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Billiards in regular polygons

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Abstract. In this paper we study billiards in regular n-sided polygons and monodromy over Teichmüller curves from an arithmetic perspective. Our main results show that the combinatorics of billiards in a periodic direction s is controlled by the projection of s to a finite projective line.

This arithmetic control coexists, experimentally, with chaotic behavior that first emerges when $n = 12$.

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1. Introduction

In this paper we study the dynamics of billiards in a regular polygon P_n in a given periodic direction s. Drawing on Teichmüller theory and the adelic analysis of triangle groups developed in [Mc5], we show that fundamental *combinatorial properties* of these trajectories are controlled by the *arithmetic* of s; more precisely, by its reduction to a point on the finite projective line $\mathbb{P}^1(\mathcal{O}_K/2)$. Parallel results are obtained for closed geodesics on the associated flat Riemann surface $(X, |\omega|)_n$.

This section presents an overview of the topics to be discussed. For a recent survey on billiards and Teichmüller curves, see [Mc4].

Billiards. Let $P_n \subset \mathbb{C}$ denote a regular polygon with $n \geq 3$ sides, normalized as in Figure 1.1 and with its edge midpoints (m_i) labeled. Let $T_s(x) \subset P_n$ denote the unique billiard trajectory passing through $x \in P_n$ in the direction $s \in \mathbb{C}^*$. We refer to $T_s(x)$ as a trajectory with *slope* s. By convention, a trajectory proceeds both forwards and backwards, stopping only if it hits a vertex. We say $T_s(x)$ is a *vertex connection* if it joins two vertices of P_n.

A trajectory $T_s(x)$ is *closed* (or periodic) if it is an immersed copy of S^1, disjoint from the vertices of P_n. The closed trajectories fall into parallel families, each

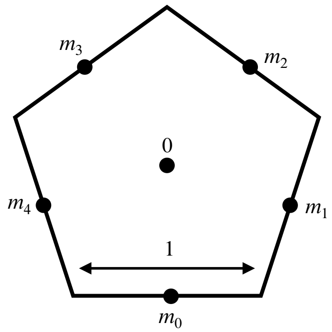

Figure 1.1
The regular pentagon $P_5 \subset \mathbb{C}$, with its edge midpoints labeled.

containing a midpoint trajectory $T_s(m_i)$, so we concentrate on these. We write

$$T_s(m_i) \cong T_s(m_j) \tag{1.1}$$

if $T_s(m_i) = \alpha \cdot T_s(m_j)$ for some $\alpha \in \mathrm{Isom}(P_n)$.

Cusps. Remarkably, by [V1], the *directions* of closed billiard trajectories in P_n correspond to the *cusps* $[s] \in \mathbb{P}^1(\mathcal{O}_K)$ of a lattice

$$\Delta_n \subset \mathrm{SL}_2(\mathcal{O}_K) \subset \mathrm{SL}_2(\mathbb{R})$$

with a model over the algebraic integers; more precisely, over the integers in the real subfield K of $L = \mathbb{Q}(\zeta_n)$, $\zeta_n = \exp(2\pi i/n)$ (see Section 3). When $K \neq \mathbb{Q}$, Δ_n is a prototypical example of a thin subgroup of $\mathrm{SL}_2(\mathcal{O}_K)$, of interest in its own right.

Following [Mc5], we say a function $f(s)$ on the cusps of Δ_n is a *congruence invariant* if it only depends on the projection $[s]_d$ of $[s]$ to a finite projective line of the form $\mathbb{P}^1(\mathcal{O}_K/d)$.

Rigidity. Our main results on billiards, stated in detail in Section 2, show that the midpoint trajectories $T_s(m_i)$ have a rigid combinatorial structure, largely governed by congruence invariants of $[s]$. Here is representative case (cf. Theorem 2.2 below).

Theorem 1.1. *Suppose n is odd. Then for each periodic direction s:*

(1) *There is a unique index k such that $T_s(m_k)$ is a vertex connection.*

(2) *The remaining midpoint trajectories $T_s(m_i)$ are closed.*

(3) *We have $T_s(m_{k+\ell}) \cong T_s(m_{k-\ell})$ for all ℓ.*

(4) *The special index k is a congruence invariant: it is determined by the condition that*

$$[s]_2 = [\zeta_n^k]_2 \quad \text{in } \mathbb{P}^1(\mathcal{O}_K/2). \tag{1.2}$$

Example: $n = 11$. See Figure 1.2 for an example of the midpoint trajectories at a fixed slope s in a regular hendecagon. Although there are 11 edge midpoints, only 6 distinct trajectories appear. The vertex connection $T_s(m_0)$ stands out, since it is much simpler than the rest. The shortest closed trajectory is $T_s(m_5)$, and the longest is $T_s(m_3)$. In general, the lengths of the closed trajectories in P_{11} satisfy $|T_s(m_{k+\ell})| = C(s)|\sin(2\pi\ell/11)|$; see Theorem 2.3 and the remarks in Section 7.

New phenomena appear when n is even; for detailed statements, see Section 2 and the discussion of the remarkable case $n = 12$ below.

Flat surfaces. We now turn to results in Teichmüller theory.

By an unfolding construction (Section 3) the regular polygon $P_n \subset \mathbb{C}$ determines a singular flat metric $|\omega|$ on a compact, hyperelliptic Riemann surface X of genus g. When n is even,

$$(X, |\omega|)_n = (P_n, |dz|)/\sim$$

is simply obtained by identifying opposite edges of P_n; when n is odd, one must first double P_n across an edge to obtain a polygon with an even number of sides.

Just as the Euclidean metric on $\mathbb{C}$ comes from the 1-form dz, the flat metric on X is associated to a holomorphic 1-form $\omega \in \Omega(X)$ with periods in $\mathbb{Z}[\zeta_n]$. The vertices of P_n give rise to the zeros $Z(\omega)$, which in turn produce cone-like singularities for the metric. The phase of ω determines the slopes of geodesics.

The passage from P_n to $(X, \omega)_n$ has the following useful features.

- First, *billiard trajectories* on P_n go over to *geodesics* on the flat surface $(X, |\omega|)_n$.
- Second, we have a natural map

  $$m_i \mapsto w_i,$$

  two-to-one when n is even and otherwise injective, from the *edge midpoints* of P_n into the *Weierstrass points* $W(X)$ of X.
- Most importantly, *hidden symmetries* of billiards in P_n are revealed by the group

  $$\mathrm{Aff}^+(X, \omega)_n \cong \Delta_n \subset \mathrm{SL}_2(\mathcal{O}_K)$$

  of *real-affine* symmetries of $(X, \omega)_n$.

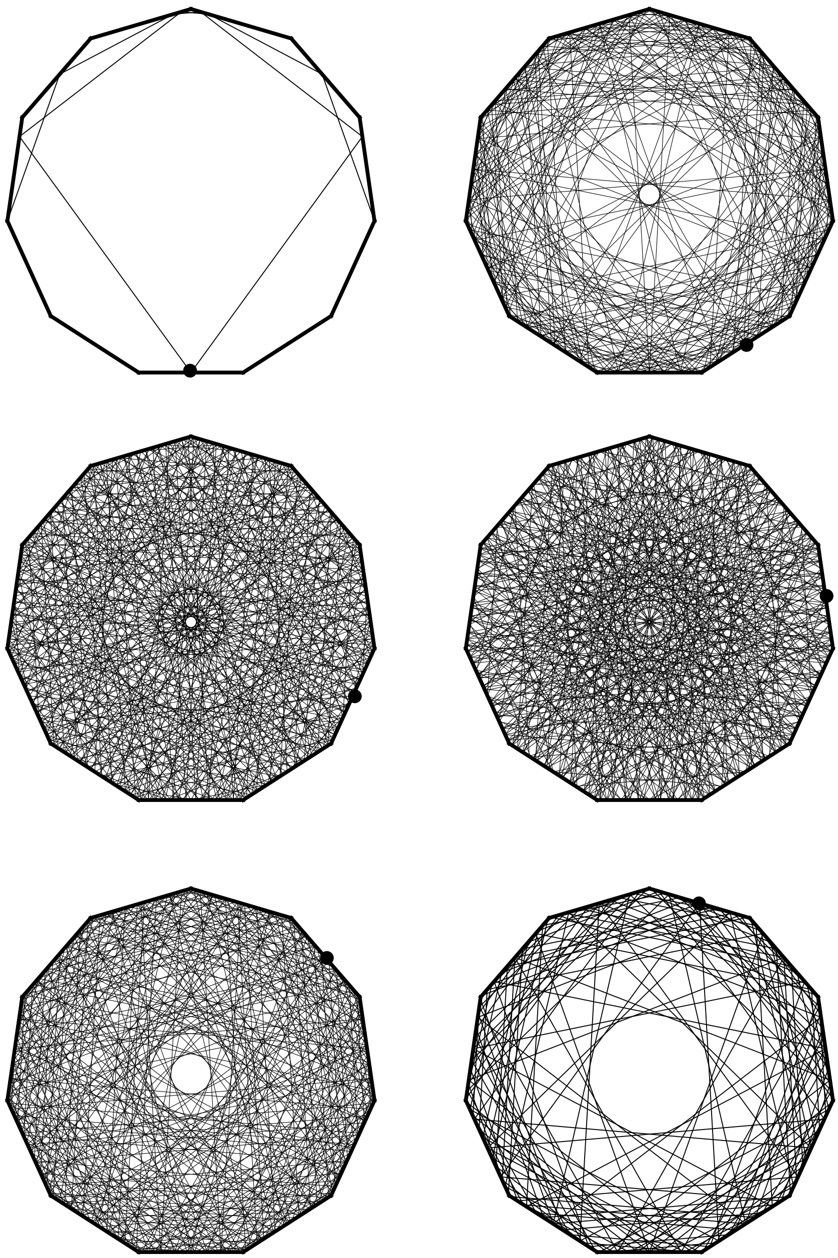

Figure 1.2
Billiard trajectories $T_s(m_i) \subset P_{11}$, $i = 0$ to 5.

Example: The square torus. Geometrically, $\mathrm{Aff}^+(X,\omega)_n$ is the group of orientation-preserving homeomorphisms $\phi : X \to X$ that send geodesics to geodesics. The corresponding matrix $D\phi \in \mathrm{SL}_2(\mathcal{O}_K)$ records the derivative of ϕ in local coordinates where $\omega = dz$.

In the familiar case $n = 4$, the unfolding construction relates billiards in a square to geodesics on the flat torus $X = \mathbb{C}/\mathbb{Z}[i]$, and $\mathrm{Aff}^+(X,\omega)$ is essentially $\mathrm{SL}_2(\mathbb{Z})$.

Closed geodesics and renormalization. Let $G_s(x)$ denote the unique geodesic on the flat surface $(X, |\omega|)_n$ passing through $x \in X$. By convention, a geodesic stops if it hits a zero of ω; otherwise, it continues in both directions.

In Section 4, we give a complete description of the behavior of the periodic geodesics $G_s(w_i)$ through the Weierstrass points of X. Here is a representative result, parallel to Theorem 1.1.

Theorem 1.2. *Let s be a periodic slope for $(X,\omega)_n$ and suppose n is odd. Then:*

(1) *There is a unique $k \in \mathbb{Z}/n$ such that $G_s(w_k)$ is a zero connection.*

(2) *Every other geodesic $G_s(w_i)$ is closed.*

(3) *We have $G_s(w_i) = G_s(w_j) \iff i + j = 2k \bmod n$.*

(4) *The special index k is again characterized by equation* (1.2).

See Corollaries 4.2 and 4.4 below.

The great advantage of passing from P_n to $(X,\omega)_n$ is that, because of its large group of real-affine symmetries, the study of closed geodesics can be reduced to just two cases, $s = 1$ and (when n is even) $s = 1 + \zeta_n$.

Unfolding reversed. In Appendix A, we develop and review the *general theory of unfolding* for a polygon P with angles in $\pi\mathbb{Q}$. The key construction is an intermediate surface (Y,η) equipped with a pair of locally isometric maps,

$$\begin{array}{ccc} (Y,|\eta|) & \xrightarrow{p} & (X,|\omega|) \\ {\scriptstyle\pi}\big\downarrow & & \\ (P,|dz|), & & \end{array}$$

to mediate between X and P. Using this construction, in Section 5, we show:

$$T_s(m_i) \cong T_s(m_j) \iff G_s(w_i) = G_s(w_j).$$

With these bridges in place, our main results on billiards follow rapidly from the theorems on closed geodesics sketched above; their proofs appear in Section 6.

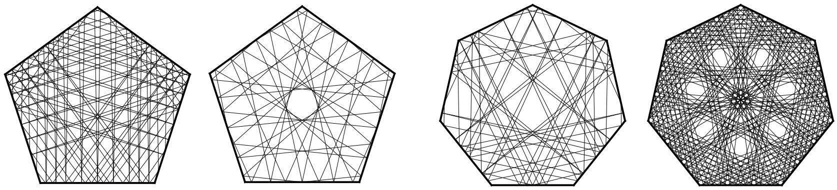

Figure 1.3
Dihedral versus reflective symmetry, for $n = 5$ and $n = 7$.

Dihedral versus reflective symmetry. Using the results on unfolding from Appendix A, we also show:

Theorem 1.3. *Suppose $n = p$ is prime, and let $\pi = 1 - \zeta_p$. Then the closed midpoint trajectories in P_n at slope $s \in \mathbb{Q}(\zeta_n)$ have full dihedral symmetry $\iff$ the valuation $v_\pi(s) = 0 \bmod 2$. Otherwise, the symmetry group is generated by a single reflection.*

For examples and a more complete discussion, see Figure 1.3 and Theorem 2.4. The proof appears in Section 7.

Monodromy over Teichmüller curves. For a somewhat different perspective on the results above, we recall that $(X, \omega)_n$ generates, via the Beltrami differential $\mu = \omega/\overline{\omega}$, a global deformation of X swept out by an isometrically immersed *Teichmüller curve*

$$f : V = \mathbb{H}/\Delta_n \to \mathcal{M}_g.$$

See [V1] and [Mc4, Section 3]. Parallel transport along loops in V provides a natural monodromy representation

$$\pi_1(V) \to \mathrm{Sym}\, W(X),$$

and in Section 3, we will show:

Theorem 1.4. *The image of $\pi_1(V)$ is a dihedral group, acting by rotations and reflections on the cyclically ordered Weierstrass points (w_i) labeled by the edge midpoints of P_n.*

(The one or two Weierstrass points not included among the (w_i) are fixed.)

It is remarkable that this cyclic structure on $W(X)$ is preserved along $f(V)$, even though the $\mathbb{Z}/n$ symmetry of X itself is lost under deformation. In contrast, the monodromy group for a typical family of hyperelliptic curves is the full symmetric group, $\mathrm{Sym}\, W(X) \cong S_{2g+2}$.

Action on homology. The action of $\pi_1(V)$ on $W(X)$ can be also related to its linear action on the homology of X (Section 3). In fact, when n is odd, we have compatible dihedral actions of $\pi_1(V)$ on each term in the sequence

$$W(X) \hookrightarrow \mathrm{Jac}(X)[2] \cong H_1(X, \mathbb{Z}/2) \to \mathcal{O}_L/2.$$

(See Remark 1 at the end of Section 3). The dihedral action on $\mathcal{O}_L/2$ gives rise to a purely arithmetic invariant,

$$\delta : \{\text{cusps of } \Delta_n\} \to \mathbb{Z}/n,$$

providing a link between $W(X)$ and the cusps of Δ_n that allows us to establish equations such as (1.2). (Relations between cusps invariant such as $\delta(s)$ and $[s]_2$ are summarized in Appendix B.)

Transition to chaos. An unexpected outcome of our investigation is that, for many $n \geq 12$, certain combinatorial features of billiards in P_n appear to behave chaotically; in particular, they are almost certainly not congruence invariants.

For a concrete example, suppose $n = 12$. Then every $s \in L^\times$ is a periodic direction, by Proposition 2.1. Let

$$\xi(s) = \begin{cases} +1 & \text{if } T_s(m_0) \text{ is a vertex connection, and} \\ -1 & \text{if not.} \end{cases}$$

Finally for $k \in \mathbb{Z}$, let $c_k = \xi(s_k)$, where $s_k = 1 + 2k\zeta_{12}$ satisfies

$$[s_k]_2 = [1]_2.$$

In [Mc5, Theorem 1.13] we show that, when $n = 12$, the location k of the vertex connection is *not* a cusp invariant. If it were, the sequence c_k would be a periodic. But experimentally, the opposite is true: the running sums of c_k seem to behave like a random walk on $\mathbb{Z}$, perhaps with an initial positive bias (see Figure 1.4).

Questions. We conclude with some directions for further investigation.

(1) Is the sequence c_k associated to billiards in P_{12} as above close to random? For example:

Does $(1/N)\sum_{k=1}^{N} c_k \to 0$*?*

Similar questions arise for P_n whenever n is even and $n \neq 2p^i$ for some prime p. For a discussion directly in terms of triangle groups, and an adelic perspective, see [Mc5].

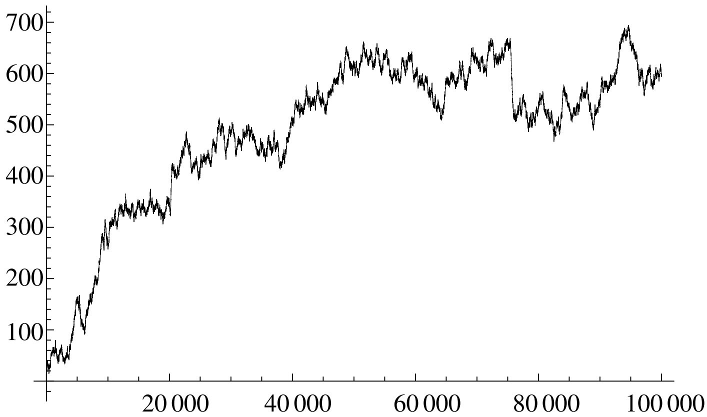

Figure 1.4
The running sum of c_k for billiards in P_{12}.

(2) Let $f : V \to \mathcal{M}_g$ be a Teichmüller curve generated by a primitive 1-form (X, ω). For all but two known examples (arising from E_7 and E_8, see [Mc4, Section 6]), there exists a canonical *Prym involution* J generating the center of $\mathrm{Aff}^+(X, \omega)$. Letting $\mathrm{Fix}(J)$ denote its set of fixed points, it is natural to ask:

How does $\mathrm{Aff}^+(X, \omega)$ *act on* $\mathrm{Fix}(J)$*?*

When X is hyperelliptic, J is the hyperelliptic involution and $\mathrm{Fix}(X) = W(X)$, the case studied here.

The question above is answered for the Weierstrass curves $W_D \subset \mathcal{M}_g$, $g = 2, 3, 4$, in [GP]; see also [Fr]. It can also be answered for the Bouw–Möller curves V_{pq}, using the explicit generators for their Veech groups given in [Ho]; for example, when $p \neq q$, the action is given by a subgroup of $D_p \times D_q$.

(3) Continuing as above, given a prime power ℓ^k, one can ask:

How does $\mathrm{Aff}^+(X, \omega)$ *act on* $H^1(X, \mathbb{Z}/\ell^k)$*?*

The study of these ℓ-adic representations is essentially the same as the study of the monodromy action of $\pi_1(V)$ on cohomology, for the family of curves of genus g it sweeps out in $\mathcal{M}_g$. The simplest case, $\ell^k = 2$, is studied here and in [Mc5]; for more complete results, see [FL].

(4) We can always realize $\mathrm{SL}(X, \omega)$ as a lattice in $\mathrm{SL}_2(\mathcal{O}_K)$ for some totally real field K, and one can ask:

What is the closure of $\mathrm{SL}(X, \omega)$ *in the profinite group* $\mathrm{SL}_2(\widehat{\mathcal{O}}_K)$*?*

The closure always has finite index; for more details, the relation to congruence invariants, and analogies with the Galois representations attached to elliptic curves, see [Mc5, Section 1].

Added in proof. A complete answer to the last question above for $\mathrm{SL}(X, \omega) = \Delta_n$ is obtained in [Cal].

Notes and references. The connection between billiards in regular polygons and Teichmüller curves was discovered and developed by Veech in [V1, V2]. A detailed study of billiards in the pentagon and its variants is given in [DL]; for the octagon, see [SU2, SU1]. The associated Teichmüller curves are now considered part of the Bouw–Möller series; see [BM] and [Mc4, Section 7]. Special cases of Appendix A appear in many of these works and elsewhere. For a recent survey of Teichmüller curves, see [Mc4].

This paper relies on the results of [Mc5], and is a sequel to [Mc2, Mc3].

I would like to thank D. Davis for many useful discussions, leading to the problem of classifying midpoint trajectories in the pentagon. See [ELM] for an algorithm to compute the answer for given s. The present paper provides, in Theorems 2.2 and 2.3, a closed-form solution in terms of the congruence invariant $[s]_2$.

A note on the figures. The billiard paths $T_s(m_i)$ shown in Figure 1.2 correspond to $s = \sum_0^{n-2} s_i \zeta_n^i$, where $(s_i) = (5, 6, 6, 6, 6, 6, 6, 4, -2)$, while the four examples in Figure 1.3 correspond to $(s_i) = (19, -2, 2, 21)$, $(4, 3, 0, 0)$, $(-3, -3, 3, 11, 13, 7)$ and $(-10, -3, 6, 6, 6, 6)$.

2. Billiards: Statements

In this section we present our main results on periodic billiards in P_n. To handle parity conditions smoothly, throughout this paper we adopt the convention that

$$m = n \textit{ when } n \textit{ is odd, and } n/2 \textit{ when } n \textit{ is even.}$$

Algebra. Let A be a commutative ring, with unit group $A^\times$. The shorthand A/a denotes the quotient of A by the principal ideal $(a) = aA$. The projective line over A is given by

$$\mathbb{P}^1(A) = \big\{[a : b] : aA + bA = A\big\}.$$

Here $[a : b] = [c : d] \iff (a, b) = (uc, ud)$ for some $u \in A^\times$. Any ring homomorphism $A \to B$ determines a natural map $\mathbb{P}^1(A) \to \mathbb{P}^1(B)$.

Given a free A-module V of rank two, we let

$$\mathbb{P}(V) = \big\{x \in V : V = Ax \oplus Ay \text{ for some } y \in V\big\}/A^\times. \tag{2.1}$$

Each choice of a free basis for V over A determines an isomorphism $\mathbb{P}(V) \cong \mathbb{P}^1(A)$.

Cyclotomic fields. Let $\zeta_n = \exp(2\pi i/n)$ and let $\varepsilon_n = \zeta_n + 1/\zeta_n$. As in the Introduction, we let

$$K = \mathbb{Q}(\varepsilon_n)$$

denote the totally real subfield of the cyclotomic field $L = \mathbb{Q}(\zeta_n)$. For brevity of notation, we have suppressed the dependence of K and L on n. The corresponding rings of integers are given by

$$\mathcal{O}_K = \mathbb{Z}[\varepsilon_n] \quad \text{and} \quad \mathcal{O}_L = \mathbb{Z}[\zeta_n];$$

see [Wa, Theorem 2.6 and Proposition 2.16].

Projective lines. Using the fact that $L = K \oplus K\zeta_n$, we have a natural map $s \mapsto [s]$ from $L^\times$ to $\mathbb{P}^1(K)$, characterized by

$$[a + b\zeta_n] = [a : b] \in \mathbb{P}^1(K)$$

for all $a, b \in K$. Given $d \neq 0$ in $\mathcal{O}_K$, we have a further reduction map,

$$\mathbb{P}^1(\mathcal{O}_K) \to \mathbb{P}^1(\mathcal{O}_K/d) \quad \text{denoted by } [s] \mapsto [s]_d,$$

taking values in a finite projective line.

Periodic directions. For later reference we recall:

Proposition 2.1. *We have* $[K : \mathbb{Q}] \leq 2 \iff n = 3, 4, 5, 6, 8, 10$ *or* $12 \iff$ *every* $s \in L^\times$ *specifies a periodic direction for billiards in* P_n.

These are among the most interesting values of n, since periodic slopes for these n are easy to produce. See e.g. [Mc3].

Billiards with an odd number of sides. We can now state our first result.

Theorem 2.2. *Let* $n \geq 3$ *be odd, and let* $s \in L^\times$ *be a periodic direction for billiards in* P_n. *Then there is a unique* $k \in \mathbb{Z}/n$ *such that*

$$[s]_2 = [\zeta_n^k]_2 \quad \text{in } \mathbb{P}^1(\mathcal{O}_K/2).$$

The trajectory $T_s(m_k)$ *is a vertex connection, and*

$$T_s(m_i) \cong T_s(m_j) \iff i + j = 2k \bmod n.$$

Here is a complement in the case where n is prime.

Theorem 2.3. *Suppose, in addition, that n is prime. Then*

$$\operatorname{length}(T_s(m_{k+\ell})) = C(s) \cdot |\sin(2\pi\ell/n)|$$

for $\ell = 1, 2, \ldots, n-1$.

Corollary 4.3 gives a similar statement for the lengths of closed geodesics, valid for all n.

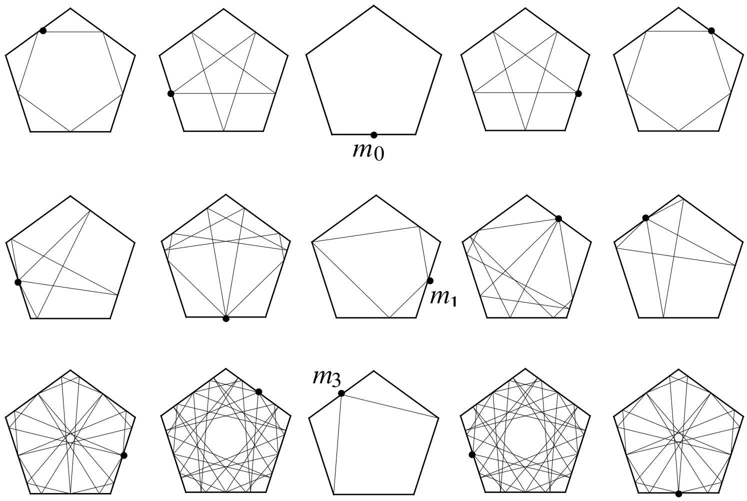

FIGURE 2.1
Midpoint trajectories in the regular pentagon.

Example: The pentagon. To illustrate these results, suppose $n = 5$. Then by Proposition 2.1, every $s \in L^\times = \mathbb{Q}(\zeta_5)^\times$ is a periodic direction. Rescaling if necessary, we may assume that

$$s = a + (b + c\gamma)\zeta_5,$$

where $\gamma = (1 + \sqrt{5})/2$ is the golden mean, and $a, b, c \in \mathbb{Z}$ are relatively prime. Then

$$[s]_2 = [\zeta_5] \quad \text{if } a = 0 \bmod 2;$$

otherwise, $a = 1 \bmod 2$ and

$$[s]_2 = [\zeta_5^k], \quad \text{where } k = \begin{cases} 0 & \text{if } (b, c) = (0, 0) \bmod 2, \\ 2 & \text{if } (b, c) = (1, 1) \bmod 2, \\ 3 & \text{if } (b, c) = (1, 0) \bmod 2, \text{ and} \\ 4 & \text{if } (b, c) = (0, 1) \bmod 2. \end{cases}$$

(In this case, $\mathbb{P}^1(\mathcal{O}_K/2) \cong \mathbb{P}^1\mathbb{F}_4$ has exactly 5 points, all of the form $[\zeta_5^k]$.)

The midpoint trajectories for $s = 1$, $s = 2 + 3\zeta_5$, and $s = 1 - (1 + 2\gamma)\zeta_5$ are shown in the three rows of Figure 2.1. These directions correspond to $k = 0, 1$, and 3 respectively. In each row, we have reordered the midpoints cyclically so that the vertex connection $T_s(m_k)$ occurs in the middle column. Consistent with Theorems 2.2 and 2.3, every closed trajectory occurs twice, up to isometry, with the short trajectories in columns 1 and 5 and the long trajectories in columns 2 and 4.

Dihedral versus reflection symmetry. Let us continue with the assumption that $n = p$ is prime. Then there are only two possibilities for the (isometric) symmetry group G of a closed trajectory $T_s(m_i)$; either $G \cong \mathbb{Z}/2$, or $G \cong D_n$ (the full dihedral group of symmetries of P_n). Both types appear in Figure 2.1.

Which alternative holds only depends on s. They can be distinguished using the valuation v_π on $L^\times$ associated to the prime $\pi = 1 - \zeta_p$, given explicitly by

$$v_\pi(s) = \sup\left\{i \in \mathbb{Z} : s \in \pi^i\, \mathcal{O}_L\right\}.$$

Theorem 2.4. *Let $n = p$ be prime, and let $s \in \mathbb{Q}(\zeta_p)$ be a periodic direction. Then the closed midpoint trajectories in direction s have full dihedral symmetry if and only if*

$$v_\pi(s) = 0 \bmod 2.$$

The condition above has two other equivalent formulations: letting $\rho = \pi\overline{\pi} = 2 - \varepsilon_n$ for $a, b \in K$, we have

$$v_\pi(s) = 0 \bmod 2 \iff [s]_\rho \neq [\pi]_\rho \iff s = a + b\pi, v_\rho(a) \leq v_\rho(b).$$

In particular, all these conditions are congruence invariants.

The pentagon, reprise. For $n = 5$, we can assume that $s = a + (b + c\sqrt{5})\zeta_5$ with $a, b, c \in \mathbb{Z}$. Using the fact that $v_\rho(x) = v_5(N_{\mathbb{Q}}^K(x))$, we find:

> *The closed midpoint trajectories $T_s(m_i)$ have dihedral symmetry* $\iff$
> $v_5((a + b)^2 - 5c^2) \leq v_5(a^2)$.

For example, when $s = 2 + 3\zeta_5$ we only have reflection symmetry, since $v_5(5^2) > v_5(2^2)$; but when $s = 1 - (2 + \sqrt{5})\zeta_5$ we have dihedral symmetry, since $v_5(4) \leq v_5(1)$. These trajectories are shown in the middle and bottom rows of Figure 2.1, respectively.

Billiards with an even number of sides. We now turn to the case where n is even. Several new features arise.

(1) The polygon P_n is symmetric under rotation by 180°. Thus $T_s(m_i) \cong T_s(m_j)$ whenever $i = j \bmod n/2$.

(2) The barycenter $c_0 \in P_n$ plays a special role. We refer to a midpoint trajectory passing through c_0 as a *central loop*.

(3) We say a trajectory $T_s(m_i)$ is *special* if it is a vertex connection or a central loop. Otherwise, it is *standard*.

(4) As we will see, it can be tricky to distinguish between the two types of special connections. However this can be accomplished when $n = 2p^i$ and $p \geq 2$ is prime.

The case $n = 4g + 2$. In this case, $\zeta_n = -\zeta_{2g+1}$, and thus $[\zeta_n^{2g+1}]_2 = [1]_2$.

Theorem 2.5. *Let $n = 4g + 2$, and let s be a periodic slope for P_n. Then there is a unique $k \in \mathbb{Z}/(2g+1)$ such that $[s]_2 = [\zeta_n^k]_2$, and we have*

$$T_s(m_i) \cong T_s(m_j) \iff i + j = 2k \bmod 2g+1.$$

The trajectory $T_s(m_i)$ is special iff $i = k \bmod 2g+1$.

The next result describes the special trajectory in many cases, including the case $n = 10$.

Theorem 2.6. *Suppose $n = 4g + 2 = 2p^i$ for some prime $p \geq 3$, with k as above. Let $v_\pi(s)$ denote the valuation of s at the prime $1 - \zeta_{p^i} \in \mathcal{O}_L$. Then:*

$$T_s(m_k) \text{ is } \begin{cases} \text{a vertex connection} & \text{if } v_\pi(s) \text{ is even, and} \\ \text{a central loop} & \text{if } v_\pi(s) \text{ is odd.} \end{cases}$$

The case $n = 4g$. Now assume $n = 0 \bmod 4$.

Theorem 2.7. *Let $n = 4g$, and let s be a periodic direction in P_n. Then there is a unique pair (ε, k), with $\varepsilon \in \{0, 1\}$, such that*

$$[s/(1+\zeta_n)^\varepsilon]_2 = [\zeta_n^k]_2.$$

Here $k \in \mathbb{Z}/2g$ if n is a power of two, and otherwise, $k \in \mathbb{Z}/g$.

The midpoint trajectories satisfy

$$T_s(m_i) \cong T_s(m_j) \iff i + j = 2k + \varepsilon \bmod 2g,$$

and $T_s(m_i)$ is special $\iff \varepsilon = 0$ and $i = k \bmod g$.

Theorem 2.8. *If $\varepsilon = 0$, n is a power of two, and k is as above, then*

$$T_s(m_i) \text{ is } \begin{cases} \text{a vertex connection} & \iff i = k \bmod 2g, \text{ and} \\ \text{a central loop} & \iff i = k + g \bmod 2g. \end{cases}$$

Example for $n = 8$. Let $s = 4 + 2\sqrt{2} + \zeta_8$. Since $4 + 2\sqrt{2} = 0$ in $\mathcal{O}_L/2$, we find $[s]_2 = [\zeta_8]_2$; thus $(\varepsilon, k) = (0, 1)$, and $T_s(m_i)$ is special $\iff i = 1 \bmod 2$. In fact, by Theorem 2.8, $T_s(m_1)$ is a vertex connection and $T_s(m_3)$ is a central loop; see Figure 2.2 (left).

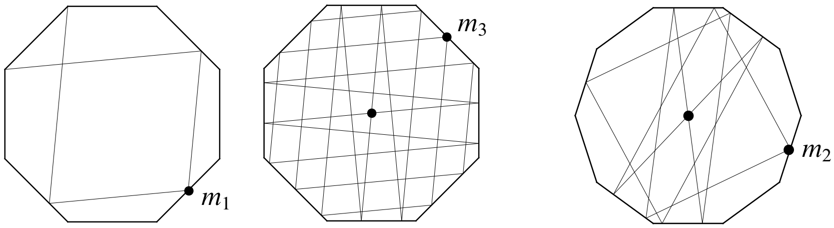

Figure 2.2
Three special trajectories: A vertex connection $T_s(m_1)$ and a central loop $T_s(m_3)$ in the octagon, and a central loop $T_s(m_2)$ in the decagon.

Example for $n = 10$. Let $s = 1 + (4 - \gamma)\zeta_{10}$. Then $[s]_2 = [\zeta_{10}^2]_2$, and

$$v_\pi(s) = v_5(N_{\mathbb{Q}}^L(s)) = v_5(205) = 1,$$

so $T_s(m_2)$ is a central connection, by Theorem 2.6. See Figure 2.2 (right).

Remarks. Methods to easily compute $[s]_2$ and $[s]_\pi$ are discussed in [Mc5, Section 4].

3. The Riemann surface of a regular polygon

In this section we turn to the study of the holomorphic 1-form $(X, \omega)_n$ naturally attached to the regular polygon P_n. As usual, $m = n$ if n is odd, and $n/2$ if n is even. Let

$$D_m = \langle r, t\rangle \subset \mathrm{Sym}(\mathbb{Z}/m) \tag{3.1}$$

denote the dihedral group, generated by $r(x) = x + 1$ and $t(x) = -x$. Note that $|D_m| = 2m$ if $m > 2$, but $D_2 \cong \mathbb{Z}/2$.

We begin by defining the form $(X, \omega)_n$, its group of affine symmetries, and the Weierstrass points $w_i \in W(X)$ labeled by the edge midpoints $m_i \in P_n$. We then show:

Theorem 3.1. *The image of* $\mathrm{Aff}^+(X, \omega)_n$ *in* $\mathrm{Sym}\, W(X)$ *is the dihedral group* D_m. *More precisely, the standard generators* ρ *and* τ *of* $\mathrm{Aff}^+(X, \omega)$ *satisfy*

$$\rho(w_i) = w_{i+1} \quad \textit{and} \quad \tau(w_i) = w_{-i},$$

while the remaining points of $W(X)$ *are fixed by both* ρ *and* τ.

We then introduce the group $\Delta_n \subset \mathrm{SL}_2(\mathcal{O}_K)$ studied in [Mc5], and show:

Theorem 3.2. *The action of* $\mathrm{Aff}^+(X,\omega)$ *on the space of relative periods of* ω,

$$\mathcal{O}_L = \mathcal{O}_K \oplus \mathcal{O}_K(1+\zeta_n),$$

determines a natural isomorphism

$$D : \mathrm{Aff}^+(X,\omega) \cong \Delta_n \subset \mathrm{SL}_2(\mathcal{O}_K).$$

Finally we relate the invariant $\delta(s) \in \mathbb{Z}/m$ for cusps of Δ_n, introduced in [Mc5], to the action of $\mathrm{Aff}^+(X,\omega)$ on $W(X)$.

Theorem 3.3. *Given* $\phi \in \mathrm{Aff}^+(X,\omega)$, *let* $[s] = [D\phi(1)] \in \mathbb{P}(\mathcal{O}_L)$, *and let* $k = \delta(s)$. *Then* $\phi(w_0) = w_k$.

Theorem 3.4. *If* n *is even,* $[s] = [D\phi(1+\zeta_n)]$, *and* $k = \delta(s)$, *then*

$$\phi(\{w_0, w_1\}) = \{w_k, w_{k+1}\}.$$

The last two results connect the combinatorics of closed geodesics and billiard trajectories to the function $\delta(s)$, which can itself be expressed in terms of congruence invariants using the results of [Mc5]; see Appendix B.

Preliminaries. Let $(X,\omega) \in \Omega\mathcal{M}_g$ be a nonzero holomorphic 1-form on a compact Riemann surface of genus g, with zero set $Z(\omega)$. The 1-form ω provides, away from its zeros, local *linear charts*

$$z : U \to \mathbb{C} \cong \mathbb{R}^2,$$

characterized by $dz = \omega$. A (real) *affine automorphism* of (X,ω) is homeomorphism $\phi : X \to X$ that preserves $Z(\omega)$ and, in linear charts, has the local form

$$\phi(z) = Az + b.$$

Here $A \in \mathrm{GL}_2(\mathbb{R})$ is an automorphism of $\mathbb{C} \cong \mathbb{R}^2$ as a *real* vector space, and $b \in \mathbb{C}$. The linear map $A = D\phi$ is an invariant of ϕ, with determinant ± 1.

We define the *affine group* by

$$\mathrm{Aff}^+(X,\omega) = \{\phi : \det D\phi = 1\}.$$

Its image under D is a countable group,

$$\mathrm{SL}(X,\omega) \subset \mathrm{SL}_2(\mathbb{R}).$$

Let $\mathrm{Aut}(X,\omega)$ denote the group of holomorphic automorphisms of X satisfying $\phi^*(\omega) = \omega$. We then have an exact sequence

$$1 \to \mathrm{Aut}(X,\omega) \to \mathrm{Aff}^+(X,\omega) \xrightarrow{D} \mathrm{SL}(X,\omega) \to 1.$$

The polygonal 1-form. Via the unfolding construction reviewed in Appendix A, the regular polygon P_n naturally determines a holomorphic 1-form on a compact Riemann surface of genus g, given concretely by

$$(X,\omega)_n = \begin{cases} (P_n, dz)/\sim & \text{for } n \text{ even, and} \\ (P_n, dz) \sqcup (-P_n, dz)/\sim & \text{for } n \text{ odd.} \end{cases}$$

The equivalence relation identifies parallel edges by translation. The relationship between n and g depends on the value of $n \bmod 4$; it is given by $n = 2g+1$, $4g$ or $4g+2$.

Zeros. The natural map

$$\iota : P_n \to X$$

sends the vertices of P_n to the zero set $Z(\omega)$, which consists of two points $\{z_1, z_2\}$ when $n = 4g+2$, and otherwise a single point $\{z_0\}$. Since the relative homology of X is generated by the edges of P, the relative periods of ω are given by

$$\text{(3.2)} \qquad \operatorname{Per}_{\mathrm{rel}}(\omega) = \left\{ \int_C \omega : C \in H_1(X, Z(\omega); \mathbb{Z}) \right\} = \mathbb{Z}[\zeta_n] = \mathcal{O}_L.$$

Weierstrass points. The hyperelliptic involution $\eta : X \to X$ rotates $\iota(P_n)$ by 180 degrees, exchanging P_n and $-P_n$ when n is odd. Its fixed points $W(X)$ are the *Weierstrass points* of X; they consist of the m points

$$w_i = \iota(m_i), \quad i \in \mathbb{Z}/m,$$

arising from the edge midpoints of P_n; the zero z_0 of ω, when $n \neq 4g+2$; and the image of the *center* of P_n,

$$c_0 = \iota(0),$$

when n is even. In brief, we have

$$W(X) = \{w_0, \ldots, w_{m-1}\} \cup \begin{cases} \{z_0\} & \text{when } n = 2g+1, \\ \{c_0\} & \text{when } n = 4g+2, \text{ and} \\ \{c_0, z_0\} & \text{when } n = 4g. \end{cases}$$

Note that opposite midpoints of P_n are identified in X when n is even.

The cases $n = 3, 4,$ and 6. For these values of n, X has genus $g = 1$ and ω has no zeros. In this case, we adopt the convention that $Z(\omega)$ still corresponds to the vertices of P_n, and regard these points as zeros of ω with *multiplicity zero*. With this convention, all values $n \geq 3$ can be treated in a uniform fashion, provided we also require that the affine automorphisms of (X, ω) preserve the sets $Z(\omega)$ and $\{w_0, \ldots, w_{m-1}\}$ coming from the vertices and edge-midpoints of P_n. This condition is automatic when $g \geq 2$.

Affine automorphisms. It is easy to see that (with the convention above) $\mathrm{Aut}(X,\omega)_n$ is trivial, and hence we have an isomorphism

$$\mathrm{Aff}^+(X,\omega)_n \cong \mathrm{SL}(X,\omega)_n;$$

equivalently, any affine automorphism ϕ is unique determined by its derivative $D\phi$.

Generators. The affine group of $(X,\omega)_n$ is generated by a rotation ρ and a twist τ, uniquely characterized by

$$D\rho = \begin{pmatrix} \cos 2\pi/n & -\sin 2\pi/n \\ \sin 2\pi/n & \cos 2\pi/n \end{pmatrix} \quad \text{and} \quad D\tau = \begin{pmatrix} 1 & 2\cot(\pi/n) \\ 0 & 1 \end{pmatrix} \tag{3.3}$$

with respect to the basis $\mathbb{C} = \mathbb{R} \oplus \mathbb{R} \cdot i$; cf. [V1, Theorem 1.1]. The rotation ρ is compatible, under the map $\iota : P_n \to X$, with a rigid rotation of P_n through angle $2\pi/n$; it is an isometry for the metric $|\omega|$, and its derivative is given by

$$D\rho(s) = \zeta_n s \quad \text{for all } s \in \mathbb{C}. \tag{3.4}$$

The map τ is a composition of Dehn twists along the horizontal cylinders of (X,ω); its derivative is a shear, characterized by

$$D\tau(1) = 1 \quad \text{and} \quad D\tau(\zeta_n) = 2 + \varepsilon_n + \zeta_n. \tag{3.5}$$

Proof of Theorem 3.1. The statement $\rho(w_i) = w_{i+1}$ is immediate by the geometry of the rotation. To understand the action of τ, it is useful to observe that when $n = 2g+1$ is odd, X decomposes into g horizontal cylinders C_i, each of the same modulus. The hyperelliptic involution η acts by a flip on each cylinder, fixing the pair of Weierstrass points w_i and w_{-i} on the core curve of C_i. The map τ fixes ∂C_i and performs a right Dehn twist; thus it exchanges w_i and w_{-i}, as claimed above.

A similar argument applies when n is even, and it is readily verified that both ρ and τ fix the points z_0 and c_0 when they are defined. ∎

The triangle group $\boldsymbol{\Delta_n}$. Next we recall definitions and results from [Mc5]. Let

$$\Delta_n = \langle T, U \rangle \subset \mathrm{SL}_2(\mathcal{O}_K)$$

denote the discrete group generated by

$$T = \begin{pmatrix} 1 & 2+\varepsilon_n \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad U = -\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}. \tag{3.6}$$

Geometrically, Δ_n is isomorphic to the triangle group $\Delta(2,n,\infty)$ when n is odd, and $\Delta(n/2,\infty,\infty)$ when n is even. We can alternatively write

$$\Delta_n = \langle R, T \rangle,$$

where $R = UT$ has order n.

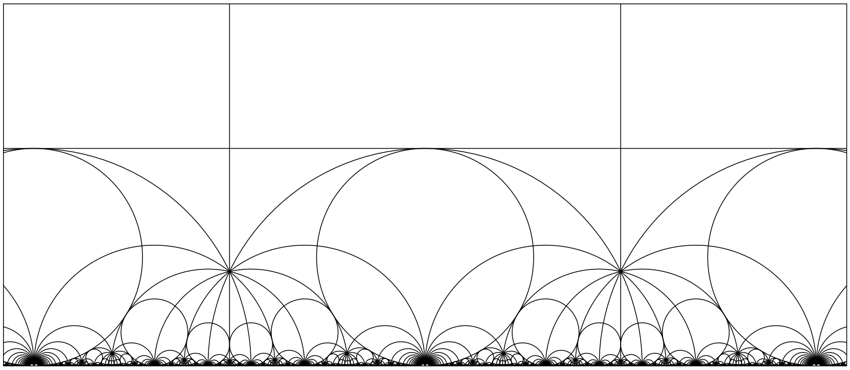

FIGURE 3.1
Tiling of $\mathbb{H}$ by triangular fundamental domains for Δ_7. The horoballs rest on the cusps of Δ_7, which are periodic directions for billiards in a regular heptagon.

Cusps. The *cusps* of Δ_n (its parabolic fixed points) coincide with the orbits of 0 and ∞ in $\mathbb{P}^1(\mathcal{O}_K)$. When n is odd, these orbits coincide; for even n, they are disjoint. See Figure 3.1 for the case $n = 7$.

Conjugation by

$$S = \begin{pmatrix} 0 & \varepsilon_{2n} \\ -\varepsilon_{2n}^{-1} & 0 \end{pmatrix}$$

normalizes Δ_n, and sends (T, U) to $(-U, -T)$, since $\varepsilon_{2n}^2 = 2 + \varepsilon_n$. When n is odd, S belongs to Δ_n. When n is even, it interchanges the cusps at 0 and ∞.

The dihedral invariant. By [Mc5, Corollary 1.3], there is a unique map

$$\delta : (\Delta_n \cdot \infty) \to \mathbb{Z}/m,$$

the *dihedral invariant*, such that

$$\delta(\infty) = 0, \quad \delta(Rx) = \delta(x) + 1 \quad \text{and} \quad \delta(Tx) = -\delta(x). \tag{3.7}$$

It follows that $\delta(Ux) = 1 - \delta(x)$ on $(\Delta_n \cdot \infty)$.

When n is even, we extend its definition to the remaining cusps of Δ_n by requiring that

$$\delta(Sx) = -\delta(x)$$

for all cusps x. The restriction of $\delta(x)$ to $(\Delta_n \cdot 0)$ is then characterized by the conditions

$$\delta(0) = 0, \quad \delta(Rx) = \delta(x) + 1 \quad \text{and} \quad \delta(Tx) = 1 - \delta(x). \tag{3.8}$$

It follows that when n is even, $\delta(Ux) = -\delta(x)$ on $(\Delta_n \cdot 0)$.

Action on periods. The natural action of $\mathrm{SL}(X,\omega)_n$ on $\mathbb{C}$ preserves the group of relative periods $\mathcal{O}_L$ of ω, since

$$\int_{\phi(C)} \omega = D\phi\Big(\int_C \omega\Big).$$

Since $D\phi$ is $\mathbb{R}$-linear, it preserves the structure of $\mathcal{O}_L$ as an $\mathcal{O}_K$-module. Thus, upon fixing the basis

$$\mathcal{O}_L = \mathcal{O}_K \oplus \mathcal{O}_K(1+\zeta_n), \tag{3.9}$$

we can regard the derivative as a map

$$D : \mathrm{Aff}^+(X,\omega) \to \mathrm{SL}_2(\mathcal{O}_K).$$

Proof of Theorem 3.2. Using equations (3.4) and (3.5), it is readily verified that the generators of $\mathrm{Aff}^+(X,\omega)$ satisfy $D\rho = R$ and $D\tau = T$; thus D gives an isomorphism between $\mathrm{Aff}^+(X,\omega) \cong \mathrm{SL}(X,\omega)$ and Δ_n. ∎

Remark: Bases for $\mathbb{C}$. It is useful to regard $D\phi : \mathbb{C} \to \mathbb{C}$ as a linear map on an abstract vector space over $\mathbb{R}$. Thus the realization of $\Delta_n = \mathrm{SL}(X,\omega)_n$ as a matrix group in $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathcal{O}_K)$ depends on the choice of a basis for $\mathbb{C}/\mathbb{R}$. The basis $(1, i)$ gives the matrices in equation (3.3), while the basis $(1, 1+\zeta_n)$ gives equation (3.6).

The second matrix representation is more useful for us, since it is adapted to the geometry and arithmetic of $(X,\omega)_n$.

Projective lines. Let us regard L and $\mathcal{O}_L$ as free, rank two modules over K and $\mathcal{O}_K$ respectively, and define the corresponding projective lines

$$\mathbb{P}(L) \quad \text{and} \quad \mathbb{P}(\mathcal{O}_L)$$

by equation (2.1).

The choice of basis for $\mathcal{O}_L$ over $\mathcal{O}_K$ given in equation (3.9) also provides a basis for L/K and hence an isomorphism

$$\mathbb{P}(L) \cong \mathbb{P}^1(K).$$

The group $\mathrm{SL}(X,\omega) \cong \Delta_n$ acts on $\mathbb{P}(L)$, preserving its integral points $\mathbb{P}(\mathcal{O}_L) \cong \mathbb{P}^1(\mathcal{O}_K)$.

Any $s \in L^\times$ determines a point $[s] \in \mathbb{P}(L)$, and $[s] = [a:b]$ in $\mathbb{P}^1(K)$ if $s = a + b(1+\zeta_n)$ with $a, b \in K$. For example, the fixed points of T and U are given by

$$[1] = [1:0] \quad \text{and} \quad [1+\zeta_n] = [0:1]$$

respectively. Since these lie in $\mathbb{P}^1(\mathcal{O}_K)$, so does every other cusp of Δ_n.

In these coordinates, the dihedral invariant $\delta(x)$ is naturally defined on the set of cusps

$$\Delta_n \cdot \{[1], [1+\zeta_n]\} \subset \mathbb{P}(\mathcal{O}_L).$$

For simplicity, we adopt the notational convention that

$$\delta(s) = \delta([s])$$

whenever $s \in L^\times$ and $[s]$ represents a cusp of Δ_n. Thus $\delta(ks) = \delta(s)$ for all $k \in K^\times$.

Proof of Theorem 3.3. Define a map

$$\delta' : \Delta_n \to \mathbb{Z}/m$$

by $\delta'(D\phi) = k$ whenever $\phi(w_0) = w_k$. Since the stabilizer of $[1] \in \mathbb{P}(\mathcal{O}_L)$ is generated by $T = D\tau$, and $\tau(w_0) = w_0$, we find that δ' descends to a function on

$$\Delta_n/\langle T\rangle \cong (\Delta_n \cdot [1]).$$

In view of Theorem 3.1, it is clear that

$$\delta'(1) = 0, \quad \delta'(Rx) = \delta'(x) + 1 \quad \text{and} \quad \delta'(Tx) = -\delta'(x).$$

By (3.7), these properties also characterize $\delta(x)$, so the two functions agree. ∎

Proof of Theorem 3.4. Let $\upsilon = \rho\tau^{-1}$. Note that $D\upsilon = U$ and $\upsilon(w_i) = w_{1-i}$. Since U generates the stabilizer of $[1+\zeta_n]$ in Δ_n, the function $\delta' : \Delta_n \to \mathbb{Z}/m$, defined by

$$\delta'(D\phi) = k \iff \phi(\{w_0, w_1\}) = \{w_k, w_{k+1}\},$$

descends to a map

$$\delta' : (\Delta_n \cdot [1+\zeta_n]) \to \mathbb{Z}/m.$$

By Theorem 3.1 again, it is clear that

$$\delta'(1+\zeta_n) = 0, \quad \delta'(Ux) = -\delta'(x) \quad \text{and} \quad \delta'(Tx) = 1 - \delta'(x).$$

By equation (3.8), these properties also characterize δ, so the two functions agree. ∎

Remarks. (1) Theorem 3.3 describes, for n odd, the behavior of the map from cusps to Weierstrass points given by

$$p : [s] \mapsto w_k \in W(X), \quad \text{where } \delta(s) = k. \tag{3.10}$$

This map can be described functorially as follows. First, we have a natural map

$$p_1 : \{\text{cusps of } \Delta_n\} = (\Delta_n \cdot 1) \subset \mathcal{O}_L/(\pm 1) \to \mathcal{O}_L/2.$$

Here we have implicitly used the fact that each cusp $[s] \in \mathbb{P}(\mathcal{O}_L)$ has a natural lift to $\mathcal{O}_L$, well defined up to sign (rather that just up to a unit in $\mathcal{O}_K$). Second, we have a natural map

$$p_2 : W(X) \to \mathrm{Jac}(X)[2] \cong H_1(X, \mathbb{Z}/2) \to \mathcal{O}_L/2,$$

given by first sending w to the divisor class $[w - z_0]$, and then evaluating $2\int_{z_0}^{w} \omega$. (When n is prime, the last arrow above is an isomorphism.) The map p_2 is injective, and we have $p = p_2^{-1} \circ p_1$. A similar discussion underlies Theorem 3.4.

(2) It is known that when $\mathrm{SL}(X, \omega)$ is nonelementary, its trace field and its invariant trace field coincide [Ho, Lemma 10.1]. Thus to prove that ρ and τ generate $\mathrm{Aff}^+(X, \omega)$, it suffices to observe that $\Delta(2, n, \infty)$ is a maximal discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$, and that any extension of $\Delta(n/2, \infty, \infty)$ has larger trace field.

4. Classification of closed geodesics

In this section we describe the combinatorics of closed geodesics on $(X, \omega)_n$ in terms of the cusp invariants $\delta(s)$ and $\kappa(s)$.

Cusps. The fact that $\mathrm{SL}(X, \omega)_n$ has only one or two cusps means, remarkably, that all closed geodesics on X for the metric $|\omega|$ are affinely equivalent to one or two models. To distinguish the two models that arise when n is even, we define the *class invariant* by

$$\kappa(s) = \begin{cases} 0 & \text{if } [s] \in \Delta_n \cdot [1], \text{ and} \\ 1 & \text{if } [s] \in \Delta_n \cdot [1 + \zeta_n]. \end{cases} \tag{4.1}$$

We also adopt the convention that $\kappa(s) = 0$ when n is odd.

Geodesics. Given $x \in X - Z(\omega)$ and $s \in L^\times$, we let $G_s(x) \subset X$ denote the complete geodesic through x in the direction s. A geodesic ends if it encounters a zero of ω.

We will only consider *periodic directions*, where $[s]$ is a cusp of $\mathrm{SL}(X, \omega)_n$. In this case $G_s(x)$ is either

- a *closed geodesic*, homeomorphic to S^1 and disjoint from $Z(\omega)$, or
- a *zero connection*, beginning and ending in $Z(\omega)$.

For n even, we say a closed geodesic $G_s(w_i)$ is a *central loop* if it contains the Weierstrass point $c_0 \in X$ coming from the center of P_n.

We let $|G| = \int_G |\omega|$ denote the length of a geodesic with respect to the singular flat metric determined by ω.

Statement of results. The first result reduces the study of an arbitrary direction s to the easily analyzed cases $s = 1$ and (if n is even) $s = 1 + \zeta_n$.

Theorem 4.1. *Let $s \in L^\times$ be a periodic direction for $(X, \omega)_n$. Let $k = \delta(s)$ and let $t = (1 + \zeta_n)^\varepsilon$, where $\varepsilon = \kappa(s)$. Then there exists an affine automorphism ϕ of $(X, \omega)_n$ such that*

$$G_s(w_{k+i}) = \phi(G_t(w_i))$$

for all $i \in \mathbb{Z}/m$.

Our main results are corollaries of this basic theorem.

Corollary 4.2. *For all i, j,*

$$G_s(w_i) = G_s(w_j) \iff i + j = 2\delta(s) + \kappa(s) \bmod m.$$

Corollary 4.3. *There exists a constant $C(s)$ such that for all i we have*

$$|G_s(w_{k+i})| = C(s) \cdot |G_t(w_i)|,$$

where $k = \delta(s)$.

Corollary 4.4. *For all n, s and i:*

$$G_s(w_i) \text{ is a zero connection } \iff i = \delta(s) \text{ and } \kappa(s) = 0.$$

If $n = 4g + 2$, then

$$G_s(w_i) \text{ is a central loop } \iff \kappa(s) = 1 \text{ and } i = \delta(s) + g + 1.$$

If $n = 4g$, then

$$G_s(w_i) \text{ is a central loop } \iff \kappa(s) = 0 \text{ and } i = \delta(s) + g.$$

In all other cases, $G_s(w_i)$ is a closed geodesic.

Proof of Theorem 4.1. First suppose that n is odd. Let $s \in L^\times$ be a periodic direction. Since Δ_n has only one type of cusp, we have $[s] \in \Delta_n \cdot [1]$. Thus there exists an affine automorphism ϕ of $(X, \omega)_n$ such that $D\phi \cdot [1] = [s]$. By Theorem 3.3, we have $\phi(w_0) = w_k$, where $k = \delta(s)$.

Since ϕ acts on $W(X)$ by a dihedral transformation (Theorem 3.1), we also have $\phi(w_1) = w_{k\pm 1}$. Replacing ϕ with $\phi \circ \tau$ if necessary, we can assume that $\phi(w_1) = \phi(w_{k+1})$. It then follows that

$$\phi(w_i) = w_{k+i}$$

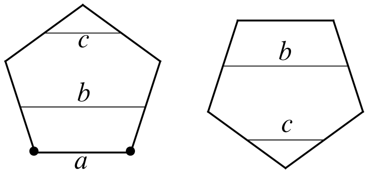

FIGURE 4.1
For $n = 2g + 1$, every closed geodesic is equivalent to one in the direction $s = 1$.

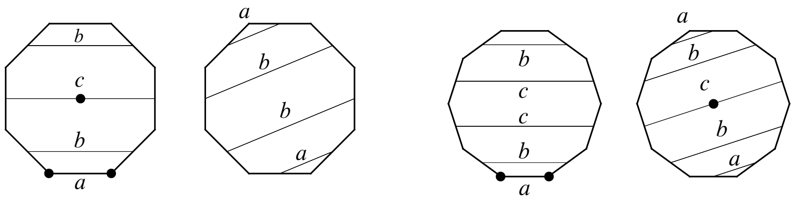

FIGURE 4.2
For $n = 4g$ or $4g + 2$, every closed geodesic is equivalent to one in the direction $s = 1$ or $s = 1 + \zeta_n$.

for all $i \in \mathbb{Z}/m$. Then, since ϕ sends geodesics to geodesics, and maps [1] to $[s]$, we have

$$G_s(w_{k+i}) = \phi(G_1(w_i)) \tag{4.2}$$

for all i.

Next suppose that n is even. If $\kappa(s) = 0$, then $[s] \in \Delta_n \cdot [1]$, and the same argument applies. Now suppose $\kappa(s) = 1$. Then $[s] \in (\Delta_n \cdot [1 + \delta_n])$. Choose $\phi \in \mathrm{Aff}^+(X, \omega)$ so that $D\phi \cdot [1 + \zeta_n] = [s]$. Then by Theorem 3.4, we have

$$\phi(\{w_0, w_1\}) = \{w_k, w_{k+1}\}.$$

Recall that $\upsilon = \rho\tau^{-1}$ swaps w_0 and w_1, while $D\upsilon = U$ fixes $[1 + \zeta_n]$. Thus, upon replacing ϕ with $\phi \circ \upsilon$ if necessary, we can assume that ϕ sends w_0 to w_k and w_1 to w_{k+1}. It then follows that $\phi(w_i) = w_{i+k}$ for all i, and the proof is completed as above. ■

Proof of Corollary 4.2. In view of Theorem 4.1, it suffices to verify this result in the cases $s = 1$ and $s = 1 + \zeta_n$. Inspecting Figures 4.1 and 4.2, we see that $G_1(w_i) = G_1(w_j)$ iff $i + j = 0 = \kappa(1)$, and $G_{1+\zeta_n}(w_i) = G_{1+\zeta_n}(w_j) \iff i + j = 1 = \kappa(t)$, where $t = 1 + \zeta_n$. ■

Proof of Corollary 4.3. The Corollary follows from the fact that the affine transformation ϕ stretches all vectors in the direction t by the same factor $C(s)$. ■

Proof of Corollary 4.4. Since any affine transformation ϕ stabilizes $Z(\omega)$, and fixes $c_0 \in X$ when n is even, if G is a zero connection, a central loop or a closed geodesic, then so is $\phi(G)$. Thus it suffices, by Theorem 4.1, to prove these Corollaries in the cases $s = 1$ and (when n is even) $s = 1 + \zeta_n$. The proof is readily completed by inspecting the representative examples in Figures 4.1 and 4.2, where $n = 5, 8$, and 10. ■

Remarks. Corollary 4.3 shows that the *ratios* of lengths of geodesics can be predicted using $\delta(s)$ and $\kappa(s)$.

Actual lengths are harder to control. We have $C(s) = O(1 + h(s)^d)$ when $d = [K : \mathbb{Q}] \leq 2$, but already for $n = 7$, there is no known explicit bound for $C(s)$ in terms of the height $h(s)$. See [Mc3] and [Mc4, Question 3.6].

5. From geodesics to billiard trajectories

To relate the previous results to billiards, in this section we will prove:

Theorem 5.1. *Let s be a periodic slope for billiards in P_n. Then $T_s(m_i) \cong T_s(m_j) \iff G_s(w_i) = G_s(w_j)$.*

The proof will use Theorem A.1 from the Appendix and special properties of regular polygons. We recall from equation (1.1) that $T \cong T'$ means $T = \alpha \cdot T'$ for some isometry α of P_n.

Trajectories in P_n. Following Appendix A, we let $G \subset \mathrm{O}(2)$ denote the group generated by the linear parts of reflections through the sides of P_n, and let $H = G \cap \mathrm{Isom}(P_n)$.

Lemma 5.2. *The closed trajectory $T_s(m_i)$ is invariant under the unique reflection $r \in \mathrm{Isom}(P_n)$ fixing m_i.*

Proof. Observe that $T_s(m_i)$ returns to m_i in the direction $r \cdot [s]$ when it closes; thus $r \cdot T_s(m_i) = T_{r \cdot s}(m_i)$ is simply the same trajectory traced in the opposite direction. ■

Although it is not needed in the sequel, we add:

Lemma 5.3. *The closed trajectory $T_s(m_i)$ has at most 4 segments incident to m_i.*

See Figure 2.1 for examples where 4 segments actually occur.

Proof. Let δ be the path obtained by following $T_s(m_i)$ from its start in direction $[s] \in \mathbb{RP}^1$ to its first return to m_i, in direction $[t]$. If $[t] = r \cdot [s]$, then the trajectory closes and it has at most two segments incident to m_i. Otherwise, the rest of the trajectory follows the path $r \cdot \delta$ in reverse, departing in direction $r \cdot [t]$ and returning in direction $r \cdot [s]$. Thus $T_s(m_i)$ has at most 4 segments incident to m_i. ■

Lemma 5.4. *We have* $T_s(m_i) \cong T_s(m_j) \iff h \cdot T_s(m_i) = h \cdot T_s(m_j)$ *for some* $h \in H$.

Proof. One implication is immediate since we have $H \subset \mathrm{Isom}(P_n)$. For the other, suppose $\alpha \cdot T_s(m_i) = T_s(m_j)$ for some $\alpha \in \mathrm{Isom}(P_n)$. We wish to show that we can choose $\alpha \in H$.

By Lemma 5.2, the unique reflection $r \in \mathrm{Isom}(P_n)$ fixing m_i also stabilizes $T_s(m_i)$. When $n \not\equiv 0 \bmod 4$, one can readily check that $\mathrm{Isom}(P_n) = H \cup rH$, so either α or $r\alpha$ is in H and the proof is complete (cf. Table A.1).

Now assume $n = 4g$ for some g. In this case we have $G = H \cong D'_{4g}$ has index 2 in $\mathrm{Isom}(P_n) \cong D_{4g}$, and $r \in H$, so a different argument is required. We first note that the directions of segments appearing in $T_s(m_j)$ lie in

$$G \cdot [s] = H \cdots [s] \subset \mathbb{RP}^2,$$

and $\alpha \cdot [s]$ is among them, since $\alpha \cdot T_s(m_i) = T_s(m_j)$. Thus $h\alpha \cdot [s] = [s]$ for some $h \in H$. This implies $\alpha \in H$ unless $[s]$ has a nontrivial stabilizer in $\mathrm{Isom}(P_n)$. Up to rotation, the only such slopes are $s = 1$ or $s = 1 + \zeta_n$; these cases can be checked directly. ■

Lemma 5.5. *If a closed geodesic satisfies* $g \cdot G_s(w_i) = G_s(w_j)$ *for some* $g \in G$*, then* $G_s(w_i) = G_s(w_j)$.

Proof. Clearly, $g(s) = \pm s$. The case where $g = I$ is the identity is trivial. If $g = -I$, it acts by the hyperelliptic involution on (X, ω), which stabilizes *all* closed geodesics, so $g \cdot G_s(w_i) = G_s(w_i)$ as desired. Finally, there are the finitely many cases where $g(s)$ is reflection fixing $[s] \in \mathbb{RP}^1$. Conjugating by an automorphism of $(X, |\omega|)$, we can assume that $s = 1$; and then one can verify by inspection that all closed geodesics of the form $G_1(w_i)$ are invariant under reflection (see Figure 5.1). ■

Lemma 5.6. *Given a periodic direction* s *and an edge midpoint* $m_i \in P_n$*, there exists a unique geodesic* $\widetilde{\gamma} \subset Y$ *such that*

$$\pi(\widetilde{\gamma}) = T_s(m_i) \quad \textit{and} \quad p(\widetilde{\gamma}) = G_s(w_i).$$

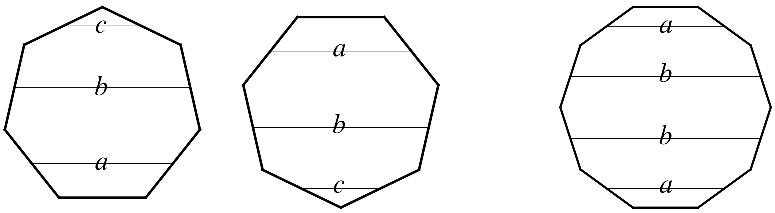

Figure 5.1
The closed geodesics $G_1(w_i)$ on $(X,\omega)_n, n = 7$, and 8. In both cases parallel sides are identified.

Proof. This geodesic is given simply by $\widetilde{\gamma} = G_s(\iota_Y(m_i))$, where $\iota_Y : P_n \to Y$ is the natural inclusion. ∎

Proof of Theorem 5.1. Given s and m_i, construct $T_s(m_i)$, $\widetilde{\gamma}$ and $G_s(w_i)$ as in the preceding Lemma. Then by Theorem A.1 and Lemmas 5.4 and 5.5, we have

$$\begin{aligned} T_s(m_i) \cong T_s(m_j) &\iff (H \cdot T_s(m_i)) = (H \cdot T_s(m_j)) \\ &\iff (G \cdot G_s(w_i)) = (G \cdot G_s(w_j)) \\ &\iff G_s(w_i) = G_s(w_j). \end{aligned}$$ ∎

6. Billiards: Proofs

In this section we establish Theorems 2.2, 2.5, and 2.7. These results describe the behavior of periodic billiard in P_n when $n = 2g + 1$, $4g + 2$, and $4g$ respectively. We also prove Theorems 2.6 and 2.8, which allow one to distinguish between vertex connections and central loops when $n = 2p^i$, $p \geq 2$ a prime. The proofs combine:

(1) the analysis of closed geodesics in $(X,\omega)_n$ in Section 4,

(2) the description of the relationship between $(X,\omega)_n$, $(Y,\eta)_n$ and P_n developed in Appendix A, and

(3) the results on congruence invariants from [Mc5] summarized in Appendix B.

The setup. We continue in the setting of Appendix A. The polygon P_n determines a natural 1-form $(Y,\eta)_n$ equipped with projections

$$\pi : Y \to P_n \quad \text{and} \quad \chi : Y \to X$$

that mediate between X and P_n. The periodic directions s for P_n and $(X,\omega)_n$ coincide, and by Lemma 5.6 for each $T_s(m_i)$ we have a unique geodesic $\widetilde{\gamma} \subset Y$ such that $\pi(\widetilde{\gamma}) = T_s(m_i)$ and $\chi(\widetilde{\gamma}) = G_s(w_i)$.

Vertex connections and central loops. It is easy to see that, with appropriate conventions for $n = 3, 4$, and 6, the locus $Z(\eta) \subset Y$ coincides with the preimage of the vertices of P_n under π, and with $\chi^{-1}(Z(\omega))$. Consequently,

$$T_s(m_i) \textit{ is a vertex connection} \iff G_s(w_i) \textit{ is a zero connection.}$$

Otherwise, $T_s(m_i)$ is closed. Since $\pi^{-1}(0) = \chi^{-1}(c_0)$, we also have (for n even):

$$T_s(m_i) \textit{ is a central loop} \iff G_s(w_i) \textit{ is a central loop.}$$

Proof of Theorem 2.2. Assume $n = 2g + 1$ is odd. Let $s \in L^\times$ be a periodic direction for billiards in P_n, and suppose $[s]_2 = [\zeta_n^k]_2$, where $k \in \mathbb{Z}/n$. Then $\delta(s) = k$ by Appendix B (see the first line in Table B.1). Consequently, by Corollaries 4.4 and 4.2, $G_s(w_k)$ is a zero connection, the rest of the geodesics $G_s(w_i)$ are closed, and

$$G_s(w_i) = G_s(w_j) \iff i + j = 2\delta(s) = 2k \bmod n.$$

By the remarks above, $T_s(w_k)$ is a zero connection and the remaining trajectories $T_s(w_i)$ are closed; and $T_s(w_i) \cong T_s(w_j) \iff i + j = 2k \bmod n$ by Theorem 5.1. ■

Proof of Theorem 2.5. Assume $n = 4g + 2$ and $[s]_2 = [\zeta_n^k]$. Then $m = 2g + 1$, and

$$\delta(s) = k + g\kappa(s) \bmod 2g + 1 \tag{6.1}$$

by the second line in Table B.1. Thus by Corollary 4.2, we have

$$\begin{aligned} G_s(w_i) = G_s(w_j) \iff i + j &= 2\delta(s) + \kappa(s) \\ &= 2k + (2g + 1)\kappa(s) = 2k \bmod 2g + 1. \end{aligned}$$

Thus Theorem 5.1 implies $T_s(m_i) \cong T_w(m_j) \iff i + j = 2k \bmod 2g + 1$. By Corollary 4.4, if $\kappa(s) = 0$ then $G_s(m_k)$ is a zero connection, while if $\kappa(s) = 1$, then $G_s(m_j)$ is a central loop, where

$$j = \delta(s) + g + 1 = k \bmod 2g + 1,$$

using (6.1) again. So in either case, $T_s(m_i)$ is special $\iff i = k \bmod 2g + 1$. ■

Proof of Theorem 2.6. We continue with the case $n = 4g + 2$. Suppose in addition that $n = 2p^i$, where p is an odd prime. Let $\pi = 1 - \zeta_{p_i}$ and let $\varepsilon = v_\pi(s) \bmod 2$. Then by the last line in Table B.1, $\kappa(s) = \varepsilon$; and by Corollary 4.4, the value of $\kappa(s)$ distinguishes between the two types of special connections that $T_s(m_k)$ can be. ■

Proof of Theorem 2.7. Assume $n = 4g$, and $[s/(1+\zeta_n)^\varepsilon]_2 = [\zeta_n^k]_2$, where $\varepsilon \in \{0,1\}$ and $k \in \mathbb{Z}/m$. Then by the third line in Table B.1, $\kappa(s) = \varepsilon$, and by the two lines that follow,

$$2k = 2\delta(s) \bmod 2g.$$

(Note that k is only determined mod g when n is not a power of two.) Thus Corollary 4.2 and Theorem 5.1 imply:

$$T_s(m_i) \cong T_s(m_j) \iff i + j = 2\delta(s) + \kappa(s) = 2k + \varepsilon \bmod 2g.$$

If $\varepsilon = 0$ then, by Corollary 4.4, $T_s(m_i)$ is special $\iff i = k \bmod g$; while if $\varepsilon = 1$, the same Corollary shows there are no special trajectories. ■

Proof of Theorem 2.8. We continue with the case $n = 4g$. Suppose in addition that n is a power of two. Then k is well defined modulo $2g$, and $\delta(s) = k$. Suppose $\varepsilon = 0$. Then by Corollary 4.4, $T_s(m_i)$ is a vertex connection $\iff i = k \bmod 2g$, and a central loop $\iff i = k + g \bmod 2g$. ■

7. Dihedral versus reflective symmetry

In this section we prove Theorems 2.3 and 2.4 on symmetries and lengths of closed billiard trajectories.

Throughout we assume that $n = p \geq 3$ *is prime*. Let $\pi = 1 - \zeta_p \in \mathcal{O}_L$. We continue to work in the setting of Appendix A, with $(X,\omega)_n$ and $(Y,\eta)_n$ the forms associated to P_n. Since n is odd, $\chi : Y \to X$ is a cyclic covering map (see Theorem A.6 and Table A.1).

The main result of this section, from which the others readily follow, is:

Theorem 7.1. *Let $\gamma = G_s(w_i)$ be a closed geodesic on X. Then*

$$\gamma \text{ lifts to } Y \iff v_\pi(s) = 1 \bmod 2.$$

Proof of Theorem 3.1. Fix a periodic slope s for billiards in P_n, and let $T_s(m_i)$ be a closed trajectory with symmetry group $S \subset \mathrm{O}(2)$.

Recall that the group $G \times H \cong D_p \times \mathbb{Z}/p$ acts on Y. Let $\widetilde{\gamma} \subset Y$ be the unique closed geodesic such that

$$\chi(\widetilde{\gamma}) = \gamma = G_s(w_i) \quad \text{and} \quad \pi(G_s(w_i)) = T_s(m_i),$$

and let $G' \times \{1\}$ and $\{1\} \times H'$ be the stabilizers of $\widetilde{\gamma}$ in $G \times \{1\}$ and $\{1\} \times H$ respectively. Since $H \cong \mathbb{Z}/p$, Theorem 7.1 implies that

$$|H'| = \deg(\widetilde{\gamma} \to \gamma) = \begin{cases} p & \text{if } v_\pi(s) \text{ is even, and} \\ 1 & \text{if } v_\pi(s) \text{ is odd.} \end{cases} \tag{7.1}$$

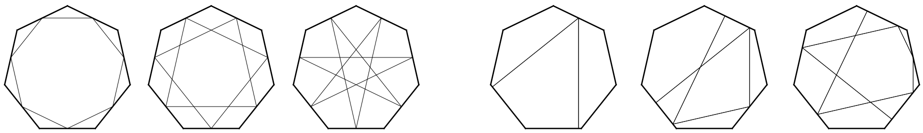

FIGURE 7.1
Closed midpoint trajectories in P_7 with $s = 1$ and $s = \zeta_7 - 1/\zeta_7$.

Now assume G' is trivial. Then by Theorem A.5 (3), we have $H' = S \cap H = S \cap \mathrm{SO}(2)$. Since S always contains a reflection, this implies that $S \cong D_{2p}$ when $v_\pi(s)$ is even, and $S \cong \mathbb{Z}/2$ when $v_\pi(s)$ is odd.

Finally assume G' is nontrivial. In this case, by Theorem A.5 (1), $|G'| = 2$ and s is fixed (up to sign) by a reflection in D_n. Up to a rotation, there are only 2 cases to consider: $s = 1$ (horizontal) and $s = \zeta_n - \zeta_n^{-1}$ (vertical). These can be settled by inspection (see Figure 7.1 for the case $n = 7$): we have dihedral symmetry in the first case and reflective symmetry in the second, consistent with the fact that $v_\pi(1) = 0$ and $v_\pi(\zeta_n - \zeta_n^{-1}) = 1$. ∎

Proof of Theorem 2.3. The proof is similar. By equation (7.1), $|H'|$ only depends on s. One can also check, using Theorem A.5, that $|G'|$ only depends on s as well. Thus, by Corollaries 4.3 and A.4, for $1 \le \ell \le n-1$, we have

$$|T_s(m_{k+\ell})| = (|H'|/|G'|)|G_s(m_{k+\ell})| = C'(s) \cdot |G_1(w_\ell)| = C(s) \cdot |\sin(2\pi\ell/n)|.$$

The final equality comes from trigonometry in P_n. ∎

The arithmetic of lifting. We proceed to the proof of Theorem 7.1. Note that $N^L_{\mathbb{Q}}(\pi) = p$, and thus $\pi\,\mathcal{O}_L$ is a prime ideal of index p in $\mathcal{O}_L$.

Lemma 7.2. *An oriented loop $\gamma \subset X$ lifts to Y if and only if*

$$\int_\gamma \omega \in \pi\,\mathcal{O}_L\,.$$

Proof. Recall that $\chi : Y \to X$ is cyclic covering map of degree p, and $\eta = \chi^*(\omega)$. By the theory of covering spaces, γ lifts iff its period, $\int_\gamma \omega$, lies in

$$J = \left\{ \int_C \eta : C \in H_1(Y, \mathbb{Z}) \right\} \subset \mathcal{O}_L \cong H_1(X, \mathbb{Z}).$$

Evidently, we have $[\mathcal{O}_L : J] = \deg(\chi) = p$; and since the action of $G = D_n \subset \mathrm{O}(2)$ on Y includes an automorphism sending η to $\zeta_n \eta$, J is an ideal in $\mathcal{O}_L$. It follows that $J = \pi\,\mathcal{O}_L$, since this is the only ideal of index p. ∎

Lemma 7.3. *The period $s = \int_C \omega$ of any oriented closed geodesic C on $(X, \omega)_n$ is part of a basis for $\mathcal{O}_L$ over $\mathcal{O}_K$.*

Proof. Since $\mathrm{Aff}^+(X, \omega)$ acts transitively on the space of periodic directions, and $\mathcal{O}_K$-linearly on $\mathcal{O}_L$, it suffices to treat the case where $C = C_k = G_1(w_k)$ for some index $1 \le k \le g$.

We claim that $s_k = \int_{C_k} \omega$ is a unit in $\mathcal{O}_K$ for all k. To see this, recall that $1 + \zeta_p = 1 - \zeta_{2p}$ is a unit in $\mathcal{O}_L$, since p is odd; see e.g. [Mc5, equation (2.2)]. Suitably oriented, it is easy to check that

$$s_g = \zeta_p^g + \zeta_p^{g+1} = \zeta_p^g(1 + \zeta_p),$$

and hence s_g is a unit. Now for $1 \le k \le g$, we also have

$$\frac{s_k}{s_1} = \frac{\sin(2\pi k/p)}{\sin(2\pi/p)} = \frac{\zeta_p(1 - \zeta_p^{2k})}{\zeta_p^k(1 - \zeta_p^2)} = \zeta_p^{1-k}(1 + \zeta_p^2 + \cdots + \zeta_p^{2k-2}).$$

The third fraction above shows that $N(s_k/s_1) = 1$, and the final expression shows that $s_k/s_1 \in \mathcal{O}_K$.

Thus s_k is a unit for all k as claimed. It follows that

$$\mathcal{O}_L = \mathcal{O}_K \oplus \mathcal{O}_K \, \zeta_n = \mathcal{O}_K \, s_k \oplus \mathcal{O}_K \, \zeta_n,$$

completing the proof. ∎

Corollary 7.4. *The period $s = \int_C \omega \in \mathcal{O}_L$ satisfies $v_\pi(s) = 0$ or 1.*

Proof. Suppose $v_\pi(s) \ge 2$. As we have just seen, there exists an $s' \in \mathcal{O}_L$ such that

$$\mathcal{O}_L = \mathcal{O}_K \, s \oplus \mathcal{O}_K \, s'. \tag{7.2}$$

Thus we can write $\pi = as + bs'$ with $a, b \in \mathcal{O}_K$. We must have $v_\pi(s') = 0$; otherwise every element of $\mathcal{O}_L$ would be divisible by π. It follows that

$$v_\pi(b) = v_\pi(bs') = v_\pi(\pi - as) = v_\pi(\pi) = 1;$$

but this contradicts the fact that $v_\pi(b)$ is even when $b \in \mathcal{O}_K$. ∎

Proof of Theorem 7.1. Choose an orientation for $\gamma = G_s(w_i)$, and let $t = \int_\gamma \omega$. Since $s/t \in K^\times$, we have $v_\pi(t) = v_\pi(s) \bmod 2$, and by Lemma 7.2 and Corollary 7.4, γ lifts iff $v_\pi(t) = 1$. ∎

Remark: Lengths of vertex connections. Continuing with the assumption that n is prime, suppose $T_s(m_k)$ is a vertex connection. Then one can check that the adjacent closed trajectory satisfies

$$|T_s(m_{k+1})| = 4n\cos^2(\pi/n)|T_s(m_k)|,$$

provided we have dihedral symmetry and s is not parallel to one of the sides of P_n. For example, $T_s(m_1)$ is about 40 times longer than the vertex connection $T_s(m_0)$ in Figure 1.2. In the case of reflective symmetry, the factor $4n$ should be replaced by 4.

A. General polygons and Riemann surfaces

Let $P \subset \mathbb{C}$ be a compact, polygonal region with internal angles in $\pi\mathbb{Q}$. (Usually P is a topological disk, but we do not require this.) For convenience, assume the barycenter of P is zero.

In this section we give a systematic account of the connection between billiards in P and geodesics for a pair of naturally associated 1-forms, (Y,η) and (X,ω). The example of a regular polygon is discussed briefly at the end.

Statement of results. Let $G \subset \mathrm{O}(2)$ be the finite group generated by the linear parts of reflections through the sides of P, and let

$$H = G \cap \mathrm{Isom}(P) = \{g \in G : g(P) = P\}.$$

We will construct from P a pair of holomorphic 1-forms, (Y,η) and (X,ω), together with an isometric group action of $G \times H$ on $(Y,|\eta|)$. This action yields a pair of locally isometric quotient maps

$$\text{(A.1)} \qquad \begin{tikzcd} (Y,|\eta|) \arrow[r, "p"] \arrow[d, "\pi"'] & (X,|\omega|) \cong Y/(\{1\}\times H) \\ (P,|dz|) \cong Y/(G\times\{1\}) & \end{tikzcd}$$

Since the factors of $G \times H$ commute, they descend to give groups actions of H on P and of G on X.

Correspondences, lengths, and symmetries. We begin by stating the main results of this section.

Theorem A.1. *There is a natural bijection*

$$\text{(A.2)} \qquad (H\cdot T) \leftrightarrow (G\times H\cdot\widetilde{\gamma}) \leftrightarrow (G\cdot\gamma)$$

between orbits of closed billiard trajectories T on P, and orbits of valid closed geodesics $\widetilde{\gamma}$ and γ on Y and X.

The three orbits above correspond to one another exactly when their representatives can be chosen so that $\pi(\widetilde{\gamma}) = T$ and $\chi(\widetilde{\gamma}) = \gamma$.

Here γ and $\widetilde{\gamma}$ are *valid* if the corresponding trajectory T avoids the vertices of P.

Corollary A.2. *A closed trajectory $T \subset P$, up to the action of H, naturally determines a closed geodesic γ on (X, ω), up to the action of G.*

Corollary A.3. *The directions $s \in \mathbb{C}^*$ of closed trajectories in T and closed geodesics on (X, ω) coincide.*

Using the fact that $G \times H$ acts isometrically on $(Y, |\eta|)$, we have:

Corollary A.4. *The lengths of T, $\widetilde{\gamma}$, and γ satisfy*

$$|\widetilde{\gamma}| = |G'| \cdot |T| = |H'| \cdot |\gamma|,$$

where $G' \times \{1\}$ and $\{1\} \times H'$ are the stabilizers of $\widetilde{\gamma}$ in $G \times \{1\}$ and $\{1\} \times H$ respectively.

More information on G' and H' is provided by:

Theorem A.5. *Assume $\pi(\widetilde{\gamma}) = T$ and $\chi(\widetilde{\gamma}) = \gamma$, and the slope of $\widetilde{\gamma}$ is given by $[s] \in \mathbb{P}^1(\mathbb{R})$. Then:*

(1) *We have $|G'| = 2$ if T has an odd number of segments, or if some segment meets an edge of P in a 90 degree angle. Otherwise $|G'| = 1$.*

(2) *We have $h \cdot T = T$ for all $h \in H'$.*

(3) *If the stabilizer of $[s]$ in G is trivial, then $H' = \{h \in H : h \cdot T = T\}$.*

For examples of case (1) above, see Figure 7.1.

From polygons to surfaces. We begin with the construction of (Y, η) and (X, ω) from P. The first form is given by

$$(Y, \eta) = \left(\bigsqcup_{g \in G} Y_g = (g \cdot P, dz) \right) \Big/ \sim .$$

Here the equivalence relation $\sim$ identifies an edge e of $g \cdot P$ isometrically with the edge $r \cdot e$ of $r \cdot g \cdot P$, where $r \in G$ is the unique reflection fixing the direction of e. The surface Y is obtained by unfolding or developing the polygon P by reflection through its sides.

The second form is defined by

$$(X,\omega) = \Big(\bigsqcup_{[g]\in G/H} X_{[g]} = (g\cdot P, dz) \Big) \Big/ \sim .$$

Note that $g \cdot P$ depends only on the coset $[g] = gH$, since $h \cdot P = P$ for all $h \in H$. The equivalence relation above is similar to that for (Y, η).

Folding and projections. Next we describe the faithful, isometric action of $G \times H$ on $(Y, |\eta|)$ and the locally isometric maps π and p.

The *folding map* $\pi : Y \to P$ is defined by

$$\pi(y) = g^{-1} \cdot y \in Y_e \cong P$$

for $y \in Y_g \cong g \cdot P$. This map is a local isometry from $|\eta|$ to $|dz|$, except that it folds or creases Y over the edges of P.

The *projection map* $\chi : Y \to X$ sends (Y_g, η) isomorphically to $(X_{[g]}, \omega)$ for all $g \in G$. It is a holomorphic map, satisfying $\chi^*(\omega) = \eta$, and hence a local isometry, except it may have branch points (see Theorem A.6).

Sections. We note that $\pi : Y \to P$ has a natural section

$$\iota_Y : P \cong Y_e \subset Y.$$

Similarly, the composition $\pi \circ \iota_Y$ gives a natural map

$$\iota_X : P \to X_{[e]};$$

however, ι_X need not be injective on ∂P.

Group actions. Next we define a natural inclusion

$$G \times H \subset \mathrm{Isom}(Y, |\eta|),$$

sending (g, h) to $\psi_g \circ \phi_h$. Here ψ_g sends $Y_{g'}$ isometrically to $Y_{gg'}$ with derivative $D\psi_g = g$, and ϕ_h sends $Y_{g'}$ to $Y_{g'h^{-1}}$ with derivative $D\phi_h = I$.

We emphasize that the derivative map

$$D : G \times H \subset \mathrm{Isom}(Y, |\eta|) \to \mathrm{O}(2)$$

sends $G \times \{1\}$ bijectively to G, but it collapses $\{1\} \times H$ to the identity.

Since the factors of $G \times H$ commute, they descend to give group actions on P and X satisfying

$$\pi(\phi_h(y)) = h \cdot \pi(y) \quad \text{and} \quad \chi(\psi_g(y)) = g \cdot \chi(y) \tag{A.3}$$

for all $y \in Y$ and $(g, h) \in G \times H$.

The fibers of π and p are simply the orbits of $G \times \{1\}$ and $\{1\} \times H$ respectively, and thus P and X can be regarded as quotients of Y as in equation (A.1).

Remark on branching. To describe the holomorphic map $\chi : Y \to X$ in more detail, let us say a vertex v of P with internal angle θ is *odd* if rotation by angle θ has odd order k in SO(2), *and* there is a reflection $h \in H$ fixing v. Then k copies of P come together over p in X, but $2k$ copies come together in Y. As a result, we have:

Theorem A.6. *The holomorphic map $\chi : Y \to X$ is a regular branched covering, with deck group H. It has simple critical points exactly at those $y \in Y$ such that $\pi(y)$ is an odd vertex of P.*

Proof of Theorem A.1. Let $T \subset P$ be a closed trajectory. We claim that there is a closed geodesic $\widetilde{\gamma}$ on Y such that

$$\pi^{-1}(T) = \bigcup (G \times \{1\}) \cdot \widetilde{\gamma}. \tag{A.4}$$

Indeed, $\pi^{-1}(T)$ is a finite union of closed geodesics; if we let $\widetilde{\gamma}$ be one of them, then its projection to P is a closed billiard trajectory, and hence $\pi(\widetilde{\gamma}) = T$. Since G acts transitively on the fibers of π, we have (A.4).

Let $\gamma = \chi(\widetilde{\gamma})$. Applying H to both sides of equation (A.4), we find

$$\pi^{-1}\Big(\bigcup H \cdot T\Big) = \bigcup (G \times H) \cdot \widetilde{\gamma};$$

and by a similar reasoning, we have

$$\chi^{-1}\Big(\bigcup G \cdot \gamma\Big) = \bigcup (G \times H) \cdot \widetilde{\gamma}.$$

Thus T up to the action of H, and γ up to the action of G, uniquely determine $\widetilde{\gamma}$ up to the action of $G \times H$. The correspondence can be inverted by applying π or p to both sides, so it is bijective. ■

Proof of Theorem A.5. We treat each statement in turn.

(1) Suppose T has k segments, and T does not meet any edge of P in a 90 degree angle. Orient T, and construct its lift to a closed geodesic $\widetilde{\gamma} \subset Y$ one segment at a time, starting in Y_e. After traversing T once, we obtain a geodesic segment $\delta \subset Y$ running from Y_e to Y_g, where g is a product of k reflections, stabilizing the direction of δ. If k is even, then $\det(g) = 1$ and so $g = e$; hence $\delta = \widetilde{\gamma}$ closes, and we have $|\widetilde{\gamma}| = |T|$. Otherwise, $g^2 = e$, so $\widetilde{\gamma} = \delta \cup g \cdot \delta$ gives a closed geodesic that double covers T; in this case, $G' = \langle g \rangle$ and $|\widetilde{\gamma}| = 2|T|$.

Now suppose T does meet an edge in a 90 degree angle. Then the billiard trajectory T is traced twice before it closes. As a result, $\widetilde{\gamma}$ double covers T, and we have $|G'| = 2$ in this case as well.

(2) By equation (A.3), for all $h \in H'$, we have

$$h \cdot T = \pi(\phi_h(\widetilde{\gamma})) = \pi(\widetilde{\gamma}) = T.$$

<table>
<tr><th>Sides</th><th>G</th><th>H</th><th>Stratum of (X, ω)</th><th>Stratum of (Y, η)</th><th>$|V_{\text{odd}}|$</th></tr>
<tr><td>$n = 2g + 1$</td><td>D_n</td><td>$\mathbb{Z}/n$</td><td rowspan="2">$(2g - 2)^1$</td><td rowspan="2">$(2g - 2)^n$</td><td rowspan="2">0</td></tr>
<tr><td>$n = 4g$</td><td rowspan="2">D'_n</td><td rowspan="2">D'_n</td></tr>
<tr><td>$n = 4g + 2$</td><td>$(g - 1)^2$</td><td>$(2g - 1)^n$</td><td>n</td></tr>
</table>

Table A.1
Groups and holomorphic forms associated to P_n.

(3) Suppose the stabilizer of $[s]$ in G is trivial. Then $\widetilde{\gamma} \subset Y$ is the unique closed geodesic with slope $[s]$ mapping by π to T. But if $h \cdot T = T$, then $\phi_h(\widetilde{\gamma})$ is another such geodesic. Thus $\phi_h(\widetilde{\gamma}) = \widetilde{\gamma}$ and hence $h \in H'$. Conversely, if $h \in H'$ then $h \cdot T = T$ by (2). ■

Example: The regular polygon P_n. Information on the groups and holomorphic forms associated to the regular polygon $P = P_n$ is summarized in Table A.1. For n even, $D'_n \subset D_n$ denotes the subgroup of index 2 generated by $\langle r^2, t \rangle$. (This is the same as $D_m = D_{n/2}$ except when $n = 4$.)

The genus g of X and the number of sides n of P_n are related by $n = 2g + 1$, $n = 4g + 2$ or $n = 4g$, depending on the value of n mod 4. A form lies in the *stratum* a^b if it has b zeros, each of multiplicity a. The last column gives the number of odd vertices of P_n; these produce branch points for the map $\chi : Y \to X$.

For example, when $P = P_6$ is a regular hexagon, X has genus 1, Y has genus 4, and $p : Y \to X$ is a degree 6 regular branched covering, with 6 critical points on Y and 2 critical values on X.

Remarks. The unfolding of a rational polygon P to obtain the large surface (Y, η) is discussed, from different perspectives, in [KZ] and [V1]. The quotient 1-form (X, ω), which is often simpler, is defined in [Mc1, Section 9].

B. Congruence invariants

Table B.1 below summarizes the main results on the calculation of the invariants $\delta(s)$ and $\kappa(s)$ we quote from [Mc5].

Each row reads as follows: given a value $n \geq 3$ satisfying a condition at the left, a solution to the congruence equation in the middle, with $\varepsilon \in \{0, 1\}$ and $k \in \mathbb{Z}/m$, gives the information on $\delta(s)$ and $\kappa(s)$ stated at the right. One equation may apply to several values of n. As usual, $m = n$ if n is odd, and $n/2$ if n is even.

Sides	Equation	Invariants
$n = 2g+1$	$[s]_2 = [\zeta_n^k]_2$	$\delta(s) = k$
$n = 4g+2$		$\delta(s) = k + g\kappa(s)$
$n = 4g$	$[s/(1+\zeta_n)^\varepsilon]_2 = [\zeta_n^k]_2$	$\kappa(s) = \varepsilon$
$n = 2^a$		$\delta(s) = k$
$n = 4g \neq 2^a$		$2\delta(s) = 2k$
$n = 2p^i$	$\varepsilon = v_\pi(s) \bmod 2, \pi = 1 - \zeta_{p^i}$	$\kappa(s) = \varepsilon$

Table B.1
Calculation of $\delta(s)$ and $\kappa(s)$.

Here is a more detailed account of the table's entries. It is useful to recall that, by [Mc5, Theorem 3.2], if $[s]$ is a cusp of Δ_n, then

$$\text{(B.1)} \qquad \begin{aligned} \kappa((1+\zeta_n)s) &= \kappa(s) + 1 \bmod 2, \\ \delta((1+\zeta_n)s) &= \delta(s) + \kappa(s) \bmod m. \end{aligned}$$

(1) $n = 2g+1$. In this case, there is a unique $k \in \mathbb{Z}/n$ such that $[s]_2 = [\zeta_n^k]_2$ in $\mathbb{P}(\mathcal{O}_L/2)$, and $\delta(s) = k$ by [Mc5, Corollary 1.5].

(2) $n = 4g+2$. When $\kappa(s) = 0$, the same Corollary shows $\delta(s) = k$ as desired. Now suppose $\kappa(s) = 1$, and $[s]_2 = [\zeta_n^k]_2$. Since $[1+\zeta_n]_2 = [\zeta_n^{g+1}]_2$ for this value of n, we have
$$[(1+\zeta_n)s]_2 = [\zeta_n^{k+g+1}]_2.$$
Using equation (B.1), it follows that
$$\delta((1+\zeta)s) = k + g + 1 = \delta(s) + \kappa(s) = \delta(s) + 1,$$
and hence $\delta(s) = k + g \bmod 2g+1$.

(3) $n = 4g$. For this value of n, there is a unique $\varepsilon \in \{0,1\}$ such that
$$[s/(1+\zeta_n)^\varepsilon] = [\zeta_n^k]_2$$
for some $k \in \mathbb{Z}/m$, and $\kappa(s) = \varepsilon$ by [Mc5, Theorem 1.7].

(4) $n = 2^a$. When n is a power of 2, k is unique, and the conclusion $\delta(s) = k$ follows from [Mc5, Corollary 1.5] when $\kappa(s) = 0$. The same conclusion holds when $\kappa(s) = 1$ since, setting $t = s/(1+\zeta_n)$, we have
$$\delta(s) = \delta((1+\zeta_n)t) = \delta(t) = k$$
by equation (B.1).

(5) $n = 4g \neq 2^a$. When $n = 4g$ is not power of 2, we have $[\zeta_n^g]_2 = [1]_2$, and hence k is only well defined modulo $g = n/4$. But we can still conclude that $2k = 2\delta(s) \in \mathbb{Z}/m$, by [Mc5, Theorem 1.6] and equation (B.1).

(6) $n = 2p^i$. Here p is an odd prime, and $v_\pi(s)$ is the valuation of s at the prime $\pi = 1 - \zeta_{p^i} \in \mathcal{O}_L$. We have $\kappa(s) = v_\pi(s) \bmod 2$ by [Mc5, Theorem 1.8]. Note that $v_\pi(k) = 0 \bmod 2$ for all $k \in K$, so that $v_\pi(s) \bmod 2$ only depends on $[s] \in \mathbb{P}(\mathcal{O}_L)$.

Added in proof. It is now known that Table B.1 describes all the cases where $\delta(s)$ and $\kappa(s)$ are congruence invariants. The only case where $\kappa(s)$ is *not* a congruence invariant is when $n = 4g + 2 \neq 2p^i$; and the only case where $\delta|(\Delta_n \cdot \infty)$ is not a congruence invariant is when $n = 4g \neq 2^a$. See [Cal, Theorem 1.4].

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