

The geometry of conjugation in Euclidean isometry groups

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Abstract. We describe the geometry of conjugation within any split subgroup H of the full isometry group G of n -dimensional Euclidean space. We prove that, for any $h \in H$, the conjugacy class $[h]_H$ of h is described geometrically by the move-set of its linearization, while the set of elements conjugating h to a given $h' \in [h]_H$ is described by the fix-set of the linearization of h' . Examples include all affine Coxeter groups, certain crystallographic groups, and the group G itself.

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1. Introduction

Group theory has a long history of studying conjugacy classes and the conjugation problem. It turns out that conjugacy classes in the full isometry group G of n -dimensional Euclidean space $\mathbb{E}^n$, as well as in all split subgroups H of G , have a simple and beautiful geometric description.

Theorem 1.1. *Let H be any subgroup of the full isometry group of $\mathbb{E}^n$ which admits a semi-direct product splitting into an $O(n)$ -part and a translation part. Then, the following statements hold.*

- (1) *The conjugacy class of any $h = t^\lambda w \in H$ is determined by the move-set of its linearization w . As a set, the class $[h]$ is obtained by first translating h by all elements of $\text{Mod}_H(w) \subseteq \text{Mov}(w)$ and then conjugating the so-obtained collection $t^{\text{Mod}_H(w)}h$ by all elements of $O(n) \cap H$.*
- (2) *The coconjugation set $C(h, h') = \{k \in H \mid khk^{-1} = h'\}$ for any $h, h' \in H$ has a closed-form description in terms of data involving the move-set $\text{Mov}(w')$ of the linearization w' of h' , and its shape is described by translates of the orthogonal space $\text{Fix}(w')$.*

Precise statements of Theorem 1.1 are given in Theorems 2.2 and 2.13. An important special case, which provided our original motivation, is when H is an affine Coxeter group; we give additional results in this setting in our companion paper [9].

Our point of view on Euclidean isometries is influenced by the works of McCammond and his coauthors [1, 5–7]. We were surprised to discover that our approach to conjugation seems to be new, even for the full isometry group G .

1.1. Structure of the paper. Section 2 contains the formal statements of our main results, illustrated by many examples. Section 3 concerns conjugacy classes and contains the proof of Theorem 2.2. We consider the coconjugation problem in Section 4, where we prove Theorem 2.13. An algorithm for solving the coconjugation problem, and hence the conjugacy problem, is contained in Section 5.

2. Results and examples

It is classical that the full isometry group G of $\mathbb{E}^n$ splits as a semidirect product $G = T \rtimes O(n)$, where $T \cong \mathbb{R}^n$ is the translation subgroup of G and $O(n)$ is the group of orthogonal transformations. We consider subgroups H of G which respect this splitting; that is, where $H = T_H \rtimes H_0$ for $T_H = T \cap H$ and $H_0 = H \cap O(n)$. For any such $H \leq G$ and for all $h, h' \in H$, we write

$$\begin{aligned} [h]_H &= \{khk^{-1} \mid k \in H\} \quad \text{for the conjugacy class of } h \in H, \\ C_H(h, h') &= \{k \in H \mid khk^{-1} = h'\} \quad \text{for the coconjugation set (from } h \text{ to } h'). \end{aligned}$$

In particular, $C_H(h, h)$ is the centralizer of h in H , which we also denote by $C_H(h)$.

For any $\lambda \in \mathbb{R}^n$, we write t^λ for the translation of $\mathbb{E}^n$ by the vector λ . For any split $H \leq G$, we define $L_H = \{\lambda \in \mathbb{R}^n \mid t^\lambda \in T_H\}$ and observe that L_H is naturally a $\mathbb{Z}$ -module. Then, any $h \in H$ can be expressed uniquely as $h = t^\lambda h_0$, where $\lambda \in L_H$ and $h_0 \in H_0$. We call t^λ the *translation part* and h_0 the *spherical part* of h . For any $\lambda \in L_H$ and $h_0 \in H_0$, we have $h_0 t^\lambda h_0^{-1} = t^{h_0 \lambda}$.

2.1. Conjugation. Following Brady–McCammond [1] (but identifying $\mathbb{E}^n$ with $\mathbb{R}^n$), we recall that the *move-set* and *fix-set* of any isometry $g \in G$ are the affine subspaces of $\mathbb{R}^n$ given by, respectively,

$$\text{Mov}(g) = \{y \in \mathbb{R}^n \mid gx = x + y \text{ for some } x \in \mathbb{R}^n\} = (g - \text{I})\mathbb{R}^n = \text{Im}(g - \text{I})$$

and

$$\text{Fix}(g) = \{x \in \mathbb{R}^n \mid gx = x\} = \text{Ker}(g - \text{I}).$$

For example, if $r \in G$ is a reflection, then $\text{Mov}(r)$ is the line through the origin orthogonal to the affine hyperplane $\text{Fix}(r)$. If $g_0 \in O(n)$, then $\text{Mov}(g_0)$ and $\text{Fix}(g_0)$ are both linear subspaces, and $\mathbb{R}^n$ has orthogonal decomposition $\mathbb{R}^n = \text{Mov}(g_0) \oplus \text{Fix}(g_0)$ (see [5, Remark 1.8]). For any $\lambda \in \mathbb{R}^n$ and any $g_0 \in O(n)$, by [5, Proposition 1.21], we have

$$\text{Mov}(t^\lambda g_0) = \lambda + \text{Mov}(g_0).$$

We now introduce an H -adapted version of the move-set, which we call the *mod-set*.

Definition 2.1 (Mod-set). Let $H = T_H \rtimes H_0$ be a split group of Euclidean isometries. For any $h \in H$, the *mod-set* (with respect to H) of h is defined by

$$\text{MOD}_H(h) = (h - I)L_H.$$

Note that since $L_H = -L_H$, we could equally well have defined $\text{MOD}_H(h) = (I - h)L_H$. We show in Lemma 3.2 that, for any $\lambda \in L_H$ and any $h_0 \in H_0$,

$$\text{MOD}_H(t^\lambda h_0) = \lambda + \text{MOD}_H(h_0).$$

It is immediate from definitions that $\text{MOD}_H(h) \subseteq \text{Mov}(h)$, and we prove in Lemma 3.3 that $\text{MOD}_H(h)$ is contained in the intersection $\text{Mov}(h) \cap L_H$. If $h_0 \in H_0$, then $\text{MOD}(h_0)$ is a submodule of L_H , and hence a submodule of $\text{Mov}(h_0) \cap L_H$.

Our first main result says that the mod-set is the key to describing the conjugacy class $[h]_H$.

Theorem 2.2 (Closed form of conjugacy classes). Let $H = T_H \rtimes H_0$ be a split group of Euclidean isometries. Let $h = t^\lambda h_0 \in H$, where $\lambda \in L_H$ and $h_0 \in H_0$. Then, the conjugacy class of h in H satisfies

$$(2.1.1) \quad [h]_H = \bigcup_{u \in H_0} u(t^{\text{MOD}_H(h_0)} h)u^{-1}$$

and also

$$(2.1.2) \quad [h]_H = \bigcup_{u \in H_0} t^{u(\lambda + \text{MOD}_H(h_0))} u h_0 u^{-1} = \bigcup_{u \in H_0} t^{u \text{MOD}_H(h)} u h_0 u^{-1}.$$

In words, the two equalities of Theorem 2.2 tell us that $[h]_H$ is obtained by, respectively,

- (1) first translating h by all elements of $\text{MOD}_H(h_0)$ and then conjugating the so-obtained collection $t^{\text{MOD}_H(h_0)} h$ by all elements of H_0 ; or
- (2) for each $u \in H_0$, translating the u -conjugate of the spherical part h_0 of h by the set $t^{u(\lambda + \text{MOD}_H(h_0))} = t^{u \text{MOD}_H(h)}$.

This second description in particular implies that the conjugacy class of every element $h = t^\lambda h_0 \in H$ is determined by the conjugacy class of h_0 in H_0 , together with the collection of images of mod-sets $u(\lambda + \text{MOD}_H(h_0)) = u\text{MOD}_H(h)$, as u runs through the elements of H_0 .

As an easy consequence of Theorem 2.2 and the fact that $\text{MOD}_H(h) \subseteq \text{Mov}(h) \cap L_H$, we obtain that (the translation parts of) conjugacy classes lie along move-sets.

Corollary 2.3 (Conjugacy classes and move-sets). *Let $H = T_H \rtimes H_0$ be a split group of Euclidean isometries. Let $h = t^\lambda h_0 \in H$, where $\lambda \in L_H$ and $h_0 \in H_0$. Then,*

$$(2.1.3) \quad [h]_H \subseteq \bigcup_{u \in H_0} u(t^{\text{Mov}(h_0) \cap L_H} h)u^{-1}$$

and also

$$(2.1.4) \quad [h]_H \subseteq \bigcup_{u \in H_0} t^{u(\text{Mov}(h) \cap L_H)} u h_0 u^{-1}.$$

Our second main definition is motivated by these containments.

Definition 2.4 (Filling). Let $H = T_H \rtimes H_0$ be a split group of Euclidean isometries. We say that $h \in H$ *fills its move-set*, or that *filling occurs for h* , if

$$\text{MOD}_H(h) = \text{Mov}(h) \cap L_H.$$

We prove in Proposition 3.4 that filling occurs (for both h and h_0) if and only if the containments in Corollary 2.3 are equalities.

Example 2.5. Let H be the wallpaper group **cmm**, denoted by 2^*22 in orbifold notation. Then, H is split, H_0 is the Klein four-group generated by two commuting reflections, say, s_1 and s_2 , and H is generated by s_1 , s_2 , and a 180° rotation, say, ρ , about a point not on any reflection axis. The group H induces the tessellation of $\mathbb{E}^2$ by triangles depicted in Figures 1 and 2, and L_H is the lattice of heavy dots in these figures. There is a natural bijection between the elements of H and the tiles in these tessellations, and we identify each element of H with its corresponding tile. A few tiles are labeled in Figure 1.

The conjugacy classes in H are as follows. In Figure 1, each set of tiles of the same color is a conjugacy class $[t^\lambda s_1]_H$. The mod-set $\text{MOD}_H(s_1) \subseteq L_H$ is the set of large gray dots along the horizontal axis, and the move-set $\text{Mov}(s_1)$ is this horizontal axis. The horizontal lines in the figure are the sets $\text{Mov}(t^\lambda s_1)$. If $\lambda \in \text{Mov}(s_1)$, then $\text{Mov}(t^\lambda s_1) = \text{Mov}(s_1)$ is H_0 -invariant, and the conjugacy class $[t^\lambda s_1]_H$ is the set of gray triangles along the horizontal axis. For $\lambda \notin \text{Mov}(s_1)$, the line $s_2 \text{Mov}(t^\lambda s_1) = s_2 s_1 \text{Mov}(t^\lambda s_1)$ is distinct from $\text{Mov}(t^\lambda s_1)$, and so, $[t^\lambda s_1]_H$ is a pair of horizontal

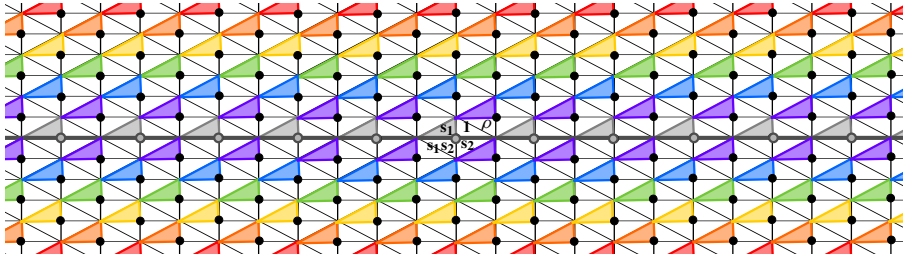


FIGURE 1
Conjugacy classes $[t^\lambda s_1]_H$ in the wallpaper group **cmm**.

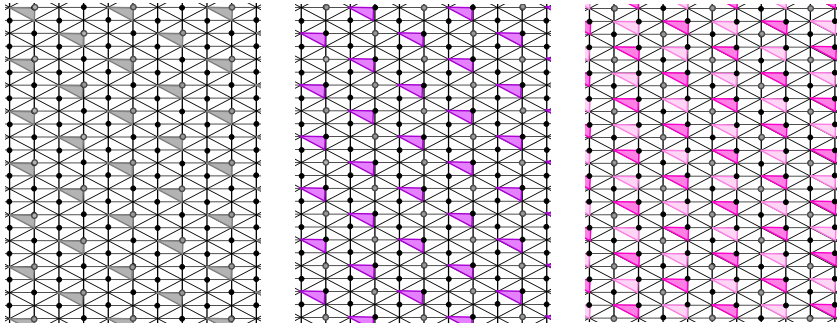


FIGURE 2
Conjugacy classes $[t^\lambda s_1 s_2]_H$ in the wallpaper group **cmm**.

“lines” of triangles (of the same color). The description of the conjugacy classes $[t^\lambda s_2]_H$ is similar, just involving vertical “lines” of triangles instead. Note that s_1 and s_2 both fill their move-sets.

Figure 2 depicts the conjugacy classes $[t^\lambda s_1 s_2]_H$. Since $s_1 s_2 = -I$, the mod-set $\text{MOD}_H(s_1 s_2)$ equals $2L_H$, shown by large gray dots, and $\text{Mov}(s_1 s_2) = \mathbb{R}^2$. Thus, $s_1 s_2$ does not fill its move-set. The left of Figure 2 shows the conjugacy class $[t^\lambda s_1 s_2]_H$ for any $\lambda \in \text{MOD}_H(s_1 s_2)$, while the center and right show the cases $\lambda \in L_H \setminus \text{MOD}_H(s_1 s_2)$. In the center, the set $\text{MOD}_H(t^\lambda s_1 s_2) = \lambda + \text{MOD}_H(s_1 s_2)$ is H_0 -invariant, while on the right, the H_0 -orbit of $\text{MOD}_H(t^\lambda s_1 s_2)$ has 2 elements, corresponding to the dark and light pink triangles.

Finally, for any $\lambda \in L_H$, the class $[t^\lambda]_H$ is just the finite set $\{t^{h_0 \lambda} \mid h_0 \in H_0\}$.

Remark 2.6. We point out the importance of the subgroup H_0 in our descriptions of conjugacy classes. For example, the spherical part of $h \in H$ completely determines the mod-set appearing in the first equality of Theorem 2.2, and in both equalities in its statement, we are conjugating just by elements of H_0 . This importance of H_0 is

perhaps unsurprising if we consider the boundary at infinity $\partial\mathbb{E}^n \cong \mathbb{S}^{n-1}$, that is, the set of all parallelism classes of rays in $\mathbb{E}^n$. The group H acts on $\partial\mathbb{E}^n$ with kernel its translation subgroup T_H , and so, all we are seeing at infinity is the induced action of H_0 .

Remark 2.7. An important class of subgroups of G are the *crystallographic* groups. These can be defined as the discrete subgroups of G which act cocompactly on $\mathbb{E}^n$; a crystallographic group is sometimes called *symmorphic* if it splits. Thus, a split subgroup $H = T_H \rtimes H_0$ of G is crystallographic exactly when L_H is a lattice in $\mathbb{R}^n$ and H_0 is finite. The 17 wallpaper groups, 13 of which split, are the 2-dimensional crystallographic groups. Any crystallographic group $H \leq G$ is of finite index in a split crystallographic $H' \leq G$ (see, for instance, [10, p. 316]), and then obviously, every H -conjugacy class is contained in some H' -conjugacy class. Hence, Theorem 2.2 provides, up to finite index, a description of all conjugacy classes in all crystallographic groups.

Remark 2.8. The *affine Coxeter groups*, all of which split, are the crystallographic groups which are generated by reflections in the faces of a convex polytope (compare [3, Definition 6.4.4]). As explained in [9, Appendix B], for $n = 2, 3$, every n -dimensional crystallographic group is finite index in some affine Coxeter group, but for $n \geq 4$, there are examples of crystallographic groups $H \leq G = \text{Isom}(\mathbb{E}^n)$ which are not contained in any affine Coxeter subgroup of G .

Example 2.9. Let $H = G = \text{Isom}(\mathbb{E}^2)$ be the full isometry group of the Euclidean plane. Let $\lambda \in \mathbb{R}^2$ be nonzero, let $r \in \text{O}(2)$ be the unique linear reflection which fixes λ , and let g be the glide-reflection $g = t^\lambda r$. Then, the conjugacy class $[g]_G$ is the disjoint union of all lines which are tangent to the circle of radius $\|\lambda\|$, as depicted in Figure 3. More precisely, if ℓ is a line tangent to this circle, then the point p of ℓ corresponds to the element $t^p r_\ell$ of $[g]_G$, where $r_\ell \in \text{O}(2)$ is the unique linear reflection preserving ℓ .

Note that each same-color pair of “lines” of tiles in Figure 1 can be viewed as a “discrete shadow” of a pair of actual lines in Figure 3.

Now, take any $\lambda \in \mathbb{R}^2$ and let $g_0 = -I$. Then, $g = t^\lambda g_0$ is the rotation by 180° about the point $\frac{1}{2}\lambda$, and all such rotations are conjugate in G . Hence, the three distinct conjugacy classes of **cm** seen in Figure 2 are all discretizations of a single conjugacy class in G .

For translations, given any $\lambda \in \mathbb{R}^2$, we can identify the conjugacy class $[t^\lambda]_G = \{t^{u\lambda} : u \in \text{O}(2)\}$ with the circle of radius $\|\lambda\|$. Thus, the finitely many conjugates of any translation in **cm** are again just a discrete glimpse of its full conjugacy class in G .

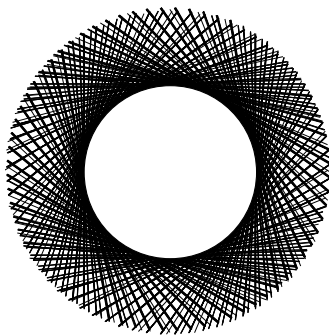


FIGURE 3

The conjugacy class of a glide-reflection in $\text{Isom}(\mathbb{E}^2)$.

We can generalize the previous paragraph as follows: the conjugacy classes of translations in $G = \text{Isom}(\mathbb{E}^n)$ partition $\mathbb{R}^n$ into the set of spheres centered at the origin, and for any split crystallographic subgroup $H \leq G$, the conjugacy classes of translations in H pick out finitely many points on each of a discrete set of these spheres.

Remark 2.10. The full isometry group G of $\mathbb{E}^n$ is a Lie group, and the conjugation action of G on itself is smooth. It follows that conjugacy classes in G , which are the orbits of this action, are immersed submanifolds of G . We note that the conjugation action of G on itself is not proper, since the entire non-compact group G fixes its identity element. There is considerable literature on conjugacy classes in compact Lie groups, such as $O(n)$, where the conjugation action is proper and there is a classical relationship to representation theory (see, for example, [2, Chapter IV]). However, we do not know of any work on conjugacy classes in G itself from a Lie-theoretic point of view. Our proofs use only the algebraic structure of G , as a semidirect product.

We now continue the description of our main results. The equalities in Theorem 2.2 suggest that a conjugacy class naturally decomposes into a family of subsets. We explore this phenomenon by considering the *components* of the conjugacy class $[h]_H$; that is, the subsets of $[h]_H$ of the form $u(t^{\text{Mod}_H(h_0)}h)u^{-1}$, where $u \in H_0$. Write $\text{Comp}_H(h)$ for the set of components of $[h]_H$. By definition, the group H_0 acts transitively by conjugation on $\text{Comp}_H(h)$.

Theorem 2.11 (Components). *Let $H = T_H \rtimes H_0$ be a split group of Euclidean isometries. Let $h = t^\lambda h_0 \in H$, where $\lambda \in L_H$ and $h_0 \in H_0$. Then, the following statements hold..*

- (1) *The conjugation action of the group T_H induces a transitive action by translation on the elements of each component of $[h]_H$.*

- (2) *Linearization induces a natural surjection from $\text{Comp}_H(h)$ to $\text{Comp}_H(h_0)$.*
 (3) *There is a natural bijection between $\text{Comp}_H(h_0)$ and $[h_0]_{H_0}$.*

Part (1) of Theorem 2.11 completes a “global” picture of the action of $H = T_H \rtimes H_0$ on itself by conjugation: the elements of H_0 permute the components of any conjugacy class, while the translation subgroup T_H acts transitively within each component. In Proposition 3.9, we determine the stabilizers of components under this H_0 -action. Parts (2) and (3) of Theorem 2.11 emphasize again the importance of H_0 for the description of conjugacy classes in H (see Remark 2.6). In situations where conjugacy classes in H_0 are well understood, such as when $H_0 = \text{O}(n)$, we can thus obtain much information about the geometry of conjugacy classes in H by “lifting” from H_0 . We give some additional results for the special case $H = G$ in Proposition 3.11.

2.2. Coconjugation. In our final main result, we turn to the question of which $k \in H$ conjugate a given $h \in H$ to some h' in its conjugacy class. We refer to this question as the *coconjugation problem*. That is, given $h, h' \in H$, we will determine the (possibly empty) set $C_H(h, h')$ of elements $k \in H$ such that $khk^{-1} = h'$. As we will show, the solution to the coconjugation problem in H crucially involves the fix-sets of elements of H_0 ; that is, the orthogonal complement of the spaces ruling the shape of the conjugacy class itself.

For any $h' \in [h]_H$, the coconjugation set $C_H(h, h')$ is equal to $k C_H(h)$ for any $k \in H$ such that $khk^{-1} = h'$. One could hence say that it is enough to consider centralizers to fully solve the coconjugation problem. However, in Theorem 2.13 below, we provide an intrinsic description of the coconjugation set that does not require prior knowledge of the centralizer, nor the determination of a conjugating element k as used above. Instead, the disjoint union in Theorem 2.13 is parametrized by the following explicitly defined subset of the coconjugation set $C_{H_0}(h_0, h'_0)$.

Definition 2.12 (Translation-compatible part of the coconjugation set). Let $H = T_H \rtimes H_0$ be a split group of Euclidean isometries, let $\lambda, \lambda' \in L_H$, and let $h_0, h'_0 \in H_0$. The *translation-compatible part* of $C_{H_0}(h_0, h'_0)$ is defined by

$$(2.2.1) \quad C_{H_0}^{\lambda, \lambda'}(h_0, h'_0) = \{u \in C_{H_0}(h_0, h'_0) \mid \lambda' - u\lambda \in \text{Mod}_H(h'_0)\}.$$

Theorem 2.13 (Coconjugation). Let $H = T_H \rtimes H_0$ be a split group of Euclidean isometries. Let $h = t^\lambda h_0$ and $h' = t^{\lambda'} h'_0$ be elements of H , where $\lambda, \lambda' \in L_H$ and $h_0, h'_0 \in H_0$. Then,

$$(2.2.2) \quad C_H(h, h') \neq \emptyset \Leftrightarrow C_{H_0}^{\lambda, \lambda'}(h_0, h'_0) \neq \emptyset.$$

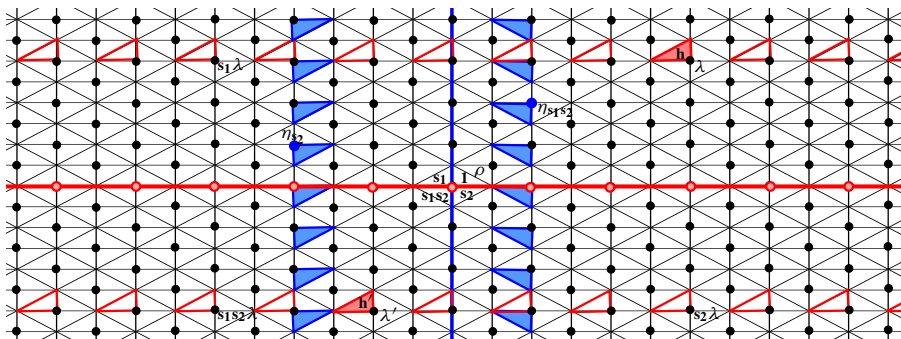


FIGURE 4

Moreover, if these sets are nonempty, then

$$(2.2.3) \quad C_H(h, h') = \bigsqcup_{u \in C_{H_0}^{\lambda, \lambda'}(h_0, h'_0)} t^{\eta_u + (F_{IX}(h'_0) \cap L_H)} u,$$

where, for each u , the element $\eta_u \in L_H$ is a particular solution to the equation

$$(2.2.4) \quad \lambda' - u\lambda = (I - h'_0)\eta.$$

In the special case that $\text{Fix}(h_0) = \{0\}$, we have that

$$\eta_u = (I - h'_0)^{-1}(\lambda' - u\lambda)$$

is the unique solution to (2.2.4), and $C_H(h, h')$ is in bijection with $C_{H_0}^{\lambda, \lambda'}(h_0, h'_0)$.

Geometrically, (2.2.3) means that the coconjugation set $C_H(h, h')$ lies along translates of the fix-set $\text{Fix}(h'_0)$, and so, it is orthogonal to $\text{Mov}(h'_0)$. The reason for this appearance of the fix-set in our description of coconjugation sets is that we are solving equation (2.2.4), and $\text{Fix}(h'_0) = \text{Ker}(\text{I} - h'_0)$. In the special case that $h = h'$, Theorem 2.13 yields a new geometric description of the centralizer $C_H(h)$.

When nonemptiness of the set $C_{H_0}^{\lambda, \lambda'}(h_0, h'_0)$ can be determined, the equivalence (2.2.2) in Theorem 2.13 provides an algorithm to solve the conjugation problem in H . If, in addition, all elements of $C_{H_0}^{\lambda, \lambda'}(h_0, h'_0)$ and all solutions to equation (2.2.4) can be computed, we obtain an algorithm which lists all elements of the coconjugation set.

Example 2.14. In H the wallpaper group **cmm**, let $h = t^\lambda s_1$ and $h' = t^{\lambda'} s_1$ be the conjugates shaded red in Figure 4 (compare Figure 1). Here, $\text{Fix}(s_1)$ is the vertical blue line and $\text{Mov}(s_1)$ is the horizontal red line. It turns out that $C_{H_0}^{\lambda, \lambda'}(s_1, s_1) = \{s_2, s_1 s_2\}$, and we can choose $\eta_{s_2}, \eta_{s_1 s_2} \in L_H$ as indicated by blue dots.

Remark 2.15. The proofs in this work use only the semidirect product structure of split subgroups of G . We initially proved a version of Theorem 2.2, investigated the filling property, and identified some elements of coconjugation sets for certain “standard representatives” in affine Coxeter groups, as part of our study of affine Deligne–Lusztig varieties (see [8, Section 5.1]). We then established the results described above for arbitrary elements of affine Coxeter groups and subsequently realized that our proofs extended immediately to all split $H \leq G$. In [9], we discuss the results of the present work in relation to the literature on Coxeter groups.

Remark 2.16. In our companion paper [9], for certain split crystallographic groups (see Remark 2.7), we refine the relationships between mod-sets and move-sets observed above. We regard the lattice L_H as a free $\mathbb{Z}$ -module of rank n and prove in [9, Theorem 1.6 and Corollary 1.7] that for any split crystallographic $H = T_H \rtimes H_0$ which is contained in an affine Coxeter group, and all $h_0 \in H_0$:

- (1) the rank of $\text{MOD}_H(h_0)$ equals the dimension of the move-set $\text{Mov}(h_0)$;
- (2) $\text{MOD}_H(h_0)$ is a finite-index submodule of $\text{Mov}(h_0) \cap L_H$; and
- (3) h_0 fills its move-set if and only if $L_H/\text{MOD}_H(h_0)$ is torsion-free.

Part (1) here tells us that the move-set can be viewed as the “enveloping subspace” of the mod-set, while part (2) implies that the containments in Corollary 2.3 are equalities up to finite index. Part (3) then provides a mechanism (quotients of free $\mathbb{Z}$ -modules) for determining which elements fill their move-sets (see Definition 2.4 above).

Our proofs of (1)–(3) in [9] use properties of affine Coxeter groups beyond their semidirect product structure, including their close relationship to finite Weyl groups. We do not know if (1)–(3) hold for split crystallographic groups which are not contained in affine Coxeter groups (see Remark 2.8).

3. Geometry of conjugacy classes

In this section, we describe the geometry of conjugacy classes of elements of split subgroups $H = T_H \rtimes H_0$ of the isometry group G of $\mathbb{E}^n$. We prove Theorem 2.2 in Section 3.1. Components and the effect of conjugation on them are described in Section 3.2, where we prove Theorem 2.11. We close the discussion by considering the special case $H = G$ in Section 3.3.

3.1. Geometric description of conjugacy classes. The mod-set (see Definition 2.1) is the key player in our geometric description of conjugacy classes. As noted after this definition, we have $\text{MOD}_H(h) = (I - h)L_H$. In the remainder of this paper, we will often work with this formulation, as it is the one which arises in our results.

We first record two easy lemmas concerning mod-sets; then, we use these together with some elementary computations to prove Theorem 2.2.

Lemma 3.1. *For all $h \in H$ and all $u \in H_0$, we have $u\text{MOD}_H(h) = \text{MOD}_H(uhu^{-1})$.*

Proof. Since L_H is H_0 -invariant, we have

$$u(\mathbf{I} - h)L_H = u(\mathbf{I} - h)u^{-1}L_H = (\mathbf{I} - uhu^{-1})L_H,$$

which establishes the result. $\blacksquare$

Lemma 3.2. *For all $\lambda \in L_H$ and all $h_0 \in H_0$, we have $\text{MOD}_H(t^\lambda h_0) = \lambda + \text{MOD}_H(h_0)$.*

Proof. For any $\mu \in L_H$, we have

$$\lambda + (h_0 - \mathbf{I})\mu = (\lambda + h_0\mu) - \mu = t^\lambda h_0\mu - \mu = (t^\lambda h_0 - \mathbf{I})\mu.$$

The result follows. $\blacksquare$

Proof of Theorem 2.2. Fix $h = t^\lambda h_0$ and let $k = t^\eta u$ be any element of H , where $\eta \in L_H$ and $u \in H_0$. Compute that

$$\begin{aligned} khk^{-1} &= t^\eta u(t^\lambda h_0)u^{-1}t^{-\eta} \\ &= t^\eta(ut^\lambda u^{-1})(uh_0u^{-1})t^{-\eta} \\ &= t^\eta t^{u\lambda}((uh_0u^{-1})t^{-\eta}(uh_0u^{-1})^{-1})uh_0u^{-1} \\ &= t^\xi uh_0u^{-1}, \end{aligned}$$

where $\xi = u\lambda + (\mathbf{I} - uh_0u^{-1})\eta$. Hence, $[h]_H$ is of the form

$$[h]_H = \bigcup_{u \in H_0} \{t^\xi uh_0u^{-1} \mid \xi \in u\lambda + \text{MOD}_H(uh_0u^{-1})\}.$$

Now, $\text{MOD}_H(uh_0u^{-1}) = u\text{MOD}_H(h_0)$ by Lemma 3.1, while for any $\mu \in \text{MOD}_H(h_0)$, we have $u(t^\mu h)u^{-1} = t^{u(\lambda + \mu)}uh_0u^{-1}$. Combining these yields Equation (2.1.1), and then using Lemma 3.2, we obtain Equation (2.1.2). $\blacksquare$

We now relate conjugacy classes to filling (see Definition 2.4). We first observe the following lemma.

Lemma 3.3. *For all $h \in H$, we have $\text{MOD}_H(h) \subseteq \text{Mov}(h) \cap L_H$.*

Proof. By definitions, $\text{MOD}_H(h) \subseteq \text{Mov}(h)$. Now, L_H is an H -invariant $\mathbb{Z}$ -module, so $(h - \mathbf{I})L_H \subseteq L_H$. The result follows. $\blacksquare$

Proposition 3.4 (Conjugacy classes and filling). *For all $h = t^\lambda h_0 \in H$, the following are equivalent:*

- (1) h fills its move-set; that is, $\text{Mod}_H(h) = \text{Mov}(h) \cap L_H$;
- (2) h_0 fills its move-set; that is, $\text{Mod}_H(h_0) = \text{Mov}(h_0) \cap L_H$;
- (3) $[h]_H = \bigcup_{u \in H_0} u(t^{\text{Mov}(h_0) \cap L_H} h)u^{-1}$; and
- (4) $[h]_H = \bigcup_{u \in H_0} t^{u(\text{Mov}(h) \cap L_H)} u h_0 u^{-1}$.

Proof. We have $\text{Mod}_H(h) = \lambda + \text{Mod}_H(h_0)$ by Lemma 3.3,

$$\text{Mov}(h) = \lambda + \text{Mov}(h_0)$$

by [5, Proposition 1.21], and $(\lambda + \text{Mov}(h_0)) \cap L_H = \lambda + (\text{Mov}(h_0) \cap L_H)$. Hence, (1) and (2) are equivalent. Similarly, (3) and (4) are equivalent.

Theorem 2.2 gives us that (2) implies (3) (for instance), and we complete the proof by showing that (3) implies (2). Suppose (3) holds, and let $\mu \in \text{Mov}(h_0) \cap L_H$. Then, there is a $k = t^\eta u \in H$, with $\eta \in L_H$ and $u \in H_0$, such that $khk^{-1} = u(t^\mu h)u^{-1}$. As in the proof of Theorem 2.2, we compute that $khk^{-1} = t^\xi u h_0 u^{-1}$, where $\xi = u\lambda + (I - u h_0 u^{-1})\eta$. On the other hand, $u(t^\mu h)u^{-1} = u t^\mu u^{-1} u t^\lambda h_0 u^{-1} = t^{u\mu + u\lambda} u h_0 u^{-1}$. Hence,

$$u\lambda + (I - u h_0 u^{-1})\eta = u\lambda + u\mu.$$

Thus, $u\mu$ is an element of $\text{Mod}_H(u h_0 u^{-1})$. But by Lemma 3.1, we have

$$\text{Mod}_H(u h_0 u^{-1}) = u \text{Mod}_H(h_0),$$

so $\mu \in \text{Mod}_H(h_0)$, as required. ■

3.2. Components of conjugacy classes and action by conjugation. In this section, we define components and prove part (1) of Theorem 2.11, as well as describing the stabilizers for the action by conjugation on components, in Proposition 3.9.

Definition 3.5 (Component). Let $h = t^\lambda h_0 \in H$, where $\lambda \in L_H$ and $h_0 \in H_0$. The *base component* of h is the subset of H given by

$$\text{Base}_H(h) = t^{\text{Mod}_H(h_0)} h = t^{\lambda + \text{Mod}_H(h_0)} h_0.$$

A *component* of $[h]_H$ is any of the base components of its elements. We write $\text{Comp}_H(h)$ for the set of components of $[h]_H$, and $\# \text{Comp}_H(h)$ for the cardinality of this set.

We start by collecting some first properties of components.

Lemma 3.6 (Shape of components). *Let $h = t^\lambda h_0 \in H$, where $\lambda \in L_H$ and $h_0 \in H_0$. Then,*

- (1) $\text{Base}_H(h) = t^\lambda \text{Base}_H(h_0)$;
- (2) $u \text{Base}_H(h) u^{-1} = \text{Base}_H(u h u^{-1})$ for all $u \in H_0$; and
- (3) every component of $[h]_H$ is of the form

$$u \text{Base}_H(h) u^{-1} = t^{u\lambda + u \text{Mod}_H(h_0)} u h_0 u^{-1} = t^{u\lambda} \text{Base}_H(u h_0 u^{-1})$$

for some $u \in H_0$.

Proof. Item (1) is clear from the definition. A straightforward computation using Lemma 3.1 implies (2). Combining the first two items with Theorem 2.2, the third item follows. ■

We now consider the effect of conjugating by a translation in H .

Lemma 3.7. *Let $h = t^\lambda h_0 \in H$, where $\lambda \in L_H$ and $h_0 \in H_0$.*

- (1) *For all $\eta \in L_H$ and all $k \in \text{Base}_H(h)$, one has*

$$t^\eta k t^{-\eta} = t^{\eta'} k,$$

where $\eta' = (I - h_0)\eta \in \text{Mod}_H(h_0)$. In particular, $t^\eta k t^{-\eta} \in \text{Base}_H(h)$.

- (2) *For all $\eta \in L_H$ and $u \in H_0$,*

$$t^\eta (u \text{Base}_H(h) u^{-1}) t^{-\eta} = u \text{Base}_H(h) u^{-1}.$$

- (3) *For all $h', h'' \in u \text{Base}_H(h) u^{-1}$, there is an $\eta \in L_H$ such that*

$$t^\eta h' t^{-\eta} = h''.$$

Proof. To see (1), let $k = t^{\lambda+\mu} h_0$ for $\mu \in \text{Mod}_H(h_0)$ and compute $t^\eta (t^{\lambda+\mu} h_0) t^{-\eta} = t^{\lambda+\mu} t^\eta t^{-h_0 \eta} h_0 = t^{\lambda+\mu+(I-h_0)\eta} h_0 = t^{(I-h_0)\eta} k$, as required. For (2), by Lemma 3.6, it suffices to consider $t^\eta \text{Base}_H(h) t^{-\eta}$. The result then follows from (1).

As a preparation for (3), observe the following: an arbitrary element h' of the component $u \text{Base}_H(h) u^{-1}$ has the form $h' = t^{u(\lambda+\mu')} u h_0 u^{-1}$, where $\mu' \in \text{Mod}_H(h_0)$. Then, a similar computation to that in the proof of part (1) establishes the following equation: for all $\eta \in L_H$ and all $h' \in \text{Base}_H(u h u^{-1})$, we have

$$(3.2.1) \quad t^\eta h' t^{-\eta} = t^{\eta'} h',$$

where $\eta' = (I - u h_0 u^{-1})\eta \in \text{Mod}_H(u h_0 u^{-1})$. In particular, $t^\eta h' t^{-\eta} \in \text{Base}_H(u h u^{-1})$.

For (3), since L_H is H_0 -invariant, it suffices to consider the base component. So, let $h' = t^{\lambda+\mu'} h_0$ and $h'' = t^{\lambda+\mu''} h_0$ be elements of $\text{Base}_H(h_0)$, where $\mu', \mu'' \in \text{Mod}_H(h_0)$. Then,

$$h''(h')^{-1} = t^{\lambda+\mu''} h_0 h_0^{-1} t^{-(\lambda+\mu')} = t^{\mu''-\mu'},$$

and so, putting $\eta' = \mu'' - \mu'$, we have $h'' = t^{\eta'} h'$. But then, $\eta' \in \text{Mod}_H(h_0)$, and so, $\eta' = (I - h_0)\eta$ for some $\eta \in L_H$. Combining this with Equation (3.2.1) completes the proof. ■

The next result establishes Theorem 2.11 (1).

Corollary 3.8 (Conjugation action). *Let $h = t^\lambda h_0 \in H$, where $\lambda \in L_H$ and $h_0 \in H_0$.*

- (1) *The conjugation action of H_0 on H induces a transitive action of H_0 on the set $\text{Comp}_H(h)$.*
- (2) *The conjugation action of T_H on H stabilizes each component of $[h]_H$ setwise and acts transitively when restricted to any of the components.*

Proof. It is immediate from Theorem 2.2 and Definition 3.5 that the conjugacy class $[h]_H$ is the union of its components, and that H_0 acts transitively by conjugation on the elements of $\text{Comp}_H(h)$. This establishes (1). Item (2) follows from Lemma 3.7. ■

We close this section by describing the stabilizers of components under the conjugation action of H_0 on $\text{Comp}_H(h)$. An alternative point of view here is that we are attempting to describe the conjugacy class $[h]_H$ as a disjoint union of components. Recall from Theorem 2.2 and Definition 3.5 that

$$(3.2.2) \quad [h]_H = \bigcup_{u \in H_0} u \text{Base}_H(h) u^{-1}.$$

So, in order to write this union as a disjoint union, we need to understand which conjugates of the base component are equal. It suffices, as done in the next proposition, to find conditions that characterize the case where $u \text{Base}_H(h) u^{-1} = \text{Base}_H(h)$. That is, we will determine the stabilizer of $\text{Base}_H(h)$ in H_0 .

Proposition 3.9. *Let $h = t^\lambda h_0 \in H$, where $\lambda \in L_H$ and $h_0 \in H_0$. Then, for all $u \in H_0$,*

$$u \text{Base}_H(h) u^{-1} = \text{Base}_H(h) \Leftrightarrow \begin{cases} (1) & u \in C_{H_0}(h_0), \quad \text{and} \\ (2) & (I - u)\lambda \in \text{Mod}_H(h_0). \end{cases}$$

Proof. We compute

$$u \text{Base}_H(h) u^{-1} = t^{u\lambda} u \text{Base}_H(h_0) u^{-1} = t^{u\lambda} \text{Base}_H(uh_0u^{-1}).$$

Hence, $u \text{ Base}_H(h)u^{-1} = \text{Base}_H(h)$ is equivalent to

$$t^{u\lambda} \text{ Base}_H(uh_0u^{-1}) = t^\lambda \text{ Base}_H(h_0)$$

and hence to

$$t^{u\lambda + \text{MOD}_H(uh_0u^{-1})} uh_0u^{-1} = t^{\lambda + \text{MOD}_H(h_0)} h_0.$$

This equation holds if and only if both $uh_0u^{-1} = h_0$ (and hence, (1) is satisfied) and the following equation hold

$$(3.2.3) \quad u\lambda + \text{MOD}_H(uh_0u^{-1}) = \lambda + \text{MOD}_H(h_0).$$

Under condition (1), equation (3.2.3) is equivalent to (2). ■

We note that condition (2) in Proposition 3.9 implies that the mod-sets of u and h_0 intersect, and that if u fixes λ , then (2) holds. It seems difficult to make any further general observations on this condition.

The next result completes the proof of Theorem 2.11.

Lemma 3.10. *Let $h = t^\lambda h_0 \in H$, where $\lambda \in L_H$ and $h_0 \in H_0$.*

(1) *The linearization map sending $\lambda + \text{MOD}_H(h_0)$ to $\text{MOD}_H(h_0)$ induces a surjection*

$$\text{Comp}[h]_H \twoheadrightarrow \text{Comp}[h_0]_H.$$

Hence, $\# \text{Comp}[h]_H \geq \# \text{Comp}[h_0]_H$.

(2) *There is a natural bijection between the components of $[h_0]_H$ and the elements of the spherical conjugacy class $[h_0]_{H_0}$. Hence,*

$$\# \text{Comp}[h_0]_H = \#\{uh_0u^{-1} \mid u \in H_0\}.$$

Proof. Part (1) follows from the definition of components (see Definition 3.5), and part (2) follows from this definition and Lemma 3.6 with $h = h_0$. ■

Figure 2 shows that the inequality in Lemma 3.10(1) can be either strict or an equality, depending upon some fine behavior. It would be interesting to see whether one can characterize those elements for which equality holds.

3.3. Components in G . We now consider components in the special case $H = G$.

Proposition 3.11. *Let $g = t^\lambda g_0 \in G$, where $\lambda \in \mathbb{R}^n$ and $g_0 \in \text{O}(n)$.*

(1) *If $\text{Mov}(g_0) = \mathbb{R}^n$, then $[g]_G = [g_0]_G$ and the components of this conjugacy class are the sets of the form*

$$\{t^\mu u g_0 u^{-1} \mid \mu \in \mathbb{R}^n, u \in \text{O}(n)\}.$$

- (2) If $\text{Mov}(g_0)$ is m -dimensional with $0 < m < n$, and $\lambda \notin \text{Mov}(g_0)$, then the fibers of the surjection

$$\text{Comp}_G(g) \twoheadrightarrow \text{Comp}_G(g_0)$$

induced by linearization are all of cardinality ≥ 2 . Moreover, in this case, the sets $\text{Comp}_G(g)$ and $\text{Comp}_G(g_0)$ both have cardinality $\aleph_1$.

Proof. If $\text{Mov}(g_0) = \mathbb{R}^n$, then we have

$$u(\lambda + \text{Mov}(g_0)) = \mathbb{R}^n$$

as well, for any $u \in \text{O}(n)$ and any $\lambda \in \mathbb{R}^n$, including $\lambda = 0$. Hence, $[g]_G = [g_0]_G$ has components as stated in (1).

In case (2), we have that $\lambda + \text{Mov}(g_0)$ is an m -dimensional affine subspace which is distinct from $\text{Mov}(g_0)$. Now, $\text{O}(n)$ contains the isometry $-I$, while $-\lambda \neq \lambda$ since $\lambda \neq 0$. Hence, $\lambda + \text{Mov}(g_0) \neq (-I)(\lambda + \text{Mov}(g_0))$, and it follows that the components $\text{Base}_G(g)$ and $(-I)\text{Base}_G(g)(-I)^{-1}$ are distinct (even though the elements of these components have the same spherical part, since $-I$ is central in $\text{O}(n)$). Hence, conjugating these two components by any element of H_0 also results in two distinct components. As linearization sends both $\text{Base}_G(g)$ and $(-I)\text{Base}_G(g)(-I)^{-1}$ to $\text{Base}_G(g_0)$, the map $\text{Comp}_G(g) \rightarrow \text{Comp}_G(g_0)$ induced by linearization has fibers of cardinality ≥ 2 .

For the final claim in (2), by Lemma 3.10, it suffices to see that g_0 has $\aleph_1$ -many distinct conjugates in $\text{O}(n)$. Since $0 < m < n$, we have that $\text{O}(n)$ acts transitively on the set of m -dimensional subspaces of $\mathbb{E}^n$, which has cardinality $\aleph_1$. On the other hand, $u\text{Mov}(g_0) = \text{Mov}(ug_0u^{-1})$ by Lemma 3.1. The result follows. ■

Remark 3.12. If $\text{Mov}(g_0) = \mathbb{R}^n$, then by Proposition 3.11 (1), for any $\lambda \in \mathbb{R}^n$, we can view each component of $[t^\lambda g_0]_G = [g_0]_G$ as a “sheet” of $\mathbb{R}^n$ with the corresponding $\text{O}(n)$ -conjugate of g_0 sitting at each point, and the conjugation action of $\text{O}(n)$ permuting these sheets. Alternatively, we can view the entire conjugacy class $[t^\lambda g_0]_G$ as a single copy of $\mathbb{R}^n$ with the entire $\text{O}(n)$ -conjugacy class of g_0 sitting at each point.

Example 3.13. We generalize the first paragraph of Example 2.9. Let $r \in \text{O}(n)$ be a reflection so that $\text{Fix}(r)$ is a linear hyperplane, and write $p_r : \mathbb{R}^n \rightarrow \text{Fix}(r)$ for the orthogonal projection onto this hyperplane. Then, for all $\lambda \in \mathbb{R}^n$ not in the line through the origin $\text{Mov}(r)$, the components of the class $[t^\lambda r]_G$ are the lines tangent to the sphere in $\mathbb{R}^n$ of radius $\|p_r(\lambda)\| > 0$. The map $\text{Comp}_G(t^\lambda r) \twoheadrightarrow \text{Comp}_G(r)$ from Proposition 3.11 (2) has fibers in bijection with the points of a sphere of dimension $n - 2$, so this surjection is 2-to-1 when $n = 2$ and $\aleph_1$ -to-1 for all $n \geq 3$.

4. Geometry of coconjugation sets

We now prove Theorem 2.13. Some key observations are gathered in the next result.

Proposition 4.1. *Let $h = t^\lambda h_0$ and $h' = t^{\lambda'} h'_0$ be elements of H , where $\lambda, \lambda' \in L_H$ and $h_0, h'_0 \in H_0$. Then, for all $u \in H_0$, we have the following.*

(1) *For any $\eta \in L_H$,*

$$k = t^\eta u \in C_H(h, h') \Leftrightarrow u \in C_{H_0}(h_0, h'_0) \quad \text{and} \quad \lambda' - u\lambda = (I - h'_0)\eta.$$

(2) *There exists at least one $\eta \in L_H$ such that $\lambda' - u\lambda = (I - h'_0)\eta$ if and only if $\lambda' - u\lambda \in \text{Mod}_H(h'_0)$.*

(3) *Let $\eta_0 \in L_H$ and suppose that $t^{\eta_0} u \in C_H(h, h')$. Then, for all $\eta \in L_H$, we have*

$$k = t^\eta u \in C_H(h, h') \Leftrightarrow \eta \in \eta_0 + (\text{Fix}(h'_0) \cap L_H).$$

Proof. Let η and k be as in (1). By the same computation as in the proof of Theorem 2.2, we have $k h k^{-1} = t^{u\lambda + (I - u h_0 u^{-1})\eta} u h_0 u^{-1}$. Thus, $k \in C_H(h, h')$ if and only if $u \in C_{H_0}(h_0, h'_0)$ and $\lambda' = u\lambda + (I - h'_0)\eta$. Part (1) follows.

Part (2) is immediate from the definition $\text{Mod}_H(h'_0) = (I - h'_0)L_H$.

To prove item (3), observe that by part (1) we have

$$(4.0.1) \quad k = t^\eta u \in C_H(h, h') \Leftrightarrow \lambda' - u\lambda = (I - h'_0)\eta.$$

By our assumption, we have that $\lambda' - u\lambda = (I - h'_0)\eta_0$. We subtract this formula from Equation (4.0.1) and obtain

$$0 = (I - u)\lambda - (I - u)\lambda = (I - h'_0)\eta - (I - h'_0)\eta_0,$$

which is equivalent to the fact that

$$(\eta - \eta_0) \in \text{Ker}(I - h'_0) = \text{Fix}(h_0).$$

Since $\eta - \eta_0 \in L_H$, we obtain item (3). ■

Recall from Definition 2.12 that the translation-compatible part of the coconjugation set is given by

$$C_{H_0}^{\lambda, \lambda'}(h_0, h'_0) = \{u \in C_{H_0}(h_0, h'_0) \mid \lambda' - u\lambda \in \text{Mod}_H(h'_0)\}.$$

This definition is motivated by Proposition 4.1.

Proof of Theorem 2.13. The equivalence (2.2.2) is immediate from Proposition 4.1 and the definition of the set $C_{H_0}^{\lambda, \lambda'}(h_0, h'_0)$.

We may henceforth assume that $C_{H_0}^{\lambda, \lambda'}(h_0, h'_0) \neq \emptyset$. Then, since h_0 and h'_0 are conjugate in H_0 , we have $\text{Fix}(h_0) = \{0\}$ if and only if

$$\text{Fix}(h'_0) = \{0\}.$$

We obtain the given shape of $C_H(h, h')$ as a consequence of Proposition 4.1. Combining this with the fact that $\text{Fix}(h'_0) = \{0\}$ is equivalent to $(I - h'_0)$ being invertible implies the statement about the case of the fixed point set being empty. ■

5. Algorithmic solution to the (co)conjugacy problem

We now use the results of Section 4 to sketch an algorithm to solve the conjugacy problem and to compute all (co)conjugation sets. We restrict to H split crystallographic so that H_0 is finite and the lattice L_H may be regarded a free $\mathbb{Z}$ -module of rank n (see Remark 2.7).

Algorithm 5.1. Let $h = t^\lambda h_0$ and $h' = t^{\lambda'} h'_0$ be elements of H , where $\lambda, \lambda' \in L_H$ and $h_0, h'_0 \in H_0$. We want to determine $C_H(h, h')$.

- (1) Determine whether h and h' are conjugate:
 - (a) If $C_{H_0}(h_0, h'_0) = \emptyset$, then $C_H(h, h') = \emptyset$.
 - (b) If $\lambda' - k_0 \lambda \notin \text{Mod}_H(h'_0)$ for all $k_0 \in C_{H_0}(h_0, h'_0)$, then $C_H(h, h') = \emptyset$.
- (2) If neither of the above two cases appeared, then h and h' are conjugate (by Proposition 4.1). In this case, $C_H(h, h')$ is obtained as in Theorem 2.13.

We have not implemented this algorithm, nor do we know its complexity, and we expect that addressing these questions would be a substantial endeavor. Part of the algorithm involves just the finite group H_0 , and so, it could be done by brute force if necessary. However, a serious implementation requires an efficient solution to the conjugacy problem in H_0 , followed by computation of the spherical coconjugation set $C_{H_0}(h_0, h'_0)$ (if it is known to be nonempty). Even for H_0 a finite Weyl group, carrying out these steps would likely involve similar efforts to those recently undertaken for finite groups of Lie type in the monograph [4]. We then need to know whether, for some $u \in C_{H_0}(h_0, h'_0)$, the $\mathbb{Z}$ -linear equation

$$\lambda' - u\lambda = (I - h'_0)\eta$$

has a solution in L_H . That is, we need to determine whether certain $\mathbb{Z}$ -linear equations have any integral solution. Finding the entire coconjugation set $C_H(h, h')$ then requires

finding all integral solutions to these $\mathbb{Z}$ -linear equations. There are various algorithms for such questions, which will require separate analysis to the work required for just H_0 .

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