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Short note On equal products of consecutive integers

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Abstract. In 1970, Macleod and Barrodale [Canad. Math. Bull. 13 (1970), 255–259] conjectured that the equation $(x+1)(x+2)\cdots(x+k)=(y+1)(y+2)\cdots(y+l)$ has no solutions in the positive integers if $k=2$ and $l\ge 4$. In 1990, Saradha and Shorey [Proc. Indian Acad. Sci. Math. Sci. 100 (1990), no. 2, 107–132, Theorem 3] showed that this equation has no solutions in the positive integers if $l=2k$. However, Saradha and Shorey's proof is quite involved. In this paper, we give an elementary proof to Saradha and Shorey's theorem.

The goal of this paper is to give a new and elementary proof of the following theorem.

Theorem 1 (Saradha–Shorey [2]). *The equation*

$$(x+1)(x+2)\cdots(x+n)=(y+1)(y+2)\cdots(y+2n)\quad(n\ge 2)\tag{1}$$

has no solutions in the positive integers.

1 Some lemmas

Recall that, for every non-zero integer m, $v_2(m)$ denotes the greatest integer p such that 2^p divides m.

Lemma 1. *For all positive integers n, we have $v_2((n+1)\cdots(2n))=n$.*

Proof. Let $A_n=(n+1)(n+2)\cdots(2n)$. Direct calculation shows that

$$v_2(A_1)=1,\quad v_2(A_2)=2,\quad v_2(A_3)=3.$$

Assume that $v_2(A_n)=n$ for a positive integer n. Since

$$\begin{aligned}A_{n+1}&=(n+2)(n+2)\cdots(n+1+n+1)\\&=2(n+1)(n+2)\cdots(2n)(2n+1)\\&=2(2n+1)A_n,\end{aligned}$$

it follows that $v_2(A_{n+1})=1+v_2(A_n)=n+1$. Lemma 1 follows from the induction principle. ■

Lemma 2. *Let $x_1, x_2, \ldots, x_n$ be real numbers in $(0, 1/2)$. Let $x = (x_1 + x_2 + \cdots + x_n)/n$ and $y = \max\{x_1^2, x_2^2, \ldots, x_n^2\}$. Then*

$$1 - x \geq \sqrt[n]{(1-x_1)(1-x_2)\cdots(1-x_n)} > 1 - x - 2y. \tag{2}$$

The equality on the left side of (2) holds if and only if $x_1 = x_2 = \cdots = x_n$.

Proof. Applying the AM-GM inequality, we have

$$\sqrt[n]{(1-x_1)(1-x_2)\cdots(1-x_n)} \leq \frac{1 - x_1 + 1 - x_2 + \cdots + 1 - x_n}{n} = 1 - x;$$

the equality holds if and only if $x_1 = x_2 = \cdots = x_n$. Applying the GM-HM inequality, we have

$$\frac{1}{1-x_1} + \frac{1}{1-x_2} + \cdots + \frac{1}{1-x_n} \geq \frac{n}{\sqrt[n]{(1-x_1)(1-x_2)\cdots(1-x_n)}}. \tag{3}$$

Notice that, for all $x \in (0, 1/2)$, we have

$$\frac{1}{1-x} = 1 + x + \frac{x^2}{1-x} < 1 + x + 2x^2. \tag{4}$$

From (3) and (4), we have

$$\begin{aligned}
&\sqrt[n]{(1-x_1)(1-x_2)\cdots(1-x_n)} \\
&\geq \frac{n}{\frac{1}{1-x_1} + \frac{1}{1-x_2} + \cdots + \frac{1}{1-x_n}} \\
&> \frac{n}{(1 + x_1 + 2x_1^2) + (1 + x_2 + 2x_2^2) + \cdots + (1 + x_n + 2x_n^2)} \\
&= \frac{1}{1 + x + \frac{2(x_1^2 + \cdots + x_n^2)}{n}} \geq \frac{1}{1 + x + 2y} > 1 - x - 2y.
\end{aligned}$$

■

Lemma 3. *Let m be an odd positive integer. Let $a_1, a_2, \ldots, a_m$ be n pairwise distinct positive integers such that $a_1 + a_2 + \cdots + a_m = mz$, where z is a positive integer. Then*

$$a_1 a_2 \ldots a_m \leq z \prod_{i=1}^{\frac{m-1}{2}} (z^2 - i^2).$$

Proof. There are only a finite number of tuples of distinct positive integers $(a_1, a_2, \ldots, a_m)$ such that

$$a_1 + a_2 + \cdots + a_m = mz.$$

Therefore, there exists a tuple, with the abuse of notation, called $(x_1, x_2, \ldots, x_m)$ with $x_1 < x_2 < \cdots < x_m$ such that $x_1 x_2 \ldots x_m$ is the maximum. We claim that $x_{i+1} - x_i = 1$ for all $i = 1, 2, \ldots, m-1$. Indeed, if $x_{i+1} - x_i \geq 2$, then considering the tuple $(y_1, y_2, \ldots, y_m)$

with $y_j = x_j$ for $j \neq i, i+1$, $y_i = x_i + 1$ and $y_{i+1} = x_{i+1} - 1$, we obtain

$$y_i y_{i+1} = x_i x_{i+1} + x_{i+1} - x_i - 1 \geq x_i x_{i+1} + 1 > x_i x_{i+1}.$$

Hence

$$y_1 y_2 \cdots y_m > x_1 x_2 \cdots x_m,$$

a contradiction. Therefore, $x_i = x_1 + i - 1$ for all $i = 1, 2, \ldots, m$. Since

$$x_1 + x_2 + \cdots + x_m = mz$$

and m is odd, we have

$$x_1 = z - \frac{m-1}{2}, \quad x_i = z - \frac{m-1}{2} + i - 1 \quad \text{for all } i = 1, 2 \ldots, m.$$

Hence

$$x_1 x_2 \cdots x_m = z \prod_{i=1}^{\frac{m-1}{2}} (z^2 - i^2). \qquad \blacksquare$$

2 A proof of Theorem 1

Macleod and Barrodale [1, Theorems 1, 4, 5, 6] showed that equation (1) has no solutions in the positive integer when $n = 2, 3, 4, 5$. So we only need to consider the case $n \geq 6$. Assume that there exist positive integers x, y satisfying (1). Let k be the unique positive integer in $\{1, 2, \ldots, n\}$ such that

$$v_2(x+k) = \max_{1 \leq j \leq n} \{v_2(x+j)\}.$$

Then, for all $i \neq k$ and $1 \leq i \leq n$,

$$v_2(x+i) = v_2(x+k+(i-k)) = v_2(i-k).$$

Hence

$$\begin{aligned} v_2((x+1)(x+2)\cdots(x+n)) &= v_2(x+k) + \sum_{1 \leq i \leq n, i \neq k} v_2(i-k) \\ &\leq v_2(x+k) + v_2((n-1)!). \end{aligned} \tag{5}$$

Therefore,

$$\begin{aligned} v_2(x+k) &\geq v_2((x+1)(x+2)\cdots(x+n)) - v_2((n-1)!) \\ &= v_2((y+1)(y+2)\cdots(y+2n)) - v_2((n-1)!) \\ &\geq v_2((2n)!) - v_2((n-1)!) \end{aligned} \tag{6}$$

where we have used the fact that $(2n)! \mid (y+1)(y+2)\cdots(y+2n)$. On the other hand, it follows from Lemma 1 that

$$v_2((2n)!) - v_2((n-1)!) = v_2(n) + v_2((n+1)(n+2)\cdots(2n)) = v_2(n) + n. \tag{7}$$

Combining (5), (6), and (7) gives $x+k \geq 2^{v_2(x+k)} \geq 2^{n+v_2(n)} \geq 2^n$. Since $k \leq n$, it follows that $x > 2^n - n$. Then

$$\begin{aligned}(y+2n)^{2n} &> (y+1)(y+2)\cdots(y+2n)\\ &= (x+1)(x+2)\cdots(x+n)\\ &> ((2^n-n)+1)((2^n-n)+2)\cdots((2^n-n)+n)\\ &> (2^n-n)^n.\end{aligned}$$

Thus $y > \sqrt{2^n-n} - 2n$. Let $x+n=a$ and let $y+n+1/2=b$. By taking the n-th root on both sides of (1), we have

$$\begin{aligned}&a\sqrt[n]{\left(1-\frac{1}{a}\right)\left(1-\frac{2}{a}\right)\cdots\left(1-\frac{n}{a}\right)}\\ &\qquad = b^2\sqrt[n]{\left(1-\frac{1}{4b^2}\right)\left(1-\frac{3^2}{4b^2}\right)\cdots\left(1-\frac{(2n-1)^2}{4b^2}\right)}.\end{aligned} \tag{8}$$

Let A be the common positive value of both sides of (8). Applying Lemma 2 to the left side of (8) gives

$$a\left(1-\frac{1}{n}\left(\frac{1}{a}+\frac{2}{a}+\cdots+\frac{n}{a}\right)\right) > A > a\left(1-\frac{1}{n}\left(\frac{1}{a}+\frac{2}{a}+\cdots+\frac{n}{a}\right)-\frac{2}{a^2}\right).$$

Hence

$$a-\frac{n+1}{2} > A > a-\frac{n+1}{2}-\frac{2}{a}. \tag{9}$$

Applying Lemma 2 to the right side of (8) and using the identity

$$1+3^2+\cdots+(2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

gives

$$b^2\left(1-\frac{1}{n}\cdot\frac{n(2n-1)(2n+1)}{12b^2}\right) > A > b^2\left(1-\frac{1}{n}\cdot\frac{n(2n-1)(2n+1)}{12b^2}-\frac{2}{4b^4}\right),$$

which is equivalent to

$$b^2-\frac{4n^2-1}{12} > A > b^2-\frac{4n^2-1}{12}-\frac{1}{2b^2}. \tag{10}$$

Since $n \geq 6$, we have

$$b^2 = (y+n+1/2)^2 > (\sqrt{2^n-n}-n+1/2)^2 > 4. \tag{11}$$

We also have

$$\begin{aligned}b^2-\frac{4n^2-1}{12} &= \left(y+n+\frac{1}{2}\right)^2-\frac{4n^2-1}{12}\\ &= (y+n)^2+(y+n)+\frac{1-n^2}{3}.\end{aligned} \tag{12}$$

Case 1: n is even. Since $a = x + n > n \cdot 2^n > 16$ and $(n+1)/2 \notin \mathbb{Z}^+$, it follows from (9) that

$$\{A\} \in \left(\frac{1}{2} - \frac{1}{8}, \frac{1}{2}\right), \tag{13}$$

where $\{A\}$ is the fractional part of A.

Case 1.1: $3 \nmid n$. Then $3 \mid 1 - n^2$. It then follows from (10), (11), and (12) that

$$\{A\} \in \left(1 - \frac{1}{8}, 1\right). \tag{14}$$

Case 1.2: $3 \mid n$. Then $3 \mid n^2$. It then follows from (10), (11), and (12) that

$$\{A\} \in \left(\frac{1}{3} - \frac{1}{8}, \frac{1}{3}\right). \tag{15}$$

Both (14) and (15) contradict (13).

Case 2: n is odd.

Case 2.1: $3 \mid n$. Since $a = x + n > n \cdot 2^n > 16$ and $(n+1)/2 \in \mathbb{Z}^+$, it follows from (9) that

$$\{A\} \in \left(1 - \frac{1}{8}, 1\right). \tag{16}$$

Since $3 \mid n$, it follows from (10), (11), and (12) that

$$\{A\} \in \left(\frac{1}{3} - \frac{1}{8}, \frac{1}{3}\right),$$

which contradicts (16).

Case 2.2: $3 \nmid n$. Since n is odd, from (9), we have

$$[A] = a - \frac{n+1}{2} - 1.$$

Since $\gcd(6, n) = 1$, from (10) and (12), we have

$$[A] = b^2 - \frac{4n^2 - 1}{12} - 1.$$

Therefore,

$$a - \frac{n+1}{2} = b^2 - \frac{4n^2 - 1}{12}. \tag{17}$$

Note that $a = x + n$, so it follows from (12) and (17) that

$$x + \frac{n-1}{2} = y^2 + (2n+1)y + \frac{2n^2 + 3n + 1}{3}. \tag{18}$$

Let $z = x + \frac{n-1}{2}$. By (18), we have

$$\begin{aligned}
&(y + i)(y + 2n + 1 - i) \\
&\quad = (y^2 + (2n+1)y + i(2n + 1 - i)) \\
&\quad = z - \frac{2n^2 + 3n + 1}{3} + i(2n + 1 - i) \quad \text{for all } i = 1, 2, \ldots, n.
\end{aligned}$$

Write (1) as

$$\prod_{i=1}^{n}\Bigl(z-\frac{n-1}{2}+i\Bigr)=\prod_{i=1}^{n}\Bigl(z-\frac{2n^2+3n+1}{3}+i(2n+1-i)\Bigr). \tag{19}$$

Since

$$\sum_{i=1}^{n}\Bigl(z-\frac{2n^2+3n+1}{3}+i(2n+1-i)\Bigr)=nz$$

and the numbers $z-\frac{2n^2+3n+1}{3}+i(2n+1-i)$, with $i=1,2,\ldots,n$, are pairwise distinct, it follows from Lemma 3 that the left-hand side of (19) (as n is odd) is less than or equal to

$$z\prod_{i=1}^{\frac{n-1}{2}}(z^2-i^2),$$

which is less than the right-hand side of (19). Therefore, (19) has no solutions in the positive integers, and Theorem 1 is proved.

References

[1] R. A. MacLeod and I. Barrodale, On equal products of consecutive integers. *Canad. Math. Bull.* **13** (1970), 255–259 Zbl 0206.05602 MR 265282

[2] N. Saradha and T. N. Shorey, On the ratio of two blocks of consecutive integers. *Proc. Indian Acad. Sci. Math. Sci.* **100** (1990), no. 2, 107–132 Zbl 0716.11017 MR 1069698

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