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Koszul duality and the Poincaré–Birkhoff–Witt theorem

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Abstract. Using a homotopy introduced by de Wilde and Lecomte and homological perturbation theory for A_∞-algebras, we give an explicit proof that the universal enveloping algebra UL of a differential graded Lie algebra L is Koszul, via an explicit contracting homotopy from the cobar construction ΩCL of the Chevalley–Eilenberg chain coalgebra CL of L to UL. This may be viewed as an extension of the Poincaré–Birkhoff–Witt theorem to L_∞-algebras.

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In memoriam of Murray Gerstenhaber (1927–2024)

If L is a differential graded (dg) Lie algebra, there is a codifferential on the exterior coalgebra ΛL defined using the Lie bracket and differential of L. With this differential, ΛL becomes a cocommutative dg coalgebra CL, called the Chevalley–Eilenberg coalgebra of L.

There is a quasi-isomorphism of cocommutative dg bialgebras $f : \Omega CL \to UL$ from the cobar construction ΩCL of CL to the universal enveloping algebra UL, split by a morphism of cocommutative dg coalgebras $g : UL \to \Omega CL$. Applying the functor of primitives P to f, we obtain a quasi-isomorphism of dg Lie algebras $Pf : P\Omega CL \to L$: this functorial resolution of L was introduced by Quillen [14].

More generally, if L is an L_∞-algebra, there is a codifferential on ΛL defined using the higher brackets of L, which makes ΛL into a cocommutative dg coalgebra CL. A natural choice for the universal enveloping algebra of L is ΩCL (Hinich and Schechtman [10]). This is a cocommutative dg bialgebra, and as we saw above, in the special case that L is a dg Lie algebra, it is quasi-isomorphic to the universal enveloping algebra UL.

It is an interesting problem to exhibit this quasi-isomorphism through an explicit contracting homotopy. In the abelian case, this becomes the problem of finding an explicit contracting homotopy from $\Omega\Lambda V$, where V is a cochain complex, to SV, the dg symmetric algebra generated by V. A contracting homotopy from ΩCL to UL is then obtained by homological perturbation theory (Baranovsky [1]). The resulting identification of the complexes underlying SL and UL may be viewed as an extension of the Poincaré–Birkhoff–Witt theorem to L_∞-algebras.

Baranovsky demonstrated the existence of a contracting homotopy for $\Omega\Lambda V$, without giving an explicit formula for it. In fact, such an explicit homotopy may be extracted from the work of de Wilde and Lecomte [2]. They construct a homotopy in the dual situation, contracting the bar construction BSV of the symmetric algebra SV to its Koszul dual, the exterior coalgebra ΛV (though they restrict attention to the case that V is a vector space). Dualizing, we obtain a contracting homotopy from the cobar construction $\Omega\Lambda V$ to the symmetric algebra SV. This homotopy is natural: it is invariant under automorphisms of V. De Wilde and Lecomte construct this homotopy by a recursive procedure: one of the new results in our paper is an explicit formula (4.3) for the resulting homotopy.

Another approach to the construction of a contracting homotopy is due to Halbout [9]. His homotopy extends to more general function algebras, such as real (or complex) analytic functions. We will see that the homotopy of de Wilde and Lecomte extends to Halbout's setting as well.

After the first version of this work was completed, an alternative contracting homotopy was discovered by Dippell et al. [3]. An advantage of their homotopy over the one that we use is that its origin is conceptually clearer: it is obtained by combining contracting homotopies of the bar construction and the Koszul complex, in an abstraction of Van Est's method (see Meinrenken and Salazar [13]). For completeness, we give a brief review of their construction in the appendix.

The articles of de Wilde and Lecomte [2], Halbout [9], and Dippel et al. [3] consider the more general setting of Hochschild complexes of chains and cochains with values in modules over the symmetric algebra SV. We have not pursued these applications in this article, but the respective generalizations are easily guessed from the homotopy formulas that we present here.

The original motivation of this paper was to make the A_∞-morphism in Lemma 19 of Tsygan [15] explicit: this is the last step in his construction of a Gauss–Manin connection on periodic cyclic homology of a deformation of A_∞-algebras at the chain level, and the only step for which no explicit formula is stated. We return to this application in the last section of this paper.

Summary of paper. Section 1 reviews the definitions of the symmetric algebra SV and exterior coalgebra ΛV.

Section 2 discusses operations on the bar construction BA of an A_∞-algebra A, parametrized by the bar construction $\mathrm{BG}(A)$ of its Gerstenhaber algebra $G(A) = C^*(A, A)$.

Section 3 reviews homological perturbation theory for complexes and using the tensor trick of [8, 11] for A_∞-algebras.

Section 4 studies the contracting homotopy from BSV to ΛV.

Section 5 studies the dual contracting homotopy from $\Omega\Lambda V$ to SV, and by homological perturbation theory, from ΩCL to UL for an L_∞-algebra L. The same method is used in Section 6 to make the formula for the A_∞-morphism in [15, Lemma 19] explicit.

In an appendix, we present the alternative contracting homotopy of [3].

1. The symmetric algebra and exterior coalgebra of a cochain complex

We work with complexes over a field $\mathbb{F}$ of characteristic zero, graded cohomologically, so that the differential has degree 1. We use the notation sV for the suspension of a complex: $(sV)^i = V^{i+1}$. All tensor products are over $\mathbb{F}$, unless otherwise indicated.

Koszul duality is streamlined by working in the category of non-unital dg algebras and A_∞-algebras and non-counital coalgebras. For this reason, we define the symmetric algebra SV of a complex V to be

$$SV = \bigoplus_{k=1}^{\infty} S^k V,$$

and the exterior coalgebra ΛV of V to be

$$\Lambda V = \bigoplus_{k=1}^{\infty} \Lambda_k V,$$

where $\Lambda_k V \cong S^k sV$.

To a non-counital dg coalgebra C we may associate a counital coalgebra

$$C_+ = C \oplus \mathbb{F},$$

with counit $\varepsilon : C_+ \to \mathbb{F}$ equal to projection on the summand $\mathbb{F}$ and codifferential extended to C_+ by vanishing on $\mathbb{F} \subset C_+$.The coproduct Δ_+ on C_+ is given by the formula

$$\Delta_+ a = a \otimes 1 + \Delta a + 1 \otimes a - \varepsilon(a)\, 1 \otimes 1.$$

The coproduct Δ on the non-counital coalgebra is sometimes called the reduced coproduct. We make use of Sweedler's notation, abbreviating the coproduct of a coalgebra

$$\Delta a = \sum_i a_i^{(1)} \otimes a_i^{(2)}$$

to

$$\Delta a = a^{(1)} \otimes a^{(2)}.$$

An element $x \in C$ of a non-counital dg coalgebra is primitive if its coproduct $\Delta x \in C \otimes C$ vanishes: the set $PC \subset C$ of primitive elements is a subcomplex of C.

Likewise, if A is a dg algebra, let $A^+ = A \oplus \mathbb{F}$, with product

$$(a + \alpha 1)(b + \beta 1) = (ab + \alpha b + \beta a) + (\alpha\beta)1,$$

and differential $\delta(a + \alpha 1) = \delta a$. In particular,

$$SV^+ = \bigoplus_{k=0}^{\infty} S^k V,$$

and

$$\Lambda V_+ = \bigoplus_{k=0}^{\infty} \Lambda_k V.$$

Working with non-unital algebras and non-counital coalgebras has the disadvantage of rendering the definition of a bialgebra less intuitive. The compatibility between the product and coproduct may be written as

$$\Delta(ab) = (a \otimes 1 + \Delta a + 1 \otimes a)(b \otimes 1 + \Delta b + 1 \otimes b) - ab \otimes 1 - 1 \otimes ab.$$

In Sweedler notation, this becomes

$$\begin{aligned}
&(ab)_{(1)} \otimes (ab)_{(2)} - a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)} \\
&\quad = a \otimes b + a_{(1)}b \otimes a_{(2)} + a_{(1)} \otimes a_{(2)}b + ab_{(1)} \otimes b_{(2)} + b_{(1)} \otimes ab_{(2)} + b \otimes a,
\end{aligned}$$

or in the graded case,

$$\begin{aligned}
(ab)_{(1)} \otimes (ab)_{(2)} &- (-1)^{|a_{(2)}||b_{(1)}|} a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)} \\
&= a \otimes b + (-1)^{|a_{(2)}||b|} a_{(1)}b \otimes a_{(2)} + a_{(1)} \otimes a_{(2)}b \\
&\quad + ab_{(1)} \otimes b_{(2)} + (-1)^{|a||b_{(1)}|} b_{(1)} \otimes ab_{(2)} + (-1)^{|a||b|} b \otimes a.
\end{aligned}$$

Both SV and ΛV are bialgebras: the coproduct of SV is the coshuffle coproduct, characterized by $PSV = V = S^1 V$, and the product of ΛV is the wedge product $a \wedge b : \Lambda_k V \otimes \Lambda_\ell V \to \Lambda_{k+\ell} V$.

A twisting cochain from a dg coalgebra C to a dg algebra A is a morphism $\mu : C \to A$ of degree 1 satisfying the Maurer–Cartan equation $d\mu + \mu^2 = 0$, where μ^2 is the composition $C \to C^{\otimes 2} \xrightarrow{\mu\otimes\mu} A^{\otimes 2} \to A$.

A coderivation of a coalgebra C is a map $\delta : C \to C$ such that

$$\Delta\delta = (\delta \otimes 1 + 1 \otimes \delta)\Delta.$$

A codifferential is a coderivation δ of degree 1 such that $\delta^2 = 0$.

An L_∞-structure on a complex L is a codifferential δ on ΛL, that is, a coderivation of degree 1 whose square vanishes. We have

$$\delta = d + \sum_{k=2}^{\infty} \delta_k,$$

where δ_k is the component of δ that maps $\Lambda_\ell L$ to $\Lambda_{\ell-k+1} L$. These coderivations are given by the formulas

$$d(v_1 \wedge \cdots \wedge v_\ell) = \sum_{j=1}^{\ell} (-1)^{|v_1|+\cdots+|v_{j-1}|+j-1} v_1 \wedge \cdots \wedge dv_j \wedge \cdots \wedge v_\ell,$$

$$\delta_k(v_1 \wedge \cdots \wedge v_\ell) = \frac{1}{\ell!}\binom{\ell}{k} \sum_{\pi\in S_\ell} \operatorname{sgn}(\pi)\, (-1)^{\sum_{i=1}^{k}(k-i)|v_i|}[v_{\pi(1)}, \dots, v_{\pi(k)}] \wedge v_{\pi(k+1)} \wedge \cdots \wedge v_{\pi(\ell)},$$

where $[v_1, \dots, v_k]$ is a graded antisymmetric k-linear bracket of degree $2-k$, and $\operatorname{sgn}(\pi)$ is the sign associated to the action of the permutation π on $sv_1 \otimes \cdots \otimes sv_\ell$:

$$\operatorname{sgn}(\pi) = (-1)^{\sum_{\{i<j \mid \pi(i)>\pi(j)\}}(|v_{\pi(i)}|+1)(|v_{\pi(j)}|+1)}. \tag{1.1}$$

The coderivation d may be viewed as the coderivation δ_1 associated to the 1-linear bracket $[v] = dv$.

The formula $\delta^2 = 0$ imposes quadratic equations among the brackets and the differential d, which in the case where $[v_1, \dots, v_k]$ vanishes for $k > 2$ become the usual axioms for a dg Lie algebra. This codifferential makes ΛL into a dg coalgebra denoted by CL, the (reduced) Chevalley–Eilenberg complex of chains of the L_∞-algebra L. In the special case that L is a Lie algebra, this recovers the complex introduced by Chevalley and Eilenberg.

2. The bar construction of an A_∞-algebra

The bar construction of a complex A is the graded vector space

$$BA = \bigoplus_{k=1}^{\infty} B_k A,$$

where $B_k A = (sA)^{\otimes k}$. We denote the tensor product

$$sa_1 \otimes \cdots \otimes sa_k \in B_k A$$

by the bar notation $[a_1|\cdots|a_k]$ of Eilenberg and Mac Lane (from which the construction derives its name). For $0 \le j \le k$, let

$$\omega_j = |a_1| + \cdots + |a_j| - j;$$

this is the degree of the element $[a_1|\cdots|a_j]$ of $B_j A$.

The bar construction is a dg coalgebra, with coproduct $\Delta : BA \to BA \otimes BA$ and codifferential $\delta : BA \to BA$ given by the formulas

$$\Delta[a_1|\cdots|a_k] = \sum_{0<j<k} [a_1|\cdots|a_j] \otimes [a_{j+1}|\cdots|a_k],$$
$$\delta[a_1|\cdots|a_k] = \sum_{0\le j\le k} (-1)^{\omega_{j-1}} [a_1|\cdots|da_j|\cdots|a_k].$$

The counital form of the bar construction is

$$B_+A = BA \oplus \mathbb{F} = \bigoplus_{k=0}^{\infty} B_k A.$$

The counit $\varepsilon : B_+A \to \mathbb{F}$ projects from B_+A to $B_0 A \cong \mathbb{F}$, with basis vector $[\,]$.

A Hochschild cochain $D \in C^*(A, A)$ is a map $D : B_+A \to A$. Denote the component of D in $\mathrm{Hom}(B_k A, A)$ by $D_{(k)}$. We denote the graded vector space $C^*(A, A)$ of Hochschild cochains by $G(A)$.[1]

There is a bijection between coderivations δ of BA and Hochschild cochains $D \in G(A)$ such that $D_{(0)} = 0$, given by composition with projection from BA to $B_1 A \cong sA$ followed by the degree one isomorphism from sA to A. We denote the coderivation corresponding to a Hochschild cochain D by $\delta(D)$. In particular, $|\delta(D)| = |D| - 1$. The coderivation $\delta(D)$ is given by the formula

$$\delta(D)[a_1|\cdots|a_k] = \sum_{0\le i<j\le k} (-1)^{(|D|-1)\omega_i} [a_1|\cdots|D[a_{i+1}|\cdots|a_j]|\cdots|a_k].$$

Gerstenhaber [4] introduced the bilinear operation

$$(D \circ E)[a_1|\cdots|a_k] = \sum_{0\le i<j\le k} (-1)^{(|E|-1)\omega_i} D[a_1|\cdots|E[a_{i+1}|\cdots|a_j]|\cdots|a_k]$$

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[1] The letter G stands for Gerstenhaber [4], who made the first close study of the algebraic properties of $G(A)$.

on $G(A)$. The graded commutator

$$[D, E] = D \circ E - (-1)^{(|D|-1)(|E|-1)} E \circ D$$

is a graded Lie bracket on $G(A)$, called the Gerstenhaber bracket. We have

$$[\delta(D), \delta(E)] = \delta([D, E]).$$

Definition 1. An A_∞-algebra structure on a graded vector space A is a codifferential δ on BA.

Denote the Hochschild cochain associated to the A_∞-algebra structure by $m \in G(A)$ so that $\delta = \delta(m)$. Since $\delta(m)^2 = \delta(m \circ m)$, the equation $\delta^2 = 0$ amounts to the sequence of quadratic relations $m \circ m = 0$ among the homogeneous components $\{m_{(k)}\}_{k>0}$ of m. The differential

$$\delta D = [m, D]$$

on $G(A)$ makes the Gerstenhaber algebra of an A_∞-algebra into a dg Lie algebra.

The twisting cochain $BA \to A$ given by projection from BA to $B_1 A \cong sA$ followed by the degree one morphism from sA to A is the universal twisting cochain: the twisting cochains from a dg coalgebra C to A are in bijection with morphisms of dg coalgebras from C to BA.

Definition 2. A morphism of A_∞-algebras is a morphism of the associated dg coalgebras $\mathbf{f} : BA_1 \to BA_2$.

The components $\mathbf{f}_{(k)} : B_k A_1 \to A_2$ of a morphism $\mathbf{f} : BA_1 \to BA_2$ of A_∞-algebras are the compositions of the map

$$B_k A_1 \hookrightarrow BA_1 \xrightarrow{\mathbf{f}} BA_2$$

with the universal twisting cochain $BA_2 \to A_2$. Together, these determine $\mathbf{f}$, by the formula

$$\mathbf{f}[a_1|\cdots|a_k] = \sum_{\ell=0}^{\infty} \sum_{0<j_1<\cdots<j_\ell<k} (-1)^{\omega_{j_1}+\cdots+\omega_{j_\ell}} \times [\mathbf{f}_{(j_1)}[a_1|\cdots|a_{j_1}]|\mathbf{f}_{(j_2-j_1)}[a_{j_1+1}|\cdots|a_{j_2}]\cdots|\mathbf{f}_{(k-j_\ell)}[a_{j_\ell+1}|\cdots|a_k]].$$

A quasi-isomorphism of A_∞-algebras $\mathbf{f} : A_1 \to A_2$ is a morphism of A_∞-algebras such that the linear component $\mathbf{f}_{(1)} : sA_1 \to A_2$ (or rather, the associated morphism from A_1 to A_2) is a quasi-isomorphism.

Dg algebras are special cases of A_∞-algebras, with $m_{(1)}(a_1) = da_1$ and

$$m_{(2)}(a_1, a_2) = (-1)^{|a_1|} a_1 a_2, \quad a_1, a_2 \in A.$$

The differential of the bar construction of a dg algebra is the coderivation

$$\delta = \delta_1 + \delta_2 = \delta(m_{(1)}) + \delta(m_{(2)}),$$

where

$$\delta_1[a_1|\cdots|a_k] = \sum_{j=1}^{k}(-1)^{\omega_{j-1}}\,[a_1|\cdots|da_j|\cdots|a_k]$$

is the coderivation associated to the differential d on the complex A, and δ_2 is the coderivation

$$\delta_2[a_1|\cdots|a_k] = \sum_{j=1}^{k-1}(-1)^{\omega_j+1}\,[a_1|\cdots|a_ja_{j+1}|\cdots|a_k]$$

associated to the product on A.

We learned the following result from Tsygan [15]. The associated dg bialgebra structure on $\mathrm{BG}(A)$ was discovered by Getzler and Jones [6]. A dg (left) module in coalgebras M for a dg bialgebra H is a (left) module for H in the monoidal category of dg coalgebras. In other words, there is an associative action

$$H \otimes M \to M$$

which is a morphism of dg coalgebras.

Proposition 1. *Let A be an A_∞-algebra. There are a unique dg bialgebra structure on $H = \mathrm{BG}(A)$ and an H-module in dg coalgebras structure on $M = BA$ such that the action of $[D] \in \mathrm{BG}(A)$ on BA is the graded coderivation $\delta(D)$ associated to $D \in G(A)$.*

Proof. The compatibility of the action of $\mathrm{BG}(A)$ on BA with the coproducts determines the formula for the action of the element $[D_1|\cdots|D_n] \in B_nG(A)$ on $[a_1|\cdots|a_k] \in B_kA$:

$$\begin{aligned}
&[D_1|\cdots|D_n] \bullet [a_1|\cdots|a_k] \\
&\quad = \sum_{0\le i_1<j_1\le\cdots\le i_n<j_n\le k} (-1)^{\sum_{\ell=1}^{n}(|D_\ell|-1)\omega_{i_\ell}} \\
&\qquad \times [a_1|\cdots|a_{i_1}|D_1[a_{i_1+1}|\cdots|a_{j_1}]|\cdots|a_{i_n}|D_n[a_{i_n+1}|\cdots|a_{j_n}]|\cdots|a_k].
\end{aligned}$$

This gives an injection of graded vector spaces

$$\mathrm{BG}(A) \hookrightarrow \mathrm{End}(BA).$$

It may be checked that this subspace is closed under the differential $\mathrm{ad}(\delta)$ on $\mathrm{End}(BA)$, giving rise to a codifferential on $\mathrm{BG}(A)$, and in particular, an A_∞-algebra structure on $G(A)$. This subspace is also closed under composition, giving rise to the product on $\mathrm{BG}(A)$ and making it into a dg bialgebra. ∎

The differential on $\mathrm{BG}(A)$ induces an A_∞-algebra structure on $G(A)$, given by the element

$$M[D_1|\cdots|D_n] = \begin{cases} [m, D_1], & n = 1, \\ m\{D_1, \dots, D_n\}, & n > 1, \end{cases}$$

of $G(G(A))$. In the special case that A is a dg algebra, $G(A)$ is a dg algebra, with Gerstenhaber's cup product

$$D_1 \cup D_2 = (-1)^{|D_1|} m\{D_1, D_2\}.$$

The product on $\mathrm{BG}(A)$ is determined by its components $m_{k,\ell}$, which are the compositions of the product

$$B_k G(A) \otimes B_\ell G(A) \to \mathrm{BG}(A)$$

with the universal twisting cochain $\mathrm{BG}(A) \to G(A)$. The map

$$m_{k,\ell} : (sG(A))^{\otimes k} \otimes (sG(A))^{\otimes \ell} \to G(A)$$

vanishes unless $k = 1$, and the operations $m_{1,\ell}$ are the brace operations

$$m_{1,\ell}\big([D] \otimes [E_1|\cdots|E_\ell]\big) = D\{E_1, \dots, E_\ell\}$$

introduced in [5], given by the formula

$$\begin{aligned} &D\{E_1, \dots, E_\ell\}[a_1|\cdots|a_k] \\ &= \sum_{0=j_1\le k_1\le\cdots\le j_\ell\le k_\ell\le k} (-1)^{\sum_{i=1}^{\ell}(|E_i|-1)\omega_{j_i}} \\ &\qquad\qquad \times D[a_1|\cdots|E_1[a_{j_1+1}|\cdots|a_{k_1}]|\cdots|E_\ell[a_{j_\ell+1}|\cdots|a_{k_\ell}]|\cdots|a_k]. \end{aligned}$$

The operation $m_{1,1}$ is Gerstenhaber's operation $D \circ E$.

3. Contractions

A *weak contraction* is a pair of complexes (X, δ) and (Y, ∂), together with morphisms of complexes $f : X \to Y$ and $g : Y \to X$ and a homotopy $h : X \to s^{-1}X$ of degree -1 such that $fg = 1_Y$ and $gf = 1_X - [\delta, h] = 1 - (\delta h + h\delta)$:

$$h \circlearrowleft X \underset{g}{\overset{f}{\rightleftarrows}} Y.$$

Hence, $p = gf$ and $1 - p = \delta h + h\delta$ are idempotents.

If the weak contraction (X, Y, f, g, h) satisfies the additional equations

$$fh = 0, \quad hg = 0, \quad h^2 = 0,$$

we call it a *contraction*. (These are also referred to as the side conditions of homological perturbation theory.) If we replace the homotopy h of a weak contraction by

$$\hat{h} = (1 - p)h\delta h\delta h(1 - p) = \tilde{h}\delta\tilde{h},$$

where $\tilde{h} = (1 - p)h(1 - p)$, we obtain a contraction $(X, Y, \hat{h}, f, g)$.

Let $\operatorname{Cone}(f) = X \oplus sY$ be the mapping cone of $f : X \to Y$, and let u be a formal commuting variable of degree 2. We may assemble the data of a contraction into a curved Maurer–Cartan element $\mathcal{A}$ on $\operatorname{Cone}(f)[u]$, with curved differential $\mathcal{D} + \mathcal{A}$, where

$$\mathcal{D} = \begin{bmatrix} \delta & 0 \\ 0 & -\partial \end{bmatrix}$$

and

$$\mathcal{A} = \begin{bmatrix} uh & ug \\ f & 0 \end{bmatrix},$$

and curvature $(\mathcal{D} + \mathcal{A})^2 = u$.

A perturbation of a contraction is a Maurer–Cartan element $\mu \in \operatorname{Hom}(X, X)$ of degree 1 such that $1 + \mu h$ is invertible. Equivalently, we may assume that $1 + h\mu$ is invertible, since

$$(1 + h\mu)^{-1} = 1 - h(1 + \mu h)^{-1}\mu$$

and

$$(1 + \mu h)^{-1} = 1 - \mu(1 + h\mu)^{-1}h,$$

or that $1 + h\mu + \mu h$ is invertible, since

$$(1 + h\mu + \mu h)^{-1} = (1 + h\mu)^{-1}(1 + \mu h)^{-1}$$

and

$$(1 + \mu h)^{-1} = (1 + h\mu)(1 + h\mu + \mu h)^{-1}.$$

A perturbation gives rise to a new contraction $(X_*, Y_*, h_*, f_*, g_*)$. Here, X_* has the same underlying graded vector space as V, with differential $\delta_* = \delta + \mu$, and W_* has the same underlying graded vector space as W, with differential

$$\partial_* = \partial + f(1 + \mu h)^{-1}\mu g.$$

The remaining data are given by the formulas

$$f_* = f(1+\mu h)^{-1}, \quad g_* = (1+h\mu)^{-1}g,$$
$$h_* = h(1+\mu h)^{-1} = (1+h\mu)^{-1}h.$$

In terms of the representation of a contraction as a curved differential $\mathcal{D}+\mathcal{A}$ on the mapping cone $C(f)$, the deformed curved differential equals $\mathcal{D}+M_*+\mathcal{A}_*$, where

$$M_* = \begin{bmatrix} \mu & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{A}_* = \begin{bmatrix} uh_* & ug_* \\ f_* & 0 \end{bmatrix} = u\mathcal{A}(u+M_*\mathcal{A})^{-1}.$$

For more on contractions and their perturbations, see [7]. (What that paper refers to as strong deformation retract data are what we call weak contractions.)

Suppose that A is an A_∞-algebra with differential δ. Let $\delta_\mu = \delta + \delta(\mu)$ be the associated codifferential on BA, where $\mu \in G(A)$. Kadeishvili [12] showed that if (A, Z, f, g, h) is a contraction, there is a natural A_∞-algebra structure on Z, corresponding to a codifferential D_* on Z, and A_∞ quasi-isomorphism $\mathbf{g}_* : BZ \to BA$ whose linearization is g. This A_∞-algebra structure and A_∞-morphism are constructed by solving a fixed-point equation.

Gugenheim, Lambe and Stasheff [8] introduced a different approach to homological perturbation theory for A_∞-algebras, which they named the tensor trick: they consider homological perturbation theory for the bar construction BA. In this way, they also obtain a left inverse $\mathbf{f}_* : BA \to BZ$ to the quasi-isomorphism $\mathbf{g}_* : BZ \to BA$, with linearization f. (See also Huebschmann and Kadeishvili [11].) We now review their results. Note that [12] and [8] restrict their attention to the case $\mu_{(1)} = 0$: we state our results in the more general setting in which $1 + \mu_{(1)}h : A \to A$ is invertible. In other words, we allow the differential underlying the A_∞-algebra structure on A to differ by a Maurer–Cartan element from the original differential on A.

The A_∞-algebra structure on Z and A_∞-morphism from Z to A constructed by the tensor trick agree with those constructed by Kadeishvili: this may be proved by showing that they solve the same fixed-point equations. We will not need this identification in this paper.

The tensor trick is as follows. The contraction (A, Z, f, g, h) induces a contraction $(BA, BZ, \mathbf{f}, \mathbf{g}, \mathbf{h})$ of the bar construction BA associated to the underlying complex of A with the A_∞-algebra structure with all higher brackets set equal to zero. The homotopy $\mathbf{h}$ is given by the formula

$$\mathbf{h}[a_1|\cdots|a_k] = \sum_{j=1}^{k} (-1)^{\omega_{j-1}} [pa_1|\cdots|pa_{j-1}|ha_j|a_{j+1}|\cdots|a_k],$$

and the morphisms $\mathbf{f} : BA \to BZ$ and $\mathbf{g} : BZ \to BA$ are given by the formulas

$$\mathbf{f}[a_1|\cdots|a_k] = [fa_1|\cdots|fa_k]$$

and

$$\mathbf{g}[a_1|\cdots|a_k] = [ga_1|\cdots|ga_k].$$

Thus, $\mathbf{f}$ and $\mathbf{g}$ are morphisms of coalgebras:

$$\Delta\mathbf{f} = (\mathbf{f} \otimes \mathbf{f})\Delta$$

and

$$\Delta\mathbf{g} = (\mathbf{g} \otimes \mathbf{g})\Delta.$$

The idempotent

$$\mathbf{p}[a_1|\cdots|a_k] = \mathbf{gf}[a_1|\cdots|a_k] = [pa_1|\cdots|pa_k]$$

is also a morphism of coalgebras. However, the homotopy operator $\mathbf{h}$ is not a coderivation: rather, it satisfies the formula

$$\Delta\mathbf{h} = (\mathbf{h} \otimes 1 + \mathbf{p} \otimes \mathbf{h})\Delta.$$

Lemma 1. *If* $1 + \delta(\mu_{(1)})h : A \to A$ *is invertible, then* $1 + \delta(\mu)\mathbf{h} : BA \to BA$ *is invertible.*

Proof. On $B_k A$, we have

$$(1 + \delta(\mu)\mathbf{h})^{-1} = \sum_{\ell=0}^{\infty} \sum_{j_1+\cdots+j_\ell<k} (-1)^\ell (1 + \delta(\mu_{(1)})\mathbf{h})^{-1} \delta(\mu_{(j_1+1)})\mathbf{h} \dots \big(1 + \delta(\mu_{(1)})\mathbf{h}\big)^{-1} \delta(\mu_{(j_\ell+1)})\mathbf{h}\big(1 + \delta(\mu_{(1)})\mathbf{h}\big)^{-1}.$$

Thus, it suffices to prove that $1 + \delta(\mu_{(1)})\mathbf{h} : B_k A \to B_k A$ is invertible.

On $B_k A$, we have

$$\delta(\mu_{(1)})\mathbf{h} = \sum_{i,j=1}^{k} \alpha_{ij},$$

where

$$\begin{aligned} \alpha_{ij} &= \big(1^{\otimes i-1} \otimes \delta(\mu_{(1)}) \otimes 1^{\otimes k-i}\big)\big(p^{\otimes j-1} \otimes h \otimes 1^{\otimes k-j}\big) \\ &= \begin{cases} p^{\otimes i-1} \otimes \delta(\mu_{(1)})p \otimes p^{\otimes j-i-1} \otimes h \otimes 1^{\otimes k-j}, & i < j, \\ p^{\otimes j-1} \otimes \delta(\mu_{(1)})h \otimes 1^{\otimes k-j}, & i = j, \\ -p^{\otimes j-1} \otimes h \otimes 1^{\otimes i-j-1} \otimes \delta(\mu_{(1)}) \otimes 1^{\otimes k-i}, & i > j. \end{cases} \end{aligned}$$

Since $\alpha_{pq}\alpha_{ij} = 0$ unless $i < j$ or $q \geq i = j$, and $\alpha_{pq} \cdots \alpha_{ij} = 0$ if $i < j \leq q$, we see that

$$\begin{aligned}(1 + \delta(\mu_{(1)})\mathbf{h})^{-1} &= \Big(1 - \sum_{i>j} \alpha_{ij}\Big)\Big(1 + \sum_{i \leq j} \alpha_{ij}\Big)^{-1} \\ &= \Big(1 - \sum_{i>j} \alpha_{ij}\Big) \sum_{\ell=0}^{k-1} (-1)^\ell \sum_{\substack{j_1 < \cdots < j_\ell \\ i_1 < j_1, \ldots, i_\ell < j_\ell}} \beta\alpha_{i_1 j_1}\beta \cdots \beta\alpha_{i_\ell j_\ell}\beta,\end{aligned}$$

where $\beta = (1 + \alpha_{kk})^{-1} \cdots (1 + \alpha_{11})^{-1}$. But $1 + \alpha_{jj} : B_k A \to B_k A$ is invertible:

$$(1 + \alpha_{jj})^{-1} = 1 - p^{\otimes j-1} \otimes \delta(\mu_{(1)})h\big(1 + \delta(\mu_{(1)})h\big)^{-1} \otimes 1^{\otimes k-j}.$$

Hence, β is invertible, completing the proof. ■

By this lemma, if $1 + \delta(\mu_{(1)})h : A \to A$ is invertible, $\delta(\mu)$ induces a deformed contraction

$$\mathbf{f}_* = \mathbf{f}(1 + \delta(\mu)\mathbf{h})^{-1} : BA \to BZ, \quad \mathbf{g}_* = (1 + \mathbf{h}\delta(\mu))^{-1}\mathbf{g} : BZ \to BA,$$
$$\mathbf{h}_* = \mathbf{h}(1 + \delta(\mu)\mathbf{h})^{-1} : BA \to BA.$$

The deformed differential ∂_* on BZ is given by the formula

$$\partial_* = \mathbf{f}\delta\mathbf{g} + \mathbf{f}\delta(\mu)(1 + \mathbf{h}\delta(\mu))^{-1}\mathbf{g},$$

and the deformed idempotent $\mathbf{p}_*$ on BA by the formula

$$\mathbf{p}_* = (1 + \mathbf{h}\delta(\mu))^{-1}\mathbf{p}(1 + \delta(\mu)\mathbf{h})^{-1}.$$

Proposition 2. *The maps $\mathbf{f}_*$, $\mathbf{g}_*$, and $\mathbf{p}_* = \mathbf{g}_*\mathbf{f}_*$ are morphisms of coalgebras, and ∂_* is a coderivation.*

Proof. Using the formulas $\mathbf{hh}_* = 0$, $\mathbf{hp}_* = 0$, $\mathbf{ph}_* = 0$, and $\mathbf{pp}_* = \mathbf{p}(1 + \delta(\mu)\mathbf{h})^{-1}$, we see that

$$\begin{aligned}&(\mathbf{h} \otimes 1 + \mathbf{p} \otimes \mathbf{h})(\delta(\mu) \otimes 1 + 1 \otimes \delta(\mu))(\mathbf{h}_* \otimes 1 + \mathbf{p}_* \otimes \mathbf{h}_*) \\ &\quad = \mathbf{h}\delta(\mu)\mathbf{h}_* \otimes 1 + \mathbf{h}\delta(\mu)\mathbf{p}_* \otimes \mathbf{h}_* + \mathbf{p}\delta(\mu)\mathbf{h}_* \otimes \mathbf{h} + \mathbf{pp}_* \otimes \mathbf{h}\delta(\mu)\mathbf{h}_* \\ &\quad = (\mathbf{h} - \mathbf{h}_*) \otimes 1 + (\mathbf{p}(1 + \delta(\mu)\mathbf{h})^{-1} - \mathbf{p}_*) \otimes \mathbf{h}_* \\ &\qquad + (\mathbf{p} - \mathbf{p}(1 + \delta(\mu)\mathbf{h})^{-1}) \otimes \mathbf{h} + \mathbf{p}(1 + \delta(\mu)\mathbf{h})^{-1} \otimes (\mathbf{h} - \mathbf{h}_*) \\ &\quad = (\mathbf{h} \otimes 1 + \mathbf{p} \otimes \mathbf{h}) - (\mathbf{h}_* \otimes 1 + \mathbf{p}_* \otimes \mathbf{h}_*).\end{aligned}$$

It follows that

$$\big(1\otimes 1+(\mathbf{h}\otimes 1+\mathbf{p}\otimes\mathbf{h})(\delta(\mu)\otimes 1+1\otimes\delta(\mu))\big)^{-1} = 1\otimes 1-(\mathbf{h}_*\otimes 1+\mathbf{p}_*\otimes\mathbf{h}_*)(\delta(\mu)\otimes 1+1\otimes\delta(\mu)),$$

proving that

$$\begin{aligned}\Delta(1+\mathbf{h}\delta(\mu))^{-1} &= \big(1\otimes 1+(\mathbf{h}\otimes 1+\mathbf{p}\otimes\mathbf{h})(\delta(\mu)\otimes 1+1\otimes\delta(\mu))\big)^{-1}\Delta \\ &= \big(1\otimes 1-(\mathbf{h}_*\otimes 1+\mathbf{p}_*\otimes\mathbf{h}_*)(\delta(\mu)\otimes 1+1\otimes\delta(\mu))\big)\Delta.\end{aligned}\tag{3.1}$$

We see that

$$\begin{aligned}\Delta\mathbf{g}_* &= \Delta(1+\mathbf{h}\delta(\mu))^{-1}\mathbf{g} \\ &= \big(1\otimes 1-(\mathbf{h}_*\otimes 1+\mathbf{p}_*\otimes\mathbf{h}_*)(\delta(\mu)\otimes 1+1\otimes\delta(\mu))\big)(\mathbf{g}\otimes\mathbf{g})\Delta \\ &= (\mathbf{g}_*\otimes\mathbf{g}_*)\Delta.\end{aligned}$$

Since $\mathbf{fh}=0$, we see that

$$\begin{aligned}\mathbf{fg}_* &= \mathbf{f}(1+\mathbf{h}\delta)^{-1}\mathbf{g} \\ &= \mathbf{f}\big(1-\mathbf{h}(1+\delta\mathbf{h})^{-1}\delta\big)\mathbf{g} \\ &= \mathbf{fg}=1.\end{aligned}$$

It follows that $\partial_*=\mathbf{f}\delta\mathbf{g}+\mathbf{f}\delta(\mu)\mathbf{g}_*$ is a coderivation: certainly, $\mathbf{f}\delta\mathbf{g}$ is a coderivation, while

$$\begin{aligned}\Delta\mathbf{f}\delta(\mu)\mathbf{g}_* &= (\mathbf{f}\otimes\mathbf{f})(\delta(\mu)\otimes 1+1\otimes\delta(\mu))(\mathbf{g}_*\otimes\mathbf{g}_*)\Delta \\ &= (\mathbf{f}\delta(\mu)\mathbf{g}_*\otimes\mathbf{fg}_*+\mathbf{fg}_*\otimes\mathbf{f}\delta(\mu)\mathbf{g}_*)\Delta \\ &= (\mathbf{f}\delta(\mu)\mathbf{g}_*\otimes 1+1\otimes\mathbf{f}\delta(\mu)\mathbf{g}_*)\Delta.\end{aligned}$$

The same calculation as for the proof of (3.1) shows that

$$\Delta(1+\delta(\mu)\mathbf{h})^{-1} = \big(1\otimes 1-(\delta(\mu)\otimes 1+1\otimes\delta(\mu))(\mathbf{h}_*\otimes 1+\mathbf{p}_*\otimes\mathbf{h}_*)\big)\Delta.$$

It follows that

$$\begin{aligned}\Delta\mathbf{f}_* &= \Delta\mathbf{f}(1+\delta(\mu)\mathbf{h})^{-1} \\ &= (\mathbf{f}\otimes\mathbf{f})\big(1\otimes 1-(\delta(\mu)\otimes 1+1\otimes\delta(\mu))(\mathbf{h}_*\otimes 1+\mathbf{p}_*\otimes\mathbf{h}_*)\big)\Delta \\ &= \big(\mathbf{f}_*\otimes\mathbf{f}_*\big)\Delta.\end{aligned}$$ ∎

Corollary 1. *Let $\delta_\mu=\delta+\delta(\mu)$, $\mu\in G(A)$, be a codifferential on BA corresponding to an A_∞-algebra structure on A such that*

$$1+\delta(\mu_{(1)})h : A\to A$$

is invertible. The codifferential ∂_* on BZ constructed by the tensor trick induces an A_∞-algebra structure on Z, and $\mathbf{f}_*:BA\to BZ$ and $\mathbf{g}_*:BZ\to BA$ are quasi-isomorphisms of A_∞-algebras.

4. The bar construction of a symmetric algebra

The bar construction BA of a dg algebra A is a graded commutative bialgebra: the product on BA is the shuffle product

$$\begin{aligned}&[a_1|\cdots|a_k] \sqcup\!\sqcup [a_{k+1}|\cdots|a_{k+\ell}]\\&\quad= \sum_{\sigma\in S(k,\ell)} (-1)^{\sum_{i=1}^{k}(\omega_{k+\sigma(i)-i}-\omega_k)(|a_i|-1)}[a_{\sigma^{-1}(1)}|\cdots|a_{\sigma^{-1}(k+\ell)}],\end{aligned}$$

where $S(k,\ell)$ is the set of shuffles,

$$S(k,\ell) = \{\sigma \in S_{k+\ell} \mid \sigma(i) < \sigma(j) \text{ if } 1 \le i < j \le k \text{ or } k+1 \le i < j \le k+\ell\}.$$

The codifferential δ of BA is a derivation with respect to the shuffle product if and only if A is graded commutative.

The calculation of the cohomology of BSV is one of the fundamental results of homological algebra. The subspace $sV \subset B_1SV$ generates a dg commutative subalgebra ΛV of the dg commutative bialgebra BSV. Since elements of sV are primitive in BSV, the inclusion $g : \Lambda V \hookrightarrow BSV$ is a morphism of dg commutative bialgebras.

In this section, we construct a contracting homotopy $h : B_kSV \to B_{k+1}SV$, which shows that g is a quasi-isomorphism. Let $\{x^\alpha\}$ be a homogeneous basis of V, let $x^\alpha \in G(SV)$ be the associated Hochschild zero-cochain, and let $\overline{\partial}_\alpha \in G(SV)$ be the Hochschild one-cochain

$$(-1)^{|x^\alpha|}\overline{\partial}_\alpha a = \partial_\alpha a - \varepsilon(\partial_\alpha a),$$

where $\partial_\alpha a$ is the graded partial derivative of a with respect to x^α and $\varepsilon(\partial_\alpha a)$ is the constant term of $\partial_\alpha a$.

Let $\rho \in G(SV)$ be the Hochschild one-cochain which acts by multiplication by k on S^kV. Let

$$\tau[a] = \sum_\alpha \varepsilon(\partial_\alpha a)x^\alpha \in G(A)$$

be the Hochschild one-cochain that projects from SV to $V = S^1V \subset SV$. We also denote the coderivations $\delta(\rho)$ and $\delta(\tau) : BSV \to BSV$ by ρ and τ. Since the product in SV is homogeneous of degree 0, it is clear that ρ commutes with the differential δ. We have

$$\sum_\alpha x^\alpha \cup \overline{\partial}_\alpha = \rho - \tau \in G(SV). \tag{4.1}$$

The intersection of ΛV with B_kSV is the image of the projection

$$p_k[a_1|\cdots|a_k] = \frac{1}{k!}[\tau a_1] \sqcup\!\sqcup \cdots \sqcup\!\sqcup [\tau a_k]$$

on $B_k SV$. Note that p is a morphism of dg algebras, though not of coalgebras. Denote the map $g^{-1} \circ p$ from BSV to ΛV by f: thus, $fg = 1_{\Lambda V}$, and $gf = p$.

Define the operators $\xi : B_k SV \to B_{k+1} SV$ and $\lambda : B_k SV \to B_k SV$ by the formulas

$$\begin{aligned}
\xi[a_1|\cdots|a_k] &= \sum_\alpha [x^\alpha \mid \overline{\partial}_\alpha] \bullet [a_1|\cdots|a_k] \\
&= \sum_{0\le i<j\le k} \sum_\alpha (-1)^{\omega_i + |x^\alpha|(\omega_{j-1}-\omega_i+1)} \\
&\qquad \times [a_1|\cdots|a_i|x^\alpha|\cdots|\partial_\alpha a_j - \varepsilon(\partial_\alpha a_j)|\cdots|a_k], \\
\lambda[a_1|\cdots|a_k] &= [a_1|\cdots|a_{k-1}] \sqcup\!\sqcup [\tau a_k].
\end{aligned}$$

By inspection, it is clear that the operators ξ and λ both commute with ρ.

We learned the following result from [2].

Proposition 3. *We have* $[\delta, \xi] = \rho - \lambda$.

Proof. Introduce the matrix for the differential $d : V \to V$ of V:

$$dx^\alpha = \sum_\beta M^\alpha_\beta x^\beta.$$

The differential $[\delta, \xi]$ of ξ is the sum of $[\delta_1, \xi]$ and $[\delta_2, \xi]$. The first of these vanishes:

$$\sum_\alpha \delta_1[x^\alpha|\overline{\partial}_\alpha] = \sum_{\alpha,\beta}\big([M^\alpha_\beta x^\beta \mid \overline{\partial}_\alpha] - [x^\alpha \mid M^\beta_\alpha \overline{\partial}_\beta]\big) = 0.$$

The second equals the action of the element

$$\sum_\alpha \delta_2[x^\alpha|\overline{\partial}_\alpha] = \sum_\alpha \big([x^\alpha \cup \overline{\partial}_\alpha] + [m_{(2)} \circ x^\alpha \mid \overline{\partial}_\alpha] - (-1)^{|x^\alpha|}[x^\alpha \mid [m_{(2)}, \overline{\partial}_\alpha]]\big)$$

of $\mathrm{BG}(SV)$ on BSV. The first term equals $\rho - \tau$ by (4.1), while the second term vanishes since SV is graded commutative. The two-cochain $[m_{(2)}, \overline{\partial}_\alpha] \in G(SV)$ equals

$$\big([m_{(2)}, \overline{\partial}_\alpha]\big)[a_1|a_2] = -(-1)^{|x^\alpha|}\, \varepsilon(\partial_\alpha a_1)a_2 + (-1)^{|a_1||x^\alpha|} a_1 \varepsilon(\partial_\alpha a_2).$$

It follows that

$$\begin{aligned}
&\sum_\alpha [x^\alpha \mid [m_{(2)}, \overline{\partial}_\alpha]] \bullet [a_1|\cdots|a_k] \\
&\quad = \sum_{1\le i\le j<k} (-1)^{(|a_j|+1)(\omega_{j-1}-\omega_{i-1})}[a_1|\cdots|a_{i-1}|\tau a_j|a_i|\cdots|\hat{a}_j|\cdots|a_k] \\
&\qquad - \sum_{1\le i\le j<k} (-1)^{(|a_{j+1}|+1)(\omega_j-\omega_{i-1})}[a_1|\cdots|a_{i-1}|\tau a_{j+1}|a_i|\cdots|\hat{a}_{j+1}|\cdots|a_k] \\
&\quad = (\tau - \lambda)[a_1|\cdots|a_k].
\end{aligned}$$

The result follows. ∎

Corollary 2. *We have* $[\delta, \lambda] = 0$.

Proof. We have

$$[\delta, \lambda] = [\delta, \rho - (\rho - \lambda)] = [\delta, \rho] - [\delta, [\delta, \xi]].$$

The second term vanishes since $\delta^2 = 0$. ∎

Proposition 4. (1) *We have* $\xi^2 = 0$.

(2) *The operator* ξ *is a derivation of the shuffle product on* BSV.

Proof. We have

$$\Delta\xi = \sum_{\alpha} [x^\alpha] \otimes [\overline{\partial}_\alpha]. \tag{4.2}$$

Squaring both sides of (4.2), we see that $\xi^2 \in \mathrm{BG}(SV)$ is primitive: $\Delta\xi^2 = 0$. We prove that $\xi^2 f = 0$ for $f \in B_k SV$ by induction on k: it is clearly true for $k = 1$. By the induction hypothesis, $\xi^2 f \in B_{k+2} SV$ is primitive. But the space of primitive elements of SV equals $B_1 SV$; hence, $\xi^2 f = 0$.

Let $\zeta(f, g) = \xi(f ш g) - (\xi f) ш g - (-1)^{|f|} f ш (\xi g)$. By (4.2), we see that, for all $f, g \in BSV$,

$$\begin{aligned}\Delta\zeta(f, g) = (-1)^{|f^{(2)}||g^{(1)}|}\big(\zeta\big(f^{(1)}, g^{(1)}\big) \otimes \big(f^{(2)} ш g^{(2)}\big) \\ + (-1)^{|f^{(1)}|+|g^{(1)}|}\big(f^{(1)} ш g^{(1)}\big) \otimes \zeta\big(f^{(2)}, g^{(2)}\big)\big).\end{aligned}$$

Here, the possible cross-terms cancel because $[x^\alpha] \bullet (f ш g) = [x^\alpha] ш (f ш g)$ and

$$[\overline{\partial}_\alpha] \bullet (f ш g) = ([\overline{\partial}_\alpha] \bullet f) ш g + (-1)^{|x^\alpha||f|} f ш ([\overline{\partial}_\alpha] \bullet g).$$

We prove that $\zeta(f, g) = 0$ for $f \in B_k SV$ and $g \in B_\ell SV$ by induction on $k + \ell$; it is clearly true for $k + \ell = 2$. By the induction hypothesis, $\zeta(f, g) \in B_{k+\ell+1} SV$ is primitive; hence, it vanishes. ∎

Corollary 3. *We have* $[\xi, \lambda] = 0$.

Proof. We have

$$[\xi, \lambda] = [\xi, \rho - (\rho - \lambda)] = [\xi, \rho] - [\xi, [\delta, \xi]].$$

The first term has been seen to vanish, while the second term equals $\frac{1}{2}[\delta, \xi^2] = 0$. ∎

Corollary 4. *The operator* λ *is a derivation with respect to the shuffle product:*

$$\lambda(x ш y) = \lambda x ш y + x ш \lambda y.$$

Proof. Since SV is a graded commutative algebra, δ is a graded derivation with respect to the shuffle product, hence so is the graded commutator $[\delta, \xi]$. But it is clear that ρ is a derivation with respect to the shuffle product, and the result follows. ■

Lemma 2. *The descending factorial*

$$(\lambda)_j = \lambda(\lambda-1)\cdots(\lambda-j+1)$$

is given by the formula

$$(\lambda)_j[a_1|\cdots|a_k] = [a_1|\cdots|a_{k-j}] ⧢ [\tau a_{k-j+1}] ⧢ \cdots ⧢ [\tau a_k].$$

Proof. This is proved by induction on j, using the explicit formula for λ, the formula $\lambda[x^\alpha] = [x^\alpha]$, and the fact that λ is a derivation for the shuffle product. ■

Corollary 5. *Let λ_k and p_k be the restrictions of λ and p to B_kSV. Then,*

$$(\lambda_k)_k = k!\, p_k,$$

while $(\lambda_k)_i = 0$ for $i > k$.

In particular, we recover one of the results of de Wilde and Lecomte, that the minimal polynomial of λ_k divides $\lambda_k(\lambda_k - 1)\cdots(\lambda_k - k)$, and λ_k is a semisimple endomorphism whose spectrum is contained in (and in fact equals) $\{0, \ldots, k\}$. The eigenspace $\{\lambda_k = k\}$ is $\Lambda_k V \subset B_kSV$. The spectrum of the restriction ρ_k of ρ to B_kSV equals $\{k, k+1, \ldots\}$, and the subspace $\{\lambda_k = k\}$ is a subset of $\{\rho_k = k\}$. Since ρ and λ commute, the eigenvalues of $\rho_k - \lambda_k$ are nonnegative integers, and the kernel of $\rho_k - \lambda_k$ is the subspace on which $\rho_k = \lambda_k = k$, namely, $\Lambda^k V$.

We now modify the operator ξ to obtain a homotopy

$$h = (\rho - \lambda)^{-1}\xi.$$

This operator is defined because $p\xi = 0$, and the eigenvalues of $\rho - \lambda$ on the image of the idempotent $1 - p$ are strictly positive. It is clear that $h^2 = 0$, since $\xi^2 = 0$. It follows that $hp = ph = 0$.

We have proved the following theorem.

Theorem 1. *The morphism of dg commutative bialgebras $g : \Lambda V \hookrightarrow BSV$, the morphism of dg commutative algebras $f : BSV \to \Lambda V$, and the homotopy $h = (\rho - \lambda)^{-1}\xi$ form a contraction $(BSV, \Lambda V, f, g, h)$*

In the remainder of this section, we give an explicit formula for the restriction $h_k : B_kSV \to B_{k+1}SV$ of the homotopy h to B_kSV; it is a finite sum, though the number of terms increases with k.

Proposition 5. *We have*

$$h_k[a_1|\cdots|a_k] = \sum_{j=1}^{k} \frac{\xi[a_1|\cdots|a_j]}{\rho(\rho+1)\cdots(\rho+k-j)} ш [\tau a_{j+1}] ш \cdots ш [\tau a_k]. \tag{4.3}$$

Proof. Since the eigenvalues of the action of ρ on $B_{k+1}SV$ lie in $\{k+1, k+2, \ldots\}$, the polynomial $(\rho)_{j+1}$ is invertible on $B_{k+1}SV$ for $0 \le j \le k$. We have

$$\begin{aligned}(\rho-\lambda)(\rho)_{j+1}^{-1}(\lambda)_j\xi &= \big((\rho-j)-(\lambda-j)\big)(\rho)_{j+1}^{-1}(\lambda)_j\xi \\ &= (\rho)_j^{-1}(\lambda)_j\xi - (\rho)_{j+1}^{-1}(\lambda)_{j+1}\xi.\end{aligned}$$

It follows that

$$\begin{aligned}(\rho-\lambda)\sum_{j=0}^{k}(\rho)_{j+1}^{-1}(\lambda)_j\xi &= \sum_{j=0}^{k}\big((\rho)_j^{-1}(\lambda)_j\xi - (\rho)_{j+1}^{-1}(\lambda)_{j+1}\xi\big) \\ &= \xi - (\rho)_{k+1}^{-1}(\lambda)_{k+1}\xi.\end{aligned}$$

Since $(\lambda)_{k+1}$ vanishes on B_kSV, we see that $(\lambda)_{k+1}\xi = \xi(\lambda)_{k+1} = 0$, proving the formula

$$h_k = \sum_{j=0}^{k}(\rho)_{j+1}^{-1}(\lambda)_j\xi.$$

Since ξ is a graded derivation for the shuffle product, we see that

$$\xi(\lambda)_j[a_1|\cdots|a_k] = \xi[a_1|\cdots|a_{k-j}] ш [\tau a_{k-j+1}] ш \cdots ш [\tau a_k].$$

Since

$$\begin{aligned}&\rho\big([a_1|\cdots|a_{k-j}] ш [\tau a_{k-j+1}] ш \cdots ш [\tau a_k]\big) \\ &\quad = \big((\rho+j)[a_1|\cdots|a_{k-j}]\big) ш [\tau a_{k-j+1}] ш \cdots ш [\tau a_k],\end{aligned}$$

the proposition follows. ■

Halbout [9] and Dippel et al. [3] have studied contracting homotopies on BSV that extend to other classes of function algebras, such as smooth and analytic functions, when $\mathbb{F} = \mathbb{R}$. We see from the above formula for h that it may also be applied in these more general cases. We discuss only the smooth case, but other function algebras are handled in the same way.

Let $U \subset V$ be a star-shaped open subset of the finite-dimensional real vector space V. Substitute for the algebra SV the algebra $C^\infty_*(U)$ of smooth functions on U vanishing at $0 \in U$, and for the bar construction the direct sum

$$B_\infty C^\infty_*(U) = \bigoplus_{k=1}^{\infty} s^k C^\infty_*(U,k),$$

where $C^\infty_*(U,k)$ is the subspace of smooth functions on U^k that vanish at $(u_1,\dots,u_k)\in U^k$ if any of the u_i equal $0\in U$. The homotopy h defines a continuous linear morphism from $C^\infty_*(U,k)$ to $C^\infty_*(U,k+1)$; the definitions of the analogues of τ and $\overline{\partial}_\alpha$ are clear, and the operation $(\rho+\ell)^{-1}$ is given by the integral

$$(\rho+\ell)^{-1} f(u_1,\dots,u_k) = \int_0^1 f(tu_1,\dots,tu_k)t^{\ell-1}\,dt.$$

This clearly defines a continuous linear map from $C^\infty_*(U,k)$ to itself if $\ell>0$, while for $\ell=0$, we have

$$\begin{aligned}\rho^{-1} f(u_1,\dots,u_k) &= \sum_{i=1}^k u_i\cdot(\rho+1)^{-1}\int_0^1 (\partial_{u_i} f)(su_1,\dots,su_k)\,ds \\ &= \sum_{i=1}^k u_i\cdot\int_0^1\int_0^1 (\partial_{u_i} f)(stu_1,\dots,stu_k)\,ds\,dt \in C^\infty_*(U,k).\end{aligned}$$

Since B_kSV is dense in $s^kC^\infty_*(U,k)$, it is clear that all of the formulas that we have proved in the algebraic setting hold for function algebras.

5. The cobar construction of an exterior coalgebra

In this section, we translate our results on the bar construction of a symmetric algebra to the dual situation, the cobar construction of an exterior coalgebra: we construct a contraction from the cobar construction $\Omega\Lambda V$ of an exterior coalgebra to the symmetric algebra SV. Using homological perturbation theory, this contraction is deformed to a contraction from the cobar construction ΩCL of the Chevalley–Eilenberg complex of a dg Lie algebra L to the universal enveloping algebra UL of L: the proof follows Baranovsky [1], with the difference that we work with an explicit, and easily computable, formula for the homotopy contraction from $\Omega\Lambda V$ to SV.

Let C be a dg coalgebra. The cobar construction of C is a dg algebra whose underlying graded algebra is the tensor algebra

$$\Omega C = \bigoplus_{k=1}^\infty (s^{-1}C)^{\otimes k}.$$

We denote the element $s^{-1}a_1\otimes\cdots\otimes s^{-1}a_k$ by $\langle a_1|\cdots|a_k\rangle$. The differential of ΩC is

$$\begin{aligned}&\delta\langle a_1|\cdots|a_k\rangle \\ &\quad= \sum_{j=1}^k (-1)^{\omega_{j-1}}\langle a_1|\cdots|da_j|\cdots|a_k\rangle + \sum_{j=1}^k (-1)^{\omega_j+1}\langle a_1|\cdots|a_j^{(1)}|a_j^{(2)}|\cdots|a_k\rangle.\end{aligned}$$

The cobar algebra ΩC is a graded cocommutative bialgebra, with coproduct given by a sum over coshuffles:

$$\langle a_1|\cdots|a_k\rangle \mapsto \sum_{0<\ell<k}\sum_{\sigma\in S(\ell,k-\ell)} (-1)^{\sum_{i=1}^{\ell}(\omega_{\ell+\sigma(i)-i}-\omega_\ell)(|a_i|-1)} \times \langle a_{\sigma(1)}|\cdots|a_{\sigma(\ell)}\rangle \otimes \langle a_{\sigma(\ell+1)}|\cdots|a_{\sigma(k)}\rangle.$$

The differential δ on ΩC is a coderivation for this coproduct if and only if the graded coalgebra C is cocommutative. The subcomplex $P\Omega C$ of primitive elements of ΩC is the Harrison chain complex of C: it is the Koszul dual dg Lie algebra to C.

As we have seen in the last section, the natural inclusion $g : \Lambda V \hookrightarrow BSV$ of the exterior coalgebra of a cochain complex V into the bar construction BSV of its symmetric algebra is a morphism of dg commutative bialgebras. Similarly, there is a surjective morphism of dg cocommutative bialgebras $f : \Omega\Lambda V \to SV$, induced by projecting the generators $s\Lambda V$ of $\Omega\Lambda V$ to the summand $s\Lambda_1 V \cong V$, followed by inclusion as the generators V of SV. This morphism has a section $g : SV \to \Omega\Lambda V$, which is the morphism of dg cocommutative coalgebras induced by the inclusions

$$S^k V \hookrightarrow V^{\otimes k} \cong \Omega^k \Lambda_1 V \hookrightarrow \Omega^k \Lambda V.$$

Let ρ be the coderivation of ΛV which acts by multiplication by k on the subspace $\Lambda_k V \subset \Lambda V$. This coderivation extends to a graded derivation of $\Omega\Lambda V$:

$$\rho\langle a_1|\cdots|a_k\rangle = \sum_{j=1}^{k}\langle a_1|\cdots|\rho a_j|\cdots|a_k\rangle.$$

We see that $[\delta,\rho] = 0$ on $\Omega\Lambda V$.

Let τ be the projection τ from ΛV to $\Lambda_1 V \cong s^{-1}V$. Define maps $\xi : \Omega^k\Lambda V \to \Omega^{k-1}\Lambda V$ and $\lambda : \Omega^k\Lambda V \to \Omega^k\Lambda V$ by the formulas

$$\xi\langle a_1|\cdots|a_k\rangle = \sum_{1\le i<j\le k} (-1)^{\omega_{i-1}+|a_i|(\omega_{j-1}-\omega_i+1)}\langle a_1|\cdots|\hat{a}_i|\cdots|\tau a_i \wedge a_j|\cdots|a_k\rangle,$$

$$\lambda\langle a_1|\cdots|a_k\rangle = \sum_{j=1}^{k}(-1)^{(|a_j|-1)(\omega_k-\omega_j+1)}\,\langle|a_1|\cdots|\hat{a}_j|\cdots|a_k|\tau a_j\rangle.$$

The operators ξ and λ both commute with ρ.

The proofs of the last section apply to any additive symmetric monoidal category tensored over the symmetric monoidal category of finite-dimensional cochain complexes. (The proofs take place in finite truncations $\bigoplus_{1\le k\le N} B_k A$ and $\bigoplus_{1\le k\le N} S_k V$ of the bar construction and symmetric algebra, so the proofs deal only in finite sums.)

The opposite of the category of cochain complexes is such a category, and this substitution has the effect of exchanging the symmetric algebra with the exterior coalgebra (after suspension of V), and the bar construction with the cobar construction. In this way, we obtain the following dual results.

(a) $[\delta, \xi] = \rho - \lambda$.

(b) ξ and λ are (graded) coderivations with respect to the shuffle coproduct on $\Omega\Lambda V$.

(c) $\xi^2 = 0$.

(d) $[\delta, \lambda] = [\xi, \lambda] = 0$.

(e) Let ρ_k and λ_k be the restrictions of λ and ρ to $\Omega^k\Lambda V$. The operators ρ_k and λ_k are commuting semisimple operators.

(f) The eigenvalues of ρ_k lie in $\{k, k+1, \ldots\}$.

(g) The eigenvalues of λ_k lie in $\{0, \ldots, k\}$, $(\lambda_k)_k = k!\, p_k$, and $(\lambda_k)_i = 0$ for $i > k$.

We have proved the following theorem.

Theorem 2. *The morphism of dg cocommutative coalgebras $g : SV \hookrightarrow \Omega\Lambda V$, the morphism of dg commutative bialgebras $f : \Omega\Lambda V \to SV$, and the homotopy*

$$h = (\rho - \lambda)^{-1}\xi$$

form a contraction $(\Omega\Lambda V, SV, f, g, h)$.

The restriction $h_k : \Omega^{k+1}\Lambda V \to \Omega^k\Lambda V$ of the homotopy h to $\Omega^{k+1}\Lambda V$ (that is, corestriction to $\Omega^k\Lambda V$) is again given by a finite sum

$$h_k = \sum_{j=0}^{k} (\rho)_{j+1}^{-1} (\lambda)_j \xi.$$

Now, suppose that L is an L_∞-algebra. The differential δ of CL is the sum of the differential d induced by the differential on the underlying cochain complex L and the contribution of the brackets $[v_1, \ldots, v_k]$, $k \geq 2$. Denote the codifferential $\delta - d$ of CL by μ: it is a Maurer–Cartan element in the dg Lie algebra of coderivations of ΛL and induces a Maurer–Cartan element on $\Omega\Lambda L$, deforming it to the dg algebra ΩCL. Applying the formulas of homological perturbation theory to the contraction h from $\Omega\Lambda L$ to the subcomplex SL, we obtain a contraction

$$h_* = h(1 + \mu h)^{-1} : \Omega CL \to s^{-1}\Omega CL$$

from ΩCL to SL, with

$$f_* = f(1 + \mu h)^{-1} : \Omega CL \to SL.$$

Since $\mu g = 0$, the morphism $g = g_*$ of SL into ΩCL and the differential $d_* = d$ of SL are not deformed.

By the tensor trick, we obtain a contraction $\mathbf{h}_*$ from $B\Omega CL$ to BSL, which induces a codifferential on BSL, that is, an A_∞-algebra structure on SL. This A_∞-algebra structure was introduced by Baranovsky [1] (though without an explicit choice of contraction h from $\Omega\Lambda L$ to SL); he calls it the universal enveloping A_∞-algebra of the L_∞-algebra L. In this way, we obtain an analogue of the Poincaré–Birkhoff–Witt Theorem for L_∞-algebras. In the case where L is a dg Lie algebra, we now identify this A_∞-algebra structure on SL with the usual enveloping algebra, following Baranovsky.

The differential on $B\Omega CL$ is a sum

$$\delta_B + \delta_\Omega + \delta_1 + \mu,$$

where δ_B is the codifferential on $B\Omega CL$ induced by the product of ΩCL, δ_Ω is the differential on ΩCL induced by the coproduct of CL, δ_1 is induced by the differential on L, and μ is the coderivation on CL corresponding to the brackets $[x_1, \dots, x_k]$, $k \geq 2$, on L. In applying the tensor trick, the contraction $\mathbf{h}$ is associated to the complex $B\Omega CL$ with differential $\delta_\Omega + \delta_1$, and it is perturbed by $\delta_B + \mu$.

Since $\delta_\Omega \mathbf{g} = \mu g = 0$, the codifferential on BSL induced by the tensor trick is

$$\begin{aligned}\mathbf{f}\big(\delta_\Omega + \delta_1\big)\mathbf{g} &+ \mathbf{f}(1 + (\delta_B + \mu)\mathbf{h})^{-1}(\delta_B + \mu)\mathbf{g} \\ &= \delta_1 + \sum_{k=2}^{\infty} \mathbf{f}\big(-(1 + \mu\mathbf{h})^{-1}\delta_B\mathbf{h}\big)^{k-2}(1 + \mu\mathbf{h})^{-1}\delta_B\mathbf{g}.\end{aligned}$$

The k-linear bracket m_k of the A_∞-algebra structure induced on SL by this codifferential is contributed by the summand indexed by k.

Lemma 3. *If L is a dg Lie algebra, that is, $\delta_k = 0$ for $k > 2$, then this sum simplifies to*

$$\delta_1 + \mathbf{f}(1 + \mu\mathbf{h})^{-1}\delta_B\mathbf{g}.$$

Proof. There is a decreasing filtration on CL by subspaces

$$F^k CL = \bigoplus_{\ell \geq k+1} C_\ell L.$$

This induces decreasing filtrations on ΩCL and $B\Omega CL$. The operator $\mathbf{h}$ has degree 1 for this filtration, the operator δ_B has degree 0, the operator μ has degree -1, and the morphism $\mathbf{f}$ vanishes on $F^1 B\Omega CL$. It follows that the operator $(1 + \mu\mathbf{h})^{-1}\delta_B\mathbf{h}$ raises filtration degree, and the result follows. ∎

It follows that the A_∞-algebra structure on SL induced by the contraction on $B\Omega CL$ is a dg algebra structure: we denote the resulting product by $x * y$, and this deformation of SL by S_*L. To identify the product on S_*L, we consider the decreasing filtration on ΩCL induced by the decreasing filtration

$$G_k CL = \bigoplus_{\ell \le k} C_\ell L.$$

This filtration is preserved by $\mathbf{h}$ and δ_B and lowered by μ. The morphisms f and g are compatible with this filtration and the decreasing filtration

$$G_k SL = \bigoplus_{\ell \le k} S^\ell L.$$

The contribution of the summand $\mathbf{f}\delta_B\mathbf{g}$ to the deformed codifferential on BSL corresponds to the original product on SL, while the contribution of the remainder of the codifferential

$$\mathbf{f}(1+\mu\mathbf{h})^{-1}\delta_B\mathbf{g} - \mathbf{f}\delta_B\mathbf{g} = -\mathbf{f}(1+\mu\mathbf{h})^{-1}\mu\mathbf{h}\delta_B\mathbf{g}$$

maps $G_k SL$ to $G_{k-1}SL$. Thus, the induced product on SL may be characterized by its value on

$$[x|y] \in sL \otimes sL \subset B_2 SL.$$

This is calculated as follows:

$$\begin{aligned}
&\mathbf{f}(1+\mu\mathbf{h})^{-1}\mu\mathbf{h}\delta_B\mathbf{g}[x|y] \\
&\quad = \frac{1}{2}\mathbf{f}(1+\mu\mathbf{h})^{-1}\mu\mathbf{h}\delta_B\big([\langle x\rangle|\langle y\rangle] + (-1)^{|x||y|}[\langle y\rangle|\langle x\rangle]\big) \\
&\quad = \frac{1}{2}\mathbf{f}(1+\mu\mathbf{h})^{-1}\mu\mathbf{h}\big((-1)^{|x|+1}[\langle x|y\rangle] - (-1)^{|x||y|+|y|+1}[\langle y|x\rangle]\big) \\
&\quad = \frac{1}{2}\mathbf{f}(1+\mu\mathbf{h})^{-1}\mu\big((-1)^{|x|+1}[\langle x\wedge y\rangle] - (-1)^{|x||y|+|y|+1}[\langle y\wedge x\rangle]\big) \\
&\quad = \frac{1}{2}\mathbf{f}(1+\mu\mathbf{h})^{-1}\big(-[\langle [x,y]\rangle] + (-1)^{|x||y|}[\langle [y,x]\rangle]\big) \\
&\quad = -\mathbf{f}[\langle [x,y]\rangle] = -[x,y].
\end{aligned}$$

We have proved the following variant of the Poincaré–Birkhoff–Witt theorem. (We have used a different normalization of the codifferential on CL to the one in Baranovsky's paper.)

Theorem 3. *Let L be a dg Lie algebra. There is an isomorphism of dg algebras from the universal enveloping algebra UL to S_*L, defined on $L \subset UL$ by $x \mapsto \frac{1}{2}x$.*

6. Application to the non-commutative Gauss–Manin connection

Let $E = Ss^{-1}\mathbb{F} \cong \mathbb{F} \oplus \varepsilon\mathbb{F}$, where ε is a variable of degree 1 and square zero. Let $\partial_\varepsilon = \partial/\partial\varepsilon$ be the graded derivation of Λ, of degree -1:

$$\partial_\varepsilon(a + \varepsilon b) = b.$$

Denote by L_ε the dg Lie algebra $L \otimes E$. Extend ∂_ε to a coderivation of the dg coalgebra CL_ε.

Let $\mathbb{F}[u]$ be the polynomial algebra generated by an element u of degree 2. Let $\Omega_u CL_\varepsilon = (\Omega CL_\varepsilon)[u]$. Deform the differential of the dg bialgebra $\Omega_u CL_\varepsilon$ by the locally finite sum

$$\nu = \nu_1 + \nu_2 + \cdots,$$

where

$$\nu_\ell\langle a_1|\cdots|a_k\rangle = \sum_{j=1}^{k}(-1)^{\omega_{j-1}} u\langle a_1|\cdots|\partial_\varepsilon a_j^{(1)}|\cdots|\partial_\varepsilon a_j^{(\ell)}|\cdots|a_k\rangle.$$

Write $\nu = \nu_1 + \nu_+$. The differential of $\Omega_u CL_\varepsilon$ is given by the formula

$$\begin{aligned}
&\delta\langle a_1|\cdots|a_k\rangle \\
&= \sum_{j=1}^{k}(-1)^{\omega_{j-1}}\langle a_1|\cdots|da_j|\cdots|a_k\rangle + \sum_{j=1}^{k}(-1)^{\omega_j+1}\langle a_1|\cdots|a_j^{(1)}|a_j^{(2)}|\cdots|a_k\rangle \\
&\quad + u\sum_{\ell=1}^{\infty}\sum_{j=1}^{k}(-1)^{\omega_{j-1}}\langle a_1|\cdots|\partial_\varepsilon a_j^{(1)}|\cdots|\partial_\varepsilon a_j^{(\ell)}|\cdots|a_k\rangle.
\end{aligned}$$

This is the dg algebra denoted by $B^{\mathrm{tw}}\mathfrak{g}^\bullet[u,\varepsilon]$ in [15].

In this section, using the methods of Section 5, we construct an explicit A_∞ quasi-isomorphism from UL to $\Omega_u CL_\varepsilon$, reproving [15, Lemma 19]: this proof yields an explicit formula for the twisting cochain realizing the Gauss–Manin connection on periodic cyclic homology whose existence was proved loc. cit.

Define an operator $\varepsilon : CL_\varepsilon \to CL_\varepsilon$ of degree 1 by the formula

$$\varepsilon\big(x_1 \wedge \cdots \wedge x_n\big) = \sum_{i=1}^{n}(-1)^{|x_1|+\cdots+|x_{j-1}|+j-1} x_1 \wedge \cdots \wedge \varepsilon x_j \wedge \cdots \wedge x_n.$$

Extend ε to an operator $\varepsilon : \Omega_u CL_\varepsilon \to \Omega_u CL_\varepsilon$ of degree 1 by the formula

$$\varepsilon\langle a_1|\cdots|a_k\rangle = \sum_{j=1}^{k}(-1)^{\omega_{j-1}+1}\langle a_1|\cdots|\varepsilon a_j\,|\cdots|a_k\rangle.$$

We have $\varepsilon \circ \partial_\varepsilon + \partial_\varepsilon \circ \varepsilon = \rho$.

Let $\partial_u = \partial/\partial u$. The operator

$$\xi_\varepsilon = \xi + \varepsilon\partial_u$$

satisfies

$$(\delta_\Omega + \delta_1 + \nu_1)\xi_\varepsilon + \xi_\varepsilon(\delta_\Omega + \delta_1 + \nu_1) = \rho(u\partial_u + 1) + \varepsilon\partial_\varepsilon - \lambda.$$

Form the locally finite sum,

$$h_\varepsilon = \sum_{j=0}^{\infty}(\rho(u\partial_u + 1) + \varepsilon\partial_\varepsilon)_{j+1}^{-1}(\lambda)_j\xi_\varepsilon.$$

Let p_ε be the projection from $\Omega_u CL_\varepsilon$ to SL, identified with the zero eigenspace of $\rho(u\partial_u + 1) + \varepsilon\partial_\varepsilon - \lambda$. Then, we have

$$(\delta_\Omega + \delta_1 + \nu_1)h_\varepsilon + h_\varepsilon(\delta_\Omega + \delta_1 + \nu_1) = p_\varepsilon.$$

The associated map $g_\varepsilon : SL \to \Omega_u CL_\varepsilon$ is given by the same formula as $g : SL \to \Omega CL$, and $f_\varepsilon : \Omega_u CL_\varepsilon \to SL$ is given by setting $u = \varepsilon = 0$ and then applying the map $f : \Omega CL \to SL$. In this way, we obtain a contraction $(\Omega_u CL_\varepsilon, SL, h_\varepsilon, f_\varepsilon, g_\varepsilon)$.

Let $B_u\Omega_u CL_\varepsilon$ be the bar construction of $\Omega_u CL_\varepsilon$ as a dg algebra over $\mathbb{F}[u]$: we have

$$B_u\Omega_u CL_\varepsilon = \bigoplus_{k=1}^{\infty}(s\Omega CL_\varepsilon)^{\otimes k}[u].$$

The differential on $B_u\Omega_u CL_\varepsilon$ is a sum

$$\delta_B + \delta_\Omega + \delta_1 + \mu + \nu = (\delta_\Omega + \delta_1 + \nu_1) + (\delta_B + \mu + \nu_+).$$

Since $\mu\mathbf{g}_\varepsilon = \nu\mathbf{g}_\varepsilon = 0$, the codifferential on BSL induced by the tensor trick takes the form

$$\begin{aligned}\mathbf{f}_\varepsilon(\delta_\Omega + \delta_1 + \nu_1)\mathbf{g}_\varepsilon &+ \mathbf{f}_\varepsilon(1 + (\delta_B + \mu + \nu_+)\mathbf{h}_\varepsilon)^{-1}(\delta_B + \mu + \nu_+)\mathbf{g}_\varepsilon \\ &= \delta_1 + \sum_{k=2}^{\infty}\mathbf{f}_\varepsilon\big(-(1 + (\mu + \nu_+)\mathbf{h}_\varepsilon)^{-1}\delta_B\mathbf{h}_\varepsilon\big)^{k-2}(1 + (\mu + \nu_+)\mathbf{h}_\varepsilon)^{-1}\delta_B\mathbf{g}_\varepsilon.\end{aligned}$$

A similar argument to Lemma 3 shows that this sum simplifies to

$$\delta_1 + \mathbf{f}(1 + \mu\mathbf{h})^{-1}\delta_B\mathbf{g}.$$

Theorem 3 identifies the resulting A_∞-algebra structure on SL with the universal enveloping algebra UL. The A_∞ quasi-isomorphism from UL to $\Omega_u CL_\varepsilon$ that proves [15, Lemma 19] is given by the morphism of dg coalgebras

$$\mathbf{h}_\varepsilon(1 + (\delta_B + \mu + \nu_+)\mathbf{h}_\varepsilon)^{-1}\delta_B\mathbf{g}_\varepsilon : BUL \to B_u\Omega_u CL_\varepsilon,$$

or equivalently, by the twisting cochain $BUL \to \Omega_u CL_\varepsilon$ given by composition with the universal twisting cochain $B_u\Omega_u CL_\varepsilon \to \Omega_u CL_\varepsilon$.

A. An alternative contracting homotopy

In this appendix, we present the contracting homotopy of Dippell et al. [3]. Because this construction is based on the Koszul complex of the graded vector space V, we use the counital exterior algebra $\Lambda V_+ = \Lambda V \oplus \mathbb{F}$. The counit $\varepsilon : \Lambda V_+ \to \mathbb{F}$ projects from ΛV_+ to $\Lambda_0 V \cong \mathbb{F}$, with basis vector 1. We will construct a homotopy k from the counital bar complex $B_+SV = BSV \oplus \mathbb{F}$ to ΛV_+.

If A is a dg algebra and M is a left dg A-module, the bar construction $B_+(A, M)$ of A with coefficients in M has an underlying graded vector space $B_+A \otimes M$. Its differential $\check{\partial}$ the sum of the differentials on BA and M together with a single additional term capturing the left action of A on M:

$$\begin{aligned}
&\check{\partial}[a_1|\cdots|a_k] \otimes m \\
&= \sum_{i=1}^{k-1}(-1)^{\omega_i+1}\,[a_1|\cdots|a_ia_{i+1}|\cdots|a_k] \otimes m + (-1)^{\omega_k+1}\,[a_1|\cdots|a_{k-1}] \otimes a_k m \\
&\quad + \sum_{i=1}^{k}(-1)^{\omega_{i-1}}\,[a_1|\cdots|da_i|\cdots|a_k] \otimes m + (-1)^{\omega_k}\,[a_1|\cdots|a_k] \otimes dm.
\end{aligned}$$

Consider the left A-module $A^+ = A \oplus \mathbb{F}$, where the action of $a \in A$ on $A \subset A^+$ is left multiplication, while the action of $a \in A$ on $\mathbb{F} \subset A^+$ takes $1 \in \mathbb{F}$ to $a \in A \subset A^+$. Denote the augmentation from A^+ to $\mathbb{F}$ with kernel A by $\varepsilon : A^+ \to \mathbb{F}$. Denote the projection $a - \varepsilon(a) : A^+ \to A$ by $\bar{a}$.

The complex $B_+(A, A^+)$ is contractible, with contracting homotopy

$$\check{\mathsf{h}}[a_1|\cdots|a_k] \otimes a = (-1)^{\omega_k+|a|}[a_1|\cdots|a_k|a - \varepsilon(a)] \otimes 1.$$

We have

$$(\check{\partial}\check{\mathsf{h}} + \check{\mathsf{h}}\check{\partial})[a_1|\cdots|a_k] \otimes a = [a_1|\cdots|a_k] \otimes a - \delta_{k,0}\,[\,] \otimes \varepsilon(a).$$

The Koszul complex of V is the left SV-module $SV^+ \otimes \Lambda V_+$. Its differential is

$$\partial(a \otimes b) = (-1)^{|a|} \sum_{\alpha} a x^{\alpha} \otimes \iota_{\alpha} b,$$

where $a \in SV^+$, $b \in \Lambda V^+$, and ι_α is the linear operator of degree $-|x^\alpha| - 1$ given by the formula

$$\iota_\alpha\big(sx^{\alpha_1}\cdots sx^{\alpha_q}\big) = \sum_{i=1}^{q}(-1)^{(|x^\alpha|+1)(|x^{\alpha_1}|+\cdots+|x^{\alpha_{i-1}}|+i-1)}\delta_\alpha^{\alpha_i}\,\big(sx^{\alpha_1}\cdots\widehat{sx}^{\alpha_i}\cdots sx^{\alpha_q}\big).$$

The Koszul complex is contractible, with contracting homotopy

$$\mathsf{h}(a \otimes b) = \rho^{-1} \sum_{\alpha} (-1)^{|a|(|x^\alpha|+1)} \partial_\alpha a \otimes s x^\alpha b,$$

where $\rho = p + q$ on $S^p V \otimes \Lambda_q V$. We have

$$(\partial \mathsf{h} + \mathsf{h} \partial)(a \otimes b) = \varepsilon(a) \otimes \varepsilon(b).$$

The differential $\check{\partial}$ on $B_+(SV, KV)$ equals

$$\begin{aligned} \check{\partial}[a_1|\cdots|a_k] \otimes a \otimes b = {} & \sum_{i=1}^{k-1} (-1)^{\omega_i+1} [a_1|\cdots|a_i a_{i+1}|\cdots|a_k] \otimes a \otimes b \\ & + (-1)^{\omega_k+1} [a_1|\cdots|a_{k-1}] \otimes a_k a \otimes b \\ & + \sum_{i=1}^{k} (-1)^{\omega_{i-1}} [a_1|\cdots|da_i|\cdots|a_k] \otimes a \otimes b \\ & + (-1)^{\omega_k} [a_1|\cdots|a_k] \otimes \big(da \otimes b + (-1)^{|a|} a \otimes db \big). \end{aligned}$$

The homotopy

$$\check{\mathsf{h}}[a_1|\cdots|a_k] \otimes a \otimes b = (-1)^{\omega_k+|a|} [a_1|\cdots|a_k|a - \varepsilon(a)] \otimes 1 \otimes b$$

on $B_+(SV, KV)$ satisfies

$$(\check{\partial}\check{\mathsf{h}} + \check{\mathsf{h}}\check{\partial})[a_1|\cdots|a_k] \otimes a \otimes b = [a_1|\cdots|a_k] \otimes a \otimes b - \delta_{k0} [\,] \otimes \varepsilon(a) \otimes b.$$

We have morphisms of complexes

$$(\Lambda V_+, d) \xrightarrow{\check{\mathsf{g}}} (B_+(SV, KV), \check{\partial}) \xrightarrow{\check{\mathsf{f}}} (\Lambda V_+, d)$$

given by the formulas

$$\check{\mathsf{g}} b = [\,] \otimes 1 \otimes b$$

and

$$\check{\mathsf{f}}[a_1|\cdots|a_k] \otimes a \otimes b = \delta_{k0}\, \varepsilon(a)\, b$$

such that $\check{\partial}\check{\mathsf{h}} + \check{\mathsf{h}}\check{\partial} = 1 - \check{\mathsf{g}}\check{\mathsf{f}}$ and $\check{\mathsf{f}}\check{\mathsf{g}} = 1$. It is easily checked that $\check{\mathsf{h}}^2 = 0$, $\check{\mathsf{f}}\check{\mathsf{h}} = 0$, and $\check{\mathsf{h}}\check{\mathsf{g}} = 0$; thus, we obtain a contraction

$$\check{\mathsf{h}} \circlearrowright (B_+(SV, KV), \check{\partial}) \underset{\check{\mathsf{g}}}{\overset{\check{\mathsf{f}}}{\rightleftarrows}} (\Lambda V_+, d).$$

Via the isomorphism $B_+(SV, KV) \cong B_+SV \otimes KV$, the differential ∂ on KV induces a differential ∂ on $B_+(SV, KV)$,

$$\partial[a_1|\cdots|a_k] \otimes a \otimes b = (-1)^{\omega_k+|a|} \sum_{\alpha} [a_1|\cdots|a_k] \otimes ax^\alpha \otimes \iota_\alpha b.$$

The homotopy

$$\mathsf{h}[a_1|\cdots|a_k] \otimes a \otimes b = (-1)^{\omega_k} \sum_{\alpha} (-1)^{|a|(|x^\alpha|+1)} [a_1|\cdots|a_k] \otimes \rho^{-1}(\partial_\alpha a \otimes sx^\alpha b)$$

on $B_+(SV, KV)$ satisfies

$$(\partial\mathsf{h} + \mathsf{h}\partial)[a_1|\cdots|a_k] \otimes a \otimes b = [a_1|\cdots|a_k] \otimes \big(a \otimes b - \varepsilon(a) \otimes \varepsilon(b)\big).$$

We have morphisms of complexes

$$(B_+SV, 0) \xrightarrow{\ \mathsf{g}\ } (B_+(SV, KV), \partial) \xrightarrow{\ \mathsf{f}\ } (B_+SV, 0)$$

given by the formulas $\mathsf{g}[a_1|\cdots|a_k] = [a_1|\cdots|a_k] \otimes 1 \otimes 1$ and

$$\mathsf{f}[a_1|\cdots|a_k] \otimes a \otimes b = \varepsilon(a)\varepsilon(b)[a_1|\cdots|a_k]$$

such that $\partial\mathsf{h} + \mathsf{h}\partial = 1 - \mathsf{gf}$ and $\mathsf{fg} = 1$. It is easily checked that $\mathsf{h}^2 = 0$, $\mathsf{fh} = 0$, and $\mathsf{hg} = 0$, and we obtain a contraction

$$\mathsf{h} \circlearrowright (B_+(SV, KV), \partial) \underset{\mathsf{g}}{\overset{f}{\rightleftarrows}} (B_+SV, 0).$$

By the formula $\check{\partial}\partial + \partial\check{\partial} = 0$, we may equally well think of ∂ as a perturbation of the differential $\check{\partial}$ on $B(SV, KV)$, or of $\check{\partial}$ as a perturbation of the differential ∂. Homological perturbation theory applies to both perturbations, since both $1 + \partial\check{\mathsf{h}}$ and $1 + \check{\partial}\mathsf{h}$ are invertible, and we obtain a pair of contractions

$$\check{\mathsf{h}}_* \circlearrowright (B_+(SV, KV), \check{\partial} + \partial) \underset{\check{\mathsf{g}}_*}{\overset{\check{\mathsf{f}}_*}{\rightleftarrows}} (\Lambda V_+, d)$$

$$\mathsf{h}_* \circlearrowright (B_+(SV, KV), \partial + \check{\partial}) \underset{\mathsf{g}_*}{\overset{\mathsf{f}_*}{\rightleftarrows}} (B_+SV, \delta)$$

given by the formulas

$$\check{\mathsf{h}}_* = \check{\mathsf{h}}(1 + \partial\check{\mathsf{h}})^{-1}, \qquad \check{\mathsf{f}}_* = \check{\mathsf{f}} = \check{\mathsf{f}}(1 + \partial\check{\mathsf{h}})^{-1}, \qquad \check{\mathsf{g}}_* = (1 + \check{\mathsf{h}}\partial)^{-1}\check{\mathsf{g}},$$

$$\mathsf{h}_* = \mathsf{h}(1 + \check{\partial}\mathsf{h})^{-1}, \qquad \mathsf{f}_* = \mathsf{f} = \mathsf{f}(1 + \check{\partial}\mathsf{h})^{-1}, \qquad \mathsf{g}_* = (1 + \mathsf{h}\check{\partial})^{-1}\mathsf{g}.$$

The formula $\check{\mathsf{f}}\partial = 0$ shows that $\check{\mathsf{f}}_* = \check{\mathsf{f}}$ and also that $\check{\mathsf{f}}\partial(1 + \check{\mathsf{h}}\partial)^{-1}\check{\mathsf{g}} = 0$, so the differential on ΛV_+ is unperturbed. Similarly, the formula $\mathsf{f}\check{\partial}\mathsf{h} = 0$ shows that $\mathsf{f}\check{\partial}(1 + \mathsf{h}\check{\partial})^{-1}\mathsf{g}$ equals the differential δ on B_+SV, and $\mathsf{f}_* = \mathsf{f}$.

The following result is due to Dippell et al. [3]. (See also Meinrenken and Salazar [13].) Let $f_+ : B_+SV \to \Lambda V_+$ and $g_+ : \Lambda V_+ \to B_+SV$ be the counital versions of the morphisms f and g from Section 4.

Proposition 6. *There is a weak contraction*

$$H = \mathsf{f}_*\check{\mathsf{h}}_*\mathsf{g}_* \circlearrowleft (B_+SV, \delta) \underset{g_+}{\overset{f_+}{\rightleftarrows}} (\Lambda V_+, d).$$

Proof. By straightforward calculation, we see that $f_+ = \check{\mathsf{f}}_*\mathsf{g}_* = \check{\mathsf{f}}(1 + \mathsf{h}\check{\partial})^{-1}\mathsf{g}$ and $g_+ = \mathsf{f}_*\check{\mathsf{g}}_* = \mathsf{f}(1 + \check{\mathsf{h}}\partial)^{-1}\check{\mathsf{g}}$. We have

$$\begin{aligned} \delta H &= \delta\big(\mathsf{f}_*\check{\mathsf{h}}_*\mathsf{g}_*\big) = \mathsf{f}_*(\check{\partial} + \partial)\check{\mathsf{h}}_*\mathsf{g}_* \\ &= \mathsf{f}_*(1 - \check{\mathsf{g}}_*\check{\mathsf{f}}_*)\mathsf{g}_* - \mathsf{f}_*\check{\mathsf{h}}_*(\check{\partial} + \partial)\mathsf{g}_* = 1 - g_+ f_+ - H\delta, \end{aligned}$$

showing that H gives a weak contraction. ■

We close the appendix with an explicit formula for the homotopy H. Using this formula, it may be proved that $Hg_+ = 0$, $f_+H = 0$, and $H^2 = 0$, and hence that k defines a contraction from B_+SV to ΛV_+. Since $\mathsf{h}^2 = 0$ and $\mathsf{hg} = 0$, we have

$$\mathsf{h}\check{\partial}(-\mathsf{h}\check{\partial})^{k-j}\mathsf{g} = [\mathsf{h}, \check{\partial}](\mathsf{h}\check{\partial})^{k-j}\mathsf{g}.$$

It is easily checked that

$$\begin{aligned} &[\mathsf{h}, \check{\partial}][a_1|\cdots|a_k] \otimes a \otimes b \\ &\quad = \sum_\alpha (-1)^{\omega_{k-1}+(|x^\alpha|+1)(|a_k|+|a|)} [a_1|\cdots|a_{k-1}] \otimes \rho^{-1}\big((\partial_\alpha a_k)a \otimes sx^\alpha b\big). \end{aligned}$$

It follows that

$$\begin{aligned} &(-\mathsf{h}\check{\partial})^{k-j}\mathsf{g}[a_1|\cdots|a_k] \\ &\quad = (-1)^{\omega_k-\omega_j} \sum_{\alpha_{j+1},\dots,\alpha_k} \prod_{j<p\le q\le k} \frac{(-1)^{(|x^{\alpha_p}|+1)|\partial_{\alpha_q}a_q|}}{\|a_q\|} \\ &\qquad \times [a_1|\cdots|a_j] \otimes (\partial_{\alpha_{j+1}}a_{j+1})\cdots(\partial_{\alpha_k}a_k) \otimes sx^{\alpha_{j+1}}\cdots sx^{\alpha_k}. \end{aligned}$$

Let $\operatorname{sgn}(\pi)$ be the sign (1.1) determined by the Koszul sign convention for the action of the symmetric group on $\Lambda_{k-j}V$. We conclude that the homotopy k is given by the explicit formula

$$
\begin{aligned}
H &= \sum_{j=0}^{k} \mathsf{f}\check{\mathsf{h}}(\partial\check{\mathsf{h}})^{k-j}(\mathsf{h}\check{\partial})^{k-j}\mathsf{g} \\
&= \sum_{j=0}^{k}(-1)^{\omega_j} \sum_{\alpha_{j+1},\dots,\alpha_k} \prod_{j<p\le q\le k} \frac{(-1)^{(|x^{\alpha_p}|+1)|\partial_{\alpha_q}a_q|}}{\|a_q\|} \\
&\quad \times \sum_{\pi\in S_{k-j}} \operatorname{sgn}(\pi)[a_1|\cdots|a_j|A_j|x^{\alpha_{j+\pi(1)}}|\cdots|x^{\alpha_{j+\pi(k-j)}}],
\end{aligned}
$$

where $A_j = (\partial_{\alpha_{j+1}}a_{j+1})\cdots(\partial_{\alpha_k}a_k) - \varepsilon((\partial_{\alpha_{j+1}}a_{j+1})\cdots(\partial_{\alpha_k}a_k))$.

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