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Obstructions to almost complex structures following Massey

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Abstract. We provide proofs of two theorems stated by Massey in 1961, concerning the obstructions to finding complex structures on real vector bundles. In addition, we determine the second obstruction to a complex structure on a rank six orientable real vector bundle. The obstructions are fractional parts of integral Stiefel–Whitney classes, and a fourth of an appropriate combination of Pontryagin, Chern, and Euler classes.

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1. Introduction

In a 1961 paper, Massey [15] studied the following question: When does a real vector bundle admit the structure of a complex vector bundle? In particular, when does a manifold admit an almost complex structure?

This problem has a long and rich history. As a starting point, one observes that on a complex vector bundle, certain relations must hold on the level of characteristic classes. First of all, since the Chern classes c_i reduce modulo two to the even-indexed Stiefel–Whitney classes w_{2i}, one sees that the odd-indexed integral Stiefel–Whitney classes W_{2i+1} must vanish, being the integral Bockstein of w_{2i}, and in particular, w_{2i+1} must vanish. Furthermore, the Pontryagin classes are related to the Chern classes by the formula

$$1 - p_1 + p_2 - \cdots = (1 - c_1 + c_2 - \cdots)(1 + c_1 + c_2 + \cdots),$$

yielding

$$(-1)^k p_k = \sum_{i+j=2k} (-1)^i c_i c_j.$$

In addition, the top-degree Chern class c_n on a complex rank n vector bundle coincides with its Euler class.

With only this in hand, one can already show that spheres S^{4k} in dimensions divisible by four admit no almost complex structure: being stably parallelizable, the Pontryagin classes vanish, and hence, so do the Chern classes, which would imply, in particular, that the Euler characteristic vanishes. To additionally rule out almost complex structures on spheres of dimension $4k+2$, for $k \geq 2$, Borel–Serre employed the cup product and Steenrod algebra structure on the cohomology of the classifying space BU [3, Section IV.15]. Alternatively, one can reach the same conclusion using the integrality of the Chern character from K-theory to rational cohomology for spheres, due to Bott periodicity. Further, K-theoretic computation shows that a product of even-dimensional spheres is almost complex exactly when it is a product of copies of $S^2, S^6, S^2 \times S^4$ [21, Section 3]. Computations involving the Chern character were also used by Massey [16] to prove that quaternionic projective spaces $\mathbb{HP}^n$, for $n \geq 2$, are not almost complex. In low dimensions, we have a complete understanding of when a given orientable manifold admits an almost complex structure. For example, in dimension 6, the sole condition is that $W_3 = 0$ [7, Section 6]. The criterion in dimension 4 is a theorem of Wu [26, Théorème IV.10].

In many cases, what is preventing the existence of an almost complex structure is the inability to select classes c_i which would be the Chern classes of the putative almost complex structure, and hence, satisfy the above relations. To systematically investigate this observation, one uses obstruction theory, whereby one identifies sufficient, not just necessary, conditions to finding an almost complex structure. By the time of Massey's paper [15], it was understood how to relate the obstructions arising in the problem of lifting through the map $BU(n) \to BSO(2n)$ to the cohomology classes classifying the stages of its Moore–Postnikov system. This approach was developed by Hermann [10, 11], whose technique Massey employed [15, Section 3]. Finding a section of the $SO(2n)/U(n)$ bundle over a certain skeleton of the CW complex is tantamount to lifting the classifying map of the tangent bundle to a certain stage of the Moore–Postnikov system of the map $BU(n) \to BSO(2n)$.

Massey, thus, observed that even though the existence of classes c_i, reducing mod 2 to w_{2i}, such that $1 - p_1 + p_2 - \cdots = (1 - c_1 + c_2 - \cdots)(1 + c_1 + c_2 + \cdots)$, is a necessary condition for having an almost complex structure, the actual obstructions in the relevant degrees are generally classes such that the vanishing of some nontrivial multiple of them corresponds to the identities $W_{2i+1} = 0$ and $(-1)^k p_k = \sum_{i+j=2k}(-1)^i c_i c_j$. More precisely, we have the following theorem.

Theorem ([15, Theorems I and II]). Let $\xi \to X$ be an orientable real vector bundle of rank $2n$, whose structure group we assume has been reduced to $SO(2n)$, over a CW complex X. Consider the associated $SO(2n)/U(n)$ bundle θ, sections of which correspond to reductions of the structure group of ξ to $U(n)$. If s is a section of θ

over the q-skeleton of X, then the obstruction to extending over the $(q+1)$-skeleton is a cohomology class $\mathfrak{o}_{q+1} \in H^{q+1}(X; \pi_q SO(2n)/U(n))$. We have the following statements.

(1) If $q = 4k+2$ and $q < 2n-1$, then

$$W_{4k+3}(\xi) = \begin{cases} (2k)!\mathfrak{o}_{4k+3}, & k \text{ even}, \\ \frac{1}{2}(2k)!\mathfrak{o}_{4k+3}, & k \text{ odd}. \end{cases}$$

(2) Assume $n = 2k$ and $q = 2n-1 = 4k-1$. The obstruction $\mathfrak{o}_{2n} = \mathfrak{o}_{4k}$ is an integral class if k is odd, while if k is even, it is a pair consisting of an integral class and a mod 2 class. If k is odd, then

$$\sum_{i+j=2k} (-1)^i c_i c_j - (-1)^k p_k = 4\mathfrak{o}_{2n},$$

while for k even, this same formula holds with $\mathfrak{o}_{2n}$ replaced by its integral component. In the formula, $c_0, \ldots, c_{n-1}$ are the Chern classes of the induced $U(n)$ bundle over the $(2n-1)$-skeleton of X, while c_n denotes the Euler class of ξ.

In full generality, this does not address all the obstructions to finding an almost complex structure, and it is unclear whether the remaining obstructions are easily expressed in terms of characteristic classes.

Given that all the odd-indexed integral Stiefel–Whitney classes vanish on an almost complex manifold, one might wonder whether the classes W_{4k+1} appear in certain obstructions. However, as we will see in Proposition 1, their vanishing is ensured by the vanishing of the classes W_{4k+3}.

As an example of statement (1) above, the obstruction to extending an almost complex structure on the ten-skeleton of a twelve-manifold over its eleven-skeleton is not W_{11}, but rather a class $\mathfrak{o}_{11}$ satisfying $W_{11} = 24\mathfrak{o}_{11}$. Likewise, as an example of statement (2), four times the obstruction to extending an almost complex structure on the three-skeleton of a four-manifold over the four-skeleton (i.e., the whole manifold) is $p_1 - c_1^2 + 2e$, where e is the Euler class. Of course, on an orientable four-manifold, this class vanishes if and only if $p_1 - c_1^2 + 2e$ vanishes, which is part of the aforementioned theorem of Wu, but on a general CW complex this need not be the case.

Though Massey included remarks in [15] on the ingredients of the proofs of the above results, the proofs are not provided, and we take this opportunity to do so. Massey also addressed a particular obstruction not covered by the above, namely, the degree eight obstruction to extending an almost complex structure on the seven-skeleton over the eight-skeleton, in the case of $2n \geq 10$ [15, Theorem III]. This mod 2 obstruction to a stable almost complex structure, in the case of closed manifolds, was later studied in detail by Thomas [24, Theorem 1.2] and Heaps [9, Section 1], and we will touch upon it only briefly in the last section.

It is wholly possible that some of the arguments we write out here for proving [15, Theorems I and II] were not the ones Massey had in mind, particularly since we do not make use of any of the general statements (a)–(f) in [15, Section 3] listed as ingredients of the proofs of [15, Theorems I–III], though perhaps these were mostly intended for the proof of [15, Theorem III]. Our method, which only employs technology known at the time of [15], is inspired by an argument of Kervaire used in [14, Lemma 1.1], which is mentioned in [15, Remark 3].

2. Moore–Postnikov systems

We begin with some general topological preliminaries that we will make use of heavily, namely, Postnikov systems of spaces, and their relative versions which we refer to as Moore–Postnikov systems.

2.1. The Postnikov system of a space. For the content of this subsection, we refer the reader to, e.g., [25, Chapter IX]. Given a simply connected CW complex F, there is a sequence of spaces F_n, $n \geq 2$, maps $F \xrightarrow{g_n} F_n$, and maps $F_n \xrightarrow{f_n} F_{n-1}$, as in Figure 1, such that

- $\pi_{>n} F_n = 0$,
- the map $F \xrightarrow{g_n} F_n$ induces an isomorphism on $\pi_{\leq n}$,
- the map $F_n \xrightarrow{f_n} F_{n-1}$ is a principal fibration with fiber $K(\pi_n F, n)$, classified by a map $F_{n-1} \to K(\pi_n F, n+1)$. The corresponding element in $H^{n+1}(F_{n-1}; \pi_n)$ is called the k-invariant in degree $n+1$,
- $f_n \circ g_n$ is homotopic to g_{n-1}.

This data is referred to as a *Postnikov system* for the space F. We refer to $F \xrightarrow{g_n} F_n$ as the nth Postnikov map of F. Denoting the homotopy fiber of $F \xrightarrow{g_n} F_n$ by F_n', we have $\pi_{\leq n} F_n' = 0$ and the map $F_n' \to F$ induces an isomorphism on $\pi_{>n}$. A map $X \to F_{n-1}$ lifts through f_n if and only if the pullback of the k-invariant in $H^{n+1}(F_{n-1}; \pi_n)$ to $H^{n+1}(X; \pi_n)$ is the zero class.

2.2. The Moore–Postnikov system of a map. For the content of this subsection, we refer the reader to [11], in particular, Section 7, and to [17, Chapter II.8], [25, Chapter IX.6], and [1, Chapter 5.3].

$$\begin{tikzcd} & F \arrow[d, "g_n"] \\ & F_n \arrow[d, "f_n"] \\ & \vdots \arrow[d, "f_5"] \\ K(\pi_4 F, 4) \arrow[r] & F_4 \arrow[d, "f_4"] \\ K(\pi_3 F, 3) \arrow[r] & F_3 \arrow[d, "f_3"] \\ & F_2 = K(\pi_2 F, 2) \end{tikzcd}$$

FIGURE 1
The Postnikov system of a space.

Consider a fibration consisting of simply connected CW complexes $F \to E \xrightarrow{p} B$. Define $E_1 = B$. There are spaces E_n, $n \geq 2$, and fibrations $F_n \to E_n \xrightarrow{p_n} B$, $F'_n \to E \xrightarrow{h_n} E_n$, and $K(\pi_n F, n) \to E_n \xrightarrow{e_n} E_{n-1}$ such that

- the diagram

$$\begin{tikzcd} E \arrow[r, "h_n"] \arrow[d, "p"] & E_n \arrow[d, "p_n"] \\ B \arrow[r, "\mathrm{id}"] & B \end{tikzcd} \tag{2.1}$$

  commutes (up to homotopy), and the induced map of fibers $F \to F_n$ is an nth Postnikov map for F as described in Section 2.1. In particular, the Moore–Postnikov system of a constant map to a point induces a Postnikov system for its domain.
- $E_n \xrightarrow{e_n} E_{n-1}$ is a principal $K(\pi_n F, n)$ fibration such that the diagram

$$\begin{tikzcd} F_n \arrow[r, "f_n"] \arrow[d] & F_{n-1} \arrow[d] \\ E_n \arrow[r, "e_n"] \arrow[d, "p_n"] & E_{n-1} \arrow[d, "p_{n-1}"] \\ B \arrow[r, "\mathrm{id}"] & B \end{tikzcd}$$

is homotopy commutative, where f_n is the induced map of homotopy fibers. Note that if $\pi_n F = 0$, then $E_n \xrightarrow{e_n} E_{n-1}$ is a homotopy equivalence.

We can complete diagram (2.1) to a three-by-three diagram with the rows and columns being fibrations, see, e.g., [19, Section 3.2], and from the long exact sequences in homotopy groups, we see that the induced map from the homotopy fiber of g_n (as in Section 2.1) to the homotopy fiber F_n' of h_n is a homotopy equivalence, justifying the notation F_n' for the homotopy fiber of h_n,

$$\begin{tikzcd}
F_n' \arrow[r] \arrow[d, "\sim"] & F \arrow[r, "g_n"] \arrow[d] & F_n \arrow[d] \\
F_n' \arrow[r] \arrow[d] & E \arrow[r, "h_n"] \arrow[d, "p"] & E_n \arrow[d, "p_n"] \\
* \arrow[r] & B \arrow[r, "\mathrm{id}"] & B
\end{tikzcd}$$

Recall that the homotopy groups of F_n' are trivial in degrees $\leq n$, while they are isomorphic to those of F in degrees $> n$. In particular, the map $E \xrightarrow{h_n} E_n$ induces an isomorphism on $\pi_{\leq n}$ and a surjection on π_{n+1}, so h_n^* is an isomorphism on cohomology in degrees $\leq n$ and injective in degree $n+1$.

The data of the fibrations $F_n \to E_n \xrightarrow{p_n} B$, $F_n' \to E \xrightarrow{h_n} E_n$, and $K(\pi_n F, n) \to E_n \xrightarrow{e_n} E_{n-1}$ satisfying the above conditions is called a *Moore–Postnikov system* for $F \to E \xrightarrow{p} B$. The principal $K(\pi_n F, n)$ fibration $E_n \xrightarrow{e_n} E_{n-1}$ is classified by a map $E_{n-1} \to K(\pi_n F, n+1)$, with the corresponding cohomology class κ of E_{n-1} referred to as the k-invariant in degree $n+1$ of the Moore–Postnikov system. The k-invariant pulls back to that of the Postnikov system of F, i.e., the diagram

$$\begin{tikzcd}
F_n \arrow[r] \arrow[d, "f_n"] & E_n \arrow[d, "e_n"] \\
F_{n-1} \arrow[r] \arrow[d] & E_{n-1} \arrow[d] \\
K(\pi_n F, n+1) \arrow[r] & K(\pi_n F, n+1)
\end{tikzcd}$$

is homotopy commutative.

Since E_n is the homotopy fiber of the map $E_{n-1} \to K(\pi_n F, n+1)$, we have $e_n^*(\kappa) = 0$. In fact, from the Serre long exact sequence, we have that κ generates the kernel of $H^{n+1}(E_{n-1}; \pi_n F) \xrightarrow{e_n^*} H^{n+1}(E_n; \pi_n F)$.

2.3. Naturality of the Moore–Postnikov system. Naturality of the ordinary Postnikov system (i.e., the Moore–Postnikov system of the map to a point) is discussed in detail in [13]. For an explicit treatment of naturality of the relative case, we point the reader to [1, (5.3.5)]; see also [11, Section 3].

Suppose that we have a fibration $F^1 \to E^1 \xrightarrow{p_1} B^1$ mapping to another fibration $F^2 \to E^2 \xrightarrow{p_2} B^2$, i.e., a homotopy commutative diagram

$$\begin{tikzcd} F^1 \arrow[r] \arrow[d] & F^2 \arrow[d] \\ E^1 \arrow[r] \arrow[d, "p_1"] & E^2 \arrow[d, "p_2"] \\ B^1 \arrow[r] & B^2 \end{tikzcd}$$

Then, we can find Moore–Postnikov systems $\{E_n^1\}$ for $E^1 \to B^1$ and $\{E_n^2\}$ for $E^2 \to B^2$ and a homotopy commutative diagram as in Figure 2, where the columns in the back are the induced Postnikov systems for F^1 and F^2.

3. More preliminaries and a proof of Ehresmann's theorem

Let X be a CW complex, and let ξ be an oriented real rank $2n$ vector bundle over X, whose structure group we have reduced to $SO(2n)$ without loss of generality. Let $X \xrightarrow{f} BSO(2n)$ be the homotopy class of maps classifying this bundle up to isomorphism. The problem of determining whether ξ admits an almost complex structure is equivalent to the problem of lifting f through the natural map $BU(n) \to BSO(2n)$.

$$\begin{tikzcd} & BU(n) \arrow[d] \\ X \arrow[ur, dashed] \arrow[r, "f"] & BSO(2n) \end{tikzcd}$$

This is an obstruction-theoretic problem, which we address by decomposing the map $BU(n) \to BSO(2n)$ into a Moore–Postnikov system and determining the obstructions to lifting stage-by-stage.

3.1. Some relevant homotopy groups. We list here the homotopy groups of the space $SO(2n)/U(n)$ relevant for our lifting problem. First of all, since $U(n+1)$ acts transitively on S^{2n+1} with isotropy $U(n)$, we have the fiber bundle $U(n) \to U(n+1) \to S^{2n+1}$, giving us a fibration $S^{2n+1} \to BU(n) \to BU(n+1)$. From here one sees that the natural map $BU(n) \to BU(n+1)$ induces an isomorphism on $\pi_{\le 2n}$. Hence, the homotopy groups $\pi_{\le 2n} BU(n)$ are stable, and in particular, isomorphic,

$$\begin{tikzcd}[column sep=small]
F^1 \arrow[rrr] \arrow[d] \arrow[dr] & & & F^2 \arrow[d] \arrow[dr] & & \\
\vdots \arrow[dd] & E^1 \arrow[rrr] \arrow[d] & & \vdots \arrow[dd] & E^2 \arrow[d] & \\
& \vdots \arrow[dd] & & & \vdots \arrow[dd] & \\
F^1_3 \arrow[rrr] \arrow[dr] \arrow[ddd] & & & F^2_3 \arrow[dr] \arrow[ddd] & & \\
& E^1_3 \arrow[rrr] \arrow[dr] \arrow[ddd] & & & E^2_3 \arrow[dr] \arrow[ddd] & \\
& & K(\pi_4 F^1, 5) \arrow[rrr] & & & K(\pi_4 F^2, 5) \\
F^1_2 \arrow[rrr] \arrow[dr] & & & F^2_2 \arrow[dr] & & \\
& E^1_2 \arrow[rrr] \arrow[dr] \arrow[dd] & & & E^2_2 \arrow[dr] \arrow[dd] & \\
& & K(\pi_3 F^1, 4) \arrow[rrr] & & & K(\pi_3 F^2, 4) \\
& B^1 \arrow[rrr] \arrow[dr] & & & B^2 \arrow[dr] & \\
& & K(\pi_2 F^1, 3) \arrow[rrr] & & & K(\pi_2 F^2, 3)
\end{tikzcd}$$

FIGURE 2
Naturality of Moore–Postnikov systems.

via the map induced by the inclusion, to $\pi_{\leq 2n} BU$. The homotopy groups of BU are known by Bott's periodicity theorem: concretely, $\pi_{2k} BU \cong \mathbb{Z}$ and $\pi_{2k+1} BU = 0$ for all k. Similarly, one obtains that the groups $\pi_{\leq 2n-1} BSO(2n)$ are stable. The homotopy groups of the stable space BO are eightfold-periodic: starting from $\pi_1 BO$, they read $\mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}$. The homotopy groups of BSO differ only in π_1, where we have $\pi_1 BSO = 0$.

From the map of fibrations

$$\begin{tikzcd}
SO(2n)/U(n) \arrow[r] \arrow[d] & SO/U \arrow[d] \\
BU(n) \arrow[r] \arrow[d] & BU \arrow[d] \\
BSO(2n) \arrow[r] & BSO
\end{tikzcd}$$

and the induced map of long exact sequences in homotopy groups, we see that the groups $\pi_{\leq 2n-2}SO(2n)/U(n)$ are stable using the five lemma, i.e., they are isomorphic via the natural map to those of SO/U. A part of Bott's periodicity theorem also gives us that SO/U has the homotopy type of a connected component of the based loop space ΩO. Hence, its first several homotopy groups, starting with π_1, are given by

$$0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \dots .$$

In particular, we have

$$\pi_{4k+2}SO/U \cong \mathbb{Z}.$$

The first unstable homotopy group of $SO(2n)/U(n)$, namely, $\pi_{2n-1}SO(2n)/U(n)$, depends on the mod 4 class of n, and is given by

$$\pi_{2n-1}(SO(2n)/U(n)) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2, & n \equiv 0 \text{ mod } 4, \\ \mathbb{Z}_{(n-1)!}, & n \equiv 1 \text{ mod } 4, \\ \mathbb{Z}, & n \equiv 2 \text{ mod } 4, \\ \mathbb{Z}_{\frac{1}{2}(n-1)!}, & n \equiv 3 \text{ mod } 4. \end{cases} \tag{3.1}$$

This is summarized in [15, Lemma 1], and the reference to (then forthcoming) work of Bruno Harris therein is to [8].

3.2. The first obstruction to an almost complex structure. Since $SO(2)/U(1)$ is a point, there are no obstructions to reducing the structure group. From now on, we suppose $n \geq 2$. Then, $\pi_2 SO(2n)/U(n)$ is stable, and hence, isomorphic to $\mathbb{Z}$. Therefore, the first k-invariant in the Moore–Postnikov system for $BU(n) \to BSO(2n)$ is a cohomology class in $H^3(BSO(2n); \mathbb{Z})$.

Since $BSO(2n)$ is simply connected, by the Hurewicz theorem, we have that

$$H_2(BSO(2n); \mathbb{Z}) \cong \pi_2(BSO(2n); \mathbb{Z}) \cong \mathbb{Z}_2,$$

and that $\pi_3(BSO(2n)) = 0$ surjects onto $H_3(BSO(2n); \mathbb{Z})$, i.e., the latter group is trivial. By the universal coefficient theorem, we have $H^3(BSO(2n); \mathbb{Z}) \cong \mathbb{Z}_2$. The non-zero element in this group is the third integral Stiefel–Whitney class $W_3 = \beta w_2$. Here, and throughout, β denotes the Bockstein morphism associated to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$.

Denote by E_2 the second stage of the Moore–Postnikov system for $BU(n) \to BSO(2n)$, as in Section 2.2. Since $BU(n) \to E_2$ has two-connected fiber, it induces an injection on $H^3(-; \mathbb{Z})$. As $H^3(BU(n); \mathbb{Z}) = 0$, we have $H^3(E_2; \mathbb{Z}) = 0$. The k-invariant for the fibration $E_2 \to BSO(2n)$ generates the kernel of the induced map on $H^3(-; \mathbb{Z})$, and therefore, it is the class W_3. We note that W_3 of an orientable bundle vanishes precisely when w_2 has an integral lift, i.e., when w_2 is the mod 2 reduction of an integral cohomology class, which would be the first Chern class of the almost complex structure we would obtain if we could lift to $BU(n)$. The homotopy fiber of $BSO(2n) \xrightarrow{W_3} K(\mathbb{Z}, 3)$ can be identified with $B\mathrm{Spin}^c(2n)$, the classifying space of the Lie group $\mathrm{Spin}^c(2n)$, though we will not make use of this here.

Note that W_3 of the tangent bundle of an orientable four-manifold vanishes, see [22]. This is not true of higher-dimensional manifolds: for example, in dimension five, the Wu manifold $M := SU(3)/SO(3)$ has non-zero W_3. In particular, the six-manifold $S^1 \times M$ does not admit an almost complex structure.

3.3. The second obstruction to an almost complex structure. Supposing W_3 of the bundle is 0, we now have to distinguish three cases: $n = 2$, $n = 3$, and $n \geq 4$. If $n = 2$, then the second (and, in the case of X having dimension ≤ 4, last) obstruction to lifting to $BU(2)$ lies in $H^4(X; \pi_3 SO(4)/U(2)) \cong H^4(X; \mathbb{Z})$. Since $W_3 = 0$ we can choose a lift $X \to E_2$ of the map τ, which yields an integral lift c of w_2 on X. The second obstruction is then obtained by pulling back the k-invariant $E_2 \to K(\mathbb{Z}, 4)$ to X via this lift (see, e.g., [25, Chapter IX.7]). Different choices of lift will generally give different classes in $H^4(X; \mathbb{Z})$. A theorem of Wu [26, Théorème IV.10] tells us that, on a closed orientable four-manifold, this obstruction class vanishes if and only if $c^2 - 2e - p_1$ vanishes. More generally, for an $SO(4)$ bundle over a CW complex, $c^2 - 2e - p_1$ is four times this second obstruction; see Theorem 6.

For $n = 3$, we have $SO(6)/U(3) = \mathbb{CP}^3$, so the second obstruction lies in the group $H^8(X; \pi_7 \mathbb{CP}^3) \cong H^8(X; \mathbb{Z})$, and we will address this case later, in Proposition 10. In particular, if X is a six-dimensional CW complex, this and all further obstructions vanish, leaving W_3 as the sole obstruction, recovering an observation of Ehresmann [7, Section 6]. Let us now focus on the case $n \geq 4$, where the second obstruction to lifting to $BU(n)$ is stable, and lies in $H^7(X; \pi_6 SO(2n)/U(n)) \cong H^7(X; \mathbb{Z})$. By the obstruction being stable, we mean that it is obtained by pulling back the corresponding obstruction for lifting the stabilized real bundle $X \to BSO$ through $BU \to BSO$, where one has the commutative diagram

$$\begin{array}{ccc} SO(2n)/U(n) & \longrightarrow & SO/U \\ \downarrow & & \downarrow \\ BU(n) & \longrightarrow & BU \\ \downarrow & & \downarrow \\ BSO(2n) & \longrightarrow & BSO \end{array}$$

We will not make use of this observation, so we will not go into the details.

As with W_3, we see that since W_7 pulls back to the zero class in $BU(n)$, we must have $W_7 = 0$ for a complex vector bundle (since c_3 will be an integral lift of w_6). However, it may be the case that when pulling back W_7 to E_2, the class becomes divisible by an integer ≥ 2, and the relevant k-invariant is in fact a "fractional part" of W_7. Even though this does not happen in the case of W_7 as we will now see, later on we will observe that this phenomenon does occur in higher degrees, for $W_{11}, W_{15}, \ldots$. One might also find it curious that W_5 has been skipped; as soon as $W_3 = 0$, we have that $W_5 = 0$. Let us record a proof of this phenomenon here; the method will also become relevant again in Section 5.

Proposition 1. *Let $\xi \to X$ be an orientable real vector bundle such that $W_{4k-1} = 0$ for $k \leq m$. Then, $W_{4m+1} = 0$.*

We will use the following formula [23, Theorem C] attributed to Wu[1], involving the Pontryagin square operation $H^{2m}(X;\mathbb{Z}_2) \xrightarrow{\mathfrak{P}} H^{4m}(X;\mathbb{Z}_4)$, which for orientable bundles (i.e., $w_1 = 0$) becomes

$$\mathfrak{P}(w_{2m}) = \rho_4(p_m) + \theta_2\left(\sum_{j=0}^{m-1} w_{2j}w_{4m-2j}\right). \tag{3.2}$$

Here, p_m denotes the Pontryagin class, ρ_4 denotes mod 4 reduction, and θ_2 denotes the map on cohomology $H^{4m}(X;\mathbb{Z}_2) \to H^{4m}(X;\mathbb{Z}_4)$ induced by the inclusion $\mathbb{Z}_2 \hookrightarrow \mathbb{Z}_4$. Before proceeding with the proof of Proposition 1, we record the following lemma.

Lemma 2. *Let $\tilde{u}$ be an integral lift of a mod 2 cohomology class u. Then, $\mathfrak{P}(u) = \rho_4(\tilde{u}^2)$.*

Proof. By construction, the Pontryagin square satisfies

$$\mathfrak{P}(\rho_2^4(x)) = x^2$$

for any mod 4 class x, where ρ_2^4 denotes the mod 2 map $H^*(-;\mathbb{Z}_4) \to H^*(-;\mathbb{Z}_2)$; see, e.g., [4, (1.3)]. Note that $\rho_2^4(\rho_4(\tilde{u})) = \rho_2(\tilde{u}) = u$. Therefore,

$$\mathfrak{P}(u) = \mathfrak{P}(\rho_2^4(\rho_4(\tilde{u}))) = \rho_4(\tilde{u})^2 = \rho_4(\tilde{u}^2). \qquad \blacksquare$$

Proof of Proposition 1. Since the classes $W_{\leq 4m-1}$ vanish, the classes $w_{\leq 4m-2}$ are the mod 2 reductions of integral classes. Let us denote by c_k some fixed integral lift of w_{2k}, for $k \leq 2m-1$. Since c_m is an integral lift of w_{2m}, Lemma 2 gives us

$$\mathfrak{P}(w_{2m}) = \rho_4(c_m^2).$$

[1]In the notation of loc. cit., W_i denotes the mod 2 Stiefel–Whitney class w_i.

Now, consider the following commutative diagram of short exact sequences of abelian groups:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{mod}\ 2"] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{mod}\ 4"] & \mathbb{Z}_2 \arrow[r] \arrow[d, "="] & 0 \arrow[d] \\
0 \arrow[r] & \mathbb{Z}_2 \arrow[r] & \mathbb{Z}_4 \arrow[r] & \mathbb{Z}_2 \arrow[r] & 0
\end{tikzcd}$$

and the following commutative diagram coming from the induced map of long exact sequences in cohomology:

$$\begin{tikzcd}
H^{4m-1}(X;\mathbb{Z}_2) \arrow[r, "\beta"] \arrow[d, "="] & H^{4m}(X;\mathbb{Z}) \arrow[r, "\times 2"] \arrow[d, "\rho_2"] & H^{4m}(X;\mathbb{Z}) \arrow[r, "\rho_2"] \arrow[d, "\rho_4"] & H^{4m}(X;\mathbb{Z}_2) \arrow[d, "="] \\
H^{4m-1}(X;\mathbb{Z}_2) \arrow[r, "Sq^1"] & H^{4m}(X;\mathbb{Z}_2) \arrow[r, "\theta_2"] & H^{4m}(X;\mathbb{Z}_4) \arrow[r, "\rho_2^4"] & H^{4m}(X;\mathbb{Z}_2)
\end{tikzcd}$$

In particular, from the commutativity of the middle square, for any class x we have

$$\theta_2(\rho_2(x)) = \rho_4(2x). \tag{3.3}$$

Now, since $\sum_{j=1}^{m-1} w_{2j} w_{4m-2j} = \rho_2(\sum_{j=1}^{m-1} c_j c_{2m-j})$, we conclude that

$$\theta_2\left(\sum_{j=1}^{m-1} w_{2j} w_{4m-2j}\right) = \rho_4\left(2\sum_{j=1}^{m-1} c_j c_{2m-j}\right).$$

Recall $\mathfrak{P}(w_{2m}) = \rho_4(c_m^2)$. Since

$$\theta_2(w_{4m}) = \mathfrak{P}(w_{2m}) - \rho_4(p_m) - \theta_2\left(\sum_{j=1}^{m-1} w_{2j} w_{4m-2j}\right),$$

we, therefore, have

$$\theta_2(w_{4m}) = \rho_4\left(c_m^2 - p_m - 2\sum_{j=1}^{m-1} c_j c_{2m-j}\right).$$

Since $\rho_2^4(\theta_2(w_{4m})) = 0$, by exactness, we have

$$\rho_2\left(c_m^2 - p_m - 2\sum_{j=1}^{m-1} c_j c_{2m-j}\right) = 0,$$

using the right-most square of the diagram. Therefore, there is a class $x \in H^{4m}(X;\mathbb{Z})$ such that

$$2x = c_m^2 - p_m - 2\sum_{j=1}^{m-1} c_j c_{2m-j}.$$

Then, $\theta_2(\rho_2(x)) = \rho_4(2x) = \theta_2(w_{4m})$. Therefore, $\rho_2(x) + w_{4m} \in \ker(\theta_2) = \mathrm{im}(Sq^1)$. Hence, there is some $y \in H^{4m-1}(X; \mathbb{Z}_2)$ such that $Sq^1(y) = \rho_2(x) + w_{4m}$. Since $Sq^1 = \rho_2 \circ \beta$, we have $w_{4m} = \rho_2(x + \beta(y))$. That is, $x + \beta(y)$ is an integral lift of w_{4m}, and hence, $W_{4m+1} = 0$. ■

We return to the second obstruction to lifting through $BU(n) \to BSO(2n)$, for $n \geq 4$. We will identify it as W_7 in another way in Section 4, as part of proving the more general Theorem 4. However, here we address it directly, as Massey points out [15, Remark 1], this obstruction was known earlier, yet stated without proof, by Ehresmann [7].

Denote by p the map $E_2 \to BSO(2n)$. To show that the k-invariant $E_2 \to K(\mathbb{Z}, 7)$ is p^*W_7, we calculate the relevant cohomology of E_2 from the Serre spectral sequence associated to the fibration $K(\mathbb{Z}, 2) \to E_2 \to BSO(2n)$. For the integral cohomology of $BSO(2n)$, we point the reader to [5, Theorem 1.6] for a detailed treatment. In Figure 3, we have the relevant terms of the third page of the spectral sequence for $n \geq 5$. The only difference in the case of $n = 4$ is that the Euler class appears in degree eight. However, it does not affect our computation, and so, our conclusion will hold in this case as well.

Note that since the cohomology of the fiber is concentrated in even degrees, only the odd-indexed pages of the spectral sequence can have non-trivial differentials. On the third page, since $p^*W_3 = 0$, we must have $d_3(\alpha) = W_3$. Note that then $d_3(\alpha^2) = 2\alpha W_3 = 0$, so α^2 together with 2α survives to the fourth, and hence, fifth page. We have $d_3(\alpha^3) = 3W_3\alpha^2 = W_3\alpha^2$, $d_3(W_3\alpha) = W_3^2$, and $d_3(W_5\alpha) = W_3W_5$. Moving on to the fifth page of the spectral sequence, we have $d_5(\alpha^2) = W_5$; see Figure 4.

Indeed, first of all, since the third through fifth homotopy groups of $SO(2n)/U(n)$ are trivial, we have that $E_5 \xrightarrow{e_5} E_4 \xrightarrow{e_4} E_3 \xrightarrow{e_3} E_2$ is a composition of homotopy equivalences. Therefore, $BU(n) \to E_2 \simeq E_5$ is an isomorphism on $H^{\leq 5}(-; \mathbb{Z})$. Since the group $H^5(BU(n); \mathbb{Z})$ is trivial, we have $p^*W_5 = 0$, and hence, it must be at this page that W_5 is killed, i.e., $d_5(\alpha^2) = p^*W_5$. On the seventh page, we have $d_7(2\alpha^3) = p^*\beta(w_2w_4)$; see Figure 5.

Namely, since $p^*W_3 = p^*W_5 = 0$, the classes p^*w_2 and p^*w_4 have integral lifts in $H^*(E_2; \mathbb{Z})$. Hence, $p^*(w_2w_4)$ has an integral lift as well, so $p^*\beta(w_2w_4) = \beta(p^*(w_2w_4)) = 0$. The only possible differential that could kill $\beta(w_2w_4)$ is the one displayed in Figure 5.

We conclude that $H^7(E_2; \mathbb{Z}) = \{0, p^*W_7\}$. Since $BU(n) \to E_6$ induces an injection on $H^7(-; \mathbb{Z})$, it must be that p^*W_7 pulls back to be zero in E_6, and W_7 vanishes when pulled back to $BU(n)$. We conclude that the k-invariant $E_2 \to K(\mathbb{Z}, 7)$ is p^*W_7.

Theorem 3 (Ehresmann, [7, Section 6]). *The second obstruction to reducing the structure group of an $SO(2n)$ bundle to $U(n)$, where $n \geq 4$ is W_7.*

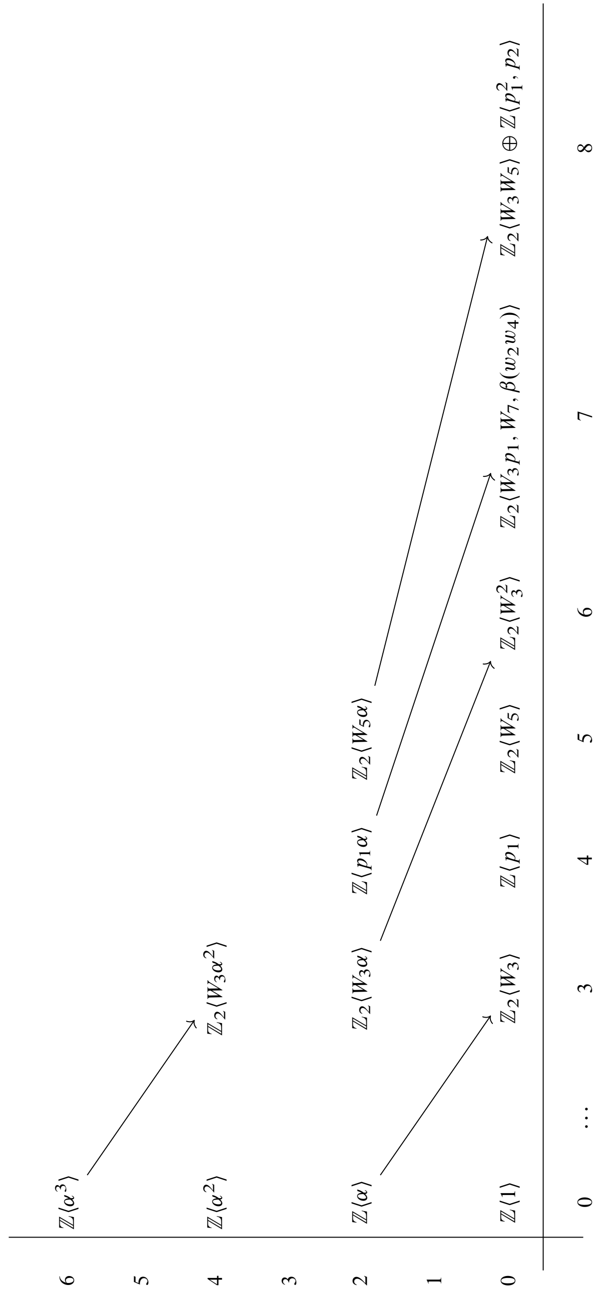

Figure 3
The third page of the spectral sequence for $K(\mathbb{Z},2) \to E_2 \to BSO(2n)$, $n \geq 5$.

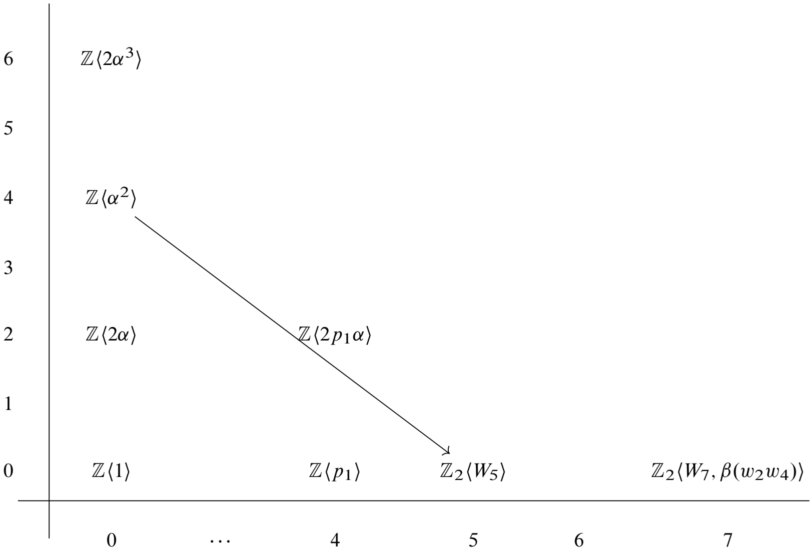

FIGURE 4
The fifth page of the spectral sequence for $K(\mathbb{Z}, 2) \to E_2 \to BSO(2n)$, $n \geq 4$.

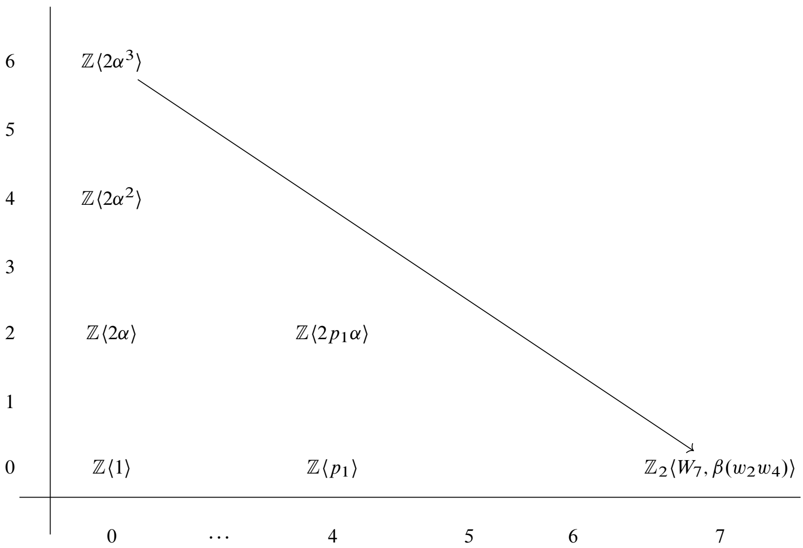

FIGURE 5
The seventh page of the spectral sequence for $K(\mathbb{Z}, 2) \to E_2 \to BSO(2n)$, $n \geq 4$.

Note that if $n = 3$, then W_7 automatically vanishes, as the Euler class is an integral lift of w_6.

4. Proof of Massey's Theorem I

We now provide the details for Massey's generalization of the above result on W_7.

Theorem 4 ([15, Theorem I]). *Let $\theta \to X$ be the $SO(2n)/U(n)$ bundle associated to an oriented rank $2n$ real vector bundle ξ over a CW complex X, and let s be a section of the bundle over the $(4k+2)$-skeleton of X. If $4k+3 < 2n$, then*

$$W_{4k+3}(\xi) = \begin{cases} (2k)!\,\mathfrak{o}_{4k+3}, & k \text{ even}, \\ \frac{1}{2}(2k)!\,\mathfrak{o}_{4k+3}, & k \text{ odd}, \end{cases}$$

where $\mathfrak{o}_{4k+3}$ denotes the obstruction[2] *to extending s over the $(4k+3)$-skeleton of X.*

Before embarking on the proof, we want to point out that the assumption $4k+3 < 2n$ will be used (in the proof of Lemma 5 (b)) to identify the homotopy groups of $SO(2n)/U(n)$ in degrees $\leq 4k+2$ with those of SO/U which are known by Bott periodicity. We want to relate the obstruction $\mathfrak{o} = \mathfrak{o}_{4k+3}$, which is the pullback of a k-invariant in the Moore–Postnikov system for $BU(n) \to BSO(2n)$, to the integral Stiefel–Whitney class W_{4k+3}. The latter naturally appears as a k-invariant in the Moore–Postnikov system for $BSO(4k+2) \to BSO(2n)$ as the primary obstruction to finding $2n-4k-2$ linearly independent sections of a rank $2n$ oriented real bundle [20, Section 38]. Now, there is no natural map between $U(n)$ and $SO(4k+2)$ in either direction, so we consider an intermediate space, the intersection of $U(n)$ and $SO(4k+2)$, namely, $U(2k+1)$. The inclusion of $U(2k+1)$ into $U(n)$ and $SO(4k+2)$ gives us the following commutative diagram:

$$\begin{tikzcd}
SO(2n)/U(n) \arrow[d] & & & & SO(2n)/SO(4k+2) \arrow[d] \\
BU(n) \arrow[d] & & SO(2n)/U(2k+1) \arrow[ull, "q"'] \arrow[urr, "r"] \arrow[d] & & BSO(4k+2) \arrow[d] \\
BSO(2n) & & BU(2k+1) \arrow[ull] \arrow[urr] \arrow[d] & & BSO(2n) \\
& & BSO(2n) \arrow[ull, "\mathrm{id}"] \arrow[urr, "\mathrm{id}"'] & &
\end{tikzcd} \tag{4.1}$$

We focus for the moment on the induced maps q and r on fibers.

[2]Note that the obstruction is only well defined up to sign, and we make a particular choice in the above statement.

Lemma 5. *We have the following statements.*

(a) *The map q induces an isomorphism on $\pi_{\leq 4k+2}$.*

(b) *π_{4k+2} of each of $SO(2n)/U(n)$, $SO(2n)/U(2k+1)$, and $SO(2n)/SO(4k+2)$ is isomorphic to $\mathbb{Z}$.*

(c) *$SO(2n)/SO(4k+2)$ is $(4k+1)$-connected, and r induces multiplication by ℓ on π_{4k+2}, up to sign, where $\ell = (2k)!$ if k is even, and $\ell = \frac{1}{2}(2k)!$ if k is odd.*

Proof. (a) We have the fiber bundle

$$U(n)/U(2k+1) \to SO(2n)/U(2k+1) \xrightarrow{q} SO(2n)/U(n).$$

The complex Stiefel manifold $U(n)/U(2k+1)$ is $(4k+2)$-connected, so q induces an isomorphism on $\pi_{\leq 4k+2}$.

(b) Since $4k+2 \leq 2n-2$, we have $\pi_{4k+2}SO(2n)/U(n) \cong \pi_{4k+2}SO/U \cong \mathbb{Z}$. From part (a), we conclude that $\pi_{4k+2}SO(2n)/U(2k+1) \cong \mathbb{Z}$. Finally, the Stiefel manifold $SO(2n)/SO(4k+2)$ is well known to be $(4k+1)$-connected with

$$\pi_{4k+2}SO(2n)/SO(4k+2) \cong \mathbb{Z}.$$

(c) We have the fiber bundle

$$SO(4k+2)/U(2k+1) \to SO(2n)/U(2k+1) \xrightarrow{r} SO(2n)/SO(4k+2) \tag{4.2}$$

and the following part of the associated long exact sequence in homotopy groups:

$$\begin{array}{c} \pi_{4k+2}SO(2n)/U(2k+1) \\ \downarrow r_* \\ \pi_{4k+2}SO(2n)/SO(4k+2) \\ \downarrow \\ \pi_{4k+1}SO(4k+2)/U(2k+1) \\ \downarrow \\ \pi_{4k+1}SO(2n)/U(2k+1) \end{array}$$

By part (b), the first two groups are isomorphic to $\mathbb{Z}$. The third group is the first unstable homotopy group of $SO(4k+2)/U(2k+1)$ listed in (3.1). Namely, it is isomorphic to $\mathbb{Z}_\ell$, where $\ell = (2k)!$ if k is even, and $\ell = \frac{1}{2}(2k)!$ if k is odd. The fourth group is, by part (a), isomorphic to $\pi_{4k+1}SO(2n)/U(n) \cong \pi_{4k+1}SO/U = 0$, as $4k+1 < 2n-2$. Therefore, the above exact sequence is given by

$$\mathbb{Z} \xrightarrow{r_*} \mathbb{Z} \to \mathbb{Z}_\ell \to 0,$$

and so, the induced map $\mathbb{Z} \xrightarrow{r_*} \mathbb{Z}$ on π_{4k+2} is, up to sign, multiplication by ℓ. ∎

Consider now the following subdiagrams of diagram (4.1):

$$\begin{tikzcd} BU(2k+1) \arrow[r] \arrow[d] & BU(n) \arrow[d] \\ BSO(2n) \arrow[r, "\mathrm{id}"] & BSO(2n) \end{tikzcd} \tag{4.3}$$

inducing the map $SO(2n)/U(2k+1) \xrightarrow{q} SO(2n)/U(n)$ on fibers, and

$$\begin{tikzcd} BU(2k+1) \arrow[r] \arrow[d] & BSO(4k+2) \arrow[d] \\ BSO(2n) \arrow[r, "\mathrm{id}"] & BSO(2n) \end{tikzcd} \tag{4.4}$$

inducing the map $SO(2n)/U(2k+1) \xrightarrow{r} SO(2n)/SO(4k+2)$ on fibers. Let

$$F^1 = SO(2n)/U(2k+1), \quad F^2 = SO(2n)/U(n), \quad F^3 = SO(2n)/SO(4k+2)$$

so that $F^1 \xrightarrow{q} F^2$ and $F^1 \xrightarrow{r} F^3$.

Induced lift and effect on the k-invariant in diagram (4.3). Consider the induced map of Moore–Postnikov systems corresponding to diagram (4.3):

$$\begin{tikzcd} F_2^1 \arrow[dr] & \vdots \arrow[d] & F_2^2 \arrow[dr] & \vdots \arrow[d] \\ & E_2^1 \arrow[rr] \arrow[d] & & E_2^2 \arrow[d] \\ & BSO(2n) \arrow[rr] & & BSO(2n) \end{tikzcd}$$

Since $F^1 \xrightarrow{q} F^2$ induces isomorphisms on $\pi_{\leq 4k+2}$ by Lemma 5 (a), from the commutative diagram

$$\begin{tikzcd} F^1 \arrow[r, "q"] \arrow[d, "g_s^1"] & F^2 \arrow[d, "g_s^2"] \\ F_s^1 \arrow[r] & F_s^2 \end{tikzcd}$$

and from $\pi_{>s} F_s^i = 0$, $i = 1, 2$ (recall Section 2.1), we see that the maps $F_s^1 \to F_s^2$ are weak homotopy equivalences for $s \leq 4k+2$. From the induced map of long exact sequences in homotopy groups associated to the diagram

$$\begin{tikzcd}
F_s^1 \arrow[r] \arrow[d] & F_s^2 \arrow[d] \\
E_s^1 \arrow[r] \arrow[d] & E_s^2 \arrow[d] \\
E_{s-1}^1 \arrow[r] & E_{s-1}^2
\end{tikzcd}$$

repeated application of the five lemma shows that the map $E_s^1 \to E_s^2$ is a weak homotopy equivalence for $s \leq 4k+2$.

By assumption, we have a lift of the map $X \to BSO(2n)$ up to E_{4k+1}^2. Choosing a homotopy inverse of $E_{4k+1}^1 \to E_{4k+1}^2$ then gives us a lift to E_{4k+1}^1 in the left-hand Moore–Postnikov system.

Our obstruction $\mathfrak{o}$ in the statement of Theorem 4 is (the pullback to X of) the k-invariant, which takes the form $E_{4k+1}^2 \to K(\mathbb{Z}, 4k+3)$ by Lemma 5 (b). Since the homotopy fiber of a k-invariant in a Moore–Postnikov system is the next stage in the system, from the induced map on long exact sequences in homotopy groups for the diagram

$$\begin{tikzcd}
E_{4k+2}^1 \arrow[rr] \arrow[d] & & E_{4k+2}^2 \arrow[d] & \\
E_{4k+1}^1 \arrow[rr] \arrow[dr] & & E_{4k+1}^2 \arrow[dr, "\mathfrak{o}"] & \\
& K(\mathbb{Z}, 4k+3) \arrow[rr] & & K(\mathbb{Z}, 4k+3)
\end{tikzcd}$$

we have that the bottom map $K(\mathbb{Z}, 4k+3) \to K(\mathbb{Z}, 4k+3)$ is a homotopy equivalence by the five lemma. Therefore, it induces an isomorphism $H^{4k+3}(X; \mathbb{Z}) \to H^{4k+3}(X; \mathbb{Z})$; let us denote the preimage of $\mathfrak{o}$ under this isomorphism simply by $\mathfrak{o}$.

Induced lift and effect on the *k*-invariant in diagram (4.4). Now, we turn to the map of Moore–Postnikov systems induced by the diagram (4.4). This time, as opposed to diagram (4.3), the induced map of homotopy fibers is not highly connected. Regardless, our chosen lift of $X \to BSO(2n)$ through E_{4k+1}^1 will induce a lift of $X \to BSO(2n)$ through E_{4k+1}^3 by post-composition. (In fact, this holds for an even more immediate reason, namely, that $E_{4k+1}^3 \to BSO(2n)$ is a homotopy equivalence. For uniformity of notation, we will put this observation on hold for now.)

$$\begin{tikzcd}
F^1_{4k+2} \arrow[rr] \arrow[dr] & & F^3_{4k+2} \arrow[dr] & & \\
& E^1_{4k+2} \arrow[rr] \arrow[d] & & E^3_{4k+2} \arrow[d] & \\
& E^1_{4k+1} \arrow[rr] \arrow[dr] \arrow[dd] & & E^3_{4k+1} \arrow[dr] \arrow[dd] & \\
& & K(\mathbb{Z}, 4k+3) \arrow[rr] & & K(\mathbb{Z}, 4k+3) \\
& \vdots \arrow[d] & & \vdots \arrow[d] & \\
& BSO(2n) \arrow[rr] & & BSO(2n) &
\end{tikzcd}$$

In the diagram

$$\begin{tikzcd}
F^1 \arrow[r, "r"] \arrow[d, "g^1_{4k+2}"] & F^3 \arrow[d, "g^3_{4k+2}"] \\
F^1_{4k+2} \arrow[r] & F^3_{4k+2}
\end{tikzcd}$$

the vertical arrows induce isomorphisms on $\pi_{\leq 4k+2}$ and r induces the map $\mathbb{Z} \xrightarrow{\times \ell} \mathbb{Z}$ on π_{4k+2} by Lemma 5 (c). Therefore, the map $F^1_{4k+2} \to F^3_{4k+2}$ induces $\mathbb{Z} \xrightarrow{\times \ell} \mathbb{Z}$ on π_{4k+2}, up to sign,

$$\begin{tikzcd}
K(\mathbb{Z}, 4k+2) \arrow[rr] \arrow[dr] \arrow[dd] & & F^1_{4k+2} \arrow[dr] \arrow[dd] & \\
& K(\mathbb{Z}, 4k+2) \arrow[rr] \arrow[dd] & & F^3_{4k+2} \arrow[dd] \\
E^1_{4k+2} \arrow[rr, "\mathrm{id}"] \arrow[dr] \arrow[dd] & & E^1_{4k+2} \arrow[dr] \arrow[dd] & \\
& E^3_{4k+2} \arrow[rr, "\mathrm{id}"] \arrow[dd] & & E^3_{4k+2} \arrow[dd] \\
E^1_{4k+1} \arrow[rr] \arrow[dr] & & BSO(2n) \arrow[dr, "\mathrm{id}"] & \\
& E^3_{4k+1} \arrow[rr] & & BSO(2n)
\end{tikzcd} \tag{4.5}$$

Since the fibers F^1_{4k+1} and F^3_{4k+1} of the maps $E^1_{4k+1} \to BSO(2n)$ and $E^3_{4k+1} \to BSO(2n)$ have $\pi_{>4k+1} = 0$, we see that the morphisms $E^i_{4k+1} \to BSO(2n)$, for

$i = 1, 3$ induce injections on π_{4k+2} and isomorphisms on $\pi_{\geq 4k+3}$. Now consider part of the map between long exact sequences in homotopy induced either by the back face (for $i = 1$) or the front face (for $i = 3$) of diagram (4.5):

$$\begin{array}{ccc}
\pi_{4k+3}E^i_{4k+2} & \xrightarrow{\sim} & \pi_{4k+3}E^i_{4k+2} \\
\downarrow & & \downarrow \\
\pi_{4k+3}E^i_{4k+1} & \xrightarrow{\sim} & \pi_{4k+3}BSO(2n) \\
\downarrow & & \downarrow \\
\pi_{4k+2}K(\mathbb{Z}, 4k+2) & \longrightarrow & \pi_{4k+2}F^i_{4k+2} \\
\downarrow & & \downarrow \\
\pi_{4k+2}E^i_{4k+2} & \xrightarrow{\sim} & \pi_{4k+2}E^i_{4k+2} \\
\downarrow & & \downarrow \\
\pi_{4k+2}E^i_{4k+1} & \hookrightarrow & \pi_{4k+2}BSO(2n)
\end{array}$$

From the five lemma, we conclude that the induced maps $\pi_{4k+2}K(\mathbb{Z}, 4k+2) \to \pi_{4k+2}F^i_{4k+2}$, for $i = 1, 3$, are isomorphisms $\mathbb{Z} \to \mathbb{Z}$. From the top face of diagram (4.5), we conclude that the induced map

$$\pi_{4k+2}K(\mathbb{Z}, 4k+2) \to \pi_{4k+2}K(\mathbb{Z}, 4k+2)$$

is the map $\mathbb{Z} \xrightarrow{\times \ell} \mathbb{Z}$ given by multiplication by ℓ, up to sign.

Now, recall that the fiber of the map $BSO(4k+2) \to BSO(2n)$, namely, the Stiefel manifold $SO(2n)/SO(4k+2)$, is $(4k+1)$-connected, and

$$\pi_{4k+2}SO(2n)/SO(4k+2) \cong \mathbb{Z}.$$

Therefore, $E^3_{4k+1} \to BSO(2n)$ is a homotopy equivalence, and since $BSO(4k+2) \to E^3_{4k+2}$ induces an injection on $H^{4k+3}(-;\mathbb{Z})$, we see that the k-invariant $E^3_{4k+1} \to K(\mathbb{Z}, 4k+3)$ is the integral Stiefel–Whitney class W_{4k+3}. Recall, from the Serre long exact sequence for the fibration

$$E^3_{4k+2} \xrightarrow{e_{4k+2}} E^3_{4k+1} \xrightarrow{\kappa} K(\mathbb{Z}, 4k+3),$$

we have that the kernel of e^*_{4k+2} in degree $4k+3$ consists of multiples of the k-invariant κ. Therefore, $W_{4k+3} = m\kappa$ for some integer m. Since W_{4k+3} has non-zero mod 2 reduction, namely, w_{4k+3}, we see that m is odd. Now, note that $m(\kappa - W_{4k+3}) = 0$ since W_{4k+3} is two-torsion. As the integral cohomology of $BSO(2n)$ contains no odd torsion [5], we see that $\kappa = W_{4k+3}$.

Consider now at last the map of fibrations

$$\begin{tikzcd}
K(\mathbb{Z}, 4k+2) \arrow[r] \arrow[d] & K(\mathbb{Z}, 4k+2) \arrow[d] \\
E^1_{4k+2} \arrow[r] \arrow[d] & E^3_{4k+2} \arrow[d] \\
E^1_{4k+1} \arrow[r, "\varphi_{4k+1}"] & E^3_{4k+1}
\end{tikzcd}$$

where φ_{4k+1} is the induced map between the stages E^1_{4k+1} and E^3_{4k+1} of the Moore–Postnikov systems. The k-invariants are given by the transgression τ of the corresponding fiber generator $\iota \in H^{4k+2}(K(\mathbb{Z}, 4k+2), 4k+2)$ in the Serre spectral sequence. Since the map $K(\mathbb{Z}, 4k+2) \to K(\mathbb{Z}, 4k+2)$ on fibers induces multiplication by $\pm\ell$ on $H^{4k+2}(-;\mathbb{Z})$, by naturality of the transgression, we have

$$\varphi^*_{4k+1} W_{4k+3} = \varphi^*_{4k+1}(\tau(\iota)) = \tau(\pm\ell\iota) = \pm\ell\tau(\iota) = \pm\ell\mathfrak{o}.$$

Recall that $\ell = (2k)!$ if k is even, and $\ell = \frac{1}{2}(2k)!$ if k is odd. Since the obstruction $\mathfrak{o}$ is only well defined up to sign, due to the non-trivial automorphism of $K(\mathbb{Z}, 4k+3)$ corresponding to multiplication by -1, this proves Theorem 4.

5. Proof of Massey's Theorem II

Suppose $\xi \to X$ is an $SO(4k)$-bundle with a lift to $BU(2k)$ over the $(4k-1)$-skeleton of X. The obstruction to extending the section over the $4k$-skeleton is the pullback of the k-invariant $E_{4k-2} \to K(\pi_{4k-1}SO(4k)/U(2k), 4k)$ in the Moore–Postnikov system of $BU(2k) \to BSO(4k)$. If k is odd, then

$$\pi_{4k-1}SO(4k)/U(2k) \cong \mathbb{Z},$$

and if k is even, then $\pi_{4k-1}SO(4k)/U(2k) \cong \mathbb{Z} \oplus \mathbb{Z}_2$.

As a warm-up, we recover the following refinement of Wu's theorem [26, Theorem IV.10], proved also in [12, Section 4.6], which is the $k = 1$ case of [15, Theorem II].

Theorem 6. *Suppose $\xi \to X$ is an $SO(4)$ bundle over a CW complex X, and that we have reduced the structure group over the three-skeleton of X to $U(2)$. The second obstruction $\mathfrak{o}$ to reducing the structure group to $U(2)$ satisfies*

$$p_1 - c_1^2 + 2e = 4\mathfrak{o}.$$

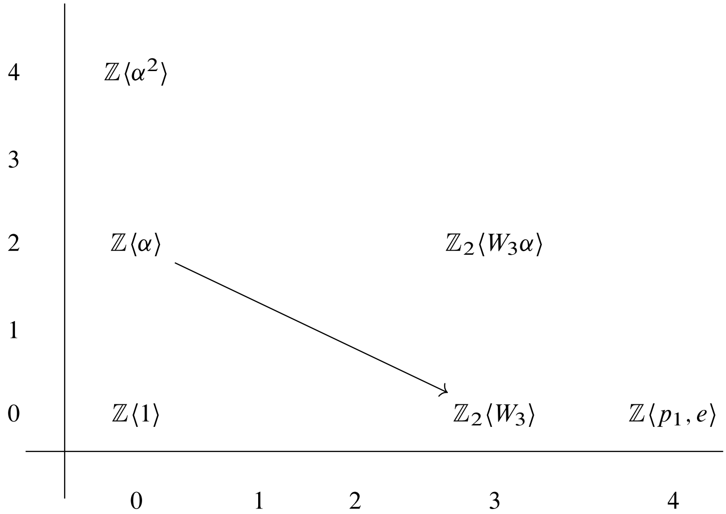

FIGURE 6
The third page of the spectral sequence for $K(\mathbb{Z},2) \to E_2 \to BSO(4)$.

Proof. Consider the beginning of the Moore–Postnikov system for the map $BU(2) \to BSO(4)$,

$$\begin{tikzcd} & BU(2) \arrow[d] & \\ & \vdots \arrow[d] & \\ & E_3 \arrow[d] & \\ K(\mathbb{Z},2) \arrow[r] & E_2 \arrow[r, "\kappa"] \arrow[d] & K(\mathbb{Z},4) \\ & BSO(4) \arrow[r, "W_3"] & K(\mathbb{Z},3) \end{tikzcd}$$

We compute $H^4(E_2;\mathbb{Z})$ using the Serre spectral sequence. (Recall $E_2 = B\mathrm{Spin}^c(4)$ for a presentation of the cohomology of $B\mathrm{Spin}^c(n)$ for all n, we refer the reader to [6].) The third page of the spectral sequence in integral cohomology associated to $K(\mathbb{Z},2) \to E_2 \to BSO(4)$ is shown in Figure 6. We have $d_3(\alpha) = W_3$, and so, $d_3(2\alpha) = 0$ and $d_3(\alpha^2) = 2\alpha W_3 = 0$.

Denote the inclusion $K(\mathbb{Z},2) \to E_2$ by i. We conclude now from the spectral sequence that there exists a unique class $\gamma \in H^2(E_2;\mathbb{Z})$ such that $i^*\gamma = 2\alpha$, and that there exists a class $\beta \in H^4(E_2;\mathbb{Z})$ such that $i^*\beta = \alpha^2$ and $\{p_1, e, \beta\}$ is a $\mathbb{Z}$-basis for $H^4(E_2;\mathbb{Z}) \cong \mathbb{Z}^3$. Now, γ^2 is a linear combination of p_1, e, and β, but since $i^*\gamma^2 = 4\alpha^2 = 4i^*\beta$ and $i^*p_1 = i^*e = 0$, we see that

$$\gamma^2 = ap_1 + be + 4\beta$$

for some integers a, b.

Since $BU(2) \xrightarrow{h_2} E_2$ induces an isomorphism on $H^2(-;\mathbb{Z})$ and $H^2(-;\mathbb{Z}_2)$, we have $h_2^*\gamma = \pm c_1$, and hence, $\rho_2(\gamma) = w_2$. The Pontryagin square formula (3.2) for $m = 1$ reads

$$\mathfrak{P}(w_2) = \rho_4(p_1) + \theta_2(w_4).$$

By Lemma 2, we have $\mathfrak{P}(w_2) = \rho_4(\gamma^2)$, and by equation (3.3), we have $\theta_2(w_4) = \rho_4(2e)$ since $\rho_2(e) = w_4$. Thus, we have

$$\rho_4(\gamma^2) = \rho_4(p_1) + \rho_4(2e).$$

Therefore, $\gamma^2 - p_1 - 2e = (a-1)p_1 + (b-2)e + 4\beta$ is divisible by four, so we have that $a-1$ and $b-2$ are divisible by four. By a change of basis preserving p_1 and e and relabeling $\beta + \frac{a-1}{4}p_1 + \frac{b-2}{4}e$ to β, we have

$$\gamma^2 = p_1 + 2e + 4\beta.$$

Now, the k-invariant κ equals $kp_1 + \ell e + m\beta$, for some integers k, ℓ, m. Pulling this class back by the map $BU(2) \xrightarrow{h_2} E_2$ must yield zero. Now, we have $4\kappa = 4kp_1 + 4\ell e + m(\gamma^2 - p_1 - 2e)$, and so,

$$0 = 4h_2^*\kappa = 4k(c_1^2 - 2c_2) + 4\ell c_2 + m(c_1^2 - (c_1^2 - 2c_2) - 2c_2) = 4kc_1^2 + (4\ell - 8k)c_2.$$

Therefore, since $\{c_1^2, c_2\}$ is a $\mathbb{Z}$-basis for $H^4(BU(2);\mathbb{Z}) \cong \mathbb{Z}^2$, it follows that $k = \ell = 0$. We conclude $\kappa = m\beta$, and so, $m\beta$ generates the kernel of the map $H^4(E_2;\mathbb{Z}) \xrightarrow{e_3^*} H^4(E_3;\mathbb{Z})$. The map $BU(2) \to E_3$ induces an injection on $H^4(-;\mathbb{Z})$. If $m = 0$, then $H^4(E_3;\mathbb{Z})$ would be isomorphic to $\mathbb{Z}^3$, which cannot inject into $H^4(BU(2);\mathbb{Z}) \cong \mathbb{Z}^2$. If $|m| > 1$, β would pull back to be a torsion class in the torsion-free group $H^4(BU(2);\mathbb{Z})$. Therefore, $\kappa = \pm\beta$. That is, the k-invariant κ satisfies (up to sign, as it is only well defined up to sign)

$$4\kappa = \gamma^2 - p_1 - 2e.$$

Pulled back to X, this becomes $4\mathfrak{o} = c_1^2 - p_1 - 2e$, as we have a lift all the way to $BU(2)$ over the three-skeleton of X, and $h_2^*\gamma = \pm c_1$. ∎

Example 7. Consider the complex projective plane $\overline{\mathbb{CP}^2}$ with its opposite orientation. Choose a generator α for $H^2(\overline{\mathbb{CP}^2};\mathbb{Z}) \cong \mathbb{Z}$. Then, $-\alpha^2$ is the oriented generator for $H^4(\overline{\mathbb{CP}^2};\mathbb{Z}) \cong \mathbb{Z}$, i.e., $\langle -\alpha^2, [\overline{\mathbb{CP}^2}]\rangle = 1$. Suppose we have a reduction of structure group for the tangent bundle to $U(2)$ over the three-skeleton. We have $p_1 = 3\alpha^2$; furthermore,

$$w_2(\overline{\mathbb{CP}^2}) = \rho_2(\alpha),$$

so $c_1 = (2m+1)\alpha$ for some integer m. The Euler class is given by $e = -3\alpha^2$. Then,

$$2e - c_1^2 + p_1 = (-6 - (2m+1)^2 + 3)\alpha^2,$$

which evaluates to $4(m^2+m+1)$ when paired with the fundamental class $[\overline{\mathbb{CP}^2}]$. This is divisible by four, but non-zero for all m. The obstruction $\mathfrak{o}$ is, up to sign, $(m^2+m+1)\alpha^2$.

By contrast, suppose we have a reduction to $U(2)$ of the structure group of the tangent bundle to $\mathbb{CP}^2$ over its three-skeleton. Then, $c_1 = (2m+1)\alpha$ as before, but $\langle \alpha^2, [\mathbb{CP}^2]\rangle = 1$ and $e = 3\alpha^2$. We have

$$\langle 2e - c_1^2 + p_1, [\mathbb{CP}^2]\rangle = 4(2 - m^2 - m).$$

Therefore, the obstruction $\mathfrak{o}$ vanishes for $m = -2$ and $m = 1$, i.e., $c_1 = \pm 3\alpha$.

Generally, for any k, consider the class $\sum_{i=0}^{2k}(-1)^i c_i c_{2k-i} - (-1)^k p_k$. Here, c_i, for $0 \leq i \leq 2k-1$, denotes the Chern classes of the $U(n)$ bundle induced over the $(4k-1)$-skeleton of X, while c_{2k} denotes the Euler class of ξ.

Theorem 8 ([15, Theorem II]). *Suppose $n = 2k$. Let $\theta \to X$ be the $SO(2n)/U(n)$ bundle associated to an oriented rank $2n$ real vector bundle ξ over X, and let s be a section of the bundle over the $(2n-1)$-skeleton of X. The obstruction $\mathfrak{o}_{2n} = \mathfrak{o}_{4k}$ is an integral class if k is odd, while if k is even, it is a pair consisting of an integral class and a mod 2 class. If k is odd, then*

$$\sum_{i+j=2k} (-1)^i c_i c_j - (-1)^k p_k = 4\mathfrak{o}_{2n},$$

while for k even, this same formula holds with $\mathfrak{o}_{2n}$ replaced by its integral component[3]. *In the formula, $c_0, \ldots, c_{n-1}$ are the Chern classes of the induced $U(n)$ bundle over the $(2n-1)$-skeleton of X, while c_n denotes the Euler class of ξ.*

[3]The integral component is only well defined once a choice of isomorphism between $\pi_{2n-1}SO(2n)/U(n)$ and $\mathbb{Z} \oplus \mathbb{Z}_2$ is made. However, up to sign as usual, $4\mathfrak{o}_{2n}$ does not depend on this choice.

Proof. Recall equation (3.2) involving the Pontryagin square:

$$\mathfrak{P}(w_{2k}) = \rho_4(p_k) + \theta_2\left(\sum_{i=0}^{k-1} w_{2i}w_{4k-2i}\right).$$

As argued in Proposition 1, we have that $\theta_2(\sum_{i=0}^{k-1} w_{2i}w_{4k-2i})$ is the mod 4 reduction of $2\sum_{i=0}^{k-1} c_i c_{2k-i}$, or equivalently, the mod 4 reduction of $\sum_{i=0}^{2k}(-1)^i c_i c_{2k-i} - (-1)^k c_k^2$.

Now, consider

$$\rho_4\left(\sum_{i=0}^{2k}(-1)^i c_i c_{2k-i} - (-1)^k p_k\right),$$

which we rewrite as

$$\rho_4\left(\sum_{i=0}^{2k}(-1)^i c_i c_{2k-i} - (-1)^k c_k^2\right) + \rho_4\big((-1)^k c_k^2 - (-1)^k p_k\big).$$

Since $\mathfrak{P}(w_{2k}) = \rho_4(c_k^2)$ by Lemma 2, using the above, this is equal to

$$\theta_2\left(\sum_{i=0}^{k-1} w_{2i}w_{4k-2i}\right) + (-1)^k\mathfrak{P}(w_{2k}) - (-1)^k\rho_4(p_k),$$

which we rewrite as

$$(-1)^k\left[\mathfrak{P}(w_{2k}) - \rho_4(p_k) + (-1)^k\theta_2\left(\sum_{i=0}^{k-1} w_{2i}w_{4k-2i}\right)\right].$$

Since θ_2 is the map on cohomology induced by the map $\mathbb{Z}_2 \to \mathbb{Z}_4$ sending [1] to [2], we have $\theta_2 = -\theta_2$. Therefore, this last expression above vanishes regardless of the parity of k.

Now, the k-invariant $E_{4k-2} \xrightarrow{\kappa} K(\pi_{4k-1}SO(4k)/U(2k), 4k)$ is either an integral cohomology class, or a pair consisting of an integral cohomology class and a mod 2 cohomology class. In either case, let $\mathfrak{o}$ denote the integral component. Consider the Serre long exact sequence in integral cohomology for the fibration $E_{4k-1} \xrightarrow{e_{4k-2}} E_{4k-2} \xrightarrow{\kappa} K(\pi_{4k-1}SO(4k)/U(2k), 4k)$, namely,

$$H^{4k}(K(\pi_{4k-1}SO(4k)/U(2k), 4k); \mathbb{Z}) \xrightarrow{\kappa^*} H^{4k}(E_{4k-2}; \mathbb{Z}) \xrightarrow{e_{4k-2}^*} H^{4k}(E_{4k-1}; \mathbb{Z}).$$

Note that $H^{4k}(K(\mathbb{Z}_2, 4k); \mathbb{Z}) = 0$, and hence, regardless of the parity of k, we have

$$H^{4k}(K(\pi_{4k-1}SO(4k)/U(2k), 4k); \mathbb{Z}) \cong H^{4k}(K(\mathbb{Z}, 4k); \mathbb{Z}) \cong \mathbb{Z},$$

generated by the class ι_{4k} generating $H^{4k}(K(\mathbb{Z}, 4k); \mathbb{Z})$. From the long exact sequence we, thus, see that the kernel of e_{4k-2}^* on $H^{4k}(-; \mathbb{Z})$ is the subgroup of $H^{4k}(E_{4k-2}; \mathbb{Z})$ generated by $\kappa^*\iota_{4k}$, which we denote by $\mathfrak{o}$ (this class pulls back to $\mathfrak{o}_{4k}$ on X).

Since the map $BU(2k) \to E_{4k-1}$ induces an injection on $H^{4k}(-; \mathbb{Z})$, and the class $\sum_{i=0}^{2k}(-1)^i c_i c_{2k-i} - (-1)^k p_k \in H^{4k}(E_{4k-2}; \mathbb{Z})$ pulls back in order to be 0 in $H^{4k}(BU(2k); \mathbb{Z})$, we have

$$\sum_{i=0}^{2k}(-1)^i c_i c_{2k-i} - (-1)^k p_k \in \ker(e_{4k-2}^*),$$

and hence, there is some integer ℓ such that

$$\sum_{i=0}^{2k}(-1)^i c_i c_{2k-i} - (-1)^k p_k = \ell\mathfrak{o}.$$

Before proceeding, we show that $\mathfrak{o}$ generates a $\mathbb{Z}$ summand in $H^{4k}(E_{4k-2}; \mathbb{Z})$. First, if there were a non-zero integer m such that $m\mathfrak{o} = 0$, then we would have

$$m\left(\sum_{i=0}^{2k}(-1)^i c_i c_{2k-i} - (-1)^k p_k\right) = 0 \in H^{4k}(E_{4k-2}; \mathbb{Z}).$$

Now, consider the classifying map for the tangent bundle of the sphere $S^{4k} \to BSO(4k)$. Since S^{4k} can be built with one 0-cell and one $4k$-cell, this map lifts to E_{4k-2}. Pulling back $\sum_{i=0}^{2k}(-1)^i c_i c_{2k-i} - (-1)^k p_k$ to S^{4k} yields twice the Euler class (p_k is trivial as the sphere is stably parallelizable), i.e., the element $4 \in H^{4k}(S^{4k}; \mathbb{Z}) \cong \mathbb{Z}$. If $m(\sum_{i=0}^{2k}(-1)^i c_i c_{2k-i} - (-1)^k p_k) = 0$, then $4m = 0$, a contradiction. Secondly, we observe that $\mathfrak{o}$ is a primitive class. Suppose that $\mathfrak{o} = m\alpha$, for some $\alpha \in H^{4k}(E_{4k-2}; \mathbb{Z})$. Then, as $\mathfrak{o}$ is non-zero by the previous argument, α would pull back to be a non-zero class of order m in $H^{4k}(BU(2k); \mathbb{Z})$. Since the integral cohomology of $BU(2k)$ is torsion-free, we conclude that $m = \pm 1$.

Since the mod 4 reduction of $\sum_{i=0}^{2k}(-1)^i c_i c_{2k-i} - (-1)^k p_k$ is zero, we conclude that

$$\sum_{i=0}^{2k}(-1)^i c_i c_{2k-i} - (-1)^k p_k = 4m\mathfrak{o}$$

for some integer m. Pulling back again to the sphere S^{4k}, we see that $4m\mathfrak{o}_{4k} = 4 \in H^{4k}(S^{4k}; \mathbb{Z})$, so $m = \pm 1$ (as $\mathfrak{o}_{4k}$ is only defined up to sign). ■

Example 9. Consider the quaternionic projective plane $\mathbb{HP}^2$, and suppose we have a reduction of structure group to $U(4)$ of its tangent bundle over its seven-skeleton. To see that such a reduction indeed exists, first note that the homotopy groups of $SO(8)/U(4)$ in degrees one through six are isomorphic to those of SO/U, which are 0, $\mathbb{Z}$, 0, 0, 0, $\mathbb{Z}$. By Theorem 4, the two obstructions we encounter are W_3 and W_7, which vanish for degree reasons.

Now, Theorem 8 tells us that the obstruction to extending this over the eight-skeleton satisfies $2e + c_2^2 - p_2 = 4\mathfrak{o}_8$. First of all, the Pontryagin classes of $\mathbb{HP}^2$ are obtained from the formula $1 + p_1 + p_2 = \frac{(1+u)^6}{1+4u}$, where u denotes a particular choice of generator for $H^4(\mathbb{HP}^2;\mathbb{Z})$; see [16, Section 2]. This yields $p_1 = 2u$ and $p_2 = 7u^2$. From the general identity $p_1 = c_1^2 - 2c_2$, since c_1 necessarily vanishes, we conclude $c_2 = -u$. We, therefore, have

$$\langle 2e + c_2^2 - p_2, [\mathbb{HP}^2]\rangle = 0,$$

and since $H^8(\mathbb{HP}^2;\mathbb{Z})$ is torsion-free, we conclude $\mathfrak{o}_8 = 0$. This same calculation holds for $\overline{\mathbb{HP}^2}$. We emphasize that $\mathbb{HP}^2$ and $\overline{\mathbb{HP}^2}$ do not admit almost complex structures, as proved in [16]. In fact, Massey shows that $\mathbb{HP}^n$ does not even admit a stable almost complex structure, for all $n \geq 2$. The point here is that the integral part of the obstruction to extending the almost complex structure over the eight-skeleton of $\mathbb{HP}^2$ vanishes, but the $\mathbb{Z}_2$ part does not. Theorem 8 does not detect the latter. For some further discussion on this, see Section 6 below.

We now address the second obstruction to reducing the structure group of an $SO(6)$ bundle to $U(3)$.

Proposition 10. *The second obstruction $\mathfrak{o}$ to reducing the structure group of an $SO(6)$ bundle to $U(3)$ satisfies $-2c_1c_3 + c_2^2 - p_2 = 4\mathfrak{o}$.*

Proof. First of all, from the well-known identification of $SO(6)/U(3)$ with $\mathbb{CP}^3$, we see that after π_2, the first non-zero homotopy group is $\pi_7 SO(6)/U(3) \cong \mathbb{Z}$.

Next, consider the fiber bundle $SO(6)/U(3) \to SO(8)/U(4) \to S^6$. Since S^6 admits an almost complex structure, this bundle has a section, and hence, the long exact sequence in homotopy groups splits. Therefore, $\pi_7 SO(6)/U(3) \cong \mathbb{Z}$ maps isomorphically onto the $\mathbb{Z}$ summand of $\pi_7 SO(8)/U(4) \cong \mathbb{Z} \oplus \mathbb{Z}_2$.

Now, we look at the map of Moore–Postnikov systems induced by the map of fibrations

$$\begin{tikzcd}
SO(6)/U(3) \arrow[r] \arrow[d] & SO(8)/U(4) \arrow[d] \\
BU(3) \arrow[r] \arrow[d] & BU(4) \arrow[d] \\
BSO(6) \arrow[r] & BSO(8)
\end{tikzcd}$$

The relevant part for us will be the following map of fibrations:

$$\begin{array}{ccc} K(\mathbb{Z},7) & \longrightarrow & K(\mathbb{Z}\oplus\mathbb{Z}_2,7) \\ \downarrow & & \downarrow \\ E_7^1 & \longrightarrow & E_7^2 \\ \downarrow & & \downarrow \\ E_6^1 & \longrightarrow & E_6^2 \end{array}$$

Arguing as in the last part of Section 4, involving diagram (4.4), we see that the map $K(\mathbb{Z},7) \to K(\mathbb{Z}\oplus\mathbb{Z}_2,7)$ induces the inclusion of the $\mathbb{Z}$ summand on π_7, as $SO(6)/U(3) \to SO(8)/U(4)$ does. Therefore, by naturality of the transgression for the Serre spectral sequence with $\mathbb{Z}$ coefficients again, and since $H^8(K(\mathbb{Z},8);\mathbb{Z}) \cong H^8(K(\mathbb{Z}\oplus\mathbb{Z}_2,8);\mathbb{Z})$ induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}\oplus\mathbb{Z}_2$ of the left summand, the k-invariant $E_6 \to K(\mathbb{Z},8)$ is the pullback of the class $\mathfrak{o}_8$ in Theorem 8. In particular, four times the k-invariant is the pullback of $2e - 2c_1c_3 + c_2^2 - p_2$. Since the Euler class e of $BSO(8)$ pulls back to zero in $BSO(6)$, we have that four times the k-invariant (i.e., four times the second obstruction $\mathfrak{o}$) is $-2c_1c_3 + c_2^2 - p_2$. ∎

Example 11. There are examples of $SO(6)$ bundles over eight-manifolds for which the first obstruction to reducing the structure group to $U(3)$ vanishes, but the second obstruction does not—note, over an eight-dimensional base, the second obstruction is also the last obstruction. For example, by [2, Lemma B.1], there exists an orientable rank 6 bundle $E \to S^8$ such that $p_2 \neq 0$. Note that the first obstruction, namely, W_3, vanishes as $H^3(S^8;\mathbb{Z}) = 0$. Due to the cohomology of S^8, the second obstruction $\mathfrak{o}$ satisfies $4\mathfrak{o} = -p_2 \neq 0$, so $\mathfrak{o} \neq 0$.

6. Remarks on Massey's Theorem III

In [15, Theorem III], the degree eight obstruction $\mathfrak{o}_8 \in H^8(-;\mathbb{Z}_2)$ to a stable almost complex structure is discussed. Concretely, suppose we have an $SO(2n)$-bundle over a CW complex X, such that $n \geq 5$, with a section s of the associated $SO(2n)/U(n)$-bundle over the seven-skeleton of X. Denoting the total space of the $SO(2n)/U(n)$-bundle by E, there are classes $u \in H^2(E;\mathbb{Z})$ and $v \in H^6(E;\mathbb{Z})$ such that $s^*u = 0$, $s^*v = 0$, and u, v restrict to generators x, y of the degree ≤ 8 cohomology of $SO(2n)/U(n)$. The obstruction $\mathfrak{o}_8$ is then claimed to be a certain class b_8 pulled back from X that appears in the expansion of $Sq^2\rho_2(v)$ in terms of the cohomology of X

and $SO(2n)/U(n)$. Namely,

$$Sq^2\rho_2(v) = p^*b_8 + (p^*b_6)\rho_2(u) + (p^*b_4)\rho_2(u)^2 \\ + (p^*b_2)\rho_2(u)^3 + (p^*b_2')\rho_2(v) + \rho_2(u)^4.$$

Such an equation holds since x and y transgress to W_3 and W_7, which are assumed to vanish, and hence, the spectral sequence collapses in sufficiently low degree.

We have the relation $Sq^2\rho_2(y) = \rho_2(x)^4$, which can be seen by combining [18, Theorem 6.11] and [18, Theorem 6.6]. It is clear that the vanishing of the obstruction $\mathfrak{o}_8$ implies the vanishing of b_8. Indeed, if the obstruction vanishes, then one can extend s over the eight-skeleton of X, in which case we can apply the pullback by this extended section to the above equation, obtaining $b_8 = 0$ since $s^*u = s^*v = 0$.

In the cases of most interest, namely, stable almost complex structures on closed eight-manifolds, this obstruction was addressed by Heaps [9] and Thomas [24]. In [9], the obstruction itself is not explicitly identified, but it is shown (making use of another theorem of Massey that $W_7 = 0$ for the tangent bundle of a closed orientable eight-manifold) that a closed orientable eight-manifold admits a stable almost complex structure if and only if $W_3 = 0$ and w_8 is in the image of $Sq^2\rho_2$ [9, Corollary 1.3]. Thomas [24, Theorem 1.2] showed that a general real orientable vector bundle ξ over a CW complex admits a stable almost complex structure if and only if $W_3(\xi) = 0$, $W_7(\xi) = 0$, and a certain secondary cohomology operation Ω vanishes on $w_8(\xi) + w_4(\xi)^2 + w_2(\xi)^2 w_4(\xi)$.

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