1-\varepsilon}$, the commutator $[M_f,(-\Delta)^{\frac{\varepsilon}{2}}]\in\mathcal{L}_p(L_2(\mathbb{R}^d,w(x)\,dx))$ yields that f is constant. Indeed, in this case, the commutator $[M_f,(-\Delta)^{\frac{\varepsilon}{2}}]$ belongs to the separable part of the ideal $\mathcal{L}_{\frac{d}{1-\varepsilon},\infty}(L_2(\mathbb{R}^d,w(x)\,dx))$. This means $f\in\dot{W}^1_{\frac{d}{1-\varepsilon}}(\mathbb{R}^d)$ (by Theorem 1.2 (ii)) and that the limit in Theorem 1.3 is zero. By Theorem 1.3, the Sobolev semi-norm of f is 0, which further yields the constancy of f.

Let us now discuss the proofs of these theorems. The approach used in [18] based on the double operator integrals does not seem applicable in the weighted setting. Our proofs of Theorems 1.1 and 1.2 are essentially motivated by [10, 16, 42]. Generally speaking, Rochberg and Semmes [42] described Schatten $\mathcal{L}_{p,q}(L_2(\mathbb{R}^d))$ properties of a commutator $[M_f,T]$ (where T is a singular integral operator) in terms of the membership of f in suitable oscillatory spaces $\mathrm{Osc}_{p,q}$. The identification $\mathrm{Osc}_{d,\infty}=\dot{W}^1_d$ is due to Connes, Sullivan and Teleman [10]. Recently, Frank [16] provided a new elementary proof of the latter identification.

We adapt the approach of Rochberg and Semmes to the weighted setting and characterize those f satisfying $[M_f,(-\Delta)^{\frac{\varepsilon}{2}}]\in\mathcal{L}_{\frac{d}{1-\varepsilon},\infty}(L_2(\mathbb{R}^d,w(x)\,dx))$ in terms of certain oscillation spaces. However, those oscillation spaces *differ* from the ones which appear in [16]. Hence, another result identifying the Sobolev and oscillation spaces is required, see Theorem 3.1 below.

Similarly, we characterize those f satisfying $[M_f,(-\Delta)^{\frac{\varepsilon}{2}}]\in\mathcal{L}_{p,\infty}(L_2(\mathbb{R}^d,w(x)\,dx))$, $p>\frac{d}{1-\varepsilon}$, in terms of an oscillation space. After that, we identify those oscillation spaces with fractional Sobolev spaces, see Theorem 3.2.

We expect that (both statements and proofs of) Theorems 3.1 and 3.2 allow further generalization and applications. For example, Frank's characterization [16] was recently extended into metric space and Muckenhoupt weight setting, see [24, 44].

As for the proof of Theorem 1.3, we apply the approximation argument, which is based on an observation that for each weight $w\in A_2(\mathbb{R}^d)$, we have $\log(w)\in\mathrm{BMO}(\mathbb{R}^d)$. This allows us to show Theorem 6.1, which is the key result to prove Theorem 1.3. Indeed, it is the crucial ingredient in the proof of the approximation presented in Theorem 7.1. Theorem 1.3 immediately follows from Theorem 7.1 combined with the asymptotic result (1.3) of Frank, Sukochev and Zanin.

The following commentary serves as a supplement to the discussion above.

Remark 1.4. (i) The study of commutators in harmonic analysis goes back at least to Nehari [39] and Calderón [5]. It has deep connections and applications to non-commutative analysis, complex analysis, operator theory, martingale theory and PDEs, see [6, 8, 27, 40]. The recent advance of Schatten properties of commutators defined on Heisenberg group, stratified Lie group or metric spaces can be found in [13, 14, 34] and so on.

(ii) The "cut-off phenomenon" also appears in other many other problems, see, for instance, [2, 11, 15].

(iii) Note that (1.1) and (1.2) also hold for $\varepsilon \in (-d/2, 0)$, see [18, 25]. Our theorems are not in this range because, in the proofs of Theorems 3.1 and 3.2, we used Theorem 2.11, which limits us.

(iv) Even though for singular integral operators T, the boundedness of commutators $[M_f, T]$ is well studied in two-weight setting [4,22,28], it seems that, until now, the boundedness for $[M_f, (-\Delta)^{\frac{\varepsilon}{2}}]$ is only investigated in the one weighted case, see, for instance, Theorem 1.4 in [7].

The organization of the paper is as follows. Basic symbols and notations used throughout the paper are collected in Section 2. In Section 3, we describe Sobolev spaces $\dot{W}^1_p(\mathbb{R}^d)$ and $\dot{W}^s_p(\mathbb{R}^d)$ in terms of oscillations of a function on cubes. These results are useful to prove our main results. We provide the proofs of Theorem 1.1 and Theorem 1.2 in Sections 4 and 5. Finally, in the last two sections, we prove our third main result Theorem 1.3.

2. Preliminaries

In this section, we collect basic notations and results used throughout the paper.

2.1. Standard conventions

Throughout this paper, $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ denote the integer set, nonnegative integer set and positive integer set, respectively. We denote by c some absolute positive constant, which can vary from line to line, and we denote by c_K the constant depending only on K. The symbol $A \lesssim B$ stands for the inequality $A \leqslant cB$ or $A \leqslant c_K B$. If we write $A \approx B$, then it stands for $A \lesssim B \lesssim A$.

Denote a measure space by (X, ν). Define the non-increasing rearrangement function of a measurable function f as follows:

$$\mu_{L_\infty(X,\nu)}(t, f) := \inf\{\lambda > 0 : \nu(x \in X : |f(x)| > \lambda) \leq t\}, \quad t > 0.$$

We usually write $\mu(t, f)$ whenever there is no confusion. For each $p > 0$, define

$$\|f\|_{L_{p,\infty}} = \sup_{t>0} t^{\frac{1}{p}} \mu_{L_\infty(X,\nu)}(t, f).$$

The set of all measurable function f with $\|f\|_{L_{p,\infty}} < \infty$ is called a Lorentz space, which is denoted by $L_{p,\infty}(X, \nu)$.

Throughout, we work on Euclidean space $\mathbb{R}^d$ for some fixed $d \in \mathbb{N}$ equipped with Lebesgue measure m. Let $\mathcal{D}$ be the collection of cubes of the shape

$$2^r([0,1)^d + n), \quad n \in \mathbb{Z}^d, \, r \in \mathbb{Z}.$$

For every cube $I \in \mathcal{D}$, we denote by $m(I)$ its Lebesgue measure, and by $s(I)$ its side length. It is immediate that $m(I) = s(I)^d$. For any $x \in \mathbb{R}^d$ and $t > 0$, set

$$B(x,t) := \{y \in \mathbb{R}^d : |y - x| < t\}.$$

Denote by $L_1^{\mathrm{loc}}(\mathbb{R}^d)$ the set of all locally integrable functions on $\mathbb{R}^d$. For each $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$ and each cube $I \subset \mathbb{R}^d$ or ball $B \subset \mathbb{R}^d$, write

$$f_I = \frac{1}{m(I)} \int_I f(x)\,dx, \quad f_B = \frac{1}{m(B)} \int_B f(x)\,dx.$$

Given $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$, let

$$(\sigma_t f)(x) := f\Big(\frac{x}{t}\Big), \quad t > 0,\ x \in \mathbb{R}^d,$$

be the dilation function of f.

The space of all functions whose k-th derivative exists and is continuous on $\mathbb{R}^d$ is denoted by $C^k(\mathbb{R}^d)$, $k \in \mathbb{N}$, and the space of all infinitely differentiable functions on $\mathbb{R}^d$ by $C^\infty(\mathbb{R}^d)$. Similarly, we define $C^k(B)$ and $C^\infty(B)$ on every ball B, respectively. The space of smooth functions with compact support on $\mathbb{R}^d$ is denoted by $C_c^\infty(\mathbb{R}^d)$. Moreover, for each measurable function f defined on measure space (X, ν), we define

$$(L_{p,\infty})_0(X,\nu) := \big\{f \in L_{p,\infty} : \lim_{u\to\infty} u^{\frac{1}{p}} \mu_{L_\infty(X,\nu)}(u, f) = 0\big\}, \quad 0 < p < \infty.$$

2.2. A_2 weights

The notion of usual A_p weights defined on $\mathbb{R}^d$ is well known, see, e.g., Definition 9.1.3 in [20]. To simplify the discussion, we here use an equivalent definition of A_2 weights which is taken from Exercise 9.2.3 (a) in [20].

Recall that

$$\|\phi\|_{\mathrm{BMO}(\mathbb{R}^d)} = \sup_I \frac{1}{|I|} \int_I |\phi - \phi_I|,$$

where the supremum is taken over all cubes in $\mathbb{R}^d$.

Definition 2.1. Denote by $\Phi(\mathbb{R}^d)$ the space consisting of all $\phi \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$ such that

$$\|\phi\|_{\Phi(\mathbb{R}^d)} = \sup_I \frac{1}{m(I)} \int_I e^{|\phi(x)-\phi_I|}\,dx < \infty,$$

where the supremum is taken over all cubes $I \subset \mathbb{R}^d$.

Clearly, if $\phi \in \Phi(\mathbb{R}^d)$, then also $\phi \in \mathrm{BMO}(\mathbb{R}^d)$. Conversely, by the John–Nirenberg inequality, for every $\phi \in \mathrm{BMO}(\mathbb{R}^d)$, there is a $0 \neq c \in \mathbb{R}$ such that $c\phi \in \Phi(\mathbb{R}^d)$.

Now, we give the definition of A_2 weight as follows.

Definition 2.2 (Exercise 9.2.3 (a) in [20]). For $\phi \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$, we say e^ϕ is an A_2 weight if and only if $\phi \in \Phi(\mathbb{R}^d)$. Denote by $A_2(\mathbb{R}^d)$ the collection of A_2 weights on $\mathbb{R}^d$, that is,

$$A_2(\mathbb{R}^d) = \{e^\phi : \phi \in \Phi(\mathbb{R}^d)\}.$$

Recall that for each $w \in A_2(\mathbb{R}^d)$ there exists $r > 1$ such that $w^r \in A_2(\mathbb{R}^d)$, see, for instance, Theorem 9.2.5 in [20]. Hence, the following lemma holds.

Lemma 2.3. *If $\phi \in \Phi(\mathbb{R}^d)$, then there exists a constant $r > 1$ such that $r\phi \in \Phi(\mathbb{R}^d)$.*

2.3. Schatten classes

For a separable Hilbert space H, denote by $\mathcal{L}_\infty(H)$ the space of all bounded linear operators on H. Let $\mathcal{K}(H)$ denote the ideal of compact operators on H. Given $T \in \mathcal{K}(H)$, define the singular value function $\mu(\cdot, T)$ by setting

$$\mu(t, T) = \inf\{\|T - R\|_\infty : \operatorname{rank}(R) \le t\}, \quad t \ge 0.$$

For $0 < p < \infty$, the Schatten class $\mathcal{L}_p(H)$ is the set of operators $T \in \mathcal{K}(H)$ such that $(\mu(n, T))_{n \in \mathbb{Z}_+} \in l_p$. If $1 \le p < \infty$, define the $\mathcal{L}_p(H)$ norm by

$$\|T\|_{\mathcal{L}_p(H)} := \|(\mu(n, T))_{n \in \mathbb{Z}_+}\|_{l_p} = \Big(\sum_{n=0}^{\infty} \mu(n, T)^p\Big)^{\frac{1}{p}}.$$

Analogously, for $0 < p < \infty$, the weak Schatten class $\mathcal{L}_{p,\infty}(H)$ is the set of operators $T \in \mathcal{K}(H)$ such that $(\mu(n, T))_{n \in \mathbb{Z}_+} \in l_{p,\infty}$. We equip $\mathcal{L}_{p,\infty}(H)$ with the quasi-norm

$$\|T\|_{\mathcal{L}_{p,\infty}(H)} = \sup_{n \ge 0} (n+1)^{\frac{1}{p}} \mu(n, T).$$

2.4. Nearly weakly orthonormal sequences

The following definition is a slightly modified formula of (1.4) in p. 239 of [42].

Definition 2.4. Consider a sequence of functions $\{e_I\}_{I \in \mathcal{D}} \subset L_2(\mathbb{R}^d)$. We say that the sequence $\{e_I\}_{I \in \mathcal{D}}$ is nearly weakly orthonormal (NWO) if, for every $f \in L_2(\mathbb{R}^d)$, we have the estimate $\|g\|_2 \le \|f\|_2$, where the function g is defined by the formula

$$g(t) = \sup_{I \ni t} m(I)^{-\frac{1}{2}} |\langle f, e_I \rangle|, \quad t \in \mathbb{R}^d,$$

where the supremum is taken over all cubes $I \in \mathcal{D}$.

For $I \in \mathcal{D}$, denote by JI, $J \in 2\mathbb{N} + 1$, the dilation of the cube I with respect to its center. In order to facilitate our calculation, we need the following results.

Lemma 2.5 ([42], p. 287). *Let $r > 1$ and $\eta \in 2\mathbb{N} + 1$. If $\{e_I\}_{I \in \mathcal{D}}$ is such that:*

(i) *e_I is supported in ηI for every $I \in \mathcal{D}$,*

(ii) *$\|e_I\|_{L_{2r}(\mathbb{R}^d)} \le m(I)^{\frac{1}{2r} - \frac{1}{2}}$ for every $I \in \mathcal{D}$,*

then there exists a constant $c_{\eta,r,d}$ such that the sequence $c_{\eta,r,d}\{e_I\}_{I \in \mathcal{D}}$ is NWO.

Lemma 2.6. *If $\{e_I\}$ and $\{f_I\}$ are two NWO sequences, then*

$$\|\{\langle Te_I, f_I \rangle\}_{I \in \mathcal{D}}\|_{l_p(\mathcal{D})} \le c_p \|T\|_{\mathcal{L}_p(L_2(\mathbb{R}^d))}, \quad 1 < p < \infty,$$

and

$$\|\{\langle Te_I, f_I \rangle\}_{I \in \mathcal{D}}\|_{l_{p,\infty}(\mathcal{D})} \le c_p \|T\|_{\mathcal{L}_{p,\infty}(L_2(\mathbb{R}^d))}, \quad 1 < p < \infty.$$

Proof. The first inequality is stated in p. 242 of [42], and the second one follows from the first inequality by real interpolation. ∎

The following lemma is taken from Corollary 2.8 in [42].

Lemma 2.7. *If $\{e_I\}_{I\in\mathcal{D}}$ and $\{f_I\}_{I\in\mathcal{D}}$ are NWO sequences and if*

$$K(x,y)=\sum_{I\in\mathcal{D}}\lambda_I e_I(x)f_I(y),\quad x,y\in\mathbb{R}^d,$$

then, for the operator T with integral kernel K, we have the estimates

$$\|T\|_{\mathcal{L}_p(L_2(\mathbb{R}^d))}\le c_{p,d}\|\{\lambda_I\}_{I\in\mathcal{D}}\|_{l_p(\mathcal{D})},\quad 0<p<\infty,$$

and

$$\|T\|_{\mathcal{L}_{p,\infty}(L_2(\mathbb{R}^d))}\le c_{p,d}\|\{\lambda_I\}_{I\in\mathcal{D}}\|_{l_{p,\infty}(\mathcal{D})},\quad 0<p<\infty.$$

The following result is a consequence of Lemma 2.5 and Lemma 2.7.

Corollary 2.8. *Let $r>1$ and $\eta\in 2\mathbb{N}+1$. Let $\{e_I\}_{I\in\mathcal{D}},\{f_I\}_{I\in\mathcal{D}}\subset L_{2r}(\mathbb{R}^d)$. Suppose that e_I and f_I are supported in ηI for every $I\in\mathcal{D}$. If*

$$K(x,y)=\sum_{I\in\mathcal{D}}\lambda_I e_I(x)f_I(y),\quad x,y\in\mathbb{R}^d,$$

then, for the operator T with integral kernel K and $0<p<\infty$, we have the estimates

$$\|T\|_{\mathcal{L}_p(L_2(\mathbb{R}^d))}\le c_{p,r,\eta,d}\Big\|\big\{\lambda_I m(I)^{1-\frac{1}{r}}\|e_I\|_{L_{2r}(\mathbb{R}^d)}\|f_I\|_{L_{2r}(\mathbb{R}^d)}\big\}_{I\in\mathcal{D}}\Big\|_{l_p(\mathcal{D})}$$

and

$$\|T\|_{\mathcal{L}_{p,\infty}(L_2(\mathbb{R}^d))}\le c_{p,r,\eta,d}\Big\|\big\{\lambda_I m(I)^{1-\frac{1}{r}}\|e_I\|_{L_{2r}(\mathbb{R}^d)}\|f_I\|_{L_{2r}(\mathbb{R}^d)}\big\}_{I\in\mathcal{D}}\Big\|_{l_{p,\infty}(\mathcal{D})}.$$

Proof. For each x and y, write

$$K(x,y)=\sum_{I\in\mathcal{D}}\lambda_I m(I)^{1-\frac{1}{r}}\|e_I\|_{L_{2r}(\mathbb{R}^d)}\|f_I\|_{L_{2r}(\mathbb{R}^d)}\frac{m(I)^{\frac{1}{2r}-\frac{1}{2}}e_I(x)}{\|e_I\|_{L_{2r}(\mathbb{R}^d)}}\,\frac{m(I)^{\frac{1}{2r}-\frac{1}{2}}f_I(y)}{\|f_I\|_{L_{2r}(\mathbb{R}^d)}}.$$

By Lemma 2.5,

$$\Big\{\frac{m(I)^{\frac{1}{2r}-\frac{1}{2}}e_I(x)}{\|e_I\|_{L_{2r}(\mathbb{R}^d)}}\Big\}_{I\in\mathcal{D}}\quad\text{and}\quad\Big\{\frac{m(I)^{\frac{1}{2r}-\frac{1}{2}}f_I(y)}{\|f_I\|_{L_{2r}(\mathbb{R}^d)}}\Big\}_{I\in\mathcal{D}}$$

are both NWO sequences associated with a constant. Thus, the desired assertion follows from Lemma 2.7. ∎

Corollary 2.9. *Let $r>1$ and $\eta\in 2\mathbb{N}+1$. Let $\{e_I\}_{I\in\mathcal{D}},\{f_I\}_{I\in\mathcal{D}}\subset L_{r+1}(\mathbb{R}^d)$. Suppose that e_I and f_I are supported in ηI for every $I\in\mathcal{D}$. Set*

$$K(x,y)=\sum_{I\in\mathcal{D}}\lambda_I e_I(x)f_I(y),\quad x,y\in\mathbb{R}^d.$$

If

$$\big\{\lambda_I m(I)^{1-\frac{2}{r+1}}\|e_I\|_{L_{r+1}(\mathbb{R}^d)}\|f_I\|_{L_{r+1}(\mathbb{R}^d)}\big\}_{I\in\mathcal{D}}\in(l_{p,\infty})_0(\mathcal{D}),$$

then the operator T with integral kernel K belongs to $(\mathcal{L}_{p,\infty})_0(L_2(\mathbb{R}^d))$.

Proof. Let $\{G_j\}_{j\geq 0}$ be an increasing sequence of subsets in $\mathcal{D}$ such that $\mathrm{card}(G_j) = j$ for $j \geq 0$ and $\bigcup_{j\geq 0} G_j = \mathcal{D}$ (here $\mathrm{card}(\cdot)$ denotes the cardinality of a set). Let

$$K_j(x,y) = \sum_{I\in G_j} \lambda_I e_I(x) f_I(y), \quad x,y\in\mathbb{R}^d,$$

and let T_j be the operator with the integral kernel K_j. Clearly, $\mathrm{rank}(T_j) \leq j$ and the integral kernel of $T - T_j$ is given by the formula

$$(x,y) \mapsto \sum_{I\in\mathcal{D}\setminus G_j} \lambda_I e_I(x) f_I(y), \quad x,y\in\mathbb{R}^d.$$

Applying Corollary 2.8 with $(r+1)/2$ instead of r, we obtain

$$\begin{aligned}&\|T - T_j\|_{\mathcal{L}_{p,\infty}(L_2(\mathbb{R}^d))}\\ &\quad\leq c_{p,\frac{r+1}{2},\eta,d} \Big\| \big\{\lambda_I m(I)^{1-\frac{2}{r+1}} \|e_I\|_{L_{r+1}(\mathbb{R}^d)} \|f_I\|_{L_{r+1}(\mathbb{R}^d)}\big\}_{I\in\mathcal{D}\setminus G_j} \Big\|_{l_{p,\infty}(\mathcal{D})}.\end{aligned}$$

By assumption, the right-hand side tends to 0 as $j\to\infty$. Hence, so does the left-hand side. Since $\mathrm{rank}(T_j) \leq j$ for every $j \geq 0$, the assertion follows. ■

2.5. Sobolev spaces

The basic notions and results of Sobolev spaces are available in, e.g., [1, 12, 31].

In this paper, ∇ is the gradient of $\mathbb{R}^d$, that is,

$$(\nabla f)(x_1, x_2, \ldots, x_d) := \Big(\frac{\partial f}{i\partial x_1}, \frac{\partial f}{i\partial x_2}, \ldots, \frac{\partial f}{i\partial x_d}\Big).$$

In addition, given a set $I \subset \mathbb{R}^d$, we denote by $C(I)$ the vector space of all continuous bounded functions on I.

Definition 2.10. Let $1 \leq p < \infty$ and let $s \in (0,1)$.

(i) The homogeneous Sobolev space $\dot{W}_p^1(\mathbb{R}^d)$ consists of functions $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$ with all partial derivatives in $L_p(\mathbb{R}^d)$. It is equipped with the natural semi-norm

$$\|f\|_{\dot{W}_p^1(\mathbb{R}^d)} = \||\nabla f|\|_{L_p(\mathbb{R}^d)}.$$

(ii) The inhomogeneous Sobolev space $W_p^1(\mathbb{R}^d)$ consists of $f \in L_p(\mathbb{R}^d)$ such that $f \in \dot{W}_p^1(\mathbb{R}^d)$. It is equipped with the natural norm

$$\|f\|_{W_p^1(\mathbb{R}^d)} = \|f\|_{L_p(\mathbb{R}^d)} + \|f\|_{\dot{W}_p^1(\mathbb{R}^d)}.$$

(iii) The homogeneous Sobolev space $\dot{W}_p^s(\mathbb{R}^d)$ consists of $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$ such that

$$\|f\|^p_{\dot{W}_p^s(\mathbb{R}^d)} \overset{\mathrm{def}}{=} \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|f(x)-f(y)|^p\,dx\,dy}{|x-y|^{sp+d}} < \infty.$$

(iv) The inhomogeneous Sobolev space $W_p^s(\mathbb{R}^d)$ consists of $f \in L_p(\mathbb{R}^d)$ such that $f \in \dot{W}_p^s(\mathbb{R}^d)$. It is equipped with the natural norm

$$\|f\|_{W_p^s(\mathbb{R}^d)} = \|f\|_{L_p(\mathbb{R}^d)} + \|f\|_{\dot{W}_p^s(\mathbb{R}^d)}.$$

Similarly, we define homogeneous Sobolev spaces on every ball B.

To prove our main theorems, we need the following result. It is certainly known to experts, however we were unable to find the particular statement in the literature and have included the proof for the reader's convenience.

Theorem 2.11 (Sobolev embedding theorem). *Let B be a unit ball in $\mathbb{R}^d$. If $f \in \dot{W}_p^s(B)$, $s \in (0,1]$, and $ps > d$, then $f \in C(B)$ and*

$$\|f - f_B\|_{L_\infty(B)} \leq c_{p,s,d}\|f\|_{\dot{W}_p^s(B)}.$$

Proof. We prove the assertion for $s \in (0,1)$. The proof for $s = 1$ is similar.

If $x, y \in B$, then $|x - y| \leq 2$. Thus,

$$\begin{aligned}\|f\|_{\dot{W}_p^s(B)}^p &= \int_{B\times B} \frac{|f(x) - f(y)|^p dx\, dy}{|x-y|^{sp+d}} \\ &\geq 2^{-sp-d} \int_{B\times B} |f(x) - f(y)|^p \,dx\,dy \\ &= 2^{-sp-d} \int_B \Big(\int_B |f(x) - f(y)|^p \,dy \Big)\, dx.\end{aligned}$$

Clearly,

$$\frac{1}{m(B)} \int_B |f(x) - f(y)|^p \,dy \geq |f(x) - f_B|^p.$$

Thus,

$$\|f\|_{\dot{W}_p^s(B)}^p \geq 2^{-sp-d} m(B) \|f - f_B\|_{L_p(B)}^p.$$

Hence,

$$\|f - f_B\|_{W_p^s(B)} \leq c_{p,s,d}^{(1)} \|f\|_{\dot{W}_p^s(B)}.$$

By the Sobolev extension theorem (see, e.g., Theorem 8.4 in [32]), there exists a function g such that $g|_B = f - f_B$ and

$$\|g\|_{W_p^s(\mathbb{R}^d)} \leq c_{p,s,d}^{(2)} \|f - f_B\|_{W_p^s(B)} \leq c_{p,s,d}^{(1)}\, c_{p,s,d}^{(2)} \|f\|_{\dot{W}_p^s(B)}.$$

The assertion follows now from the Morrey's version of the Sobolev embedding theorem (see Theorem 7.23 in [32]). ∎

3. Oscillation description of Sobolev spaces

In this section, we establish characterizations of the Sobolev spaces $\dot{W}_p^1(\mathbb{R}^d)$ and $\dot{W}_p^s(\mathbb{R}^d)$ in terms of oscillations of a function on cubes. These results will be used to prove our main results Theorem 1.1 and Theorem 1.2. To prove Theorem 3.1 and Theorem 3.2, in next two subsections, we first prove their continuous versions, and then in Section 3.3 we provide the detailed proofs. Our proof in this section is motivated by Frank [16].

Theorem 3.1. *Let $p > d$, $\eta \in 2\mathbb{N} + 1$ and $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$. The following conditions are equivalent*:

(i) $f \in \dot{W}_p^1(\mathbb{R}^d)$,

(ii) $\{m(I)^{\frac{1}{p}-\frac{1}{d}-1}\|f - f_{\eta I}\|_{L_1(\eta I)}\}_{I\in\mathcal{D}} \in l_{p,\infty}(\mathcal{D})$,

(iii) $\{m(I)^{\frac{1}{p}-\frac{1}{d}}\|f - f_{\eta I}\|_{L_\infty(\eta I)}\}_{I\in\mathcal{D}} \in l_{p,\infty}(\mathcal{D})$.

Furthermore, there exists positive constant $c_{p,d,\eta}$ such that

$$\begin{aligned} c_{p,d,\eta}^{-1}&\left\|\left\{m(I)^{\frac{1}{p}-\frac{1}{d}}\|f - f_{\eta I}\|_{L_\infty(\eta I)}\right\}_{I\in\mathcal{D}}\right\|_{l_{p,\infty}(\mathcal{D})} \\ &\le \|f\|_{\dot{W}_p^1(\mathbb{R}^d)} \le c_{p,d,\eta}\left\|\left\{m(I)^{\frac{1}{p}-\frac{1}{d}-1}\|f - f_{\eta I}\|_{L_1(\eta I)}\right\}_{I\in\mathcal{D}}\right\|_{l_{p,\infty}(\mathcal{D})}. \end{aligned}$$

Theorem 3.2. *Let $p > d$, $s \in (d/p, 1)$, $\eta \in 2\mathbb{N} + 1$ and $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$. The following conditions are equivalent*:

(i) $f \in \dot{W}_p^s(\mathbb{R}^d)$,

(ii) $\{m(I)^{\frac{1}{p}-\frac{s}{d}-1}\|f - f_{\eta I}\|_{L_1(\eta I)}\}_{I\in\mathcal{D}} \in l_p(\mathcal{D})$,

(iii) $\{m(I)^{\frac{1}{p}-\frac{s}{d}}\|f - f_{\eta I}\|_{L_\infty(\eta I)}\}_{I\in\mathcal{D}} \in l_p(\mathcal{D})$.

Furthermore, there exists positive constant $c_{p,s,d,\eta}$ such that

$$\begin{aligned} c_{p,s,d,\eta}^{-1}&\left\|\left\{m(I)^{\frac{1}{p}-\frac{s}{d}}\|f - f_{\eta I}\|_{L_\infty(\eta I)}\right\}_{I\in\mathcal{D}}\right\|_{l_p(\mathcal{D})} \\ &\le \|f\|_{\dot{W}_p^s(\mathbb{R}^d)} \le c_{p,s,d,\eta}\left\|\left\{m(I)^{\frac{1}{p}-\frac{s}{d}-1}\|f - f_{\eta I}\|_{L_1(\eta I)}\right\}_{I\in\mathcal{D}}\right\|_{l_p(\mathcal{D})}. \end{aligned}$$

This result may be known to the experts, but no direct proof exists in the literature.

3.1. Oscillation description of Sobolev spaces: critical case

In this subsection, we prove a continuous version of Theorem 3.1.

Theorem 3.3. *Let $p > d$. For every $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$, the following conditions are equivalent*:

(i) $f \in \dot{W}_p^1(\mathbb{R}^d)$,

(ii) *the function*

$$\mathfrak{m}_f : (x, t) \mapsto t^{\frac{d}{p}-d-1}\|f - f_{B(x,t)}\|_{L_1(B(x,t))}, \quad x \in \mathbb{R}^d,\ t > 0,$$

belongs to $L_{p,\infty}(\mathbb{R}^d \times \mathbb{R}_+, \frac{dx\,dt}{t^{d+1}})$,

(iii) *the function*

$$\mathfrak{M}_f : (x, t) \mapsto t^{\frac{d}{p}-1}\|f - f_{B(x,t)}\|_{L_\infty(B(x,t))}, \quad x \in \mathbb{R}^d,\ t > 0,$$

belongs to $L_{p,\infty}(\mathbb{R}^d \times \mathbb{R}_+, \frac{dx\,dt}{t^{d+1}})$.

Furthermore, there exists positive constant $c_{p,d}$ such that

$$c_{p,d}^{-1}\|\mathfrak{M}_f\|_{L_{p,\infty}(\mathbb{R}^d\times\mathbb{R}_+,\frac{dx\,dt}{t^{d+1}})} \le \|f\|_{\dot{W}_p^1(\mathbb{R}^d)} \le c_{p,d}\|\mathfrak{m}_f\|_{L_{p,\infty}(\mathbb{R}^d\times\mathbb{R}_+,\frac{dx\,dt}{t^{d+1}})}.$$

The implication (iii) $\Rightarrow$ (ii) is obvious. In the sequel of this subsection, we prove the implications (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (i), respectively.

We begin with the following lemmas.

Lemma 3.4. *Let $r > d$, and let B be a ball in $\mathbb{R}^d$. Then*

$$\|f - f_B\|_{L_\infty(B)} \le c_{r,d}\, m(B)^{\frac{1}{d}-\frac{1}{r}} \|f\|_{\dot{W}^1_r(B)}.$$

Proof. Let $f \in \dot{W}^1_r(B)$. The assertion is translation invariant and, therefore, we may assume $B = B(0,a)$ for some $a > 0$. Take h such that $f = \sigma_a h$. It is immediate that

$$|\nabla f|(x) = \frac{1}{a}|\nabla h|\Big(\frac{x}{a}\Big), \quad x \in \mathbb{R}^d.$$

Thus,

$$\begin{aligned}\|f\|_{\dot{W}^1_r(B)} &= \frac{1}{a}\Big(\int_{B(0,a)} |\nabla h|^r\Big(\frac{x}{a}\Big)\,dx\Big)^{\frac{1}{r}} \\ &\overset{x=ay}{=} a^{\frac{d}{r}-1}\Big(\int_{B(0,1)} |\nabla h|^r(y)\,dy\Big)^{\frac{1}{r}} = \frac{m(B)^{\frac{1}{r}-\frac{1}{d}}}{m(B(0,1))^{\frac{1}{r}-\frac{1}{d}}}\|h\|_{\dot{W}^1_r(B(0,1))}.\end{aligned}$$

From this, using Theorem 2.11, we have

$$\begin{aligned}\|f - f_B\|_{L_\infty(B)} &= \|h - h_{B(0,1)}\|_{L_\infty(B(0,1))} \\ &\le c^{(2)}_{r,d}\|h\|_{\dot{W}^1_r(B(0,1))} = c^{(3)}_{r,d}\, m(B)^{\frac{1}{d}-\frac{1}{r}}\|f\|_{\dot{W}^1_r(B)}.\end{aligned}$$

The proof is complete. ∎

Lemma 3.5. *For $0 < q < \infty$, let $g \in L_q(\mathbb{R}^d)$, and set*

$$h_{q,g} : (x,t) \mapsto g(x)t^{\frac{d}{q}}, \quad x \in \mathbb{R}^d,\ t > 0.$$

We have

$$\begin{aligned}\|h_{q,g}\|_{L_{q,\infty}(\mathbb{R}^d\times\mathbb{R}_+,\frac{dx\,dt}{t^{d+1}})} &= d^{-\frac{1}{q}}\|g\|_{L_q(\mathbb{R}^d)}, \\ \lim_{u\to\infty} u^{\frac{1}{q}}\mu_{L_\infty(\mathbb{R}^d\times\mathbb{R}_+,\frac{dx\,dt}{t^{d+1}})}(u,h_{q,g}) &= d^{-\frac{1}{q}}\|g\|_{L_q(\mathbb{R}^d)}.\end{aligned}$$

Proof. Denote for brevity $d\nu = \frac{dx\,dt}{t^{d+1}}$. We have

$$\begin{aligned}\nu(\{(x,t) : |h_{q,g}(x,t)| > \lambda\}) &= \nu(\{(x,t) : t > \lambda^{\frac{q}{d}}|g(x)|^{-\frac{q}{d}}\}) \\ &= \int_{\mathbb{R}^d}\Big(\int_{\lambda^{\frac{q}{d}}|g(x)|^{-\frac{q}{d}}}^{\infty} \frac{dt}{t^{d+1}}\Big)\,dx \\ &= \frac{\lambda^{-q}}{d}\int_{\mathbb{R}^d}|g(x)|^q\,dx = \frac{1}{d}\,\lambda^{-q}\|g\|^q_{L_q(\mathbb{R}^d)}.\end{aligned}$$

This is, obviously, equivalent to

$$\mu_{L_\infty(\mathbb{R}^d\times\mathbb{R}_+,\frac{dx\,dt}{t^{d+1}})}(u,h_{q,g}) = d^{-\frac{1}{q}}\|g\|_{L_q(\mathbb{R}^d)}\,u^{-\frac{1}{q}}, \quad u > 0.$$

The desired assertions follow. ∎

We are now in the position to prove the implication (i) $\Rightarrow$ (iii) in Theorem 3.3.

Proof of Theorem 3.3 (i) $\Rightarrow$ (iii). Fix $p \in (d, \infty)$. Let $r = (p+d)/2 \in (d, p)$ and $q = p/r \in (1, \infty)$. Set $g = |\nabla f|^r$ and

$$Mg(x) = \sup_{t>0} \frac{1}{m(B(x,t))} \int_{B(x,t)} g(y)\, dy, \quad x \in \mathbb{R}^d.$$

By Lemma 3.4, for each $x \in \mathbb{R}^d$ and $t > 0$, we have

$$\begin{aligned} \mathfrak{M}_f(x,t) &\le c_{r,d}^{(1)} t^{\frac{d}{p}} \cdot \Big(\frac{1}{m(B(x,t))} \int_{B(x,t)} g(y)\, dy \Big)^{\frac{1}{r}} \\ &\le c_{r,d}^{(1)} t^{\frac{d}{p}} (Mg)^{\frac{1}{r}}(x) = c_{r,d}^{(1)} h_{q,Mg}^{\frac{1}{r}}(x,t), \end{aligned}$$

where the last equality is simply the notation from Lemma 3.5. Thus, it follows from Lemma 3.5 that

$$\begin{aligned} \|\mathfrak{M}_f\|_{L_{p,\infty}(\mathbb{R}^d \times \mathbb{R}_+, \frac{dx\,dt}{t^{d+1}})} &\le c_{r,d}^{(1)} \|h_{q,Mg}^{\frac{1}{r}}\|_{L_{p,\infty}(\mathbb{R}^d \times \mathbb{R}_+, \frac{dx\,dt}{t^{d+1}})} \\ &= c_{r,d}^{(1)} \|h_{q,Mg}\|^{\frac{1}{r}}_{L_{q,\infty}(\mathbb{R}^d \times \mathbb{R}_+, \frac{dx\,dt}{t^{d+1}})} \\ &= c_{r,d}^{(1)} (d^{-\frac{1}{q}} \|Mg\|_{L_q(\mathbb{R}^d)})^{\frac{1}{r}} \\ &\le c_{r,q,d}^{(2)} [\|g\|_{L_q(\mathbb{R}^d)}]^{\frac{1}{r}} = c_{r,q,d}^{(2)} \|f\|_{\dot{W}_p^1(\mathbb{R}^d)}, \end{aligned}$$

where the second inequality is due to the Hardy–Littlewood maximal inequality. This completes the proof. ∎

We now turn to the proof of the implication (ii) $\Rightarrow$ (i) in Theorem 3.3. Let us begin with the following lemmas.

Lemma 3.6. *Let $0 < p < \infty$, let $f \in C^\infty(\mathbb{R}^d)$, and let $g = |\nabla f|$. In the notations of Lemma 3.5, for every ball $B \subset \mathbb{R}^d$, there exists a constant c_d depending only on the dimension d such that*

$$\mathfrak{m}_f - c_d h_{p,g} \in (L_{p,\infty})_0\Big(B \times \mathbb{R}_+, \frac{dx\,dt}{t^{d+1}}\Big).$$

Proof. To lighten the notations, we let $B = B(0,1)$ and assume $\|f\|_{C^2(B(0,2))} = 1$. Let $x \in B(0,1)$, $t \in (0,1)$ and $y \in B(x,t)$. It is easy to see that

$$\big| |f(y) - f(x)| - |(\nabla f)(x) \cdot (y-x)| \big| \le c_d^{(1)} t^2.$$

Thus,

$$\Big| \int_{B(x,t)} |f(y) - f(x)|\, dy - \int_{B(x,t)} |(\nabla f)(x) \cdot (y-x)|\, dy \Big| \le c_d^{(1)} m(B(0,1)) t^{d+2}.$$

However,

$$\int_{B(x,t)} |(\nabla f)(x) \cdot (y-x)|\, dy = c_d\, g(x) t^{d+1}.$$

Hence,

$$\Big| \int_{B(x,t)} |f(y) - f(x)|\, dy - c_d\, g(x) t^{d+1} \Big| \le c_d^{(2)} \|f\|_{C^2(B(0,2))} t^{d+2}.$$

Multiplying by $t^{d/p-d-1}$, we obtain

$$\mathfrak{m}_f(x,t) - c_d h_{p,g}(x,t) = O(t^{\frac{d}{p}+1}), \quad x \in B(0,1),\ t \in (0,1).$$

In particular,

$$\mathfrak{m}_f - c_d h_{p,g} \in L_p\Big(B(0,1) \times (0,1), \frac{dx\,dt}{t^{d+1}}\Big) \subset (L_{p,\infty})_0\Big(B(0,1) \times (0,1), \frac{dx\,dt}{t^{d+1}}\Big).$$

On the other hand, since the set $B \times (1,\infty)$ has finite measure, it follows immediately that

$$\mathfrak{m}_f, h_{p,g} \in (L_{p,\infty})_0\Big(B(0,1) \times (1,\infty), \frac{dx\,dt}{t^{d+1}}\Big).$$

Combining these inclusions, we obtain

$$\mathfrak{m}_f - c_d h_{p,g} \in (L_{p,\infty})_0\Big(B(0,1) \times \mathbb{R}_+, \frac{dx\,dt}{t^{d+1}}\Big).$$

This completes the proof. ∎

Lemma 3.7. *Let $0 < p < \infty$ and $f \in C^\infty(\mathbb{R}^d)$. For every ball $B \subset \mathbb{R}^d$, we have*

$$\lim_{u\to\infty} u^{\frac{1}{p}} \mu_{L_\infty(B\times\mathbb{R}_+, \frac{dx\,dt}{t^{d+1}})}(u, \mathfrak{m}_f) = c_{d,p} \| |\nabla f| \|_{L_p(B)}.$$

Proof. The assertion follows by combining Lemma 3.5 and Lemma 3.6. ∎

Lemma 3.8. *Fix a positive function $\phi \in C_c^\infty(\mathbb{R}^d)$. For each $n \in \mathbb{N}$, define the mapping $A_{n,\phi}\colon L_1^{\mathrm{loc}}(\mathbb{R}^d) \to L_1^{\mathrm{loc}}(\mathbb{R}^d)$ by setting*

$$A_{n,\phi} : f \mapsto n^{-d} \sum_{k\in\mathbb{Z}^d} \phi\Big(\frac{k}{n}\Big) T_{k/n} f,$$

where $T_{k/n} f(x) = f(x + k/n)$. We have

(i) *the operators $\{A_{n,\phi}\}_{n\in\mathbb{N}}$ are uniformly bounded on $L_1(\mathbb{R}^d)$,*

(ii) *$A_{n,\phi} f \to \phi * f$ in $L_1^{\mathrm{loc}}(\mathbb{R}^d)$ as $n \to \infty$,*

(iii) *$\mathfrak{m}_{A_{n,\phi} f} \to \mathfrak{m}_{\phi * f}$ pointwise as $n \to \infty$.*

Proof. By the triangle inequality, we have

$$\|A_{n,\phi} f\|_{L_1(\mathbb{R}^d)} \le n^{-d} \sum_{k\in\mathbb{Z}^d} \phi\Big(\frac{k}{n}\Big) \cdot \|f\|_{L_1(\mathbb{R}^d)}.$$

Since

$$\sup_{x\in\mathbb{R}^d} \phi(x) e^{|x|^2} < \infty,$$

it follows that

$$n^{-d}\sum_{k\in\mathbb{Z}^d}\phi\Big(\frac{k}{n}\Big)\le\sup_{x\in\mathbb{R}^d}\phi(x)e^{|x|^2}\cdot n^{-d}\sum_{k\in\mathbb{Z}^d}e^{-\frac{|k|^2}{n^2}}\le\sup_{x\in\mathbb{R}^d}\phi(x)e^{|x|^2}\cdot(1+\sqrt{\pi})^d.$$

This yields the first assertion.

To see the second assertion, it suffices to prove that $A_{n,\phi}f\to\phi*f$ in $L_1([-m,m]^d)$ for every $m\in\mathbb{N}$. Without loss of generality, m is so large that ϕ is supported in $[-m,m]^d$. If $\phi(k/n)\neq 0$, then $k/n\in[-m,m]^d$. Set $f_m=f\chi_{[-2m,2m]^d}$. On the cube $[-m,m]^d$, we have $A_{n,\phi}f=A_{n,\phi}f_m$ and $\phi*f=\phi*f_m$.

Fix $\varepsilon>0$ and choose $\psi\in C_c^\infty(\mathbb{R}^d)$ such that $\|f_m-\psi\|_{L_1(\mathbb{R}^d)}<\varepsilon$. We have

$$\begin{aligned}
&\|A_{n,\phi}f-\phi*f\|_{L_1([-m,m]^d)}\\
&\quad=\|A_{n,\phi}f_m-\phi*f_m\|_{L_1([-m,m]^d)}\\
&\quad\le\|A_{n,\phi}f_m-\phi*f_m\|_{L_1(\mathbb{R}^d)}\\
&\quad\le\|A_{n,\phi}(f_m-\psi)\|_{L_1(\mathbb{R}^d)}+\|A_{n,\phi}\psi-\phi*\psi\|_{L_1(\mathbb{R}^d)}+\|\phi*(f_m-\psi)\|_{L_1(\mathbb{R}^d)}\\
&\quad\le\big(\|\phi\|_{L_1(\mathbb{R}^d)}+\sup_{n\in\mathbb{N}}\|A_{n,\phi}\|_{L_1(\mathbb{R}^d)\to L_1(\mathbb{R}^d)}\big)\varepsilon+\|A_{n,\phi}\psi-\phi*\psi\|_{L_1(\mathbb{R}^d)}.
\end{aligned}$$

Clearly, $A_{n,\phi}\psi\to\phi*\psi$ in $L_1(\mathbb{R}^d)$. Thus,

$$\limsup_{n\to\infty}\|A_{n,\phi}f-\phi*f\|_{L_1([-m,m]^d)}\le\big(\|\phi\|_{L_1(\mathbb{R}^d)}+\sup_{n\in\mathbb{N}}\|A_{n,\phi}\|_{L_1(\mathbb{R}^d)\to L_1(\mathbb{R}^d)}\big)\varepsilon.$$

Since $\varepsilon>0$ is arbitrarily small, the second assertion follows.

If $h_n\to h$ in $L_1^{\mathrm{loc}}(\mathbb{R}^d)$, then $\mathfrak{m}_{h_n}\to\mathfrak{m}_h$ pointwise. The third assertion follows now from the second one. ■

Now we finish the proof of Theorem 3.3.

Proof of Theorem 3.3 (ii) $\Rightarrow$ (i). Since ϕ is compactly supported, it follows that the sum in the definition of $A_{n,\phi}$ is actually finite. It is now immediate that

$$0\le\mathfrak{m}_{A_{n,\phi}f}\le n^{-d}\sum_{k\in\mathbb{Z}^d}\phi\Big(\frac{k}{n}\Big)\mathfrak{m}_{T_{k/n}f}.$$

Recall that $L_{p,\infty}(\Omega)$ allows an equivalent norm for $p>1$ (see, e.g., p. 15 of [21]).[1] Thus,

$$\|\mathfrak{m}_{A_{n,\phi}f}\|_{L_{p,\infty}(\mathbb{R}^d\times\mathbb{R}_+,\frac{dx\,dt}{t^{d+1}})}\le\frac{p}{p-1}\,n^{-d}\sum_{k\in\mathbb{Z}^d}\phi\Big(\frac{k}{n}\Big)\|\mathfrak{m}_{T_{k/n}f}\|_{L_{p,\infty}(\mathbb{R}^d\times\mathbb{R}_+,\frac{dx\,dt}{t^{d+1}})}.$$

By translation invariance, we have

$$\|\mathfrak{m}_{A_{n,\phi}f}\|_{L_{p,\infty}(\mathbb{R}^d\times\mathbb{R}_+,\frac{dx\,dt}{t^{d+1}})}\le\frac{p}{p-1}\,n^{-d}\sum_{k\in\mathbb{Z}^d}\phi\Big(\frac{k}{n}\Big)\cdot\|\mathfrak{m}_f\|_{L_{p,\infty}(\mathbb{R}^d\times\mathbb{R}_+,\frac{dx\,dt}{t^{d+1}})}.$$

[1] We have that

$$\|f\|_{L_{p,\infty}}\le\|\mathcal{C}\mu(f)\|_{L_{p,\infty}(0,\infty)}\le\frac{p}{p-1}\|f\|_{L_{p,\infty}},$$

where $\mathcal{C}\mu(f)(t)=\frac{1}{t}\int_0^t\mu(s,f)\,ds$.

Since $L_{p,\infty}$ possesses the Fatou property,[2] it follows from Lemma 3.8 that

$$\|\mathfrak{m}_{\phi*f}\|_{L_{p,\infty}(\mathbb{R}^d\times\mathbb{R}_+,\frac{dx\,dt}{t^{d+1}})} \le \frac{p}{p-1}\|\phi\|_{L_1(\mathbb{R}^d)}\|\mathfrak{m}_f\|_{L_{p,\infty}(\mathbb{R}^d\times\mathbb{R}_+,\frac{dx\,dt}{t^{d+1}})}.$$

Applying Lemma 3.7 to the function $\phi * f$, we obtain, for every ball $B\subset\mathbb{R}^d$,

$$\begin{aligned} c_{d,p}\||\nabla(\phi*f)|\|_{L_p(B)} &= \lim_{u\to\infty} u^{\frac{1}{p}}\mu_{L_\infty(B\times\mathbb{R}_+,\frac{dx\,dt}{t^{d+1}})}(u,\mathfrak{m}_{\phi*f}) \\ &\le \|\mathfrak{m}_{\phi*f}\|_{L_{p,\infty}(\mathbb{R}^d\times\mathbb{R}_+,\frac{dx\,dt}{t^{d+1}})} \\ &\le \|\phi\|_{L_1(\mathbb{R}^d)}\|\mathfrak{m}_f\|_{L_{p,\infty}(\mathbb{R}^d\times\mathbb{R}_+,\frac{dx\,dt}{t^{d+1}})}. \end{aligned}$$

Since the right-hand side does not depend on the choice of the ball B, it follows that

$$c_{d,p}\||\nabla(\phi*f)|\|_{L_p(\mathbb{R}^d)} \le \|\phi\|_{L_1(\mathbb{R}^d)}\|\mathfrak{m}_f\|_{L_{p,\infty}(\mathbb{R}^d\times\mathbb{R}_+,\frac{dx\,dt}{t^{d+1}})}.$$

The assertion follows now from Lemma 6.5.4 in [36]. ∎

3.2. Continuous oscillation description of Sobolev spaces: subcritical case

The main result of this subsection is a continuous version of Theorem 3.2.

Theorem 3.9. *Let $p>d$, $s\in(d/p,1)$. For every $f\in L_1^{\mathrm{loc}}(\mathbb{R}^d)$, the following conditions are equivalent*:

(i) $f\in\dot{W}_p^s(\mathbb{R}^d)$,

(ii) *the function*

$$\mathfrak{m}_f:(x,t)\mapsto t^{\frac{d}{p}-d-s}\|f-f_{B(x,t)}\|_{L_1(B(x,t))},\quad x\in\mathbb{R}^d,\ t>0,$$

belongs to $L_p(\mathbb{R}^d\times\mathbb{R}_+,\frac{dx\,dt}{t^{d+1}})$,

(iii) *the function*

$$\mathfrak{M}_f:(x,t)\mapsto t^{\frac{d}{p}-s}\|f-f_{B(x,t)}\|_{L_\infty(B(x,t))},\quad x\in\mathbb{R}^d,\ t>0,$$

belongs to $L_p(\mathbb{R}^d\times\mathbb{R}_+,\frac{dx\,dt}{t^{d+1}})$.

Furthermore, there exists positive constant $c_{p,d}$ such that

$$c_{p,d}^{-1}\|\mathfrak{M}_f\|_{L_p(\mathbb{R}^d\times\mathbb{R}_+,\frac{dx\,dt}{t^{d+1}})} \le \|f\|_{\dot{W}_p^s(\mathbb{R}^d)} \le c_{p,d}\|\mathfrak{m}_f\|_{L_p(\mathbb{R}^d\times\mathbb{R}_+,\frac{dx\,dt}{t^{d+1}})}.$$

The implication (iii) ⇒ (ii) is obvious. In what follows, we prove the implications (i) ⇒ (iii) and (ii) ⇒ (i), respectively.

[2] To our surprise, we were unable to find this fact in the literature. The proof is provided below. Suppose that $\|x_n\|_{p,\infty}\le 1$ for every $n\in\mathbb{N}$ and let $x_n\to x$ pointwise. On a set of finite measure, pointwise convergence yields convergence in measure, which, in turn, yields a pointwise convergence of decreasing rearrangements. For every set A of finite measure, we have $x_n\chi_A\to x\chi_A$ in measure and, therefore, $\mu(x_n\chi_A)\to\mu(x\chi_A)$ pointwise (except, possibly at a countable set). On the other hand, $\mu(t,x_n\chi_A)\le\mu(t,x_n)\le t^{-1/p}$. Thus, $\mu(t,x\chi_A)\le t^{-1/p}$ for every $t>0$. In other words, $\|x\chi_A\|_{p,\infty}\le 1$ for every set A of finite measure. Hence, $\|x\|_{p,\infty}\le 1$.

We begin with the following result, which is motivated by Lemma 3.4.

Lemma 3.10. *Let $d < p < \infty$ and $s \in (d/p, 1)$. Let also B be a ball in $\mathbb{R}^d$. Then every $f \in \dot{W}_p^s(B)$ is bounded on B and*

$$\|f - f_B\|_{L_\infty(B)} \le c_{p,s,d}\, m(B)^{\frac{s}{d}-\frac{1}{p}} \|f\|_{\dot{W}_p^s(B)}.$$

Proof. The assertion is translation invariant and, therefore, we may assume $B = B(0,a)$. Let function h be such that $f(x) = h(x/a)$ for each $x \in \mathbb{R}^d$. Then

$$\begin{aligned}\|f\|_{\dot{W}_p^s(B)} &= \Big(\int_{B(0,a)\times B(0,a)} \frac{|h(\frac{x}{a}) - h(\frac{y}{a})|^p\, dx\, dy}{|x-y|^{sp+d}}\Big)^{\frac{1}{p}}\\ &= a^{\frac{d}{p}-s}\Big(\int_{B(0,1)\times B(0,1)} \frac{|h(u)-h(v)|^p\, du\, dv}{|u-v|^{sp+d}}\Big)^{\frac{1}{p}}\\ &= \frac{m(B)^{\frac{1}{p}-\frac{s}{d}}}{m(B(0,1))^{\frac{1}{p}-\frac{s}{d}}}\|h\|_{\dot{W}_p^s(B(0,1))},\end{aligned}$$

where we took $x = au$ and $y = av$ in the second equality. From this, by Theorem 2.11, we get

$$\begin{aligned}\|f - f_B\|_{L_\infty(B)} &= \|h - h_{B(0,1)}\|_{L_\infty(B(0,1))}\\ &\le c_{p,s,d}^{(2)}\|h\|_{\dot{W}_p^s(B(0,1))} = c_{p,s,d}^{(3)}\, m(B)^{\frac{s}{d}-\frac{1}{p}}\|f\|_{\dot{W}_p^s(B)}.\end{aligned}$$

The proof is complete. ∎

We are now ready to prove Theorem 3.9 (i) ⇒ (iii).

Proof of Theorem 3.9 (i) ⇒ (iii). Fix $\varepsilon \in (d/p, s)$. By Lemma 3.10, we have

$$\mathfrak{M}_f(x,t) \le c_{p,\varepsilon,d}^{(1)}\, t^{\varepsilon - s}\|f\|_{\dot{W}_p^\varepsilon(B(x,t))}.$$

Thus,

$$\begin{aligned}&\|\mathfrak{M}_f\|^p_{L_p(\mathbb{R}^d\times\mathbb{R}_+, \frac{dx\,dt}{t^{d+1}})}\\ &\le (c_{p,\varepsilon,d}^{(1)})^p\int_{\mathbb{R}^d\times\mathbb{R}_+}\Big(\int_{B(x,t)\times B(x,t)} \frac{|f(u)-f(v)|^p\, du\, dv}{|u-v|^{\varepsilon p+d}}\Big)t^{(\varepsilon-s)p}\frac{dx\,dt}{t^{d+1}}\\ &\le (c_{p,\varepsilon,d}^{(1)})^p\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|f(u)-f(v)|^p}{|u-v|^{\varepsilon p+d}}\Big(\int_0^\infty\Big(\int_{B(u,t)\cap B(v,t)} dx\Big)t^{(\varepsilon-s)p-d-1}\, dt\Big)\, du\, dv.\end{aligned}$$

By translation and dilation invariance,

$$\int_0^\infty\Big(\int_{B(u,t)\cap B(v,t)} dx\Big)t^{(\varepsilon-s)p-d-1}\, dt = c_{\varepsilon,s,d,p}|u-v|^{(\varepsilon-s)p}.$$

Thus,

$$\|\mathfrak{M}_f\|^p_{L_p(\mathbb{R}^d\times\mathbb{R}_+, \frac{dx\,dt}{t^{d+1}})} \le (c_{p,\varepsilon,d}^{(1)})^p\, c_{\varepsilon,s,d,p}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|f(u)-f(v)|^p}{|u-v|^{sp+d}}\, du\, dv.$$

This completes the proof. ∎

It remains to show the implication (ii) $\Rightarrow$ (i) of Theorem 3.9.

Let $\phi \in C^\infty(\mathbb{R}^d)$ be compactly supported in $B(0,1)$. Define

$$F(x,t) = t^{-d}(f * \sigma_t\phi)(x), \quad x \in \mathbb{R}^d,\ t > 0. \tag{3.1}$$

Later, in the proof of Theorem 3.9 (ii) $\Rightarrow$ (i), we take ϕ to be radial (see, e.g., p. 90 of [21]). However, this restriction plays no role in Lemma 3.11.

Lemma 3.11. *For every $x \in \mathbb{R}^d$ and for every $t > 0$, we have*

$$\Big|\frac{\partial F}{\partial t}(x,t)\Big| \le 2dt^{-d-1}\|f - f_{B(x,t)}\|_{L_1(B(x,t))}\|\phi\|_{C^1(\mathbb{R}^d)}.$$

Proof. Fix $r > t$. We have

$$\begin{aligned} t^{-d}\int_{B(x,r)}(f(y) - f_{B(x,r)})\phi\Big(\frac{x-y}{t}\Big)\,dy &= t^{-d}\int_{\mathbb{R}^d}(f(y) - f_{B(x,r)})\phi\Big(\frac{x-y}{t}\Big)\,dy \\ &= F(x,t) - f_{B(x,r)}\int_{\mathbb{R}^d}\phi(y)\,dy. \end{aligned}$$

It follows that

$$\frac{\partial F}{\partial t}(x,t) = \int_{B(x,r)}(f(y) - f_{B(x,r)})\,\frac{\partial}{\partial t}\Big(t^{-d}\phi\Big(\frac{x-y}{t}\Big)\Big)\,dy.$$

Thus,

$$\Big|\frac{\partial F}{\partial t}(x,t)\Big| \le \int_{B(x,r)}|f(y) - f_{B(x,r)}|\,dy \cdot \sup_{y\in B(x,r)}\frac{\partial}{\partial t}\Big(t^{-d}\phi\Big(\frac{x-y}{t}\Big)\Big).$$

Clearly,

$$\sup_{y\in B(x,r)}\Big|\frac{\partial}{\partial t}\Big(t^{-d}\phi\Big(\frac{x-y}{t}\Big)\Big)\Big| \le d\,\|\phi\|_{C^1(\mathbb{R}^d)}(t^{-d-1} + rt^{-d-2}).$$

Thus,

$$\Big|\frac{\partial F}{\partial t}(x,t)\Big| \le d\,\|\phi\|_{C^1(\mathbb{R}^d)}\int_{B(x,r)}|f(y) - f_{B(x,r)}|\,dy \cdot (t^{-d-1} + rt^{-d-2}).$$

Passing $r \downarrow t$, we complete the proof. ∎

We are ready to finish the proof of Theorem 3.9.

Proof of Theorem 3.9 (ii) $\Rightarrow$ (i). Let ϕ be radial, and let F be as in (3.1). It follows from Lemma 3.11 that

$$\Big\|\frac{\partial F}{\partial t}\Big\|_{L_p(\mathbb{R}^d\times\mathbb{R}_+,t^{p-ps-1}dx\,dt)} \le 4d\,\|\mathfrak{m}_f\|_{L_p(\mathbb{R}^d\times\mathbb{R}_+,t^{-d-1}dx\,dt)}\|\phi\|_{C^1(\mathbb{R}^d)}, \tag{3.2}$$

where $s \in (d/p, 1)$.

Combining Proposition 2.2.11 and Corollary 2.2.12 in [21], we note that $\widehat{\phi} \in \mathcal{S}(\mathbb{R}^d)$ is radial. Define even functions $h, g \in \mathcal{S}(\mathbb{R})$ such that

$$h(|x|) = (2\pi)^{\frac{d}{2}}\widehat{\phi}(|x|), \quad x \in \mathbb{R}^d, \quad g(t) = th'(t), \quad t \in \mathbb{R}.$$

Note that

$$\begin{aligned} F(x,t) = t^{-d}(f * \sigma_t\phi)(x) &= \int_{\mathbb{R}^d} \phi(t^{-1}z) f(x-z)\,dz \cdot t^{-d} \\ &\overset{z=yt}{=} \int_{\mathbb{R}^d} \phi(y) f(x-yt)\,dy = \int_{\mathbb{R}^d} \phi(y)(e^{-iy\cdot t\nabla}) f(x)\,dy \\ &= (2\pi)^{\frac{d}{2}} (\widehat{\phi}(t\nabla) f)(x) = (h(t\Delta^{\frac{1}{2}}) f)(x). \end{aligned}$$

Thus,

$$\frac{\partial F}{\partial t}(t,x) = (\Delta^{\frac{1}{2}} h'(t\Delta^{\frac{1}{2}}) f)(x) = t^{-1}(g(t\Delta^{\frac{1}{2}}) f)(x),$$

which, together with (3.2), implies

$$\Big(\int_0^\infty \| g(t\Delta^{\frac{1}{2}}) f \|_{L_p(\mathbb{R}^d)}^p \frac{dt}{t^{ps+1}} \Big)^{\frac{1}{p}} \leq 4d \| \mathfrak{m}_f \|_{L_p(\mathbb{R}^d \times \mathbb{R}_+, t^{-d-1} dx\,dt)} \|\phi\|_{C^1(\mathbb{R}^d)}.$$

The assertion follows now from the homogeneous version of Triebel's theorem (see Theorem A.1). ■

3.3. Proofs of Theorems 3.1 and 3.2

We provide the proofs of our main results (Theorems 3.1 and 3.2) in this subsection. Recall that for any $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$, the dilation function of f is given by

$$(\sigma_t f)(x) := f\Big(\frac{x}{t}\Big), \quad t > 0,\ x \in \mathbb{R}^d.$$

Now, let us first show several lemmas.

Lemma 3.12. *Let h be a Borel measurable function on $\mathbb{R}^d \times \mathbb{R}_+$. For each $I \in \mathcal{D}$, set*

$$a(h,I) = \operatorname*{ess\,sup}_{\substack{x\in I \\ t\in[\frac{s(I)}{2}, s(I))}} |h(x,t)|, \quad b(h,I) = \operatorname*{ess\,inf}_{\substack{x\in I \\ t\in[\frac{s(I)}{2}, s(I))}} |h(x,t)|,$$

where $s(I)$ denotes the side length of I. Then, for each $t > 0$,

$$\mu_{l_\infty(\mathcal{D})}(\{b(h,I)\}_{I\in\mathcal{D}}) \leq \mu_{L_\infty(\mathbb{R}^d\times\mathbb{R}_+, \frac{dx\,dt}{t^{d+1}})}(h) \leq \sigma_{2^d}\mu_{l_\infty(\mathcal{D})}(\{a(h,I)\}_{I\in\mathcal{D}}).$$

Proof. Assume without loss of generality that $h \geq 0$. For brevity, denote by ν the measure $\frac{dx\,dt}{t^{d+1}}$. For each $I \in \mathcal{D}$, set

$$A_I = I \times \Big[\frac{s(I)}{2}, s(I)\Big).$$

It is immediate that $\{A_I\}_{I\in\mathcal{D}}$ are pairwise disjoint, $\mathbb{R}^d \times \mathbb{R}_+ = \bigcup_{I\in\mathcal{D}} A_I$, and

$$\nu(A_I) = \frac{2^d - 1}{d} \quad \text{for all } I \in \mathcal{D}. \tag{3.3}$$

Hence, for every Borel measurable set $B \subset \mathbb{R}^d \times \mathbb{R}_+$, we have

$$\nu(B) = \sum_{I \in \mathcal{D}} \nu(B \cap A_I).$$

In particular,

$$\nu(\{(x,t) : h(x,t) \geq \alpha\}) = \sum_{I \in \mathcal{D}} \nu(\{(x,t) \in A_I : h(x,t) \geq \alpha\}).$$

On the other hand side, by the definition of $a(h, I)$ and (h, I), for each $I \in \mathcal{D}$, we have

$$\begin{aligned} \nu(\{(x,t) \in A_I : h(x,t) \geq \alpha\}) &\leq \nu(\{(x,t) \in A_I : a(h,I) \geq \alpha\}), \\ \nu(\{(x,t) \in A_I : h(x,t) \geq \alpha\}) &\geq \nu(\{(x,t) \in A_I : b(h,I) \geq \alpha\}). \end{aligned}$$

Thus,

$$\sum_{\substack{I \in \mathcal{D} \\ b(h,I) \geq \alpha}} \nu(A_I) \leq \nu(\{(x,t) : h(x,t) \geq \alpha\}) \leq \sum_{\substack{I \in \mathcal{D} \\ a(h,I) \geq \alpha}} \nu(A_I),$$

which, together with (3.3), implies

$$\sum_{\substack{I \in \mathcal{D} \\ b(h,I) \geq \alpha}} 1 \leq \nu(\{(x,t) : h(x,t) \geq \alpha\}) \leq 2^d \sum_{\substack{I \in \mathcal{D} \\ a(h,I) \geq \alpha}} 1.$$

According to the definition of the decreasing rearrangement function, the desired assertion follows immediately. ∎

Lemma 3.13. *Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$, $\eta \in 2\mathbb{N} + 1$, and let the function a be as in Lemma 3.12.*

(i) *If $\mathfrak{m}_f$ is as in Theorem 3.3, then for each $I \in \mathcal{D}$,*

$$a(\mathfrak{m}_f, I) \leq c_{d,p}\, m(I)^{\frac{1}{p} - \frac{1}{d} - 1} \|f - f_{\eta I}\|_{L_1(\eta I)},$$

and consequently,

$$\|\mathfrak{m}_f\|_{L_{p,\infty}(\mathbb{R}^d \times \mathbb{R}_+, \frac{dx\,dt}{t^{d+1}})} \leq 2^{d+2} \left\| \left\{ m(I)^{\frac{1}{p} - \frac{1}{d} - 1} \|f - f_{\eta I}\|_{L_1(\eta I)} \right\}_{I \in \mathcal{D}} \right\|_{l_{p,\infty}(\mathcal{D})}.$$

(ii) *If $\mathfrak{m}_f$ is as in Theorem 3.9, then for each $I \in \mathcal{D}$,*

$$a(\mathfrak{m}_f, I) \leq c_{s,d,p}\, m(I)^{\frac{1}{p} - \frac{s}{d} - 1} \|f - f_{\eta I}\|_{L_1(\eta I)},$$

and consequently,

$$\|\mathfrak{m}_f\|_{L_p(\mathbb{R}^d \times \mathbb{R}_+, \frac{dx\,dt}{t^{d+1}})} \leq 2^{d+2} \left\| \left\{ m(I)^{\frac{1}{p} - \frac{s}{d} - 1} \|f - f_{\eta I}\|_{L_1(\eta I)} \right\}_{I \in \mathcal{D}} \right\|_{l_p(\mathcal{D})}.$$

Proof. We only prove (ii). The proof of (i) follows mutatis mutandis.

Take $I \in \mathcal{D}$. If $x \in I$ and $t \in [s(I)/2, s(I))$, then $B(x,t) \subset 3I \subset \eta I$, and hence

$$\|f_{B(x,t)} - f_{\eta I}\|_{L_1(B(x,t))} \leq \|f - f_{\eta I}\|_{L_1(B(x,t))} \leq \|f - f_{\eta I}\|_{L_1(\eta I)}.$$

By the triangle inequality, $x \in I$ and $t \in [s(I)/2, s(I))$, we further have

$$\|f - f_{B(x,t)}\|_{L_1(B(x,t))} \le 2\|f - f_{\eta I}\|_{L_1(\eta I)}.$$

Therefore, for each $x \in I$ and $t \in [s(I)/2, s(I))$, we have

$$\mathfrak{m}_f(x,t) = t^{\frac{d}{p}-d-s}\|f - f_{B(x,t)}\|_{L_1(B(x,t))} \le 2^{s+d+1-\frac{1}{p}} m(I)^{\frac{1}{p}-\frac{s}{d}-1}\|f - f_{\eta I}\|_{L_1(\eta I)}.$$

Since I is taken arbitrarily, the last inequality means

$$a(\mathfrak{m}_f, I) \le c_{s,d,p}\, m(I)^{\frac{1}{p}-\frac{s}{d}-1}\|f - f_{\eta I}\|_{L_1(\eta I)} \quad \text{for all } I \in \mathfrak{D}.$$

This is the first assertion of (ii), and the second inequality of (ii) follows now from Lemma 3.12 (with $h = \mathfrak{m}_f$ there). ∎

Lemma 3.14. *Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$, and let $\eta \in 2\mathbb{N} + 1$.*

(i) *If $\mathfrak{M}_f$ is as in Theorem 3.3, then*

$$\Big\|\big\{m(I)^{\frac{1}{p}-\frac{1}{d}}\|f - f_{\eta I}\|_{L_\infty(\eta I)}\big\}_{I\in\mathfrak{D}}\Big\|_{l_{p,\infty}(\mathfrak{D})} \le c_{p,d,\eta}\|\mathfrak{M}_f\|_{L_{p,\infty}(\mathbb{R}^d\times\mathbb{R}_+,\frac{dx\,dt}{t^{d+1}})}.$$

(ii) *If $\mathfrak{M}_f$ is as in Theorem 3.9, then*

$$\Big\|\big\{m(I)^{\frac{1}{p}-\frac{s}{d}}\|f - f_{\eta I}\|_{L_\infty(\eta I)}\big\}_{I\in\mathfrak{D}}\Big\|_{l_p(\mathfrak{D})} \le c_{p,s,d,\eta}\|\mathfrak{M}_f\|_{L_p(\mathbb{R}^d\times\mathbb{R}_+,\frac{dx\,dt}{t^{d+1}})}.$$

Proof. We only prove (ii), and (i) can be proved by the same argument.

Take $I \in \mathfrak{D}$. If $x \in I$ and $t \in [s(I)/2, s(I))$, then $\eta I \subset B(x, 2Jt)$, and hence

$$\|f_{\eta I} - f_{B(x,2Jt)}\|_{L_\infty(\eta I)} \le \|f - f_{B(x,2Jt)}\|_{L_\infty(\eta I)} \le \|f - f_{B(x,2Jt)}\|_{L_\infty(B(x,2Jt))}.$$

By the triangle inequality, we have

$$\|f - f_{\eta I}\|_{L_\infty(\eta I)} \le 2\|f - f_{B(x,2Jt)}\|_{L_\infty(B(x,2Jt))}.$$

Thus, for each $x \in I$ and $t \in [s(I)/2, s(I))$, we have

$$\begin{aligned}\mathfrak{M}_f(x, 2Jt) &= (2Jt)^{\frac{d}{p}-s}\|f - f_{B(x,2Jt)}\|_{L_\infty(B(x,2Jt))}\\ &\ge \frac{1}{2}\min\{J^{\frac{d}{p}-1}, (2J)^{\frac{d}{p}-s}\}\|f - f_{\eta I}\|_{L_\infty(\eta I)}\, m(I)^{\frac{1}{p}-\frac{s}{d}}.\end{aligned}$$

Let $h(x,t) = \mathfrak{M}_f(x, 2Jt)$. The last inequality implies that

$$m(I)^{\frac{1}{p}-\frac{s}{d}}\|f - f_{\eta I}\|_{L_\infty(\eta I)} \le c_{p,s,d,\eta}\, b(h, I) \quad \text{for all } I \in \mathfrak{D}.$$

Therefore,

$$\begin{aligned}\Big\|\big\{m(I)^{\frac{1}{p}-\frac{s}{d}}\|f - f_{\eta I}\|_{L_\infty(\eta I)}\big\}_{I\in\mathfrak{D}}\Big\|_{l_p(\mathfrak{D})} &\le c_{p,s,d,J}\|\{b(h,I)\}_{I\in\mathfrak{D}}\|_{l_p(\mathfrak{D})}\\ &\le c_{p,s,d,\eta}\|h\|_{L_p(\mathbb{R}^d\times\mathbb{R}_+,\frac{dx\,dt}{t^{d+1}})},\end{aligned}$$

where the second inequality is due to Lemma 3.12. Note that

$$\|h\|_{L_p(\mathbb{R}^d\times\mathbb{R}_+,\frac{dx\,dt}{t^{d+1}})} \le c_K\|\mathfrak{M}_f\|_{L_p(\mathbb{R}^d\times\mathbb{R}_+,\frac{dx\,dt}{t^{d+1}})}.$$

The desired assertion follows. ∎

Now we are ready to prove Theorems 3.1 and 3.2.

Proof of Theorem 3.1. Note that the implication (iii) $\Rightarrow$ (ii) is obvious. For the implication (ii) $\Rightarrow$ (i), if (ii) holds, then, by Lemma 3.13, we have $\mathfrak{m}_f \in L_p$. Hence, by Theorem 3.3, we deduce that $f \in \dot{W}^1_p(\mathbb{R}^d)$, that is, (i) holds. Similarly, the implication (i) $\Rightarrow$ (iii) follows from Theorem 3.3 and Lemma 3.14. ∎

Proof of Theorem 3.2. Note that the implication (iii) $\Rightarrow$ (ii) is obvious. And the implication (ii) $\Rightarrow$ (i) follows from Theorem 3.9 and Lemma 3.13. The implication (i) $\Rightarrow$ (iii) follows from Theorem 3.9 and Lemma 3.14. ∎

4. Proof of Theorem 1.1 (ii) and Theorem 1.2 (ii)

In this section, we begin to prove our main results, Theorems 1.1 and 1.2. To lighten the notations, we introduce some symbols (recall that $\Phi(\mathbb{R}^d)$ was given in Definition 2.1). For each measurable function $f:\mathbb{R}^d \to \mathbb{C}$, $\Re f$ denotes the real part of f.

Notation 4.1. For every $I \in \mathcal{D}$ and a given measurable function $f:\mathbb{R}^d \to \mathbb{C}$, let $c_I(f)$ be a real number such that $m(F_I), m(G_I) \geq \frac{1}{2} m(3I)$, where

$$F_I = \{x \in 3I : (\Re f)(x) \geq c_I(f)\}, \quad G_I = \{x \in 3I : (\Re f)(x) \leq c_I(f)\}.$$

Notation 4.2. Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$ and let $\phi \in \Phi(\mathbb{R}^d)$. For $\varepsilon \in (0,1)$, denote

$$T_f = [M_f, (-\Delta)^{\frac{\varepsilon}{2}}], \quad T_{f,\phi} = M_{e^{\frac{1}{2}\phi}}[M_f, (-\Delta)^{\frac{\varepsilon}{2}}]M_{e^{-\frac{1}{2}\phi}}.$$

With the above symbols at hand, we present several useful lemmas.

Lemma 4.3. *If $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$ and $I \in \mathcal{D}$, then*

$$\|\Re f - (\Re f)_{3I}\|_{L_1(3I)} \leq c_{d,\varepsilon}\, m(I)^{\frac{\varepsilon}{d}} |\langle T_f \chi_{F_I}, \chi_{G_I}\rangle|.$$

Proof. If $x \in F_I$ and $y \in G_I$, then it is obvious that

$$\max\{|(\Re f)(x) - c_I(f)|, |(\Re f)(y) - c_I(f)|\} \leq (\Re f)(x) - (\Re f)(y).$$

Also recall that the integral kernel of $(-\Delta)^{\frac{\varepsilon}{2}}$ is

$$(x,y) \mapsto c^{(1)}_{d,\varepsilon}|x-y|^{-\varepsilon-d}, \quad x,y \in \mathbb{R}^d.$$

Now, we write

$$\begin{aligned}
&\Big|\int_{F_I \times G_I} (f(x) - f(y))|x-y|^{-d-\varepsilon}\,dx\,dy\Big| \\
&\quad \geq \Big|\Re\Big(\int_{F_I \times G_I} (f(x) - f(y))|x-y|^{-d-\varepsilon}\,dx\,dy\Big)\Big| \\
&\quad = \Big|\int_{F_I \times G_I} ((\Re f)(x) - (\Re f)(y))|x-y|^{-d-\varepsilon}\,dx\,dy\Big| \\
&\quad \geq \int_{F_I \times G_I} |(\Re f)(x) - c_I(f)| \cdot |x-y|^{-d-\varepsilon}\,dx\,dy.
\end{aligned}$$

On the other hand, it is clear that

$$|x-y|^{-d-\varepsilon} \geq (\sqrt{d})^{-d-\varepsilon}\, m(3I)^{-1-\frac{\varepsilon}{d}} \quad \text{for all } x, y \in 3I.$$

Thus,

$$\begin{aligned}
&\Big|\int_{F_I\times G_I} (f(x)-f(y))|x-y|^{-d-\varepsilon}\, dx\, dy\Big| \\
&\quad \geq (3\sqrt{d})^{-d-\varepsilon}\, m(I)^{-1-\frac{\varepsilon}{d}} \int_{F_I\times G_I} |(\Re f)(x)-c_I(f)|\, dx\, dy \\
&\quad \geq c^{(2)}_{d,\varepsilon}\, m(I)^{-\frac{\varepsilon}{d}} \|\Re f - c_I(f)\|_{L_1(F_I)}.
\end{aligned}$$

Similarly, we have

$$\Big|\int_{F_I\times G_I} (f(x)-f(y))|x-y|^{-d-\varepsilon}\, dx\, dy\Big| \geq c^{(2)}_{d,\varepsilon}\, m(I)^{-\frac{\varepsilon}{d}} \|\Re f - c_I(f)\|_{L_1(G_I)}.$$

Summing these inequalities and taking into account that

$$\|\Re f - (\Re f)_{3I}\|_{L_1(3I)} \leq 2\|\Re f - c_I(f)\|_{L_1(3I)},$$

we conclude that

$$\Big|\int_{F_I\times G_I} (f(x)-f(y))|x-y|^{-d-\varepsilon}\, dx\, dy\Big| \geq \frac{1}{2}\, c^{(2)}_{d,\varepsilon}\, m(I)^{-\frac{\varepsilon}{d}} \|\Re f - (\Re f)_{3I}\|_{L_1(3I)}.$$

This is exactly the desired assertion. ∎

Lemma 4.4. *If $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$, $\phi \in \Phi(\mathbb{R}^d)$ and $I \in \mathfrak{D}$, then*

$$\|\Re f - (\Re f)_{3I}\|_{L_1(3I)} \leq c_{d,\varepsilon}\, m(I)^{\frac{\varepsilon}{d}} |\langle T_{f,\phi}(e^{\frac{1}{2}(\phi-\phi_{3I})}\chi_{F_I}), e^{\frac{1}{2}(\phi_{3I}-\phi)}\chi_{G_I}\rangle|.$$

Proof. By Lemma 4.3, we have

$$\begin{aligned}
\|\Re f - (\Re f)_{3I}\|_{L_1(3I)} &\leq c_{d,\varepsilon}\, m(I)^{\frac{\varepsilon}{d}} |\langle T_f\chi_{F_I}, \chi_{G_I}\rangle| \\
&= c_{d,\varepsilon}\, m(I)^{\frac{\varepsilon}{d}} |\langle T_{f,\phi}(e^{\frac{1}{2}\phi}\chi_{F_I}), e^{-\frac{1}{2}\phi}\chi_{G_I}\rangle| \\
&= c_{d,\varepsilon}\, m(I)^{\frac{\varepsilon}{d}} |\langle T_{f,\phi}(e^{\frac{1}{2}(\phi-\phi_{3I})}\chi_{F_I}), e^{\frac{1}{2}(\phi_{3I}-\phi)}\chi_{G_I}\rangle|.
\end{aligned}$$

∎

Lemma 4.5. *If $\phi \in \Phi(\mathbb{R}^d)$, then the two sequences of functions*

$$\{m(I)^{-\frac{1}{2}} e^{\frac{1}{2}(\phi-\phi_{3I})}\chi_{F_I}\}_{I\in\mathfrak{D}}, \quad \{m(I)^{-\frac{1}{2}} e^{\frac{1}{2}(\phi_{3I}-\phi)}\chi_{G_I}\}_{I\in\mathfrak{D}}$$

become NWO when multiplied by a constant c_ϕ.

Proof. Let $r>1$ be from Lemma 2.3. For each $I \in \mathfrak{D}$, we clearly have

$$\begin{aligned}
\|m(I)^{-\frac{1}{2}} e^{\frac{1}{2}(\phi-\phi_{3I})}\chi_{F_I}\|_{L_{2r}(\mathbb{R}^d)} &\leq m(I)^{-\frac{1}{2}} \|e^{\frac{1}{2}|\phi-\phi_{3I}|}\|_{L_{2r}(3I)} \\
&= m(I)^{-\frac{1}{2}} \Big(\int_{3I} e^{r|\phi-\phi_{3I}|}\Big)^{\frac{1}{2r}} \\
&\leq m(I)^{-\frac{1}{2}} (m(3I)\cdot \|r\phi\|_{\Phi(\mathbb{R}^d)})^{\frac{1}{2r}} \\
&\leq c_\phi\, m(I)^{\frac{1}{2r}-\frac{1}{2}}.
\end{aligned}$$

It follows from Lemma 2.5 that the first sequence is NWO. The other one can be proved similarly. ■

Lemma 4.6. *Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$ and $\phi \in \Phi(\mathbb{R}^d)$. If $T_{f,\phi} \in \mathcal{L}_p(L_2(\mathbb{R}^d))$, with $1 < p < \infty$, then*

$$\left\| \left\{ m(I)^{-1-\frac{\varepsilon}{d}} \|f - f_{3I}\|_{L_1(3I)} \right\}_{I \in \mathcal{D}} \right\|_{l_p(\mathcal{D})} \le c_{\phi,p,d,\varepsilon} \|T_{f,\phi}\|_{\mathcal{L}_p(L_2(\mathbb{R}^d))}.$$

If $T_{f,\phi} \in \mathcal{L}_{p,\infty}(L_2(\mathbb{R}^d))$, with $1 < p < \infty$, then

$$\left\| \left\{ m(I)^{-1-\frac{\varepsilon}{d}} \|f - f_{3I}\|_{L_1(3I)} \right\}_{I \in \mathcal{D}} \right\|_{l_{p,\infty}(\mathcal{D})} \le c_{\phi,p,d,\varepsilon} \|T_{f,\phi}\|_{\mathcal{L}_{p,\infty}(L_2(\mathbb{R}^d))}.$$

Proof. We only prove the first assertion as the second one follows mutatis mutandis.

Using Lemma 4.4, we write

$$\begin{aligned} &\left\| \left\{ \|\Re f - (\Re f)_{3I}\|_{L_1(3I)} m(I)^{-1-\frac{\varepsilon}{d}} \right\}_{I \in \mathcal{D}} \right\|_{l_p(\mathcal{D})} \\ &\quad \le c_{d,\varepsilon} \left\| \left\{ \langle T_{f,\phi}(m(I)^{-\frac{1}{2}} e^{\frac{1}{2}(\phi - \phi_{3I})} \chi_{F_I}),\, m(I)^{-\frac{1}{2}} e^{\frac{1}{2}(\phi_{3I} - \phi)} \chi_{G_I} \rangle \right\}_{I \in \mathcal{D}} \right\|_{l_p(\mathcal{D})}. \end{aligned}$$

It follows now from Lemma 4.5 and Lemma 2.6 that

$$\left\| \left\{ \|\Re f - (\Re f)_{3I}\|_{L_1(3I)}\, m(I)^{-1-\frac{\varepsilon}{d}} \right\}_{I \in \mathcal{D}} \right\|_{l_p(\mathcal{D})} \le \frac{1}{2} c_{\phi,p,d,\varepsilon} \|T_{f,\phi}\|_{\mathcal{L}_p(L_2(\mathbb{R}^d))}.$$

Replacing f with if, we obtain

$$\left\| \left\{ \|\Im f - (\Im f)_{3I}\|_{L_1(3I)}\, m(I)^{-1-\frac{\varepsilon}{d}} \right\}_{I \in \mathcal{D}} \right\|_{l_p(\mathcal{D})} \le \frac{1}{2} c_{\phi,p,d,\varepsilon} \|T_{if,\phi}\|_{\mathcal{L}_p(L_2(\mathbb{R}^d))},$$

where $\Im f$ denotes the imaginary part of f. Observe that $T_{if,\phi} = iT_{f,\phi}$. Hence,

$$\|T_{if,\phi}\|_{\mathcal{L}_p(L_2(\mathbb{R}^d))} = \|T_{f,\phi}\|_{\mathcal{L}_p(L_2(\mathbb{R}^d))},$$

and consequently the desired assertion follows now from the triangle inequality. ■

The next fact follows from the simple observation that $M_{w^{1/2}}\colon L_2(\mathbb{R}^d, w(x)\,dx) \mapsto L_2(\mathbb{R}^d)$ is a unitary operator, see, for instance, Lemma 2.7 in [29] and Lemma 2.7 in [19].

Fact 4.7. If w is an A_2 weight which is almost everywhere non-zero, then the following hold:

(i) $T \in \mathcal{L}_\infty(L_2(\mathbb{R}^d, w(x)\,dx))$ if and only if $M_{w^{1/2}} T M_{w^{-1/2}} \in \mathcal{L}_\infty(L_2(\mathbb{R}^d))$. Moreover,

$$\|T\|_{\mathcal{L}_\infty(L_2(\mathbb{R}^d, w(x)\,dx))} = \|M_{w^{1/2}} T M_{w^{-1/2}}\|_{\mathcal{L}_\infty(L_2(\mathbb{R}^d))}.$$

(ii) For each $p > 0$, $T \in \mathcal{L}_p(L_2(\mathbb{R}^d, w(x)\,dx))$ if and only if

$$M_{w^{1/2}} T M_{w^{-1/2}} \in \mathcal{L}_p(L_2(\mathbb{R}^d)).$$

Moreover,

$$\|T\|_{\mathcal{L}_p(L_2(\mathbb{R}^d, w(x)\,dx))} = \|M_{w^{1/2}} T M_{w^{-1/2}}\|_{\mathcal{L}_p(L_2(\mathbb{R}^d))}.$$

(iii) For each $p > 0$, $T \in \mathcal{L}_{p,\infty}(L_2(\mathbb{R}^d, w(x)\,dx))$ if and only if

$$M_{w^{1/2}} T M_{w^{-1/2}} \in \mathcal{L}_{p,\infty}(L_2(\mathbb{R}^d)).$$

Moreover,

$$\|T\|_{\mathcal{L}_{p,\infty}(L_2(\mathbb{R}^d, w(x)\,dx))} = \|M_{w^{1/2}} T M_{w^{-1/2}}\|_{\mathcal{L}_{p,\infty}(L_2(\mathbb{R}^d))}.$$

Now we are ready to prove part (ii) in Theorems 1.1 and 1.2.

Proof of Theorem 1.1 (ii). Let $\varepsilon \in (0,1)$ and let $p > d/(1-\varepsilon)$, which in turn implies $\varepsilon < 1 - d/p$. Since $w \in A_2(\mathbb{R}^d)$, we can write $w = e^{\phi}$ with $\phi \in \Phi(\mathbb{R}^d)$. For $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$, assume that $T_f = [M_f, (-\Delta)^{\frac{\varepsilon}{2}}] \in \mathcal{L}_p(L_2(\mathbb{R}^d, w(x)\,dx))$. Then $T_{f,\phi} \in \mathcal{L}_p(L_2(\mathbb{R}^d))$ by the last fact. Now item (ii) of Theorem 1.1 follows by combining Lemma 4.6 and Theorem 3.2 (with $s = \varepsilon + d/p$). ∎

Proof of Theorem 1.2 (ii). Let $\varepsilon \in (0,1)$. For $w \in A_2(\mathbb{R}^d)$, write $w = e^{\phi}$ with $\phi \in \Phi(\mathbb{R}^d)$. For $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$, assume that $T_f = [M_f, (-\Delta)^{\frac{\varepsilon}{2}}] \in \mathcal{L}_{\frac{d}{1-\varepsilon},\infty}(L_2(\mathbb{R}^d, w(x)\,dx))$. Then $T_{f,\phi} \in \mathcal{L}_{\frac{d}{1-\varepsilon},\infty}(L_2(\mathbb{R}^d))$ by the above fact. Now the desired assertion follows by combining Lemma 4.6 and Theorem 3.1 (with $p = d/(1-\varepsilon)$ there). ∎

5. Proof of Theorem 1.1 (i) and Theorem 1.2 (i)

In this section, we finish the proofs of Theorems 1.1 and 1.2. To this end, we first recall the Whitney decomposition (for convenience, we provide a proof of it in Appendix B). In what follows, for each $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we write

$$|x|_\infty = \sup_{1 \le k \le d} |x_k|.$$

Lemma 5.1. *Let*

$$\Omega = \{(x,y) : x, y \in \mathbb{R}^d,\ x \neq y\}.$$

There exists $\mathcal{F} \subset \mathcal{D} \times \mathcal{D}$ such that:

(i) *the sets $\{I \times J\}_{I \times J \in \mathcal{F}}$ are pairwise disjoint,*

(ii) $\bigcup_{I \times J \in \mathcal{F}} I \times J = \Omega$,

(iii) *$s(I) = s(J)$ for every $I \times J \in \mathcal{F}$,*

(iv) *for every $I \times J \in \mathcal{F}$, we have*

$$|x - y|_\infty \in [2s(I), 10s(I)), \quad x \in I,\ y \in J.$$

Lemma 5.2. *For each $\varepsilon \in (0,1)$, there exists a rapidly decreasing*[3] *sequence $\{\lambda_k\}_{k \in \mathbb{Z}^d}$ such that*

$$|x|^{-d-\varepsilon} = \sum_{k \in \mathbb{Z}^d} \lambda_k e_k(x), \quad |x|_\infty \in [\tfrac{2}{5}, 2).$$

Here, $e_k(x) = e^{i\langle x,k\rangle}$, $x, k \in \mathbb{R}^d$.

[3] This means: for every $n \in \mathbb{N}$, $\lambda_k = O(|k|^{-n})$.

Proof. Let ϕ be a smooth function compactly supported in $(-2/5, 2/5)^d$. Let ψ be a smooth function compactly supported in $(-\pi, \pi)^d$ such that $\psi = 1$ on $(-2, 2)^d$. The function

$$\theta : x \mapsto (1 - \phi(x))\psi(x)|x|^{-d-\varepsilon}, \quad x \in \mathbb{R}^d,$$

is smooth and compactly supported in $(-\pi, \pi)^d$. Thus, the Fourier coefficients $\{\lambda_k\}_{k \in \mathbb{Z}^d}$ of $\theta|_{(-\pi,\pi)^d}$ decay faster than any power. We clearly have

$$|x|^{-d-\varepsilon} = \sum_{k \in \mathbb{Z}^d} \lambda_k e_k(x), \quad x \in (-2, 2)^d \setminus (-2/5, 2/5)^d. \qquad \blacksquare$$

Next, with the help of Lemma 5.1, we decompose the commutator $T_{f,\phi}$ (see Notation 4.2) into good shape. We shall use the following notations. The number 21 below is not that important, and actually, we only need an odd integer which is large enough.

Notation 5.3. Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$, $\phi \in \Phi(\mathbb{R}^d)$, and let $\mathcal{F}$ be as in Lemma 5.1. For each $I \in \mathcal{D}$, set

$$J_I = \bigcup_{J \in \mathcal{D}: I \times J \in \mathcal{F}} J.$$

For each $k \in \mathbb{Z}^d$ and $I \in \mathcal{D}$, denote

$$a_{k,I} = (f - f_{21I})e^{\frac{1}{2}\phi} e_{\frac{k}{5s(I)}} \chi_I, \quad b_{k,I} = e^{-\frac{1}{2}\phi} e_{-\frac{k}{5s(I)}} \chi_{J_I},$$
$$c_{k,I} = e^{\frac{1}{2}\phi} e_{\frac{k}{5s(I)}} \chi_I, \quad d_{k,I} = (f - f_{21I})e^{-\frac{1}{2}\phi} e_{-\frac{k}{5s(I)}} \chi_{J_I}.$$

Denote by A_k and B_k the integral operators with integral kernels, respectively,

$$(x, y) \mapsto \sum_{I \in \mathcal{D}} m(I)^{-1-\frac{\varepsilon}{d}} a_{k,I}(x) b_{k,I}(y), \quad (x, y) \mapsto \sum_{I \in \mathcal{D}} m(I)^{-1-\frac{\varepsilon}{d}} c_{k,I}(x) d_{k,I}(y).$$

Lemma 5.4. *Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$ and let $\phi \in \Phi(\mathbb{R}^d)$. Let the sequence $\{\lambda_k\}_{k \in \mathbb{Z}^d}$ be as in Lemma 5.2. We have*

$$T_{f,\phi} = c_{d,\varepsilon} \sum_{k \in \mathbb{Z}^d} \lambda_k (A_k - B_k).$$

Proof. Recall that the integral kernel of the operator $(-\Delta)^{\frac{\varepsilon}{2}}$ is given by the formula

$$(x, y) \mapsto c_{d,\varepsilon}^{(1)} |x - y|^{-d-\varepsilon}.$$

Hence, the integral kernel of $T_{f,\phi} := M_{e^{\phi/2}} [M_f, (-\Delta)^{\frac{\varepsilon}{2}}] M_{e^{-\phi/2}}$ is given by the formula

$$K_{f,\phi} : (x, y) \mapsto c_{d,\varepsilon}^{(1)} e^{\frac{1}{2}\phi(x)} \frac{f(x) - f(y)}{|x - y|^{d+\varepsilon}} e^{-\frac{1}{2}\phi(y)}.$$

We claim that for each $(x, y) \in \Omega$ (the set Ω is from Lemma 5.1),

$$e^{\frac{1}{2}\phi(x)} \frac{f(x) - f(y)}{|x - y|^{d+\varepsilon}} e^{-\frac{1}{2}\phi(y)}$$
$$= 5^{-d-\varepsilon} \sum_{k \in \mathbb{Z}^d} \lambda_k \sum_{I \in \mathcal{D}} m(I)^{-1-\frac{\varepsilon}{d}} (a_{k,I}(x) b_{k,I}(y) - c_{k,I}(x) d_{k,I}(y)).$$

The right-hand side formula will be denoted by $\mathrm{RHS}(x,y)$. The desired assertion of the lemma (with $c_{d,\varepsilon} = 5^{-d-\varepsilon} c^{(1)}_{d,\varepsilon}$) would follow from the claim.

For each $I \in \mathcal{D}$, if $x \in I$ and $y \in J_I$, then (provided that $J_I \neq \varnothing$) it follows from Lemma 5.1 (iv) that

$$\left|\frac{x-y}{5s(I)}\right|_\infty \in [2/5, 2),$$

and consequently, by Lemma 5.2, we have

$$\sum_{k\in\mathbb{Z}^d} \lambda_k e_k\Big(\frac{x-y}{5s(I)}\Big) = \left|\frac{x-y}{5s(I)}\right|^{-d-\varepsilon}.$$

Note that for each $k \in \mathbb{Z}^d$ and $I \in \mathcal{D}$,

$$\begin{aligned} &a_{k,I}(x)b_{k,I}(y) - c_{k,I}(x)d_{k,I}(y) \\ &\qquad = e^{\frac{1}{2}\phi(x)}[f(x)-f(y)]\, e^{-\frac{1}{2}\phi(y)} \cdot \chi_I(x)\chi_{J_I}(y) \cdot e_k\Big(\frac{x-y}{5s(I)}\Big). \end{aligned}$$

If $f(x) = f(y)$, it is clearly that the claim holds true. If $f(x) \neq f(y)$, then, for each $(x,y) \in \Omega$, we have

$$\begin{aligned} \frac{\mathrm{RHS}(x,y)}{e^{\frac{1}{2}\phi(x)}[f(x)-f(y)]e^{-\frac{1}{2}\phi(y)}} &= 5^{-d-\varepsilon} \sum_{k\in\mathbb{Z}^d} \lambda_k \sum_{I\in\mathcal{D}} m(I)^{-1-\frac{\varepsilon}{d}} \chi_I(x)\chi_{J_I}(y)\cdot e_k\Big(\frac{x-y}{5s(I)}\Big) \\ &= 5^{-d-\varepsilon} \sum_{I\in\mathcal{D}} \left|\frac{x-y}{5s(I)}\right|^{-d-\varepsilon} \cdot m(I)^{-1-\frac{\varepsilon}{d}} \chi_I(x)\chi_{J_I}(y) \\ &= |x-y|^{-d-\varepsilon} \sum_{I\in\mathcal{D}} s(I)^{d+\varepsilon} m(I)^{-1-\frac{\varepsilon}{d}} \chi_I(x)\chi_{J_I}(y) \\ &= |x-y|^{-d-\varepsilon} \sum_{I\in\mathcal{D}} \chi_I(x)\chi_{J_I}(y) \\ &= |x-y|^{-d-\varepsilon} \sum_{I\times J\in\mathcal{F}} \chi_I(x)\chi_J(y) = |x-y|^{-d-\varepsilon}. \end{aligned}$$

This proves the claim, and hence the proof of the lemma is complete. ∎

Lemma 5.5. *Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$ and let $\phi \in \Phi(\mathbb{R}^d)$. For each $I \in \mathcal{D}$ and $k \in \mathbb{Z}^d$, the functions $a_{k,I}$, $b_{k,I}$, $c_{k,I}$ and $d_{k,I}$ are supported in $21I$. Moreover, if $r > 1$ is as in Lemma 2.3, then*

$$\begin{aligned} \|a_{k,I}\|_{L_{2r}(\mathbb{R}^d)} &\leq m(21I)^{\frac{1}{2r}}\, e^{\frac{1}{2}\phi_{21I}} \|f - f_{21I}\|_{L_\infty(21I)} \|r\phi\|^{\frac{1}{2r}}_{\Phi(\mathbb{R}^d)}, \\ \|b_{k,I}\|_{L_{2r}(\mathbb{R}^d)} &\leq m(21I)^{\frac{1}{2r}}\, e^{-\frac{1}{2}\phi_{21I}} \|r\phi\|^{\frac{1}{2r}}_{\Phi(\mathbb{R}^d)}, \\ \|c_{k,I}\|_{L_{2r}(\mathbb{R}^d)} &\leq m(21I)^{\frac{1}{2r}}\, e^{\frac{1}{2}\phi_{21I}} \|r\phi\|^{\frac{1}{2r}}_{\Phi(\mathbb{R}^d)}, \\ \|d_{k,I}\|_{L_{2r}(\mathbb{R}^d)} &\leq m(21I)^{\frac{1}{2r}}\, e^{-\frac{1}{2}\phi_{21I}} \|f - f_{21I}\|_{L_\infty(21I)} \|r\phi\|^{\frac{1}{2r}}_{\Phi(\mathbb{R}^d)}. \end{aligned}$$

Proof. Take $I \in \mathcal{D}$ and $k \in \mathbb{Z}^d$. By definition (see Notation 5.3), $a_{k,I}$ and $c_{k,I}$ are supported in I, hence in $21I$. By definition, $b_{k,I}$ and $d_{k,I}$ are supported in J_I. By Lemma 5.1 (iv), $J_I \subset 21I$, and thus $b_{k,I}$ and $d_{k,I}$ are supported in $21I$. By Lemma 2.3, we have

$$\begin{aligned} e^{-\frac{1}{2}\phi_{21I}}\|a_{k,I}\|_{L_{2r}(21I)} &\leq \|f - f_{21I}\|_{L_\infty(21I)}\|e^{\frac{1}{2}(\phi-\phi_{21I})}\|_{L_{2r}(21I)} \\ &= \|f - f_{21I}\|_{L_\infty(21I)} \cdot \Big(\int_{21I} e^{r|\phi-\phi_{21I}|}\Big)^{\frac{1}{2r}} \\ &\leq \|f - f_{21I}\|_{L_\infty(21I)} \cdot (m(21I)\|r\phi\|_{\Phi(\mathbb{R}^d)})^{\frac{1}{2r}}. \end{aligned}$$

This delivers the estimate for $a_{k,I}$. The estimates for $b_{k,I}$, $c_{k,I}$ and $d_{k,I}$ are similar. ∎

Lemma 5.6. *Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$ and let $\phi \in \Phi(\mathbb{R}^d)$. We have*

$$\begin{aligned} &\max\{\|A_k\|_{\mathcal{L}_{\frac{d}{1-\varepsilon},\infty}(L_2(\mathbb{R}^d))}, \|B_k\|_{\mathcal{L}_{\frac{d}{1-\varepsilon},\infty}(L_2(\mathbb{R}^d))}\} \\ &\qquad \leq c_{\phi,d,\varepsilon}\Big\|\big\{m(I)^{-\frac{\varepsilon}{d}}\|f - f_{21I}\|_{L_\infty(21I)}\big\}_{I\in\mathcal{D}}\Big\|_{l_{\frac{d}{1-\varepsilon},\infty}(\mathcal{D})}. \end{aligned}$$

Similarly, for $p > d/(1-\varepsilon)$, we have

$$\begin{aligned} &\max\{\|A_k\|_{\mathcal{L}_p(L_2(\mathbb{R}^d))}, \|B_k\|_{\mathcal{L}_p(L_2(\mathbb{R}^d))}\} \\ &\qquad \leq c'_{\phi,d,\varepsilon}\Big\|\big\{m(I)^{-\frac{\varepsilon}{d}}\|f - f_{21I}\|_{L_\infty(21I)}\big\}_{I\in\mathcal{D}}\Big\|_{l_p(\mathcal{D})}. \end{aligned}$$

Proof. Let $r > 1$ be as in Lemma 2.3. Using Corollary 2.8 and Lemma 5.5, we write

$$\begin{aligned} \|A_k\|_{\mathcal{L}_{\frac{d}{1-\varepsilon},\infty}} &\leq c_{\frac{d}{1-\varepsilon},r,d}\Big\|\big\{m(I)^{-\frac{\varepsilon}{d}-\frac{1}{r}}\|a_{k,I}\|_{L_{2r}(\mathbb{R}^d)}\|b_{k,I}\|_{L_{2r}(\mathbb{R}^d)}\big\}_{I\in\mathcal{D}}\Big\|_{l_{\frac{d}{1-\varepsilon},\infty}(\mathcal{D})} \\ &\leq 21^{\frac{d}{r}}c_{\frac{d}{1-\varepsilon},r,d}\|r\phi\|_{\Phi(\mathbb{R}^d)}^{\frac{1}{r}}\Big\|\big\{m(I)^{-\frac{\varepsilon}{d}}\|f - f_{21I}\|_{L_\infty(21I)}\big\}_{I\in\mathcal{D}}\Big\|_{l_{\frac{d}{1-\varepsilon},\infty}(\mathcal{D})}. \end{aligned}$$

The other inequalities can be proved similarly. ∎

Lemma 5.7. *Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$ and let $\phi \in \Phi(\mathbb{R}^d)$. Then*

$$\|T_{f,\phi}\|_{\mathcal{L}_{\frac{d}{1-\varepsilon},\infty}(L_2(\mathbb{R}^d))} \leq c_{\phi,d,\varepsilon}\|f\|_{\dot{W}^1_{\frac{d}{1-\varepsilon}}(\mathbb{R}^d)}.$$

Similarly, for $p > d/(1-\varepsilon)$, we have

$$\|T_{f,\phi}\|_{\mathcal{L}_p(L_2(\mathbb{R}^d))} \leq c_{\phi,d,\varepsilon}\|f\|_{\dot{W}_p^{\varepsilon+\frac{d}{p}}}.$$

Proof. Recall that $\mathcal{L}_{\frac{d}{1-\varepsilon},\infty}$ is a Banach space (equivalence constant between the norm and the quasi-norm is at most 2). We can, therefore, use the triangle inequality and write, using Lemma 5.4,

$$\|T_{f,\phi}\|_{\mathcal{L}_{\frac{d}{1-\varepsilon},\infty}} \leq 2c_{d,\varepsilon}\sum_{k\in\mathbb{Z}^d}\lambda_k(\|A_k\|_{\frac{d}{1-\varepsilon},\infty} + \|B_k\|_{\frac{d}{1-\varepsilon},\infty}).$$

Using Lemma 5.6, we write

$$\begin{aligned} &\|T_{f,\phi}\|_{\mathcal{L}_{\frac{d}{1-\varepsilon},\infty}(L_2(\mathbb{R}^d))} \\ &\qquad \leq 4c_{d,\varepsilon}c'_{\phi,d,\varepsilon}\sum_{k\in\mathbb{Z}^d}\lambda_k \cdot \Big\|\big\{m(I)^{-\frac{\varepsilon}{d}}\|f - f_{21I}\|_{L_\infty(21I)}\big\}_{I\in\mathcal{D}}\Big\|_{l_{\frac{d}{1-\varepsilon},\infty}(\mathcal{D})}. \end{aligned}$$

By Theorem 3.1,

$$\|T_{f,\phi}\|_{\mathcal{L}_{\frac{d}{1-\varepsilon},\infty}(L_2(\mathbb{R}^d))} \le 4c'_{d,\varepsilon}c'_{\phi,d,\varepsilon}\sum_{k\in\mathbb{Z}^d}\lambda_k \cdot c''_{d,\varepsilon}\|f\|_{\dot{W}^1_{\frac{d}{1-\varepsilon}}(\mathbb{R}^d)}.$$

Taking into account that $\sum_{k\in\mathbb{Z}^d}\lambda_k$ depends only on d and ε, we complete the proof of the first inequality. Similarly, by using Theorem 3.2, the second inequality can be also proved. ∎

Now we turn back to prove part (i) of Theorems 1.1 and 1.2.

Proof of Theorem 1.1 (i). Write $w = e^{\phi}$ with $\phi \in \Phi(\mathbb{R}^d)$. The assertion follows by combining Lemma 5.7 and Fact 4.7 (ii). ∎

Proof of Theorem 1.2 (i). Write $w = e^{\phi}$ with $\phi \in \Phi(\mathbb{R}^d)$. The assertion follows by combining Lemma 5.7 and Fact 4.7 (iii). ∎

6. From BMO to the separable part of $l_{1,\infty}$

The main result of this section is Theorem 6.1, which will be used to prove Theorem 1.3. In this (and the next) section, for each sequence $a \in l_\infty$, we simply write $\mu(a)$ instead of $\mu_{l_\infty}(a)$. For each $n \in \mathbb{Z}$, we denote

$$\mathcal{D}_n = \{I \in \mathcal{D} : m(I) = 2^{-nd}\}.$$

Theorem 6.1. *Let $I_0 \in \mathcal{D}$ and let $\phi \in \mathrm{BMO}(\mathbb{R}^d)$. Then*

$$\{\|\phi - \phi_{21I}\|_{L_1(21I)}\}_{\substack{I\in\mathcal{D}\\ I\subset I_0}} \in (l_{1,\infty})_0(\mathcal{D}).$$

In this section, the cardinality of a finite set A is denoted by $\mathrm{card}(A)$. To prove this theorem, we need several lemmas.

Lemma 6.2. *Fix $I_0 \in \mathcal{D}_0$. Let $A \subset \mathcal{D}$ such that $I \subset I_0$ for every $I \in A$. If*

$$\mathrm{card}(A \cap \mathcal{D}_n) = o(2^{nd}), \quad n \to \infty,$$

then $\{m(I)\}_{I\in A} \in (l_{1,\infty})_0(\mathcal{D})$.

Proof. Fix $k \in \mathbb{N}$. By the assumption, there exists $n(k) \in \mathbb{Z}_+$ such that

$$\mathrm{card}(A \cap \mathcal{D}_n) \le 2^{(n-k)d}, \quad n \ge n(k).$$

Without loss of generality, we may assume that $n(k) \ge k$. Define a set A_k such that $A \subset A_k \subset \mathcal{D}$, $I \subset I_0$ for every $I \in A_k$, and

$$\mathrm{card}(A_k \cap \mathcal{D}_n) = 2^{(n-k)d}, \quad n \ge n(k).$$

Define a set $B_k \subset \mathcal{D}$ such that $I \subset I_0$ for every $I \in B_k$, such that $B_k \cap \mathcal{D}_n = A_k \cap \mathcal{D}_n$ for $n \ge n(k)$, and $B_k \cap \mathcal{D}_n = \varnothing$ for $n < k$. It is immediate that

$$\mu(\{m(I)\}_{I\in B_k}) = 2^{-kd}\mu(\{m(I)\}_{\substack{I\in\mathcal{D}\\ I\subset I_0}}).$$

Thus,

$$\limsup_{t\to\infty} t\mu(t,\{m(I)\}_{I\in B_k}) \le 2^{-kd}\|\{m(I)\}_{\substack{I\in\mathcal{D}\\ I\subset I_0}}\|_{l_{1,\infty}}.$$

Note that $A \subset A_k$ and the sets A_k and B_k differ only by finitely many elements. Thus,

$$\limsup_{t\to\infty} t\mu(t,\{m(I)\}_{I\in A}) \le \limsup_{k\to\infty} t\mu(t,\{m(I)\}_{I\in A_k}) \le 2^{-kd}\|\{m(I)\}_{\substack{I\in\mathcal{D}\\ I\subset I_0}}\|_{l_{1,\infty}}.$$

Letting $k \to \infty$, we infer

$$\limsup_{t\to\infty} t\mu(t,\{m(I)\}_{I\in A}) = 0.$$

This is exactly the required assertion. ∎

Lemma 6.3. *Fix $I_0 \in \mathcal{D}_0$. Let $\{a_I\}_{I\in\mathcal{D}}$ be a bounded sequence. If*

$$\sum_{\substack{I\in\mathcal{D}_n\\ I\subset I_0}} |a_I| = o(2^{nd}), \quad n\to\infty,$$

then

$$\{a_I m(I)\}_{\substack{I\in\mathcal{D}\\ I\subset I_0}} \subset (l_{1,\infty})_0(\mathcal{D}).$$

Proof. Without loss of generality, assume that $0 \le a_I \le 1$ for every $I \in \mathcal{D}$ and $a_I = 0$ for every $I \not\subset I_0$. Fix $\varepsilon > 0$ and consider the set

$$A = \{I \in \mathcal{D} : I \subset I_0,\ a_I > \varepsilon\}.$$

It is immediate that

$$\begin{aligned}&\limsup_{t\to\infty} t\mu(t,\{a_I m(I)\}_{\substack{I\in\mathcal{D}\\ I\subset I_0}})\\ &\quad\le 2\limsup_{t\to\infty} t\mu(t,\{m(I)\}_{I\in A}) + 2\varepsilon \limsup_{t\to\infty} t\mu\big(t,\{m(I)\}_{\substack{I\in\mathcal{D}\\ I\subset I_0}}\big).\end{aligned}$$

By the definition of A, we have

$$\mathrm{card}(A\cap\mathcal{D}_n) = \sum_{I\in A\cap\mathcal{D}_n} 1 \le \varepsilon^{-1}\sum_{I\in A\cap\mathcal{D}_n} a_I \le \varepsilon^{-1}\sum_{\substack{I\in\mathcal{D}_n\\ I\subset I_0}} a_I = o(2^{nd}), \quad n\to\infty.$$

Hence, the set A satisfies the assumptions in Lemma 6.2. Applying Lemma 6.2, we conclude that

$$\limsup_{t\to\infty} t\mu(t,\{a_I m(I)\}_{\substack{I\in\mathcal{D}\\ I\subset I_0}}) \le 2\varepsilon \limsup_{t\to\infty} t\mu(t,\{m(I)\}_{\substack{I\in\mathcal{D}\\ I\subset I_0}}).$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\limsup_{t\to\infty} t\mu(t,\{a_I m(I)\}_{\substack{I\in\mathcal{D}\\ I\subset I_0}}) = 0.$$

This is exactly the required assertion. ∎

Lemma 6.4. *Fix $I_0 \in \mathfrak{D}_0$. For every $\phi \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$, we have*

$$\sum_{\substack{I \in \mathfrak{D}_n \\ I \subset I_0}} \|\phi - \phi_{21I}\|_{L_1(21I)} = o(1), \quad n \to \infty.$$

Proof. Without loss of generality, we take $I_0 = [0,1]^d$.

Fix $\varepsilon > 0$ and choose $\psi \in \mathrm{Lip}([-2,2]^d)$ such that $\|\phi - \psi\|_{L_1([-2,2]^d)} \leq \varepsilon$. Obviously,

$$\|\phi - \phi_{21I}\|_{L_1(21I)} \leq 2\|\phi - \psi\|_{L_1(21I)} + \|\psi - \psi_{21I}\|_{L_1(21I)}.$$

Hence,

$$\sum_{\substack{I \in \mathfrak{D}_n \\ I \subset I_0}} \|\phi - \phi_{21I}\|_{L_1(21I)} \leq 2 \sum_{\substack{I \in \mathfrak{D}_n \\ I \subset I_0}} \|\phi - \psi\|_{L_1(21I)} + \sum_{\substack{I \in \mathfrak{D}_n \\ I \subset I_0}} \|\psi - \psi_{21I}\|_{L_1(21I)}.$$

Note that

$$\sum_{\substack{I \in \mathfrak{D}_n \\ I \subset I_0}} \|\phi - \psi\|_{L_1(21I)} \leq 21^d \|\phi - \psi\|_{L_1([-2,2]^d)} \leq 21^d \varepsilon.$$

Thus,

$$\sum_{\substack{I \in \mathfrak{D}_n \\ I \subset I_0}} \|\phi - \phi_{21I}\|_{L_1(21I)} \leq 2 \cdot 21^d \varepsilon + \sum_{\substack{I \in \mathfrak{D}_n \\ I \subset I_0}} \|\psi - \psi_{21I}\|_{L_1(21I)}.$$

We have

$$\|\psi - \psi_{21I}\|_{L_1(21I)} \leq \frac{1}{m(21I)} \int_{21I \times 21I} |\psi(y) - \psi(z)| \, dy \, dz.$$

Clearly,

$$|\psi(y) - \psi(z)| \leq \|\psi\|_{\mathrm{Lip}} \cdot |y - z|_\infty \leq 21 \cdot 2^{-n} \|\psi\|_{\mathrm{Lip}}, \quad y, z \in 21I.$$

Thus,

$$\|\psi - \psi_{21I}\|_{L_1(21I)} \leq 21^{d+1} \|\psi\|_{\mathrm{Lip}} \cdot 2^{-n(d+1)}.$$

Consequently,

$$\sum_{\substack{I \in \mathfrak{D}_n \\ I \subset I_0}} \|\phi - \phi_{21I}\|_{L_1(21I)} \leq 2 \cdot 21^d \varepsilon + 21^{d+1} \|\psi\|_{\mathrm{Lip}} \cdot 2^{-n}.$$

Passing to the limit $n \to \infty$, we obtain

$$\limsup_{n \to \infty} \sum_{\substack{I \in \mathfrak{D}_n \\ I \subset I_0}} \|\phi - \phi_{21I}\|_{L_1(21I)} \leq 2 \cdot 21^d \varepsilon.$$

Since $\varepsilon > 0$ is arbitrarily small, the assertion follows. ∎

Now we are ready to prove Theorem 6.1.

Proof of Theorem 6.1. We may assume without loss of generality that $I_0 \in \mathcal{D}_0$. Set

$$a_I = \frac{1}{m(I)} \|\phi - \phi_{2^1 I}\|_{L_1(2^1 I)}, \quad I \in \mathcal{D},\ I \subset I_0.$$

Since $\phi \in \mathrm{BMO}(\mathbb{R}^d)$, it follows that the sequence $\{a_I\}_{I\in\mathcal{D}}$ is bounded. By Lemma 6.4, the sequence $\{a_I\}_{I\in\mathcal{D}}$ satisfies the conditions in Lemma 6.3. Then the desired assertion follows from Lemma 6.3. The proof is complete. ∎

7. Proof of Theorem 1.3

In this section, we calculate the asymptotics of the singular values. To this end, we first prove the following theorem. Recall that the T_f and $T_{f,\phi}$ denote the commutator operator, see Notation 4.2 for details.

Theorem 7.1. *If $f \in C_c^\infty(\mathbb{R}^d)$ and $\phi \in \Phi(\mathbb{R}^d)$, then*

$$T_{f,\phi} - T_f \in (\mathcal{L}_{\frac{d}{1-\varepsilon},\infty})_0(L_2(\mathbb{R}^d)).$$

We first show a decomposition of $T_{f,\phi} - T_f$.

Notation 7.2. Let $f \in C_c^\infty(\mathbb{R}^d)$ and let $\phi \in \Phi(\mathbb{R}^d)$. Denote

$$\begin{aligned}
e'_{k,I} &= (f - f_{2^1 I})(e^{\frac{1}{2}(\phi-\phi_{2^1 I})} - 1)e_{\frac{k}{5s(I)}}\chi_I, \quad e''_{k,I} = e^{\frac{1}{2}(\phi-\phi_{2^1 I})}e_{\frac{k}{5s(I)}}\chi_I,\\
g'_{k,I} &= (f - f_{2^1 I})e_{\frac{k}{5s(I)}}\chi_I, \quad g''_{k,I} = (e^{\frac{1}{2}(\phi-\phi_{2^1 I})} - 1)e_{\frac{k}{5s(I)}}\chi_I,\\
f'_{k,I} &= e^{\frac{1}{2}(\phi_{2^1 I}-\phi)}e_{-\frac{k}{5s(I)}}\chi_{J_I}, \quad f''_{k,I} = (f - f_{2^1 I})(e^{\frac{1}{2}(\phi_{2^1 I}-\phi)} - 1)e_{-\frac{k}{5s(I)}}\chi_{J_I},\\
h'_{k,I} &= (e^{\frac{1}{2}(\phi_{2^1 I}-\phi)} - 1)e_{-\frac{k}{5s(I)}}\chi_{J_I}, \quad h''_{k,I} = (f - f_{2^1 I})e_{-\frac{k}{5s(I)}}\chi_{J_I}.
\end{aligned}$$

The symbol J_I used above is from Notation 5.3.

Denote by C'_k, D'_k, C''_k and D''_k the integral operators with integral kernels

$$\begin{aligned}
(x,y) &\mapsto \sum_{I\in\mathcal{D}} m(I)^{-1-\frac{\varepsilon}{d}} e'_{k,I}(x) f'_{k,I}(y), \quad (x,y) \mapsto \sum_{I\in\mathcal{D}} m(I)^{-1-\frac{\varepsilon}{d}} g'_{k,I}(x) h'_{k,I}(y),\\
(x,y) &\mapsto \sum_{I\in\mathcal{D}} m(I)^{-1-\frac{\varepsilon}{d}} e''_{k,I}(x) f''_{k,I}(y), \quad (x,y) \mapsto \sum_{I\in\mathcal{D}} m(I)^{-1-\frac{\varepsilon}{d}} g''_{k,I}(x) h''_{k,I}(y).
\end{aligned}$$

Lemma 7.3. *If $f \in C_c^\infty(\mathbb{R}^d)$ and $\phi \in \Phi(\mathbb{R}^d)$, then*

$$T_{f,\phi} - T_f = \sum_{k\in\mathbb{Z}^d} \lambda_k (C'_k + D'_k - C''_k - D''_k).$$

Here, the sequence $\{\lambda_k\}_{k\in\mathbb{Z}^d}$ is given by Lemma 5.2.

Proof. The assertion follows from Lemma 5.4. ∎

We next study more properties related to C_k', D_k', C_k'' and D_k''. We begin with the following lemma.

Lemma 7.4. *Let $J \in \mathcal{D}$ and let $\phi \in \mathrm{BMO}(\mathbb{R}^d)$. For every $p > 1$, $r > 0$, we have*

$$\{m(I)^{\frac{1}{p}-\frac{1}{r+1}}\|\phi - \phi_{21I}\|_{L_{r+1}(21I)}\}_{\substack{I\in\mathcal{D}\\ I\subset J}} \in (l_{p,\infty})_0(\mathcal{D}).$$

Proof. It is immediate that

$$\begin{aligned}
&\|\phi - \phi_{21I}\|_{L_{r+1}(21I)}\\
&\quad\le \|\phi - \phi_{21I}\|_{L_1(21I)}^{\frac{1}{2r+1}}\|\phi - \phi_{21I}\|_{L_{2r+2}(21I)}^{\frac{2r}{2r+1}}\\
&\quad= m(21I)^{\frac{r}{(r+1)(2r+1)}}\|\phi - \phi_{21I}\|_{L_1(21I)}^{\frac{1}{2r+1}}\cdot(m(21I)^{-\frac{1}{2r+2}}\|\phi - \phi_{21I}\|_{L_{2r+2}(21I)})^{\frac{2r}{2r+1}}\\
&\quad\le 21^{\frac{dr}{(r+1)(2r+1)}}m(I)^{\frac{1}{r+1}-\frac{1}{2r+1}}\|\phi - \phi_{21I}\|_{L_1(21I)}^{\frac{1}{2r+1}}\cdot(c_{2r+2}\|\phi\|_{\mathrm{BMO}(\mathbb{R}^d)})^{\frac{2r}{2r+1}}.
\end{aligned}$$

Here, the very last inequality follows from the John-Nirenberg theorem (see, e.g., Corollary 7.1.8 in [20]).

Hence, for each $I \in \mathcal{D}$, we have

$$\begin{aligned}
&m(I)^{\frac{1}{p}-\frac{1}{r+1}}\|\phi - \phi_{21I}\|_{L_{r+1}(21I)}\\
&\qquad\le c_{\phi,r}m(I)^{\frac{1}{p}-\frac{1}{2r+1}}\|\phi - \phi_{21I}\|_{L_1(21I)}^{\frac{1}{2r+1}}\\
&\qquad= c_{\phi,r}[m(I)^{\frac{1}{p}}]^{1-\frac{1}{2r+1}}\cdot[m(I)^{\frac{1}{p}-1}\|\phi - \phi_{21I}\|_{L_1(21I)}]^{\frac{1}{2r+1}}\\
&\qquad= c_{\phi,r}[m(I)]^{(1-\frac{1}{2r+1})\cdot\frac{1}{p}}\\
&\qquad\quad\cdot[m(I)^{-1}\|\phi - \phi_{21I}\|_{L_1(21I)}]^{\frac{1}{2r+1}\cdot(1-\frac{1}{p})}[\|\phi - \phi_{21I}\|_{L_1(21I)}]^{\frac{1}{2r+1}\cdot\frac{1}{p}}\\
&\qquad\le c_{\phi,r}[m(I)]^{(1-\frac{1}{2r+1})\cdot\frac{1}{p}}\cdot\|\phi\|_{\mathrm{BMO}(\mathbb{R}^d)}^{\frac{1}{2r+1}\cdot(1-\frac{1}{p})}[\|\phi - \phi_{21I}\|_{L_1(21I)}]^{\frac{1}{2r+1}\cdot\frac{1}{p}}\\
&\qquad= c_{\phi,p,r}(m(I)^{1-\frac{1}{2r+1}}\|\phi - \phi_{21I}\|_{L_1(21I)}^{\frac{1}{2r+1}})^{\frac{1}{p}}.
\end{aligned}$$

Note that

$$\{m(I)\}_{\substack{I\in\mathcal{D}\\ I\subset J}} \in l_{1,\infty}(\mathcal{D}).$$

By Theorem 6.1, we know that

$$\{\|\phi - \phi_{21I}\|_{L_1(21I)}\}_{\substack{I\in\mathcal{D}\\ I\subset J}} \in (l_{1,\infty})_0(\mathcal{D}).$$

The assertion follows immediately from the last argument (and the Hölder inequality). ∎

Lemma 7.5. *Let $f \in C_c^\infty(\mathbb{R}^d)$ and let $\phi \in \mathrm{BMO}(\mathbb{R}^d)$. For every $r > 0$ and for every $p > d$, we have*

$$\left\{m(I)^{\frac{1}{p}-\frac{1}{d}-\frac{1}{r+1}}\|f - f_{21I}\|_{L_\infty(21I)}\|\phi - \phi_{21I}\|_{L_{r+1}(21I)}\right\}_{I\in\mathcal{D}} \in (l_{p,\infty})_0(\mathcal{D}).$$

Proof. Fix unit cubes $J_1, \ldots, J_K \in \mathcal{D}$ such that f is supported in $\bigcup_{k=1}^K J_k$.

If $f - f_{21I} \not\equiv 0$ on $21I$, then also $f \not\equiv 0$ on $21I$. This means $21I \cap J_k \neq \varnothing$ for some $1 \leq k \leq K$. Hence, there exist $1 \leq k \leq K$ and $l \in \{-10, \dots, 10\}^d$ such that $(I + ls(I)) \cap J_k \neq \varnothing$. This either means $I + ls(I) \subset J_k$ (if $s(I) \leq 1$) or $J_k \subset I + ls(I)$ (if $s(I) > 1$). If $s(I) \leq 1$, then there exist[4] $l \in \{-10, \dots, 10\}^d$ and $1 \leq k \leq K$ such that $I \subset J_k + l$. If $s(I) \geq 1$, then $J_k \subset I + ls(I)$ for some $l \in \{-10, \dots, 10\}^d$ and for some $1 \leq k \leq K$.

Set

$$A_{k,l} = \{I \in \mathcal{D} : I \subset J_k + l\}, \quad B_k = \{I \in \mathcal{D} : J_k \subset I + ls(I)\}, \quad 1 \leq k \leq K, \, l \in \mathbb{Z}^d.$$

In particular, for each $I \in \mathcal{D}$, we have

$$\begin{aligned} m(I)^{\frac{1}{p}-\frac{1}{d}-\frac{1}{r+1}} &\|f - f_{21I}\|_{L_\infty(21I)} \|\phi - \phi_{21I}\|_{L_{r+1}(21I)} \\ \leq &\sum_{\substack{1 \leq k \leq K \\ l \in \{-10,\dots,10\}^d}} m(I)^{\frac{1}{p}-\frac{1}{d}-\frac{1}{r+1}} \|f - f_{21I}\|_{L_\infty(21I)} \|\phi - \phi_{21I}\|_{L_{r+1}(21I)} \chi_{A_{k,l}}(I) \\ &+ \sum_{\substack{1 \leq k \leq K \\ l \in \{-10,\dots,10\}^d}} m(I)^{\frac{1}{p}-\frac{1}{d}-\frac{1}{r+1}} \|f - f_{21I}\|_{L_\infty(21I)} \|\phi - \phi_{21I}\|_{L_{r+1}(21I)} \chi_{B_{k,l}}(I). \end{aligned}$$

If $J_k \subset I + ls(I)$, then

$$\|f - f_{21I}\|_{L_\infty(21I)} \leq \|f\|_{L_\infty(21I)} + |f_{21I}| \leq 2\|f\|_{L_\infty(\mathbb{R}^d)},$$

and, according to the John–Nirenberg theorem (see, e.g., Corollary 7.1.8 in [20]),

$$m(I)^{-\frac{1}{r+1}} \|\phi - \phi_{21I}\|_{L_{r+1}(21I)} \leq c_{r+1} \|\phi\|_{\mathrm{BMO}(\mathbb{R}^d)}.$$

If $I \subset J_k + l$, then

$$\|f - f_{21I}\|_{L_\infty(21I)} \leq 21 m(I)^{\frac{1}{d}} \|f\|_{\mathrm{Lip}(\mathbb{R}^d)}.$$

Hence, for each $I \in \mathcal{D}$, we have

$$\begin{aligned} m(I)^{\frac{1}{p}-\frac{1}{d}-\frac{1}{r+1}} &\|f - f_{21I}\|_{L_\infty(21I)} \|\phi - \phi_{21I}\|_{L_{r+1}(21I)} \\ &\leq 21\|f\|_{\mathrm{Lip}(\mathbb{R}^d)} \sum_{\substack{1 \leq k \leq K \\ l \in \{-10,\dots,10\}^d}} m(I)^{\frac{1}{p}-\frac{1}{r+1}} \|\phi - \phi_{21I}\|_{L_{r+1}(21I)} \chi_{A_{k,l}}(I) \\ &\quad + 2c_{r+1}\|f\|_{L_\infty(\mathbb{R}^d)} \|\phi\|_{\mathrm{BMO}(\mathbb{R}^d)} \sum_{\substack{1 \leq k \leq K \\ l \in \{-10,\dots,10\}^d}} m(I)^{\frac{1}{p}-\frac{1}{d}} \chi_{B_{k,l}}(I). \end{aligned}$$

It follows from Lemma 7.4 that every summand

$$\big\{m(I)^{\frac{1}{p}-\frac{1}{r+1}} \|\phi - \phi_{21I}\|_{L_{r+1}(21I)} \chi_{A_{k,l}}(I)\big\}_{I \in \mathcal{D}} \in (l_{p,\infty})_0(\mathcal{D}).$$

On the other hand, every summand $\{m(I)^{\frac{1}{p}-\frac{1}{d}} \chi_{B_{k,l}}(I)\}_{I \in \mathcal{D}} \in (l_{p,\infty})_0(\mathcal{D})$ according to the formula

$$\mu_{l_\infty(\mathcal{D})}(\{m(I)^{\frac{1}{p}-\frac{1}{d}} \chi_{B_{k,l}}(I)\}_{I \in \mathcal{D}}) = \{2^{m(\frac{d}{p}-1)}\}_{m \geq 0}, \quad 1 \leq k \leq K, \, l \in \mathbb{Z}^d.$$

The desired assertion follows by combining the above arguments. ■

[4] This l can be different from l in the preceding sentence.

Lemma 7.6. *Let $f \in C_c^\infty(\mathbb{R}^d)$ and let $\phi \in \Phi(\mathbb{R}^d)$. Let $r > 1$ be as in Lemma 2.3 and let $n \in \mathbb{N}$. Then*

$$\max\{\|e'_{k,I}\|_{L_{r+1}(\mathbb{R}^d)}\|f'_{k,I}\|_{L_{r+1}(\mathbb{R}^d)}, \|e''_{k,I}\|_{L_{r+1}(\mathbb{R}^d)}\|f''_{k,I}\|_{L_{r+1}(\mathbb{R}^d)},$$
$$\|g'_{k,I}\|_{L_{r+1}(\mathbb{R}^d)}\|h'_{k,I}\|_{L_{r+1}(\mathbb{R}^d)}, \|g''_{k,I}\|_{L_{r+1}(\mathbb{R}^d)}\|h''_{k,I}\|_{L_{r+1}(\mathbb{R}^d)}\}$$
$$\leq c_\phi\|f - f_{21I}\|_{L_\infty(21I)} \cdot (e^{-\frac{r-1}{2(r+1)}n} \cdot m(I)^{\frac{2}{r+1}} + e^n\|\phi - \phi_{21I}\|_{L_{r+1}(21I)} \cdot m(I)^{\frac{1}{r+1}}),$$

where

$$c_\phi = 21^{\frac{2d}{r+1}}\|r\phi\|_{\Phi(\mathbb{R}^d)}^{\frac{2}{r+1}}.$$

Proof. Note that the following elementary estimate holds:

$$|e^{\frac{t}{2}} - 1| \leq e^{-\frac{r-1}{2(r+1)}n}e^{\frac{r}{r+1}|t|} + e^n|t|, \quad t \in \mathbb{R}.$$

Hence,

$$|e'_{k,I}|, |f''_{k,I}| \leq |f - f_{21I}|(e^{-\frac{r-1}{2(r+1)}n}e^{\frac{r}{r+1}|\phi-\phi_{21I}|} + e^n|\phi - \phi_{21I}|),$$
$$|f'_{k,I}|, |e''_{k,I}| \leq e^{\frac{1}{2}|\phi-\phi_{21I}|},$$

and

$$|g'_{k,I}|, |h''_{k,I}| \leq |f - f_{21I}|, \quad |h'_{k,I}|, |g''_{k,I}| \leq e^{-\frac{r-1}{2(r+1)}n}e^{\frac{r}{r+1}|\phi-\phi_{21I}|} + e^n|\phi - \phi_{21I}|.$$

Also recall that the support of the related functions are in $21I$ as proved in Lemma 5.5. The assertion follows now from the triangle inequality and the definition of the $\|\cdot\|_{\Phi(\mathbb{R}^d)}$. ∎

Lemma 7.7. *Let $f \in C_c^\infty(\mathbb{R}^d)$ and let $\phi \in \Phi(\mathbb{R}^d)$. Let $r > 1$ be as in Lemma 2.3. For every $p > d$, we have*

$$\{m(I)^{\frac{1}{p}-\frac{1}{d}-\frac{2}{r+1}}\|e'_{k,I}\|_{L_{r+1}(\mathbb{R}^d)}\|f'_{k,I}\|_{L_{r+1}(\mathbb{R}^d)}\}_{I\in\mathcal{D}} \in (l_{p,\infty})_0(\mathcal{D}),$$
$$\{m(I)^{\frac{1}{p}-\frac{1}{d}-\frac{2}{r+1}}\|g'_{k,I}\|_{L_{r+1}(\mathbb{R}^d)}\|h'_{k,I}\|_{L_{r+1}(\mathbb{R}^d)}\}_{I\in\mathcal{D}} \in (l_{p,\infty})_0(\mathcal{D}),$$
$$\{m(I)^{\frac{1}{p}-\frac{1}{d}-\frac{2}{r+1}}\|e''_{k,I}\|_{L_{r+1}(\mathbb{R}^d)}\|f''_{k,I}\|_{L_{r+1}(\mathbb{R}^d)}\}_{I\in\mathcal{D}} \in (l_{p,\infty})_0(\mathcal{D}),$$
$$\{m(I)^{\frac{1}{p}-\frac{1}{d}-\frac{2}{r+1}}\|g''_{k,I}\|_{L_{r+1}(\mathbb{R}^d)}\|h''_{k,I}\|_{L_{r+1}(\mathbb{R}^d)}\}_{I\in\mathcal{D}} \in (l_{p,\infty})_0(\mathcal{D}).$$

Proof. We only consider the first assertion. The argument for others follows mutatis mutandis.

By Lemma 7.6, we have

$$c_\phi^{-1}\limsup_{t\to\infty} t^{\frac{1}{p}}\mu\big(t, \{m(I)^{\frac{1}{p}-\frac{1}{d}-\frac{2}{r+1}}\|e'_{k,I}\|_{L_{r+1}(\mathbb{R}^d)}\|f'_{k,I}\|_{L_{r+1}(\mathbb{R}^d)}\}_{I\in\mathcal{D}}\big)$$
$$\leq \limsup_{t\to\infty} t^{\frac{1}{p}}\mu\big(t, e^{-\frac{r-1}{2(r+1)}n}\{m(I)^{\frac{1}{p}-\frac{1}{d}}\|f - f_{21I}\|_{L_\infty(21I)}\}_{I\in\mathcal{D}}$$
$$+ e^n\{m(I)^{\frac{1}{p}-\frac{1}{d}-\frac{1}{r+1}}\|f - f_{21I}\|_{L_\infty(21I)}\|\phi - \phi_{21I}\|_{L_{r+1}(21I)}\}_{I\in\mathcal{D}}\big)$$
$$\leq 2e^{-\frac{r-1}{2(r+1)}n}\limsup_{t\to\infty} t^{\frac{1}{p}}\mu\big(t, \{m(I)^{\frac{1}{p}-\frac{1}{d}}\|f - f_{21I}\|_{L_\infty(21I)}\}_{I\in\mathcal{D}}\big)$$
$$+ 2e^n \cdot \limsup_{t\to\infty} t^{\frac{1}{p}}\mu\big(t, \{m(I)^{\frac{1}{p}-\frac{1}{d}-\frac{1}{r+1}}\|f - f_{21I}\|_{L_\infty(21I)}\|\phi - \phi_{21I}\|_{L_{r+1}(21I)}\}_{I\in\mathcal{D}}\big).$$

By Lemma 7.5, the second limit on the right-hand side is 0. Therefore,

$$\begin{aligned} c_\phi^{-1} \limsup_{t\to\infty} t^{\frac{1}{p}} \mu\big(t, \big\{m(I)^{\frac{1}{p}-\frac{1}{d}-\frac{2}{r+1}} \|e'_{k,I}\|_{L_{r+1}(\mathbb{R}^d)} \|f'_{k,I}\|_{L_{r+1}(\mathbb{R}^d)}\big\}_{I\in\mathcal{D}}\big) \\ \leq 2e^{-\frac{r-1}{2(r+1)}n} \big\| \big\{m(I)^{\frac{1}{p}-\frac{1}{d}} \|f - f_{21I}\|_{L_\infty(21I)}\big\}_{I\in\mathcal{D}} \big\|_{l_{p,\infty}(\mathcal{D})} \\ \overset{\text{Th. 3.1}}{\leq} e^{-\frac{r-1}{2(r+1)}n} \cdot c_{p,d} \|f\|_{\dot{W}^1_p(\mathbb{R}^d)}. \end{aligned}$$

Since $n \in \mathbb{N}$ is arbitrarily large, the assertion follows. ∎

Lemma 7.8. *Let $f \in C_c^\infty(\mathbb{R}^d)$ and let $\phi \in \Phi(\mathbb{R}^d)$. We have*

$$C'_k, D'_k, C''_k, D''_k \in (\mathcal{L}_{\frac{d}{1-\varepsilon},\infty})_0(L_2(\mathbb{R}^d)).$$

Proof. The assertion follows from Notation 7.2, Corollary 2.9 (applied with $p = d/(1-\varepsilon)$) and Lemma 7.7. ∎

Lemma 7.9. *Let $f \in C_c^\infty(\mathbb{R}^d)$ and let $\phi \in \Phi(\mathbb{R}^d)$. For every $k \in \mathbb{Z}^d$, we have*

$$\|\Lambda_k\|_{\mathcal{L}_{\frac{d}{1-\varepsilon},\infty}(L_2(\mathbb{R}^d))} \leq c'''_{\phi,d,\varepsilon} \big\| \big\{m(I)^{-\frac{\varepsilon}{d}} \|f - f_{21I}\|_{L_\infty(21I)}\big\}_{I\in\mathcal{D}} \big\|_{l_{\frac{d}{1-\varepsilon},\infty}(\mathcal{D})},$$

where $\Lambda \in \{C', D', C'', D''\}$.

Proof. The argument follows that in Lemma 5.6 mutatis mutandis. ∎

Combining the last several lemmas, we can show Theorem 7.1.

Proof of Theorem 7.1. The assertion follows by combining Lemmas 7.3, 7.8 and 7.9. ∎

Now, let us turn to prove Theorem 1.3.

Proof of Theorem 1.3. For $w \in A_2(\mathbb{R}^d)$, write $w = e^\phi$ for some $\phi \in \Phi(\mathbb{R}^d)$. Suppose first $f \in C_c^\infty(\mathbb{R}^d)$. Let the commutators T_f and $T_{f,\phi}$ be as in Notation 4.2. It follows from Theorem 1.1 in [18] (see also (1.3)) that

$$\lim_{t\to\infty} t^{\frac{1-\varepsilon}{d}} \mu(t, T_f) = c_{d,\varepsilon} \|f\|_{\dot{W}^1_{\frac{d}{1-\varepsilon}}(\mathbb{R}^d)}.$$

By Theorem 7.1, we have

$$T_{f,\phi} - T_f \in (\mathcal{L}_{\frac{d}{1-\varepsilon},\infty})_0.$$

Using a standard lemma due to Birman and Solomyak (see, e.g., Lemma 3.1 in [17]), we conclude that there exists a limit

$$\lim_{t\to\infty} t^{\frac{1-\varepsilon}{d}} \mu(t, T_{f,\phi}) = c_{d,\varepsilon} \|f\|_{\dot{W}^1_{\frac{d}{1-\varepsilon}}(\mathbb{R}^d)}.$$

Let $f \in \dot{W}^1_{\frac{d}{1-\varepsilon}}(\mathbb{R}^d)$. By Theorem 11.43 in [31], there exists a sequence $\{f_n\}_{n\geq 0} \subset C_c^\infty(\mathbb{R}^d)$ such that $f_n \to f$ in $\dot{W}^1_{\frac{d}{1-\varepsilon}}(\mathbb{R}^d)$. By Theorem 1.2 (and Fact 4.7), we have $T_{f_n,\phi} \to T_{f,\phi}$ in $\mathcal{L}_{\frac{d}{1-\varepsilon},\infty}$. By the preceding paragraph, for every $n \geq 0$, there exists a limit

$$\lim_{t\to\infty} t^{\frac{1-\varepsilon}{d}} \mu(t, T_{f_n,\phi}) = c_{d,\varepsilon} \|f\|_{\dot{W}^1_{\frac{d}{1-\varepsilon}}(\mathbb{R}^d)}.$$

Using another standard lemma due to Birman and Solomyak (see, for instance, Lemma 3.2 in [17]), we conclude that there exists a limit

$$\lim_{t\to\infty} t^{\frac{1-\varepsilon}{d}}\mu(t, T_{f,\phi}) = c_{d,\varepsilon}\|f\|_{\dot{W}^1_{\frac{d}{1-\varepsilon}}(\mathbb{R}^d)}.$$

The assertion now follows from Fact 4.7. ∎

A. Homogeneous version of Triebel's theorem

The following theorem (for inhomogeneous spaces) can be found in p. 132 of [43]. The proof for homogeneous spaces is known to experts, but seems not to be available in the literature. We provide a proof for convenience of the reader. The idea of the argument below is taken from Theorem 2.34 in [3].

Theorem A.1. *Let $g \in \mathcal{S}(\mathbb{R})$ be even function such that $g \not\equiv 0$. For every $s \in (0,1)$, Sobolev–Slobodeckii semi-norm $f \mapsto \|f\|_{\dot{W}^s_p(\mathbb{R}^d)}$ is controlled by the semi-norm*

$$f \mapsto \Big(\int_0^\infty \|g(t\Delta^{\frac12})f\|^p_{L_p(\mathbb{R}^d)} \frac{dt}{t^{ps+1}}\Big)^{\frac1p}.$$

The following three lemmas are standard.

Lemma A.2. *If $f \in L_p(\mathbb{R}^d)$ is such that $\widehat{f}$ is supported in $B(0,1)^c$, then*

$$\|(1+t^2\Delta)^{-1}f\|_{L_p(\mathbb{R}^d)} \le \frac{c(p,d)}{1+t^2}\|f\|_{L_p(\mathbb{R}^d)}.$$

Lemma A.3. *If $f \in L_p(\mathbb{R}^d)$ is such that $\widehat{f}$ is supported in $B(0,1)$, then*

$$\|\Delta^{\frac12}f\|_{L_p(\mathbb{R}^d)} \le c(p,d)'\|f\|_{L_p(\mathbb{R}^d)}.$$

Lemma A.4. *Let $\psi \in C_c^\infty(\mathbb{R}^d)$. If $f \in L_p(\mathbb{R}^d)$, then*

$$\Big\|\psi\Big(\frac{\Delta^{\frac12}}{2^j}\Big)f\Big\|_{L_p(\mathbb{R}^d)} \le c(\psi,p,d)\|f\|_{L_p(\mathbb{R}^d)}.$$

The next lemma follows from Lemma A.3 and Lemma A.2 by dilation.

Lemma A.5. *If $f \in L_p(\mathbb{R}^d)$ is such that $\widehat{f}$ is supported in the annulus $\{2^j \le |x| \le 2^{j+1}\}$, then*

$$\|(1+t^2\Delta)^{-1}f\|_{L_p(\mathbb{R}^d)} \le \frac{c(p,d)}{1+t^2\cdot 2^{2j}}\|f\|_{L_p(\mathbb{R}^d)}$$

and

$$\|\Delta^{\frac12}f\|_{L_p(\mathbb{R}^d)} \le c(p,d)'2^{j+1}\|f\|_{L_p(\mathbb{R}^d)}.$$

Lemma A.6. *Let $g \in \mathcal{S}(\mathbb{R})$ be an even function. If*

$$\int_0^\infty \frac{g(\frac{t}{\lambda})\,dt}{(1+t^2)^2} = 0$$

for every $\lambda > 0$, then $g = 0$.

Proof. Passing $\lambda \to \infty$, we obtain $g(0) = 0$. Since $g \in \mathcal{S}(\mathbb{R})$ is an even function, it follows that $g(t) = O(t^2)$, $t \in (0,1)$.

We have

$$\int_0^\infty \frac{\lambda^2 g(t)\,dt}{(\lambda^2+t^2)^2} = 0, \quad \lambda > 0.$$

Multiplying by $\cos(\lambda s)$ and integrating by λ from 0 to ∞, we obtain

$$\int_0^\infty \cos(\lambda s)\Big(\int_0^\infty \frac{\lambda^2 g(t)\,dt}{(\lambda^2+t^2)^2}\Big)\,d\lambda = 0, \quad s > 0.$$

Since $g(t) = O(t^2)$, $t \in (0,1)$, it follows from Fubini's theorem that

$$\int_0^\infty g(t)\Big(\int_0^\infty \frac{\lambda^2 \cos(\lambda s)\,d\lambda}{(\lambda^2+t^2)^2}\Big)\,dt = 0, \quad s > 0.$$

It is immediate that

$$\int_0^\infty \frac{\lambda^2 \cos(\lambda s)\,d\lambda}{(\lambda^2+t^2)^2} \overset{\lambda = tu}{=} \frac{1}{t}\int_0^\infty \frac{u^2\cos(ust)du}{(u^2+1)^2} = \frac{\pi}{4t}(1-st)e^{-st}.$$

Thus,

$$\int_0^\infty \frac{g(t)}{t}\,e^{-st}\,dt + \int_0^\infty g(t)\,d(e^{-st}) = 0, \quad s > 0.$$

Integrating by parts and taking into account that $g(0) = 0$, we obtain

$$\int_0^\infty \Big(\frac{g(t)}{t} - g'(t)\Big)e^{-st}\,dt = 0, \quad s > 0.$$

Define the odd Schwartz function h by setting $h(t) = g(t)/t - g'(t)$, $t \in \mathbb{R}$. The Laplace transform of h is zero. Since the Laplace transform is injective on odd Schwartz functions, it follows that $h = 0$. Solving the differential equation $g(t)/t = g'(t)$, $t \in \mathbb{R}$, we obtain $g(t) = ct$, $t \in \mathbb{R}$. Since g is a Schwartz function, the assertion follows. ∎

Let $\psi \in C^\infty(\mathbb{R})$ be supported on $(1/2, 2)$ and such that

$$\sum_{j\in\mathbb{Z}} \psi\Big(\frac{t}{2^j}\Big) = 1, \quad t > 0.$$

It is a standard fact (see, e.g., Definition 2.15 and Theorem 2.36 in [3]) that the Sobolev–Slobodeckii space $\dot{W}^s_p(\mathbb{R}^d)$ admits an equivalent semi-norm

$$f \mapsto \Big(\sum_{j\in\mathbb{Z}} 2^{jsp}\Big\|\psi\Big(\frac{\Delta^{\frac{1}{2}}}{2^j}\Big)f\Big\|^p_{L_p(\mathbb{R}^d)}\Big)^{\frac{1}{p}}.$$

Proof of Theorem A.1. By Lemma A.6, there exists $\lambda > 0$ such that

$$\int_0^\infty \frac{(\sigma_\lambda g)(t)\,dt}{(1+t^2)^2} \neq 0.$$

It does not really matter if we prove the assertion for g or for $\sigma_\lambda g$. Thus, we may assume without loss of generality that

$$\int_0^\infty \frac{g(t)\,dt}{(1+t^2)^2} = 1.$$

We have

$$\int_0^\infty \frac{g(t\Delta^{\frac12})}{(1+t^2\Delta)^2}\,dt = \Delta^{-\frac12}\int_0^\infty \frac{g(t)}{(1+t^2)^2}\,dt = \Delta^{-\frac12}.$$

Thus,

$$\psi\Big(\frac{\Delta^{\frac12}}{2^j}\Big)f = \int_0^\infty \frac{\Delta^{\frac12}}{(1+t^2\Delta)^2}\,g(t\Delta^{\frac12})\psi\Big(\frac{\Delta^{\frac12}}{2^j}\Big)f\,dt.$$

By the triangle inequality, we have

$$\Big\|\psi\Big(\frac{\Delta^{\frac12}}{2^j}\Big)f\Big\|_{L_p(\mathbb{R}^d)} \le \int_0^\infty \Big\|\frac{\Delta^{\frac12}}{(1+t^2\Delta)^2}\,g(t\Delta^{\frac12})\psi\Big(\frac{\Delta^{\frac12}}{2^j}\Big)f\Big\|_{L_p(\mathbb{R}^d)}\,dt.$$

Next,

$$\begin{aligned}
&\Big\|\frac{\Delta^{\frac12}}{(1+t^2\Delta)^2}\,g(t\Delta^{\frac12})\psi\Big(\frac{\Delta^{\frac12}}{2^j}\Big)f\Big\|_{L_p(\mathbb{R}^d)}\\
&\overset{\text{Lem. A.5}}{\le} c(p,d)'2^{j+1}\Big\|\frac{1}{(1+t^2\Delta)^2}\,g(t\Delta^{\frac12})\psi\Big(\frac{\Delta^{\frac12}}{2^j}\Big)f\Big\|_{L_p(\mathbb{R}^d)}\\
&\overset{\text{Lem. A.5}}{\le} c(p,d)'c(p,d)^2\frac{2^{j+1}}{(1+t^2\cdot 2^{2j})^2}\Big\|g(t\Delta^{\frac12})\psi\Big(\frac{\Delta^{\frac12}}{2^j}\Big)f\Big\|_{L_p(\mathbb{R}^d)}\\
&\overset{\text{Lem. A.4}}{\le} c(p,d)'c(p,d)^2c(\psi,p,d)\frac{2^{j+1}}{(1+t^2\cdot 2^{2j})^2}\|g(t\Delta^{\frac12})f\|_{L_p(\mathbb{R}^d)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\Big\|\psi\Big(\frac{\Delta^{\frac12}}{2^j}\Big)f\Big\|_{L_p(\mathbb{R}^d)}\\
&\quad\le 2c(p,d)'c(p,d)^2c(\psi,p,d)\cdot 2^j\cdot\int_0^\infty \frac{1}{(1+t^2\cdot 2^{2j})^2}\|g(t\Delta^{\frac12})f\|_{L_p(\mathbb{R}^d)}\,dt.
\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}
\Big\|\psi\Big(\frac{\Delta^{\frac12}}{2^j}\Big)f\Big\|_{L_p(\mathbb{R}^d)} &\le 2c(p,d)'c(p,d)^2c(\psi,p,d)\cdot 2^j\cdot\Big(\int_0^\infty (1+t^2\cdot 2^{2j})^{-\frac{p}{p-1}}\,dt\Big)^{1-\frac1p}\\
&\quad\cdot\Big(\int_0^\infty \frac{1}{(1+t^2\cdot 2^{2j})^p}\|g(t\Delta^{\frac12})f\|^p_{L_p(\mathbb{R}^d)}\,dt\Big)^{\frac1p}\\
&= 2c(p,d)'c(p,d)^2c(\psi,p,d)\Big(\int_0^\infty (1+t^2)^{-\frac{p}{p-1}}\,dt\Big)^{1-\frac1p}\\
&\quad\cdot 2^{\frac{j}{p}}\Big(\int_0^\infty \frac{1}{(1+t^2\cdot 2^{2j})^p}\|g(t\Delta^{\frac12})f\|^p_{L_p(\mathbb{R}^d)}\,dt\Big)^{\frac1p}\\
&= c(\psi,p,d)'2^{\frac{j}{p}}\Big(\int_0^\infty \frac{1}{(1+t^2\cdot 2^{2j})^p}\|g(t\Delta^{\frac12})f\|^p_{L_p(\mathbb{R}^d)}\,dt\Big)^{\frac1p}.
\end{aligned}$$

Now,

$$
\begin{aligned}
\|f\|_{\dot W_p^s(\mathbb{R}^d)}^p &= \sum_{j\in\mathbb{Z}} 2^{jsp}\Big\|\psi\Big(\frac{\Delta^{\frac12}}{2^j}\Big)f\Big\|_{L_p(\mathbb{R}^d)}^p \\
&\le c(\psi,p,d)'^p \sum_{j\in\mathbb{Z}} 2^{j(sp+1)}\cdot\int_0^\infty \frac{1}{(1+t^2\cdot 2^{2j})^p}\|g(t\Delta^{\frac12})f\|_{L_p(\mathbb{R}^d)}^p\,dt \\
&\le c(\psi,p,d)'^p\cdot\int_0^\infty\Big(\sum_{j\in\mathbb{Z}}\frac{2^{j(sp+1)}}{(1+t^2\cdot 2^{2j})^p}\Big)\|g(t\Delta^{\frac12})f\|_{L_p(\mathbb{R}^d)}^p\,dt.
\end{aligned}
$$

We have

$$\sum_{j\in\mathbb{Z}}\frac{2^{j(sp+1)}}{(1+t^2\cdot 2^{2j})^p}\le c(p)t^{-sp-1}.$$

Thus,

$$\|f\|_{\dot W_p^s(\mathbb{R}^d)}^p\le c(\psi,p,d)'^p c(p)\cdot\int_0^\infty\|g(t\Delta^{\frac12})f\|_{L_p(\mathbb{R}^d)}^p\frac{dt}{t^{sp+1}}. \qquad \blacksquare$$

B. Whitney decomposition

We here provide a detailed proof for Lemma 5.1.

Proof of Lemma 5.1. For each $k\in\mathbb{Z}$, set

$$\Omega_k=\{(t,s): t,s\in\mathbb{R}^d,\ 2^{k+2}\le|t-s|_\infty<2^{k+3}\}.$$

and

$$\mathcal{F}_k=\{I\times J: I,J\in\mathcal{D},\ s(I)=s(J)=2^k,\ (I\times J)\cap\Omega_k\neq\varnothing\}.$$

Set

$$\mathcal{F}'=\bigcup_{k\in\mathbb{Z}}\mathcal{F}_k.$$

We first show that

$$\Omega=\bigcup_{I\times J\in\mathcal{F}'} I\times J. \tag{B.1}$$

If $I\times J\in\mathcal{F}_k$ for some given $k\in\mathbb{Z}$, then $s(I)=s(J)=2^k$ and there exist $x_0\in I$ and $y_0\in J$ such that $|x_0-y_0|_\infty\in[2^{k+2},2^{k+3})$. If $x\in I$ and $y\in J$, then $|x-x_0|_\infty<2^k$ and $|y-y_0|_\infty<2^k$. By the triangle inequality,

$$2s(I)=2\cdot 2^k\le|x-y|_\infty<10\cdot 2^k=10s(I).$$

In particular, $I\times J\subset\Omega$. If $(x,y)\in\Omega$, then one can uniquely choose $k\in\mathbb{Z}$ such that

$$2^{k+2}\le|x-y|_\infty<2^{k+3}.$$

Now one can uniquely choose $I \in \mathcal{D}$ such that $x \in I$ and $s(I) = 2^k$, and $J \in \mathcal{D}$ such that $y \in J$ and $s(J) = 2^k$. Since $(x, y) \in I \times J$ and $(x, y) \in \Omega_k$, it follows that $I \times J \in \mathcal{F}_k$. This gives us (B.1).

Let $\mathcal{F}$ be the set of maximal elements in $\mathcal{F}'$. We shall show that $\mathcal{F}$ satisfies the desired assertions. It is clear that (iii) and (iv) are satisfied. On the other hand, the elements in $\mathcal{F}$ are pairwise disjoint. Indeed, if $Q_1, Q_2 \in \mathcal{F}$ are such that $Q_1 \cap Q_2 \neq \varnothing$, then either $Q_1 \subset Q_2$ or $Q_2 \subset Q_1$. Since both Q_1 and Q_2 are maximal elements of $\mathcal{F}'$, it follows that $Q_1 = Q_2$. We claim that every $Q \in \mathcal{F}'$ is contained in some $R \in \mathcal{F}$. Assuming that this claim has been proved, by (B.1), we have

$$\Omega = \bigcup_{Q \in \mathcal{F}'} Q = \bigcup_{Q \in \mathcal{F}} Q.$$

Now we turn to prove the claim. If $Q \in \mathcal{F}'$ is maximal, then set $R = Q$, otherwise choose $R \in \mathcal{F}'$ such that $Q \subsetneq R$. For the latter case, we have $s(R) = 2s(Q)$. Indeed, let $Q = I_1 \times J_1$ and $R = I_2 \times J_2$. We have $I_1 \subset I_2$ and $J_1 \subset J_2$. Thus,

$$2s(I_2) \leq \inf_{\substack{x \in I_2 \\ y \in J_2}} |x - y|_\infty \leq \inf_{\substack{x \in I_1 \\ y \in J_1}} |x - y|_\infty < 8s(I_1),$$

which implies $s(I_2) < 4s(I_1)$. Since $s(I_1)$ and $s(I_2)$ are powers of 2, it follows that either $s(I_2) = s(I_1)$ or $s(I_2) = 2s(I_1)$. However, by the fact $Q \subsetneq R$, we only have $s(R) = s(I_2) = 2s(I_1) = 2s(Q)$. It remains to show that $R \in \mathcal{F}$. To this end, we check that R is maximal. Actually, if there exists $S \in \mathcal{F}'$ such that $R \subsetneq S$, then, by the preceding argument, $s(S) = 2s(R)$, which further implies and $s(S) = 4s(Q)$. However, this contradicts the fact $Q \subsetneq S$ (which implies $s(S) = 2s(Q)$). Hence, R is maximal.

The proof is complete. ■

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