Elemente der Mathematik 81 (2026), 125–128
DOI 10.4171/EM/568

Short note

An elementary proof of the symmetry of power sum polynomials

Helmut Länger

Abstract. It is well known that the power sums $1^{k-1}+2^{k-1}+\cdots+n^{k-1}$ are polynomials $p_k(n)$ of degree k in n. We provide an elementary proof of the fact that the polynomial $p_k(x)$ is symmetric with respect to the line $x=-1/2$ or the point $(-1/2,0)$, depending on whether k is even or odd, respectively.

The power sums $1^{k-1}+2^{k-1}+\cdots+n^{k-1}$ are polynomials of degree k in n. These power sum polynomials were investigated by many mathematicians such as Johannes Faulhaber, Jakob Bernoulli and Leonard Euler. The coefficients of these polynomials are strongly related to Bernoulli numbers that appear, e.g., in the Taylor expansion of tangent and cotangent or when computing values of the zeta function. In [6], we presented a simple recursion for these polynomials. In the present note, we present an elementary proof of the symmetry of power sum polynomials. For other papers on these polynomials, cf., e.g., [1–4, 6–8].

In [9], some problem concerning factors of power sum polynomials was posed. In [5], the present author presented a partial solution to this problem. In the present note, we elaborate some parts of this proof.

In the following, let $\mathbb{N}$ denote the set of all positive integers and $\mathbb{N}_0$ the set of all non-negative integers and let $k\in\mathbb{N}$ and $n\in\mathbb{N}_0$. Put

$$p_k(n):=\sum_{i=1}^{n} i^{k-1}.$$

In the sequel, we use the following observation: if p,q are polynomials and $p(n)=q(n)$ for all $n\in\mathbb{N}_0$, then the polynomial $p-q$ has infinitely many zeros and hence $p-q=0$, i.e., $p(x)=q(x)$ for all $x\in\mathbb{R}$.

The aim of the present note is to present an elementary proof (avoiding Bernoulli numbers) of the following known symmetry result (cf., e.g., [8]).

Theorem 1. *For $k>1$, the power sum polynomial of degree k is symmetric with respect to the line $x=-1/2$ or the point $(-1/2,0)$ depending on whether k is even or odd, respectively.*

It should be remarked that another elementary proof of this result can be found in [3], whereas in [8], Bernoulli numbers are used in order to obtain Theorem 1. But also this proof can be modified in such a way that Bernoulli numbers are avoided.

The first eight power sum polynomials are as follows:

$$\begin{aligned}
p_1(n) &= n,\\
p_2(n) &= n(n+1)/2,\\
p_3(n) &= n(n+1)(2n+1)/6,\\
p_4(n) &= n^2(n+1)^2/4,\\
p_5(n) &= n(n+1)(2n+1)(3n^2+3n-1)/30,\\
p_6(n) &= n^2(n+1)^2(2n^2+2n-1)/12,\\
p_7(n) &= n(n+1)(2n+1)(3n^4+6n^3-3n+1)/42,\\
p_8(n) &= n^2(n+1)^2(3n^4+6n^3-n^2-4n+2)/24.
\end{aligned}$$

We start with an easy lemma.

Lemma 2 (cf. [3,4,8]). *We have*

$$p_k(n) = \left((n+1)^k - 1 - \sum_{i=1}^{k-1}\binom{k}{i-1}p_i(n)\right)/k.$$

Proof. Using a telescopic sum and the Binomial Theorem and interchanging sum signs, we obtain

$$\begin{aligned}
(n+1)^k - 1 &= \sum_{j=1}^{n}\left((j+1)^k - j^k\right) = \sum_{j=1}^{n}\sum_{i=0}^{k-1}\binom{k}{i}j^i = \sum_{i=1}^{k}\binom{k}{i-1}\sum_{j=1}^{n}j^{i-1}\\
&= \sum_{i=1}^{k}\binom{k}{i-1}p_i(n) = \sum_{i=1}^{k-1}\binom{k}{i-1}p_i(n) + kp_k(n).
\end{aligned}$$

■

Corollary 3. *The function $p_k(n)$ is a polynomial of degree k in n with leading coefficient $1/k$. This follows from Lemma 2 by induction on k.*

For the proof of the above mentioned theorem, we need a further lemma.

Lemma 4. *We have*

$$-(-n)^k = \sum_{i=1}^{k}\binom{k}{i-1}(-1)^{i-1}p_i(n).$$

Proof. Using again a telescopic sum and the Binomial Theorem and interchanging sum signs, we obtain

$$\begin{aligned}
-(-n)^k &= \sum_{j=1}^{n}\left((-j+1)^k - (-j)^k\right) = \sum_{j=1}^{n}\sum_{i=1}^{k}\binom{k}{i-1}(-j)^{i-1}\\
&= \sum_{i=1}^{k}\binom{k}{i-1}\sum_{j=1}^{n}(-j)^{i-1} = \sum_{i=1}^{k}\binom{k}{i-1}(-1)^{i-1}p_i(n).
\end{aligned}$$

■

Now we can prove our main result.

Proof of Theorem 1. Since, according to the proof of Lemma 2 and according to Lemma 4,

$$(x+1)^k - 1 = \sum_{i=1}^{k} \binom{k}{i-1} p_i(x),$$
$$-(-x)^k = \sum_{i=1}^{k} \binom{k}{i-1} (-1)^{i-1} p_i(x)$$

hold for all $x \in \mathbb{N}_0$, these equalities hold for all $x \in \mathbb{R}$. Substituting x in the first formula by $-1/2 - x$ and in the second formula by $-1/2 + x$ yields

$$(1/2-x)^k - 1 = \sum_{i=1}^{k} \binom{k}{i-1} p_i(-1/2-x),$$
$$-(1/2-x)^k = \sum_{i=1}^{k} \binom{k}{i-1} (-1)^{i-1} p_i(-1/2+x)$$

for all $x \in \mathbb{R}$. By adding, we obtain

$$-1 = \sum_{i=1}^{k} \binom{k}{i-1} \big(p_i(-1/2-x) + (-1)^{i-1} p_i(-1/2+x)\big)$$

and hence

$$0 = \sum_{i=2}^{k} \binom{k}{i-1} \big(p_i(-1/2-x) + (-1)^{i-1} p_i(-1/2+x)\big)$$

for all $x \in \mathbb{R}$. Since all binomial coefficients are different from 0, we conclude by induction on k that

$$p_k(-1/2-x) + (-1)^{k-1} p_k(-1/2+x) = 0,$$

i.e.,

$$p_k(-1/2-x) = (-1)^k p_k(-1/2+x)$$

for $k > 1$ and all $x \in \mathbb{R}$, completing the proof of the theorem. ∎

This symmetry is reflected by the following result.

Theorem 5 ([4]). *We have*

(i) p_1 *has exactly one rational zero, namely* 0. *It has multiplicity* 1.

(ii) p_2 *has exactly two rational zeros, namely* -1 *and* 0. *They have both multiplicity* 1.

(iii) *For* $k > 2$, k *odd,* p_k *has exactly three rational zeros, namely* -1, $-1/2$ *and* 0. *All of them have multiplicity* 1.

(iv) *For* $k > 2$, k *even,* p_k *has exactly two rational zeros, namely* -1 *and* 0. *They have both multiplicity* 2.

Acknowledgements. The author is grateful to the anonymous referee for her/his valuable remarks.

Funding. Support by the Austrian Science Fund (FWF), project 10.55776/PIN5424624, is gratefully acknowledged.

References

[1] B. C. Berndt, *Ramanujan's notebooks. Part I*. Springer, New York, 1985 Zbl 0555.10001 MR 781125

[2] K. Dilcher, Zeros of Bernoulli, generalized Bernoulli and Euler polynomials. *Mem. Amer. Math. Soc.* **73** (1988), no. 386, iv+94 Zbl 0645.10015 MR 938890

[3] J. Feldvoss, Über Potenzsummenpolynome. *Mitt. Math. Ges. Hamburg* **20** (2001), 115–124 Zbl 1046.11008 MR 1884397

[4] N. Kimura and H. Siebert, Über die rationalen Nullstellen der von Potenzsummen der natürlichen Zahlen definierten Polynome. *Proc. Japan Acad. Ser. A Math. Sci.* **56** (1980), no. 7, 354–356 Zbl 0462.10007 MR 593305

[5] H. Länger, Teillösung zu Problem 11. *Math. Semesterber.* **41** (1994), 92–93

[6] H. Länger, A simple recursion for power sum polynomials. *Elem. Math.* **69** (2014), no. 2, 65–67 Zbl 1319.11010 MR 3195109

[7] H. Rademacher, *Topics in analytic number theory*. Grundlehren Math. Wiss. 169, Springer, New York–Heidelberg, 1973 Zbl 0253.10002 MR 364103

[8] D. Treiber, Zur Kurvendiskussion der Potenzsummenpolynome. *Elem. Math.* **53** (1998), no. 3, 119–125 Zbl 0909.05002 MR 1656338

[9] U. Warnecke, Problem 11. *Math. Semesterber.* **30** (1983), 150–151

Helmut Länger
Institute of Discrete Mathematics and Geometry,
Faculty of Mathematics and Geoinformation
TU Wien
Wiedner Hauptstr. 8–10
A-1040 Vienna, Austria
Department of Algebra and Geometry,
Faculty of Science
Palacký University Olomouc
17. listopadu 12
CZ-771 46 Olomouc, Czech Republic
helmut.laenger@tuwien.ac.at