

Preface: In Memory of Michael Goldstein

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Wilhelm Schlag



Michael Goldstein

This half of this special issue of the Journal of Spectral Theory (JST) is dedicated to the memory of Michael Goldstein (known to his family and friends as Misha), honoring his profound contributions to and impact on the field.

Misha was born on September 13, 1952 in Tashkent (in the former USSR). He received his M.Sc in 1974 and his Ph.D in 1977, both from Tashkent State University. He received the degree of Doctor of Mathematics and Physics (Habilitation) in 1987 from Vilnius State University.

Misha held faculty appointments at Tashkent State University from 1977 until 1990, at Ben-Gurion University of the Negev in Be'er Sheva from 1990 until 1996, and then after several short-term appointments, at the University of Toronto from 2000 until he passed away on December 27, 2022.

Many of his papers appeared in prestigious journals, including the *Annals of Mathematics*, the *Journal of the American Mathematical Society*, *Acta Mathematica*, *Publications Mathématiques de l'IHÉS*, and *Inventiones Mathematicae*. He is particularly well known for his work on quasi-periodic Schrödinger operators, in both the localized regime and the delocalized regime, as well as applications of this work to the Korteweg–de Vries equation.

We are indebted to all authors who contributed a paper to this memorial special issue to show their appreciation for Misha.

Below we recount some of our personal recollections of Misha and our interactions with him.

1. Dmitry Goldstein

Michael Goldstein was born on September 13, 1952, in Tashkent. His father, Shulim Goldstein, was an electrical engineer, and his mother, Bella Goldstein, was an accountant. Michael was the eldest of three brothers. Both sides of his family came to Tashkent before or during the Second World War: his father's family moved there from Ukraine, while his mother's family was evacuated from Vitebsk shortly before the city was occupied.

At the age of eight, Michael became seriously ill with tetanus. The illness posed a grave threat to his life, but after prolonged treatment he recovered. He excelled at school and developed an early interest in the exact sciences.

After finishing school, he initially intended to enroll in an institute of communications engineering. On the advice of a friend's mother, however, he applied to the Faculty of Mechanics and Mathematics at Tashkent State University. He was admitted in 1970, and mathematics soon became the central pursuit of his life.

As a student, Michael made two study visits to Novosibirsk State University, in 1973 and 1974, together with his fellow student Eduard Sluzhevsky. At the time, Novosibirsk was one of the leading mathematical centers in the Soviet Union. During these visits, they had the opportunity to discuss mathematics with Pavel P. Belinsky and Samuel L. Krushkal. Under the guidance of Lev I. Volkovysky, Michael and

Eduard worked together on problems in complex analysis and the theory of Riemann surfaces. Their joint research resulted in the paper “Some questions on the theory of closed Riemann surfaces,” published in 1975 in *Doklady Akademii Nauk SSSR*.

Soon afterward, Michael changed the direction of his research and turned to functional analysis, operator theory, and operator algebras. In 1976, he defended his Candidate of Sciences dissertation, a degree broadly equivalent to a PhD, under the supervision of Academician Tashmukhamed A. Sarymsakov. In the following years, he taught first in the Faculty of Physics and later in the Faculty of Mathematics at Tashkent State University.

From 1977 to 1990, his principal research interests lay in functional analysis, operator theory, and operator algebras. He studied ordered involutive algebras and von Neumann algebras, conditional expectations in noncommutative algebras, positive linear mappings, and noncommutative ergodic theorems. Gradually, his interests broadened to include spectral theory, the distribution of eigenvalues of energy operators, and mathematical problems arising in quantum and statistical physics.

In the late 1980s, Michael defended his dissertation for the degree of Doctor of Physical and Mathematical Sciences, a higher doctorate roughly comparable to a habilitation, in Vilnius. Around the same time, he began to experience symptoms of a severe chronic autoimmune disease, with which he lived for the rest of his life. Periods of remission were often especially productive phases of his research, yet he continued to work in mathematics even during relapses. His close colleagues were well aware of his illness, and many supported him through the most difficult periods. Despite the limitations it imposed, Michael continued to work with extraordinary intensity and exceptional devotion to science.

In 1990, Michael made aliyah to Israel with his wife and two children. He received a Guastella Fellowship and worked in the Department of Mathematics at Ben-Gurion University of the Negev in Beersheba. During the early 1990s, he continued his research in functional analysis, operator theory, and mathematical physics. His work from this period concerned von Neumann algebras, noncommutative probability, ergodic theorems, and questions of almost-everywhere convergence for sequences of operators.

In 1994, the family moved to Toronto. The years that followed were difficult. Nevertheless, in the second half of the 1990s, his research took an important new direction: the spectral theory of Schrödinger operators became its principal focus. He worked on quasiperiodic and dynamically defined potentials, Lyapunov exponents, spectral structure, and Anderson localization.

In 1999–2000, and again in the fall of 2001, Michael Goldstein was a Member of the School of Mathematics at the Institute for Advanced Study in Princeton. There he collaborated with Jean Bourgain on the spectral theory of quasiperiodic Schrödinger

operators and Anderson localization. This collaboration marked an important stage in his scientific career and largely shaped the direction of his subsequent work.

Michael was distinguished by an uncommon degree of scientific rigor. He was uncompromising in matters of mathematical precision and intellectual honesty, and he applied the highest standards above all to his own work. He did not seek quick results and could devote many years to a difficult problem if he considered it genuinely important. Some problems occupied him for five, ten, or even more years. For him, it was essential not merely to obtain a result, but to understand the mechanism behind it and to find a proof that he regarded as fully convincing.

This scientific integrity was inseparable from his personal character. Michael was a man of profound decency, directness, responsibility, and intellectual independence. He was also deeply religious; faith occupied an important place in his life and strongly influenced his attitude toward other people, toward his own responsibilities, and toward the hardships he endured.

Family held a special place in Michael's life. His older son, Edward, became a mathematician and received his PhD from the Massachusetts Institute of Technology, while his younger son, Garry, became a physicist and received his PhD from Harvard University. Michael also shared a very close bond with his grandson Isaac, a graduate of Brandeis University who works in neuroscience research.

Even during the final weeks of his life, Michael's thoughts remained centered on mathematics. In October 2022, when he was already gravely ill, he asked me to help post his last paper on arXiv—a work that he regarded as the deepest result of his scientific career. Michael died of non-Hodgkin lymphoma on December 27, 2022.

2. Wilhelm Schlag

I met Michael Goldstein (Misha) at Princeton in 98/99. We lost Misha, who was my dear friend and collaborator, on December 27, 2022. He was an extraordinary human being—witty, warm, extremely generous, kind, and a brilliant but at times chaotic and messy mathematician. I miss him sorely and my heart still aches from his untimely passing from the side effects of his cancer treatments.

Misha left the USSR as soon as it dissolved, having been a professor in his native Tashkent, Uzbekistan, for many years. He had gotten his habilitation with Yakov Sinai in Moscow in the early 80s. Upon his arrival in Israel, Misha obtained a permanent position at Ben-Gurion University of the Negev. But his wife Bella decided to move the family to Canada after several years. So she relocated with her mother and their two sons, Gary and Eddy, to Toronto. Misha then had to abandon his tenured position at Be'er Sheva, without having secured a professorship at the University of Toronto.

His wife Bella tried to convince Misha to leave academic mathematics and become a programmer, as she had done.

But he was not only extremely stubborn; he also had severe difficulties coping with the “real” world (which he was aware of). Misha did not speak much of those years, but he occasionally shared a few facts. Apparently, he lived in a small damp basement room somewhere, living partially off money his brothers sent him and working furiously on his mathematics, or at least trying to do so. In addition, he was able to make around 10k per year as an adjunct teacher at the University of Toronto. At that point, Misha was in his early 40s.

While Misha’s life in Toronto was in a delicate balance, or perhaps off balance, Thomas Spencer invited him to the IAS in Princeton. Misha had sent Tom four long preprints, and so Tom wanted to discuss the mathematics in them and to hear from Misha directly about his work. In addition, Yakov Sinai (his former PhD advisor from MGU) invited Misha to the university to spend some time there. This is where I met Misha in early ‘99, as far as I recall. I had been talking to Sinai about Anderson localization, so I was introduced to Misha and we started discussing the quasi-periodic Schrödinger cocycle.

Misha showed me a new estimate that he had proved around that time, which he called a large deviation estimate (LDE). His bounds were of the form

$$|\{\theta \in \mathbb{T} : |n^{-1} \log \|M_n(\theta, \omega, E)\| - L_n(E, \omega)| > n^{-\sigma}\}| \leq C \exp(-cn^{1-3\sigma})$$

for some small $\sigma > 0$, where $M_n \in \text{SL}(2, \mathbb{R})$ is the step- n propagator of the equation

$$\psi_{n+1} + \psi_{n-1} + V(\theta + n\omega)\psi_n = E\psi_n, \quad n \in \mathbb{Z} \quad (*)$$

with an analytic function V on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and with L_n the Lyapunov exponent in finite volume. Moreover, a Diophantine condition on ω was assumed. Misha’s original proof of this estimate used Cartan’s bound on the sub-level sets of subharmonic functions in the plane, a technique he loved. Bourgain reproved the LDE using Fourier series (without Cartan) and then they went on to write their famous paper “On nonperturbative localization with quasi-periodic potential”, *Annals of Math.* 152 (2000). It is widely acknowledged that this paper revolutionized this field, as it established localization for $(*)$ for general analytic V and almost every ω, θ (note: not every Diophantine ω). In addition to LDEs, this paper crucially uses quantitative complexity bounds on semi-algebraic sets as proved by Basu and others.

In early summer of ‘99, Misha returned to Toronto. I asked to see him in August of ‘99, which he reluctantly agreed to. He was under severe stress trying to secure a faculty position at the university there. When I arrived, he insisted that we study very small deviations in the LDE. Instead, I showed him what I had worked on over the summer, namely taking very large deviations (on the order of the Lyapunov exponent

itself) and proving exponentially small bounds on the measure (i.e., e^{-cn}). I had noted that in combination with the so-called avalanche principle expansion, a tool Misha and I had developed earlier that summer, we could derive Hölder bounds on the *integrated density of states* (the limiting distribution of the eigenvalues) and the infinite volume Lyapunov exponent. But I had gotten stuck on how to prove the *sharp LDE* since only the exponentially small bound will suffice.

Misha was immediately hooked, and we jumped into this project with great enthusiasm. In a few days, we had established an exponential bound in the LDE and proved the Hölder regularity of the IDS and some other facts. I have never seen Misha so excited and enthusiastic, it was pure joy to work with him and to experience how we made progress in strides. I will never forget this. After returning to Princeton, I wrote the paper and Bourgain received it with much interest. The fall of '99 marked the beginning of the joint work by Bourgain, Misha, and myself. Our first collaborative paper was perturbative localization for the skew shift, and others followed. The big development in early 2000 on the personal front was that Misha received the job at Toronto that he coveted.

From 2003–05, Misha and I returned to the matter of the small deviations in the LDE (one cannot go below $1/n$). We introduced a new technique based on the number of zeros of the determinant $f_n(z, E) = \det(H_{[1,n]}(z, \omega, E))$, where $z = \theta + i\eta$ is a complexification of θ with $|\eta| < \varepsilon$. Here, $H_{[1,n]}$ is the finite volume Hamiltonian associated with $(*)$. In our paper “Fine properties of the integrated density of states and a quantitative separation property of the Dirichlet eigenvalues” (*Geom. Funct. Anal.*, 2008), we showed that the Hölder exponent of the Lyapunov exponent $L(E) > 0$ is in fact no worse than

$$k_0(E) := \limsup_{N \rightarrow \infty} N^{-1} \#\{z : f_n(z, E) = 0\}. \quad (\dagger)$$

We did not state the theorem in this fashion, but instead, bounded $k_0 \leq 2d = 2 \deg V$, where V in $(*)$ is a trigonometric polynomial. Now I understand that we left gold on the table—my paper with Rui Han from December '22 shows that $k_0(E) = 2\kappa(E)$, which is Avila's acceleration. Misha and I should have realized that our $k_0(E)$ is a gold mine, but instead, we focused on the almost Lipschitz behavior of the IDS, which is the dominant feature here. The Hölder behavior occurs only at the band edges of the spectrum. I cannot go into the role of the zero counting technique here. In essence, in combination with the avalanche principle, it allows one to control the small divisors in the resolvent with energy $E + i\varepsilon$ whose trace governs the number of eigenvalues falling into the interval $[E - \varepsilon, E + \varepsilon]$. This in turn controls the regularity of the IDS.

In our paper “On resonances and the formation of gaps in the spectrum of quasi-periodic Schrödinger equations” (*Ann. Math* 173, 2011), Misha and I did employ our zero counting technique once more, in the proof of the Cantor structure of the

spectrum of $(*)$. In his Ph.D. thesis in Barcelona, Joaquim Puig achieved a major breakthrough in '04 with his proof of the Cantor spectrum for the Harper model (set $V(\theta) = 2\lambda \cos(2\pi\theta)$ in $(*)$), see his paper "Cantor spectrum for the almost Mathieu operator", *Commun. Math. Phys.* 2004. His proof is based on Eliasson's reducibility, the Johnson–Moser gap labeling theorem, and is indirect. Assuming a gap is closed leads to a contradiction (but this is delicate). Puig's paper had a major impact and solved the Diophantine case of a long-standing open problem.

Avila and Jitomirskaya built on Puig's reducibility ideas and extended the argument to cover all irrational numbers, and they also proved that all gaps are open. The latter is hard, and known as the *dry ten Martini problem*. All these results only work for $V(\theta) = 2\lambda \cos(2\pi\theta)$. Misha and my proof applies to general analytic V as long as the Lyapunov exponent is positive. But we needed to remove a set of Hausdorff dimension 0 from the Diophantine frequencies ω , in order to eliminate triple resonances (such resonances cannot occur for cosine-like potentials). It is a direct, multi-scale construction, based on the idea that a resonance (i.e., the crossing of the graph of a parametrized eigenvalue with the translation of another graph of an eigenvalue) opens a "pre-gap" at the next step of the inductive procedure. The main problem is to show that these pre-gaps do not close at subsequent steps which we achieved by the zero count (this is a bit of a long story).

Starting around 2009, Misha began his work with David Damanik, while I was more focused on PDEs. We did reconnect again on a project about multi-frequency potentials to prove that the spectrum is in fact an interval or contains an interval. This is in stark contrast to the Cantor spectrum, which is due to the fact that resonance gaps cannot open up if one has two or more frequencies (provided $V(x, y)$ is generic in a very precise sense).

3. Ilia Binder

I first met Misha in 1999 in Princeton. It was a brief encounter, and we only reconnected in 2004 after I moved to Toronto. Even then, our active collaboration did not begin until around 2010. I was a novice in the field at the time, and I remember being completely astonished by Misha's generosity. He shared his time and his mathematical ideas without any hesitation.

This exceptional generosity was always evident when we worked with younger mathematicians, such as Mircea Voda and Miliwoje Lukić. Misha's ideas deeply influenced the early papers Mircea and I wrote on quasiperiodic Jacobi matrices, starting with our 2012 paper on eigenvalue estimates. This naturally evolved into a joint collaboration between the three of us, tackling Lyapunov exponents and the Anderson

model on a strip. It was the same story across our other projects, including our work with two of the co-authors of this note.

Our work together also came with a running joke about our schedule. Misha was a devoted Judaist, which introduced a very strict boundary to our meetings. It is easy to lose track of time in the middle of a proof, but our Friday sessions were strictly conditioned on him returning home before sunset. No matter how deep we were into a calculation or how close we felt to a breakthrough, the setting sun was the one absolute force that could definitively put a stop to our mathematics for the week.

4. David Damanik

I started to collaborate with Misha in 2009. The seeds were planted during the 50th anniversary of Anderson's localization paper, celebrated at a semester program at the Issac Newton Institute in 2008. At that event, Misha asked me whether I would be interested in working with him on the Deift conjecture, which states that the Korteweg–de Vries (KdV) equation with almost periodic initial data admits global solutions that are almost periodic in space and time. It was understood from the beginning that this would be an ambitious undertaking. However, I did not foresee at the time that we would be working intensively on this project for almost ten years.

I was well aware of Misha's impressive work on ergodic Schrödinger operators with Jean Bourgain and Wilhelm Schlag, and I was at a stage in my career where embarking on a project that would likely take an extraordinary amount of time and effort felt appropriate and even exciting.

After this initial exploratory conversation, we planned to kick off this project during a visit of mine to Toronto in 2009. Upon arrival in Toronto, Misha handed me a 50-page manuscript that contained the initial steps of what he envisioned would become a proof of global solutions for small quasi-periodic initial data with analytic sampling function and Diophantine frequency. This proof would be based on a quantitative inverse spectral theory result for Schrödinger operators on the real line with potentials of this type, in which the exponential decay rate of the Fourier coefficients is connected in a two-sided way to the exponential decay rate of the length of the gaps of the spectrum. As the KdV evolution is isospectral, the latter is preserved under this evolution, and hence the former cannot deteriorate over time. Thus, a local existence result can then be uniformly iterated, resulting in the desired global existence result.

Starting in 2010, Misha would come to Rice University for numerous extended visits to advance our work on this project. Each of his visits turned into a period of intense focus and persistence, leading to steady progress over time. I admired and enjoyed the positive and calm demeanor with which he kept pushing against a wall of technical complexity. It took us until about the fall of 2012 to complete the first stage

of the project, resulting in a 185-page paper in *Publ. Math. Inst. Hautes Études Sci.* (published in 2014) that established the two-sided inverse spectral theory result, which formed the technical foundation for all future works in the series of papers coming out of this long-term project on the Deift conjecture. Coincidentally or not, just as we completed the work on this paper, I had a stroke, which unfortunately impacted my ability to advertise our result. However, by the end of 2012 we did manage to complete the second stage of the project and show the existence and uniqueness of global solutions for the KdV equation with those quasi-periodic initial data for which we had the inverse spectral theory result from the first paper. This work appeared in *J. Amer. Math. Soc.* in 2016.

For the subsequent stages, we invited Milivoje Lukić (and then later Ilia Binder) to join the collaboration. The results obtained jointly with them will be discussed in Milivoje's contribution below.

The period between 2008 and 2018, during which I interacted closely with Misha, was marked by his kindness and generosity. I view the work leading to the 2014 PMI-HES paper as a singular experience, in terms of its intensity and depth. I will forever be grateful to Misha for taking me on that ride. It was mathematical exploration and discovery in its purest form.

5. Milivoje Lukić

I first met Michael Goldstein in 2012, during one of his visits to Rice University to collaborate with David Damanik. At the time, they were completing their work on Schrödinger operators with small analytic quasiperiodic potentials (with Diophantine frequency). A fascinating aspect of this work was a two-sided link between the exponential rate of decay of the Fourier coefficients of the potential and the exponential rate of decay of the spectral gaps, which they used to prove global existence and uniqueness of solutions to the KdV equation with this class of potentials as initial data.

Michael had further goals in this direction, motivated by integrability of the KdV equation and specifically by the Deift conjecture: the question of whether KdV with almost periodic initial data has solutions almost periodic in time. At that point, in 2013, I joined the collaboration. From the start, I was impressed by Michael's work ethic and the passion with which he immersed himself in challenging projects. Michael had a key idea to use periodic approximants and continuously change the potential in a very particular way to preserve the key estimates. This entailed a great technical effort, which Michael took in stride, and it resulted in the description of the isospectral torus for small analytic quasiperiodic potentials.

In 2014, Ilia Binder joined the effort, and I started a postdoc position at University of Toronto to work with them. I remember fondly the discussions in Michael's office, trying to decide on an approach. Michael or Ilia would propose an idea enthusiastically, and the other would be skeptical. A lively debate would ensue, in which both were so persuasive that eventually they would switch sides. This intensive work led us to some interesting detours, and also to the successful resolution of our goal: the proof that KdV with small quasiperiodic initial data has a global solution almost periodic in time.

In those years and later, Michael was always generous with his time and knowledge. I am grateful for his mentorship, which made me a better mathematician. He was also supportive to other junior mathematicians, taking a genuine interest in their work and offering thoughtful advice and encouragement. He placed great value on high-quality work, and he had a contagious enthusiasm for mathematics.



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