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# On the transportation cost norm on finite metric graphs

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**Abstract.** For a finite metric graph $X = (V, E, \ell)$, where $V$ is endowed with the shortest path metric, we consider the transportation cost problem associated with the distance $d$ on $V$. Namely, for $f$ a function with total sum 0 on $V$, write $f = \sum_{a,b \in V} P(a,b)(\delta_a - \delta_b)$, where the transportation plan $P$ satisfies $P(a,b) \geq 0$ for $(a,b) \in V \times V$. The cost of $P$ is $W(P) := \sum_{a,b \in V} P(a,b)d(a,b)$ and the transportation norm of $f$ is $\|f\|_{TC} = \min_P W(P)$, where $P$ runs over all transportation plans for $f$.

In this semi-survey paper, we give short proofs for the following statements:

- There always exists an optimal transportation plan supported in $V_+ \times V_-$, where $V_+ = \{x \in V : f(x) > 0\}$ and $V_- = \{x \in V : f(x) < 0\}$. If $X$ is a metric tree, we may moreover assume that this plan involves at most $|\operatorname{Supp}(f)| - 1$ transports.
- There always exists an optimal transportation plan supported in the set of edges of $X$.
- Better, there always exists an optimal transportation plan supported in some spanning tree of $X$.

We use this to reprove known formulae for the transportation norm when $X$ is either a tree or a cycle.

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## 1. Introduction

Let $(X, d)$ be a finite metric space. The *transportation cost space* of $X$ (also known as *Arens–Eells space*, *Kantorovich–Rubinstein space*, or *Lipschitz free space*[1]) is the space $\mathcal{F}(X)$ of real-valued functions $f$ on $X$ with total mass 0, endowed with a norm defined as follows. For such an $f$, set

$$\Pi_f = \left\{ P : X \times X \to \mathbb{R}_{\geq 0} : \ f = \sum_{a,b \in X} P(a,b)(\delta_a - \delta_b) \right\},$$

[1] See [4, Definition 1.1] or [12, Chapter 3]; for historical background, see the remarks in [11, Section 10.1] and [8, Section 1.6].

where $\delta_x$ denotes the Dirac mass at $x$. Equivalently, $P \in \Pi_f$ if and only if

$$f(x) = \sum_{a \in X} \big(P(x,a) - P(a,x)\big)$$

for all $x \in X$. It is clear that $\Pi_f$ is non-empty, and does not depend on $d$. Put also

$$X_+ = \{a \in V;\, f(a) > 0\} \quad \text{and} \quad X_- = \{a \in V;\, f(a) < 0\}$$

and let also

$$\widetilde{\Pi}_f = \{P \in \Pi_f; \mathrm{Supp}(P) \subset X_+ \times X_-\}.$$

It is clear that $\widetilde{\Pi}_f$ is also non-empty. The elements of $\Pi_f$ are *transportation plans* for $f$ and those of $\widetilde{\Pi}_f$ are *simultaneous transportation plans.*

We take $d$ into account by defining the *cost* of $P \in \Pi_f$ as

$$W_X(P) = \sum_{a,b \in X} P(a,b)d(a,b).$$

Now define the *transportation cost norm* of $f$ as

$$\|f\|_{TC(X)} = \min\{W(P) : P \in \Pi_f\},$$

and the *simultaneous transportation cost norm* of $f$ as

$$\|f\|_{STC(X)} = \min\{W(P) : P \in \widetilde{\Pi}_f\}.$$

Let $P_0 \in \Pi_f$. The set $\{P \in \Pi_f; W(P) \leq W(P_0)\}$ is compact, and therefore the infimum is a minimum. When the metric space $X$ is clear from context we drop the subscript, both in $W(P)$ and in $\|f\|_{TC}$.

We recall the optimal transportation interpretation of $\|f\|_{TC}$: suppose that a mass $f(x)$ is available at points $x \in X$, where $f(x) > 0$, and a mass $-f(y)$ is in demand at points $y \in X$ with $f(y) < 0$. An element $P \in \Pi_f$ can be viewed as a *transportation plan*, i.e. a way of transporting the mass from points where it is in excess to the points where it is in demand, $W(P)$ represents the physical work associated with this transportation plan, and $\|f\|_{TC}$ represents the most economical transportation associated with $f$.

The following result is due to Bergman [1, Proposition 24] with a short proof in [12, Proposition 3.16]. We will give a new proof in Section 2.

**Theorem 1.1.** *Given any finite metric space $(X,d)$ and any $f \in TC(X)$, we have $\|f\|_{TC} = \|f\|_{STC}$.*

Next we deal with $X = (V, E, \ell)$ a finite, connected, metric graph. This means that $(V, E)$ is a simple connected graph with no loop, $\ell\colon E \to \mathbb{R}_{>0}$ is a function giving a positive length to every edge, and $V$ becomes a finite metric space with the shortest path metric. We define two more transportation cost norms on $\mathcal{F}(V)$ associated with the graph structure.

(1) Let $\sim$ denote the adjacency relation on $V$, so that $x \sim y$ if and only if $\{x, y\} \in E$. Let $\Pi_{f,E}$ denote the set of transportation plans supported in the graph of $\sim$ in $V \times V$. This corresponds to transports supported on the edges of $X$. We denote by $\|f\|_{TCE}$ the corresponding norm.

(2) Recall that a *spanning tree* in $X$ is a subtree of $X$ passing through all vertices. We denote by $\Pi_{f,T}$ the set of transportation plans $P \in \Pi_f$ for which there exists a spanning tree $T$ such that $P$ is supported on the edges of $T$. We denote by $\|f\|_{TCT}$ the corresponding norm.

Assume that $X$ is endowed with an orientation, i.e. every edge $e \in E$ has an origin $e_-$ end an extremity $e_+$. This allows us to define the boundary operator

$$\partial\colon \mathbb{R}^E \to \mathbb{R}^V\colon \delta_e \mapsto \delta_{e_+} - \delta_{e_-};$$

equivalently,

$$(\partial\phi)(v) = \sum_{e:e_+=v} \phi(e) - \sum_{e:e_-=v} \phi(e) \quad (v \in V, \phi \in \mathbb{R}^E).$$

It is classical that the image of $\partial$ is exactly the space $\mathcal{F}(V)$, so that the rank of $\partial$ is $|V| - 1$; and that the kernel of $\partial$ is the space $Z(X)$ of cycles of $X$, so that

$$\dim(Z(X)) = |E| - |V| + 1$$

(see [2, Proposition 4.3 and Theorem 4.5]). Denote by $\ell^1(E, \ell)$ the space $\mathbb{R}^E$ endowed with the norm

$$\|\phi\|_1 := \sum_{e \in E} \ell(e)|\phi(e)|.$$

Then, we have the following result.

**Theorem 1.2.** *The following statements hold.*

(1) *For $f \in \mathcal{F}(V)$, we have*

$$\|f\|_{TC} = \|f\|_{TCE} = \|f\|_{TCT}.$$

(2) *The operator $\partial\colon \mathbb{R}^E \to \mathbb{R}^V$ induces an isometric isomorphism from the quotient space $\ell^1(E, \ell)/Z(X)$ onto $TC(V)$.*

The first equality in the first part is proved in [10, after Remark 1.3]; the second equality appears as [7, Theorem 10]. The latter corresponds to the intuitive idea that, if a transportation plan involves loops in the graph, it is possible to avoid the loops without increasing the cost.

For the second part, see [11, Proposition 10.10] for usual graphs, and [9, Theorem 3.1] or [10, Proposition 1.5] for the general case; note that the proofs in [11] and [9] are indirect, establishing that the transpose operator

$$\partial^t\colon (\mathcal{F}(V))^* \to \left(\ell^1(E,\ell)/Z(X)\right)^*$$

is an isometric isomorphism. A short direct proof of Theorem 1.2 will be given in Section 3.

Section 4 will be devoted to three corollaries of Theorem 1.2.

(1) Due to the presence of a minimum either in the original definition of $\|f\|_{TCE}$, or as a quotient norm in Theorem 1.2, for a general metric graph we do not expect a closed formula for $\|f\|_{TCE}$. A notable exception to this philosophy is the case of metric *trees*: then an explicit formula for $\|f\|_{TC}$ is available. We refer to [5] for the interesting history and two proofs of the formula; see also [6, Section 2.1].

**Corollary 1.3.** *Let $X = (V, E, \ell)$ be a finite metric tree. Fix an arbitrary orientation of the edges. For $e \in E$, we define $S_e$ as the vertex set of the half-tree containing $e_+$, i.e. $S_e = \{x \in V; d(x, e_+) \leq d(x, e_-)\}$. For $f \in \mathcal{F}(V)$, we have*

$$\|f\|_{TCE} = \sum_{e \in E} |f(S_e)|\ell(e), \tag{1.1}$$

*where $f(S_e) = \sum_{x \in S_e} f(x)$.*

Our new, simple proof of (1.1) follows by combining Theorem 1.2 with injectivity of $\partial$ for trees; see Proposition 4.1 below.

(2) In Corollary 4.2, we provide a new proof of a formula for the transportation cost norm on the $N$-cycle $C_N$, originally appearing as in [3, formula (4.2)].

(3) Recall that a *bridge* in a graph is an edge whose deletion produces a graph with more connected components.[2] Deleting all bridges in a graph results in a *bridgeless* graph. We show in Corollary 4.4 that, for a general finite metric graph $X$, solving the TC problem is equivalent to solving it on the bridgeless part $X_b$ of $X$.

Finally, Section 5 discusses some algorithmic issues in the case of metric trees. For example, we provide an algorithmic way to find $P \in \tilde{\Pi}_f$ such that

$$W(P) = \|f\|_{STC} \quad \text{and} \quad |\operatorname{Supp}(P)| \leq |\operatorname{Supp}(f)| - 1.$$

[2] Equivalently, an edge is a bridge if it is not contained in any cycle in the graph.

## 2. A new proof of Theorem 1.1

The proof will follow from the following lemma.

**Lemma 2.1.** *If all triangle inequalities in $X$ are strict, i.e. if for all $x$, $y$, $z$ in $X$ with $x \neq y \neq z$, we have $d(x,z) < d(x,y) + d(y,z)$ and $W(P) = \|f\|_{TC}$, then* $\operatorname{Supp}(P) \subset X_+ \times X_-$.

*Proof.* If there exists $(x,y) \in \operatorname{Supp}(P)$ with $y \notin X_-$, then

$$0 \leq f(y) = \sum_{a \in X} \big(P(y,a) - P(a,y)\big),$$

and since $P(x,y) > 0$, there exists $z \in X$ with $P(y,z) > 0$. Let

$$m = \min\big(P(x,y), P(y,z)\big)$$

and define $Q$ by setting

$$\begin{aligned} Q(x,z) &= m + P(x,z), & Q(x,y) &= P(x,y) - m, \\ Q(y,z) &= P(y,z) - m, & Q(u,v) &= P(u,v) \end{aligned}$$

for all other points. It is readily checked that $Q \in \Pi_f$ and

$$W(P) - W(Q) = m\big(d(x,y) + d(y,z) - d(x,z)\big),$$

contradicting $W(P) = \|f\|_{TC}$.

We use similar reasoning if there exist $(x,y) \in \operatorname{Supp}(P)$ with $x \notin X_+$. ∎

*Proof of Theorem 1.1.* For $n \geq 1$, define a distance $d_n$ on $X$ by

$$d_n(x,x) = 0 \quad \text{and} \quad d_n(x,y) = \frac{1}{n} + d(x,y)$$

for $x \neq y$. Write $\|f\|_{TC_n}$ for the transportation cost norm associated with $d_n$. Let $P_n \in \Pi_f$ minimize the cost for $d_n$. By Lemma 2.1, $P_n$ is supported in $X_+ \times X_-$. By compactness, we may assume that $(P_n)_{n>0}$ converges pointwise to $P$, also supported in $X_+ \times X_-$. As $(d_n)_{n>0}$ converges pointwise to $d$, we also have

$$W(P) = \lim_{n \to \infty} W_n(P_n) = \lim_{n \to \infty} \|f\|_{TC_n}.$$

It remains to show that $W(P) = \|f\|_{TC}$. To see this, let $Q \in \Pi_f$ be such that $W(Q) = \|f\|_{TC}$. Then

$$\|f\|_{TC_n} \leq W_n(Q) \leq W(Q) + \frac{1}{n}\|Q\|_1 = \|f\|_{TC} + \frac{1}{n}\|Q\|_1,$$

hence passing to the limit $W(P) \leq \|f\|_{TC}$. The other inequality holds by definition. ∎

## 3. Towards Theorem 1.2

**3.1. The basic construction.** For every 2-element subset $\{a,b\} \subset V$, we select (arbitrarily if non-unique) a shortest length edge path $[a,b]$ joining $a$ and $b$. Let $\varepsilon_{ab} \in \mathbb{R}^E$ be the *characteristic function of* $[a,b]$, defined as

$$\varepsilon_{ab}(e) = \begin{cases} 0 & \text{if } e \notin [a,b], \\ 1 & \text{if } e \in [a,b] \text{ and } e \text{ points from } a \text{ to } b, \\ -1 & \text{if } e \in [a,b] \text{ and } e \text{ points from } b \text{ to } a. \end{cases}$$

To every $P\colon V \times V \to \mathbb{R}_{\geq 0}$, we associate $\phi_P \in \mathbb{R}^E$ defined by

$$\phi_P = \sum_{a,b\in V} P(a,b)\varepsilon_{ab}.$$

Motivation for introducing $\phi_P$ comes from the following simple but crucial observation.

**Basic observation.** For a fixed $f \in \mathcal{F}(V)$ and $P \geq 0$, we have $P \in \Pi_f$ if and only if $\partial\phi_P = -f$. Indeed,

$$\partial\phi_P = \sum_{a,b\in V} P(a,b)(\delta_b - \delta_a),$$

so this follows from the definition of $\Pi_f$.

**Lemma 3.1.** *For every $P\colon V \times V \to \mathbb{R}_{\geq 0}$, we have*

$$\|\phi_P\|_1 \leq W(P),$$

*where $\|\cdot\|_1$ denotes the norm in $\ell^1(E,\ell)$.*

*Proof.* Observe that

$$d(a,b) = \sum_{e\in E} \ell(e)|\varepsilon_{ab}(e)|$$

for every $a,b \in V$, hence

$$\begin{aligned} W(P) &= \sum_{a,b\in V}\sum_{e\in E} P(a,b)\ell(e)|\varepsilon_{ab}(e)| = \sum_{e\in E}\ell(e)\sum_{a,b\in V} P(a,b)|\varepsilon_{ab}(e)| \\ &\geq \sum_{e\in E}\ell(e)\left|\sum_{a,b\in V} P(a,b)\varepsilon_{ab}(e)\right| = \sum_{e\in E}\ell(e)|\phi_P(e)| = \|\phi_P\|_1. \end{aligned}$$ ■

**Lemma 3.2.** *For any $f \in TC(V)$ and $\phi \in \partial^{-1}(-f)$, there exists $P \in \Pi_{f,E}$ such that $\phi_P = \phi$ and $W(P) = \|\phi\|_1$.*

*Proof.* We define $\phi$ as follows:

- if $\phi(e) \geq 0$, set $P(e_-, e_+) = \phi(e)$ and $P(e_+, e_-) = 0$;
- if f $\phi(e) < 0$, set $P(e_+, e_-) = -\phi(e)$ and $P(e_-, e_+) = 0$.

Observe that $P(e_-, e_+) - P(e_+, e_-) = \phi(e)$ for every $e \in E$. Then, as $P$ is supported on edges, we have

$$\phi_P = \sum_{a,b \in V} P(a,b)\varepsilon_{ab} = \sum_{e \in E} \big(P(e_-, e_+) - P(e_+, e_-)\big)\delta_e = \sum_{e \in E} \phi(e)\delta_e = \phi.$$

Then $\partial \phi_P = -f$, so $P \in \Pi_{f,E}$ thanks to the basic observation. Finally,

$$W(P) = \sum_{a,b \in V} P(a,b)d(a,b) = \sum_{e \in E} |\phi(e)|\ell(e) = \|\phi\|_1. \qquad \blacksquare$$

**3.2. Proof of Theorem 1.2, except for the equality $\|f\|_{TC} = \|f\|_{TCT}$.** As $\partial$ realizes a linear isomorphism $\ell^1(E,\ell)/Z(X) \to \mathcal{F}(V)$, we may endow $\mathcal{F}(V)$ with the quotient norm $\|\cdot\|_{\ell^1(E,\ell)/Z(X)}$. For $f \in \mathcal{F}(V)$, we compute

$$\begin{aligned}
\|f\|_{\ell^1(E,\ell)/Z(X)} &= \inf_{\phi : \partial\phi = -f} \|\phi\|_1 \\
&\leq \inf_{P \in \Pi_f} \|\phi_P\|_1 \leq \inf_{P \in \Pi_f} W(P) \\
&= \|f\|_{TC} \leq \|f\|_{TCE},
\end{aligned}$$

where the first inequality follows from $\partial \phi_P = -f$ for $P \in \Pi_f$, and the second inequality follows from Lemma 3.1.

Now, take $\phi$ such that $\partial \phi = -f$ and $\|f\|_{\ell^1(E,\ell)/Z(X)} = \|\phi\|_1$. By Lemma 3.2, we find $P \in \Pi_{f,E}$ such that $\phi = \phi_P = \|\phi\|_1 = W(P)$. This shows that

$$\|f\|_{TCE} \leq \|f\|_{\ell^1(E,\ell)/Z(X)}.$$

At this point, we have proved

$$\|f\|_{\ell^1(E,\ell)/Z(X)} = \|f\|_{TC} = \|f\|_{TCE},$$

so that Theorem 1.2 is proved except for $\|f\|_{TC} = \|f\|_{TCT}$.

**3.3. Spanning trees and the proof of $\|f\|_{TC} = \|f\|_{TCT}$.** The next lemma appears as [7, Theorem 3 (ii)], with a very similar proof.

**Lemma 3.3.** *For every $f \in \mathcal{F}(V)$, there exists $\phi \in \partial^{-1}(f)$ realizing the minimum of $\{\|\psi\|_1 : \partial\psi = f\}$ and such that, for every cycle $C$ in $X$, the function $\phi$ vanishes on at least one edge of $C$; equivalently, $\phi$ is supported in some spanning tree $T$ of $X$.*

*Proof.* Take any $\phi_0$ realizing the minimum of the $\ell^1$-norm on $\partial^{-1}(-f)$. Let $C$ be any cycle in $X$; recall the definition of the characteristic function $\xi_C$ (taken from [2, p. 23]): Choose one cyclic orientation for $C$ and define, for $e \in E$:

$$\xi_C(e) = \begin{cases} 0 & \text{if } e \notin C, \\ 1 & \text{if } e \in C \text{ and } e \text{ is coherent with the orientation of } C, \\ -1 & \text{if } e \in C \text{ and } e \text{ is not coherent with the orientation of } C. \end{cases}$$

Define then $\phi_t = \phi_0 + t\xi_C$ for $t \in \mathbb{R}$. Observe that $\phi_t \in \partial^{-1}(-f)$ as $\partial\xi_C = 0$. The function

$$t \mapsto \|\phi_t\|_1 = \sum_{e \in E} \ell(e)|(\phi_0 + t\xi_C)(e)|$$

is convex and piecewise affine; it is affine at every $t$ such that $\phi_t$ does not vanish on $C$. So the minimum of $\|\phi_t\|_1$ is attained for a $t_0$ such that $\phi_{t_0}$ vanishes at some edge in $C$. Note that $\|\phi_{t_0}\|_1 = \|\phi_0\|_1$ by minimality of $\|\phi_0\|_1$.

Choosing now $\phi \in \partial^{-1}(-f)$ realizing the minimum of the $\ell^1$-norm and with minimal support, the above argument shows that the support of $\phi$ cannot contain any cycle, i.e. it is a forest, which is of course contained in some spanning tree. ∎

*End of proof of Theorem 1.2.* For $T$ a spanning tree in $X$, we denote by $d_T$ (resp. $\partial_T$) the distance (resp. boundary operator) of $T$. Fix $f \in \mathcal{F}(V)$. For $a, b \in V$, we have

$$d(a,b) \leq d_T(a,b),$$

hence $W_X(P) \leq W_T(P)$ for every $P \in \Pi_f$, and

$$\|f\|_{TC(X)} \leq \|f\|_{TC(T)}.$$

To show that the equality is realized by some spanning tree, by the part already proved of Theorem 1.2, we find $\phi \in \ell^1(E, \ell)$ such that

$$\partial\phi = -f \quad \text{and} \quad \|f\|_{TC(X)} = \|\phi\|_1.$$

By Lemma 3.3, we may assume that $\phi$ is supported on some spanning tree $T$ in $X$, so that the $\ell^1$-norms of $\phi$ on $E$ and $E(T)$ are equal. Since $\partial_T\phi = -f$, we have

$$\|f\|_{TC(X)} = \|\phi\|_1 = \|f\|_{TC(T)}.$$ ∎

**Remark 3.4.** Let us come to the general case, where $(X, d)$ is a finite metric space. Let $f \in \mathcal{F}(V)$. By Theorem 1.1, we already know that $\|f\|_{TC}$ does not change if we replace $(X, d)$ by $(\mathrm{Supp}(f), d)$. Consider the complete graph with vertex set $\mathrm{Supp}(f)$, whose path length is given by $d$. Theorem 1.2 then implies that there is a spanning tree realizing the optimal transportation, in particular at most $|\operatorname{Supp}(f)|-1$ transports are needed.

Note that the number $|\operatorname{Supp}(f)|-1$ is generically optimal since, writing $f=\partial(P)$, we see that the dimension over $\mathbb{Q}$ of the space of linear combinations with coefficients in $\mathbb{Q}$ of $\{f(u);\ u\in V\}$, is at most $|\operatorname{Supp}(P)|$.

**Remark 3.5.** In the proof of Lemma 3.3, we see that the derivative of $t\mapsto\|\phi_t\|_1$ at any regular point is $\sum_e \pm\xi_C(e)\ell(e)$. In a generic case (in particular, whenever all the $\ell(e)$'s are linearly independent over the rational numbers), this derivative cannot be 0 and it follows that $t\mapsto\|\phi_t\|_1$ has a unique minimum at $t=0$, so that $\phi_0$ is automatically carried by a spanning tree. We can then give another proof of the equality $\|f\|_{TC}=\|f\|_{TCT}$ in the spirit of our proof of Theorem 1.1: approximate $\ell$ by a sequence $\ell_n$ such that, for all $n$, the family $(\ell_n(e))_{e\in E}$ is linearly independent over $\mathbb{Q}$.

## 4. Corollaries of Theorem 1.2

**4.1. The case of trees.** We improve formula (1.1) by finding an explicit function $\phi_f$ with $\partial(\phi_f)=-f$, and an explicit $P\in\Pi_f$ with $W(P)=\|f\|_{TC}$.

**Proposition 4.1.** *Let $X=(V,E,\ell)$ be a finite metric tree, and $f\in\mathcal{F}(V)$.*

(1) *Set $\phi_f(e):=-f(S_e)=-\sum_{x\in S_e}f(x)$; then $\phi_f$ is the unique solution of $\partial\phi=-f$.*

(2) *We have*

$$\|f\|_{TCE}=\|\phi_f\|_1=\sum_{e\in E}|f(S_e)|\ell(e)$$

*(which is exactly formula (1.1)).*

(3) *Define $P\colon V\times V\to\mathbb{R}^+$ by $P(a,b)=|f(S_e)|$ if either*

$$e_-=a, e_+=b \quad \textit{and} \quad f(S_e)<0;$$

*or*

$$e_-=b, e_+=a \quad \textit{and} \quad f(S_e)>0;$$

*and $P(a,b)=0$ in all other cases. Then $P\in\Pi_f$ and $W(P)=\|f\|_{TC}$.*

*Proof.* (1) It is readily checked that $\partial(\phi_f)=-f$, and moreover $\partial$ is injective as $X$ is a tree.

(2) The formula immediately follows from Theorem 1.2.

(3) If we apply the proof of Lemma 3.2 to $\phi_f$, we find that $\phi_f=\phi_P$, so that $P\in\Pi_f$ and $W(P)=\|\phi_f\|_1$. ∎

**4.2. The case of cycles.** Here we provide a new proof of a formula for the transportation cost norm on the $N$-cycle $C_N$, originally appearing as [3, formula (4.2)] (see also [6, Section 2.2] for another proof).

**Corollary 4.2.** *Let $C_N$ be the $N$-cycle with vertex set $\{v_i; i \in \mathbb{Z}/N\mathbb{Z}\}$ and edge set $E = \{e_i; i \in \mathbb{Z}/N\mathbb{Z}\}$, with the edge $e_i$ joining $v_i$ and $v_{i-1}$. Let $\ell: E \to \mathbb{R}_+^*$ be a function. For $f \in \mathcal{F}(V)$ and $k \in \{0, \dots, N-1\}$, set $\alpha_k = \sum_{i=0}^{k} f(i)$. Then*

$$\|f\|_{TC} = \min_{0 \le k \le N-1} \sum_{i=0}^{N-1} |\alpha_i - \alpha_k| \ell(e_i).$$

*This minimum is $\alpha_k$ such that*

$$\sum_{\substack{i \in \{0,\dots,N-1\},\\ \alpha_i < \alpha_k}} \ell(e_i) \le \frac{1}{2} \sum_{i=0}^{N-1} \ell(e_i) \le \sum_{\substack{i \in \{0,\dots,N-1\},\\ \alpha_i \le \alpha_k}} \ell(e_i).$$

*In particular, if $\ell(e) = 1$ for every edge, we find*

$$\|f\|_{TC} = \sum_{i=0}^{N-1} |\alpha_i - \mathrm{Me}|,$$

*where* Me *denotes the median of the $\alpha_k$'s.*

*Proof.* Choose the orientation of $E$ by setting $(e_i)_+ = v_i$, and $(e_i)_- = v_{i-1}$ and put $\phi(e_i) = \alpha_i$, so that $\partial\phi = -f$. The cycle space of $C_N$ is 1-dimensional and generated by the constant function 1 on $E$. By Theorem 1.2, we know that $\|f\|_{TC}$ is the minimum of

$$q(t) = \|\phi - t\|_1 = \sum_i |\alpha_i - t| \ell(e_i).$$

For $t \notin \{\alpha_k; k \in \mathbb{Z}/N\mathbb{Z}\}$, we have

$$q'(t) = \sum_{i:\alpha_i < t} \ell(e_i) - \sum_{i:\alpha_i > t} \ell(e_i).$$

By convexity, the minimum of $q(t)$ is attained at $t = \alpha_k$, where $k$ is characterized by $q'(t) \le 0$ for $t < \alpha_k$, and $q'(t) \ge 0$ for $t \ge \alpha_k$. The conclusion follows. ∎

**Remark 4.3.** Assume $X$ has just one cycle. Write

$$V = V' \sqcup \{v_i, i \in \mathbb{Z}/N\mathbb{Z}\} \quad \text{and} \quad E = E' \sqcup \{e_i, i \in \mathbb{Z}/N\mathbb{Z}\}.$$

Note that $\{e_i,\ i \in \mathbb{Z}/N\mathbb{Z}\}$ is the bridgeless part of the graph.

Let $f \in \mathcal{F}(V)$. Choose an orientation of $E$ such that

$$(e_i)_+ = v_i \quad \text{and} \quad (e_i)_- = v_{i-1}.$$

Take $\phi \in \ell^1(E, \ell)$ be such that $\partial\phi = -f$ (e.g. by removing $e_0$). All the solutions are then $\phi - t\psi$, where $\psi(e_i) = 1$ and $\psi(e) = 0$ if $e \in E'$. The minimum is obtained with $t = \phi(e_k)$, where $\phi(e_k)$ is such that

$$\sum_{\substack{i \in \{0,\dots,N-1\}, \\ \phi(e_i) < \phi(e_k)}} \ell(e_i) \leq \frac{1}{2} \sum_{i=0}^{N-1} \ell(e_i) \leq \sum_{\substack{i \in \{0,\dots,N-1\}, \\ \phi(e_i) \leq \phi(e_k)}} \ell(e_i).$$

**4.3. An application to arbitrary graphs.** Fix an orientation of $E$; recall that, for $e \in E$, we defined $S_e = \{x \in V : d(x, e_+) < d(x, e_-)\}$.

**Corollary 4.4.** *Solving the TC problem on $X$ is equivalent to solving it on the bridgeless part $X_b$.*

*Proof.* Fix $f \in \mathcal{F}(V)$ and $\phi \in \ell^1(E)$ such that $\partial\phi = -f$. Let $B$ denote the set of bridges of $X$.

**Claim.** For $e^0 \in B$, we have $\phi(e^0) = f(S_{e^0})$.

*Proof of the claim.* Indeed, since $e^0$ disconnects $X$, we may write

$$E = \{e^0\} \cup E_+ \cup E_-,$$

where $E_+$ is the set of edges whose endpoints are both in $S_{e^0}$ and $E_-$ is the set of edges whose endpoints are both in $V \setminus S_{e^0}$.

Put $\psi = \mathbf{1}_{E_+ \cup \{e^0\}}\phi$. We see that $\partial\psi$ coincides with $\partial\phi = -f$ on $S_{e^0}$ and vanishes outside $S_{e^0} \cup e^0_-$. We find

$$0 = \sum_{v \in V} \partial\psi(v) = \sum_{v \in S_{e^0}} \partial\phi(v) + \partial\psi(e^0_-).$$

But $\partial\psi(e^0_-) = \psi(e^0) = \phi(e^0)$, which proves the claim. ∎

As a consequence of the claim, if $\partial\phi = -f$, then

$$\|\phi\|_1 = \sum_{e \in B} |f(S_e)|\ell(e) + \sum_{e \notin B} |\phi(e)|\ell(e),$$

so by Theorem 1.2, we have to minimize the quantity $\sum_{e \notin B} |\phi(e)|\ell(e)$.

We now show that this amounts to solving a TC problem on the bridgeless part $X_b$. For this, define $\phi_0 \in \ell^1(E, \ell)$ by

$$\phi_0 = \begin{cases} f(S_e) & \text{if } e \in B, \\ 0 & \text{if } e \notin B, \end{cases}$$

and set $f_0 := f - \partial\phi_0$. It is clear that the equation $\partial\phi = -f_0$ is equivalent to $\partial(\phi - \phi_0) = -f$. But if $\partial\phi = -f_0$, then by the claim, for $e \in B$, we have

$$\phi(e) = f_0(S_e) = f(S_e) - f(S_e) = 0,$$

i.e. $\phi$ is supported in $X_b$, and minimizing $\|\phi\|_1$ is equivalent to minimizing $\|\phi - \phi_0\|_1$, concluding the proof. ∎

## 5. Algorithmic aspects in the case of trees

In all of this section, $X = (V, E, \ell)$ denotes a metric tree with $|E| \geq 1$, and a function $f \in \mathcal{F}(V)$ will be fixed.

**5.1. Computing $\|f\|_{TCE}$ with a few additions.** Given $f \in \mathcal{F}(V)$, computing $\phi_f$, and then $\|f\|_{TCE} = \|\phi_f\|_1$, involves many additions (of the order of $|E||V|/2$). This can be improved.

**Proposition 5.1.** *Let $X = (V, E, \ell)$ be a metric tree with $|E| \geq 1$, and $f \in \mathcal{F}(V)$. One may compute $\|f\|_{TCE} = \|\phi_f\|_1$ with at most $2|E| - 1$ additions.*

*Proof.* We proceed by induction on $|E|$, the case $|E| = 1$ being trivial.

For $|E| > 1$, let $v$ be a leaf (= terminal vertex) of $X$ and the corresponding edge $e$ is oriented with $e_+ = v$. Then $\phi_f(e) = f(v)$.

Delete the vertex $v$ and the edge $e$ and solve the problem for the remaining tree $Y$ and $f_1(w) := f(w)$ for $w \neq e_-$ and $f_1(e_-) := f(v) + f(e_-)$ (one addition). By the induction hypothesis, $\|\phi_{f_1}\|_1$ can be computed in at most $2|E| - 3$ additions, and

$$\|\phi_f\|_1 = \|\phi_{f_1}\|_1 + |f(v)|\ell(e). \qquad ∎$$

**5.2. Finding a transportation with minimal number of transports.** Let $f \in \mathcal{F}(V)$. We assume that we have already found $\phi_f$ (with $f = \partial(\phi_f)$).

**Question 1.** *Find a solution of STC involving the minimal number of transports (namely, $\leq |\mathrm{Supp}(f)| - 1$, see Remark 3.4).*

In other words, we want to find $P \in \widetilde{\Pi}_f$ with

$$W(P) = \|f\|_{TC} \quad \text{and} \quad |\operatorname{Supp}(P)| \le |\operatorname{Supp}(f)| - 1.$$

**Remark 5.2.** If $\phi_f(e) = 0$ for some edges, we replace $E$ by the support $E'$ of $\phi_f$. Then $(V, E')$ is a forest. Treating separately each connected component, we will find an optimal transportation plan whose support has at most $|\operatorname{Supp}(f)| - k$ elements, where $k$ is the number of connected components that contain points of the support of $f$.

Let $f \in \mathcal{F}(V)$. Using Remark 5.2, we may assume that $\phi_f(e) \neq 0$ for every $e \in E$. It is convenient for what follows to change the orientation of all edges with $\phi_f(e) < 0$, and so we assume that $\phi_f(e) > 0$ for every edge.

Then a solution $P$ to Question 1 can be constructed algorithmically as follows.

(1) Choose arbitrarily $a \in V_+$.

(2) Construct a finite sequence of edges $e_1, \dots, e_k$ such that

$$(e_1)_- = a \quad \text{and} \quad (e_j)_+ = (e_{j+1})_-$$

for $1 \le j < k$, and $(e_k)_+ \in V_-$.

To show the existence of this sequence, observe that

$$f(a) = \sum_{e, e_- = a} \phi_f(e) - \sum_{e, e_+ = a} \phi_f(e).$$

It follows that there exists $e_1 \in E$ with $(e_1)_- = a$. Assume that $e_1, \dots, e_j$ are constructed, since

$$0 < f\big((e_j)_+\big) + \phi_f(e_j) \le \sum_{e, e_- = (e_j)_+} \phi_f(e),$$

we see that if $(e_j)_+ \notin V_-$, there exists $e_{j+1} \in E$ with $(e_j)_+ = (e_{j+1})_-$.

(3) Setting $b = (e_k)_+$ and $\alpha := \min\{f(a), -f(b), \min\{\phi_f(e_j) : 1 \le j \le k\}\}$, construct $P \in \widetilde{\Pi}_f$ with

$$P(a, b) = \alpha, \quad W(P) = \|f\|_{TC}, \quad \text{and} \quad |\operatorname{Supp}(P)| \le |\operatorname{Supp}(f)| - 1.$$

To do this, we proceed by induction on the number of elements of $\operatorname{Supp}(f)$.

If $\operatorname{Supp}(f)$ consists only of the two points $a, b$, then there is only one possible transportation plan with support equal to $V_+ \times V_-$, given by

$$P(x, y) = \begin{cases} f(a) & \text{if } (x, y) = (a, b), \\ 0 & \text{otherwise.} \end{cases}$$

Assume then that $\mathrm{Supp}(f)$ has $\geq 3$ points. Define:

- $\psi: E \to \mathbb{R}_+$ by setting
$$\psi(e) = \begin{cases} \alpha & \text{if } e \in \{e_1, \dots, e_k\}, \\ 0 & \text{otherwise.} \end{cases}$$
- $h = -\partial\psi$, so that $h(a) = \alpha$, $h(b) = -\alpha$, $h(x) = 0$ for $x \notin \{a, b\}$.
- $g = f - h$.

Define the plan $P_0$, by setting $P_0(a,b) = \alpha$ and $P_0(x,y) = 0$ if $(x,y) \neq (a,b)$. Then, as in the initial step, $P_0 \in \Pi_h$, and $\|\psi\|_1 = W(P_0)$

Since $\psi$ and $\phi_f - \psi$ are non-negative, $\|\phi_f\|_1 = \|\psi\|_1 + \|\phi_f - \psi\|_1$, in other words $\|f\|_{TC} = \|g\|_{TC} + \|h\|_{TC}$. We now consider three cases.

(1) If $\alpha = f(a)$, then $\mathrm{Supp}(g) \subset \mathrm{Supp}(f) \setminus \{a\}$. By the induction hypothesis, there is a transportation plan $Q \in \Pi_g$ with $W(Q) = \|g\|_{TC}$, and whose support has at most $|\mathrm{Supp}(g)| - 1$ elements. As $a \notin \mathrm{Supp}(g)$, we have $Q(a,b) = 0$.

(2) If $\alpha = -f(b)$, then $\mathrm{Supp}(g) \subset \mathrm{Supp}(f) \setminus \{b\}$. This case is similar to the previous one.

(3) If $\alpha < \min\{f(a), -f(b)\}$, then $\mathrm{Supp}(g) = \mathrm{Supp}(f)$. There exists $j$ with $\alpha = \phi_f(e_j)$. Now $\mathrm{Supp}(\phi_g) \subset E \setminus \{e_j\}$, and $(V, \mathrm{Supp}(\phi_g))$ has at least two connected components, each of which has a strictly smaller number of vertices $v$ with $g(v) \neq 0$ (as $a$ and $b$ have been disconnected). By the induction hypothesis, there is a transportation plan $Q \in \widetilde{\Pi}_g$ with $W(Q) = \|g\|_{TC}$, and whose support has at most $|\mathrm{Supp}(g)| - k$ elements, where $k$ is the number of connected components that contain points of $\mathrm{Supp}(g) = \mathrm{Supp}(f)$. Again, $Q(a,b) = 0$.

In all cases, $P = P_0 + Q$ does the job, i.e. $P \in \widetilde{\Pi}_f$, $P(a,b) = P_0(a,b)$,
$$W(P) = W(P_0) + W(Q) = \|h\|_{TC} + \|g\|_{TC} = \|f\|_{TC},$$
and $|\mathrm{Supp}(P)| \leq |\mathrm{Supp}(f)| - 1$.

**Remark 5.3.** Denote by $(V_f, E_P)$ the graph whose vertices are the points of the support of $f$ and edges are the points of the support of $P$. One may easily check that, in the above construction, if $(V_g, E_Q)$ is a forest, then $(V_f, E_P)$ is also a forest. In other words, our construction always yields a forest.

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