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Strichartz estimates for the Schrödinger equation on compact manifolds with nonpositive sectional curvature

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Abstract. We obtain improved Strichartz estimates for solutions of the Schrödinger equation on compact manifolds with nonpositive sectional curvatures which are related to the classical universal results of Burq, Gérard, and Tzvetkov (2004). More explicitly, we are able refine the arguments in the recent work of Blair and the authors (2024) to obtain no-loss $L^p_t L^q_x$-estimates on intervals of length $\log\lambda \cdot \lambda^{-1}$ for all *admissible* pairs (p, q) when the initial data have frequencies comparable to λ, which, given the role of the Ehrenfest time, is the natural analog in this setting of the universal results in Burq, Gérard, and Tzvetkov (2004). We achieve this log-gain over the universal estimates by applying the Keel–Tao theorem along with improved global kernel estimates for microlocalized operators which exploit the geometric assumptions.

In memoriam of Steve Zelditch (1953–2022)

1. Introduction

The purpose of this paper is to improve the Strichartz estimates for the Schrödinger equation on negatively curved compact manifolds of Blair and the authors [3], while at the same time simplifying the arguments there.

Let us first recall the universal estimates of Burq, Gérard, and Tzvetkov [11]. If (M^d, g) is a compact Riemannian manifold of dimension $d \geq 2$, then the main estimate in [11] is that if Δ_g is the associated Laplace–Beltrami operator and

$$u(x,t) = (e^{-it\Delta_g} f)(x)$$

is the solution of the Schrödinger equation on $M^d \times \mathbb{R}$,

$$i\partial_t u(x,t) = \Delta_g u(x,t), \quad u(x,0) = f(x), \tag{1.1}$$

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then if we define

$$\|u\|_{L_t^p L_x^q(M^d\times[0,1])} = \Big(\int_0^1 \|u(\cdot,t)\|^p_{L_x^q(M^d)}\,dt\Big)^{1/p},$$

we have the mixed-norm Strichartz estimates

$$\|u\|_{L_t^p L_x^q(M^d\times[0,1])} \lesssim \|f\|_{H^{1/p}(M^d)} \tag{1.2}$$

for all *admissible* pairs (p,q). By the latter we mean, as in Keel and Tao [17],

$$d\Big(\frac{1}{2}-\frac{1}{q}\Big) = \frac{2}{p} \tag{1.3a}$$

and

$$\begin{aligned} &2 < q \le \frac{2d}{d-2} &&\text{if } d\ge 3,\\ &2 < q < \infty &&\text{if } d = 2. \end{aligned} \tag{1.3b}$$

Also, in (1.2) H^μ denotes the standard Sobolev space

$$\|f\|_{H^\mu(M^d)} = \|(I+P)^\mu f\|_{L^2(M^d)}, \quad \text{with } P=\sqrt{-\Delta_g},$$

and "$\lesssim$" in (1.2) and in what follows denotes an inequality with an implicit, but unstated, constant C which can change at each occurrence.

Note that if e_λ is an eigenfunction of P with eigenvalue λ, i.e.,

$$-\Delta_g e_\lambda = \lambda^2 e_\lambda,$$

then

$$u(x,t) = e^{it\lambda^2} e_\lambda(x) \tag{1.4}$$

solves (1.1) with initial data $f = e_\lambda$. Thus, unlike Euclidean space, or hyperbolic space, one cannot replace $[0,1]$ by $\mathbb{R}$ in (1.2) due to the existence of eigenfunctions. Also, for the endpoint Strichartz estimates where $p=2$ and $q=2d/(d-2)$ with $d\ge 3$, the derivative loss of $1/2$ in (1.2) cannot be improved on the standard round sphere S^d, by taking the initial data f to be zonal eigenfunctions. See, e.g., [3, 11] for more details.

However, on compact manifolds with other types of geometries, (1.2) can be improved, see e.g., [10, 12, 13] for the torus case and [3] for general compact manifolds with nonpositive sectional curvatures. Also, for *admissible* pairs (p,q) other than $(2, 2d/(d-2))$, one can have better estimates than (1.2) on the sphere as well using the specific arithmetic properties of the distinct eigenvalues of the Laplacian on S^d. See, e.g., [11, Theorem 4].

As in [3], to align with the numerology in earlier related results involving eigenfunction and spectral projection estimates, in what follows, we shall always take $d = n - 1$. Our main result which improves on estimates in [3] then is the following.

Theorem 1.1. *Let M^{n-1} be a $d = n - 1 \geq 2$-dimensional compact manifold all of whose sectional curvatures are nonpositive. Then, for all* admissible *pairs (p, q),*

$$\|u\|_{L_t^p L_x^q(M^{n-1}\times[0,1])} \lesssim \|(I + P)^{1/p}(\log(2I + P))^{-1/p} f\|_{L^2(M^{n-1})}. \tag{1.5}$$

If we let the initial data $f = e_\lambda$, then by (1.4) and the special case $(p, q) = (2, 2d/(d-2))$ of (1.5), we have

$$\|e_\lambda\|_{L^{2d/(d-2)}(M^d)} \lesssim \lambda^{1/2}(\log\lambda)^{-1/2}\|e_\lambda\|_{L^2(M^d)}, \tag{1.6}$$

which gives a gain $(\log\lambda)^{-1/2}$ compared with the universal eigenfunction estimates of the second author [19]. For manifolds with nonpositive curvature, (1.6) was first proved by Hassell and Tacy [14], with similar results for all $q > 2(d+1)/(d-1)$. Thus, (1.5) is a natural generalization of (1.6) for solutions of Schrödinger equation and also provides a novel approach to get improved eigenfunction estimates.

By the Littlewood–Paley theory, we may reduce (1.5) to proving certain dyadic estimates. More explicitly, let us fix a Littlewood–Paley bump function β satisfying

$$\beta \in C_0^\infty\Big(\Big(\frac{1}{2}, 2\Big)\Big) \quad \text{and} \quad 1 = \sum_{k=-\infty}^{\infty} \beta(2^{-k}s), \quad s > 0. \tag{1.7}$$

Then, if we set

$$\beta_0(s) = 1 - \sum_{k=1}^{\infty} \beta(2^{-k}s) \in C_0^\infty(\mathbb{R}_+)$$

and

$$\beta_k(s) = \beta(2^{-k}s), \quad k = 1, 2, \dots,$$

we have (see, e.g., [20])

$$\|h\|_{L^q(M^{n-1})} \approx \Big\|\Big(\sum_{k=0}^{\infty} |\beta_k(P)h|^2\Big)^{1/2}\Big\|_{L^q(M^{n-1})}, \quad 1 < q < \infty. \tag{1.8}$$

Trivially, $\|\beta_0(P)e^{-it\Delta_g}\|_{L^2(M^{n-1})\to L_t^p L_x^q(M^{n-1}\times[0,1])} = O(1)$, and, similarly, such results where $k = 0$ is replaced by a small fixed $k \in \mathbb{N}$ are also standard. So, as noted in [11], one can use (1.8) and Minkowski's inequality to see that (1.2) follows from the uniform bounds

$$\|e^{-it\Delta_g}\beta(P/\lambda)f\|_{L_t^p L_x^q(M^{n-1}\times[0,1])} \leq C\lambda^{1/p}\|f\|_{L^2(M^{n-1})}, \quad \lambda \gg 1. \tag{1.9}$$

Burq, Gérard, and Tzvetkov proved this estimate in [11] by showing that one always has the following uniform dyadic estimates over very small intervals:

$$\|e^{-it\Delta_g}\beta(P/\lambda)f\|_{L_t^pL_x^q(M^{n-1}\times[0,\lambda^{-1}])} \le C\|f\|_{L^2(M^{n-1})}, \quad \lambda \gg 1. \tag{1.10}$$

It is not hard to see that (1.10) yields (1.9), since one can write [0, 1] as the union of $\approx \lambda$ intervals of length λ^{-1} and thus obtain (1.9) by adding up the uniform estimates on each of these subintervals that (1.10) affords. Also, the bounds in (1.10) cannot be improved on *any* manifold by taking $f(x) = f_\lambda(x) = \beta(P/\lambda)(x, x_0)$, with $\beta(P/\lambda)(x, y)$ being the kernel of the Littlewood–Paley operators. As a result, to obtain improvements such as those in (1.5), larger time intervals must be used. Specifically, we shall show that if M^{n-1} is as in Theorem 1.1, then we have the uniform bounds

$$\|e^{-it\Delta_g}\beta(P/\lambda)f\|_{L_t^pL_x^q(M^{n-1}\times[0,\log\lambda\cdot\lambda^{-1}])} \le C\|f\|_{L^2(M^{n-1})}, \quad \lambda \gg 1, \tag{1.11}$$

which is a natural extension of the uniform small-time scale estimates (1.10), and perhaps the largest one can hope to obtain such estimates in the geometry we are focusing on using available techniques, due to the role of the Ehrenfest time. Also, by the above counting arguments, one obtains (1.5) from (1.11) since [0, 1] can be covered by $\approx \lambda/\log\lambda$ intervals of length $\log\lambda\cdot\lambda^{-1}$.

The bound in (1.11) improves the result in [3] in two aspects: first, it removes the power of $\log\lambda$ loss there; and second, it includes all *admissible* pairs (p, q). The main ideas in the proof of (1.11) are similar to those in [3], both of which involve a height decomposition. The larger values can be dealt with using kernel estimates for certain *global* operators, while the smaller values involves the use of a Whitney-type decomposition and bilinear oscillatory integral estimates of Lee [18]. Compared to [3], we are able to improve the arguments used for the *diagonal* term in the Whitney decomposition, which previously relied on a microlocalized version of improved Strichartz estimates. We avoid this by applying the abstract theorem of Keel and Tao [17] to some new Banach space with L^p norm depending on the ℓ^p norm of different microlocalized pieces. The analogous dispersive estimate adapted to the new space, which is necessary to apply the Keel–Tao theorem, is obtained by using the kernel estimates involving the microlocalized operators as proved in [3].

If $\chi_{[\lambda,\lambda+(\log\lambda)^{-1}]}$ denotes the spectral projection operator for $P = \sqrt{-\Delta_g}$ associated with the interval $[\lambda, \lambda + (\log\lambda)^{-1}]$, then a simple consequence of (1.11) is that for $q_e = 2d/(d-2)$, $d = n-1 \ge 3$,

$$\|\chi_{[\lambda,\lambda+(\log\lambda)^{-1}]}f\|_{L^{q_e}(M^d)} \le C\lambda(\log\lambda)^{-1/2}\|f\|_{L^2(M^d)}, \tag{1.12}$$

assuming that M^d has nonpositive sectional curvatures. This, in turn, implies the natural sharp spectral projection estimates for these operators for all $q \ge q_e$ under these curvature assumptions, which are due to Bérard [1] for $q = \infty$ and later generalized by Hassell and Tacy [14] to $q > 2(d+1)/(d-1)$ and $d \ge 2$.

To prove (1.12), fix $a \in \mathcal{S}(\mathbb{R})$ satisfying $a(0) = 1$ and $\hat{a}(\tau) = 0$ if $|\tau| \ge 1$. Then, for q_e and $d = n-1$ as in (1.12), by Bernstein's inequality and (1.11),

$$\begin{aligned}
&\|\chi_{[\lambda,\lambda+(\log\lambda)^{-1}]} f\|_{L^{q_e}(M^{n-1})} \\
&\le \Big\| a(\lambda(\log\lambda)^{-1}t) e^{-it\Delta_g} \beta\Big(\frac{P}{\lambda}\Big)\chi_{[\lambda,\lambda+(\log\lambda)^{-1}]} f\Big\|_{L_t^\infty L_x^{q_e}(M^{n-1}\times\mathbb{R})} \\
&\le C\lambda^{1/2}(\log\lambda)^{-1/2}\Big\| a(\lambda(\log\lambda)^{-1}t) e^{-it\Delta_g} \beta\Big(\frac{P}{\lambda}\Big)\chi_{[\lambda,\lambda+(\log\lambda)^{-1}]} f\Big\|_{L_t^2 L_x^{q_e}(M^{n-1}\times\mathbb{R})} \\
&\le C\lambda^{1/2}(\log\lambda)^{-1/2}\|f\|_{L^2(M^{n-1})}.
\end{aligned}$$

Note that this argument implies that any improvements of (1.11) to include uniform bounds for intervals of size $\delta(\lambda)$ with $\delta(\lambda)/[\lambda^{-1}\cdot\log\lambda] \nearrow \infty$ would imply sharp spectral projection estimates associated with spectral windows of length $o((\log\lambda)^{-1})$ for $q \ge q_e$ under the assumption of nonpositive curvature, which, even for the case of $q = \infty$, seems difficult. There has been no such improvement of the sup-norm estimates of Bérard [1] in the last five decades, and progress of this nature has been elusive due to the role of the Ehrenfest time.

In the case of flat tori, if we fix $q = 2(n+1)/(n-1)$, recall the near optimal results of Bourgain and Demeter [10] for $n \ge 4$ and Bourgain [9] for $n = 2, 3$,

$$\begin{aligned}
&\Big\| e^{-it\Delta_g}\beta\Big(\frac{P}{\lambda}\Big) f\Big\|_{L^q_{t,x}(M^{n-1}\times[0,1])} \\
&\qquad \le C_\varepsilon \lambda^\varepsilon \|f\|_{L^2(M^{n-1})}, \quad \lambda \gg 1, \quad \text{for all } \varepsilon > 0.
\end{aligned} \tag{1.13}$$

The proof of (1.13) is based on number-theoretic methods for $n = 2, 3$, while for $n \ge 4$, the estimates were derived via ℓ^2-decoupling methods. More recently, for $n = 3$, Herr and Kwak [15] obtained a lossless version of (1.13) for any time intervals of length $(\log\lambda)^{-1}$ by using a new method based on incidence geometry, and the length of the interval ccannot be extended further by testing against

$$f = \sum_{k\in[-\lambda,\lambda]^2\cap\mathbb{Z}^2} e^{ik\cdot x}.$$

See e.g., [9] for more details.

By using the favorable properties of the universal cover $\mathbb{R}^n$ and the fact that the types of microlocal cutoffs, we shall employ commute well with Schrödinger propagators, it seems likely that we shall be able to modify the arguments in the proof of Theorem 1.1 to obtain no loss dyadic estimates on tori $\mathbb{T}^n$ on intervals of length $\lambda^{-1+\delta_n}$ for some $0 < \delta_n < 1$, which would be a natural generalization of the results in [15] for all *admissible* pairs (p,q) in any dimension $n \geq 3$. We hope to explore this problem as well as possible improved Strichartz estimates for spheres in a later work.

This paper is organized as follows. In the next section, we present the main arguments that allow us to prove Theorem 1.1. The proof requires local bilinear arguments from harmonic analysis, which are discussed in Section 3.

2. Main arguments

To start, let β be the Littlewood–Paley bump function in (1.7), and also fix

$$\eta \in C_0^\infty((-1,1)) \quad \text{with } \eta(t) = 1, |t| \leq \frac{1}{2}. \tag{2.1}$$

We then shall consider the dyadic time-localized dilated Schrödinger operators

$$S_\lambda = \eta\Big(\frac{t}{T}\Big)e^{-it\lambda^{-1}\Delta_g}\beta\Big(\frac{P}{\lambda}\Big), \tag{2.2}$$

and claim that the estimates in Theorems 1.1 is a consequence of the following.

Proposition 2.1. *Let M^d, $d = n-1 \geq 2$ be a fixed compact manifold all of whose sectional curvatures are nonpositive, and (p,q) be as in (1.3). Then, we can fix $c_0 > 0$ so that for large $\lambda \gg 1$ we have the uniform bounds*

$$\|S_\lambda f\|_{L_t^p L_x^q(M^{n-1}\times[0,T])} \leq C\lambda^{1/p}\|f\|_{L^2(M^{n-1})}, \quad \text{if } T = c_0 \log\lambda. \tag{2.3}$$

We claim that (2.3) implies Theorem 1.1. First note that, by changing scales, (2.1) and (2.3) imply that for large enough λ, we have the analog of (1.11) where the interval $[0, \log\lambda\cdot\lambda^{-1}]$ in the left is replaced by $[0, (c_0/2)\log\lambda\cdot\lambda^{-1}]$, and this of course implies (1.11) at the expense of including an additional factor of $(c_0/2)^{-1/q_c}$ in the constant in the right if $c_0 < 2$. As we indicated before, the estimate (1.11) for large λ and Littlewood–Paley theory yield Theorem 1.1, which verifies our claim regarding (2.3).

Also note that if we replace $[0,T]$ by $[0,1]$, then by (1.10) and a rescaling argument we have

$$\|S_\lambda f\|_{L_t^p L_x^q(M^{n-1}\times[0,1])} \leq C\lambda^{1/p}\|f\|_{L^2(M^{n-1})}, \tag{2.4}$$

which hold on a general smooth compact manifold.

To prove (2.3), note that since the case $(p,q) = (\infty, 2)$ is trivial, by interpolation, it suffices to consider the (p,q) pairs which satisfy

$$\begin{aligned} &(p,q) = \Big(2, \frac{2d}{d-2}\Big) && \text{if } d \geq 3, \\ &d\Big(\frac{1}{2} - \frac{1}{q}\Big) = \tfrac{2}{p}, 4 < q < \infty && \text{if } d = 2. \end{aligned} \tag{2.5}$$

The condition $q > 4$ is equivalent to $q > p$ when $d = 2$, this will allow us to simplify some of the calculations to follow.

As in [3], we need to introduce a few auxiliary operators that allow us to use bilinear techniques. First of all, we need to compose the "global operators" S_λ with related local ones. Motivated by the recent work of the authors [16], our "local" auxiliary operators will be the following "quasimode" operators adapted to the scaled Schrödinger operators $\lambda D_t + \Delta_g$,

$$\sigma_\lambda = \sigma(\lambda^{1/2}|D_t|^{1/2} - P)\tilde{\beta}\Big(\frac{D_t}{\lambda}\Big), \tag{2.6}$$

where $\sigma \in \mathcal{S}(\mathbb{R})$ satisfies

$$\sigma(0) = 1 \quad \text{and} \quad \operatorname{supp} \hat{\sigma} \subset \delta \cdot [1 - \delta_0, 1 + \delta_0] = [\delta - \delta_0\delta, \delta + \delta_0\delta], \tag{2.7}$$

with $0 < \delta, \delta_0 < 1/8$ to be specified later, and, also here $\tilde{\beta} \in C_0^\infty((1/8, 8))$ satisfies

$$\tilde{\beta} = 1 \quad \text{on} \ \Big[\frac{1}{6}, 6\Big]. \tag{2.8}$$

Here, the properties of σ, as well as the small constants δ and δ_0, are the same as those in [3], which allows us to use the bilinear oscillatory integral estimates in [3] in the next section. The smallness of δ is also related to another initial microlocalization that is needed for the bilinear arguments, as we shall describe below.

Let us write

$$I = \sum_{j=1}^{N} B_j(x, D), \tag{2.9}$$

where each $B_j \in S^0_{1,0}(M^{n-1})$ is a standard pseudo-differential operator with symbol supported in a small conic neighborhood of some $(x_j, \xi_j) \in S^*M$. The size of the support will be described later; however, these operators will not depend on our parameter $\lambda \gg 1$. Next, if $\tilde{\beta}$ is as in (2.8), then the dyadic operators

$$B = B_{j,\lambda} = B_j \circ \tilde{\beta}\Big(\frac{P}{\lambda}\Big) \tag{2.10}$$

are uniformly bounded on L^p, i.e.,

$$\|B\|_{L^p(M^{n-1}) \to L^p(M^{n-1})} = O(1) \quad \text{for } 1 \le p \le \infty. \tag{2.11}$$

Also, note that since $\sigma \in \mathcal{S}(\mathbb{R})$ a simple calculation shows that if λ_k is an eigenvalue of P

$$\Big(1 - \tilde{\beta}\Big(\frac{\lambda_k}{\lambda}\Big)\Big)\sigma(\lambda^{1/2}|\tau|^{1/2} - \lambda_k)\tilde{\beta}\Big(\frac{\tau}{\lambda}\Big) = O(\lambda^{-N}(1 + \lambda_k + |\tau|)^{-N}) \quad \text{for all } N.$$

Consequently,

$$\|\sigma_\lambda - \tilde\beta(P/\lambda)\circ\sigma_\lambda\|_{L^2(M^{n-1}\times[0,T])\to L_t^pL_x^q(M^{n-1}\times[0,T])} = O(\lambda^{-N}) \quad \text{for all } N.$$

Thus, if B_j is as in (2.9) and $B_{j,\lambda}$ is the corresponding dyadic operator in (2.10),

$$\|B_j\sigma_\lambda - B_{j,\lambda}\sigma_\lambda\|_{L^2(M^{n-1}\times[0,T])\to L_t^pL_x^q(M^{n-1}\times[0,T])} = O(\lambda^{-N}) \quad \text{for all } N, \tag{2.12}$$

since operators in $S^0_{1,0}(M^{n-1})$ are bounded on L^p for $1<p<\infty$.

We need one more result for now about these local operators.

Lemma 2.2. *If S_λ as in (2.2), σ_λ is as in (2.6) and (p,q) are as in (1.3), then*

$$\|(I-\sigma_\lambda)\circ S_\lambda f\|_{L_t^pL_x^q(M^{n-1}\times[0,T])} \le CT^{1/p-1/2}\lambda^{1/p}\|f\|_2. \tag{2.13}$$

We also have

$$\|\sigma_\lambda F\|_{L_t^pL_x^q} \le C\lambda^{1/p}\|F\|_{L^2_{t,x}}. \tag{2.14}$$

Lemma 2.2 is a simple generalization of the results in [3, Lemmas 2.2 and 2.3] to all *admissible* pairs (p,q). Using the local dyadic Strichartz estimates (1.10), the proof of Lemma 2.2 follows from the same arguments as in [3], so we skip the details here.

For a given $B=B_{j,\lambda}$ as in (2.10), let us define the microlocalized variant of σ_λ as

$$\tilde\sigma_\lambda = B\circ\sigma_\lambda, \quad B=B_{j,\lambda},$$

and the associated “semi-global” operators as

$$\widetilde S_\lambda = \tilde\sigma_\lambda\circ S_\lambda.$$

By (2.9), (2.12), and (2.13), in order to prove Proposition 2.1, it suffices to show that if $T=c_0\log\lambda$ with $c_0>0$ sufficiently small (depending on M^{n-1}), then, if all the sectional curvatures of M^{n-1} are nonpositive, for (p,q) be as in (2.5), we have

$$\|\widetilde S_\lambda f\|_{L_t^pL_x^q(M^{n-1}\times[0,T])} \le C\lambda^{1/p}\|f\|_2. \tag{2.15}$$

As we shall see, in order to prove (2.15), we shall need to take δ and δ_0 in (2.7) and (2.8) to be sufficiently small for each j; however, since, by the compactness of M^{n-1} and the arguments to follow, the sum in (2.9) can be taken to be finite, we can take these two parameters to be the minimum over what is needed for $j=1,\dots,N$.

2.1. Height decomposition

Next, we set up a variation of an argument of Bourgain [8] originally used to study Fourier transform restriction problems, and, more recently, to study eigenfunction problems in [2,7,21]. This involves splitting the estimates in Proposition 2.1 into two heights involving relatively large and small values of $|\widetilde{S}_\lambda f(x,t)|$.

To describe this, here and in what follows we shall assume, as we just did, that f is L^2-normalized, that is

$$\|f\|_2 = 1. \tag{2.16}$$

Then, we shall prove the estimates in Proposition 2.1, using very different techniques by estimating $L^p_t L^q_x$ bounds over the two regions

$$\begin{aligned} A_+ &= \{(x,t) \in M^{n-1} \times [0,T] : |\widetilde{S}_\lambda f(t,x)| \ge \lambda^{(n-1)/4+\varepsilon_1}\}, \\ A_- &= \{(x,t) \in M^{n-1} \times [0,T] : |\widetilde{S}_\lambda f(x,t)| < \lambda^{(n-1)/4+\varepsilon_1}\}. \end{aligned} \tag{2.17}$$

Due to the numerology of the powers of λ arising, the splitting occurs at height $\lambda^{(n-1)/4+\varepsilon_1}$, where $\varepsilon_1 > 0$ is a small constant that may depend on the dimension $d = n-1$. As we shall see later in (3.8) and (3.30), we can take $\varepsilon_1 = 1/100$ for $n-1 \ge 3$ while for $n-1 = 2$, the choice of ε_1 depends on the exponent q for *admissible* pairs (p,q), with $\varepsilon_1 \to 0$ as $q \to \infty$. The transition occurring at, basically, $\lambda^{(n-1)/4}$ is natural and arises due to Knapp-type phenomena, both in Euclidean problems, as well as the geometric ones that we are considering here.

Thus, to prove Proposition 2.1, it suffices to prove the analog (2.15) for the two regions in (2.17).

2.2. Estimates for relatively large values: Proof of (2.15) on the set A_+

We first note that, by Lemma 2.2 and (2.11), we have

$$\|\widetilde{S}_\lambda f\|_{L^p_t L^q_x(A_+)} \le \|BS_\lambda f\|_{L^p_t L^q_x(A_+)} + CT^{1/p-1/2}\lambda^{1/p},$$

and, since $p \ge 2$ for (p,q) as in (2.5), (2.15) would follow from

$$\|BS_\lambda f\|_{L^p_t L^q_x(A_+)} \le C\lambda^{1/p} + \frac{1}{2}\|\widetilde{S}_\lambda f\|_{L^p_t L^q_x(A_+)}. \tag{2.18}$$

To prove this, we shall adapt an argument of Bourgain [8] and more recent variants in [2,21] . Specifically, choose $g(x,t)$ such that

$$\|g\|_{L^{p'}_t L^{q'}_x(A_+)} = 1$$

and

$$\|BS_\lambda f\|_{L^p_t L^q_x(A_+)} = \iint BS_\lambda f \cdot \overline{(\mathbb{1}_{A_+} \cdot g)}\, dx\, dt.$$

Then, since we are assuming that $\|f\|_2 = 1$, by the Schwarz inequality,

$$\begin{aligned}
\|BS_\lambda f\|^2_{L^p_t L^q_x(A_+)} &= \Big(\int f(x)\cdot \overline{(S^*B^*)(\mathbb{1}_{A_+}\cdot g)(x)}\,dx\Big)^2 \\
&\le \int |S^*_\lambda B^*(\mathbb{1}_{A_+}\cdot g)(x)|^2\,dx \\
&= \iint (BS_\lambda S^*_\lambda B^*)(\mathbb{1}_{A_+}\cdot g)(x,t)\overline{(\mathbb{1}_{A_+}\cdot g)(x,t)}\,dx\,dt \\
&= \iint (B\circ L_\lambda \circ B^*)(\mathbb{1}_{A_+}\cdot g)(x,t)\overline{(\mathbb{1}_{A_+}\cdot g)(x,t)}\,dx\,dt \\
&\quad + \iint (B\circ G_\lambda \circ B^*)(\mathbb{1}_{A_+}\cdot g)(x,t)\overline{(\mathbb{1}_{A_+}\cdot g)(x,t)}\,dx\,dt \\
&= \mathrm{I}+\mathrm{II},
\end{aligned} \tag{2.19}$$

where L_λ is the integral operator with kernel equaling that of $S_\lambda S^*_\lambda$ if $|t-s|\le 1$ and 0 otherwise, i.e.,

$$L_\lambda(x,t;y,s) = \begin{cases} (S_\lambda S^*_\lambda)(x,t;y,s) & \\ \quad = \eta\Big(\dfrac{t}{T}\Big)\eta\Big(\dfrac{s}{T}\Big)\Big(\beta^2\Big(\dfrac{P}{\lambda}\Big)e^{-i(t-s)\lambda^{-1}\Delta_g}\Big)(x,y), & \text{if } |t-s|\le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.20}$$

Since $p\ge 2$, it is straightforward to see that (2.4) and (2.22) below yield

$$\|L_\lambda\|_{L^{p'}_t L^{q'}_x \to L^p_t L^q_x} = O(\lambda^{2/p}).$$

If we use this, along with Hölder's inequality and (2.11), we obtain for the term I in (2.19)

$$\begin{aligned}
|\mathrm{I}| &\le \|BL_\lambda B^*(\mathbb{1}_{A_+}\cdot g)\|_{L^p_t L^q_x}\cdot \|\mathbb{1}_{A_+}\cdot g\|_{L^{p'}_t L^{q'}_x} \\
&\lesssim \|L_\lambda B^*(\mathbb{1}_{A_+}\cdot g)\|_{L^p_t L^q_x}\cdot \|\mathbb{1}_{A_+}\cdot g\|_{L^{p'}_t L^{q'}_x} \\
&\lesssim \lambda^{2/p}\|B^*(\mathbb{1}_{A_+}\cdot g)\|_{L^{p'}_t L^{q'}_x}\cdot \|\mathbb{1}_{A_+}\cdot g\|_{L^{p'}_t L^{q'}_x} \\
&\lesssim \lambda^{2/p}\|g\|^2_{L^{p'}_t L^{q'}_x(A_+)} = \lambda^{2/p}.
\end{aligned} \tag{2.21}$$

To estimate the other term in (2.19), II, we need the following kernel bound which is [3, Proposition 4.1]:

$$|(S_\lambda S^*_\lambda)(x,t;y,s)| \le C\lambda^{(n-1)/2}|t-s|^{-(n-1)/2}\exp(C_M|t-s|), \quad \text{if } |t-s|\le 2T. \tag{2.22}$$

The proof of (2.22) follows from arguments in Bérard [1], and also in [6, 22], and other related works, which use the Hadamard parametrix and the Cartan–Hadamard theorem to lift the calculations that will be needed up to the universal cover $(\mathbb{R}^{n-1}, \tilde{g})$ of (M^{n-1}, g).

If we choose c_0 small enough so that if C_M is the constant in (2.22),

$$\exp(2C_M T) \le \lambda^{\varepsilon_1}, \quad \text{if } T = c_0 \log \lambda \text{ and } \lambda \gg 1.$$

Then, since $\eta(t) = 0$ for $|t| \ge 1$, it follows from (2.20) and (2.22) that

$$\|G_\lambda\|_{L^1(M^{n-1}\times\mathbb{R}) \to L^\infty(M^{n-1}\times\mathbb{R})} \le C\lambda^{(n-1)/2+\varepsilon_1}.$$

As a result, since, by (2.11), the dyadic operators B are bounded on L^1 and L^∞, we can repeat the arguments to estimate I and use Hölder's inequality to see that

$$\begin{aligned} |\mathrm{II}| &\le C\lambda^{(n-1)/2}\lambda^{\varepsilon_1}\|\mathbb{1}_{A_+}\cdot g\|_1^2 \\ &\le C\lambda^{n-1/2}\lambda^{\varepsilon_1}\|g\|^2_{L_t^{p'}L_x^{q'}} \cdot \|\mathbb{1}_{A_+}\|^2_{L_t^p L_x^q} = C\lambda^{(n-1)/2}\lambda^{\varepsilon_1}\|\mathbb{1}_{A_+}\|^2_{L_t^p L_x^q}. \end{aligned}$$

If we recall the definition of A_+ in (2.17), we can estimate the last factor:

$$\|\mathbb{1}_{A_+}\|^2_{L_t^p L_x^q} \le (\lambda^{(n-1)/4+\varepsilon_1})^{-2}\|\widetilde{S}_\lambda f\|^2_{L_t^p L_x^q(A_+)}.$$

Therefore,

$$|\mathrm{II}| \lesssim \lambda^{-\varepsilon_1}\|\widetilde{S}_\lambda f\|^2_{L_t^p L_x^q(A_+)} \le \left(\frac{1}{2}\|\widetilde{S}_\lambda f\|_{L_t^p L_x^q(A_+)}\right)^2,$$

assuming, as we may, that λ is large enough.

If we combine this bound with the earlier one, (2.21) for I, we conclude that (2.18) is valid, which completes the proof of (2.15) on the set A_+. ■

2.3. Estimates for relatively small values: Proof of (2.15) on the set A_-

We now turn to the proving the $L_t^p L_x^q(A_-)$ estimates in (2.15). To do this, we need to borrow the bilinear estimates from [3], which replies on the results from bilinear harmonic analysis in [18, 23].

We need to utilize a microlocal decomposition as in [3]. Recall that the symbol $B(x,\xi)$ of B in (2.10) is supported in a small conic neighborhood of some $(x_0, \xi_0) \in S^*M^{n-1}$. We may assume that its symbol has small enough support so that we may work in a coordinate chart Ω and that $x_0 = 0$, $\xi_0 = (0, \dots, 0, 1)$ and $g_{jk}(0) = \delta_k^j$ in the local coordinates. So, we shall assume that $B(x,\xi) = 0$ when x is outside a small relatively compact neighborhood of the origin or ξ is outside of a small conic neighborhood of $(0, \dots, 0, 1)$.

Next, let us define the microlocal cutoffs that we shall use. We fix a function $a \in C_0^\infty(\mathbb{R}^{2(n-2)})$ supported in $\{z : |z_j| \le 1, 1 \le j \le 2(n-2)\}$ which satisfies

$$\sum_{j \in \mathbb{Z}^{2(n-2)}} a(z-j) \equiv 1.$$

We shall use this function to build our microlocal cutoffs. By the above, we shall focus on defining them for $(y, \eta) \in S^*\Omega$ with y near the origin and η in a small conic neighborhood of $(0, \dots, 0, 1)$. We shall let

$$\Pi = \{y : y_{n-1} = 0\}$$

be the points in Ω whose last coordinate vanishes. Let $y' = (y_1, \dots, y_{n-2})$ and $\eta' = (\eta_1, \dots, \eta_{n-2})$ denote the first $n-2$ coordinates of y and η, respectively. For $y \in \Pi$ near 0 and η near $(0, \dots, 0, 1)$, we can just use the functions $a(\theta^{-1}(y', \eta') - j)$, $j \in \mathbb{Z}^{2(n-2)}$ to obtain cutoffs of scale θ.

We can then extend the definition to a neighborhood of $(0, (0, \dots, 0, 1))$ by setting for $(x, \xi) \in S^*\Omega$ in this neighborhood

$$a_j^\theta(x, \xi) = a(\theta^{-1}(y', \eta') - j) \quad \text{if } \chi_s(x, \xi) = (y', 0, \eta', \eta_{n-1}) \text{ with } s = d_g(x, \Pi).$$

Here, χ_s denotes geodesic flow in $S^*\Omega$. Thus, $a_j^\theta(x, \xi)$ is constant on all geodesics $(x(s), \xi(s)) \in S^*\Omega$ with $x(0) \in \Pi$ near 0 and $\xi(0)$ near $(0, \dots, 0, 1)$. As a result,

$$a_j^\theta(\chi_s(x, \xi)) = a_j^\theta(x, \xi)$$

for s near 0 and $(x, \xi) \in S^*\Omega$ near $(0, (0, \dots, 0, 1))$.

We then extend the definition of the cutoffs to a conic neighborhood of $(0, (0, \dots, 0, 1))$ in $T^*\Omega \backslash 0$ by setting

$$a_j^\theta(x, \xi) = a_j^\theta\Big(x, \frac{\xi}{p(x, \xi)}\Big).$$

Notice that if $(y_j', \eta_j') = \theta j$ and γ_j is the geodesic in $S^*\Omega$ passing through $(y_j', 0, \eta_j) \in S^*\Omega$ with $\eta_j \in S^*_{(y_j', 0)}\Omega$ having η_j' as its first $(n-2)$ coordinates, then

$$a_j^\theta(x, \xi) = 0 \quad \text{if } \operatorname{dist}((x, \xi), \gamma_j) \ge C_0\theta,$$

for some fixed constant $C_0 > 0$. Also, a_j^θ satisfies the estimates

$$|\partial_x^\sigma \partial_\xi^\gamma a_j^\theta(x, \xi)| \lesssim \theta^{-|\sigma| - |\gamma|}, \quad (x, \xi) \in S^*\Omega$$

related to this support property.

The a_j^θ provide "directional" microlocalization. We also need a "height" localization since the characteristics of the symbols of our scaled Schrödinger operators lie on paraboloids. The variable coefficient operators that we shall use of course are adapted to our operators and are analogs of ones that are used in the study of Fourier restriction problems involving paraboloids.

To construct these, choose $b \in C_0^\infty(\mathbb{R})$ supported in $|s| \le 1$ satisfying $\sum_{-\infty}^{\infty} b(s-\ell) \equiv 1$. We then simply define the "height operator" as follows:

$$A_\ell^\theta(P) = b(\theta^{-1}\lambda^{-1}(P - \lambda\kappa_\ell^\theta))\Upsilon\Big(\frac{P}{\lambda}\Big), \quad \kappa_\ell^\theta = 1 + \theta\ell, \quad |\ell| \lesssim \theta^{-1},$$

where if $\tilde{\beta}$ is as in (2.8), then $\Upsilon \in C_0^\infty((1/10, 10))$ satisfies

$$\Upsilon(r) = 1 \quad \text{in a neighborhood of supp } \tilde{\beta}.$$

Thus, these operators microlocalize P to intervals of size $\approx \theta\lambda$ about "heights" $\lambda\kappa_\ell^\theta \approx \lambda$. As we shall see below, different "heights" will give rise to different "Schrödinger tubes" about which the kernels of our microlocalization of the operators $\tilde{\sigma}_\lambda$ are highly concentrated. Also, standard arguments as in [20] show that if $A_\ell^\theta(x, y)$ is the kernel of this operator, then

$$A_\ell^\theta(x, y) = O(\lambda^{-N}) \quad \text{for all } N, \text{ if } d_g(x, y) \ge C_0\theta,$$

for a fixed constant if $\theta \in [\lambda^{-\delta_0}, 1]$ with, as we are assuming, $\delta_0 < 1/2$.

If $\psi(x) \in C_0^\infty(\Omega)$ equals 1 in a neighborhood of the x-support of the $B(x, \xi)$ and $A_j^\theta(x, D_x)$ is the operator with symbol

$$A_j^\theta(x, \xi) = \psi(x) a_j^\theta(x, \xi),$$

then for $\nu = (\theta j, \theta\ell) \in \theta\mathbb{Z}^{2(n-2)+1}$ we can finally define the cutoffs that we shall use:

$$A_\nu^\theta = A_j^\theta(x, D_x) \circ A_\ell^\theta(P).$$

Let us collect several basic facts about the operators A_ν^θ for later use. First, if $A_\nu^\theta(x, \xi)$ and $A_{\tilde{\nu}}^\theta(x, \xi)$ are the symbols of A_ν^θ and $A_{\tilde{\nu}}^\theta$, respectively, then

$$A_\nu^\theta(x, \xi) A_{\tilde{\nu}}^\theta(x, \xi) \equiv 0, \quad \text{if } |\nu - \tilde{\nu}| \ge C_0\theta,$$

for some uniform constant C_0. Also, the principal symbol $a_\nu^\theta(x, \xi)$ of A_ν^θ are all supported in a neighborhood of the support of $B(x, \xi)$, and satisfies

$$a_\nu^\theta(\chi_r(x, \xi)) = a_\nu^\theta(x, \xi), \quad \text{on supp } B(x, \xi) \text{ if } |r| \le 2\delta, \tag{2.23}$$

assuming that $\delta > 0$ is small.

Besides, as operators between any $L^p \to L^q$, $1 \le p, q \le \infty$, spaces we have

$$\tilde{\sigma}_\lambda = \sum_\nu \tilde{\sigma}_\lambda A_\nu^\theta + O(\lambda^{-N}) \quad \text{for all } N, \tag{2.24}$$

and the A_ν^θ are almost orthogonal in the sense that we have

$$\sum_\nu \|A_\nu^\theta h\|_{L_x^2}^2 \lesssim \|h\|_{L_x^2}^2, \tag{2.25}$$

with constants independent of $\theta \in [\lambda^{-\varepsilon_0}, 1]$.

Also, since, for each x, the symbols vanish outside of cubes of sidelength $\theta\lambda$ and $|\partial_\xi^\gamma A_\nu^\theta(x,\xi)| = O((\lambda\theta)^{-|\gamma|})$, we also have that their kernels are $O((\theta\lambda)^{n-1}(1+\theta\lambda d_g(x,y))^{-N})$ for all N and so

$$\|A_\nu^\theta\|_{L^p(M)\to L^p(M)} = O(1) \quad \text{for all } 1 \le p \le \infty. \tag{2.26}$$

By interpolation, (2.25) and (2.26) imply

$$\|A_\nu^\theta h\|_{\ell_\nu^p L^p(M)} \lesssim \|h\|_{L^p(M)} \quad \text{for all } 2 \le p \le \infty. \tag{2.27}$$

In view of (2.24), we have, for $\theta_0 = \lambda^{-\varepsilon_0}$,

$$(\tilde\sigma_\lambda H)^2 = \sum_{\nu,\tilde\nu} (\tilde\sigma_\lambda A_\nu^{\theta_0} H)\cdot(\tilde\sigma_\lambda A_{\tilde\nu}^{\theta_0} H) + O(\lambda^{-N}\|H\|_2^2). \tag{2.28}$$

Recall that in $A_\nu^{\theta_0}$, $\nu \in \theta_0\mathbb{Z}^{2(n-2)+1}$ indexes a $\lambda^{-\varepsilon_0}$-separated set in $\mathbb{R}^{2n-3}$. Here, $\varepsilon_0 < 1/2$ is small constant that we shall specify later, the choice of ε_0 depends on the dimension $d = n-1$.

We need to organize the pairs of indices $\nu, \tilde\nu$ in (2.28) as in many earlier works (see [18,23]). To this end, consider dyadic cubes, τ_μ^θ in $\mathbb{R}^{2n-3}$ of sidelength $\theta = 2^k\theta_0$, with τ_μ^θ denoting translations of the cube $[0,\theta)^{2n-3}$ by $\mu \in \theta\mathbb{Z}^{2n-3}$. Two such dyadic cubes of sidelength θ are said to be *close* if they are not adjacent but have adjacent parents of length 2θ, and, in this case, we write $\tau_\mu^\theta \sim \tau_{\tilde\mu}^\theta$. We note that close cubes satisfy $\mathrm{dist}(\tau_\mu^\theta, \tau_{\tilde\mu}^\theta) \approx \theta$, and so each fixed cube has $O(1)$ cubes which are "close" to it. Moreover, as noted in [23, p. 971], any distinct points $\nu, \tilde\nu \in \mathbb{R}^{2n-3}$ must lie in a unique pair of close cubes in this Whitney decomposition. So, there must be a unique triple $(\theta = \theta_0 2^k, \mu, \tilde\mu)$ such that $(\nu,\tilde\nu) \in \tau_\mu^\theta \times \tau_{\tilde\mu}^\theta$ and $\tau_\mu^\theta \sim \tau_{\tilde\mu}^\theta$. We remark that by choosing B to have small support we need only consider $\theta = 2^k\theta_0 \ll 1$.

Taking these observations into account, as in earlier works, we conclude that the bilinear sum (2.28) can be organized as follows:

$$\sum_{\{k\in\mathbb{N}: k\ge 10 \text{ and } \theta=2^k\theta_0\ll 1\}} \sum_{\{(\mu,\tilde\mu):\tau_\mu^\theta\sim\tau_{\tilde\mu}^\theta\}} \sum_{\{(\nu,\tilde\nu)\in\tau_\mu^\theta\times\tau_{\tilde\mu}^\theta\}} (\tilde\sigma_\lambda A_\nu^{\theta_0} H)\cdot(\tilde\sigma_\lambda A_{\tilde\nu}^{\theta_0} H)$$
$$+ \sum_{(\tau,\tilde\tau)\in\Xi_{\theta_0}} (\tilde\sigma_\lambda A_\nu^{\theta_0} H)\cdot(\tilde\sigma_\lambda A_{\tilde\nu}^{\theta_0} H), \tag{2.29}$$

where Ξ_{θ_0} indexes the remaining pairs such that $|\nu-\tilde{\nu}|\lesssim\theta_0=\lambda^{-\varepsilon_0}$, including the diagonal ones where $\nu=\tilde{\nu}$.

The key estimate that we require, which follows from bilinear harmonic analysis arguments, then is the following.

Proposition 2.3. *If $H=S_\lambda f$ is as in (2.2) and (p,q) is as in (2.5), we have*

$$\|\tilde{\sigma}_\lambda H\|_{L_t^pL_x^q(A_-)} \lesssim \|\tilde{\sigma}_\lambda A_\nu^{\theta_0}H\|_{L_t^p\ell_\nu^qL_x^q(M^{n-1}\times[0,T])}+\lambda^{1/p-}\|H\|_{L^2_{t,x}(M^{n-1}\times\mathbb{R})}. \tag{2.30}$$

The notation $\lambda^{1/p-}$ that we are using for the last term in (2.30) denotes $\lambda^{1/p-\varepsilon}$ for some unspecified $\varepsilon>0$. Note that since $\|H\|_{L^2_{t,x}}\approx T^{1/2}$ for $H=S_\lambda f$ and $T\approx\log\lambda$, the log-loss afforded by having the last term involve this norm is more than overset by the power gain $1/p-$ of λ. (2.30) is the place where we require the mixed-norm to be taken over the set A_-. As we shall see later in the proof, the upper bound of $\tilde{\sigma}_\lambda H$ on the set A_- will allow us to fully exploit the gain from bilinear estimates.

We shall postpone the proof of Proposition 2.3 until the next section. Let us now see how we can use it to prove (2.15) on the set A_-. Given (2.30), it suffices to show that when M^{n-1} has nonpositive curvature

$$\|\tilde{\sigma}_\lambda A_\nu^{\theta_0}H\|_{L_t^p\ell_\nu^qL_x^q(M^{n-1}\times[0,T])}\le C\lambda^{1/p}, \tag{2.31}$$

with $T=c_0\log\lambda$ for $c_0>0$ sufficiently small.

We shall also need the following simple lemma whose proof we postpone until the end of this subsection.

Lemma 2.4. *If $\delta>0$ in (2.7) is small enough and $\theta_0=\lambda^{-\varepsilon_0}$, then we have for B as in (2.10),*

$$\|B\sigma_\lambda A_\nu^{\theta_0}-BA_\nu^{\theta_0}\sigma_\lambda\|_{L^2_{t,x}\to L_t^pL_x^q}=O(\lambda^{1/p-1/2+2\varepsilon_0}). \tag{2.32}$$

By (2.32) along with the fact that $\ell^q\subset\ell^2$ if $q\ge 2$, we have

$$\begin{aligned}\|\tilde{\sigma}_\lambda A_\nu^{\theta_0}H\|_{L_t^p\ell_\nu^qL_x^q} &\lesssim \|BA_\nu^{\theta_0}\sigma_\lambda H\|_{L_t^p\ell_\nu^qL_x^q}+\|(B\sigma_\lambda A_\nu^{\theta_0}-BA_\nu^{\theta_0}\sigma_\lambda)H\|_{\ell_\nu^2L_t^pL_x^q}\\ &\lesssim \|BA_\nu^{\theta_0}\sigma_\lambda H\|_{L_t^p\ell_\nu^qL_x^q}+\lambda^{1/p-1/2+2\varepsilon_0}\|H\|_{\ell_\nu^2L^2_{t,x}}.\end{aligned} \tag{2.33}$$

Since the number of choices of ν is $O(\lambda^{(2n-3)\varepsilon_0})$ and H is independent of ν, we have $\|H\|_{\ell_\nu^2L^2_{t,x}}\lesssim\lambda^{(n-3/2)\varepsilon_0}\|H\|_{L^2_{t,x}}$. Thus, if we choose $\varepsilon_0<1/(2n+1)$, the second term on the right side of (2.33) is bounded by $\lambda^{1/p-}\|H\|_{L^2_{t,x}}$.

On the other hand, since we are assuming f is L^2 normalized in (2.16), by (2.11), (2.27), (2.13), and the fact that $H=S_\lambda f$, we have

$$\|BA_\nu^{\theta_0}(I-\sigma_\lambda)H\|_{L_t^p\ell_\nu^qL_x^q(M^{n-1}\times[0,T])}\le\|(I-\sigma_\lambda)H\|_{L_t^pL_x^q(M^{n-1}\times[0,T])}\le\lambda^{1/p}.$$

Thus, by (2.33), we would have (2.31) if

$$\|BA_\nu^{\theta_0}H\|_{L_t^p\ell_\nu^q L_x^q(M^{n-1}\times[0,T])}\lesssim\lambda^{1/p},$$

which, by (2.11) and the fact that $H=S_\lambda f$, is a consequence of

$$\|A_\nu^{\theta_0}S_\lambda f\|_{L_t^p\ell_\nu^q L_x^q(M^{n-1}\times[0,T])}\lesssim\lambda^{1/p}. \tag{2.34}$$

To prove (2.34), let us define

$$f\to(Wf)(x,t,\nu)=\eta\Big(\frac{t}{T}\Big)(A_\nu^{\theta_0}\circ e^{-it\lambda^{-1}\Delta_g}f)(x).$$

By applying the abstract theorem of Keel and Tao [17] and a simple rescaling argument, we would have (2.34) if

$$\|Wf(t,\cdot)\|_{\ell_\nu^2L_x^2}\le C\|f\|_{L_x^2}, \tag{2.35}$$

and

$$\|W(t)W^*(s)G\|_{\ell_\nu^\infty L_x^\infty}\le C\lambda^{(n-1)/2}|t-s|^{-(n-1)/2}\|G\|_{\ell_\nu^1L_x^1}, \tag{2.36}$$

with

$$\begin{aligned}&WW^*G(x,t,\nu)\\&\quad=\eta\Big(\frac{t}{T}\Big)\sum_{\nu'}\int_{-\infty}^{\infty}\eta\Big(\frac{s}{T}\Big)[(A_\nu^{\theta_0}e^{-i(t-s)\lambda^{-1}\Delta_g}(A_{\nu'}^{\theta_0})^*)G(\,\cdot\,,s,\nu')](x)\,ds\\&\quad=\sum_{\nu'}\iint K(x,t,\nu;y,s,\nu')G(y,s,\nu')\,dy\,ds,\end{aligned}$$

where

$$K(x,t,\nu;y,s,\nu')=\eta\Big(\frac{t}{T}\Big)(A_\nu^{\theta_0}e^{-i(t-s)\lambda^{-1}\Delta_g}(A_{\nu'}^{\theta_0})^*)(x,y)\eta\Big(\frac{s}{T}\Big).$$

It is not hard to check that (2.35) follows from (2.26) and the fact that $e^{-it\lambda^{-1}\Delta_g}$ is unitary, and (2.36) is a consequence of the kernel estimates

$$|K(x,t,\nu;y,s,\nu')|\le C\lambda^{(n-1)/2}|t-s|^{-(n-1)/2}, \tag{2.37}$$

which follows from [3, Proposition 4.2].

Here, compared with (2.22), there is no e^{CT} loss in the above kernel estimates due to the presence of the operators $A_\nu^{\theta_0}$. The proof of (2.37) follows from exploiting the directional localization in the operators $A_\nu^{\theta_0}$ through the use of Toponogov's triangle comparison theorem. See [3, Section 4], and also recent works [6, 7] for more details. Also, note that here we take $\theta_0=\lambda^{-\varepsilon_0}$, which may be much larger than $\theta_0=\lambda^{-1/8}$ as in [3]; however, the proof of the microlocalized kernel estimates extends to the larger θ_0 similarly, as long as we fix $T = c_0 \log \lambda$ for some $c_0 \ll \varepsilon_0$.

This completes the proof of (2.15) on the set A_- up to proving the crucial local estimates in Proposition 2.3, which we shall postpone to the next section.

The other task remaining to complete the proofs Theorems 1.1 is to prove the commutator estimate that we employed.

Proof of Lemma 2.4. The proof follows from similar arguments to those in [3, Lemma 2.7]; we include the details here for completeness. Recall that by (2.10), the symbol

$$B(x,\xi) = B_\lambda(x,\xi) \in S^0_{1,0}$$

vanishes when $|\xi|$ is not comparable to λ. In particular, it vanishes if $|\xi|$ is larger than a fixed multiple of λ, and it belongs to a bounded subset of $S^0_{1,0}$. Furthermore, if $a_\nu^{\theta_0}(x,\xi)$ is the principal symbol of our zero-order dyadic microlocal operators, we recall that by (2.23) we have that for $\delta > 0$ small enough

$$a_\nu^{\theta_0}(x,\xi) = a_\nu^{\theta_0}(\chi_r(x,\xi)) \quad \text{on } \operatorname{supp} B_\lambda \text{ if } |r| \le 2\delta,$$

where $\chi_r : T^*M^{n-1}\backslash 0 \to T^*M^{n-1}\backslash 0$ denotes geodesic flow in the cotangent bundle.

By Sobolev estimates for $M^{n-1} \times \mathbb{R}$, in order to prove (2.32), it suffices to show that

$$\begin{aligned} &\|(\sqrt{I + P^2 + D_t^2})^{(n-1)(1/2-1/q)+1/2-1/p}[B_\lambda \sigma_\lambda A_\nu^{\theta_0} - B_\lambda A_\nu^{\theta_0}\sigma_\lambda]\|_{L^2_{t,x}\to L^2_{t,x}} \\ &\quad = O(\lambda^{1/p-1/2+2\varepsilon_0}). \end{aligned} \tag{2.38}$$

Note that for (p,q) as in (2.5),

$$(n-1)\Big(\frac{1}{2}-\frac{1}{q}\Big) + \frac{1}{2} - \frac{1}{p} = \frac{1}{2} + \frac{1}{p},$$

and since the symbol of B_λ and σ_λ are supported in $|\xi|, |\tau| \approx \lambda$, in order to prove (2.38), it suffices to show that

$$\|B_\lambda \sigma_\lambda A_\nu^{\theta_0} - B_\lambda A_\nu^{\theta_0}\sigma_\lambda\|_{L^2_{t,x}\to L^2_{t,x}} = O(\lambda^{-1+2\varepsilon_0}). \tag{2.39}$$

To prove this, we recall that

$$\sigma_\lambda = (2\pi)^{-1}\tilde\beta\Big(\frac{D_t}{\lambda}\Big)\int \hat\sigma(r) e^{ir\lambda^{1/2}|D_t|^{1/2}} e^{-irP}\,dr,$$

and, therefore, since $e^{ir\lambda^{1/2}|D_t|^{1/2}}$ has $L^2 \to L^2$ norm one and commutes with B_λ and $A_\nu^{\theta_0}$, and since $\hat\sigma(r) = 0$, $|r| \ge 2\delta$, by Minkowski's integral inequality, we would have (2.39) if

$$\sup_{|r|\le 2\delta}\Big\|\tilde\beta\Big(\frac{D_t}{\lambda}\Big)[B_\lambda e^{-irP}A_\nu^{\theta_0} - B_\lambda A_\nu^{\theta_0} e^{-irP}]\Big\|_{L^2_{t,x}\to L^2_{t,x}} = O(\lambda^{-1+2\varepsilon_0}). \tag{2.40}$$

Next, to be able to use Egorov's theorem, we write

$$[B_\lambda e^{-irP}A_\nu^{\theta_0} - B_\lambda A_\nu^{\theta_0}e^{-irP}] = B_\lambda[(e^{-irP}A_\nu^{\theta_0}e^{irP}) - B_\lambda A_\nu^{\theta_0}]\circ e^{-irP}.$$

Since e^{-irP} also has L^2-operator norm one, we would obtain (2.40) from

$$\Big\|\tilde\beta\Big(\frac{D_t}{\lambda}\Big)B_\lambda[(e^{-irP}A_\nu^{\theta_0}e^{irP}) - A_\nu^{\theta_0}]\Big\|_{L^2_{t,x}\to L^2_{t,x}} = O(\lambda^{-1+2\varepsilon_0}),$$

which is a simple consequence of the Egorov's theorem, see e.g., Taylor [24, Section VIII.1] and [3, Lemma 2.7] for more details. ∎

3. Proof of Proposition 2.3

Let us fix $p = 2, q = q_e = 2(n-1)/(n-3)$; we shall first give the arguments for $d = n-1 \geq 4$, and later modify it for $n-1 = 3$. The case $d = n-1 = 2$ follows from similar arguments, which we shall briefly discuss at the end of this section.

To prove (2.30), the strategy is similar to [3], which is related to the ideas in Blair and Sogge [7] and earlier works, especially Tao, Vargas, and Vega [23] and Lee [18].

We first note that if δ as in (2.7) is small enough, we have

$$\tilde\sigma_\lambda - \sum_\nu \tilde\sigma_\lambda A_\nu^{\theta_0} = R_\lambda, \quad \text{where } \|R_\lambda H\|_{L^\infty_{t,x}} \lesssim \lambda^{-N}\|H\|_{L^2_{t,x}} \text{ for all } N.$$

Thus, we have

$$(\tilde\sigma_\lambda H)^2 = \sum_{\nu,\tilde\nu}(\tilde\sigma_\lambda A_\nu^{\theta_0}H)\cdot(\tilde\sigma_\lambda A_{\tilde\nu}^{\theta_0}H) + O(\lambda^{-N}\|H\|^2_{L^2_{t,x}}) \quad \text{for all } N. \tag{3.1}$$

As in earlier works, let

$$\Upsilon^{\mathrm{diag}}(H) = \sum_{(\nu,\tilde\nu)\in\Xi_{\theta_0}}(\tilde\sigma_\lambda A_\nu^{\theta_0}H)\cdot(\tilde\sigma_\lambda A_{\tilde\nu}^{\theta_0}H), \tag{3.2}$$

and

$$\Upsilon^{\mathrm{far}}(H) = \sum_{(\nu,\tilde\nu)\notin\Xi_{\theta_0}}(\tilde\sigma_\lambda A_\nu^{\theta_0}H)\cdot(\tilde\sigma_\lambda A_{\tilde\nu}^{\theta_0}H) + O(\lambda^{-N}\|H\|^2_{L^2_{t,x}}), \tag{3.3}$$

with the last term denoting the error term in (3.1). Thus,

$$(\tilde\sigma_\lambda H)^2 = \Upsilon^{\mathrm{diag}}(H) + \Upsilon^{\mathrm{far}}(H). \tag{3.4}$$

Here, the summation in $\Upsilon^{\mathrm{diag}}(H)$ is over near diagonal pairs $(\nu,\tilde\nu)\in\Xi_{\theta_0}$ as in (2.29). In particular, we have $|\nu-\tilde\nu| \leq C\theta_0$ for some uniform constant as $\nu,\tilde\nu$ range over

$\theta_0\mathbb{Z}^{(2n-3)}$. The other term $\Upsilon^{\text{far}}(H)$ is the remaining pairs, which include many which are far from the diagonal. This sum will provide the contribution to the last term in (2.30).

The two types of terms here are treated differently, as in analyzing parabolic restriction problems or spectral projection estimates.

We shall treat the first term in the right of (3.4) as in [2, 7] by using a variable coefficient variant of [23, Lemma 6.1] (see also [7, Lemma 4.2]):

Lemma 3.1. *If $\Upsilon^{\text{diag}}(H)$ is as in (3.4) and $d = n-1 \geq 4$, then we have the uniform bounds*

$$\|\Upsilon^{\text{diag}}(H)\|_{L^1_t L^{q_e/2}_x} \lesssim \|\tilde{\sigma}_\lambda A^{\theta_0}_\nu H\|^2_{L^2_t \ell^{q_e}_\nu L^{q_e}_x} + O(\lambda^{1-}\|H\|^2_{L^2_{t,x}}). \tag{3.5}$$

We also need the following estimate for $\Upsilon^{\text{far}}(H)$ which is a consequence of the bilinear oscillatory integral estimates of Lee [18] and arguments of Blair and Sogge in [4, 5, 7].

Lemma 3.2. *If $\Upsilon^{\text{far}}(H)$ is as in (3.3), and, as above, $\theta_0 = \lambda^{-\varepsilon_0}$, then, for all $\varepsilon > 0$, we have, for $H = S_\lambda f$,*

$$\iint |\Upsilon^{\text{far}}(H)|^{q/2}\,dx\,dt$$
$$\lesssim_\varepsilon \lambda^{1+\varepsilon}(\lambda^{1-\varepsilon_0})^{((n-1)/2)(q-2(n+1)/(n-1))}\|H\|^q_{L^2_{t,x}}, \quad \text{if } q = \frac{2(n+2)}{n}. \tag{3.6}$$

Let us postpone the proofs of these two lemmas for a bit and show how they can be used to obtain Proposition 2.3 if $d = n-1 \geq 4$.

If we let $q = 2(n+2)/n$ as in Lemma 3.2, we note that $q < q_e$ and also

$$|\tilde{\sigma}_\lambda H \cdot \tilde{\sigma}_\lambda H|^{q_e} \leq 2^{q/2}|\tilde{\sigma}_\lambda H \cdot \tilde{\sigma}_\lambda H|^{(q_e-q)/2} \cdot (|\Upsilon^{\text{diag}}(H)|^{q/2} + |\Upsilon^{\text{far}}(H)|^{q/2}).$$

Thus,

$$\begin{aligned}
\|\tilde{\sigma}_\lambda H\|^2_{L^2_t L^{q_e}_x(A_-)} &= \int \Big(\int |\tilde{\sigma}_\lambda H \cdot \tilde{\sigma}_\lambda H|^{q_e/2} dx\Big)^{2/q_e} dt \\
&\lesssim \int \Big(\int |\tilde{\sigma}_\lambda H \cdot \tilde{\sigma}_\lambda H|^{(q_e-q)/2} |\Upsilon^{\text{diag}}(H)|^{q/2} dx\Big)^{2/q_e} dt \\
&\quad + \int \Big(\int |\tilde{\sigma}_\lambda H \cdot \tilde{\sigma}_\lambda H|^{(q_e-q)/2} |\Upsilon^{\text{far}}(H)|^{q/2} dx\Big)^{2/q_e} dt \\
&= \mathrm{I} + \mathrm{II}.
\end{aligned} \tag{3.7}$$

To estimate II, first note that $\tilde{\sigma}_\lambda H = \widetilde{S}_\lambda f$ if $H = S_\lambda f$; we have

$$\begin{aligned}
\mathrm{II} &\lesssim \|\widetilde{S}_\lambda f\|^{2((q_e-q))/q_e}_{L^\infty(A_-)} \cdot \|\Upsilon^{\text{far}}(H)\|^{q/q_e}_{L^{q/q_e}_t L^{q/2}_x} \\
&\lesssim T^{(q_e/q-2/q)\cdot q/q_e}\|\widetilde{S}_\lambda f\|^{2(q_e-q)/q_e}_{L^\infty(A_-)} \cdot \|\Upsilon^{\text{far}}(H)\|^{q/q_e}_{L^{q/2}_t L^{q/2}_x},
\end{aligned}$$

where in the second line we used Hölder's inequality on the time interval $[0, T]$. By (3.6) and the ceiling for A_-, we have

$$\begin{aligned} \mathrm{II} \le\, & T^{(q_e/q-2/q)\cdot q/q_e}\lambda^{((n-1)/4+\varepsilon_1)(2(q_e-q)/q_e)} \\ & \cdot (\lambda^{1+\varepsilon}(\lambda^{1-\varepsilon_0})^{((n-1)/2)(q-2(n+1)/(n-1))})^{2/q_e}\|H\|_{L^2_x}^{2q/q_e}. \end{aligned} \tag{3.8}$$

If we take $\varepsilon_0, \varepsilon_1$ and ε to be small enough, e.g., $\varepsilon = \varepsilon_1 = 1/100$ and $\varepsilon_0 = 1/(2n+2)$, it is straightforward to check that

$$\mathrm{II} = O(\lambda^{1-}\|H\|_{L^2_{t,x}}^{2q/q_e}) = O(\lambda^{1-}\|H\|_{L^2_{t,x}}^2).$$

Here we also used the fact that $\|H\|_{L^2_{t,x}}^2$ dominates $\|H\|_{L^2_{t,x}}^{2q/q_e}$, since $q_e > q$ and $\|H\|_{L^2_{t,x}} \approx T$ since $H = S_\lambda f$, $\|f\|_2 = 1$ and $e^{-it\lambda^{-1}\Delta_g}$ is a unitary operator on L^2_x.

Consequently, we just need to see that $\mathrm{I}^{1/2}$ is dominated by the other term on the right-hand side of this inequality. To estimate this term, we use Hölder's inequality followed by Young's inequality and Lemma 3.1 to see that

$$\begin{aligned} \mathrm{I} =& \int\left(\int |\tilde{\sigma}_\lambda H\cdot\tilde{\sigma}_\lambda H|^{(q_e-q)/2}|\Upsilon^{\mathrm{diag}}(H)|^{q/2}dx\right)^{2/q_e}dt \\ \le& \int(\|\tilde{\sigma}_\lambda H\cdot\tilde{\sigma}_\lambda H\|_{L_x^{q_e/2}}^{(q_e-q)/2}\|\Upsilon^{\mathrm{diag}}(H)\|_{L_x^{q_e/2}}^{q/2})^{2/q_e}dt \\ \le& \|\tilde{\sigma}_\lambda H\cdot\tilde{\sigma}_\lambda H\|_{L_t^1L_x^{q_e/2}}^{(q_e-q)/q_e}\|\Upsilon^{\mathrm{diag}}(H)\|_{L_t^1L_x^{q_e/2}}^{q/q_e} \\ \le& \frac{q_e-q}{q_e}\|\tilde{\sigma}_\lambda H\|_{L_t^2L_x^{q_e}}^2 + \frac{q}{q_e}\|\Upsilon^{\mathrm{diag}}(H)\|_{L_t^1L_x^{q_e/2}} \\ \le& \frac{q_e-q}{q_e}\|\tilde{\sigma}_\lambda H\|_{L_t^2L_x^{q_e}}^2 + C(\|\tilde{\sigma}_\lambda A_\nu^{\theta_0}H\|_{L_t^2\ell_\nu^{q_e}L_x^{q_e}}^2 + \lambda^{1-}\|H\|_{L^2_{t,x}}^2). \end{aligned}$$

Since $(q_e - q)/q_e < 1$, the first term on the right can be absorbed into the left side of (3.7), and this, along with the estimate for II above, yields (2.30).

Thus, if we can prove Lemma 3.1 and Lemma 3.2, the proof of Proposition 2.3 for $d = n - 1 \ge 4$ will be complete.

Proof of Lemma 3.1. To prove (3.5), let us define slightly wider microlocal cutoffs by setting

$$\widetilde{A}_\nu^{\theta_0} = \sum_{|\mu-\nu|\le C_0\theta_0} A_\mu^{\theta_0}.$$

We can fix C_0 large enough so that

$$\|A_\nu^{\theta_0} - A_\nu^{\theta_0}\widetilde{A}_\nu^{\theta_0}\|_{L_x^p\to L_x^p} = O(\lambda^{-N}) \quad \text{for all } N, \text{ if } 1 \le p \le \infty. \tag{3.9}$$

Also, like the original operators $A_\nu^{\theta_0}$, the operators $\widetilde{A}_\nu^{\theta_0}$ are almost orthogonal,

$$\sum_\nu \|\widetilde{A}_\nu^{\theta_0} h\|_{L_x^2}^2 \lesssim \|h\|_{L_x^2}^2. \tag{3.10}$$

Since by (2.11) and (2.14), we have

$$\|\tilde{\sigma}_\lambda F\|_{L_t^2 L_x^{q_e}(M^{n-1}\times[0,T])} \le C\lambda^{1/2}\|F\|_{L_{t,x}^2},$$

we conclude that, in order to prove (3.5), we may replace $\Upsilon^{\text{diag}}(H)$ by $\widetilde{\Upsilon}^{\text{diag}}(H)$ where the latter is defined by the analog of (3.2) with $A_\nu^{\theta_0}$ and $A_{\tilde{\nu}}^{\theta_0}$ replaced by $A_\nu^{\theta_0}\widetilde{A}_\nu^{\theta_0}$ and $A_{\tilde{\nu}}^{\theta_0}\widetilde{A}_{\tilde{\nu}}^{\theta_0}$, respectively.

So, it suffices to prove

$$\begin{aligned}\Big\|\sum_{(\nu,\tilde{\nu})\in\Xi_{\theta_0}}(\tilde{\sigma}_\lambda A_\nu^{\theta_0}\widetilde{A}_\nu^{\theta_0}H)\cdot(\tilde{\sigma}_\lambda A_{\tilde{\nu}}^{\theta_0}\widetilde{A}_{\tilde{\nu}}^{\theta_0}H)\Big\|_{L_t^1 L_x^{q_e/2}}\\ \lesssim \|\tilde{\sigma}_\lambda A_\nu^{\theta_0}H\|_{L_t^2\ell_\nu^{q_e}L_x^{q_e}}^2 + O(\lambda^{1-}\|H\|_{L_{t,x}^2}^2).\end{aligned} \tag{3.11}$$

We shall need the following variant of (2.32):

$$\|[\tilde{\sigma}_\lambda A_\nu^{\theta_0} - A_\nu^{\theta_0}\tilde{\sigma}_\lambda]F\|_{L_t^2 L_x^{q_e}} \lesssim \lambda^{2\varepsilon_0}\|F\|_{L_{t,x}^2}. \tag{3.12}$$

This follows from the proof of Lemma 2.4, (2.32), and the fact that the commutator $[B, A_\nu^{\theta_0}]$ is bounded on $L_x^{q_e}(M^{n-1})$ with norm $O(\lambda^{-1+\varepsilon_0})$. By (3.10) and (3.12), we would have (3.11) if we could show that

$$\begin{aligned}\Big\|\sum_{(\nu,\tilde{\nu})\in\Xi_{\theta_0}}(A_\nu^{\theta_0}\tilde{\sigma}_\lambda\widetilde{A}_\nu^{\theta_0}H)\cdot(A_{\tilde{\nu}}^{\theta_0}\tilde{\sigma}_\lambda\widetilde{A}_{\tilde{\nu}}^{\theta_0}H)\Big\|_{L_t^1 L_x^{q_e/2}}\\ \lesssim \|\tilde{\sigma}_\lambda A_\nu^{\theta_0}H\|_{L_t^2\ell_\nu^{q_e}L_x^{q_e}}^2 + O(\lambda^{1-}\|H\|_{L_{t,x}^2}^2).\end{aligned} \tag{3.13}$$

Since the operators $A_\nu^{\theta_0}$ are time independent, we claim that it suffices to show that for arbitrary $h_\nu, h_{\tilde{\nu}}$, which may depend on ν and $\tilde{\nu}$,

$$\begin{aligned}&\Big\|\sum_{(\nu,\tilde{\nu})\in\Xi_{\theta_0}} A_\nu^{\theta_0}h_\nu\cdot A_{\tilde{\nu}}^{\theta_0}h_{\tilde{\nu}}\Big\|_{L_x^{q_e/2}}\\ &\quad\lesssim\Big(\sum_{(\nu,\tilde{\nu})\in\Xi_{\theta_0}}\|A_\nu^{\theta_0}h_\nu\cdot A_{\tilde{\nu}}^{\theta_0}h_{\tilde{\nu}}\|_{L_x^{q_e/2}}^{q_e/2}\Big)^{2/q_e}\\ &\quad\quad + O\Big(\lambda^{-N}\sum_{(\nu,\tilde{\nu})\in\Xi_{\theta_0}}\|h_\nu\|_{L_x^1}\|h_{\tilde{\nu}}\|_{L_x^1}\Big),\quad \text{for all } N.\end{aligned} \tag{3.14}$$

To verify the claim, note that if we take $h_\nu = \tilde\sigma_\lambda \widetilde{A}_\nu^{\theta_0} H$ and $h_{\tilde\nu} = \tilde\sigma_\lambda \widetilde{A}_{\tilde\nu}^{\theta_0} H$, (3.14) implies

$$\begin{aligned}
&\Big\| \sum_{(\nu,\tilde\nu)\in\Xi_{\theta_0}} (A_\nu^{\theta_0}\tilde\sigma_\lambda \widetilde{A}_\nu^{\theta_0} H)\cdot(A_{\tilde\nu}^{\theta_0}\tilde\sigma_\lambda \widetilde{A}_{\tilde\nu}^{\theta_0} H)\Big\|_{L_t^1 L_x^{q_e/2}} \\
&\quad\lesssim \int\Big(\sum_{(\nu,\tilde\nu)\in\Xi_{\theta_0}} \|(A_\nu^{\theta_0}\tilde\sigma_\lambda \widetilde{A}_\nu^{\theta_0} H)\cdot(A_{\tilde\nu}^{\theta_0}\tilde\sigma_\lambda \widetilde{A}_{\tilde\nu}^{\theta_0} H)\|_{L_x^{q_e/2}}^{q_e/2}\Big)^{2/q_e} dt \\
&\qquad\qquad + \lambda^{-N}\int \sum_{(\nu,\tilde\nu)\in\Xi_{\theta_0}} \|(A_\nu^{\theta_0}\tilde\sigma_\lambda \widetilde{A}_\nu^{\theta_0} H)\|_{L_x^1}\|(A_{\tilde\nu}^{\theta_0}\tilde\sigma_\lambda \widetilde{A}_{\tilde\nu}^{\theta_0} H)\|_{L_x^1}\,dt. \\
&\quad\lesssim \|A_\nu^{\theta_0}\tilde\sigma_\lambda \widetilde{A}_{\tilde\nu}^{\theta_0} H\|^2_{L_t^2\ell_\nu^{q_e}L_x^{q_e}} + O(\lambda^{1-}\|H\|^2_{L^2_{t,x}}).
\end{aligned}\tag{3.15}$$

Here, we used the fact that for fixed ν, the number of choices of $\tilde\nu$ is finite, and, for each pair $(\nu,\tilde\nu)\in\Xi_{\theta_0}$,

$$\begin{aligned}
&\int \|(A_\nu^{\theta_0}\tilde\sigma_\lambda \widetilde{A}_\nu^{\theta_0} H)\|_{L_x^1}\|(A_{\tilde\nu}^{\theta_0}\tilde\sigma_\lambda \widetilde{A}_{\tilde\nu}^{\theta_0} H)\|_{L_x^1}\,dt \\
&\quad\le \int \|(A_\nu^{\theta_0}\tilde\sigma_\lambda \widetilde{A}_\nu^{\theta_0} H)\|_{L_x^2}\|(A_{\tilde\nu}^{\theta_0}\tilde\sigma_\lambda \widetilde{A}_{\tilde\nu}^{\theta_0} H)\|_{L_x^2}\,dt = O(\|H\|^2_{L^2_{t,x}}).
\end{aligned}$$

If we repeat earlier arguments and use (3.9) again, we conclude that the first term on the right side of (3.15) is dominated by the right side of (3.13).

Thus, it remains to prove (3.14). By duality, if we take $r=(q_e/2)'$ so that r is the conjugate exponent for $q_e/2$, (3.14) is equivalent to

$$\begin{aligned}
&\Big| \sum_{(\nu,\tilde\nu)\in\Xi_{\theta_0}} \iint A_\nu^{\theta_0}h_\nu\cdot A_{\tilde\nu}^{\theta_0}h_{\tilde\nu}\cdot G\,dx\Big| \\
&\quad\lesssim \Big(\sum_{(\nu,\tilde\nu)\in\Xi_{\theta_0}} \|A_\nu^{\theta_0}h_\nu\cdot A_{\tilde\nu}^{\theta_0}h_{\tilde\nu}\|_{L_x^{q_e/2}}^{q_e/2}\Big)^{2/q_e} \\
&\qquad + O\Big(\lambda^{-N}\sum_{(\nu,\tilde\nu)\in\Xi_{\theta_0}} \|h_\nu\|_{L_x^1}\|h_{\tilde\nu}\|_{L_x^1}\Big) \quad \text{if } \|G\|_{L_x^r}=1.
\end{aligned}\tag{3.16}$$

To prove (3.16), note that if x and ν are fixed and $\xi\to A_\nu^{\theta_0}(x,\xi)$ does not vanish identically, then this function of ξ is supported in a cube $Q_\nu^{\theta_0}(x)\subset\mathbb{R}_\xi^{n-1}$ of sidelength $\approx\lambda^{1-\varepsilon_0}$. The cubes can be chosen so that, if $\eta_\nu(x)$ is its center, then $\partial_x^\gamma\eta_\nu(x)=O(\lambda)$ for all multi-indices γ. Keeping this in mind, it is straightforward to construct for every pair $(\nu,\tilde\nu)\in\Xi_{\theta_0}$ symbols $b_{\nu,\tilde\nu}(x,\xi)$ belonging to a bounded subset of $S^0_{1-\varepsilon_0,\varepsilon_0}$ satisfying

$$b_{\nu,\tilde\nu}(x,\eta)=1 \quad \text{if } \operatorname{dist}(\eta,\ \operatorname{supp}_\xi A_\nu^{\theta_0}(x,\xi)+\operatorname{supp}_\xi A_{\tilde\nu}^{\theta_0}(x,\xi))\le\lambda^{1-\varepsilon_0}, \tag{3.17}$$

with "+" denoting the algebraic sum. Using this and a simple integration-by-parts argument shows that, for every pair $(\nu, \tilde{\nu}) \in \Xi_{\theta_0}$,

$$\|(\mathrm{I} - b_{\nu,\tilde{\nu}}(x, D))[A_\nu^{\theta_0} h_\nu \cdot A_{\tilde{\nu}}^{\theta_0} h_{\tilde{\nu}}]\|_{L_x^\infty} \le C_N \lambda^{-N} \|h_\nu\|_{L_x^1} \|h_{\tilde{\nu}}\|_{L_x^1}, \quad \text{for all } N. \tag{3.18}$$

The symbols can also be chosen so that $b_{\nu_1,\tilde{\nu}_1}(x, \xi)$ and $b_{\nu_2,\tilde{\nu}_2}(x, \xi)$ have disjoint supports if $(\nu_j, \tilde{\nu}_j) \in \Xi_{\theta_0}$, $j = 1, 2$ and

$$\min(|(\nu_1 - \nu_2, \tilde{\nu}_1 - \tilde{\nu}_2)|, |(\nu_1 - \tilde{\nu}_2, \tilde{\nu}_1 - \nu_2)|) \ge C_2 \theta_0$$

with C_2 being a fixed constant independent of λ since all pairs in Ξ_{θ_0} are nearly diagonal. Due to this, the adjoints $b^*_{\nu,\tilde{\nu}}(x, D)$ are almost orthogonal in the sense that we have the uniform bounds

$$\sum_{(\nu,\tilde{\nu}) \in \Xi_{\theta_0}} \|b^*_{\nu,\tilde{\nu}}(x, D) h\|^2_{L_x^2} \lesssim \|h\|^2_{L_x^2}. \tag{3.19}$$

Since $\mathrm{supp}_\xi\, A_\nu^{\theta_0}(x, \xi) + \mathrm{supp}_\xi\, A_{\tilde{\nu}}^{\theta_0}(x, \xi)$ is contained in a cube of sidelength $\approx \lambda^{1-\varepsilon_0}$ and can be chosen to have center $\eta_{\nu,\tilde{\nu}}(x)$ satisfying $\partial_x^\gamma \eta_{\nu,\tilde{\nu}}(x) = O(\lambda)$, we can furthermore assume that we have the uniform bounds

$$\sup_{(\nu,\tilde{\nu}) \in \Xi_{\theta_0}} \|b^*_{\nu,\tilde{\nu}}(x, D) h\|_{L_x^\infty} \lesssim \|h\|_{L_x^\infty}. \tag{3.20}$$

We have now set up our variable coefficient version of the simple argument in [23] that will allow us to obtain (3.16). First, by (3.18), it suffices to estimate the left side of (3.16) with G replaced by $b^*_{\nu,\tilde{\nu}}(x, D)G$. By Hölder's inequality,

$$\begin{aligned} &\Big| \sum_{(\nu,\tilde{\nu}) \in \Xi_{\theta_0}} \iint A_\nu^{\theta_0} h_\nu \cdot A_{\tilde{\nu}}^{\theta_0} h_{\tilde{\nu}} \cdot (b^*_{\nu,\tilde{\nu}}(x, D)G)\, dx \Big| \\ &\quad \le \Big(\sum_{(\nu,\tilde{\nu}) \in \Xi_{\theta_0}} \|A_\nu^{\theta_0} h_\nu \cdot A_{\tilde{\nu}}^{\theta_0} h_{\tilde{\nu}}\|^{q_e/2}_{L_x^{q_e/2}} \Big)^{2/q_e} \cdot \Big(\sum_{(\nu,\tilde{\nu}) \in \Xi_{\theta_0}} \|b^*_{\nu,\tilde{\nu}}(x, D)G\|^r_{L_x^r} \Big)^{1/r}. \end{aligned}$$

Note that when $n - 1 \ge 4$, $q_e \in (2, 4]$, thus $r \in [2, \infty)$. So, if we use (3.19), (3.20), and an interpolation argument, we conclude that

$$\Big(\sum_{(\nu,\tilde{\nu}) \in \Xi_{\theta_0}} \|b^*_{\nu,\tilde{\nu}}(x, D)G\|^r_{L_x^r} \Big)^{1/r} = O(1), \tag{3.21}$$

for G as in (3.16). ∎

Proof of Lemma 3.2. To prove this, let $\alpha \in C_0^\infty((-1,1))$ with $\alpha \equiv 1$ in $(-1/2, 1/2)$; then if $\alpha_m(t) = \alpha(t-m)$, up to a $\log\lambda$ loss, it suffices to show that for $m = 1, 2, \dots, T$

$$\iint |\alpha_m(t)\Upsilon^{\mathrm{far}}(H)|^{q/2} dx\, dt$$
$$\lesssim_\varepsilon \lambda^{1+\varepsilon}(\lambda^{1-\varepsilon_0})^{((n-1)/2)(q-2(n+1)/(n-1))} \|H\|^q_{L^2_{t,x}}. \tag{3.22}$$

This follow from [3, Lemma 3.2], where we replace $\lambda^{7/8}$ there by $\lambda^{1-\varepsilon_0}$. The proof uses bilinear oscillatory integral estimates of Lee [18] and arguments of previous work of Blair and Sogge in [4, 5, 7]. And this is also where we used the condition on δ and δ_0 in (2.7), in order to apply the bilinear oscillatory integral theorems of Lee [18], see [3, Section 3] for more details.

The constants in the right side of (3.22) is better than applying classical linear estimate. Actually, one can also rewrite the right side as

$$\lambda^{(n-1)/2(q/2-1)}\lambda^{\varepsilon}(\lambda^{1/2-\varepsilon_0})^{((n-1)/2)(q-2(n+1)/n-1)}\|H\|^q_{L^2_{t,x}};$$

thus if $\varepsilon_0 \le 1/2$, we have a gain compared with the bound $\lambda^{(n-1)/2(q/2-1)}$, which follows from applying linear estimates. ∎

Modified arguments for $n-1=3$. When $n-1=3$, note that

$$q_e = \frac{2(n-1)}{n-3} = 6,$$

thus

$$r = \left(\frac{q_e}{2}\right)' = \frac{3}{2},$$

which means that we do not have (3.21) and thus Lemma 3.1 in this case, so we need to modify the arguments above. If we repeat the previous arguments, it suffices to estimate the first term

$$\mathrm{I} = \int\left(\int |\tilde\sigma_\lambda H \cdot \tilde\sigma_\lambda H|^{(q_e-q)/2}|\Upsilon^{\mathrm{diag}}(H)|^{q/2}dx\right)^{2/q_e} dt,$$

where, as in (3.2),

$$\Upsilon^{\mathrm{diag}}(H) = \sum_{(\nu,\tilde\nu)\in\Xi_{\theta_0}} (\tilde\sigma_\lambda A_\nu^{\theta_0}H)\cdot(\tilde\sigma_\lambda A_{\tilde\nu}^{\theta_0}H).$$

For later use, note that when $n-1=3$, it is not hard to check that $(\nu,\tilde\nu)\in\Xi_{\theta_0}$ implies $|\nu-\tilde\nu| \le 2^{13}\theta_0$.

Let us define

$$T_\nu H = \sum_{\tilde\nu:(\nu,\tilde\nu)\in\Xi_{\theta_0}} (\tilde\sigma_\lambda A_\nu^{\theta_0}H)(\tilde\sigma_\lambda A_{\tilde\nu}^{\theta_0}H),$$

and write

$$(\Upsilon^{\text{diag}}(H))^2 = \Big(\sum_\nu T_\nu H\Big)^2 = \sum_{\nu_1,\nu_2} T_{\nu_1} H T_{\nu_2} H. \tag{3.23}$$

Now, we can employ another Whitney-type decomposition, more explicitly, the sum in (3.23) can be organized as

$$\begin{aligned}&\Big(\sum_{\{k\in\mathbb{N}:k\geq 20 \text{ and } \theta=2^k\theta_0\ll 1\}} \sum_{\{(\mu_1,\mu_2):\tau^\theta_{\mu_1}\sim\tau^\theta_{\mu_2}\}} \sum_{\{(\nu_1,\nu_2)\in\tau^\theta_{\mu_1}\times\tau^\theta_{\mu_2}\}} + \sum_{(\nu_1,\nu_2)\in\overline{\Xi}_{\theta_0}}\Big) T_{\nu_1} h T_{\nu_2} h \\ &\quad = \overline{\Upsilon}^{\text{far}}(H) + \overline{\Upsilon}^{\text{diag}}(H),\end{aligned} \tag{3.24}$$

where $\tau^\theta_{\mu_1} \sim \tau^\theta_{\mu_2}$ means they are adjacent cubes of distance $\approx \theta$ and $\overline{\Xi}_{\theta_0}$ denotes the remaining pairs not included in the first sum. Here, the *diagonal* set $\overline{\Xi}_{\theta_0}$ is much larger than Ξ_{θ_0}, and it is not hard to check that $(\nu_1,\nu_2)\in\overline{\Xi}_{\theta_0}$ implies $|\nu_1-\nu_2|\leq 2^{23}\theta_0$.

By (3.24), we have

$$\begin{aligned}\mathrm{I} &= \int\Big(\int |\tilde\sigma_\lambda H\cdot\tilde\sigma_\lambda H|^{(q_e-q)/2}|\Upsilon^{\text{diag}}(H)|^{q/2}dx\Big)^{2/q_e}dt \\ &\lesssim \int\Big(\int |\tilde\sigma_\lambda H\cdot\tilde\sigma_\lambda H|^{(q_e-q)/2}|\overline{\Upsilon}^{\text{diag}}(H)|^{q/4}dx\Big)^{2/q_e}dt \\ &\quad + \int\Big(\int |\tilde\sigma_\lambda H\cdot\tilde\sigma_\lambda H|^{(q_e-q)/2}|\overline{\Upsilon}^{\text{far}}(H)|^{q/4}dx\Big)^{2/q_e}dt \\ &= A + B.\end{aligned}$$

For the diagonal term A, note that when $n-1=3$, $q_e/4\in[1,2]$, if we repeat the proof of Lemma 3.1, it is not hard to show the following analog of (3.5):

$$\|\overline{\Upsilon}^{\text{diag}}(H)\|^{1/2}_{L_t^{1/2}L_x^{q_e/4}} \lesssim \|\tilde\sigma_\lambda A_\nu^{\theta_0} H\|^2_{L_t^2\ell_\nu^{q_e}L_x^{q_e}} + O(\lambda^{1-}\|H\|^2_{L^2_{t,x}}). \tag{3.25}$$

To prove this, we just need to define the auxiliary operator $b_{\nu_1,\nu_2}(x,D)$ in (3.17) such that its frequency support is essentially the algebraic sum of the frequency support of four nearby operators $A_\nu^{\theta_0}(x,D)$, and the remaining arguments can be carried over in the same way.

By (3.25), if we use Hölder's inequality followed by Young's inequality,

$$\begin{aligned}A &= \int\Big(\int |\tilde\sigma_\lambda H\cdot\tilde\sigma_\lambda H|^{(q_e-q)/2}|\overline{\Upsilon}^{\text{diag}}(H)|^{q/4}dx\Big)^{2/q_e}dt \\ &\leq \int\big(\|\tilde\sigma_\lambda H\cdot\tilde\sigma_\lambda H\|^{(q_e-q)/2}_{L_x^{q_e/2}}\|\overline{\Upsilon}^{\text{diag}}(H)\|^{q/4}_{L_x^{q_e/4}}\big)^{2/q_e}dt \\ &\leq \|\tilde\sigma_\lambda H\cdot\tilde\sigma_\lambda H\|^{(q_e-q)/q_e}_{L_t^1L_x^{q_e/2}}\|\overline{\Upsilon}^{\text{diag}}(H)\|^{q/2q_e}_{L_t^{1/2}L_x^{q_e/4}}\end{aligned}$$

$$\le \frac{q_e - q}{q_e} \|\tilde\sigma_\lambda H\|^2_{L^2_t L^{q_e}_x} + \frac{q}{q_e} \|\bar\Upsilon^{\mathrm{diag}}(H)\|^{1/2}_{L^{1/2}_t L^{q_e/4}_x}$$
$$\le \frac{q_e - q}{q_e} \|\tilde\sigma_\lambda H\|^2_{L^2_t L^{q_e}_x} + C(\|\tilde\sigma_\lambda A^{\theta_0}_\nu H\|^2_{L^2_t \ell^{q_e}_\nu L^{q_e}_x} + \lambda^{1-}\|H\|^2_{L^2_{t,x}}).$$

which is what we desired, since the first term in the right can be absorbed to the left side of (3.7).

To control the off-diagonal term B, first we note that by the Cauchy–Schwarz inequality,

$$|T_{\nu_1} h T_{\nu_2} H| \le C\Big(\sum_{\tilde\nu_1 : |\tilde\nu_1 - \nu_1| \le 2^{13}\theta_0} |\tilde\sigma_\lambda A^{\theta_0}_{\nu_1} H|^2\Big)\Big(\sum_{\tilde\nu_2 : |\tilde\nu_2 - \nu_2| \le 2^{13}\theta_0} |\tilde\sigma_\lambda A^{\theta_0}_{\nu_2} H|^2\Big). \tag{3.26}$$

Since for fixed ν_1, ν_2, the number of choices of $\tilde\nu_1, \tilde\nu_2$ is finite, (3.26) implies that

$$\sum_{\{(\mu_1,\mu_2):\tau^\theta_{\mu_1} \sim \tau^\theta_{\mu_2}\}} \sum_{\{(\nu_1,\nu_2)\in \tau^\theta_{\mu_1} \times \tau^\theta_{\mu_2}\}} |T_{\nu_1} H T_{\nu_2} H|$$
$$\lesssim \sum_{\{(\mu_1,\mu_2):\tau^\theta_{\mu_1} \sim \tau^\theta_{\mu_2}\}} \sum_{\{(\nu_1,\nu_2)\in \tilde\tau^\theta_{\mu_1} \times \tilde\tau^\theta_{\mu_2}\}} |A^{\theta_0}_{\nu_1} \tilde\sigma_\lambda H|^2 \cdot |A^{\theta_0}_{\nu_2} \tilde\sigma_\lambda H|^2.$$

Here, $\tilde\tau^\theta_{\mu_1}$ and $\tilde\tau^\theta_{\mu_2}$ are the cubes with the same centers but $11/10$ times the side length of $\tau^\theta_{\mu_1}$ and $\tau^\theta_{\mu_2}$, respectively; we used the fact that the side length of $\tau^\theta_{\mu_1}$ is $\ge 2^{20}\theta_0$, so $0.1 *$ side length $\gg 2^{13}\theta_0$.

Furthermore, we have for a given fixed $c_0 = 2^{-m_0}$, $m_0 \in \mathbb{N}$, and pair of dyadic cubes $\tau^\theta_{\mu_1}, \tau^\theta_{\mu_2}$ with $\tau^\theta_{\mu_1} \sim \tau^\theta_{\mu_2}$ and $\theta = 2^k\theta_0$,

$$\sum_{(\nu_1,\nu_2)\in \tilde\tau^\theta_{\mu_1} \times \tilde\tau^\theta_{\mu_2}} |\tilde\sigma_\lambda A^{\theta_0}_{\nu_1} H|^2 \cdot |\tilde\sigma_\lambda A^{\theta_0}_{\nu_2} H|^2$$
$$= \sum_{(\nu_1,\nu_2)\in \tilde\tau^\theta_{\mu_1} \times \tilde\tau^\theta_{\mu_2}} \sum_{\substack{\tau^{c_0\theta}_{\mu_1'} \cap \bar\tau^\theta_{\mu_1} \neq \emptyset \\ \tau^{c_0\theta}_{\mu_2'} \cap \bar\tau^\theta_{\mu_2} \neq \emptyset}} |\tilde\sigma_\lambda A^{c_0\theta}_{\mu_1'} A^{\theta_0}_{\nu_1} H|^2 \cdot |\tilde\sigma_\lambda A^{c_0\theta}_{\mu_2'} A^{\theta_0}_{\nu_2} H|^2 + O(\lambda^{-N}\|H\|_2^4), \tag{3.27}$$

if $\bar\tau^\theta_{\mu_1}$ and $\bar\tau^\theta_{\mu_2}$ are the cubes with the same centers but $12/10$ times the side length of $\tau^\theta_{\mu_1}$ and $\tau^\theta_{\mu_2}$, respectively, so that we have $\mathrm{dist}(\bar\tau^\theta_{\mu_1}, \bar\tau^\theta_{\mu_2}) \ge \theta/2$ when $\tau^\theta_{\mu_1} \sim \tau^\theta_{\mu_2}$. This follows from the fact that for c_0 small enough the product of the symbol of $A^{c_0\theta}_{\mu_1'}$ and $A^{\theta_0}_{\nu_1}$ vanishes identically if $\tau^{c_0\theta}_{\mu_1'} \cap \bar\tau^\theta_{\mu_1} = \emptyset$ and $\nu_1 \in \tilde\tau^\theta_{\mu_1}$, since $\theta = 2^k\theta_0$ with $k \ge 20$. Also, notice that we then have for a fixed $c_0 = 2^{-m_0}$ which is sufficiently small,

$$\mathrm{dist}(\tau^{c_0\theta}_{\mu_1'}, \tau^{c_0\theta}_{\mu_2'}) \in [4^{-1}\theta, 4^2\theta], \quad \text{if } \tau^{c_0\theta}_{\mu_1'} \cap \bar\tau^\theta_{\mu_1} \neq \emptyset \text{ and } \tau^{c_0\theta}_{\mu_2'} \cap \bar\tau^\theta_{\mu_2} \neq \emptyset.$$

Also, for each μ_1, there are $O(1)$ terms μ_1' with $\tau^{c_0\theta}_{\mu_1'} \cap \bar\tau^\theta_{\mu_1} \neq \emptyset$, if c_0 is fixed.

If the cubes $\tau^{c_0\theta}_{\mu_1'}$, $\tau^{c_0\theta}_{\mu_2'}$ satisfy the conditions above, we have the following bilinear-type estimate which is a consequence of [3, Proposition 3.3]:

$$\begin{aligned}\iint &|\tilde\sigma_\lambda A^{c_0\theta}_{\mu_1} H_1 \cdot \tilde\sigma_\lambda A^{c_0\theta}_{\mu_2} H_2|^{q/2}\, dx\, dt \\ &\lesssim \lambda^{1+\varepsilon}(2^k\theta_0\lambda)^{((n-1)/2)(q-2(n+1)/(n-1))}\|H_1\|^{q/2}_{L^2_{t,x}}\|H_2\|^{q/2}_{L^2_{t,x}}.\end{aligned} \tag{3.28}$$

The proof of (3.28) is obtained by using the Hadamard parametrix to rewrite the operators $\tilde\sigma_\lambda A^{c_0\theta}_\nu$, $\nu = \mu_1, \mu_2$ as oscillatory integral operators with explicit kernels and then applying Lee's oscillatory integral estimate [18]. The assumption that c_0 is small enough is crucial to ensure the separation conditions in [18, Theorem 1.1], see the proof of [3, Proposition 3.3] for more details.

Note that by (3.24) and (3.27), we have

$$\begin{aligned}&|\overline\Upsilon^{\text{far}}(H)| \\ &\le \sum_{\{k\in\mathbb{N}:k\ge 20 \text{ and } \theta=2^k\theta_0\ll 1\}}\ \sum_{\{(\mu_1,\mu_2):\tau^\theta_{\mu_1}\sim\tau^\theta_{\mu_2}\}}\ \sum_{(\nu_1,\nu_2)\in\bar\tau^\theta_{\mu_1}\times\bar\tau^\theta_{\mu_2}}\ \sum_{\substack{\tau^{c_0\theta}_{\mu_1'}\cap\bar\tau^\theta_{\mu_1}\neq\emptyset \\ \tau^{c_0\theta}_{\mu_2'}\cap\bar\tau^\theta_{\mu_2}\neq\emptyset}} \mathfrak{S} \\ &\quad + O(\lambda^{-N}\|H\|_2^4),\end{aligned} \tag{3.29}$$

where

$$\mathfrak{S} := |\tilde\sigma_\lambda A^{c_0\theta}_{\mu_1'} A^{\theta_0}_{\nu_1} H|^2 \cdot |\tilde\sigma_\lambda A^{c_0\theta}_{\mu_2'} A^{\theta_0}_{\nu_2} H|^2.$$

It is not hard to see that the number of terms in the sum is $O(\lambda^{(2n-3)\varepsilon_0}) = O(\lambda^{5\varepsilon_0})$, and by (3.28), we have for each term in the right side of (3.29),

$$\begin{aligned}&\int\Big(\int |\tilde\sigma_\lambda H\cdot\tilde\sigma_\lambda H|^{(q_e-q)/2}||\tilde\sigma_\lambda A^{c_0\theta}_{\mu_1'} A^{\theta_0}_{\nu_1} H|^2 \cdot |\tilde\sigma_\lambda A^{c_0\theta}_{\mu_2'} A^{\theta_0}_{\nu_2} H|^2|^{q/4} dx\Big)^{2/q_e} dt \\ &\lesssim \|\widetilde S_\lambda f\|^{2(q_e-q)/q_e}_{L^\infty(A_-)} \cdot \|(\tilde\sigma_\lambda A^{c_0\theta}_{\mu_1'} A^{\theta_0}_{\nu_1} H)\cdot(\tilde\sigma_\lambda A^{c_0\theta}_{\mu_2'} A^{\theta_0}_{\nu_2} H)\|^{q/q_e}_{L_t^{q/q_e}L_x^{q/2}} \\ &\lesssim T^{(q_e/q-2/q)\cdot q/q_e}\|\widetilde S_\lambda f\|^{2(q_e-q)/q_e}_{L^\infty(A_-)} \cdot \|(\tilde\sigma_\lambda A^{c_0\theta}_{\mu_1'} A^{\theta_0}_{\nu_1} H)\cdot(\tilde\sigma_\lambda A^{c_0\theta}_{\mu_2'} A^{\theta_0}_{\nu_2} H)\|^{q/q_e}_{L_t^{q/2}L_x^{q/2}}, \\ &\lesssim T^{(q_e/q-2/q)\cdot q/q_e}\lambda^{((n-1)/4+\varepsilon_1)(2(q_e-q)/q_e)} \\ &\quad\cdot(\lambda^{1+\varepsilon}(2^k\lambda^{1-\varepsilon_0})^{((n-1)/2)(q-2(n+1)/(n-1))})^{2/q_e}\|H\|^{2q/q_e}_{L^2_x}.\end{aligned} \tag{3.30}$$

As in (3.8), if we take $\varepsilon, \varepsilon_0$, and ε_1 to be small enough, e.g., $\varepsilon = \varepsilon_0 = \varepsilon_1 = 1/100$, it is straightforward to check that the right side of (3.30) is $O(\lambda^{1-6\varepsilon_0}\|H\|^2_{L^2_{t,x}})$, using the fact that $\|H\|^2_{L^2_{t,x}}$ dominates $\|H\|^{2q/q_e}_{L^2_{t,x}}$, since we are assuming f is L^2 normalized. This implies that

$$B = \int\Big(\int |\tilde\sigma_\lambda H\cdot\tilde\sigma_\lambda H|^{(q_e-q)/2}|\overline\Upsilon^{\text{far}}(H)|^{q/4}dx\Big)^{2/q_e} dt \lesssim \lambda^{1-6\varepsilon_0+5\varepsilon_0}\|H\|^2_{L^2_{t,x}},$$

which finishes the proof of Proposition 2.3 for $n-1=3$.

Remarks about $n-1=2$. The arguments for $n-1=2$ is similar to the case $n-1=3$. Recall that when $n-1=3$, we are essentially doing another round of bilinear estimates within the diagonal terms in the Whitney decomposition due to the fact that $q_e = 2(n-1)/(n-3) = 6 > 4$. When $n-1=2$, q can be arbitrary large, if $q \in [2^{k+1}, 2^{k+2}]$ for some $k \in \mathbb{N}^+$, then one can repeat the above arguments k times, the resulting diagonal term will involve a product of 2^{k+1} terms of type $\tilde{\sigma}_\lambda A_\nu^{\theta_0} H$, and it satisfies the natural analog of Lemma 3.1 since $q/2^{k+1} \in [1,2]$. And there are off-diagonal terms each time we run the Whitney decomposition, those terms can be treated as in (3.30) by using bilinear estimates. The main difference is, as $q \to \infty$, unlike (3.8) and (3.30), we have to take ε_0 and ε_1 to be small enough depending on q, instead of some fixed small constant.

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