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On a nonlocal superconductivity problem

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Abstract. This paper investigates degenerate nonlocal free boundary problems arising in the context of superconductivity, extending the nonlocal counterpart to the work of Caffarelli and Salazar (2002) and Caffarelli, Salazar and Shahgholian (2004) in the local setting. In these models, no partial differential equation governs the moving sets where the gradient vanishes, meaning that test functions are only required to have a nonzero gradient. Our main results provide interior gradient Hölder regularity estimates for viscosity solutions.

1. Introduction

In this paper, we investigate regularity estimates of solutions to nonlocal free boundary problems, which emerge in the study of nonlocal superconductivity and degenerate diffusion problems.

The significance of these models lies in their application to finance, particularly in scenarios involving jump processes. In this case, diffusion occurs at points where the cost function is not maximized, meaning that diffusion can only be inferred at non-critical points. Secondly, within the context of superconductivity, the problem serves as a nonlocal variant of the stationary equation in the mean-field model for superconducting vortices, see [7, 13] and [17].

Local versions of this problem were explored by Caffarelli, Salazar, and Shahgholian in [9, 10], where they investigate fully nonlinear elliptic PDEs of the form

$$F(x, D^2u) = g(x, u) \quad \text{in } B_1 \cap \Omega, \tag{1.1}$$

for $\Omega = \{|Du| \neq 0\}$. See also [18], for a broader class of free boundary problems.

Building on the framework of the aforementioned works, we formulate the problem studied in this paper. Given a smooth function f, we consider an unknown pair (u, Ω), where u is a function defined in $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ is an open set, satisfying in the viscosity sense

$$\begin{cases} (-\Delta)^s u = f & \text{in } B_1 \cap \Omega, \\ |Du| = 0 & \text{in } B_1 \setminus \Omega. \end{cases} \tag{1.2}$$

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Here, $(-\Delta)^s$ denotes the fractional Laplacian (see Section 2 for details). Problem (1.2) constitutes a genuine free boundary problem, where the nonlocal diffusion properties break down near $\partial\Omega$, while the complementary set retains a local character, marked by the presence of critical points.

1.1. Challenges for the nonlocal setting

The nonlocal setting involves specific features that do not arise in the local framework, even in the simplest case $f = 0$, as critical points have a stronger influence on the system due to the nonlocal nature of the problem.

In the local setting, the breakthrough work of Imbert and Silvestre [20] shows that functions which are harmonic on the set of non-critical points are, in fact, harmonic throughout the entire domain. A slightly more technical statement is as follows: given an open set $\Omega \subset \mathbb{R}^n$, such that $\{|Du| \neq 0\} \subseteq \Omega$, the equation

$$\Delta u = 0, \quad \text{in } \Omega \cap B_1, \tag{1.3}$$

holds if and only if u satisfies $\Delta u = 0$ in B_1.

However, in the nonlocal scenario, significant challenges arise, and such equivalence generally fails to hold. In fact, for the homogeneous problem

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } B_1 \cap \Omega, \\ |Du| = 0 & \text{in } B_1 \setminus \Omega, \end{cases} \tag{1.4}$$

one might be tempted to infer that solutions of (1.4) are s-harmonic in B_1; however, this is not true in general. Indeed, explicit counterexamples show that such an equivalence fails in the nonlocal setting, see, for instance, [2, 16]. The reverse implication is trivial in both cases. In light of this, we observe that problem (1.2) remains significant even in the homogeneous case, further motivating our investigation of the regularity estimates of solutions in the nonlocal setting.

One of the key components of the approach adopted by Caffarelli and Salazar in the local case [9] is that solutions to problem (1.1) actually satisfy a uniformly elliptic PDE with a bounded right-hand side throughout the entire domain. However, in a nonlocal setting, such a reduction is generally not available. The nature of the fractional Laplacian requires integration throughout the space $\mathbb{R}^n$, and thus the behavior of the solution outside the domain B_1 can heavily influence, as solutions are only s-integrable in $\mathbb{R}^n \setminus B_1$, any attempt to extend the PDE globally in B_1 would introduce irregular or singular terms.

1.2. Main results and consequences

Our main goal is to develop a De Giorgi-type gradient oscillation method tailored to the nonlocal setting, drawing inspiration from elliptic degeneracy scenarios as studied in [19, 22].

We shall consider solutions (u, Ω) of problem (1.4), using this problem as our primary prototype throughout the paper – further discussions for broader settings, shall be discussed in Section 6.

Here is our main result.

Theorem 1.1. *For $u \in C(B_1) \cap L^\infty(\mathbb{R}^n)$ and $\Omega \subset \mathbb{R}^n$ an open set, assume that (u, Ω) solves (1.4), for some $s \in (1/2, 1)$. Then, u is locally $C^{1,\alpha}$, for some universal $\alpha \in (0, 1)$, depending only on n and s. Furthermore, there exists C, depending on n and s, such that*

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq C\|u\|_{L^\infty(\mathbb{R}^n)}.$$

The strategy relies on a positivity argument applied to the derivatives, coupled with a discrete normalization scheme that controls the nonvanishing derivatives. At each iteration step, nonlocal contributions are carefully handled to ensure that the structure of the problem is preserved throughout the process. In this context, if for some direction the gradient is large in measure at a given step, we consider the PDE region to apply an appropriate rescaling combined with a small perturbation argument, leading to an improved regularity (Proposition 4.1). Conversely, if the gradient remains small in measure and in every direction for an infinite number of steps, this reveals the presence of critical points, allowing us to iteratively refine the oscillation (Proposition 3.1), overcoming the absence of PDE information at these points.

In our next result, we derive similar results to those obtained for the problem (1.2), provided that f is sufficiently smooth. Moreover, the estimate in Theorem 1.1 is refined by replacing the global L^∞ bound for u in $\mathbb{R}^n$ with the weaker and more natural L^1_s-norm, better suited to the nonlocal context.

Theorem 1.2. *For $u \in C(B_1) \cap L^1_s(\mathbb{R}^n)$ and $\Omega \subset \mathbb{R}^n$ an open set, assume that (u, Ω) solves (1.2), for some $s \in (1/2, 1)$. Then, u is locally $C^{1,\alpha}$, for some universal $\alpha \in (0, 1)$, depending only on n and s. Furthermore, there exists C, depending on n and s, such that*

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq C\big(\|u\|_{L^\infty(B_1)} + \|u\|_{L^1_s(\mathbb{R}^n)} + \|f\|_{C^{0,1}(B_1)}\big).$$

Since our methods are purely nonlinear and naturally extend to a broader class of nonlocal operators, we dedicate Section 6 to focus on a class of fully nonlinear operators. Moreover, we emphasize that our results remain stable as $s \to 1$, seamlessly recovering the local framework classically studied in [9] and [10].

The regularity assumption on f is natural in view of the structure of the argument. Indeed, at a certain stage, L^ε-estimates must be applied, which are currently available only under uniform L^∞ bounds on $\partial_i f$, since the PDE is differentiated in the region $B_1 \cap \Omega$. We nevertheless believe that this condition could be weakened. However, the primary focus of the present paper is the development of sharp estimates in the homogeneous setting, which we expect to constitute a key ingredient for extending the theory to weaker regularity assumptions on f. Establishing such extensions is left for future investigation.

Obstacle-type problems of the form

$$\min\{-(-\Delta)^s v, v - \varphi\} = 0 \quad \text{in } B_1, \tag{1.5}$$

has been investigated in [1, 6, 8, 21]. In contrast with this class of free boundary problems, problem (1.2) presents additional difficulties: no supersolution condition is imposed throughout the entire domain B_1, and no lower bound is enforced by an obstacle. From this perspective, we observe that problem (1.5) can be considered in the following class of problems

$$(-\Delta)^s u = f \quad \text{in } B_1 \cap \Omega, \ \Omega \supseteq \{Du \neq D\varphi\}. \tag{1.6}$$

In light of this, from Theorem 1.2 we derive the following consequence.

Corollary 1.1. *Let $s \in (1/2, 1)$, and assume that $u \in C(B_1) \cap L^1_s(\mathbb{R}^n)$ solves (1.6). Then, there exist constants $\alpha \in (0,1)$ and $C > 0$, depending only on n and s, such that*

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|u\|_{L^1_s(\mathbb{R}^n)} + \|f\|_{C^{0,1}(B_1)} + \|(-\Delta)^s\varphi\|_{C^{0,1}(B_1)}).$$

Theorem 1.1 further allows us to observe that the derivatives of solutions to (1.4) are viscosity solutions within the framework developed by Ros-Oton and Serra [24], see also [1, 23]. From this, assuming that Ω is a $C^{1,\mu}$ domain, optimal $C^{1,s}$ regularity for solutions can be established.

The organization of the paper goes as follows: in Section 2, we establish basic definitions and known results. In Section 3, we provide the positive argument for derivatives. In Section 4, we discuss a small perturbation approach. We conclude the proof of the main results in Section 5, and provide further extensions in Section 6.

2. Preliminaries

In this section, we introduce the basic definitions and results to be used throughout the paper, along with an auxiliary problem that shall be crucial for the analysis.

To address problem (1.4), we begin by presenting concepts related to s-harmonic functions and their equivalences. A function u, sufficiently smooth and defined in $\mathbb{R}^n$, is called a s-harmonic function in a domain $\mathcal{O} \subset \mathbb{R}^n$ if

$$(-\Delta)^s u(x) := C_{n,s} \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy = 0, \tag{2.1}$$

for every $x \in \mathcal{O}$, where $C_{n,s}$ is a normalizing constant depending on dimension and s, see [15]. For the sake of simplicity, we will adopt the following notation: $\Delta^s := -(-\Delta)^s$. The class of s-harmonic functions arises, for instance, as the Euler–Lagrange equations associated with minimizers of the functional

$$[u]^2_{H^s(\mathbb{R}^n)} := \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|y-x|^{n+2s}}\, dx\, dy \longrightarrow \min, \quad u \in H^s(\mathbb{R}^n),$$

see [25] for details.

We remark that, for the concept of s-harmonic functions in the context of viscosity solutions, the most convenient way to define the fractional Laplacian is through formula (2.1), where we replace u by C^2 touching functions near the integral singularity; see Definition 2.2 in [11]. For a further discussion, we refer to [5].

To address gradient regularity for solutions to nonlocal equations, it is essential to establish a convenient prescription for growth at infinity. Thus, we define the space of functions whose growth is appropriately controlled. Specifically, we say that a function $u\colon \mathbb{R}^n \to \mathbb{R}$ is in $L^1_s(\mathbb{R}^n)$ if it satisfies

$$\|u\|_{L^1_s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \frac{|u(y)|}{1+|y|^{n+2s}}\, dy < \infty.$$

Next, we state Lipschitz estimates of solutions to (1.4). The proof is a careful adaptation of the Ishii–Lions method (see [4, 5, 14]), and follows from the fact that solutions of (1.4) solve in particular equation $|Du|\Delta^s u = 0$. For a detailed proof, we refer to [16].

Proposition 2.1. *Let $u \in L^1_s(\mathbb{R}^n)$ be a solution to (1.4). Then, there is a constant C, depending on dimension and s, such that*

$$\|Du\|_{L^\infty(B_{3/4})} \le C\big(\|u\|_{L^\infty(B_1)} + \|u\|_{L^1_s(\mathbb{R}^n)}\big).$$

Next, we study an auxiliary problem that will be required to obtain gradient oscillation estimates. For $\xi \in \mathbb{S}^{n-1}$ and $\nu \in \mathbb{R}$, we consider the following problem:

$$\Delta^s u = 0 \quad \text{in } \{|\nu Du + \xi| \neq 0\} \cap B_1. \tag{2.2}$$

Solutions are understood in the viscosity sense, for test functions φ that touch u at point x, satisfying $|\nu D\varphi(x) + \xi| > 0$, see Definition 1.3 in [16]. Additionally, we observe that solutions for (2.2) are expected to be in $L^1_s(\mathbb{R}^n)$, as they further satisfy the inequality

$$|u(x)| \le \max\{1, |x|^{1+\alpha_1}\} \quad \text{in } \mathbb{R}^n, \tag{2.3}$$

for some $0 < \alpha_1 < 2s - 1$. In particular, condition (2.3) implies

$$\begin{aligned} \|u\|_{L^1_s(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \frac{|u(y)|}{1 + |y|^{n+2s}}\, dy \\ &\le |B_1| + \int_{\mathbb{R}^n \setminus B_1} \frac{|y|^{1+\alpha_1}}{|y|^{n+2s}}\, dy = |B_1| + \frac{|\mathbb{S}^{n-1}|}{2s - 1 - \alpha_1}. \end{aligned} \tag{2.4}$$

Lipschitz estimates for solutions of (2.2) are available, with the advantage that they are uniform with respect to $\xi \in \mathbb{S}^{n-1}$ (see Lemma 2.2 in [16]). For the reader's convenience, we restate the result below.

Proposition 2.2. *Let u be a solution to (2.2), for $\xi \in \mathbb{S}^{n-1}$. If*

$$|u(x)| \le \max\{1, |x|^{1+\alpha_1}\} \quad \textit{in } \mathbb{R}^n,$$

then there is a constant $\Lambda = \Lambda(n, s)$ such that

$$\|Du\|_{L^\infty(B_{3/4})} \le \Lambda,$$

provided ν is universally small enough.

We conclude this section by recording the following L^ε-estimate, which is stated to the fractional Laplacian setting in Theorem 10.4 of [11] and provides a quantitative control on the measure of sublevel sets.

Lemma 2.1. *Let $r > 0$ and $z \in \mathbb{R}^n$. Consider $u \ge 0$ in $\mathbb{R}^n$ and $\Delta^s u \le C_0$ in $B_{2r}(z)$. Then, there are universal constants $C > 0$ and $\varepsilon > 0$ such that*

$$|\{u < t\} \cap B_r(z)| \le C r^n (u(z) + C_0 r^{2s})^\varepsilon t^{-\varepsilon} \quad \forall t > 0.$$

3. Nonlocal positivity argument

In this section, we develop a positivity argument to improve the oscillation of Du in dyadic balls. The idea is that if the derivative is small in a region of positive measure, then it oscillates in a controlled fashion in a smaller portion of the region. In what follows, we provide a subsolution information for derivatives of a given solution.

Lemma 3.1. *Let $\eta\colon \mathbb{R}^n \to [0,1]$ be a smooth cut-off function satisfying*

$$\eta = 1 \quad \text{in } B_1, \quad \text{and} \quad \eta = 0 \quad \text{in } \mathbb{R}^n \setminus B_2.$$

If u is a solution to (1.4) and $e \in \mathbb{S}^{n-1}$, then $v = \eta(\partial_e u - \mu)_+$ solves

$$\Delta^s v \geq -C \|u\|_{L^1_s(\mathbb{R}^n)} \quad \text{in } B_{1/2},$$

for any $\mu \in (0,1)$ and where C is a dimensional constant.

Proof. Recall that u is s-harmonic in $\Omega \cap B_1$ and thus smooth in this region. First, let us consider $x \in \Omega \cap B_1$. We proceed through a cut-off argument. Let $e \in \mathbb{S}^{n-1}$, and define

$$w^h(z) := \frac{u(z+he) - u(z)}{h}.$$

For h small enough, it follows that $x + he \in \Omega \cap B_1$. Consequently, we have

$$\Delta^s w^h(x) = 0.$$

Now, for η as in the statement of Lemma 3.1, we write

$$w^h = \eta w^h + (1-\eta) w^h =: w_1 + w_2.$$

Hence, we obtain

$$\Delta^s w_1(x) = f(x), \quad \text{for } f := -\Delta^s w_2.$$

Using change of variables, we obtain

$$\begin{aligned}
f(x) &= \int_{\mathbb{R}^n} \frac{w_2(y)}{|y-x|^{n+2s}}\, dy \\
&= \int_{\mathbb{R}^n} (1-\eta(y)) \frac{(u(y+he) - u(y))}{h} \frac{1}{|y-x|^{n+2s}}\, dy \\
&= \int_{\mathbb{R}^n} u(y) \Big(\frac{g(y-he) - g(y)}{h} \Big)\, dy,
\end{aligned}$$

where

$$g(z) := (1-\eta(z))\, |z-x|^{-(n+2s)}.$$

By the mean value theorem, we have

$$|g(y-he) - g(y)| \leq |Dg(y+\theta e)|\, h, \quad \text{for } \theta \in (0,h).$$

By direct computations, we observe that

$$\begin{aligned}|Dg(z)| &\le |D\eta(z)|\,|z-x|^{-(n+2s)} + (n+2s)(1-\eta(z))\,|z-x|^{-(n+1+2s)}\\ &\le C\,|z-x|^{-(n+2s)}\chi_{B_2\setminus B_1} + (n+2)\,|z-x|^{-(n+1+2s)}\chi_{\mathbb{R}^n\setminus B_1}.\end{aligned}$$

Since $x \in B_{1/2}$ and $z \in \mathbb{R}^n \setminus B_1$, we have $|z-x| \ge \frac{1}{2}|z|$, and so

$$|Dg(z)| \le C\,|z|^{-(n+2s)}\chi_{\mathbb{R}^n\setminus B_1}.$$

As a consequence,

$$|f(x)| \le C\int_{\mathbb{R}^n\setminus B_{1/2}} u(z)\,\frac{1}{|z|^{n+2s}} \le C(n)\|u\|_{L^1_s(\mathbb{R}^n)}.$$

Thus, we have obtained

$$|\Delta^s w_1| \le C(n)\|u\|_{L^1_s(\mathbb{R}^n)} \quad \text{in } \Omega \cap B_1,$$

where $w_1 = \eta w^h$. Passing to the limit as $h \to 0$, we obtain

$$|\Delta^s(\eta\partial_e u)| \le C\,\|u\|_{L^1_s(\mathbb{R}^n)} \quad \text{in } B_1 \cap \Omega.$$

Now observe that if $v = \eta(\partial_e u - \mu)_+$ and $x \in \{v > 0\} \cap B_1$, then

$$\begin{aligned}\Delta^s v(x) &= \int_{\mathbb{R}^n} \frac{(\partial_e u(y)-\mu)_+ - (\partial_e u(x)-\mu)}{|y-x|^{n+2s}}\,dy\\ &\ge \int_{\mathbb{R}^n} \frac{\eta(y)\,\partial_e u(y) - \partial_e u(x) + (1-\eta(y))\,\mu}{|y-x|^{n+2s}}\,dy.\end{aligned}$$

Since $1-\eta \ge 0$, it then follows that

$$\Delta^s v(x) \ge \Delta^s(\eta\,\partial_e u)(x) \ge -C\,\|u\|_{L^1_s(\mathbb{R}^n)},$$

where we have also used that since $x \in \{v > 0\}$, then $\partial_e u(x) \ne 0$. As a consequence, it follows that $Du(x) \ne 0$ and so $x \in \Omega \cap B_1$. If $x \in \{v = 0\}$, then it is straightforward to see that

$$\Delta^s v(x) = \int_{\mathbb{R}^n} \frac{v(y)}{|y-x|^{n+2s}}\,dy \ge 0.$$

In any case, it holds

$$\Delta^s v \ge -C\,\|u\|_{L^1_s(\mathbb{R}^n)} \quad \text{in } B_{1/2}.$$

■

Now, we present the gradient improvement of oscillation estimates. For simplicity, we assume throughout this section that u is a solution to (1.4) satisfying $u(0) = 0$.

Lemma 3.2. *Assume u satisfies $\|Du\|_{L^\infty(B_{1/2})} \le 1$ and*

$$|u(x)| \le \max\{1, |x|^{1+\alpha_1}\} \quad \textit{in } \mathbb{R}^n.$$

Given $\mu, \delta \in (0,1)$, there exist positive parameters $\mu_\star$ and $r_\star$, depending only on n, s, μ and δ, such that the following holds: for a given $e \in \mathbb{S}^{n-1}$, if

$$|\{x \in B_{r_\star} \mid Du(x) \cdot e \leq \delta\}| > \mu |B_{r_\star}|,$$

then

$$Du \cdot e \leq 1 - \mu_\star \quad \textit{in } B_{r_\star/4}.$$

Proof. Let η be as specified in the assumptions of Lemma 3.1. Defining

$$w := (Du \cdot e - \delta)_+,$$

we apply Lemma 3.1 and conclude that w is a solution to

$$\Delta^s(\eta w) \geq -C_1 \|u\|_{L^1_s(\mathbb{R}^n)} \geq -C_0 \quad \text{in } B_{1/2}.$$

In the second inequality, we follow (2.4), for C_0 a constant depending only on n and s.

Thanks to $\|Du\|_{L^\infty(B_{1/2})} \leq 1$, we observe that $w \leq 1 - \delta$ in $B_{1/2}$. This implies that function

$$\overline{w} := (1 - \delta - w)_+$$

satisfies

$$\Delta^s(\eta \overline{w}) \leq C_0 \quad \text{in } B_{1/2}.$$

Moreover,

$$|\{x \in B_{r_\star} \mid \overline{w} \geq 1 - \delta\}| > \mu |B_{r_\star}|.$$

We now observe that for any $y \in B_{r_\star/4}$, the inclusion $B_{r_\star} \subset B_{3r_\star/2}(y)$ holds. Consequently, Lemma 2.1 can be applied to yield

$$\begin{aligned}
\mu |B_{r_\star}| &< |\{x \in B_{r_\star} \mid \overline{w}(x) \geq 1 - \delta\}| \leq |\{x \in B_{3r_\star/2}(y) \mid \overline{w}(y) \geq 1 - \delta\}| \\
&\leq C r_\star^n (\overline{w}(x) + C_0 r_\star^{2s})^\varepsilon (1 - \delta)^{-\varepsilon},
\end{aligned}$$

for $y \in B_{r_\star/4}$. Rearranging terms,

$$c(1 - \delta)\mu^{1/\varepsilon} \leq \overline{w}(y) + r_\star^{2s} C_0,$$

for some $c > 0$. Next, we take $r_\star$ small depending on δ, μ, c and C_0, so that

$$\frac{c(1 - \delta)\mu^{1/\varepsilon}}{2} \leq \overline{w} \quad \text{in } B_{r_\star/4}.$$

Finally, by the definition of $\overline{w}$, we conclude that

$$Du \cdot e \leq 1 - \mu_\star \quad \text{in } B_{r_\star/4},$$

for $\mu_\star := 2^{-1} c \mu^{1/\varepsilon}(1 - \delta)$. ∎

We now apply an iterative method to establish gradient control within dyadic balls. Unlike local cases, special care is needed to ensure that the growth at infinity for rescaled functions is maintained, which evidences the nonlocal influence in the argument. For notational simplicity, let us define $I_k := \{0, 1, \ldots, k\}$.

Proposition 3.1. *Assume u satisfies $\|Du\|_{L^\infty(B_{1/2})} \le 1$ and*

$$|u(x)| \le \max\{1, |x|^{1+\alpha_1}\} \quad \text{in } \mathbb{R}^n.$$

Given $\mu, \delta \in (0,1)$, there exist positive parameters $r_\star$, λ and α, depending only on n, s, μ and δ, such that the following holds: given $k > 0$ integer, assume that

$$\inf_{e\in\mathbb{S}^{n-1}} |\{x \in B_{r_\star\lambda^i} \mid Du(x)\cdot e \le \delta\lambda^{\alpha i}\}| > \mu |B_{r_\star\lambda^i}|, \tag{3.1}$$

for each $i \in I_k$. Then

$$\|Du\|_{L^\infty(B_{r_\star\lambda^{i+1}})} \le \lambda^{\alpha(i+1)}. \tag{3.2}$$

Proof. Initially, we consider $\lambda > 0$ small enough satisfying

$$\max\left(\lambda\frac{2}{r_\star}, \lambda^{\alpha_1/2}\left(\frac{4}{r_\star}\right)^{1+\alpha_1}\right) \le 1. \tag{3.3}$$

For $\mu_\star$ as in Lemma 3.2, we consider

$$v_{i+1}(x) := \frac{v_i(\lambda x)}{\lambda(1-\mu_\star)},$$

for each $i \in I_k$, where $v_0 = u$.

Next, we claim that if $v_j(0) = 0$, $\|Dv_j\|_{L^\infty(B_{1/2})} \le 1$, and

$$|v_j(x)| \le \max\{1, |x|^{1+\alpha_1}\} \quad \text{in } \mathbb{R}^n \tag{3.4}$$

holds for $j = i \le k$, then the same holds for $j = i+1$. Indeed, we easily have $v_{i+1}(0) = 0$. Considering

$$\alpha := \frac{\ln(1-\mu_\star)}{\ln(\lambda)}, \tag{3.5}$$

we observe that

$$v_i(x) = \frac{u(\lambda^i x)}{\lambda^{i(1+\alpha)}}.$$

By further adjusting λ, we may assume $\alpha \le \alpha_1/2$. Additionally, note that v_i solves (1.4). By assumption (3.1), we notice that

$$\inf_{e\in\mathbb{S}^{n-1}} |\{x \in B_{r_\star} \mid Dv_i(x)\cdot e \le \delta\}| > \mu|B_{r_\star}|.$$

From the choice in (3.5), Lemma 3.2 applied to v_i yields

$$\|Dv_i\|_{L^\infty(B_{r_\star/4})} \le 1-\mu_\star = \lambda^\alpha. \tag{3.6}$$

Hence, we have that the estimate above implies

$$\|Dv_{i+1}\|_{L^\infty(B_{1/2})} \le \|Dv_{i+1}\|_{L^\infty(B_{\lambda^{-1}r_\star/4})} \le 1.$$

Finally, we shall conclude estimate (3.4). From the latter estimate, we use that $v_{i+1}(0) = 0$ to obtain

$$|v_{i+1}(x)| \leq \max\{1, |x|^{1+\alpha_1}\}, \quad \text{for } x \in B_{\lambda^{-1} r_\star/4}.$$

For the complementary set $\mathbb{R}^n \setminus B_{\lambda^{-1} r_\star/4}$, we use (3.4) for $j = i$ to derive

$$\begin{aligned} |v_{i+1}(x)| &= \lambda^{-(1+\alpha)} |v_i(\lambda x)| \leq \lambda^{-(1+\alpha)} \max\{1, \lambda^{1+\alpha_1} |x|^{1+\alpha_1}\} \\ &= \lambda^{\alpha_1 - \alpha} \max\{\lambda^{-(1+\alpha_1)}, |x|^{1+\alpha_1}\} \end{aligned}$$

and so,

$$|v_{i+1}(x)| \leq \lambda^{\alpha_1 - \alpha} \left(\frac{4}{r_\star}\right)^{1+\alpha_1} |x|^{1+\alpha_1}.$$

Using that $\alpha_1 - \alpha \geq \alpha_1/2$, and inequality (3.3), we get

$$|v_{i+1}(x)| \leq \max\{1, |x|^{1+\alpha_1}\},$$

as claimed.

Finally, we apply the claim recursively for each $i \in I_k$, where (3.6) gives

$$\|Du\|_{L^\infty(B_{r_\star \lambda^{i+1}})} \leq \lambda^{\alpha(i+1)}. \qquad \blacksquare$$

4. Oscillation estimates

In this section, we establish gradient oscillation estimates for solutions under small flatness assumptions.

Proposition 4.1. *Let u be a solution to (1.4) satisfying*

$$|u(x)| \leq \max\{1, |x|^{1+\alpha_1}\}, \quad \text{for } x \in \mathbb{R}^n.$$

There exist parameters $\lambda_\star$ and C, depending only on n and s, such that the following holds: suppose there exists an affine function $\ell(x) := a + \xi \cdot x$, with $\xi \in \mathbb{S}^{n-1}$, such that

$$\|u - \ell\|_{L^\infty(B_1)} \leq \lambda_\star^2. \tag{4.1}$$

Then, we have

$$|Du(x) - Du(0)| \leq C|x|^{2s-1}, \quad \text{for } x \in B_{1/2}.$$

Proof. For $\beta < \alpha_1$, define

$$w(x) := \frac{[u - \ell](\lambda_\star x)}{\lambda_\star^{1+\beta}}.$$

for $\lambda_\star$ to be chosen later. First, we concentrate our analysis to show that

$$|w(x)| \leq \max\{1, |x|^{1+\alpha_1}\} \quad \text{in } \mathbb{R}^n.$$

Indeed, by (4.1) we easily have $|w(x)| \leq 1$ in $B_{\lambda_\star^{-1}}$.

For $|x| > \lambda_\star^{-1}$, we obtain

$$\begin{aligned}|w(x)| &\leq \lambda_\star^{-(1+\beta)}\left(|u(\lambda_\star x)| + |a| + |\lambda_\star x|\right)\\ &\leq \lambda_\star^{-(1+\beta)}\left(\max\{1, \lambda_\star^{1+\alpha_1}|x|^{1+\alpha_1}\} + 2 + \lambda_\star|x|\right)\\ &\leq \lambda_\star^{\alpha_1-\beta}\left(\max\{\lambda_\star^{-(1+\alpha_1)}, |x|^{1+\alpha_1}\} + 2\lambda_\star^{-(1+\alpha_1)} + \lambda_\star^{-\alpha_1}|x|\right)\\ &\leq 4\lambda_\star^{\alpha_1-\beta}|x|^{1+\alpha_1}.\end{aligned}$$

Assuming $4\lambda_\star^{\alpha_1-\beta} \leq 1$, this implies that $|w(x)| \leq |x|^{1+\alpha_1}$ for $x \in \mathbb{R}^n \setminus B_{\lambda_\star^{-1}}$.

Secondly, we claim that

$$B_{3/4} \subseteq \{x \in B_{3/4} \mid \xi + \lambda_\star^\beta Dw(x) \neq 0\}.$$

Indeed, since w promptly satisfies (2.2), we take $\lambda_\star^\beta$ even smaller, and apply Proposition 2.2, deriving

$$\|Dw\|_{L^\infty(B_{3/4})} \leq \Lambda,$$

for some universal $\Lambda > 0$. Hence, we observe that

$$|\xi + \lambda_\star^\beta Dw(x)| \geq 1 - \lambda_\star^\beta \Lambda \geq \frac{1}{2},$$

for each $x \in B_{3/4}$.

In view of this, w is s-harmonic in $B_{3/4}$. By classical gradient regularity estimates,

$$|Dw(x) - Dw(0)| \leq C|x|^{2s-1} \quad \text{for } x \in B_{1/2}, \tag{4.2}$$

for some $C > 0$ depending only on n and s. Scaling back w to u, we have

$$|Du(x) - Du(0)| \leq C|x|^{2s-1} \quad \text{for } x \in B_{\lambda_\star/2}.$$

Finally, for $x \in B_{1/2} \setminus B_{\lambda_\star/2}$, we conclude that

$$|Du(x) - Du(0)| \leq \frac{4}{\lambda_\star}\|Du\|_{L^\infty(B_{1/2})}|x|^{2s-1}. \qquad \blacksquare$$

Before proceeding with the proof of the main theorem, we conclude this section by showing that, in a neighborhood of a nondegenerate point, a given Lipschitz function is close to an affine function with a unit slope. This result follows from a standard argument using Sobolev's inequality.

Lemma 4.1. *Let $v \in W^{1,\infty}(B_2)$ be such that $v(0) = 0$ and $\|Dv\|_{L^\infty(B_2)} \leq 1$. Given $\varepsilon > 0$, there exist μ and δ, depending on ε and n, such that if*

$$|\{x \in B_1 \mid Dv(x) \cdot e \leq \delta\}| \leq \mu|B_1|,$$

for some $e \in \mathbb{S}^{n-1}$, then there exist $a \in [-1, 1]$ and $\xi \in \mathbb{S}^{n-1}$ such that

$$\|v - \ell\|_{L^\infty(B_1)} \leq \varepsilon,$$

for $\ell(x) := a + \xi \cdot x$.

Proof. Let

$$a = \frac{1}{|B_1|} \int_{B_1} v.$$

From Sobolev's inequality,

$$\left|v(x) - (a + e \cdot x)\right|^{2n} \leq C(n) \int_{B_1} |Dv(x) - e|^{2n} dx.$$

For simplicity, define

$$\mathcal{A} = \left\{x \in B_1 \mid Dv(x) \cdot e \leq \delta\right\}.$$

By assumption, we have $|\mathcal{A}| \leq \mu |B_1|$, and so

$$\begin{aligned}\int_{B_1} |Dv(x) - e|^{2n} dx &= \int_{\mathcal{A}} |Dv(x) - e|^{2n}\, dx + \int_{B_1 \setminus \mathcal{A}} |Dv(x) - e|^{2n}\, dx \\ &\leq 4^n |\mathcal{A}| + 4^n |B_1| (1 - \delta)^{2n} \\ &\leq C(n)(\mu + (1 - \delta)^{2n}) \leq \varepsilon^{2n},\end{aligned}$$

provided δ and μ are carefully chosen. Hence, it follows that

$$\|v - (a + e \cdot x)\|_{L^\infty(B_1)} \leq \varepsilon.$$

In addition, since $v(0) = 0$ and $\|Dv\|_{L^\infty(B_2)} \leq 1$, this implies that $a \in [-1, 1]$. ■

5. Gradient regularity estimates

In this section, we build on the results from Sections 3 and 4 to prove Theorem 1.1, applying a dichotomy argument that considers possible degeneracy contexts.

Proof of Theorem 1.1. For

$$K := 2\|u\|_{L^\infty(\mathbb{R}^n)} + \|Du\|_{L^\infty(B_{3/4})} \quad \text{and} \quad x_0 \in B_{1/2},$$

denote

$$v(x) := \frac{u(x_0 + 4^{-1}x) - u(x_0)}{K}.$$

Let $\lambda_\star$ be as defined in Proposition 4.1. Set $\varepsilon = \lambda_\star^2$ in the assumptions of Lemma 4.1, and let μ and δ be the corresponding parameters from that result. By applying μ and δ in Proposition 3.1, we consider parameters $r_\star$, λ, and α.

For each nonnegative integer i, let

$$\mathcal{A}(i) := \inf_{e \in \mathbb{S}^{n-1}} |\{x \in B_{r_\star \lambda^i} \mid Dv(x) \cdot e \leq \delta \lambda^{\alpha i}\}|,$$

and define

$$\iota := \inf\{i \in \mathbb{N} \mid \mathcal{A}(i) \leq \mu |B_{r_\star \lambda^i}|\}. \tag{5.1}$$

We analyze the proof into two cases.

Case $\iota = \infty$. From Proposition 3.1,

$$\|Dv\|_{L^\infty(B_{r_\star \lambda^{i+1}})} \leq \lambda^{\alpha(i+1)}, \quad \text{for each } i \in \mathbb{N}.$$

In particular, this implies that $Dv(0) = 0$. Additionally, for each $x \in B_{r_\star \lambda}$, there exists $j = j(x) \in \mathbb{N}$ such that

$$r_\star \lambda^{j+1} \leq |x| \leq r_\star \lambda^j.$$

Hence, we obtain

$$|Dv(x)| \leq \lambda^{\alpha j} \leq (r_\star \lambda)^{-\alpha} |x|^\alpha. \tag{5.2}$$

For $x \in B_2 \setminus B_{r_\star \lambda}$, we use that $\|Dv\|_{L^\infty(B_2)} \leq 1$ to get

$$|Dv(x)| \leq 1 \leq (r_\star \lambda)^{-\alpha} |x|^\alpha.$$

Therefore,

$$|Dv(x)| \leq C|x|^\alpha \quad \text{for } x \in B_2.$$

Case $\iota < \infty$. Immediately, for some $e \in \mathbb{S}^{n-1}$, we have

$$|\{x \in B_1 \mid Dw(x) \cdot e \leq \delta\}| \leq \mu |B_1|, \tag{5.3}$$

provided

$$w(x) := \frac{v(r_\star \lambda^\iota x)}{r_\star \lambda^{\iota(1+\alpha)}}.$$

According to the proof of Proposition 3.1, we derive that $\|Dw\|_{L^\infty(B_2)} \leq 1$ and

$$|w(x)| \leq \max\{1, |x|^{1+\alpha_1}\} \quad \text{for } x \in \mathbb{R}^n.$$

In the sequel, by taking $\varepsilon = \lambda_\star^2$, we apply Lemma 4.1 for w, obtaining so

$$\|w - \ell\|_{L^\infty(B_1)} \leq \lambda_\star^2,$$

for some affine function ℓ with $|D\ell| = 1$. From Proposition 4.1, we obtain

$$|Dw(x) - Dw(0)| \leq C|x|^{2s-1} \quad \text{for } x \in B_{1/2},$$

and so,

$$|Dv(y) - Dv(0)| \leq C|y|^\alpha \quad \text{for } y \in B_{r_\star \lambda^\iota}.$$

We use the fact that (5.1) holds up to index $\iota - 1$, yielding

$$\|Dv\|_{L^\infty(B_{r_\star \lambda^j})} \leq \lambda^{\alpha j}, \quad \text{for } j = 0, \ldots, \iota.$$

Consequently, for each $y \in B_{r_\star} \setminus B_{r_\star \lambda^\iota}$,

$$|Dv(y) - Dv(0)| \leq 2\|Dv\|_{L^\infty(B_{r_\star \lambda^j})} \leq 2\lambda^{\alpha j} \leq C|y|^\alpha,$$

for $j = j(y) \in \{0, 1, \ldots, \iota - 1\}$, satisfying

$$r_\star \lambda^{j+1} \leq |y| \leq r_\star \lambda^j.$$

This provides the desired estimate in $B_{r_\star}$. Since Dv is normalized, we argue as before to extend the estimate for Dv up to B_2. From this, we obtain

$$|Du(z) - Du(x_0)| \leq C\left(\|u\|_{L^\infty(\mathbb{R}^n)} + \|Du\|_{L^\infty(B_{3/4})}\right) \quad \text{for each } z \in B_{1/2}(x_0). \qquad \blacksquare$$

Let us briefly explain how to extend Theorem 1.1 to the case with nonhomogeneous right-hand side and bounded solutions.

Proposition 5.1. *Let $u \in C(B_1) \cap L^\infty(\mathbb{R}^n)$ be a viscosity solution to (1.2), for some $s \in (1/2, 1)$. Then, u is locally $C^{1,\alpha}$, for some universal $\alpha \in (0,1)$, depending only on n and s. Furthermore, there exists $C = C(n,s)$, such that*

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{C^{0,1}(B_1)}\big).$$

Following the program developed here, the first step is to adapt Lemma 3.1. The proof will be the same, except that the difference quotient w^h will solve

$$\Delta^s w^h = f_h := \frac{f(x+h) - f(x)}{h}.$$

Lipschitz continuity of the right-hand side plays a role in controlling the L^∞ size of f_h.

Lemma 5.1. *Let $\eta\colon \mathbb{R}^n \to [0,1]$ be a smooth cut-off function satisfying*

$$\eta = 1 \quad \textit{in } B_1, \quad \textit{and} \quad \eta = 0 \quad \textit{in } \mathbb{R}^n \setminus B_2.$$

If u is a solution to (1.2) and $e \in \mathbb{S}^{n-1}$, then $v = \eta(\partial_e u - \mu)_+$ solves

$$\Delta^s v \geq -C\big(\|u\|_{L^1_s(\mathbb{R}^n)} + \|Df\|_{L^\infty(B_1)}\big) \quad \textit{in } B_{1/2},$$

for any $\mu \in (0,1)$ and where C is a dimensional constant.

The rest of the program now has to consider this new ingredient, which can be done by following the ideas developed in [3].

The proof of Theorem 1.2 now follows through a cut-off argument. It a consequence of Proposition 5.1.

Proof of Theorem 1.2. Define

$$v := u\chi_{B_1} \quad \text{and} \quad w := u(1 - \chi_{B_1}).$$

Since $u = v + w$, it follows that within $\Omega \cap B_1$,

$$f = \Delta^s u = \Delta^s v + \Delta^s w.$$

Thus, if we denote

$$g := -\Delta^s w,$$

we get that the function v solves (1.2). Observe also that for points in $B_{3/4}$, we have

$$|Dg(x)| \leq (n+2s)\int_{\mathbb{R}^n \setminus B_1} |u(y)||y-x|^{-n-2s-1}\,dy \leq C\|u\|_{L^1_s(\mathbb{R}^n)},$$

and we can apply Proposition 5.1 with B_1 replaced by $B_{3/4}$. ∎

6. Fully nonlinear operators

In this final section, we briefly outline how our results extend to a broader class of equations. Specifically, we consider the class $\mathcal{L}_1(s)$, first introduced in [11], which consists of kernels $\mathcal{K}\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ satisfying

$$\Lambda^{-1} \le \mathcal{K}(y)|y|^{n+2s} \le \Lambda, \quad \text{and} \quad |D\mathcal{K}(y)| \le \Lambda |y|^{-n-2s-1}$$

for all $y \in \mathbb{R}^n \setminus \{0\}$. A nonlocal operator $\mathcal{I}$ is said to be elliptic with respect to the class $\mathcal{L}_1(s)$ if it satisfies the following inequality:

$$M^-_{\mathcal{L}_1(s)}[w](x) \le \mathcal{I}[u+w](x) - \mathcal{I}[u](x) \le M^+_{\mathcal{L}_1(s)}[w](x), \tag{6.1}$$

where the extremal operators are defined as

$$M^-_{\mathcal{L}_1(s)}[w](x) := \inf_{\mathcal{K} \in \mathcal{L}_1(s)} L_{\mathcal{K}}[w](x)$$

and

$$M^+_{\mathcal{L}_1(s)}[w](x) := \sup_{\mathcal{K} \in \mathcal{L}_1(s)} L_{\mathcal{K}}[w](x),$$

with

$$L_{\mathcal{K}}[w](x) := \int_{\mathbb{R}^n} (w(y) - w(x))\, \mathcal{K}(y-x)\, dy.$$

This definition characterizes nonlocal ellipticity in terms of the extremal influence of the class $\mathcal{L}_1(s)$, playing a crucial role in the analysis to be developed in the following.

For this nonlocal operator $\mathcal{I}$, we consider solutions to

$$\begin{cases} \mathcal{I}u = f & \text{in } B_1 \cap \Omega, \\ |Du| = 0 & \text{in } B_1 \setminus \Omega. \end{cases} \tag{6.2}$$

Although the strategy is analogous, the estimates will now depend on the ellipticity constant Λ. To illustrate this, we state below the corresponding version of Lemma 3.1.

Lemma 6.1. *Let $\eta\colon \mathbb{R}^n \to [0,1]$ be a smooth cut-off function satisfying*

$$\eta = 1 \quad \text{in } B_1, \quad \text{and} \quad \eta = 0 \quad \text{in } \mathbb{R}^n \setminus B_2.$$

If u is a solution to (6.2) and $e \in \mathbb{S}^{n-1}$, then $v = \eta(\partial_e u - \mu)_+$ solves

$$\mathcal{M}_{\mathcal{L}_1(s)} v \ge -C\left(\|u\|_{L^1_s(\mathbb{R}^n)} + \|Df\|_{L^\infty(B_1)}\right) \quad \text{in } B_{1/2},$$

for any $\mu \in (0,1)$ and where C depends on n, s and Λ.

Proof. We can assume $\mathcal{I}[0] = 0$ for simplicity. We consider

$$w^h(x) := \frac{u(x+h) - u(x)}{h} = w_1 + w_2,$$

where $w_1 = w^h \chi_{B_1}$ and $w_2 = w^h(1 - \chi_{B_1})$. By the ellipticity assumption on $\mathcal{I}$, we have

$$\mathcal{M}_{\mathcal{L}_1(s)} w^h \ge f^h.$$

Therefore,

$$\mathcal{M}_{\mathcal{L}_1(s)} w_1 \geq -\mathcal{M}_{\mathcal{L}_1(s)} w_2 - \|Df\|_{L^\infty(B_1)}$$

As before,

$$|-\mathcal{M}_{\mathcal{L}_1(s)} w_2| \leq C(n, s, \Lambda)\|u\|_{L^1_s(\mathbb{R}^n)}.$$

Here we are using that the kernels in the class $\mathcal{L}_1(s)$ satisfies

$$|D\mathcal{K}(y)| \leq \Lambda |y|^{-n-2s-1}, \quad \text{for } y \neq 0. \qquad \blacksquare$$

The next ingredients we require are the corresponding versions of Lemma 3.2 and Proposition 4.1. Although their proofs follow the same general strategy as before, a few minor adjustments are required, and we briefly comment on them below.

First, we invoke the most general formulation of Lemma 2.1, stated in terms of the nonlocal Pucci operators (see Theorem 10.4 in [11]). With this modification, the proof of Proposition 3.1 carries over unchanged.

For the proof of the analogue of Proposition 4.1, only a minor modification is required: in estimate (4.2), the exponent $2s-1$ is replaced by a universal exponent $\overline{\alpha} \leq 2s-1$. This exponent arises from the interior regularity theory established in [12].

After these adjustments, we follow the same strategy as in the proof of Theorem 1.1, to establish the following result.

Theorem 6.1. *Let u be a viscosity solution to (6.2), for some $s \in (1/2, 1)$. Then, u is locally $C^{1,\alpha}$, for some universal $\alpha \in (0,1)$, depending only on n, s and Λ. Furthermore, there exists $C = C(n, s, \Lambda)$, such that*

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq C\big(\|u\|_{L^\infty(B_1)} + \|u\|_{L^1_s(\mathbb{R}^n)} + \|f\|_{C^{0,1}(B_1)}\big).$$

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