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Short note A generalization of the harmonic series

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Abstract. In this note, we establish a general theorem concerning the divergence of series. As an immediate application, this theorem demonstrates divergence for a well-known class of slowly diverging series.

It is well known that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges. For the popular proofs of the divergence of harmonic series, see [2]. Replacing n by a sequence of positive numbers a_n which is satisfying the property $0 < (a_{n+1} - a_n) \leq 1$, one can easily conclude that

$$\sum_{n=1}^{\infty} \frac{1}{a_n}$$

diverges as $a_{k+1} - a_1 = \sum_{n=1}^{k} (a_{n+1} - a_n) \leq k$, i.e., $\frac{1}{a_{k+1}} \geq \frac{1}{k+a_1}$.

In this note, we present a test which establishes divergence for each member of a well-known class of divergent series. One member of that class is the harmonic series itself.

Theorem 1. *Let* (a_n) *and* (t_n) *be two sequences of positive numbers satisfying*

$$0 < (a_{n+1} - a_n) \leq t_n.$$

If $\lim_{n\to\infty} a_n = \infty$, *then*

$$\sum_{n=1}^{\infty} \frac{t_n}{a_n} = \infty.$$

Proof. Since $0 < a_{n+1} \leq t_n + a_n$ and the function $\frac{1}{x}$ is convex and decreasing on $(0, \infty)$, we have

$$\ln a_{n+1} - \ln a_n = \int_{a_n}^{a_{n+1}} \frac{dx}{x} < \frac{1}{a_n} \cdot t_n \quad \text{(for a visual proof, see Figure 1)},$$

$$\text{i.e.,} \quad \ln a_{k+1} - \ln a_1 = \sum_{n=1}^{k} (\ln a_{n+1} - \ln a_n) < \sum_{n=1}^{k} \frac{t_n}{a_n},$$

which gives $\sum_{n=1}^{\infty} \frac{t_n}{a_n} = \infty$. ∎

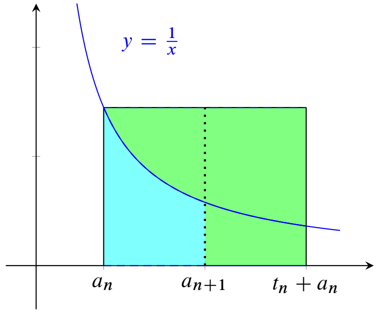

Figure 1. $\int_{a_n}^{a_{n+1}} \frac{dx}{x} < \frac{1}{a_n} \cdot t_n$

Example 1. For $\alpha \geq -1$,

$$\sum_{n=1}^{\infty} n^{\alpha} = \infty.$$

Consider $a_n = n$ and $t_n = n^{\alpha+1}$. Then $0 < a_{n+1} - a_n = 1 \leq n^{\alpha+1} = t_n$.

Example 2. For $0 < p \leq 1$,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \infty.$$

Consider $a_n = n^p$ and $t_n = 1$. Since $0 < \frac{n}{n+1} < 1$, thus $p \ln \frac{n}{n+1} < 0$ and $(1-p)\ln \frac{n}{n+1} \leq 0$. Hence, $a_n < a_{n+1}$ and $a_{n+1} \leq a_n + 1$.

Example 3. For $p \leq 1$,

$$\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^p} = \infty.$$

Consider $a_n = \ln n$ and $t_n = \frac{1}{n(\ln n)^{p-1}}$. Then $0 < a_{n+1} - a_n < \frac{1}{n} \leq \frac{1}{n(\ln n)^{p-1}}$. (Note that $(p-1)\ln(\ln n) \leq 0$ as $n \geq 3$.)

Example 4. For $p \leq 1$,

$$\sum_{n=2}^{\infty} \frac{1}{n^p \ln n} = \infty.$$

Consider $a_n = \ln n$ and $t_n = \frac{1}{n^p}$. Then $0 < a_{n+1} - a_n < \frac{1}{n} \leq \frac{1}{n^p}$. (Note that $(1-p)\ln n \geq 0$ as $n \geq 2$.)

Define $\ln^{[k]} n := \ln(\ln^{[k-1]} n)$ for $k = 1, 2, 3, \ldots$ and $\ln^{[0]} n := n$, where $n \in \mathbb{N}$.

Example 5 ([1, Corollary 1]). Let $k \in \mathbb{N}$. Choose $m \in \mathbb{N}$ such that $\ln^{[k]} m > 0$ but $\ln^{[k]}(m-1) < 0$. Then

$$\sum_{n=m}^{\infty} \frac{1}{n \ln^{[1]} n \ln^{[2]} n \ldots \ln^{[k]} n} = \infty.$$

Consider

$$a_n = \ln^{[k]} n \quad \text{and} \quad t_n = \frac{1}{n \ln^{[1]} n \ln^{[2]} n \ldots \ln^{[k-1]} n}.$$

Then, by the Mean Value Theorem,

$$a_{n+1} - a_n = \frac{1}{c \ln^{[1]} c \ln^{[2]} c \ldots \ln^{[k-1]} c} \quad \text{for some } c \in (n, n+1).$$

Thus

$$0 < a_{n+1} - a_n < \frac{1}{n \ln^{[1]} n \ln^{[2]} n \ldots \ln^{[k-1]} n}.$$

For $k = 2$, the choice of m is 3.

Example 6. In the above example, if we replace n by a strictly increasing sequence of positive numbers A_n which is tending to infinity, then we have the following.

Let $k \in \mathbb{N}$. Choose $m \in \mathbb{N}$ such that $\ln^{[k]} A_m > 0$ but $\ln^{[k]} A_{m-1} < 0$. Then

$$\sum_{n=m}^{\infty} \frac{1}{A_n \ln^{[1]} A_n \ln^{[2]} A_n \ldots \ln^{[k]} A_n} = \infty.$$

References

[1] R. Farhadian, A general approach to a familiar class of slowly diverging series. *Appl. Math. E-Notes* **24** (2024), 209–211 Zbl 07923072 MR 4831002

[2] S. J. Kifowit and T. A. Stamps, The harmonic series diverges again and again. *AMATYC Review* **27** (2006), no. 2, 31–43 (2006)

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