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Persistence and disappearance of negative eigenvalues in dimension two

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Abstract. We compute asymptotics of eigenvalues approaching the bottom of the continuous spectrum, and associated resonances, for Schrödinger operators in dimension two. We distinguish persistent eigenvalues, which have associated resonances, from disappearing ones, which do not. We illustrate the significance of this distinction by computing corresponding scattering phase asymptotics and numerical Breit–Wigner peaks. We prove all of our results for circular wells, and extend some of them to more general problems using recent resolvent techniques.

In memory of Steve Zelditch

1. Introduction

The transition between discrete and continuous spectrum at low energies poses a special challenge in two-dimensional scattering. The setting of scattering by a finite well is important for the light it sheds here, with non-analytic behavior exhibited by the simplest example: if ε is a coupling constant tending to zero, then the ground state eigenvalue goes to zero like $e^{-C/|\varepsilon|}$. This was shown in [3, 29], and in [23, Section 45, Problem 2].

The same behavior is observed for certain excited eigenvalues, while others are to leading order linear in the coupling constant. In the examples we study, the latter kind correspond to resonances near zero, and we call them *persistent*, while the former kind do not and we call them *disappearing*; see Section 1.1. There we also establish a connection between persistent eigenvalues and p-resonances and/or zero eigenvalues, and between disappearing eigenvalues and s-resonances.

We then turn to consequences for the scattering phase. In Section 1.2 we compute low energy asymptotics, and in Section 1.3 we interpret our results numerically in

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terms of Breit–Wigner peaks. These illustrate in a striking way the significance of a persistent eigenvalue for the behavior of the scattering matrix at low energies.

Throughout, we focus especially on the case of a circular well, and indicate how some results obtained in that setting by Bessel function analysis can be extended to more general problems using resolvent techniques from [6–8].

1.1. Persistence and disappearance of eigenvalues

Consider a Schrödinger operator $P = -\Delta + V$ on $\mathbb{R}^2$, where the potential function V is bounded, compactly supported, and real valued. Denote by V_ε a family of such potentials, depending smoothly on a real parameter ε. If $P_\varepsilon = -\Delta + V_\varepsilon$ has a bound state at $\varepsilon = 0$, with energy $E_0 < 0$, then at nearby values of ε there is a family of bound states with energies $E_\varepsilon < 0$ depending smoothly on ε. This is the familiar problem of perturbation theory as discussed in [23, Section 38].

The problem becomes richer if the energy level approaches 0, which is the beginning of the continuous spectrum. To analyze this case, we bring in the notion of resonance.

In polar coordinates, a general solution to $(P - \lambda^2)u = 0$ has a large r expansion of the form

$$u(r,\theta) = \sum_{\ell=-\infty}^{\infty} \big(c_\ell H_\ell^{(1)}(\lambda r) + d_\ell H_\ell^{(2)}(\lambda r)\big) e^{i\ell\theta}. \tag{1.1}$$

See Section 2.1 below for a review of Bessel functions; (1.1) follows from (2.1) and (2.2). If $\operatorname{Im}\lambda > 0$, then P has an eigenvalue at $E = \lambda^2$ if and only if there is a solution $u \not\equiv 0$ to $(P - \lambda^2)u = 0$ whose expansion (1.1) has all $d_\ell = 0$; see (2.4).

To introduce the notion of resonances and outgoing solutions of $Pu = 0$, we need to work with a larger set on which $H_\ell^{(j)}(\lambda)$, $j = 1, 2$, is single-valued. We cut $\mathbb{C}$ along the negative imaginary axis and identify this with the set $\big\{\lambda \in \mathbb{C} \setminus \{0\} : -\frac{\pi}{2} \le \arg\lambda < \frac{3\pi}{2}\big\}$. We use the natural branch of $H_\ell^{(j)}(\lambda)$ which is continuous for $-\frac{\pi}{2} \le \arg\lambda < \frac{3\pi}{2}$. As will become apparent in Section 3.2, we are essentially making a branch cut for the logarithm in a way that suits our initial focus on eigenvalues and resonances near the continuous spectrum, particularly in the fourth quadrant.

For $\lambda \in \mathbb{C}$ with $-\frac{\pi}{2} \le \arg\lambda < \frac{3\pi}{2}$, we say that a solution u to $(P - \lambda^2)u = 0$ is *outgoing* if its expansion (1.1) has all $d_\ell = 0$. By continuity, we extend this condition to $\lambda = 0$ by saying that a solution to $Pu = 0$ is *outgoing* if it has a large r expansion of the form

$$u(r,\theta) = \sum_{\ell=-\infty}^{\infty} c_\ell r^{-|\ell|} e^{i\ell\theta}. \tag{1.2}$$

We say that $\lambda \in \mathbb{C} \setminus -i(0,\infty)$ is a *resonance* and $u\colon \mathbb{R}^2 \to \mathbb{C}$ is a *resonant state* if u is outgoing, $u \not\equiv 0$, and $(P - \lambda^2)u = 0$. See Section 2.3 for further details. Note that $E = \lambda^2$ is an eigenvalue if and only if λ is a resonance and there is a resonant state $u \in L^2$. This always occurs if $\operatorname{Im}\lambda > 0$, and it can occur for $\operatorname{Im}\lambda \le 0$ only when $\lambda = 0$, and then if and only if $c_{-1} = c_0 = c_1 = 0$.

More precisely, given an outgoing solution to $Pu = 0$, one says that P has an *s-resonance*, and u is an *s-resonant state*, if $c_0 \neq 0$. One says that P has a *p-resonance*, and u is a *p-resonant state*, if $c_0 = 0$ but $|c_1| + |c_{-1}| > 0$. One says that P has a *zero eigenvalue*, and u is a *zero eigenfunction*, if $c_0 = c_{-1} = c_1 = 0$ but $u \not\equiv 0$.

Let V_ε be a family of bounded, compactly supported potentials defined for $\varepsilon \in I$, where I is an open interval containing 0. Suppose there is a family of eigenvalues E_ε of P_ε, defined for $\varepsilon \in I$, $\varepsilon < 0$, such that $E_\varepsilon \uparrow 0$ when $\varepsilon \uparrow 0$. We say that E_ε *persists* along I if there exists a continuous family of outgoing solutions $(P_\varepsilon - \lambda_\varepsilon^2)u_\varepsilon = 0$, defined for all $\varepsilon \in I$, such that $\lambda_\varepsilon^2 = E_\varepsilon$ for $\varepsilon < 0$ and $-\frac{\pi}{2} \le \arg\lambda < \frac{3\pi}{2}$.

Theorem 1. *Let V_ε be a family of compactly supported real-valued radial functions in $L^\infty(\mathbb{R}^2)$, defined for $\varepsilon \in I$, where I is an open interval containing* 0. *Suppose $\varepsilon \mapsto V_\varepsilon$ is smooth from I to $L^\infty(\mathbb{R}^2)$, and that the derivative of this map at $\varepsilon = 0$ is a nonnegative function which is not identically zero. Suppose further that there is a family of energy levels E_ε such that $E_\varepsilon \uparrow 0$ when $\varepsilon \uparrow 0$.*

(1) *If there exists a corresponding family of nonradial eigenfunctions, i.e., if P_0 has a p-resonance or zero eigenvalue, then E_ε persists.*

(2) *If every corresponding family of eigenfunctions is radial, i.e., if P_0 has an s-resonance but no p-resonance or zero eigenvalues, then E_ε does not persist.*

Our key example is the circular well, given by

$$V(r) = -a^2\mathbb{1}_\rho(r) = \begin{cases} -a^2, & r \le \rho, \\ 0, & r > \rho, \end{cases} \tag{1.3}$$

for some positive a and ρ; then we may consider $\rho > 0$ fixed and $a = a_\varepsilon = j - \varepsilon$, where j is any zero of a Bessel function. Figure 1 illustrates persistence in the $\ell = 1, 2, 3$ modes, while Figure 2 illustrates disappearance in the $\ell = 0$ mode.

So far, we have ignored multiplicities. Our next theorem takes them into account and also gives more detailed information about the behavior of the resonance near $\varepsilon = 0$. We work with joint eigenfunctions of P_ε and ∂_θ.

For real-valued V, the resonances of $-\Delta + V$ are symmetric in the sense that if $\lambda = |\lambda|e^{i\arg\lambda}$ is a resonance, so is $|\lambda|e^{i(\pi-\arg\lambda)}$. In the next theorem we concentrate on resonances in the fourth quadrant for simplicity. The function W_{-1} appearing in the statement of the theorem is a Lambert W function, see Section 2.2.

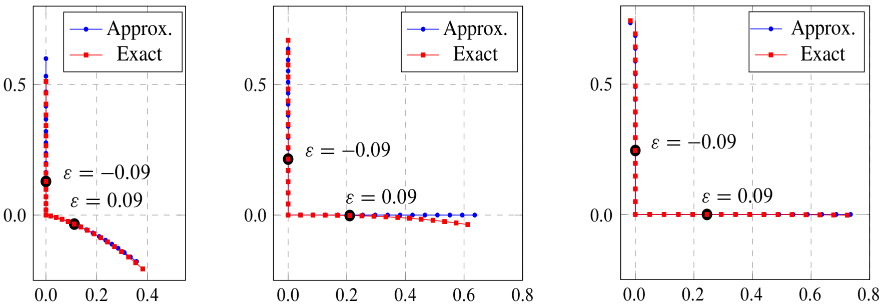

Figure 1. Plot of λ_ε in $\mathbb{C} \setminus -i(0,\infty)$ for $a^2 = j^2_{\ell-1,1} - \varepsilon$, with $\ell = 1$ (left) $\ell = 2$ (middle) and $\ell = 3$ (right). The values of ε range from $\varepsilon = -0.81$ to $\varepsilon = 0.81$. When $\varepsilon < 0$, λ_ε is on the positive imaginary axis and corresponds to an eigenvalue, when $\varepsilon > 0$, λ_ε is in the fourth quadrant and is a resonance.

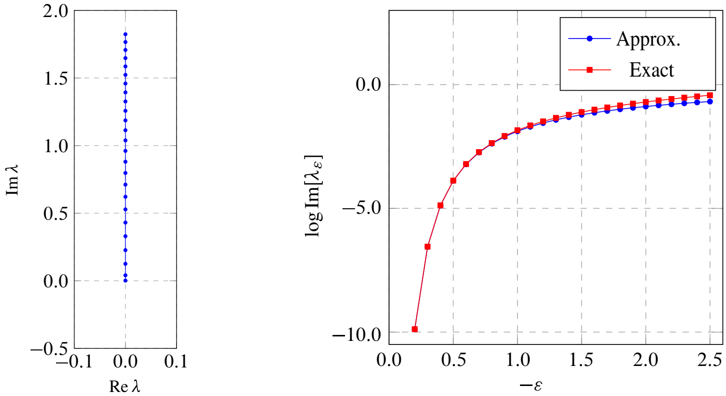

Figure 2. Plot of λ_ε for $\ell = 0$ mode, with $\varepsilon \in \{-k\Delta\varepsilon : k = 2, \ldots, 25\}$, with $\Delta\varepsilon = 0.3$ in the first plot and $\Delta\varepsilon = 0.1$ in the second plot. Approx. is $\lambda_\varepsilon = i\frac{2}{\rho}\exp\left(\frac{2}{\rho^2\varepsilon} - \gamma\right)$.

Theorem 2. *Let V_ε and E_ε satisfy the conditions of Theorem 1. Suppose for $\varepsilon < 0$, $\psi_\varepsilon \in L^2(\mathbb{R})$, $\|\psi_\varepsilon\| = 1$, $(-\Delta + V_\varepsilon)\psi_\varepsilon = E_\varepsilon\psi_\varepsilon$, $\partial_\theta\psi_\varepsilon = i\ell\psi_\varepsilon$, and $E_\varepsilon \uparrow 0$ as $\varepsilon \uparrow 0$.*

(1) *If $|\ell| > 1$, then P_0 has a zero eigenvalue and there is a constant $\alpha_0(\ell) > 0$ and an $\varepsilon_0 > 0$ so that for $0 < \varepsilon < \varepsilon_0$, P_ε has a resonance in the fourth quadrant satisfying*

$$\lambda_\varepsilon = \sqrt{\alpha_0(\ell)\varepsilon} + O(\varepsilon^{3/2}(1 + \delta_{|\ell|,2}\log\varepsilon)).$$

Counting with multiplicity, there is a resonance for each ℓ satisfying these conditions.

(2) *If $|\ell| = 1$, then P_0 has a p-resonance and there are constants $\alpha_0 > 0$, $b = \operatorname{Re} b + i\frac{\pi}{2}$ and an $\varepsilon_0 > 0$ so that for $0 < \varepsilon < \varepsilon_0$, P_ε has a resonance in the*

fourth quadrant satisfying

$$\lambda_\varepsilon = \exp\Big(b + \frac{1}{2} W_{-1}(2\varepsilon\alpha_0 e^{-2\operatorname{Re} b})\Big)(1 + O(\varepsilon)).$$

Counting with multiplicity, there is a resonance for each ℓ satisfying these conditions.

In fact, as we shall describe in Section 3.2, in the setting of Theorem 2 for each such ℓ there are many resonances which are related to the negative eigenvalue, and they exist for both negative and positive ε. Moreover, we find explicit expressions for α_0 in terms of the null space of $-\Delta + V_0$ and $(\partial_\varepsilon V_\varepsilon) \restriction_{\varepsilon=0}$ in the general case, see Lemmas 14 and 15 in Section 3.2. In case $V_\varepsilon = (-a_0^2 + \varepsilon)\mathbb{1}_\rho$, these α_0 can be written in terms of ρ and ℓ, see Lemmas 11 and 12 in Section 3.1. In addition, Lemmas 14 and 15 include a further term in the expansion for $\ell \neq \pm 2$ that helps explain the difference between the center and right graphs in Figure 1.

1.1.1. Further background and context. The paper [15] surveys some results in perturbation of eigenvalues and resonances. We mention just a few papers most closely related to studying eigenvalues and resonances of Schrödinger operators near 0, particularly in dimension two, and direct the reader to [15] for further results and references.

Klaus and Simon [21] and Holden [17] study the possible expansions of $E_\varepsilon \uparrow 0$ as $\varepsilon \uparrow 0$ in any dimension and for potentials which need not be radially symmetric. The results of [21] show that dimension 2 holds the most different possibilities. These papers do not explore whether or not the eigenvalues persist. Compared to [21], our results hold for a smaller class of potentials, but our leading terms are more detailed and cover resonances as well. The paper [25] contains some nice diagrams of numerically computed eigenvalues and resonances for the Schrödinger operator with circular well potential, $-\Delta - a^2\mathbb{1}_\rho$, on $\mathbb{R}^2$. These eigenvalues and resonances are not just those very near 0, and the diagrams indicate how the resonances and eigenvalues move as a varies. In [14], Grigis and Klopp study eigenvalues and resonances near the boundary of the continuous spectrum for perturbations of Schrödinger operators with periodic potentials. Their results do not include dimension two, but we note that their results in even dimension at least four resemble our Lemma 14, on perturbing an eigenvalue at 0.

For Schrödinger operators on $\mathbb{R}^3$ depending on a parameter, the papers [13, 28] study the transition from eigenvalues to resonances at the bottom of the continuous spectrum. In [19], Jensen and Nenciu study the perturbation of an eigenvalue at the bottom of the continuous spectrum using a time-dependent notion of resonances. Their results are quite general, and apply to Schrödinger operators in dimension 3, and to certain two channel operators in dimension 1 and 3 with potentials which decay at infinity but need not be compactly supported. They do not study the even-dimensional case. In fact, near the end of [15], Harrell asks: "Can the restriction to odd dimensions in articles such as [19] be relaxed?" Our results in Theorem 2 and Section 2.3.3 are a step towards doing so.

Most recently, perturbations of resonances at the bottom of the continuous spectrum have seen applications in the study of subwavelength resonator systems; see [1] for a review and references. In that setting, one seeks to use small objects to strongly scatter waves with comparatively large wavelengths. In [1], and in references therein, asymptotics are computed for resonances obtained by perturbing off of a resonance at zero for a certain non-self-adjoint problem.

Our results in Section 3.2 use results about the resolvent expansion at 0 for a Schrödinger operator on $\mathbb{R}^2$. Results in this direction include [5, 7, 18, 30, 31].

1.2. Scattering phase asymptotics

Theorems 1 and 2 examine the importance of the different kinds of resonance or eigenvalue at zero energy for perturbation theory. If we consider a fixed circular well, we can illustrate their importance for scattering phase asymptotics. The scattering phase is defined by

$$\sigma(\lambda) = \frac{1}{2\pi i} \log \det S(\lambda) = \sum_{\ell} \sigma_\ell(\lambda), \tag{1.4}$$

where $S(\lambda)$ is the scattering matrix and the $\sigma_\ell(\lambda)$ are the phase shifts, i.e., $e^{2\pi i \sigma_\ell(\lambda)}$ are the eigenvalues of $S(\lambda)$, normalized so that $\sigma_\ell(0) = 0$. When V is radial, the set $\{e^{i\ell\theta}\}_{\ell\in\mathbb{Z}}$ is an orthogonal basis for $S(\lambda)$, and we may take each $\sigma_\ell(\lambda)$ to be the phase shift corresponding to $e^{i\ell\theta}$. See [11, Sections 2.6 and 3.9] for a more general introduction.

Theorem 3. *Let $P = -\Delta - a^2 \mathbb{1}_\rho$ for some positive a and ρ. Let $\gamma = -\Gamma'(1) =$ 0.577 . . .. Then, as $\lambda \downarrow 0$,*

(1) *if P has no resonance at zero, then there is a real b such that*

$$\sigma'(\lambda) = \frac{-1}{\lambda} \frac{2}{4(\log \lambda + b)^2 + \pi^2} + O(\lambda);$$

(2) *if P has a p-resonance at zero, then*

$$\sigma'(\lambda) = \frac{-1}{\lambda}\Big(\frac{2}{4\big(\log \lambda + \log\big(\frac{\rho}{2}\big) + \gamma\big)^2 + \pi^2} + \frac{4}{4(\log \lambda + \log\big(\frac{\rho}{2}\big) + \gamma - \frac{1}{2})^2 + \pi^2} \Big) + O(\lambda);$$

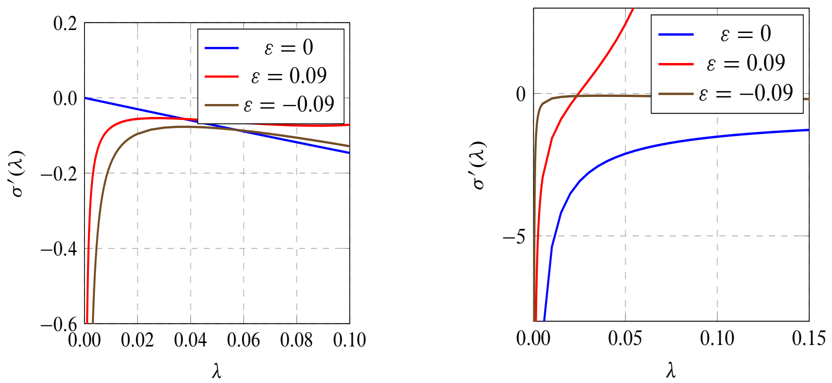

Figure 3. Graphs of $\sigma'(\lambda)$ for $-\Delta - a^2\mathbb{1}_1$, with $a^2 = j_{1,1}^2 - \varepsilon$ on the left (perturbing a zero eigenvalue) and $a^2 = j_{0,1}^2 - \varepsilon$ (perturbing a p-resonance at zero) on the right. The cases $\varepsilon \neq 0$ all correspond to case (1) of Theorem 3. Case (2) of Theorem 3 is $\varepsilon = 0$ on the right and Case (3) of Theorem 3 is $\varepsilon = 0$ on the left. The behavior for larger λ is in Figure 5.

(3) *if P has an s-resonance at zero, then*

$$\sigma'(\lambda) = -\frac{3}{2}\rho^2\lambda + O(\lambda^3 \log\lambda).$$

Theorem 23 gives an expression for the constant b, and Figure 3 provides an illustration.

1.2.1. Further background and context. The behavior near 0 of the scattering phase is largely determined by the presence or absence of different kinds of resonance or eigenvalue at 0, and this, in turn, is related to the low-energy behavior of the resolvent. This is true not only for Schrödinger operators with compactly supported potentials (without requiring the potential to be radial), but for general self-adjoint black-box compactly supported perturbations of $-\Delta$ on $\mathbb{R}^2$. In the absence of 0-energy eigenvalues or resonances, the proof of [6, Theorem 3] combined with [7, Theorem 2], shows that there is a real number b such that

$$\sigma'(\lambda) = \frac{-1}{\lambda}\frac{2}{4(\log\lambda + b)^2 + \pi^2} + O(\lambda). \tag{1.5}$$

Here, σ is the scattering phase for the black-box operator P.

For the Dirichlet Laplacian on an external domain in $\mathbb{R}^2$, the universal behavior $\sigma(\lambda) = \frac{1}{2}(\log\lambda)^{-1} + O((\log\lambda)^{-2})$ was proved first by Hassell and Zelditch in [16]. Their results were further refined in [6,24,30]. The recent work [12] provides a wealth of numerical examples.

A Schrödinger operator on $\mathbb{R}^2$ generically has neither eigenvalue nor resonance at 0, in which case again $\sigma(\lambda) = \frac{1}{2}(\log\lambda)^{-1} + O((\log\lambda)^{-2})$. This, or at least very closely related results, has been observed in [4,5]; see [2,20] for further related results and more references. Moreover, [4, 20] relate the next term in the expansion to the scattering length.

The constant b appearing in (1.5) can be identified using the asymptotic behavior of a function in the null space of P. Under the assumption that P does not have a resonance or eigenvalue at 0, there is a unique function G which is locally in the domain of P and which satisfies $PG = 0$ and $G(x) - \log|x| = O(1)$ as $|x| \to \infty$. Then

$$b = -\log 2 + \gamma + C(P), \quad \text{where } C(P) = \lim_{x\to\infty}(\log|x| - G(x)).$$

For the case of scattering by a nontrivial smooth obstacle with Dirichlet boundary conditions, $C(P)$ is the logarithm of the logarithmic capacity of the obstacle – see [6] for further discussion. For the case of a radial potential V with $P_V = -\Delta + V$, $C(P_V)$ is the negative reciprocal of the scattering length as defined in [4]. See [4, Sections II and VII] for a discussion of other definitions (both equivalent and not) of the scattering length in two dimensions.

The paper [5] obtains, for much more general potentials V, expressions for the leading term of the scattering matrix which may be used to derive the leading term of the scattering phase (with $\sigma(0) = 0$), but with worse errors than those coming from Theorem 3. The results of [5] allow the possibility of an eigenvalue or resonance at 0.

1.3. Breit–Wigner peaks

In this section we use Breit–Wigner formulas to illustrate the significance of a persistent eigenvalue for scattering.

We begin with a one-dimensional example to set the stage. Let $\sigma(\lambda)$ be the scattering phase of $-\frac{d^2}{dx^2} + V$, with V compactly supported. Let L be the length of the convex hull of the support of V, and let Res^* be the set of nonzero resonances of $-\frac{d^2}{dx^2} + V$. The Breit–Wigner formula says that

$$\sigma'(\lambda) = \frac{-L}{\pi} + \sum_{\lambda_k \in \mathrm{Res}^*} \frac{-\operatorname{Im}\lambda_k}{\pi|\lambda - \lambda_k|^2}. \tag{1.6}$$

This formula, and others like it, are deduced from factorizations of $\det S(\lambda)$: see [11, Theorem 2.20] for the full-line problem, and [22, Theorem 1.1] for the half-line problem.

A simple and striking example is $P = -\frac{d^2}{dx^2} + a\delta_1$ on $\mathbb{R}_+$, with Dirichlet boundary condition at 0. The resonances (obtained by solving $(P - \lambda^2)u = 0$ subject to $u(x) = e^{i\lambda x}$ for $x > 1$) are given in terms of the Lambert W function (see (2.6))

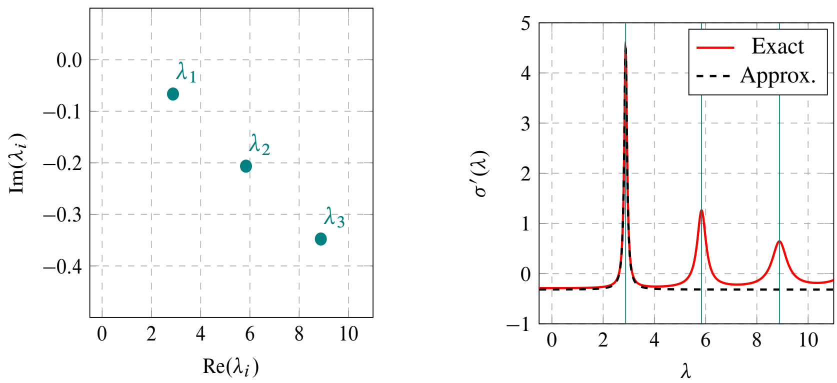

Figure 4. On the left, the first three resonances of $-\frac{d^2}{dx^2} + 10\delta_1$ on the half line. On the right, a graph of $\sigma'(\lambda)$. The vertical lines are at $\operatorname{Re}\lambda_1$, $\operatorname{Re}\lambda_2$, $\operatorname{Re}\lambda_3$, and the dashed curve is the approximation $\frac{-1}{\pi} - \frac{\operatorname{Im}\lambda_1}{\pi|\lambda-\lambda_1|^2}$.

by $\lambda_k = \frac{1}{2i}(a - W_{-k}(ae^a))$, where k varies over the nonzero integers. The derivative of the scattering phase (obtained by solving $(P - \lambda^2)u = 0$ subject to $u(x) = e^{-i\lambda x} + e^{2\pi i\sigma(\lambda)}e^{i\lambda x}$ for $x > 1$) is given by $\sigma'(\lambda) = -\frac{1}{\pi} + \frac{1}{\pi}\frac{a+\lambda^2\csc^2\lambda}{(a+\lambda\cot\lambda)^2+\lambda^2}$. Figure 4 shows the first three Breit–Wigner peaks, corresponding to the first three resonances. Moreover, at low energies, up to and including the first peak, we see that $\sigma'(\lambda) \approx \frac{-L}{\pi} - \frac{\operatorname{Im}\lambda_1}{\pi|\lambda-\lambda_1|^2}$.

In higher dimensions, Breit–Wigner formulas are more complicated. In $\mathbb{R}^d$, $d \geq 2$, work of Petkov and Zworski [27] shows that $\sigma'(\lambda) = \sum_{|\lambda_k-\lambda|<1}\frac{-\operatorname{Im}\lambda_k}{\pi|\lambda-\lambda_k|^2} + O(\lambda^{d-1})$ as $\lambda \to +\infty$. If d is odd, an analog of (1.6), with $\frac{-L}{\pi}$ replaced by a polynomial of degree $\leq d$ and with the sum $\sum\frac{-\operatorname{Im}\lambda_k}{\pi|\lambda-\lambda_k|^2}$ modified so as to converge, follows from the factorization of $\det S(\lambda)$ in [11, Theorem 3.54].

In even dimensions, the non-resonance contribution (i.e., the replacement of the term $\frac{-L}{\pi}$ in (1.6)) is not known; by Theorem 3, it is sometimes singular as $\lambda \to 0$. Moreover, it is not clear how to modify the sum $\sum\frac{-\operatorname{Im}\lambda_k}{\pi|\lambda-\lambda_k|^2}$ to account for resonances on the Riemann surface Λ (see Section 2.3.3). Nevertheless, in Figure 5, which is typical for the circular well family, we see that a resonance coming from a persistent eigenvalue produces a large Breit–Wigner peak, easily seen despite the singularity as $\lambda \to 0$. See also [12, Figure 2] for examples of Breit–Wigner peaks for obstacle scattering in $\mathbb{R}^2$, with significant peaks corresponding to resonances generated by significant trapping of the billiard flow.

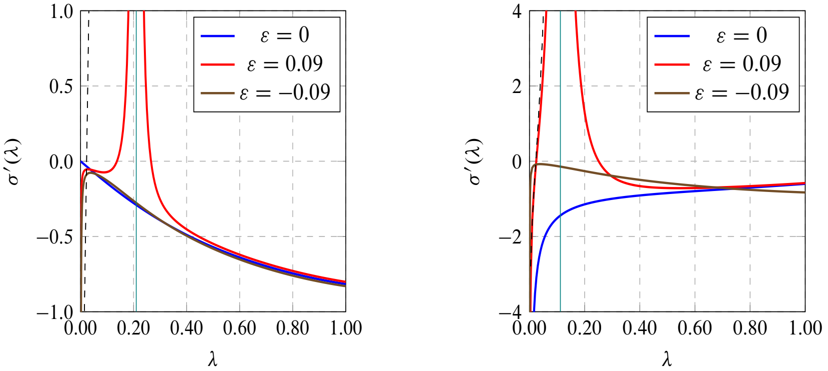

Figure 5. Graphs of $\sigma'(\lambda)$ for $-\Delta - a^2\mathbb{1}_1$, with $a^2 = j_{1,1}^2 - \varepsilon$ on the left (perturbing a zero eigenvalue) and $a^2 = j_{0,1}^2 - \varepsilon$ (perturbing a p-resonance at zero) on the right. The vertical lines are at $\operatorname{Re}\lambda_\varepsilon$, with λ_ε as in Figure 1. Each $\varepsilon = 0.09$ curve has a maximum when $\lambda \approx \operatorname{Re}\lambda_\varepsilon$, with $\sigma'(\operatorname{Re}\lambda_\varepsilon) \approx 367.37$ on the left, and $\sigma'(\operatorname{Re}\lambda_\varepsilon) \approx 17.048$ on the right. The dashed curves are Breit–Wigner approximations $\frac{-\operatorname{Im}\lambda_\varepsilon}{\pi|\lambda-\lambda_\varepsilon|^2} - 0.8\sqrt{\lambda}$ and $\frac{-\operatorname{Im}\lambda_\varepsilon}{\pi|\lambda-\lambda_\varepsilon|^2} + \log\lambda$, with the non-resonance terms $\sqrt{\lambda}$ and $\log\lambda$ (meant to be analogous to $\frac{-L}{\pi}$ in (1.6)) chosen ad hoc by eye to improve the fit.

2. Preliminaries

2.1. Bessel and Hankel functions

We use the standard notation of [26, (10.2 (ii)) and (10.4.3)]:

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{-z^2}{4}\right)^k}{k!\Gamma(\nu+k+1)}, \tag{2.1a}$$

$$Y_\ell(z) = \frac{1}{\pi}\partial_\nu J_\nu(z)\Big|_{\nu=\ell} + \frac{(-1)^n}{\pi}\partial_\nu J_{-\nu}(z)\Big|_{\nu=-\ell}, \tag{2.1b}$$

$$H_\ell^{(1)}(z) = J_\ell(z) + iY_\ell(z), \quad H_\ell^{(2)}(z) = J_\ell(z) - iY_\ell(z). \tag{2.1c}$$

If $\mathscr{C}_\ell$ is one of J_ℓ, Y_ℓ, $H_\ell^{(1)}$, or $H_\ell^{(2)}$, then $\mathscr{C}_\ell$ solves the differential equation [26, (10.2 (i))]

$$z^2\mathscr{C}_\ell''(z) + z\mathscr{C}_\ell'(z) + (z^2 - \ell^2)\mathscr{C}_\ell(z) = 0, \tag{2.2}$$

and we also have the recurrence relations [26, (10.6 (i))]

$$\begin{aligned} \mathscr{C}_{\ell-1}(z) + \mathscr{C}_{\ell+1}(z) &= \frac{2\ell}{z}\mathscr{C}_\ell(z), \quad \mathscr{C}_{\ell-1}(z) - \mathscr{C}_{\ell+1}(z) = 2\mathscr{C}_\ell'(z), \\ \mathscr{C}_\ell'(z) &= \mathscr{C}_{\ell-1}(z) - \frac{\ell}{z}\mathscr{C}_\ell(z), \quad \mathscr{C}_\ell'(z) = -\mathscr{C}_{\ell+1}(z) + \frac{\ell}{z}\mathscr{C}_\ell(z). \end{aligned} \tag{2.3}$$

As $|z| \to \infty$, $z \in \mathbb{C} \setminus -i(0,\infty)$, we have [26, (10.2.5) and (10.2.6)]

$$H_\ell^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{i(z - \frac{\ell\pi}{2} - \frac{\pi}{4})}, \qquad H_\ell^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{-i(z - \frac{\ell\pi}{2} - \frac{\pi}{4})}. \tag{2.4}$$

As $|z| \to 0$, we have [26, (10.8)]

$$Y_0(z) = \frac{2}{\pi}\Big[\log\Big(\frac{z}{2}\Big) + \gamma\Big] + O(z^2 \log z), \quad \gamma = -\Gamma'(1) \approx 0.5772, \tag{2.5a}$$

$$Y_1(z) = -\frac{2}{\pi z} + \frac{z \log z}{\pi} + O(z), \tag{2.5b}$$

$$Y_\ell(z) = -\frac{(\ell-1)! 2^\ell}{\pi z^\ell} - \frac{(\ell-2)! 2^{\ell-2}}{\pi z^{\ell-2}} + O(z^{4-\ell}(1 + \delta_{2,\ell} \log z)), \quad \ell \geq 2. \tag{2.5c}$$

2.2. Lambert W function

If $x \in \mathbb{C} \setminus \{0\}$, then $we^w = x$ has infinitely many solutions. Following [9], we denote these by $W_n(x)$, with n varying over the integers. If $n \neq 0$, then

$$W_n(x) = L_n(x) - \log(L_n(x)) + \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} c_{km} \frac{(\log(L_n(x)))^m}{L_n(x)^{k+m}}, \tag{2.6}$$

where $L_n(x) = \log x + 2\pi i n$, $\operatorname{Im} \log z \in (-\pi, \pi]$ for $z \in \mathbb{C} \setminus \{0\}$, and the c_{km} are combinatorial coefficients (see [9, (4.20)]). The remaining solution $W_0(x)$ is the analytic function with power series $\sum_{k=1}^{\infty} (-k)^{k-1} \frac{x^k}{k!}$ (see [9, (3.1)]).

Below we will mostly use just the following simple consequence of (2.6).

Lemma 4. *For $n \in \mathbb{Z} \setminus \{0\}$, one has $\lim_{\varepsilon \to 0} \operatorname{Re} W_n(\varepsilon) = -\infty$, $\lim_{\varepsilon \uparrow 0} \operatorname{Im} W_n(\varepsilon) = \pi(1 + 2n - \operatorname{sgn} n)$, and $\lim_{\varepsilon \downarrow 0} \operatorname{Im} W_n(\varepsilon) = \pi(2n - \operatorname{sgn} n)$.*

Proof. By (2.6), $\operatorname{Re} W_n(\varepsilon) \sim \log|\varepsilon| \to -\infty$ as $\varepsilon \to 0$. Meanwhile, $\operatorname{Im} L_n(\varepsilon) \to 2\pi n + \operatorname{Im} \log \varepsilon$ and $\operatorname{Im} \log(L_n(\varepsilon)) \to \pi \operatorname{sgn} n$, as desired. ∎

2.3. Resonances

Let $V : \mathbb{R}^2 \to \mathbb{R}$ be bounded and supported in $\{x : |x| \leq \rho\}$, and let $P = -\Delta + V$. We discuss separately resonances in the cut plane $\mathbb{C} \setminus -i[0,\infty)$, at zero, and on the Riemann surface of the logarithm.

2.3.1. Resonances on the cut plane. Recall that in Section 1.1 we defined a *resonance* of P to be a value of λ with $-\frac{\pi}{2} \leq \arg \lambda < \frac{3\pi}{2}$ for which there is an *outgoing* solution $u \not\equiv 0$ to $(P - \lambda^2)u = 0$, i.e., one obeying

$$u(r,\theta) = \sum_{\ell=-\infty}^{\infty} c_\ell H_\ell^{(1)}(\lambda r) e^{i\ell\theta} \quad \text{for } r > \rho,$$

where $V(x) = 0$ for $|x| > \rho$.

Lemma 5. *If* $\lambda \in \mathbb{C} \setminus -i[0,\infty)$ *is a resonance, then* $\operatorname{Im}\lambda < 0$ *or* $\operatorname{Re}\lambda = 0$.

Proof. If $\operatorname{Im}\lambda > 0$, then $u \in H^2(\mathbb{R}^2)$ by the Hankel function asymptotics (2.4) and the equation $(P - \lambda^2)u = 0$. Hence, $\int_{\mathbb{R}^2} |\nabla u|^2 + (V - \lambda^2)|u|^2 = 0$, implying that λ^2 is real and $\operatorname{Re}\lambda = 0$. By Rellich's uniqueness theorem [11, Theorem 3.33], there are no resonances in $\mathbb{R} \setminus \{0\}$. ∎

The proof of [11, Theorem 3.33] uses the following characterization of resonances, which we will also use in Section 3.2. Let $R_V(\lambda) = (P - \lambda^2)^{-1}\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the resolvent, defined when $\operatorname{Im}\lambda > 0$ and λ^2 is not an eigenvalue. It is given by $R_V(\lambda) = R_0(\lambda)(I + VR_0(\lambda))^{-1}$, where $R_0(\lambda) = (-\Delta - \lambda^2)^{-1}$ is the free resolvent and has integral kernel $\frac{i}{4}H_0^{(1)}(\lambda|x-y|)$ (see [6, Section 2.1]). Using analytic Fredholm theory as in [11, Section 3.2.1] shows that if $\chi \in C_c^\infty(\mathbb{R}^2)$ and $\chi V = \chi$, then the cutoff resolvent $\chi R_V(\lambda)\chi$ continues meromorphically to $\mathbb{C} \setminus -i[0,\infty)$, and

$$\chi R_V(\lambda)\chi = \chi R_0(\lambda)\chi(I + VR_0(\lambda)\chi)^{-1}. \tag{2.7}$$

Moreover, the *Birman–Schwinger principle* holds in the following form: λ is a pole of $\chi R_V(\lambda)\chi$ if and only if -1 is an eigenvalue of $VR_0(\lambda)\chi$, i.e., if and only if $g = -VR_0(\lambda)g$ for some $g \not\equiv 0$.

Lemma 6. *Let* $\lambda \in \mathbb{C} \setminus -i[0,\infty)$ *and* $\chi \in C_c^\infty(\mathbb{R}^2)$ *with* $\chi V = V$. *Then* λ *is a resonance if and only if* $\chi R_V(\lambda)\chi$ *has a pole at* λ.

Proof. Suppose $\chi R_V(\lambda)\chi$ has a pole at λ. Take g as in the Birman–Schwinger principle and let $u = R_0(\lambda)g$. Then $(P - \lambda^2)u = 0$ and u is outgoing in the sense of [11, Definition 3.32]. The equivalence of that definition with ours above follows from the fact that, by variation of parameters, if $g(r,\theta) = g_\ell(r)e^{i\ell\theta}$ then $[R_0(\lambda)g](r,\theta) = \frac{\pi i}{2}\int_0^\infty J_\ell(\lambda\min(r,r'))H_\ell^{(1)}(\lambda\max(r,r'))g_\ell(r')\,dr' e^{i\ell\theta}$.

Conversely, suppose $u \not\equiv 0$ is outgoing and $(P - \lambda^2)u = 0$. Let $g = (-\Delta - \lambda^2)u$. Then $VR_0(\lambda)g = Vu = (\Delta + \lambda^2)u = -g$, so $\chi R_V(\lambda)\chi$ has a pole at λ by the Birman–Schwinger principle. ∎

2.3.2. Resonances at zero. In Section 1.1 we have already extended the notion of resonance and resonant state to $\lambda = 0$, with definitions in terms of the c_ℓ of the expansion (1.2). Equivalently, we may say that P has an s-resonance at zero if $Pu = 0$ has a bounded solution which is not $O(|x|^{-1})$ as $|x| \to \infty$, that P has a p-resonance at zero if $Pu = 0$ has a solution which is $O(|x|^{-1})$ but not $O(|x|^{-2})$, and that P has an eigenvalue at zero if $Pu = 0$ has a nontrivial solution which is $O(|x|^{-2})$.

2.3.3. Resonances on the Riemann surface of the logarithm. In Section 3.2 we extend some of our results beyond the branch cut of $\mathbb{C} \setminus -i[0,\infty)$, to Λ, the Riemann surface of the logarithm. We denote points on Λ by $\lambda = |\lambda|e^{i \arg \lambda}$, but without identifying values of arg that differ by an integer multiple of 2π. (Identification of such values of arg gives the universal covering $\Lambda \to \mathbb{C} \setminus \{0\}$.) Thus, log: $\Lambda \to \mathbb{C} \setminus \{0\}$ is biholomorphic, and the functions $H_\ell^{(1)}(\lambda r)$ and $H_\ell^{(2)}(\lambda r)$ continue analytically from $\mathbb{C} \setminus -i[0,\infty)$ to Λ, where we identify $\mathbb{C} \setminus -i[0,\infty)$ with $\{\lambda \in \Lambda : -\frac{\pi}{2} < \arg \lambda < \frac{3\pi}{2}\}$. Using (2.7), this implies that $\chi R_V(\lambda)\chi$ continues meromorphically to Λ.

The same definitions of resonances and outgoing functions from Sections 1.1 and 2.3.1 still work, and Lemmas 5 and 6 extend directly.

Lemma 7. *Let $\lambda \in \Lambda$ and $\chi \in C_c^\infty(\mathbb{R}^2)$ with $\chi V = V$. Then λ is a resonance if and only if $\chi R_V(\lambda)\chi$ has a pole at λ. Moreover, there are no resonances in $\{\lambda \in \Lambda : 0 < |\arg \lambda - \frac{\pi}{2}| \le \frac{\pi}{2}\}$.*

3. Eigenvalues and resonances

In this section we study eigenvalues and resonances near 0, proving Theorems 1 and 2. We begin with the circular well case, where $(-\Delta + V - \lambda^2)u = 0$ can be solved explicitly using Bessel and Hankel functions.

3.1. The circular well

For the circular well potential, we can characterize eigenvalues and resonances as the zeros of certain functions defined using Bessel and Hankel functions. Our main tools are expansions near 0 of such functions, Taylor approximations, and Rouché's theorem. Lemmas 11 and 12 together prove Theorem 2 for the special case $V_\varepsilon = (-a_0^2 + \varepsilon)\mathbb{1}_\rho$. Combined with Lemma 10 they prove Theorem 1 for this V_ε.

We consider the circular well potential $V = V(r) = -a^2\mathbb{1}_\rho$ given by (1.3) with $a, \rho > 0$. We note that some of our results hold for other values of a, but since our main interest is tracking negative eigenvalues as they reach the continuous spectrum, it suffices to consider $a > 0$. We define

$$\mu = \mu(\lambda, a) = \sqrt{\lambda^2 + a^2}$$

where we take the square root to be positive for $\lambda > 0$ and to depend continuously on λ. By the Bessel equation and the asymptotics (2.1), (2.2), and (2.5), if $u \in L^2_{\text{loc}}(\mathbb{R}^2)$ solves $(-\Delta + V - \lambda^2)u = 0$, then

$$u(r,\theta) = \sum_{\ell=-\infty}^{\infty} b_\ell(u) J_\ell(\mu r) e^{i\ell\theta}, \quad \text{for } 0 \le r < \rho. \tag{3.1}$$

Using that u and $\partial_r u$ are continuous at $r = \rho$, this yields, when combined with (1.1),

$$b_\ell = \frac{c_\ell H_\ell^{(1)}(\lambda\rho) + d_\ell H_\ell^{(2)}(\lambda\rho)}{J_\ell(\mu\rho)} = \frac{\lambda}{\mu}\frac{c_\ell H_\ell^{(1)'}(\lambda\rho) + d_\ell H_\ell^{(2)'}(\lambda\rho)}{J_\ell'(\rho\mu)}.$$

We note the following consequence if u is nonvanishing in the ℓ-th mode:

$$c_\ell(u) \neq 0, d_\ell(u) = 0 \iff \mu H_\ell^{(1)}(\rho\lambda)J_\ell'(\rho\mu) - \lambda H_\ell^{(1)\prime}(\rho\lambda)J_\ell(\rho\mu) = 0 \quad (3.2)$$

for $\lambda \neq 0$. This shall be important for our study of eigenvalues and resonances, so we define

$$Q_\ell(\lambda, a) := \mu H_\ell^{(1)}(\rho\lambda)J_\ell'(\rho\mu) - \lambda H_\ell^{(1)\prime}(\rho\lambda)J_\ell(\rho\mu). \quad (3.3)$$

Although Q depends on ρ as well, we omit this in our notation because we will not be varying ρ. Using the recurrence relations (2.3) gives

$$Q_\ell(\lambda, a) = \mu J_{\ell-1}(\rho\mu)H_\ell^{(1)}(\lambda\rho) - \lambda J_\ell(\rho\mu)H_{\ell-1}^{(1)}(\lambda\rho), \quad (3.4)$$

which for $\ell = 0$ can also be written

$$Q_0(\lambda, a) = -\mu J_1(\rho\mu)H_0^{(1)}(\rho\lambda) + \lambda J_0(\rho\mu)H_1^{(1)}(\lambda\rho). \quad (3.5)$$

Now, we allow the depth of the well to depend on ε, so that

$$a^2 = a^2(\varepsilon) = a_0^2 - \varepsilon,\ a_0 > 0, \text{ and set } V_\varepsilon(r) = -a^2(\varepsilon)\mathbb{1}_\rho(r). \quad (3.6)$$

Notice that $\varepsilon > 0$ means the well is shallower than with $\varepsilon = 0$, and $\varepsilon < 0$ means the well is deeper than for $\varepsilon = 0$. We remark that we shall assume $a_0 > 0$ throughout this section.

Lemma 8. *Let a_ε be as in (3.6), with $a_0 > 0$. Suppose there is an $\varepsilon_0 > 0$ so that for each $\varepsilon \in (-\varepsilon_0, 0)$ there exists nontrivial $\psi_\varepsilon \in L^2(\mathbb{R}^2)$, $\partial_\theta\psi_\varepsilon = i\ell\psi_\varepsilon$, $(-\Delta - a_\varepsilon^2\mathbb{1}_\rho - E_\varepsilon)\psi_\varepsilon = 0$, and $E_\varepsilon \uparrow 0$ as $\varepsilon \uparrow 0$. Then $J_{|\ell|-1}(a_0\rho) = 0$, and if $\ell = 0$ then this means $J_1(a_0\rho) = 0$.*

We shall later see that the implication goes the other way as well.

Proof. Using the notation of (1.1), we see that we must have $d_\ell(\psi_\varepsilon) = 0$, but $c_\ell(\psi_\varepsilon) \neq 0$. Using (3.2) and (3.3), we see that this can happen if and only $Q_\ell(i\sqrt{|E_\varepsilon|}, a_\varepsilon) = 0$.

By (2.1) and (2.5), as $\lambda \to 0$ we have $|H_\ell^{(1)}(\lambda)| \propto |\lambda|^{-\ell}$ for $\ell > 0$ and $|H_0^{(1)}(\lambda)| \propto |\log\lambda|$. Using these and the continuity of J_ℓ, $J_{\ell-1}$ in (3.4) shows that we must have $J_{\ell-1}(\rho\mu(0, a_0)) = J_{\ell-1}(\rho a_0) = 0$ if $\ell \geq 0$. The result for $\ell < 0$ follows from the result for $|\ell|$. ∎

Lemma 8 has a natural interpretation in terms of a resonance or eigenvalue of $-\Delta - a_0^2 \mathbb{1}_\rho$ at 0.

Lemma 9. *For $a_0 > 0$ the operator $-\Delta - a_0^2 \mathbb{1}_\rho$*

- *has an s-resonance if and only if $J_1(\rho a_0) = 0$,*
- *has a p-resonance if and only if $J_0(\rho a_0) = 0$,*
- *has an eigenvalue 0 in the ℓth mode, $|\ell| \geq 2$, if and only if $J_{|\ell|-1}(\rho a_0) = 0$.*

Proof. This follows by using (1.2), (3.1), the fact that the solution u and $\partial_r u$ must be continuous across $r = \rho$, and (2.3). ∎

In the remainder of Section 3.1, we prove a sequence of lemmas which, among other things, compute asymptotics of eigenvalues and resonances as $\varepsilon \to 0$. Figures 1 and 2 illustrate these results numerically. Recalling that with V real, $|\lambda| e^{i \arg \lambda}$ is a resonance of $-\Delta + V$ if and only if $|\lambda| e^{i(\pi - \arg \lambda)}$ is a resonance, we focus on the region with $|\arg \lambda| \leq \frac{\pi}{2}$.

Our first result is for the case in which $-\Delta + V_0$ has an s-resonance.

Lemma 10. *Let $a_0 > 0$ and suppose for $-\varepsilon_1 < \varepsilon < 0$ there exists a nontrivial radial function $\psi_\varepsilon \in L^2(\mathbb{R}^2)$, with $(-\Delta - a_\varepsilon^2 \mathbb{1}_\rho - E_\varepsilon)\psi_\varepsilon = 0$, and $E_\varepsilon \uparrow 0$ as $\varepsilon \uparrow 0$. Then there are δ_0, $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the operator $-\Delta - a_\varepsilon \mathbb{1}_\rho$ has no resonance in $\left\{\lambda \in \mathbb{C} : 0 \leq |\lambda| < \delta_0,\ |\arg \lambda| \leq \frac{\pi}{2}\right\}$. Moreover, as $\varepsilon \uparrow 0$, $E_\varepsilon = -\frac{4}{\rho^2} e^{4/(\rho^2 \varepsilon) - 2\gamma}(1 + O(\varepsilon))$, with γ the Euler constant as in (2.5).*

We see then that in the language of the introduction a negative eigenvalue corresponding to a rotationally symmetric eigenfunction (that is, one in the $\ell = 0$ mode) does not persist.

Proof. From Lemma 9, we see that $J_1(\rho a_0) = 0$. Thus, we use Taylor polynomials to expand $Q_0(\lambda, a_\varepsilon)$ for small $|\lambda|$ and $|\varepsilon|$ with $J_1(\rho a_0) = 0$. As preliminaries, note that

$$\mu(\lambda, a_\varepsilon) = a_0 + \frac{\lambda^2 - \varepsilon}{2a_0} - \frac{1}{8a_0^3}(\lambda^2 - \varepsilon)^2 + O((\lambda^2 - \varepsilon)^3),$$

and, using $J_1(\rho a_0) = 0$, (2.3) and the Bessel differential equation (2.2),

$$\frac{d}{d\mu}(\mu J_1(\rho\mu)) \restriction_{\mu = a_0} = a_0 \rho J_1'(\rho a_0) = a_0 \rho J_0(\rho a_0),$$
$$\frac{d^2}{d\mu^2}(\mu J_1(\rho\mu)) \restriction_{\mu = a_0} = 2\rho J_1'(\rho a_0) + a_0 \rho^2 J_1''(\rho a_0) = \rho J_0(\rho a_0).$$

We note that here and below all errors are uniform in $|\arg\lambda| \le \frac{\pi}{2}$. Using these to expand $\mu J_1(\rho\mu)$ in λ and ε yields

$$\begin{aligned}\mu J_1(\rho\mu) &= a_0\rho J_0(\rho a_0)\Big(\frac{\lambda^2-\varepsilon}{2a_0} - \frac{(\lambda^2-\varepsilon)^2}{8a_0^3}\Big)\\ &\quad + \frac{\rho}{2}J_0(\rho a_0)\Big(\frac{\lambda^2-\varepsilon}{2a_0}\Big)^2 + O((\lambda^2-\varepsilon)^3)\\ &= -\frac{1}{2}J_0(\rho a_0)\rho\varepsilon + O((\lambda^2-\varepsilon)^3).\end{aligned}$$

This, along with the small λ asymptotics of $H_0^{(1)}(\rho\lambda)$ and $H_1^{(1)}(\rho\lambda)$ from (2.1) and (2.5), gives

$$\begin{aligned}Q_0(\lambda, a_\varepsilon) &= \Big(\frac{1}{2}J_0(\rho a_0)\rho\varepsilon + O((\lambda^2-\varepsilon)^3)\Big)\\ &\qquad \times \Big(1 + \frac{2i}{\pi}\log\Big(\frac{\lambda\rho}{2}\Big) + \frac{2i}{\pi}\gamma + O(\lambda^2\log\lambda)\Big)\\ &\quad + i\lambda\Big(J_0(\rho a_0) - J_0'(\rho a_0)\frac{\rho\varepsilon}{2a_0} + O(\lambda^2) + O(\varepsilon^2)\Big)\\ &\qquad \times \Big(-\frac{2}{\pi\lambda\rho} + O(\lambda\log\lambda)\Big).\end{aligned}$$

Using (2.3), $J_0'(\rho a_0) = 0$. Now, set

$$g(\lambda, a_0, \varepsilon) = J_0(\rho a_0)\Big(\frac{\rho}{2}\varepsilon\Big(1 + \frac{2i}{\pi}\log\Big(\frac{\lambda\rho}{2}\Big) + \frac{2i}{\pi}\gamma\Big) - \frac{2i}{\pi\rho}\Big)$$

which is chosen so that

$$Q_0(\lambda, a_\varepsilon) = g(\lambda, a_0, \varepsilon) + O(\lambda^2\log\lambda) + O(\varepsilon^3\log\lambda) + O(\varepsilon^2). \tag{3.7}$$

Set $\tilde\lambda_\varepsilon = \frac{2}{\rho}\exp(\frac{2}{\rho^2\varepsilon} - \gamma + \frac{i\pi}{2})$ and note that

$$g(\lambda, a_0, \varepsilon) = 0 \iff \lambda = \tilde\lambda_\varepsilon. \tag{3.8}$$

We have $\lim_{\varepsilon\uparrow 0}\tilde\lambda_\varepsilon = 0$ but $\lim_{\varepsilon\downarrow 0}|\tilde\lambda_\varepsilon| = \infty$.

First consider $\varepsilon < 0$. For $w \in \mathbb{C}$ with $|w|$ small,

$$g(\tilde\lambda_\varepsilon(1+w), a_0, \varepsilon) = iJ_0(\rho a_0)\frac{\rho}{\pi}w\varepsilon(1 + O(w)). \tag{3.9}$$

Since $\varepsilon\log\tilde\lambda_\varepsilon = O(1)$ and $\tilde\lambda_\varepsilon = O(e^{2/(\rho^2\varepsilon)})$, for $\varepsilon < 0$ with $|\varepsilon|$ sufficiently small we can apply Rouché's theorem to the pair $Q_0(\lambda, a_\varepsilon)$ and $g(\lambda, a_0, \varepsilon)$ on the circle with center $\tilde\lambda_\varepsilon$ and radius $|\tilde\lambda_\varepsilon||\varepsilon|^{1/2}$ to see that there is an $\varepsilon_0 > 0$ so that $Q_0(\lambda, a_\varepsilon)$ has a single simple zero at λ_ε satisfying $\lambda_\varepsilon - \tilde{\lambda}_\varepsilon = O(\varepsilon^{1/2}\tilde{\lambda}_\varepsilon)$ when $-\varepsilon_0 < \varepsilon < 0$. Writing $\lambda_\varepsilon = \tilde{\lambda}_\varepsilon(1 + f)$, using (3.7) and (3.9) yields

$$\begin{aligned} 0 = Q_0(\lambda_\varepsilon, a_0, \varepsilon) &= g(\tilde{\lambda}_\varepsilon(1 + f), a_0, \varepsilon) + O(\varepsilon^2) \\ &= \varepsilon J_0(\rho a_0)\frac{i\rho}{\pi} f + O(\varepsilon f^2) + O(\varepsilon^2). \end{aligned}$$

Since $f = O(\varepsilon^{1/2})$, this shows $f = O(\varepsilon)$. Since $0 < \arg \lambda_\varepsilon < \pi$, λ_ε^2 is an eigenvalue of $-\Delta - a_\varepsilon^2 \mathbb{1}_\rho$.

If $\varepsilon > 0$ and $|\arg \lambda| \leq \frac{\pi}{2}$, then

$$|g(\lambda, a_0, \varepsilon)| \geq |J_0(\rho a_0)|\Big(\frac{2}{\pi\rho} - \varepsilon\frac{\rho}{\pi}\operatorname{Re}\log\lambda\Big) + O(\varepsilon),$$

so that

$$|Q_0(\lambda, a_\varepsilon)| \geq |J_0(\rho a_0)|\Big(\frac{2}{\pi\rho} - \varepsilon\frac{\rho}{\pi}\operatorname{Re}\log\lambda\Big) + O(\varepsilon) + O(\varepsilon^3\log\lambda) + O(\lambda^2\log\lambda).$$

Hence, shrinking ε_0 if necessary, there is $\delta_0 > 0$ so that $Q_0(\lambda, a_\varepsilon) \neq 0$ for $0 < \varepsilon < \varepsilon_0$ and $\{\lambda : |\lambda| < \delta_0,\ |\arg\lambda| \leq \frac{\pi}{2}\}$, giving a resonance-free region. ■

We remark that the only place in which the assumption on the eigenvalue E_ε and eigenfunction of $-\Delta - a_\varepsilon^2\mathbb{1}_\rho$ was used was to show $J_1(\rho a_0) = 0$. Starting with that assumption, the same proof shows that $-\Delta - a_\varepsilon^2\mathbb{1}_\rho$ has an eigenvalue increasing to 0 as $\varepsilon \uparrow 0$, with expansion as given. This is also true (with obvious modifications) for the proofs of Lemmas 11 and 12.

Next we turn to the case corresponding to $-\Delta + V_0$ having eigenvalue 0. Here we will focus on eigenvalues and on resonances in the fourth quadrant, but in Section 3.2 we will show that for ε having either sign there are many more resonances related to this eigenvalue at 0.

Lemma 11. *Let $|\ell| \geq 2$, and suppose for $-\varepsilon_1 < \varepsilon < 0$ there exist nontrivial $\psi_\varepsilon \in L^2(\mathbb{R}^2)$, $\partial_\theta\psi_\varepsilon = i\ell\psi_\varepsilon$ with $(-\Delta - a_\varepsilon^2\mathbb{1}_\rho - E_\varepsilon)\psi_\varepsilon = 0$, and $E_\varepsilon \uparrow 0$ as $\varepsilon \uparrow 0$. Then there is an $\varepsilon_0 > 0$ so that for $0 < |\varepsilon| < \varepsilon_0$, $-\Delta - a_\varepsilon^2\mathbb{1}_\rho$ has a resonance satisfying $\lambda = \sqrt{\frac{\varepsilon(|\ell|-1)}{|\ell|}} + O(\varepsilon^{3/2}(1 + \delta_{|\ell|,2}\log|\varepsilon|))$. Moreover, as $\varepsilon \uparrow 0$, $E_\varepsilon = \frac{|\ell|-1}{|\ell|}\varepsilon + O(\varepsilon^2(1 + \delta_{|\ell|,2}\log|\varepsilon|))$.*

In the language of the introduction, in this case the eigenvalue persists. For $\varepsilon > 0$ the resonance is near the positive real axis, with negative imaginary part by Lemma 5.

Proof. We give the proof for $\ell \geq 2$, as the result for $-\ell$ follows from that for ℓ.

By Lemma 8, we have $J_{\ell-1}(\rho a_0) = 0$. As in the proof of Lemma 10 we expand Q_ℓ for λ and ε near 0. By (2.3) and (2.1), $J'_{\ell-1}(\rho a_0) = -J_\ell(\rho a_0)$ and $H_\ell^{(1)}(\rho\lambda) =$

$iY_\ell(\rho\lambda) + O(\lambda^\ell)$. Using this, Taylor expanding μ and $J_{\ell-1}(\mu\rho)$, and using the expansions (2.5) for Y_ℓ, $Y_{\ell-1}$ yields

$$\begin{aligned} Q_\ell(\lambda, a_\varepsilon) = -\frac{i}{\pi}\Big\{ &\Big(-J_\ell(\rho a_0) a_0 \rho \frac{\lambda^2 - \varepsilon}{2a_0} + O((\lambda^2-\varepsilon)^2)\Big) \\ &\times \Big(\Big(\frac{\rho\lambda}{2}\Big)^{-\ell}(\ell-1)! + O(\lambda^{-\ell+2})\Big) \\ &- \lambda\Big(J_\ell(\rho a_0) + O(\lambda^2 - \varepsilon)\Big) \\ &\times \Big(\Big(\frac{\rho\lambda}{2}\Big)^{-\ell+1}(\ell-2)! + O(\lambda^{-\ell+3}(1 + \delta_{\ell,2}\log\lambda))\Big)\Big\}. \end{aligned} \tag{3.10}$$

Now, set

$$g(\lambda, a_0, \varepsilon) = \frac{i\rho}{2\pi}\Big(\frac{\rho\lambda}{2}\Big)^{-\ell}(\ell-2)! J_\ell(\rho a_0)\{(\lambda^2-\varepsilon)(\ell-1) + \lambda^2\} \tag{3.11}$$

which is chosen so that

$$Q_\ell(\lambda, a_\varepsilon) = g(\lambda, a_0, \varepsilon) + O(\lambda^{-\ell+4}(1 + \delta_{2,\ell}\log\lambda)) + O(\varepsilon^2\lambda^{-\ell}) + O(\varepsilon\lambda^{-\ell+2}). \tag{3.12}$$

Set $\tilde\lambda_\varepsilon = \sqrt{\frac{\varepsilon(\ell-1)}{\ell}}$, with the branch of square root taken such that $\operatorname{Re}\tilde\lambda_\varepsilon \geq 0$ and $\operatorname{Im}\tilde\lambda_\varepsilon \geq 0$, and note that $g(\lambda, a_0, \varepsilon) = 0$ if and only if $\lambda = \pm\tilde\lambda_\varepsilon$. There is a $C_0 > 0$ so that for $|w|$ small, $|g(\pm\tilde\lambda_\varepsilon + w, a_0, \varepsilon)| \geq C_0|\tilde\lambda_\varepsilon|^{-\ell+1}|w| + O(|w|^2|\tilde\lambda_\varepsilon|^{-\ell})$. Since $|\tilde\lambda_\varepsilon| \propto |\varepsilon|^{1/2}$, we can ensure that for $|\varepsilon|$ sufficiently small

$$|Q_\ell(\tilde\lambda_\varepsilon + \varepsilon z, a_\varepsilon) - g(\tilde\lambda_\varepsilon + \varepsilon z, a_0, \varepsilon)| < |g(\tilde\lambda_\varepsilon + \varepsilon z, a_0, \varepsilon)|, \quad \text{for } |z| = 1.$$

Hence, an application of Rouché's theorem on the circle centered at $\tilde\lambda_\varepsilon$ with radius $|\varepsilon|$ shows that there is an $\varepsilon_0 > 0$ so that for $0 < |\varepsilon| < \varepsilon_0$, $Q_\ell(\lambda, a_\varepsilon)$ has a single simple zero λ_ε within $|\varepsilon|$ of $\tilde\lambda_\varepsilon$. Note that for $\varepsilon \neq 0$ small enough the disk enclosed by this circle stays away from $\lambda = 0$. This zero of Q_ℓ at λ_ε is a resonance of $-\Delta + V_\varepsilon$.

We describe how to improve the error using (3.11) and (3.12). Write $\lambda_\varepsilon = \tilde\lambda_\varepsilon + f_\varepsilon$. Then

$$\begin{aligned} 0 &= \lambda_\varepsilon^\ell Q_\ell(\lambda_\varepsilon, a_\varepsilon) \\ &= \frac{i\rho}{\pi}\Big(\frac{\rho}{2}\Big)^{-\ell}(\ell-2)! J_\ell(\rho a_0)\ell\tilde\lambda_\varepsilon f_\varepsilon + O(f_\varepsilon^2) + O(\varepsilon^2(1 + \delta_{\ell,2}\log\varepsilon)), \end{aligned}$$

using $|\tilde\lambda_\varepsilon| \propto |\varepsilon|^{1/2}$. Since $f_\varepsilon = O(\varepsilon)$, solving for f_ε yields

$$f_\varepsilon = O(\varepsilon^{3/2}(1 + \delta_{2,\ell}\log\varepsilon)).$$

For $\varepsilon < 0$, $\lambda_\varepsilon \in i(0,\infty)$, and hence λ_ε^2 is a negative eigenvalue of $-\Delta - a_\varepsilon^2 \mathbb{1}_\rho$. ∎

We now turn to a second kind of persistent eigenvalue.

Lemma 12. *Suppose that for* $-\varepsilon_1 < \varepsilon < 0$ *there exist nontrivial* $\psi_\varepsilon \in L^2(\mathbb{R}^2)$*,* $\partial_\theta \psi_\varepsilon(r,\theta) = \pm i \psi_\varepsilon$*, with* $(-\Delta - a_\varepsilon^2 \mathbb{1}_\rho - E_\varepsilon)\psi_\varepsilon = 0$*, and* $E_\varepsilon \uparrow 0$ *as* $\varepsilon \uparrow 0$*. Then there is an* $\varepsilon_0 > 0$ *so that, for* $0 < |\varepsilon| < \varepsilon_0$*,* $-\Delta - a_\varepsilon^2 \mathbb{1}_\rho$ *has a resonance satisfying*

$$\lambda = \frac{2i}{\rho} \exp\Big(\frac{1}{2} - \gamma + \frac{W_{-1}\big(\frac{\varepsilon\rho^2 e^{-1+2\gamma}}{4}\big)}{2}\Big) + o(|\varepsilon|^{3/2}).$$

Moreover, as $\varepsilon \uparrow 0$

$$\begin{aligned} E_\varepsilon &= -\frac{4}{\rho^2} \exp\Big(1 - 2\gamma + W_{-1}\Big(\frac{\varepsilon}{4}\rho^2 e^{-1+2\gamma}\Big)\Big) + o(\varepsilon^2) \\ &= \frac{-\varepsilon}{W_{-1}\big(\varepsilon\rho^2 \frac{e^{-1+2\gamma}}{4}\big)} + o(\varepsilon^2). \end{aligned}$$

Proof. We give the proof for $\ell = 1$, as the case $\ell = -1$ then follows. The difference between this case and that of Lemma 11, with $\ell \geq 2$, is that the leading term in the expansion of $H_0^{(1)}(\lambda) = H_{\ell-1}^{(1)}(\lambda)$ behaves like $\log \lambda$ rather than $\lambda^{-\ell-1}$. Using $J_0(\rho a_0) = 0$ from Lemma 8 gives (compare (3.10))

$$\begin{aligned} Q_1(\lambda, a_\varepsilon) = i\Big(&-J_1(\rho a_0) a_0 \rho \frac{\lambda^2 - \varepsilon}{2a_0} + O((\lambda^2 - \varepsilon)^2)\Big) \\ &\times \Big(-\frac{1}{\pi}\Big(\frac{\rho\lambda}{2}\Big)^{-1} + O(\lambda \log \lambda)\Big) \\ &- \lambda(J_1(\rho a_0) + O(\lambda^2 - \varepsilon)) \\ &\times \Big(1 + \frac{2i}{\pi} \log\Big(\frac{\lambda\rho}{2}\Big) + \frac{2i}{\pi}\gamma + O(\lambda^2 \log \lambda)\Big). \end{aligned}$$

Set

$$g(\lambda, a_0, \varepsilon) = \frac{2\lambda}{i\pi} J_1(\rho a_0)\Big(\log\Big(\frac{\lambda\rho}{2}\Big) - \frac{1}{2} + \frac{\pi}{2i} + \gamma + \frac{\varepsilon}{2\lambda^2}\Big) \tag{3.13}$$

so that

$$Q_1(\lambda, a_\varepsilon) = g(\lambda, a_0, \varepsilon) + O(\lambda^3 \log \lambda) + O(\varepsilon^2 \lambda^{-1}) + O(\varepsilon\lambda \log \lambda). \tag{3.14}$$

Set

$$b = -\log\Big(\frac{\rho}{2}\Big) + \frac{1}{2} + i\frac{\pi}{2} - \gamma \quad \text{and} \quad \tau = \log \lambda - b.$$

Then λ satisfies $g(\lambda, a_0, \varepsilon) = 0$ if and only if $2\tau e^{2\tau} = -\varepsilon e^{-2b}$. Solutions of this equation are given by

$$\tau = \frac{1}{2} W_n(-\varepsilon e^{-2b}) = \frac{1}{2} W_n\Big(\varepsilon \frac{\rho^2}{4} e^{-1+2\gamma}\Big).$$

The case $n = 0$ is not relevant for us, as $\lim_{\varepsilon\to 0} W_0(\varepsilon) = 0$. We want $-\frac{\pi}{2} < \operatorname{Im}\log\lambda = \operatorname{Im}\tau + \frac{\pi}{2} \leq \frac{\pi}{2}$ when ε is near 0. By Lemma 4, this holds if and only if $n = -1$. Set

$$\tilde{\lambda}_\varepsilon = \frac{2i}{\rho}\exp\Big(\frac{1}{2} - \gamma + \frac{1}{2}W_{-1}\Big(\varepsilon\frac{\rho^2}{4}e^{-1+2\gamma}\Big)\Big);$$

then $g(\tilde{\lambda}_\varepsilon, a_0, \varepsilon) = 0$ and $\tilde{\lambda}_\varepsilon \to 0$ as $\varepsilon \to 0$ by Lemma 4. We use $g'(\tilde{\lambda}_\varepsilon) \propto \log\tilde{\lambda}_\varepsilon$ to apply Rouché's theorem on the disk with center $\tilde{\lambda}_\varepsilon$ and radius $|\varepsilon|^{1/2}|\tilde{\lambda}_\varepsilon|$ to the pair Q_1 and g. This shows that for $|\varepsilon|$ sufficiently small $Q_1(\lambda, a_\varepsilon)$ has a zero at λ_ε with $|\lambda_\varepsilon - \tilde{\lambda}_\varepsilon| < |\varepsilon|^{1/2}|\tilde{\lambda}_\varepsilon|$. Using (3.13) and (3.14) allows us to improve this to an error of size $o(|\varepsilon|^{3/2})$.

When $\varepsilon < 0$, $\arg\lambda_\varepsilon = \frac{\pi}{2}$, and λ_ε^2 is an eigenvalue. When $\varepsilon > 0$, $-\frac{\pi}{2} < \arg\lambda_\varepsilon < 0$ and λ_ε is near the positive real axis and is a resonance. ∎

Lemma 15 shows that for any $n \in \mathbb{Z}\setminus\{0\}$, $\frac{2i}{\rho}\exp\big(\frac{1}{2} - \gamma + \frac{1}{2}W_n(\varepsilon\frac{\rho^2}{4}e^{-1+2\gamma})\big)$ gives the approximate location of a resonance on Λ for sufficiently small $|\varepsilon|$.

3.2. More general potentials and resonances on the Riemann surface of the logarithm

In this section we turn to more general Schrödinger operators and consider resonances and eigenvalues near 0 but with the resonances on Λ rather than on $\mathbb{C}\setminus -i[0,\infty)$; this means allowing more general values of $\arg\lambda$ than we did in Section 3.1.

Lemmas 14 and 15 show that if $-\Delta + V_0$ has either a p-resonance or an eigenvalue at 0, then for small $|\varepsilon| \neq 0$, $-\Delta + V_\varepsilon$ has many nearby resonances. Figure 6 illustrates some resonances for a family of circular wells $V_\varepsilon(r) = (-j_{0,1}^2 + \varepsilon)\mathbb{1}_1(r)$. The subset of Λ shown is larger than that shown in the illustrations of the introduction. Here the branch cut is the brown ray in the second quadrant, so that the string of resonances shown in the second quadrant (for varying values of ε) has argument between $-\frac{4\pi}{3}$ and $-\pi$. The color-coding here indicates the sign of ε. These resonances correspond to the $\ell = 1$ (or $\ell = -1$) mode and are the ones approximated in Lemma 15.

Recall that on Λ we do not identify points with arguments that differ by a multiple of 2π.

In this section we show that by using results about the structure of $R_{V_0}(\lambda)$ near $\lambda = 0$ we can obtain similar, but more general results, to those obtained in Section 3.1. In addition to allowing more general potentials V_ε, we also show that in the case of the persistent eigenvalues there are, for small $|\varepsilon|$, many associated resonances on Λ. For simplicity, we confine ourselves to the case of radial potentials. However, at the end of this section we indicate how our proofs work in more generality, provided the singularity of the resolvent R_{V_0} at $\lambda = 0$ has rank one.

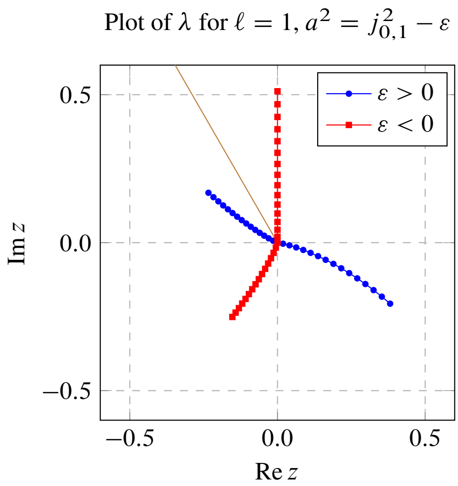

Figure 6. Plot of λ_ε in a sheet of Λ cut along the brown ray, with arguments ranging from $-\frac{4\pi}{3}$ to $\frac{2\pi}{3}$, values of ε as in Figure 1.

We make the following assumptions throughout this section.

Assumption 1. *There is an $\varepsilon_* > 0$ such that*

$$(-\varepsilon_*, \varepsilon_*) \ni \varepsilon \mapsto V_\varepsilon \in L_c^\infty(\mathbb{R}^2; \mathbb{R})$$

is C^∞ in ε, and V_ε is a radial potential for each ε.

These potentials are more general than those of Theorem 2, since there is no assumption on the sign of V_0. Our replacement is that we shall need a certain integral to be nonzero. Denote

$$\partial_\varepsilon V_\varepsilon = \dot{V}_\varepsilon, \quad \dot{V}_0 = \dot{V}_\varepsilon \restriction_{\varepsilon=0}, \quad \ddot{V}_0 = \partial_\varepsilon^2 V_\varepsilon \restriction_{\varepsilon=0},$$

and note that in the circular well example considered above in (3.6) we have $\dot{V}_0(x) = \mathbb{1}_\rho(|x|)$ and $\ddot{V}_0 = 0$.

It is easy to see using (3.15) that if $\|\chi R_{V_0}(i\kappa)\chi\|$ is bounded as $\kappa \downarrow 0$ for all $\chi \in C_c^\infty(\mathbb{R}^2)$, then there are $\varepsilon_0, \delta_0 > 0$ so that for $|\varepsilon| < \varepsilon_0$, $-\Delta + V_\varepsilon$ has no negative eigenvalues with norm less than δ_0^2. It is well known (see, e.g., [7, Theorem 1] or [5]) that if there is a $\chi \in C_c^\infty(\mathbb{R}^2)$ so that $\lim_{|\kappa\downarrow 0} \|\chi R_{V_0}(i\kappa)\chi\| = \infty$ then 0 must be an eigenvalue, an s-resonance, or a p-resonance of $-\Delta + V_0$. Hence, our hypotheses for each lemma will involve the existence of a certain kind of zero resonance or a zero eigenvalue of $-\Delta + V_0$; compare Lemma 9. A consequence of each of Lemmas 13, 14, and 15 is that then $-\Delta + V_\varepsilon$ has negative eigenvalue approaching 0 as ε goes to 0 from one direction. These observations together with Lemmas 14 and 15 prove Theorem 2. In fact, they prove much more: for $\varepsilon \neq 0$ and $|\varepsilon|$ sufficiently small, there are many resonances related to the negative eigenvalue. Combining this with Lemma 13 proves Theorem 1.

Choose $\chi \in C_c^\infty(\mathbb{R}^2; [0,1])$ radial and such that $\chi V = V$, and note that

$$(-\Delta + V_\varepsilon - \lambda^2) R_{V_0}(\lambda)\chi = \chi(I + (V_\varepsilon - V_0)R_{V_0}(\lambda)\chi).$$

Hence, for $\operatorname{Im}\lambda > 0$,

$$R_{V_\varepsilon}(\lambda)\chi = R_{V_0}(\lambda)\chi(I + (V_\varepsilon - V_0)R_{V_0}(\lambda)\chi)^{-1}, \tag{3.15}$$

and the identity extends to Λ by meromorphic continuation. As in Section 2.3, we have the following version of the Birman–Schwinger principle: R_{V_ε} has a pole at λ, a regular point of R_{V_0}, if and only if -1 is an eigenvalue of $(V_\varepsilon - V_0)R_{V_0}(\lambda)\chi$.

For $\ell \in \mathbb{Z}$, define $\mathcal{P}_\ell \colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ by

$$(\mathcal{P}_\ell f)(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(r,\theta')e^{-i\ell\theta'}\,d\theta' e^{i\ell\theta}.$$

Since V_ε is radial, (3.15) yields another Birman–Schwinger-type principle: $R_{V_\varepsilon}\mathcal{P}_\ell$ has a pole at λ, a regular point of $R_{V_0}\mathcal{P}_\ell$, if and only if -1 is an eigenvalue of $(V_\varepsilon - V_0)R_{V_0}(\lambda)\mathcal{P}_\ell\chi$. We will use expansions of $R_{V_0}(\lambda)$ from [7] to iterate this argument and replace $(V_\varepsilon - V_0)R_{V_0}(\lambda)\mathcal{P}_\ell\chi$ by a suitable rank one operator $A(\lambda,\varepsilon)$, and study the zeros of $\det(I + A(\lambda,\varepsilon))$.

Our first result in this setting parallels Lemma 10.

Lemma 13. *Let V_ε satisfy Assumption 1. Suppose $-\Delta + V_0$ has an s-resonance with corresponding resonant state $\psi(r,\theta)$ satisfying $\psi(r,\theta) = \frac{1}{\sqrt{2\pi}}$ for r sufficiently large. Suppose $\alpha_0 = \int_{\mathbb{R}^2} \dot V_0(r)|\psi(r)|^2 dx \neq 0$. There is an $\varepsilon_0 > 0$ such that, if $0 < -\alpha_0\varepsilon < |\alpha_0|\varepsilon_0$, then $-\Delta + V_\varepsilon$ has an eigenvalue at $-e^{2/(\varepsilon\alpha_0) - 2\alpha_1/\alpha_0^2}(1 + O(\varepsilon))$ for some $\alpha_1 \in \mathbb{R}$, and the corresponding eigenfunction is rotationally symmetric. Moreover, given $\varphi > 0$, there are $\varepsilon_0'(\varphi) > 0$, $\lambda_0 > 0$ such that, if $0 < \alpha_0\varepsilon < |\alpha_0|\varepsilon_0'$, then for any $\tilde\chi \in C_c^\infty(\mathbb{R}^2)$, $\tilde\chi R_{V_\varepsilon}(\lambda)\mathcal{P}_0\tilde\chi$ is bounded for $|\arg\lambda| < \varphi$, $|\lambda| < \lambda_0$.*

The proof of the lemma gives an expression for α_1. Likewise, the proofs of Lemmas 14 and 15 give expressions for the constants α_1 appearing in their statements.

Proof. By the Birman–Schwinger principle discussed above, $R_{V_\varepsilon}\mathcal{P}_0$ has a pole at λ, a regular point of $R_{V_0}\mathcal{P}_0$, if and only if -1 is an eigenvalue of $(V_\varepsilon - V_0)R_{V_0}(\lambda)\mathcal{P}_0\chi$. From the proof of [7, Theorem 2], since $-\Delta + V_0$ has an s-resonance there is an operator $B_{00}\colon L_c^2 \to H_{\mathrm{loc}}^2$ such that

$$\chi R_{V_0}(\lambda)\mathcal{P}_0\chi = -\chi\psi \otimes \psi\chi \log\lambda + \chi B_{00}\mathcal{P}_0\chi + O(\lambda^2(\log\lambda)^3), \tag{3.16}$$

where $u \otimes v$ denotes the operator taking w to $\langle w, v\rangle u$ and $\chi \in C_c^\infty(\mathbb{R}^2)$ satisfies $\chi V = V$. Set

$$\widetilde{R}_{V_0}(\lambda) = R_{V_0}(\lambda) + \psi \otimes \psi\Big(\log\lambda - i\frac{\pi}{2}\Big).$$

The operator $\widetilde{R}_{V_0}(\lambda)\mathcal{P}_0\colon L^2_c(\mathbb{R}^2) \to L^2_{\mathrm{loc}}(\mathbb{R}^2)$ is bounded at $\lambda = 0$, and is chosen to make it easy to see that the constant α_1 below is real. For $\varphi > 0$ there are $\lambda_0(\varphi)$, $\varepsilon_0'(\varphi) > 0$ such that in the region $|\arg\lambda| < \varphi$, $|\lambda| < \lambda_0$, $|\varepsilon| < \varepsilon_0'$,

$$I + (V_\varepsilon - V_0)\widetilde{R}_{V_0}(\lambda)\mathcal{P}_0\chi$$

is invertible, yielding

$$\big(I + (V_\varepsilon - V_0)R_{V_0}\mathcal{P}_0\chi\big)\big(I + (V_\varepsilon - V_0)\widetilde{R}_{V_0}\mathcal{P}_0\chi\big)^{-1} = I + A(\lambda,\varepsilon),$$

where

$$A(\lambda,\varepsilon) := -(V_\varepsilon - V_0)\psi \otimes \psi\chi\Big(\log\lambda - i\frac{\pi}{2}\Big)\big(I + (V_\varepsilon - V_0)\widetilde{R}_{V_0}\mathcal{P}_0\chi\big)^{-1}.$$

Thus, the poles of $R_{V_\varepsilon}(\lambda)\mathcal{P}_0$ for $\lambda,\ \varepsilon$ as described above equal the zeros of

$$Q(\lambda,\varepsilon) := \det(I + A(\lambda,\varepsilon)).$$

Using $\det(I + u \otimes vT) = 1 + \operatorname{tr} u \otimes vT = 1 + \langle Tu, v\rangle$ gives

$$Q(\lambda,\varepsilon) = 1 - \varepsilon\Big(\log\lambda - i\frac{\pi}{2}\Big)\big(\alpha_0 + \alpha_1\varepsilon + O(\varepsilon^2) + O(\varepsilon\lambda^2(\log\lambda)^3)\big), \qquad (3.17)$$

where

$$\alpha_1 = -\Big\langle\Big(-\frac{i\pi}{2}\psi \otimes \psi + B_{00}\Big)\dot{V}_0\psi, \dot{V}_0\psi\Big\rangle + \frac{1}{2}\langle\ddot{V}_0\psi, \psi\rangle.$$

Since for $\kappa > 0$ and $\chi \in C_c^\infty(\mathbb{R}^2;\mathbb{R})$, the operator $\chi R_{V_0}(i\kappa)\chi$ is self-adjoint we find from the expansion (3.16) that $\chi\big(-\frac{i\pi}{2}\psi \otimes \psi + B_{00}\big)\chi$ is self-adjoint, implying that α_1 is real.

The rest of the proof closely resembles that of Lemma 10. Suppose first that $\alpha_0\varepsilon > 0$. Then, since $\operatorname{Re}\log\lambda < 0$ for λ near 0, it is easy to see from (3.17) that, shrinking ε_0', λ_0 if necessary, $Q(\lambda,\varepsilon)$ has no zeros in $|\arg\lambda| < \varphi$, $|\lambda| < \lambda_0$, $0 < \alpha_0\varepsilon < |\alpha_0|\varepsilon_0'$, and hence the cutoff resolvent is bounded.

Now, consider the case $\alpha_0\varepsilon < 0$. Define

$$g(\lambda,\varepsilon) := 1 - \varepsilon\Big(\log\lambda - i\frac{\pi}{2}\Big)(\alpha_0 + \alpha_1\varepsilon),$$

and note

$$g(\lambda,\varepsilon) = 0 \iff \lambda = \tilde{\lambda}_\varepsilon,$$

where

$$\log \tilde{\lambda}_\varepsilon = i\frac{\pi}{2} + \frac{1}{\varepsilon\alpha_0 + \varepsilon^2\alpha_1}.$$

Thus, $|\tilde{\lambda}_\varepsilon| = e^{1/\alpha_0\varepsilon}e^{-\alpha_1/\alpha_0^2}(1 + O(\varepsilon))$ and $\arg\tilde{\lambda}_\varepsilon = \frac{\pi}{2}$. Apply Rouché's theorem on the circle $|\lambda - \tilde{\lambda}_\varepsilon| = |\tilde{\lambda}_\varepsilon||\varepsilon|^{1/2}$ to see that, if $0 < -\alpha_0\varepsilon$ is sufficiently small, then $Q(\lambda, \varepsilon)$ has a zero at a single point λ_ε satisfying $\log\lambda_\varepsilon = \log\tilde{\lambda}_\varepsilon + O(\varepsilon^{1/2})$. This zero is a resonance of $-\Delta + V_\varepsilon$. Using the expansion (3.17) improves the error to $\log\lambda_\varepsilon = \log\tilde{\lambda}_\varepsilon + O(\varepsilon^2)$. Since λ_ε is in the physical space $\{\lambda \in \Lambda : 0 < \arg\lambda < \pi\}$, λ_ε^2 corresponds to an eigenvalue of $-\Delta + V_\varepsilon$. ∎

To compare this result with Lemma 10, we note that if $V_0(r) = -a_0^2\mathbb{1}_\rho(r)$, with $J_1(a_0\rho) = 0$ and $\dot{V}_0 = \mathbb{1}_\rho$, then the ψ of Lemma 13 is

$$\psi(r) = (\sqrt{2\pi}J_0(\rho a_0))^{-1}J_0(ra_0) \quad \text{for } 0 < r < \rho,$$

and, by [26, (10.22.7)], $\alpha_0 = \frac{\rho^2}{2}$.

The next lemma extends Lemma 11. In addition to allowing more general potentials, it shows that on Λ there are many resonances near 0 for both positive and negative values of ε.

Lemma 14. *Suppose V_ε satisfies Assumption 1, $-\Delta + V_0$ has an eigenvalue at 0, with a corresponding eigenfunction ψ, $\partial_\theta\psi = i\ell\psi$, $|\ell| \geq 2$, and $\|\psi\|_{L^2(\mathbb{R}^2)} = 1$. Use the notation*

$$O'_m(\varepsilon, \ell) = \delta_{|\ell|,2}O_m(\varepsilon\log|\varepsilon|) + O_m(|\varepsilon|^{2-\delta})$$

for arbitrary $\delta > 0$. If $\alpha_0 := \int_{\mathbb{R}^2}|\psi(r,\theta)|^2\dot{V}_0(r)dx \neq 0$, then there is an $\alpha_1 \in \mathbb{R}$ such that, for each $m \in \mathbb{Z}$, there is an $\varepsilon_0 = \varepsilon_0(m) > 0$ such that if $0 < |\varepsilon| < \varepsilon_0$, then there is a resonance at $\lambda_\varepsilon(m)$ satisfying

$$|\lambda_\varepsilon(m)| = \sqrt{|\varepsilon\alpha_0 + \alpha_1\varepsilon^2|} + |\varepsilon|^{1/2}O'_m(\varepsilon, \ell)$$

and

$$\arg\lambda_\varepsilon(m) = \pi\left(m + \frac{1 - \operatorname{sgn}(\varepsilon\alpha_0)}{4}\right) + O'_m(\varepsilon, \ell)$$

where $\operatorname{sgn}(x) = \frac{x}{|x|}$. When $m = 0$ and $0 < -\alpha_0\varepsilon < |\alpha_0|\varepsilon_0$, the resonance corresponds to a negative eigenvalue of $-\Delta + V_\varepsilon$ at

$$\varepsilon\alpha_0 + \varepsilon^2\alpha_1 + O\big(\varepsilon^2(|\varepsilon|^{1-\delta} + \delta_{|\ell|,2}\log|\varepsilon|)\big).$$

In each case, the resonance or eigenvalue corresponds to a pole of $R_{V_\varepsilon}\mathcal{P}_\ell$.

By including the α_1 term, we see that the resonances lie quite close to $\arg\lambda = m\pi$ or $\arg\lambda = (m + \frac{1}{2})\pi$ when $|\ell| > 2$; compare the numerically computed resonances for $\ell = 2$ and $\ell = 3$ in Figure 1. If $|\ell| = 2$, we can take $\alpha_1 = 0$ since the error is larger than the correction made by including α_1.

Proof. The proof of this lemma begins in a manner similar to that of Lemma 13. We first consider the case $|\ell| > 2$. From the expansions of the resolvent from [7, Theorem 1] and using that the potential V_0 is rotationally symmetric, we see that if $|\ell| > 2$,

$$R_{V_0}(\lambda)\mathcal{P}_\ell = -\frac{1}{\lambda^2}\psi \otimes \psi + B_{0,0}\mathcal{P}_\ell + O(|\lambda|^{2-\delta}) \tag{3.18}$$

where $B_{0,0}\colon L^2_c(\mathbb{R}^2) \to H^2_{\mathrm{loc}}(\mathbb{R}^2)$ and the error is as an operator $L^2_c(\mathbb{R}^2) \to H^2_{\mathrm{loc}}(\mathbb{R}^2)$. The error is uniform in a finite range of $\arg\lambda$. By the Birman–Schwinger principle, R_{V_ε} has a pole at λ, a regular point of R_{V_0}, if and only if -1 is an eigenvalue of $(V_\varepsilon - V_0)R_{V_0}(\lambda)\mathcal{P}_\ell\chi$.

Set $\widetilde{R}_{V_0}(\lambda) = R_{V_0}(\lambda) + \lambda^{-2}\psi \otimes \psi$. For $\arg\lambda$ in a bounded set and $|\lambda|$, $|\varepsilon|$ sufficiently small, $\widetilde{R}_{V_0}(\lambda)\mathcal{P}_l$ is analytic away from $\lambda = 0$, bounded at $\lambda = 0$, and $I + (V_\varepsilon - V_0)\widetilde{R}_{V_0}(\lambda)\mathcal{P}_l\chi$ is invertible. Thus, as in the proof of Lemma 13, we reduce the problem to that of studying the zeros of

$$Q(\lambda,\varepsilon) := \det\Big(I - (V_\varepsilon - V_0)\frac{1}{\lambda^2}\psi \otimes \psi\big(I + (V_\varepsilon - V_0)\widetilde{R}_{V_0}\mathcal{P}_l\chi\big)^{-1}\Big). \tag{3.19}$$

Away from $\lambda = 0$, this is an analytic function of λ in this region. Using that we are taking the determinant of the identity plus a rank one operator as in Lemma 13, the expansion (3.18) shows that

$$Q(\lambda,\varepsilon) = 1 - \frac{\varepsilon}{\lambda^2}\big(\alpha_0 + \varepsilon\alpha_1 + O(\varepsilon^2) + O(\varepsilon|\lambda|^{2-\delta})\big), \tag{3.20}$$

where $\alpha_1 = -\langle B_{00}\dot{V}_0\psi, \dot{V}_0\psi\rangle + \frac{1}{2}\langle \ddot{V}_0\psi, \psi\rangle$. Because $B_{00}\mathcal{P}_\ell$ is self-adjoint, α_1 is real.

Set

$$g(\lambda,\varepsilon) = 1 - \frac{\varepsilon}{\lambda^2}(\alpha_0 + \varepsilon\alpha_1).$$

Fix $m \in \mathbb{Z}$, and let $\tilde{\lambda}_\varepsilon(m) \in \Lambda$ be the continuous function of ε satisfying $\tilde{\lambda}_\varepsilon^2(m) = \varepsilon\alpha_0 + \alpha_1\varepsilon^2$ (so that $g(\tilde{\lambda}_\varepsilon,\varepsilon) = 0$) and $\left|\arg\tilde{\lambda}_\varepsilon(m) - m\pi - \frac{\pi}{4}\right| \le \frac{\pi}{4}$. Notice that, if $\alpha_0\varepsilon > 0$ then $\arg\tilde{\lambda}_\varepsilon(m) = m\pi$, and if $\alpha_0\varepsilon < 0$, then $\arg\tilde{\lambda}_\varepsilon(m) = m\pi + \frac{\pi}{2}$ when $|\varepsilon|$ is sufficiently small. We identify $\left\{\lambda \in \Lambda : \left|\arg\lambda - \pi\left(m + \frac{1}{4}\right)\right| < \frac{\pi}{2}\right\}$ with a subset of $\mathbb{C}$. For $w \in \mathbb{C}$, $|w|$ small compared to $|\tilde{\lambda}_\varepsilon|$, $|(\tilde{\lambda}_\varepsilon + w)^2 g(\tilde{\lambda}_\varepsilon + w,\varepsilon)| \ge 2|\tilde{\lambda}_\varepsilon w| + O(|w|^2)$. Using this, (3.20), and Rouché's theorem applied to the pair $\lambda^2 Q$ and $\lambda^2 g$ shows that Q has a simple zero, and hence $-\Delta + V_\varepsilon$ has a resonance, at a point $\lambda_\varepsilon(m)$ satisfying $|\lambda_\varepsilon(m) - \tilde{\lambda}_\varepsilon(m)| = O(\varepsilon)$ when $|\varepsilon|$ is sufficiently small, depending on m. Using (3.20) allows us to improve the error to $O_m(|\varepsilon|^{5/2-\delta})$.

If $0 < -\alpha_0\varepsilon$ is sufficiently small, then $\arg(\lambda_\varepsilon(0)) = \frac{\pi}{2}$ and $-\Delta + V_\varepsilon$ has an eigenvalue at $(\lambda_\varepsilon(0))^2 = \varepsilon(\alpha_0 + \alpha_1\varepsilon) + O(|\varepsilon|^{3-\delta})$. This completes the proof for $|\ell| > 2$.

If $|\ell| = 2$, the expansion (3.18) must be replaced by

$$R_{V_0}(\lambda)\mathcal{P}_\ell = -\frac{1}{\lambda^2}\psi \otimes \psi + O(\log \lambda).$$

Again, for $|\lambda|$ and $|\varepsilon|$ sufficiently small, the poles of R_{V_ε} away from $\lambda = 0$ are given by the zeros of the function Q defined in (3.19), but the expansion (3.20) must be replaced by

$$Q(\lambda, \varepsilon) = 1 - \frac{\varepsilon}{\lambda^2}(\alpha_0 + O(\varepsilon \log \lambda))$$

and we take $\alpha_1 = 0$ for simplicity. The proof proceeds as in the $|\ell| > 2$ case, except that we use $g(\lambda, \varepsilon) = 1 - \varepsilon\frac{\alpha_0}{\lambda^2}$, $\tilde{\lambda}_\varepsilon(m)^2 = \varepsilon\alpha_0$, and get a worse error. ■

Suppose $\ell \geq 2$ and $V_\varepsilon = (-a_0^2 + \varepsilon)\mathbb{1}_\rho(r)$, with $J_{\ell-1}(a_0\rho) = 0$ so that $-\Delta + V_0$ has an eigenvalue at 0. Then, in the notation of Lemma 14, using [26, Section 10.22.38],

$$\psi_l(r, \theta) = \begin{cases} \sqrt{\dfrac{\ell - 1}{\pi\ell}}\dfrac{1}{\rho J_\ell(a_0\rho)}J_\ell(a_0 r)e^{i\ell\theta}, & 0 < r < \rho, \\ \sqrt{\dfrac{\ell - 1}{\pi\ell}}\dfrac{1}{\rho}\left(\dfrac{r}{\rho}\right)^{-\ell}e^{i\ell\theta}, & r \geq \rho, \end{cases}$$

and $\alpha_0 = \frac{\ell-1}{\ell}$.

The next lemma extends Lemma 12, and shows that not only does a family of negative eigenvalues in the $\ell = \pm 1$ mode persist, but that it corresponds to many resonances for both signs of ε. Since $\log \lambda$ is a natural coordinate to use on Λ and because $\log \lambda$ naturally arises in the proof of Lemma 15, we use $\log \lambda$ in the statement of the lemma.

Lemma 15. *Let V_ε satisfy Assumption 1. Suppose $-\Delta + V_0$ has a p wave resonance with corresponding resonant states $\psi_\pm$ satisfying $\psi_\pm(r, \theta) = \frac{e^{\pm i\theta}}{r\sqrt{2\pi}}$ for r sufficiently large, and define*

$$\alpha_0 := \int_{\mathbb{R}^2} \dot{V}_0(r)|\psi_\pm(r, \theta)|^2 dx$$

and

$$b = \log 2 - \gamma + i\frac{\pi}{2} + \lim_{\rho' \to \infty}\left(\int_{|x|<\rho'} |\psi_\pm|^2 dx - \log \rho'\right).$$

If $\alpha_0 \neq 0$, then for each $n \in \mathbb{Z} \setminus \{0\}$ there is an $\varepsilon_0 = \varepsilon_0(n) > 0$ so that if $0 < |\varepsilon| < \varepsilon_0$, $-\Delta + V_\varepsilon$ has a resonance at a point $\lambda_\varepsilon(n)$ satisfying

$$\log \lambda_\varepsilon(n) = b + \frac{1}{2}W_n\left(2\varepsilon(\alpha_0 + \alpha_1\varepsilon)e^{-2\,\mathrm{Re}\, b}\right) + o(|\varepsilon|).$$

When $\varepsilon\alpha_0 < 0$, the $n = -1$ resonance corresponds to $-\Delta + V_\varepsilon$ having a negative eigenvalue given by

$$(\lambda_\varepsilon(-1))^2 = e^{2b} \exp\bigl(W_{-1}(2\varepsilon(\alpha_0 + \alpha_1\varepsilon)e^{-2\operatorname{Re} b})\bigr)(1 + o(\varepsilon))$$
$$= -\frac{2\varepsilon(\alpha_0 + \alpha_1\varepsilon)(1 + o(\varepsilon))}{W_{-1}(2\varepsilon(\alpha_0 + \alpha_1\varepsilon)e^{-2\operatorname{Re} b})}.$$

In each case, $R_{V_\varepsilon}(\lambda)\mathcal{P}_{\pm 1}$ has a pole at $\lambda = \lambda_\varepsilon(n)$.

We note that, by Lemma 4, $\lim_{\alpha_0\varepsilon\uparrow 0} \operatorname{Im}\log\lambda_\varepsilon(n) = \bigl(\frac{\pi}{2}\bigr)(2 + 2n - \operatorname{sgn} n)$ and $\lim_{\alpha_0\varepsilon\downarrow 0} \operatorname{Im}\log\lambda_\varepsilon(n) = \bigl(\frac{\pi}{2}\bigr)(1 + 2n - \operatorname{sgn} n)$.

Proof. We give the proof for the $\ell = 1$ mode; the case $\ell = -1$ follows. We set $\psi = \psi_+$. From [7, Theorem 1], there is an operator $B_{00}\colon L^2_c \to H^2_{\mathrm{loc}}$ such that near $\lambda = 0$

$$\chi R_{V_0}(\lambda)\mathcal{P}_1\chi = \chi\Bigl(\frac{1}{(\log\lambda - b)\lambda^2}\psi\otimes\psi + B_{00}\mathcal{P}_1\Bigr)\chi + O((\log\lambda)^{-1}). \tag{3.21}$$

We set

$$\widetilde{R}_{V_0}(\lambda) = R_{V_0}(\lambda) - \frac{1}{\log\lambda - b}\psi\otimes\psi.$$

As in the proofs of Lemmas 14 and 13, to study the resonances with $|\arg\lambda| < \varphi$ and $0 < |\lambda|$, for $|\lambda|$, $|\varepsilon|$ sufficiently small, it suffices to study the zeros of

$$Q(\lambda,\varepsilon) := \det\Bigl(I + \frac{1}{(\log\lambda - b)\lambda^2}(V_\varepsilon - V_0)\psi\otimes\psi(I + (V_\varepsilon - V_0)\widetilde{R}_{V_0}(\lambda)\mathcal{P}_1\chi)^{-1}\Bigr).$$

Using (3.21) yields

$$Q(\lambda,\varepsilon) = 1 + \frac{\varepsilon}{(\log\lambda - b)\lambda^2}\bigl(\alpha_0 + \alpha_1\varepsilon + O(\varepsilon(\log\lambda)^{-1}) + O(\varepsilon^2)\bigr) \tag{3.22}$$

where

$$\alpha_1 = -\langle B_{00}\dot{V}_0\psi, \dot{V}_0\psi\rangle + \frac{1}{2}\langle\ddot{V}_0\psi, \psi\rangle \in \mathbb{R}.$$

Now, set

$$g(\lambda,\varepsilon) = 1 + \frac{\varepsilon}{(\log\lambda - b)\lambda^2}(\alpha_0 + \alpha_1\varepsilon). \tag{3.23}$$

Solving $g(\lambda,\varepsilon) = 0$ as in Lemma 12 leads for $n \in \mathbb{Z}\setminus\{0\}$ to solutions $\tilde{\lambda}_\varepsilon(n)$ satisfying

$$\log\tilde{\lambda}_\varepsilon(n) = b + \frac{1}{2}W_n\bigl(2\varepsilon(\alpha_0 + \alpha_1\varepsilon)e^{-2\operatorname{Re} b}\bigr).$$

Apply Rouché's theorem to the pair g, Q on the curve $|\tilde{\lambda}_\varepsilon(n) - \lambda| = |\varepsilon|^{1/4}|\tilde{\lambda}_\varepsilon(n)|$. This shows that for each $n \in \mathbb{Z}\setminus\{0\}$ there is an $\varepsilon_0(n)$ so that there is a zero $\lambda_\varepsilon(n)$ of $Q(\lambda,\varepsilon)$ within $|\varepsilon|^{1/4}|\tilde{\lambda}_\varepsilon(n)|$ of $\tilde{\lambda}_\varepsilon(n)$ when $|\varepsilon| < \varepsilon_0(n)$. Using (3.22) and (3.23) shows that we can improve the error to $\log \lambda_\varepsilon(n) - \log \tilde{\lambda}_\varepsilon(n) = o(\varepsilon)$, or $\lambda_\varepsilon(n) - \tilde{\lambda}_\varepsilon(n) = o(|\varepsilon|^{3/2})$.

If $n = -1$ and $\varepsilon\alpha_0 < 0$, then Lemma 4 shows that the resulting pole of $R_\varepsilon \mathcal{P}_1$ corresponds to an eigenvalue, since the pole lies in the physical space. ■

Above we have restricted our considerations to the radially symmetric case in an effort to make the paper as readable as possible. However, radial symmetry is not really essential to the proofs. What is essential is the ability to reduce the problem to the study of a family of operators of λ and ε with a singularity of rank 1 at $\lambda = 0$ when $\varepsilon = 0$. For the radial case, we did this by projecting $I + (V_\varepsilon - V_0)R_{V_0}(\lambda)$ onto a Fourier mode. If the operator $R_{V_0}(\lambda)$ has a singularity of rank 1 at $\lambda = 0$, there is no need for a projection, and so no need for an assumption of radial symmetry. As an example, we include the following lemma which is an analog of Lemma 14 without the assumption of radial symmetry. Analogs of Lemmas 13 and 15 are equally possible, though in general one should expect the errors to be worse than in those lemmas.

Lemma 16. *Suppose V_ε is a family of compactly supported real-valued functions in $L^\infty(\mathbb{R}^2)$ which depend in a C^∞ way on ε. Suppose 0 is a simple eigenvalue of $-\Delta + V_0$ and $-\Delta + V_0$ has neither an s-wave resonance nor a p-wave resonance, and suppose $\psi \in L^2(\mathbb{R}^2)$ satisfies $(-\Delta + V_0)\psi = 0$, $\|\psi\| = 1$. If*

$$\alpha_0 := \int_{\mathbb{R}^2} |\psi(x)|^2 \dot{V}_0(x)dx \neq 0,$$

then for each $m \in \mathbb{Z}$ there is an $\varepsilon_0 = \varepsilon_0(m) > 0$ such that if $0 < |\varepsilon| < \varepsilon_0$, then there is a resonance at $\lambda_\varepsilon(m)$, satisfying

$$|\lambda_\varepsilon(m)| = \sqrt{|\varepsilon\alpha_0|} + O_m(|\varepsilon|^{3/2}\log|\varepsilon|)$$

and

$$\arg \lambda_\varepsilon(m) = \pi\left(m + \frac{1 - \operatorname{sgn}(\varepsilon\alpha_0)}{4}\right) + O_m(|\varepsilon|\log|\varepsilon|).$$

When $m = 0$ and $0 < -\alpha_0\varepsilon < |\alpha_0|\varepsilon_0$, the resonance corresponds to a negative eigenvalue of $-\Delta + V_\varepsilon$ at $\varepsilon\alpha_0 + O(\varepsilon^2\log|\varepsilon|)$.

4. The scattering matrix and scattering phase for the circular well

In this section we return to the circular well potential, $V = -a^2 \mathbb{1}_\rho$, with $a, \rho > 0$ fixed, and study the derivative of the scattering phase, $\sigma'(\lambda)$, as $\lambda \downarrow 0$, proving Theorem 3. Because the potential is radial, the scattering matrix $S(\lambda)$ is diagonal when using the eigenfunctions $e^{i\ell\theta}$ as a basis. The eigenvalue S_ℓ of $S(\lambda)$ associated with $e^{i\ell\theta}$ is the ratio of the outgoing amplitude to the incoming amplitude for elements of the null space of $P-\lambda^2$:

$$S_\ell = \frac{c_\ell}{d_\ell} = -\frac{B_\ell^{(2)}}{B_\ell^{(1)}},$$
$$B_\ell^{(k)}(\lambda,\rho,a) := H_\ell^{(k)}(\lambda\rho) + A_\ell H_\ell^{(k)\prime}(\lambda\rho),$$
$$A_\ell(\lambda,\rho,a) := -\frac{\lambda}{\mu}\frac{J_\ell(\mu\rho)}{J_\ell'(\mu\rho)}.$$

We note that $S_{-\ell} = S_\ell$ for all (λ,ρ,a,ℓ) since $\mathcal{C}_{-\ell}(z) = (-1)^\ell \mathcal{C}_\ell(z)$. Thus, (1.4) becomes

$$\sigma(\lambda) = \frac{1}{2\pi i}\sum_{\ell=-\infty}^{\infty} \log S_\ell = \sigma_0 + 2\sum_{\ell=1}^{\infty}\sigma_\ell, \tag{4.1}$$

where the σ_ℓ are given by $\sigma_\ell(0) = 0$ and the following lemma, which among other things proves that the sum in (4.1) converges very rapidly.

Lemma 17. *For every $\ell \in \mathbb{N}_0$, $\lambda \in \Lambda$, $a > 0$, and $\rho > 0$, we have*

$$\sigma_\ell' = -\frac{a^2}{\mu^2}\frac{2}{\pi^2\lambda}\frac{1-\frac{\ell^2}{(\lambda\rho)^2}A_\ell^2}{(J_\ell(\lambda\rho)+A_\ell J_\ell'(\lambda\rho))^2 + (Y_\ell(\lambda\rho)+A_\ell Y_\ell'(\lambda\rho))^2}. \tag{4.2}$$

Moreover, given $\lambda_0 > 0$, $a_0 > 0$, and $\rho_0 > 0$, there is $\ell_0 > 0$ such that

$$|\sigma_\ell'(\lambda)| \lesssim \frac{\ell^3}{\mu^2\lambda}\Big(\frac{\lambda\rho e}{2\ell}\Big)^{2\ell}, \tag{4.3}$$

uniformly for $\ell \geq \ell_0$, $\lambda \in (0,\lambda_0]$, $a \in (0,a_0]$, and $\rho \in (0,\rho_0]$.

Proof. We have

$$2\pi i\sigma_\ell'(\lambda) = \frac{\partial_\lambda B_\ell^{(2)}}{B_\ell^{(2)}} - \frac{\partial_\lambda B_\ell^{(1)}}{B_\ell^{(1)}} = \frac{B_\ell^{(1)}\partial_\lambda B_\ell^{(2)} - B_\ell^{(2)}\partial_\lambda B_\ell^{(1)}}{B_\ell^{(1)}B_\ell^{(2)}}. \tag{4.4}$$

To simplify the denominator, we expand by writing the Hankel functions in terms of J_ℓ and Y_ℓ:

$$B_\ell^{(1)}B_\ell^{(2)} = (J_\ell(\lambda\rho)+A_\ell J_\ell'(\lambda\rho))^2 + (Y_\ell(\lambda\rho)+A_\ell Y_\ell'(\lambda\rho))^2.$$

As for the numerator we first use Bessel's equation on A_ℓ':

$$A_\ell' = \frac{-a^2}{\mu^3}\frac{J_\ell(\mu\rho)}{J_\ell'(\mu\rho)} - \frac{\lambda^2\rho}{\mu^2} + \frac{\lambda^2\rho}{\mu^2}\frac{J_\ell(\mu\rho)J_\ell''(\mu\rho)}{(J_\ell'(\mu\rho))^2}$$
$$= \frac{-1}{\mu}\frac{J_\ell(\mu\rho)}{J_\ell'(\mu\rho)} - \frac{\lambda^2\rho}{\mu^2}\Big(1+\Big(\frac{J_\ell(\mu\rho)}{J_\ell'(\mu\rho)}\Big)^2\Big(1-\frac{\ell^2}{(\mu\rho)^2}\Big)\Big).$$

Using Bessel's equation on the Hankel functions, we combine in a similar manner for $\partial_\lambda B_\ell^{(2)}$ to get

$$\partial_\lambda B_\ell^{(2)} = -\rho A_\ell\Big(1-\frac{\ell^2}{(\lambda\rho)^2}\Big)H_\ell^{(2)}(\lambda\rho) - \rho A_\ell^2\Big(1-\frac{\ell^2}{(\mu\rho)^2}\Big)H_\ell^{(2)\prime}(\lambda\rho) + \frac{a^2\rho}{\mu^2}H_\ell^{(2)\prime}(\lambda\rho).$$

Multiplying by $B_\ell^{(1)}$ gives us the expanded form

$$\begin{aligned} &B_\ell^{(1)}\partial_\lambda B_\ell^{(2)} \\ &= -\rho A_\ell\Big(1-\frac{\ell^2}{(\lambda\rho)^2}\Big)H_\ell^{(2)}(\lambda\rho)H_\ell^{(1)}(\lambda\rho) - \rho A_\ell^2\Big(1-\frac{\ell^2}{(\mu\rho)^2}\Big)H_\ell^{(1)}(\lambda\rho)H_\ell^{(2)\prime}(\lambda\rho) \\ &\quad + \frac{a^2\rho}{\mu^2}H_\ell^{(1)}(\lambda\rho)H_\ell^{(2)\prime}(\lambda\rho) - \rho A_\ell^2\Big(1-\frac{\ell^2}{(\lambda\rho)^2}\Big)H_\ell^{(1)\prime}(\lambda\rho)H_\ell^{(2)}(\lambda\rho) \\ &\quad - \rho A_\ell^3\Big(1-\frac{\ell^2}{(\mu\rho)^2}\Big)H_\ell^{(1)\prime}(\lambda\rho)H_\ell^{(2)\prime}(\lambda\rho) + A_\ell\frac{a^2\rho}{\mu^2}H_\ell^{(1)\prime}(\lambda\rho)H_\ell^{(2)\prime}(\lambda\rho). \end{aligned}$$

The product $B_\ell^{(2)}\partial_\lambda B_\ell^{(1)}$ is given by the same form as above but swapping (1) $\leftrightarrow$ (2) in the superscripts of the Hankel functions, so we get

$$\begin{aligned} &B_\ell^{(1)}\partial_\lambda B_\ell^{(2)} - B_\ell^{(2)}\partial_\lambda B_\ell^{(1)} \\ &= \Big[-\rho A_\ell^2\Big(1-\frac{\ell^2}{(\mu\rho)^2}\Big) + \frac{a^2\rho}{\mu^2} + \rho A_\ell^2\Big(1-\frac{\ell^2}{(\lambda\rho)^2}\Big)\Big] \\ &\quad \times \Big[H_\ell^{(1)}(\lambda\rho)H_\ell^{(2)\prime}(\lambda\rho) - H_\ell^{(1)\prime}(\lambda\rho)H_\ell^{(2)}(\lambda\rho)\Big] \\ &= 2i\Big[\rho A_\ell^2\Big(\frac{\ell^2}{(\mu\rho)^2} - \frac{\ell^2}{(\lambda\rho)^2}\Big) + \frac{a^2\rho}{\mu^2}\Big][J_\ell'(\lambda\rho)Y_\ell(\lambda\rho) - J_\ell(\lambda\rho)Y_\ell'(\lambda\rho)]. \end{aligned}$$

Using Wronskian identities on the remaining Bessel functions, we get

$$\begin{aligned} B_\ell^{(1)}\partial_\lambda B_\ell^{(2)} - B_\ell^{(2)}\partial_\lambda B_\ell^{(1)} &= -\frac{4i}{\pi\lambda\rho}\Big[\rho A_\ell^2\Big(\frac{\ell^2}{(\mu\rho)^2} - \frac{\ell^2}{(\lambda\rho)^2}\Big) + \frac{a^2\rho}{\mu^2}\Big] \\ &= \frac{4i}{\pi\lambda}\Big[\frac{(a\ell)^2}{(\lambda\mu\rho)^2}A_\ell^2 - \frac{a^2}{\mu^2}\Big] = -\frac{a^2}{\mu^2}\frac{4i}{\pi\lambda}\Big(1-\frac{\ell^2}{(\lambda\rho)^2}A_\ell^2\Big). \end{aligned}$$

Combining the two expressions for $B_\ell^{(1)}\partial_\lambda B_\ell^{(2)} - B_\ell^{(2)}\partial_\lambda B_\ell^{(1)}$ and $B_\ell^{(1)}B_\ell^{(2)}$ with (4.4) yields (4.2).

To prove (4.3), we use the notation $f \asymp g$ to mean $f \lesssim g$ and $g \lesssim f$. By Olver's uniform large order Bessel function asymptotics (see e.g., [10, (A.7)]), for every $x_0 > 0$ there is $\ell_0 > 0$ such that

$$J_\ell(x) \asymp xJ_\ell'(x), \quad -Y_\ell(x) \asymp xY_\ell'(x) \asymp \ell^{-1/2}\exp(\ell\xi),$$

$$\xi = \int_{x/\ell}^{1} \sqrt{t^{-2}-1}dt = \log\frac{\ell}{x} + \log 2 - 1 + O\Big(\frac{x^2}{\ell^2}\Big),$$

uniformly for $x \in (0, x_0]$ and $\ell \geq \ell_0$. Hence,

$$-A_\ell \asymp \lambda\rho,$$

and

$$|\sigma'(\lambda)| \lesssim \frac{\ell^2}{\mu^2 \lambda Y_\ell(\lambda\rho)^2} \lesssim \frac{\ell^3}{\mu^2\lambda}\Big(\frac{\lambda\rho e}{2\ell}\Big)^{2\ell}.$$

This proves (4.3). Note that, for λ, ρ, a fixed, we can get (4.3) from the simpler large order, fixed argument, asymptotics of [26, Section 10.19 (i)]. ■

We remark that (4.3) implies that

$$\sigma'(\lambda) = \sigma_0'(\lambda) + 2\sum_{\ell=1}^{\infty}\sigma_\ell'(\lambda) = \sigma_0'(\lambda) + 2\sum_{\ell=1}^{\ell_0}\sigma_\ell'(\lambda) + O(\lambda^{2\ell_0+1}) \quad \text{as } \lambda \to 0$$

when ℓ_0 is sufficiently large.

In this section we prove an expansion for $\sigma'(\lambda)$ for small λ. We will use repeatedly the general property that

$$\begin{aligned} & r(x) = o(f(x)) \\ & \implies \frac{1}{f(x)+r(x)} = \frac{1}{f(x)} - \frac{r(x)}{(f(x))^2} + O\Big(\Big(\frac{r(x)}{(f(x))^2}\Big)^2\Big). \end{aligned} \tag{4.5}$$

We begin with σ_0'.

Lemma 18. *For $\lambda \to 0$, we have the following asymptotic expressions for $\sigma_0'(\lambda)$:*

$$\sigma_0'(\lambda) = \begin{cases} -\dfrac{\rho^4}{8}\lambda^3 + O(\lambda^5\log\lambda), & J_1(a\rho) = 0, \\ \dfrac{-\frac{2}{\lambda}}{4\big(\log\frac{\lambda}{2} + C(\rho,a) + \gamma\big)^2 + \pi^2} + O\Big(\dfrac{\lambda}{(\log\lambda)^2}\Big), & J_1(a\rho) \neq 0, \end{cases}$$

where

$$C(\rho, a) = \log\rho + \frac{1}{a\rho}\frac{J_0(a\rho)}{J_1(a\rho)}.$$

Compare [6, (1.11)], for asymptotics of the derivative of the scattering phase in the Dirichlet obstacle case.

Proof. First we assume $J_1(a\rho) \neq 0$. Using

$$J_0'(z) = -J_1(z),\ Y_0'(z) = -Y_1(z) \tag{4.6}$$

and expanding the two terms in the denominator of (4.2) using (2.5) yields

$$\begin{aligned}(J_0(\lambda\rho) - A_0 J_1(\lambda\rho))^2 &= 1 + O(\lambda^2),\\ (Y_0(\lambda\rho) - A_0 Y_1(\lambda\rho))^2 &= \Big[\frac{2}{\pi}(\log\big(\lambda\rho 2\big) + \gamma) + \frac{2}{\pi a\rho}\frac{J_0(a\rho)}{J_1(a\rho)} + O(\lambda^2\log\lambda)\Big]^2\\ &= \frac{4}{\pi^2}\Big(\log\frac{\lambda}{2} + \gamma + C(\rho,a)\Big)^2 + O(\lambda^2\log^2\lambda).\end{aligned}$$

Putting this into (4.2) we get

$$\sigma_0'(\lambda) = -\frac{a^2}{\mu^2}\frac{2}{\pi^2\lambda}\frac{1}{1 + \frac{4}{\pi^2}\big(\log\frac{\lambda}{2} + \gamma + C(\rho,a)\big)^2 + O(\lambda^2\log^2\lambda)}. \tag{4.7}$$

Using (4.5) gives

$$\mu^{-2}(\lambda) = \frac{1}{a^2} - \frac{\lambda^2}{a^4} + O(\lambda^4).$$

Applying these observations to (4.7) yields

$$\begin{aligned}\sigma_0'(\lambda) &= -\frac{2}{\pi^2\lambda}\Big[\frac{1}{1 + \frac{4}{\pi^2}\big(\log\frac{\lambda}{2} + \gamma + C(\rho,a)\big)^2} + O\Big(\frac{\lambda^2}{(\log\lambda)^2}\Big)\Big]\\ &= \frac{\frac{-2}{\lambda}}{4\big(\log\frac{\lambda}{2} + \gamma + C(\rho,V)\big)^2 + \pi^2} + O\Big(\frac{\lambda}{(\log\lambda)^2}\Big).\end{aligned}$$

Now, consider the case $J_1(\rho a) = 0$. Here we again use (4.6), and multiply both the numerator and denominator of (4.2) by $\mu^2(J_1(\mu\rho))^2$ to get

$$\sigma_0'(\lambda) = -\frac{2a^2}{\pi^2\lambda}\frac{(J_1(\mu\rho))^2}{D_1(\lambda,\rho)}, \tag{4.8}$$

where

$$\begin{aligned}D_1(\lambda,\rho) = {}&(\mu J_1(\mu\rho)J_0(\lambda\rho) - \lambda J_0(\mu\rho)J_1(\lambda\rho))^2\\ &+ (\mu J_1(\mu\rho)Y_0(\lambda\rho) - \lambda J_0(\mu\rho)Y_1(\lambda\rho))^2.\end{aligned}$$

Since, in this case,

$$J_1(\mu\rho) = J_1'(a\rho)\frac{\rho\lambda^2}{2a} + O(\lambda^4) = -J_0(a\rho)\frac{\rho\lambda^2}{2a} + O(\lambda^4), \tag{4.9}$$

we get from the expansions (2.5)

$$\big(\mu J_1(\mu\rho)J_0(\lambda\rho)-\lambda J_0(\mu\rho)J_1(\lambda\rho)\big)^2=O(\lambda^4) \tag{4.10a}$$

$$\big(\mu J_1(\mu\rho)Y_0(\lambda\rho)-\lambda J_0(\mu\rho)Y_1(\lambda\rho)\big)^2=\Big(\frac{2J_0(\rho a)}{\pi\rho}+O(\lambda^2\log\lambda)\Big)^2. \tag{4.10b}$$

Using (4.9) and (4.10) in (4.8) gives the result when $J_1(a\rho)=0$. ■

We give an alternate expression for σ_ℓ'. Using the recurrence relations (2.3), we find

$$\begin{aligned}
&\mu^2(J_\ell'(\mu\rho))^2-\frac{\ell^2}{\rho^2}J_\ell(\mu\rho)^2\\
&\quad=\mu^2\Big[\Big(J_{\ell-1}(\mu\rho)-\frac{\ell}{\mu\rho}J_\ell(\mu\rho)\Big)^2-\frac{\ell^2}{(\mu\rho)^2}(J_\ell(\mu\rho))^2\Big]\\
&\quad=-\mu^2J_{\ell-1}(\mu\rho)J_{\ell+1}(\mu\rho).
\end{aligned}\tag{4.11}$$

Multiplying the numerator and denominator in (4.2) by $(\mu J_\ell'(\mu\rho))^2$ and using the identities (2.3) and (4.11), yields that for $\ell\ge 1$

$$\sigma_\ell'(\lambda)=\frac{2a^2}{\pi^2\lambda}\frac{J_{\ell-1}(\mu\rho)J_{\ell+1}(\mu\rho)}{D_2(\lambda,\rho)}, \tag{4.12}$$

where

$$\begin{aligned}
D_2(\lambda,\rho)&=(\mu J_{\ell-1}(\mu\rho)J_\ell(\lambda\rho)-\lambda J_\ell(\mu\rho)J_{\ell-1}(\lambda\rho))^2\\
&\qquad+(\mu J_{\ell-1}(\mu\rho)Y_\ell(\lambda\rho)-\lambda J_\ell(\mu\rho)Y_{\ell-1}(\lambda\rho))^2.
\end{aligned}$$

This expression shows perhaps more explicitly than equation (4.2) why we should expect different behavior in the three cases $J_{\ell+1}(a\rho)=0$, $J_{\ell-1}(\rho a)=0$, and $J_{\ell-1}(\rho a)J_{\ell+1}(\rho a)\neq 0$.

The next lemma will be used to study $\sigma_\ell'(\lambda)$ for $\ell\ge 1$ when $J_{\ell-1}(\rho a)\neq 0$.

Lemma 19. *If $\ell\ge 1$, then, if $J_{\ell-1}(\rho a)\neq 0$, then as $\lambda\to 0$*

$$\sigma_\ell'(\lambda)=\Big(\frac{\rho}{[(\ell-1)!]^2J_{\ell-1}(\rho a)}\Big(\frac{\rho\lambda}{2}\Big)^{2\ell-1}+O(\lambda^{2\ell+1})+\delta_{\ell,1}O(\lambda^3\log\lambda)\Big)J_{\ell+1}(\rho\mu).$$

Proof. The assumptions $J_{\ell-1}(\rho a)\neq 0$ and $\ell\ge 1$ ensure that the leading term in the denominator of (4.12) as $\lambda\to 0$ is determined by $\mu J_{\ell-1}(\mu\rho)Y_\ell(\lambda\rho)$. We have

$$\begin{aligned}
&\mu J_{\ell-1}(\mu\rho)Y_\ell(\lambda\rho)-\lambda J_\ell(\mu\rho)Y_{\ell-1}(\lambda\rho)\\
&\quad=aJ_{\ell-1}(a\rho)\Big(-\frac{(\ell-1)!}{\pi}\Big)\Big(\frac{2}{\rho\lambda}\Big)^\ell+O(\lambda^{-\ell+2})+\delta_{1,\ell}O(\lambda\log\lambda).
\end{aligned}$$

The asymptotics of J_ℓ, $J_{\ell-1}$ at 0 yield

$$\mu J_{\ell-1}(\mu\rho)J_\ell(\lambda\rho) - \lambda J_\ell(\mu\rho)J_{\ell-1}(\lambda\rho) = O(\lambda^\ell).$$

Using these gives, with D_2 from (4.12),

$$\big(D_2(\lambda,\rho)\big)^{-1} = \frac{\pi^2}{[aJ_{\ell-1}(\rho a)(\ell-1)!]^2}\Big(\frac{\rho\lambda}{2}\Big)^{2\ell} + O(\lambda^{2\ell+2}) + \delta_{1,\ell}O(\lambda^4\log\lambda).$$

Putting this in (4.12) proves the lemma. ∎

We prove an expansion for A_ℓ which will be helpful in the $\ell \geq 1$, $J_{\ell-1}(\rho a) = 0$ case.

Lemma 20. *For $\ell \geq 1$, if $J_{\ell-1}(a\rho) = 0$ then*

$$A_\ell(\lambda) = \frac{\lambda\rho}{\ell} - \frac{(\lambda\rho)^3}{2\ell^2} + O(\lambda^5)$$

Proof. Using the recurrence relations (2.3) we obtain

$$A_\ell(\lambda) = \frac{-\lambda\rho}{\ell}\frac{J_{\ell-1}(\mu\rho)+J_{\ell+1}(\mu\rho)}{J_{\ell-1}(\mu\rho)-J_{\ell+1}(\mu\rho)} = \frac{-\lambda\rho}{\ell}\Big(-1 + \frac{2J_{\ell-1}(\mu\rho)}{J_{\ell-1}(\mu\rho)-J_{\ell+1}(\mu\rho)}\Big).$$

We will use, again from (2.3),

$$J_{\ell-1}(a\rho) = 0 \implies J'_{\ell-1}(\rho a) = -J_\ell(\rho a) = -\frac{\rho a}{2\ell}J_{\ell+1}(\rho a).$$

Then, since, for smooth f, $f(\mu\rho) = f(a\rho) + f'(a\rho)\rho(\mu - a) + O((\mu-a)^2)$, we get

$$\begin{aligned} A_\ell(\lambda) &= \frac{-\lambda\rho}{\ell}\Big(-1 + \frac{2J'_{\ell-1}(a\rho)}{J_{\ell-1}(a\rho)-J_{\ell+1}(a\rho)}\rho(\mu-a) + O((\mu-a)^2)\Big) \\ &= \frac{-\lambda\rho}{\ell}\Big(-1 + \frac{(\rho\lambda)^2}{2\ell} + O(\lambda^4)\Big). \end{aligned}$$

∎

Lemma 21. *For $\lambda \to 0$, we have the following asymptotic expressions for σ_1':*

$$\sigma_1'(\lambda) = \begin{cases} \dfrac{\frac{-2}{\lambda}}{4\big(\log\frac{\rho\lambda}{2} - \frac{1}{2} + \gamma\big)^2 + \pi^2} + O\Big(\dfrac{\lambda}{(\log\lambda)^2}\Big), & J_0(a\rho) = 0, \\ \dfrac{\rho^4}{8}\lambda^3 + O(\lambda^5\log\lambda), & J_2(a\rho) = 0, \\ \dfrac{J_2(a\rho)}{J_0(a\rho)}\dfrac{\rho^2}{2}\lambda + O(\lambda^3\log\lambda), & \text{otherwise.} \end{cases}$$

Proof. First we consider the case $J_0(\rho a) = 0$. For this, we use for $\ell \geq 1$

$$\mathcal{C}_\ell(\lambda\rho) + \frac{\lambda\rho}{\ell}\mathcal{C}'_\ell(\lambda\rho) = \frac{\lambda\rho}{\ell}\mathcal{C}_{\ell-1}(\lambda\rho) \tag{4.13}$$

where $\mathcal{C}_\ell$ is J_ℓ or Y_ℓ. Using this and Lemma 20,

$$\begin{aligned} Y_1(\lambda\rho) + A_1 Y'_1(\lambda\rho) &= \lambda\rho Y_0(\lambda\rho) - \frac{(\lambda\rho)^3}{2} Y'_1(\lambda\rho) + O(\lambda^3) \\ &= \frac{2}{\pi}\lambda\rho\Big[\log\frac{\lambda\rho}{2} + \gamma\Big] - \frac{\lambda\rho}{\pi} + O(\lambda^3\log\lambda). \end{aligned}$$

Similarly,

$$\begin{aligned} J_1(\lambda\rho) + A_1 J'_1(\lambda\rho) &= \lambda\rho J_0(\lambda\rho) - \frac{(\lambda\rho)^3}{2} J'_1(\lambda\rho) + O(\lambda^5) \\ &= \lambda\rho + O(\lambda^3). \end{aligned}$$

Using these and Lemma (20) in (4.2) completes the proof for this case.

For the remaining cases we use Lemma 19. If $J_0(\rho a)J_2(\rho a) \neq 0$, we get the third case immediately from Lemma 19. To handle the remaining case, the second one, note that, if $\ell \geq 1$ and $J_{\ell+1}(\rho a) = 0$, then $J'_{\ell+1}(\rho a) = J_\ell(\rho a)$ and

$$J_{\ell+1}(\rho\mu) = J_\ell(\rho a)\frac{\rho\lambda^2}{2a} + O(\lambda^4) \quad \text{when } J_{\ell+1}(\rho a) = 0. \tag{4.14}$$

Moreover, since $J_2(\rho a) = 0$, $J_0(\rho a) = \frac{2}{a\rho}J_1(\rho a)$. Using these in Lemma 19 with $\ell = 1$ completes the proof of the second case. ∎

The proof of the asymptotic expansion of σ'_ℓ for $\ell \geq 2$ is similar to the $\ell = 1$ case handled by Lemma 21.

Lemma 22. *For $\lambda \to 0$, we have the following asymptotic expressions for σ'_ℓ for $\ell \geq 2$:*

$$\sigma'_\ell(\lambda) = \begin{cases} -\dfrac{\rho}{\ell[(\ell-2)!]^2}\left(\dfrac{\lambda\rho}{2}\right)^{2\ell-3} + \widetilde{O}_\ell, & J_{\ell-1}(a\rho) = 0, \\ \dfrac{\rho}{\ell[(\ell-1)!]^2}\left(\dfrac{\lambda\rho}{2}\right)^{2\ell+1} + O(\lambda^{2\ell+3}), & J_{\ell+1}(a\rho) = 0, \\ \dfrac{J_{\ell+1}(a\rho)}{J_{\ell-1}(a\rho)}\dfrac{\rho}{[(\ell-1)!]^2}\left(\dfrac{\lambda\rho}{2}\right)^{2\ell-1} + O(\lambda^{2\ell+1}), & \text{otherwise}. \end{cases}$$

where $\widetilde{O}_2 = O(\lambda^3\log\lambda)$ and $\widetilde{O}_\ell = O(\lambda^{2\ell-1})$ for $\ell \geq 3$.

Proof. We begin with the case $J_{\ell-1}(\rho a) = 0$. Using (4.13) and Lemma 20 gives

$$\begin{aligned}
&Y_\ell(\lambda\rho) + A_\ell Y_\ell'(\lambda\rho) \\
&= \frac{\lambda\rho}{\ell} Y_{\ell-1}(\lambda\rho) - \frac{(\lambda\rho)^3}{2\ell^2} Y_\ell'(\lambda\rho) + O(\lambda^{-\ell+4}) \\
&= -\frac{\lambda\rho(\ell-2)!}{\ell\pi}\Big(\frac{\lambda\rho}{2}\Big)^{-\ell+1} - \frac{(\lambda\rho)^3 \ell!}{4\pi\ell^2}\Big(\frac{\lambda\rho}{2}\Big)^{-\ell-1} + O(\lambda^{-\ell+4}) + \delta_{\ell,2} O(\lambda^2\log\lambda) \\
&= -2\frac{(\ell-2)!}{\pi}\Big(\frac{\lambda\rho}{2}\Big)^{-\ell+2} + O(\lambda^{-\ell+4}) + \delta_{\ell,2} O(\lambda^2\log\lambda).
\end{aligned}$$

Notice that $J_\ell(\lambda\rho) + A_\ell J_\ell'(\lambda\rho) = O(\lambda^\ell)$. Since $1 - \frac{\ell^2}{(\lambda\rho)^2}A_\ell^2 = \frac{(\lambda\rho)^2}{\ell} + O(\lambda^4)$ here, inserting these in (4.2) yields

$$\begin{aligned}
\sigma_\ell'(\lambda) &= -\frac{2}{\lambda}\frac{\frac{(\lambda\rho)^2}{\ell} + O(\lambda^4)}{4[(\ell-2)!]^2 + O(\lambda^2) + \delta_{\ell,2} O(\lambda^2\log\lambda)}\Big(\frac{\lambda\rho}{2}\Big)^{2\ell-4} \\
&= -\frac{\rho}{\ell[(\ell-2)!]^2}\Big(\frac{\lambda\rho}{2}\Big)^{2\ell-3} + O(\lambda^{2\ell-1}) + \delta_{\ell,2} O(\lambda^3\log\lambda)
\end{aligned}$$

For the remaining cases, $J_{\ell-1}(\rho a) \neq 0$. If, in addition, $J_{\ell+1}(\rho a) \neq 0$, we get the result in the third case immediately from Lemma 19. The second case, with $J_{\ell+1}(\rho a) = 0$, follows from combining Lemma 19 with (4.14). ■

The next theorem follows by combining the results of Lemmas 18, 21, and 22, recalling Lemma 17 justifies convergence of the series.

Theorem 23. *When $\lambda \downarrow 0$, $\sigma'(\lambda)$ has the following asymptotics:*

- *if $J_0(a\rho) = 0$,*

$$\sigma'(\lambda) = \frac{-\frac{2}{\lambda}}{4\big(\log\frac{\lambda\rho}{2} + \gamma\big)^2 + \pi^2} + \frac{-\frac{4}{\lambda}}{4\big(\log\frac{\lambda\rho}{2} - \frac{1}{2} + \gamma\big)^2 + \pi^2} + O\Big(\frac{\lambda}{(\log\lambda)^2}\Big),$$

- *if $J_1(a\rho) = 0$,*

$$\sigma'(\lambda) = -\frac{3}{2}\rho^2\lambda + O(\lambda^3\log\lambda),$$

- *otherwise,*

$$\sigma'(\lambda) = \frac{-\frac{2}{\lambda}}{4\big(\log\frac{\lambda}{2} + C(\rho,a) + \gamma\big)^2 + \pi^2} + \frac{J_2(a\rho)}{J_0(a\rho)}\rho^2\lambda + O\Big(\frac{\lambda}{(\log\lambda)^2}\Big).$$

Here

$$C(\rho,a) = \log\rho + \frac{1}{a\rho}\frac{J_0(a\rho)}{J_1(a\rho)}.$$

This theorem, when combined with Lemma 9, proves Theorem 3 and gives an explicit expression for the constants b which appear in the statement of that theorem.

5. Numerical Methods

The points in the plots for $\ell = 1, 2, 3$ in Figure 1 and 6 are the values $\{\operatorname{sgn}(k)k^2\Delta\varepsilon : k = -N, \dots, N\}$, where $\Delta\varepsilon = 0.0036$ and $N = 15$. The ε values for the $\ell = 0$ plot in Figure 2 are $\{-k\Delta\varepsilon : k = 2, \dots, N\}$, with $N = 25$ and $\Delta\varepsilon = 0.1$. The points on the blue curve of each plot are the approximate formulas for λ_ε, and the points on the red curve are the "exact" values for λ_ε obtained by applying Newton's method 20 times to locate the solution to the equation $B_\ell^{(2)}(\lambda_\varepsilon, \sqrt{a^2 - \varepsilon}, \rho) = 0$ with an initial guess given by the approximation from the blue curve.

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