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Short note

Separation of even and odd terms for solving quartic equations

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1 Introduction

The quartic equation was first solved by the Italian mathematician Ferrari in 1545. Thereafter, several methods of solving quartic equations have appeared in the literature like Descartes' method, Tschirnhaus' method, Euler's method, and so on; see [1–6].

In this note, a different method of solving quartic equation is presented, in which the variable of the quartic equation is split into a sum of two unknown numbers. The terms containing the odd powers of one unknown are collected and equated to zero, leaving behind the equation, which now contains only the even powers of the unknown. Elimination of this unknown from the two equations results in an even powered sextic equation, which is known as the resolvent cubic equation; solving it leads to determination of the two unknowns, and thereafter the solution to the given quartic equation is obtained. We solve one numerical example using the proposed method.

2 The proposed method

Since a general quartic equation $u^4 + Au^3 + Bu^2 + Cu + D = 0$ can be reduced to a depressed one (in which the third power term is absent), by means of a simple linear transformation $u = x - (A/4)$, we consider the following depressed quartic equation, without loosing any generality:

$$x^4 + ax^2 + bx + c = 0, \tag{1}$$

where a, b, and c are the coefficients in (1) and $bc \neq 0$. Splitting the variable x in (1) as the sum of two unknown numbers d and f, as shown below,

$$x = d + f, \tag{2}$$

and using it in (1) yields

$$d^4 + 4d^3 f + (6f^2 + a)d^2 + (4f^3 + 2af + b)d + f^4 + af^2 + bf + c = 0. \tag{3}$$

We now form an expression by collecting the odd power terms of d in (3), and equating it to zero:

$$4d^3 f + (4f^3 + 2af + b)d = 0. \tag{4}$$

Consequently, (3) becomes an equation containing only the even power terms of d,

$$d^4 + (6f^2 + a)d^2 + f^4 + af^2 + bf + c = 0,$$

which is a quadratic equation in d^2, solving which we obtain

$$d^2 = \frac{1}{2}[-(6f^2 + a) \pm \sqrt{(6f^2 + a)^2 - 4(f^4 + af^2 + bf + c)}]. \tag{5}$$

Observe that, from (4), an expression for d^2 is obtained as

$$d^2 = -(4f^3 + 2af + b)/4f. \tag{6}$$

Equating (5) and (6) and simplifying the resulting expression yields an even powered sextic equation in f,

$$f^6 + (a/2)f^4 + [(a^2 - 4c)/16]f^2 - (b^2/64) = 0, \tag{7}$$

which is a cubic equation in f^2, and it is known as the resolvent cubic equation. Solving (7) yields three values of f^2, and thereafter six values of f are determined. Using (6), we determine d^2 for each value of f, so two values of d are obtained for each value of f. We note that, using any one value of f^2 (and the corresponding two values of f), we obtain all the four solutions of (1) from relation (2). For example, let the three solutions of the resolvent cubic equation (7) be f_1^2, f_2^2, and f_3^2. Consider f_1^2; taking its square root, we obtain two solutions, say f_{11} and f_{12}. Using f_{11} in (6), we determine d^2; let us call it d_{11}^2. Taking square root of d_{11}^2 yields two solutions d_{111} and d_{112}. Now, using (2), we determine two solutions of the given quartic equation (1),

$$x_1 = d_{111} + f_{11} \quad \text{and} \quad x_2 = d_{112} + f_{11}.$$

Similarly, using f_{12} in (6), we determine d^2, and let us term it as d_{12}^2. Taking square root of d_{12}^2 yields two solutions d_{121} and d_{122}. Now, using (2), we determine the remaining two solutions of the given quartic equation (1),

$$x_3 = d_{121} + f_{12} \quad \text{and} \quad x_4 = d_{122} + f_{12}.$$

Notice that one solution (either f_1^2, f_2^2, or f_3^2) of the resolvent cubic equation is enough to determine all the four solutions of the given quartic equation (1). A numerical example is solved below illustrating the proposed method.

3 Numerical example

Consider the quartic equation $x^4 - 73x^2 - 252x - 180 = 0$. The resolvent cubic equation in f^2 is obtained from (7) as $f^6 - 36.5f^4 + 378.0625f^2 - 992.25 = 0$. Solving it yields three values of f^2: 20.25, 12.25, and 4. Choosing $f^2 = 4$, we obtain two values of f: ± 2, and corresponding to $f = 2$, the parameter d^2 is determined from (6) as 64, and so two values of d are obtained as 8 and -8. Hence the two solutions of $x^4 - 73x^2 - 252x - 180 = 0$ are determined from (2) as $x_1 = 10$ and $x_2 = -6$.

To determine the remaining two solutions, we choose $f = -2$, and using this value in (6), we get $d^2 = 1$ and so $d = \pm 1$. Using $d = \pm 1$ and $f = -2$ in (2), the remaining two solutions are obtained as $x_3 = -1$ and $x_4 = -3$.

Conclusion

In conclusion, this note has presented a new and simple method of solving quartic equations, in which the variable in the quartic equation is split as a sum of two unknown numbers. The even powered terms and the odd powered terms in one of the unknowns are separated, which results in a resolvent cubic equation, and thereafter leads to the solutions of the quartic equation. The method proposed here enables teachers of pre-university education with a material on classroom projects of solving quartic equations by several methods.

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