Elemente der Mathematik 81 (2026), 102–112
DOI 10.4171/EM/566

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Odd perfect numbers and their Euler prime

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1 Introduction

Let N be a positive integer and let $\sigma(N)$ denote the sum of positive divisors of N, including N itself. Then N is a perfect number if $\sigma(N) = 2N$. Euclid showed that, when $2^p - 1$ is prime, $N = 2^{p-1}(2^p - 1)$ is a perfect number. It is not difficult to prove that if $2^p - 1$ is prime, then p itself must be prime. This result is generally attributed to Cataldi–Fermat, e.g., see [7, Proposition 8]. Euler showed that if N is an even perfect number, then N must satisfy Euclid's formula, thereby completely characterizing even perfect numbers. The first few even perfect numbers include 6, 28, 496, and 8 128. Currently, 52 even perfect numbers are known, the largest being $2^{136\,279\,840}(2^{136\,279\,841} - 1)$, e.g., see [8]. It remains an open question whether there are infinitely many even perfect numbers.

While we know examples of even perfect numbers, to this day, it remains unknown whether an odd perfect number exists. Euler proved that an odd perfect number N must

Vollkommene Zahlen haben in der Zahlentheorie seit jeher eine besondere Faszination ausgeübt, nicht nur wegen ihrer eleganten Definition, sondern auch aufgrund ihrer reichen Geschichte. Während gerade vollkommene Zahlen dank klassischer Resultate von Euklid und Euler vollständig beschrieben sind, bleibt die Existenz ungerader vollkommener Zahlen bis heute ein ungelöstes und spannendes Problem. Im Laufe der Zeit wurden zahlreiche notwendige Bedingungen formuliert, die eine solche Zahl erfüllen müsste, doch ein konkretes Beispiel ist bislang nicht bekannt. Der vorliegende Beitrag nimmt die von Euler gegebene Struktur ungerader vollkommener Zahlen genauer in den Blick und untersucht, welche Auswirkungen die Wahl der sogenannten Eulerschen Primzahl auf die Gestalt und Eigenschaften solcher Zahlen hat. Die dargestellten Resultate liefern zusätzliche Einschränkungen, die das Bild möglicher ungerader vollkommener Zahlen weiter schärfen.

be of the form

$$N = p^{\alpha} Q^2, \quad \text{where } p \equiv \alpha \equiv 1 \bmod (4) \text{ and } p \text{ is a prime.} \tag{1}$$

The special prime p in representation (1) is also referred to as Euler's prime.

For the existence of an odd perfect number, many necessary conditions have been established. In particular, if such a number exists, it must be very large. For example, it is known that $N > 10^{1\,500}$, e.g., see [5]. In addition, an odd perfect number is characterized by having a large number of prime factors. Nielsen [4] has shown that N must have at least 10 distinct primes. Moreover, if $3 \nmid N$, then N must have at least 12 distinct prime factors. These are improvements on earlier results, e.g., see [1–3].

In this short paper, we examine Euler's formula (1) and discuss some of the implications that the choice of the Euler's prime p has for N. In particular, we show that if $p = 5$, then N is divisible by 3 but not by 7. Similarly, if $p = 29$, then N is divisible by 3 and 5, but not by 7, 11, and 13. We also discuss results concerning the exponent α. Specifically, we show that if $p = 5$, then $\alpha \equiv 1$ or 9 mod (12). On the other hand, if $p = 29$ and $\alpha \neq 1$, then $\alpha \geq 25$.

Particular attention is given to the case $p = 29$ and $\alpha = 25$. In this situation, we show that N has at least 12 distinct prime factors. Moreover, the largest prime factor of N is greater than $4 \cdot 10^9$, the second largest is greater than $2 \cdot 10^8$, and the third largest exceeds 10^5. We make also a list of non-admissible candidates for the Euler prime p. More specifically,

$$\begin{aligned} p \notin \{&1\,049, 4\,289, 6\,269, 8\,969, 9\,689, 13\,259, 16\,829, \\ &17\,489, 19\,139, 20\,129, 22\,229, 22\,619, 24\,419, 27\,059\}. \end{aligned}$$

Finally, we present an example of a positive odd integer N in Euler form (1) such that

$$(2 - \varepsilon)N < \sigma(N) < (2 + \varepsilon)N,$$

where $\varepsilon = 3.643 \cdot 10^{-7}$. This motivates us to introduce the notion of an ε-perfect number: given $\varepsilon > 0$, a positive integer is ε-perfect if

$$\left| \frac{\sigma(N)}{N} - 2 \right| < \varepsilon.$$

We conclude with a conjecture that, for every $\varepsilon > 0$, there exists an odd ε-perfect number.

2 Preliminaries

2.1 Non-admissible tuples

Sylvester [6] noted that if N is an odd perfect number, then N cannot be divisible by 3, 5, and 7 simultaneously. Indeed, if that were the case, then from (1), it would follow that $(5 \cdot 3^2 \cdot 7^2) \mid N$. Consequently,

$$2 = \frac{\sigma(N)}{N} \geq \frac{\sigma(5 \cdot 3^2 \cdot 7^2)}{5 \cdot 3^2 \cdot 7^2} = \left(1 + \frac{1}{5}\right)\left(1 + \frac{1}{3} + \frac{1}{3^2}\right)\left(1 + \frac{1}{7} + \frac{1}{7^2}\right) = \frac{494}{245} > 2,$$

raising a contradiction. Given a positive integer k, we say a k-tuple of odd primes

$$(p, q_1, \ldots, q_{k-1}) \quad \text{with } p \equiv 1 \bmod (4)$$

is admissible if

$$\Big(1+\frac{1}{p}\Big)\Big(1+\frac{1}{q_1}+\frac{1}{q_1^2}\Big)\cdots\Big(1+\frac{1}{q_{k-1}}+\frac{1}{q_{k-1}^2}\Big) \leq 2.$$

Hence, the triple $(5, 3, 7)$ is non-admissible. In fact, it can be shown that $(5, 3, 7)$ is the only non-admissible triple. For, if (p, q_1, q_2) is any other given triple of primes, then

$$\Big(1+\frac{1}{p}\Big)\Big(1+\frac{1}{q_1}+\frac{1}{q_1^2}\Big)\Big(1+\frac{1}{q_2}+\frac{1}{q_2^2}\Big) \leq \Big(1+\frac{1}{5}\Big)\Big(1+\frac{1}{3}+\frac{1}{3^2}\Big)\Big(1+\frac{1}{11}+\frac{1}{11^2}\Big) < 2.$$

If $(p, q_1, \ldots, q_{k-1})$ is non-admissible, then so is any tuple of greater length that contains $(p, q_1, \ldots, q_{k-1})$. For example, when $k = 4$, and considering primes up to 100 (i.e., $p, q_i < 100$), there are 47 non-admissible 4-tuples (p, q_1, q_2, q_3). However, only 16 of these (up to permutations of q_i) are new and do not include the triple $(5, 3, 7)$. These include

$$(5,3,11,13);\ (5,3,11,17);\ (5,3,11,19);\ (13,3,5,11);$$
$$(13,3,5,17);\ (13,3,5,19);\ (13,3,5,23);\ (17,3,5,11);$$
$$(17,3,5,13);\ (17,3,5,19);\ (29,3,5,11);\ (29,3,5,13);$$
$$(37,3,5,11);\ (41,3,5,11);\ (53,3,5,11);\ (61,3,5,11).$$

For any such 4-tuple, we obtain a Sylvester type result, that is, $(pq_1q_2q_3) \nmid N$. When $k = 5$, we obtain the following list of new non-admissible 5-tuples, considering primes up to 40 (i.e., $p, q_i < 40$):

$$(5,3,11,23,29);\ (5,3,11,23,31);\ (5,3,11,23,37);\ (5,3,11,29,31);$$
$$(5,3,11,29,37);\ (5,3,11,31,37);\ (5,3,13,17,19);\ (5,3,13,17,23);$$
$$(5,3,13,17,29);\ (5,3,13,17,31);\ (5,3,13,17,37);\ (5,3,13,19,23);$$
$$(5,3,13,19,29);\ (5,3,13,19,31);\ (5,3,13,19,37);\ (5,3,13,23,29);$$
$$(5,3,13,23,31);\ (5,3,13,23,37);\ (5,3,13,29,31);\ (5,3,17,19,23);$$
$$(5,3,17,19,29);\ (5,3,17,19,31);\ (13,3,5,29,31);\ (13,3,5,29,37);$$
$$(13,3,5,31,37);\ (13,3,7,11,17);\ (13,3,7,11,19);\ (13,3,7,11,23);$$
$$(13,3,7,11,29);\ (13,3,7,11,31);\ (13,3,7,11,37);\ (13,3,7,17,19);$$
$$(13,3,7,17,23);\ (17,3,5,23,29);\ (17,3,5,23,31);\ (17,3,5,23,37);$$
$$(17,3,5,29,31);\ (17,3,5,29,37);\ (17,3,5,31,37);\ (17,3,7,11,13);$$
$$(17,3,7,11,19);\ (17,3,7,11,23);\ (17,3,7,11,29);\ (17,3,7,11,31);$$
$$(17,3,7,11,37);\ (17,3,7,13,19);\ (17,3,7,13,23);\ (29,3,5,17,19);$$
$$(29,3,5,17,23);\ (29,3,5,17,31);\ (29,3,5,17,37);\ (29,3,5,19,23);$$
$$(29,3,5,19,31);\ (29,3,5,19,37);\ (29,3,5,23,31);\ (29,3,7,11,13);$$

(29, 3, 7, 11, 17); (29, 3, 7, 11, 19); (37, 3, 5, 11, 23); (37, 3, 5, 11, 31); (37, 3, 5, 13, 31); (37, 3, 5, 17, 23); (37, 3, 5, 17, 29); (37, 3, 5, 17, 31); (37, 3, 5, 19, 23); (37, 3, 5, 19, 29); (37, 3, 5, 19, 31); (37, 3, 7, 11, 13); (37, 3, 7, 11, 17); (37, 3, 7, 11, 19).

2.2 Non-feasible tuples

Given a positive integer k, we say a k-tuple of odd primes $(p, q_1, \ldots, q_{k-1})$ with $p \equiv 1 \bmod (4)$ is non-feasible if, for all possible k-tuples $(\alpha, \beta_1, \ldots, \beta_{k-1})$ of positive integers, the following inequality holds:

$$\Big(1 + \frac{1}{p} + \cdots + \frac{1}{p^{\alpha}}\Big) \prod_{i=1}^{k-1} \Big(1 + \frac{1}{q_i} + \cdots + \frac{1}{q_i^{2\beta_i}}\Big) < 2.$$

A sufficient condition to check whether a k-tuple of odd primes $(p, q_1, \ldots, q_{k-1})$ is non-feasible is the inequality

$$\frac{p}{p-1} \prod_{i=1}^{k-1} \frac{q_i}{q_i - 1} \leq 2.$$

This follows immediately from the fact that we have

$$\begin{aligned} &\Big(1 + \frac{1}{p} + \cdots + \frac{1}{p^{\alpha}}\Big) \prod_{i=1}^{k-1} \Big(1 + \frac{1}{q_i} + \cdots + \frac{1}{q_i^{2\beta_i}}\Big) \\ &\quad < \Big(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\Big) \prod_{i=1}^{k-1} \Big(1 + \frac{1}{q_i} + \frac{1}{q_i^2} + \cdots\Big) = \frac{p}{p-1} \prod_{i=1}^{k-1} \frac{q_i}{q_i - 1} \end{aligned}$$

for any k-tuple of exponents $(\alpha, \beta_1, \ldots, \beta_{k-1})$. For example, any triple (p, q_1, q_2) other than $(5, 3, 7)$, $(5, 3, 11)$, and $(5, 3, 13)$, and their permutations, is non-feasible, since

$$\frac{p}{p-1} \cdot \frac{q_1}{q_1 - 1} \cdot \frac{q_2}{q_2 - 1} \leq \frac{5}{4} \cdot \frac{3}{2} \cdot \frac{17}{16} < 1.993 < 2.$$

3 First results

Proposition 1. *Let $N = p^{\alpha} Q^2$ be an odd perfect number. If $(p, q_1, \ldots, q_m)$ is a non-admissible m-tuple such that*

$$p \equiv -1 \bmod (q_i) \quad \textit{for all } i \in I \subset \{1, 2, \ldots, m\},$$

then $q_i \mid N$ for all $i \in I$, but it is not divisible by $\prod_{j \notin I} q_j$.

Proof. Let $(p, q_1, \ldots, q_m)$ be a non-admissible m-tuple such that

$$p \equiv -1 \bmod (q_i) \quad \text{for all } i \in I \subset \{1, 2, \ldots, m\}.$$

Note that

$$\begin{aligned}\sigma(p^\alpha) &= 1 + p + p^2 + p^3 + \cdots + p^{\alpha-1} + p^\alpha \\ &= (1+p)(1 + p^2 + p^4 + \cdots + p^{\alpha-3} + p^{\alpha-1}),\end{aligned}$$

where the factorization is possible because $\alpha \equiv 1 \bmod (4)$. By multiplicativity of σ, we have $\sigma(N) = \sigma(p^\alpha)\sigma(Q^2)$, since $\gcd(p, Q) = 1$. Together with the equation $\sigma(N) = 2N$, we get that $q_i \mid N$ for all $i \in I$. Because $(p, q_1, \ldots, q_m)$ is a non-admissible m-tuple, N cannot be divisible simultaneously by all $q_j \notin I$. ∎

As a direct consequence of Proposition 1, for the first few Euler prime candidates, we have the following.

- If $p = 5$, then N is divisible by 3, but not 7.
- If $p = 13$, then N is divisible by 7, but not by $3 \cdot 5$.
- If $p = 17$, then N is divisible by 3, but not by $5 \cdot 7$.
- If $p = 29$, then N is divisible by 3 and 5, but neither by 7, 11, 13, nor by any of the products $17 \cdot 19$, $17 \cdot 23$, $17 \cdot 31$, $17 \cdot 37$, $19 \cdot 23$, $19 \cdot 31$, $19 \cdot 37$, $23 \cdot 31$.
- If $p = 37$, then N is divisible by 19, but not by any of the products $3 \cdot 5 \cdot 23$, $3 \cdot 5 \cdot 29$, $3 \cdot 5 \cdot 31$, $3 \cdot 7 \cdot 11$.

Proposition 2. *Let $N = p^\alpha Q^2$ be an odd perfect number. If $p = 5$, then the exponent α satisfies*

$$\alpha \equiv 1 \bmod (12) \quad \textit{or} \quad \alpha \equiv 9 \bmod (12).$$

Proof. Since $\alpha \equiv 1 \bmod (4)$, it follows that $\alpha \equiv 1, 5$, or $9 \bmod (12)$. Suppose $\alpha \equiv 5 \bmod (12)$, i.e., $\alpha = 12m + 5$ for some integer $m \geq 0$. Then

$$\begin{aligned}\sigma(5^{12m+5}) = (1 + 5 + 5^2 + 5^3 + 5^4 + 5^5) \\ \cdot (1 + 5^6 + 5^{12} + 5^{18} + 5^{24} + \cdots + 5^{12m-6} + 5^{12m}).\end{aligned}$$

Note that

$$1 + 5 + 5^2 + 5^3 + 5^4 + 5^5 = 3\,906 = 2 \cdot 3^2 \cdot 7 \cdot 31,$$

which implies $7 \mid \sigma(5^5)$. Since $\sigma(N) = 2N$, it follows that $7 \mid N$. However, from previous results, $7 \nmid N$ when $p = 5$, leading to a contradiction. Therefore, $\alpha \equiv 1$ or $9 \bmod (12)$. ∎

Proposition 3. *Let $(p, q_1, q_2, \ldots, q_k\}$ be a non-admissible tuple. If*

$$p \equiv -1 \bmod (q_i) \quad \textit{for all } 1 \leq i \leq k,$$

then p cannot be the Euler prime of an odd perfect number.

Proof. Suppose that $p \equiv -1 \bmod (q_i)$ for all $1 \leq i \leq k$ and that $N = p^\alpha Q^2$ is a perfect number. Then each q_i divides $\sigma(p^\alpha)$, and since $\sigma(N) = \sigma(p^\alpha)\sigma(Q^2) = 2N$, it follows that each $q_i \mid N$. However, $(p, q_1, q_2, \ldots, q_k)$ is a non-admissible tuple, contradicting the perfect number condition $\sigma(N) = 2N$. ∎

For example, the tuple $(1\,049, 5, 3, 7)$ is non-admissible. It holds that $p + 1 \equiv 0 \bmod (105)$. If $N = 1\,049^{\alpha} Q^2$, then $105 \mid N$. Therefore, N cannot be a perfect number. A further list of non-admissible Euler primes includes

$$\{4\,289, 6\,269, 8\,969, 9\,689, 13\,259, 16\,829, 17\,489,\\ 19\,139, 20\,129, 22\,229, 22\,619, 24\,419, 27\,059\}.$$

4 Odd perfect numbers of the form $N = 29^{25} Q^2$

Theorem 4. *Let $N = 29^{\alpha} Q^2$ be an odd perfect number. Then the following holds.*

(1) *N is not divisible by* 7, 11, *and* 13.

(2) *If $\alpha \neq 1$, then $\alpha \geq 25$.*

Proof. Let $p = 29$. Then $p + 1 = 30$, which implies $3 \mid \sigma(5^{\alpha})$ and $5 \mid \sigma(5^{\alpha})$. Therefore, $(3 \cdot 5) \mid \sigma(N)$, and since $\sigma(N) = 2N$, it follows that $(3 \cdot 5) \mid N$. Consequently, $7 \nmid N$. Moreover, both tuples $(29, 3, 5, 11)$ and $(29, 3, 5, 13)$ are non-admissible, which implies that N is not divisible by 11 or 13. This proves (1).

Next, let $\alpha \neq 1$. Recall that $\sigma(p^{\alpha}) = 1 + p + p^2 + \cdots + p^{\alpha}$. We analyze possible values of α modulo 4, and examine prime factorizations of $\sigma(p^{\alpha})$.

- If $\alpha = 5$,

$$1 + 29 + 29^2 + 29^3 + 29^4 + 29^5 = 21\,243\,690 = 2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 67 \cdot 271.$$

  Therefore, $(3 \cdot 5 \cdot 13) \mid N$, but $(29, 3, 5, 13)$ is non-admissible.

- If $\alpha = 9$,

$$1 + 29 + 29^2 + \cdots + 29^8 + 29^9 = 2 \cdot 3^2 \cdot 5^2 \cdot 11 \cdot 31 \cdot 401 \cdot 732\,541$$

  implies that $(3 \cdot 5 \cdot 11) \mid N$. This is also impossible as $(29, 3, 5, 11)$ is non-admissible.

- If $\alpha = 13$,

$$1 + 29 + 29^2 + \cdots + 29^{12} + 29^{13} = 10\,627\,079\,738\,421\,409\,410 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot M$$

  implies that $(3 \cdot 5 \cdot 7) \mid N$, but the triple $(5, 3, 7)$ is non-admissible.

- If $\alpha = 17$,

$$1 + 29 + 29^2 + \cdots + 29^{16} + 29^{17} \\ = (1 + 29 + 29^2 + 29^3 + 29^4 + 29^5)(1 + 29^6 + 29^{12}).$$

  The first expression on the right was shown to be divisible by $3 \cdot 5 \cdot 13$, which is non-admissible.

- If $\alpha = 21$,

$$1 + 29 + 29^2 + \cdots + 29^{20} + 29^{21} = (1 + 29)\left(\sum_{i=0}^{10} 29^i\right)\left(\sum_{i=0}^{10} (-29)^i\right).$$

Note that

$$\sum_{i=0}^{10} 29^i = 435\,732\,491\,632\,351 = 23 \cdot 18\,944\,890\,940\,537.$$

Let $\beta \geq 1$ be the exponent of 3. Then $\sigma(3^{2\beta}) \mid N$. We claim that $\beta \geq 3$. Indeed,

– if $\beta = 1$, then $\sigma(3^2) = 13$. Consequently, $13 \mid N$, but $(29, 3, 5, 13)$ is non-admissible.

– If $\beta = 2$, then $\sigma(3^4) = 11^2$. Hence, $11 \mid N$, but $(29, 3, 5, 11)$ is also non-admissible.

Therefore, $\beta \geq 3$. However, in this case, we have

$$\left(1 + \frac{1}{29} + \cdots + \frac{1}{29^{21}}\right)\left(1 + \frac{1}{3} + \cdots + \frac{1}{3^6}\right) \cdot \left(1 + \frac{1}{5} + \frac{1}{5^2}\right)\left(1 + \frac{1}{23} + \frac{1}{23^2}\right) > 2.005 > 2,$$

which contradicts that N is perfect number. This proves (2). ∎

Theorem 5. *Let $N = 29^{25}Q^2$ be an odd perfect number. Then the following holds true.*

(1) *The three largest prime divisors of N satisfy the following inequalities:*

$$\textit{largest prime factor} > 4 \cdot 10^9, \quad \textit{second largest} > 2 \cdot 10^8, \quad \textit{third largest} > 10^5.$$

(2) *The number N has the form $N = 29^{25}3^{2\beta_1}5^{2\beta_2}53^{2\beta_3} \cdot M^2$, where M is an odd positive integer and the exponents satisfy $(\beta_1, \beta_2, \beta_3) \geq (3, 3, 3)$.*

Proof. Let $N = 29^{25}Q^2$ be an odd perfect number such that $p = 29$. Suppose that $\alpha = 25$. Then we have

$$1 + 29 + 29^2 + \cdots + 29^{24} + 29^{25} = (1 + 29)\left(\sum_{i=0}^{12} 29^i\right)\left(\sum_{i=0}^{12}(-29)^i\right).$$

It holds that

$$\sum_{i=0}^{12} 29^i = 521 \cdot 148\,123 \cdot 4\,748\,492\,087$$

and

$$\sum_{i=0}^{12}(-29)^i = 53 \cdot 3\,407 \cdot 7\,489 \cdot 252\,918\,667.$$

Since $\sigma(p^\alpha) \mid \sigma(N) = 2N$, all prime divisors of these quantities divide N. Therefore, we conclude

- the largest prime factor of N is at least $4\,748\,492\,087 > 4 \cdot 10^9$,
- the second largest is at least $252\,918\,667 > 2 \cdot 10^8$,
- the third largest is at least $148\,123 > 10^5$.

This proves (1).

Now, let $\beta_1, \beta_2, \beta_3 \geq 1$ be the exponents of 3, 5, and 53 in Q^2, respectively. From the previous argument, we have $\beta_1 \geq 3$. Suppose that $\beta_2 = 1$. Then $\sigma(5^2) = 31$, so $31 \mid N$. However,

$$\left(1 + \frac{1}{29} + \cdots + \frac{1}{29^{25}}\right)\left(1 + \frac{1}{3} + \cdots + \frac{1}{3^6}\right) \cdot \left(1 + \frac{1}{5} + \frac{1}{5^2}\right)\left(1 + \frac{1}{31} + \frac{1}{31^2}\right)\left(1 + \frac{1}{53} + \frac{1}{53^2}\right) > 2.02 > 2.$$

Therefore, $\beta_2 \geq 2$. If $\beta_2 = 2$, then $\sigma(5^4) = 11 \cdot 71$. Since $(29, 3, 5, 11)$ is non-admissible, then $\beta_2 \neq 2$. Consequently, $\beta_2 \geq 3$. Next, if $\beta_3 = 1$, then $\sigma(53^2) = 7 \cdot 409$, and we know that $(5, 3, 7)$ is non-admissible. Hence, $\beta_3 \geq 2$. If $\beta_3 = 2$, then $\sigma(53^4) = 11 \cdot 131 \cdot 5\,581$. But, again, the tuple $(29, 3, 5, 11)$ is non-admissible. Therefore, $\beta_3 \geq 3$. This proves (2). ∎

Theorem 6. *Let $N = 29^{25}Q^2$ be an odd perfect number. Then N has at least 12 distinct prime factors.*

Proof. From Theorem 5, denote the prime factors of N found so far as

$$p = 29, \quad q_1 = 3, \quad q_2 = 5, \quad q_3 = 53, \quad q_4 = 521, \quad q_5 = 3\,407,$$
$$q_6 = 7\,489, \quad q_7 = 148\,123, \quad q_8 = 252\,918\,667, \quad q_9 = 4\,748\,492\,087.$$

The tuple $(p, q_1, q_2, \ldots, q_9)$ satisfies

$$\frac{p}{p-1}\prod_{i=1}^{9}\frac{q_i}{q_i - 1} < 1.984 < 2.$$

Hence, it is a non-feasible tuple. There must be at least another prime factor of N, which arises from $\sigma(q_i^{2\beta_i})$ for some $i \in \{1, 2, \ldots, 9\}$. Let r_1 be the first such additional prime factor. Then, from Theorem 5 (2), we get

$$\left(1 + \frac{1}{r_1} + \frac{1}{r_1^2}\right)\left(1 + \frac{1}{29} + \cdots + \frac{1}{29^{25}}\right)\left(1 + \frac{1}{3} + \cdots + \frac{1}{3^6}\right) \cdot \left(1 + \frac{1}{5} + \cdots + \frac{1}{5^6}\right)\left(1 + \frac{1}{53} + \cdots + \frac{1}{53^6}\right)\prod_{i=4}^{9}\left(1 + \frac{1}{q_i} + \frac{1}{q_i^2}\right) \leq 2;$$

otherwise, $(p, q_1, \ldots, q_9, r_1)$ would be non-admissible. This inequality forces $r_1 \geq 127$. Then

$$\frac{r_1}{r_1 - 1}\frac{p}{p-1}\prod_{i=1}^{9}\frac{q_i}{q_i - 1} < \frac{127}{126} \cdot 1.984 < 1.9997 < 2,$$

which implies that $(p, q_1, \ldots, q_9, r_1)$ is still a non-feasible tuple. Hence, there exists yet another prime factor r_2 that divides N. By the same arguments,

$$\left(1 + \frac{1}{r_1} + \frac{1}{r_1^2}\right)\left(1 + \frac{1}{r_2} + \frac{1}{r_2^2}\right)\left(1 + \frac{1}{29} + \cdots + \frac{1}{29^{25}}\right)\left(1 + \frac{1}{3} + \cdots + \frac{1}{3^6}\right) \cdot \left(1 + \frac{1}{5} + \cdots + \frac{1}{5^6}\right)\left(1 + \frac{1}{53} + \cdots + \frac{1}{53^6}\right)\prod_{i=4}^{9}\left(1 + \frac{1}{q_i} + \frac{1}{q_i^2}\right) \leq 2, \tag{2}$$

implying

$$\left(1+\frac{1}{r_1}+\frac{1}{r_1^2}\right)\left(1+\frac{1}{r_2}+\frac{1}{r_2^2}\right)<1.0081.$$

It is easy to check that $r_2 > r_1$; otherwise, inequality (2) would not hold. Therefore, we obtain

$$\left(1+\frac{1}{r_2}+\frac{1}{r_2^2}\right)^2<1.0081,$$

implying that $r_2 \geq 239$. Then

$$\frac{r_1}{r_1-1}\frac{r_2}{r_2-1}\frac{p}{p-1}\prod_{i=1}^{9}\frac{q_i}{q_i-1}>\frac{127}{126}\cdot\frac{239}{238}\cdot 1.983>2.007.$$

This does not violate the feasibility of the tuple $(p, q_1, \ldots, q_9, r_1, r_2)$. We may therefore conclude that N has at least 12 distinct prime factors. This completes the proof. ∎

Also, an immediate consequence from Theorem 6 is that if $p = 29$ and $\alpha = 25$, then $N > 10^{147}$.

Theorem 7. *Let $N = 29^{25}Q^2$ be an odd perfect number. Then either N has at least* 14 *distinct prime factors or else* $(3^{16}\cdot 5^{12}\cdot 53^{16}) \mid N$.

Proof. By Theorem 5 (2), we have that $N = 29^{25}3^{2\beta_1}5^{2\beta_2}53^{2\beta_3}\cdot M^2$, where M is an odd positive integer and the exponents satisfy $(\beta_1, \beta_2, \beta_3) \geq (3, 3, 3)$.

Now, suppose that $\beta_1 = \beta_2 = \beta_3 = 3$. Then

$$\sigma(3^6) = 1\,093, \quad \sigma(5^6) = 19\,531, \quad \sigma(53^6) = 29\cdot 778\,986\,167,$$

implying $(1\,093\cdot 19\,531\cdot 778\,986\,167) \mid N$. On the other hand, we have that

$$\frac{29}{28}\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{53}{52}\cdot\frac{521}{520}\cdot\frac{1\,093}{1\,092}\cdot\frac{3\,407}{3\,406}\cdot\frac{7\,489}{7\,488}\cdot\frac{19\,531}{19\,530}\cdot\frac{148\,123}{148\,122}\cdot\frac{252\,918\,667}{252\,918\,666}\cdot\frac{778\,986\,167}{778\,986\,166}\cdot\frac{4\,748\,492\,087}{4\,748\,492\,086}<1.986<2.$$

Therefore, the tuple

$$(29, 3, 5, 53, 521, 1\,093, 3\,407, 7\,489, 19\,531, 148\,123, 252\,918\,667, 778\,986\,167, 4\,748\,492\,087)$$

is non-feasible. Let q_i, $i \in \{1, 2, \ldots, 12\}$, denote the primes in the list above, except for the Euler prime $p = 29$. There must be at least another prime factor of N, which arises from $\sigma(q_i^{2\beta_i})$ for some $i \in \{1, 2, \ldots, 12\}$. Let r_1 be the first such additional prime factor. By arguments similar to the proof of Theorem 6, we must have

$$\left(1+\frac{1}{r_1}+\frac{1}{r_1^2}\right)\left(1+\frac{1}{29}+\cdots+\frac{1}{29^{25}}\right)\left(1+\frac{1}{3}+\cdots+\frac{1}{3^6}\right)\cdot\left(1+\frac{1}{5}+\cdots+\frac{1}{5^6}\right)\left(1+\frac{1}{53}+\cdots+\frac{1}{53^6}\right)\prod_{i=4}^{12}\left(1+\frac{1}{q_i}+\frac{1}{q_i^2}\right)\leq 2,$$

implying $r_1 \geq 137$. Then

$$\frac{r_1}{r_1-1}\frac{p}{p-1}\prod_{i=1}^{12}\frac{q_i}{q_i-1} > \frac{137}{136}\cdot 1.9858 > 2.0004.$$

This does not violate the feasibility of the tuple $(p, q_1, \ldots, q_{12}, r_1)$. Therefore, N has at least 14 distinct prime factors. Otherwise, if N cannot have more than 13 distinct prime factors, then at least one of β_i for $i = 1, 2, 3$ must satisfy $\beta_i \geq 4$. If $\beta_1 = 4$, then $\sigma(3^8) = 13 \cdot 757$. But the tuple $(29, 3, 5, 13)$ is non-admissible. If $\beta_2 = 4$, then $\sigma(5^8) = 19 \cdot 31 \cdot 829$. However, $(29, 3, 5, 19, 31)$ is non-admissible. If $\beta_3 = 4$, then $\sigma(53^8) = 7 \cdot 37 \cdot M$. Again, it is not possible as $(5, 3, 7)$ is non-admissible. Thus, $\beta_i \geq 5$ for all $i = 1, 2, 3$. Now, if $\beta_1 = 5$, then $\sigma(3^{10}) = 23 \cdot 3\,851$. We calculate

$$\begin{aligned}&\left(1+\frac{1}{29}+\cdots+\frac{1}{29^{25}}\right)\left(1+\frac{1}{3}+\cdots+\frac{1}{3^{10}}\right)\left(1+\frac{1}{5}+\cdots+\frac{1}{5^{10}}\right)\\ &\quad\cdot\left(1+\frac{1}{53}+\cdots+\frac{1}{53^{10}}\right)\left(1+\frac{1}{23}+\frac{1}{23^2}\right) > 2.069 > 2.\end{aligned}$$

Hence, $\beta_1 \geq 6$. For $(\beta_1, \beta_2, \beta_3) = (6, 5, 5)$, we have that

$$\sigma(3^{12}) = 797\,161, \quad \sigma(5^{10}) = 12\,207\,031, \quad \sigma(53^{10}) = 178\,250\,690\,949\,465\,223.$$

Calculations show that the tuple

$$\begin{aligned}&(29, 3, 5, 53, 521, 3\,407, 7\,489, 148\,123, 797\,161, 12\,207\,031,\\ &\quad 252\,918\,667, 4\,748\,492\,087, 178\,250\,690\,949\,465\,223)\end{aligned}$$

satisfies

$$\begin{aligned}&\frac{29}{28}\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{53}{52}\cdot\frac{521}{520}\cdot\frac{3\,407}{3\,406}\cdot\frac{7\,489}{7\,488}\cdot\frac{148\,123}{148\,122}\cdot\frac{797\,161}{797\,160}\cdot\frac{12\,207\,031}{12\,207\,030}\\ &\quad\cdot\frac{252\,918\,667}{252\,918\,666}\cdot\frac{4\,748\,492\,087}{4\,748\,492\,086}\cdot\frac{178\,250\,690\,949\,465\,223}{178\,250\,690\,949\,465\,222} < 1.984 < 2.\end{aligned}$$

So it is not a feasible tuple. There must be at least another prime factor of N not appearing in the tuple. Hence, N must have at least 14 distinct prime factors. Otherwise, $\beta_1 \geq 7$ or $\beta_2, \beta_3 \geq 6$. As before, let $\beta_1 = 7$. Then $\sigma(3^{14}) = 11^2 \cdot 13 \cdot 4\,561$. But $(29, 3, 5, 11)$ is non-admissible. Hence, $\beta_1 \geq 8$. Next, let $\beta_2 = 6$. Then $\sigma(5^{12}) = 305\,175\,781$. If $\beta_3 = 6$, then $\sigma(53^{12}) = 13 \cdot M$. This is not admissible. Hence, $\beta_3 \geq 7$. If $\beta_3 = 7$, then $\sigma(53^{14}) = 7 \cdot 11 \cdot M$. Again non-admissible, so $\beta_3 \geq 8$. This completes the proof. ∎

Corollary 8. *Let*

$$\begin{aligned}N = 29^{25}(&3^8 \cdot 5^6 \cdot 53^8 \cdot 521 \cdot 3\,407 \cdot 7\,489 \cdot 148\,123\\ &\cdot 252\,918\,667 \cdot 4\,748\,492\,087 \cdot 127 \cdot 6\,841)^2.\end{aligned}$$

Then

$$\left|\frac{\sigma(N)}{N} - 2\right| < 3.643 \cdot 10^{-7}.$$

Given $\varepsilon > 0$, a positive integer N is an ε-perfect number if

$$\left|\frac{\sigma(N)}{N} - 2\right| < \varepsilon.$$

Evidently, any perfect number is an ε-perfect number for every $\varepsilon > 0$.

Conjecture 9. *For every $\varepsilon > 0$, there exists an odd ε-perfect number.*

Corollary 8 shows that the Conjecture is true for all $\varepsilon \geq 3.643 \cdot 10^{-7}$.

References

[1] J. E. Z. Chein, *An odd perfect number has at least 8 prime factors*. PhD thesis, Pennsylvania State University, 1979; ProQuest LLC, Ann Arbor, MI, 1979 MR 2630408

[2] P. Hagis, Jr., Outline of a proof that every odd perfect number has at least eight prime factors. *Math. Comp.* **35** (1980), no. 151, 1027–1032 Zbl 0444.10004 MR 572873

[3] P. P. Nielsen, Odd perfect numbers have at least nine distinct prime factors. *Math. Comp.* **76** (2007), no. 260, 2109–2126 Zbl 1142.11086 MR 2336286

[4] P. P. Nielsen, Odd perfect numbers, Diophantine equations, and upper bounds. *Math. Comp.* **84** (2015), no. 295, 2549–2567 Zbl 1325.11009 MR 3356038

[5] P. Ochem and M. Rao, Odd perfect numbers are greater than 10^{1500}. *Math. Comp.* **81** (2012), no. 279, 1869–1877 Zbl 1263.11005 MR 2904606

[6] J. J. Sylvester, Sur l'impossibilité de l'existence d'un nombre parfait impair qui ne contient pas au moins 5 diviseurs premiers distincts, *Comptes Rendus* **106** (1888), 522–526 Zbl 20.0173.04

[7] J. Voight, *Perfect numbers: An elementary introduction*. Lecture Notes in Number Theory, University of California, Berkley, 1998

[8] GIMPS, Gimps home. https://Mersenne.org, 2025

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