J. Eur. Math. Soc. 15, 755–774
DOI 10.4171/JEMS/375

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Separable solutions of quasilinear Lane–Emden equations

Received May 25, 2011 and in revised form October 6, 2011

Abstract. For $0 < p - 1 < q$ and either $\epsilon = 1$ or $\epsilon = -1$, we prove the existence of solutions of $-\Delta_p u = \epsilon u^q$ in a cone C_S, with vertex 0 and opening S, vanishing on ∂C_S, of the form $u(x) = |x|^{-\beta}\omega(x/|x|)$. The problem reduces to a quasilinear elliptic equation on S and the existence proof is based upon degree theory and homotopy methods. We also obtain a nonexistence result in some critical case by making use of an integral type identity.

Keywords. Quasilinear elliptic equations, p-Laplacian, cones, Leray–Schauder degree

1. Introduction

It is well established that the description of the boundary behavior of positive singular solutions of Lane–Emden equations

$$-\Delta u = \epsilon u^q \tag{1.1}$$

with $q > 1$ in a domain $\Omega \subset \mathbb{R}^N$ is greatly facilitated by using specific separable solutions of this equation. This was shown in 1991 by Gmira–Véron [7] in the case $\epsilon = -1$ and more recently by Bidaut-Véron–Ponce–Véron [3] in the case $\epsilon = 1$. If the domain is assumed to be a cone $C_S = \{x \in \mathbb{R}^N \setminus \{0\} : x/|x| \in S\}$ with vertex 0 and opening $S \subsetneq S^{N-1}$ (the unit sphere in $\mathbb{R}^N$), separable solutions of (1.1) vanishing on $\partial C_S \setminus \{0\}$ are of the form

$$u(x) = |x|^{-2/(q-1)}\omega(x/|x|), \tag{1.2}$$

with ω satisfying

$$-\Delta'\omega - \ell_{q,N}\omega - \epsilon\omega^q = 0 \quad \text{in } S, \tag{1.3}$$

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Mathematics Subject Classification (2010): 35J92, 35J60, 47H11, 58C30

vanishing on ∂S and where $\ell_{q,N} = \frac{2}{q-1}\frac{2q}{q-1} - N$ and Δ' is the Laplace–Beltrami operator on S^{N-1}. To this equation is associated the functional

$$J(\phi) := \int_S \left(\frac{1}{2}|\nabla'\phi|^2 - \frac{\ell_{q,N}}{2}\phi^2 - \frac{\epsilon}{q+1}|\phi|^{q+1} \right) dv_g, \tag{1.4}$$

where ∇' is the covariant derivative on S^{N-1}. In the case $\epsilon = 1$, nonexistence of a nontrivial positive solution of (1.3) when $\ell_{q,N} \geq \lambda_S$ (the first eigenvalue of $-\Delta'$ in $W_0^{1,2}(S)$) follows by multiplying the equation by the first eigenfunction and integrating over S; existence holds when $\ell_{q,N} < \lambda_S$ and $q < (N+1)/(N-3)$ by classical variational methods, and again nonexistence holds when $q \geq (N+1)/(N-3)$ and $S \subset S_+^{N-1}$ is starshaped by using an integral identity [3, Th. 2.1, Cor. 2.1]. When $\epsilon = -1$, nonexistence of a nontrivial solution of (1.3) when $\ell_{q,N} \leq \lambda_S$ is obtained by multiplying the equation by ω and integrating over S, while existence when $\ell_{q,N} > \lambda_S$ follows by minimizing J over $W_0^{1,2}(S) \cap L^{q+1}(S)$.

In this paper we investigate similar questions for the quasilinear Lane–Emden equations

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \epsilon u^q \quad \text{in } C_S, \tag{1.5}$$

where S is a smooth subset of S^{N-1}, $q > p-1 > 0$ and $\epsilon = \pm 1$, and we look for positive solutions u, vanishing on $\partial C_S \setminus \{0\}$, of the separable form

$$u(x) = |x|^{-\beta}\omega(x/|x|). \tag{1.6}$$

It is straightforward to check that u is a solution of (1.5) provided

$$\beta = \beta_q := \frac{p}{q+1-p} \tag{1.7}$$

and ω is a positive solution of

$$-\operatorname{div}((\beta_q^2\omega^2 + |\nabla'\omega|^2)^{(p-2)/2}\nabla'\omega) - \beta_q\lambda(\beta_q)(\beta_q^2\omega^2 + |\nabla'\omega|^2)^{(p-2)/2}\omega = \epsilon\omega^q \tag{1.8}$$

in S vanishing on ∂S, where $\operatorname{div}(\cdot)$ is the divergence operator defined according to the intrinsic metric g and where we have set

$$\lambda(\beta) = \beta(p-1) + p - N. \tag{1.9}$$

If $\epsilon = 0$, it is now well-known that positive p-harmonic functions in C_S vanishing on ∂C_S exist in the form (1.6), and either they are regular at 0 and $\beta = -\tilde{\beta}_S < 0$, or they are singular and $\beta = \beta_S > 0$, where the values of $\tilde{\beta}_S$, β_S are unique. In this case $\omega = \tilde{\omega}_S$ or ω_S is a solution of

$$-\operatorname{div}((\beta^2\omega^2 + |\nabla'\omega|^2)^{(p-2)/2}\nabla'\omega) - \beta\lambda(\beta)(\beta^2\omega^2 + |\nabla'\omega|^2)^{(p-2)/2}\omega = 0 \tag{1.10}$$

in S, where $\beta = \tilde{\beta}_S$ or β_S. The existence of $(\tilde{\beta}_S, \tilde{\omega}_S)$ is due to Tolksdorf in a pioneering work [18]. Tolksdorf's method has been adapted by Véron [20] in order to prove the existence of (β_S, ω_S). Later on Porretta and Véron [13] obtained a more general proof of the existence of such couples. Notice that β_S (as well as $\tilde{\beta}_S$) is uniquely determined while ω is unique up to homothety. In both cases the proofs rely on the strong maximum principle.

When $p \neq 2$, existence of a nontrivial solution in the case $\epsilon = 1$ is obtained in [2] when $N = 2$ and $\beta_q < \beta_S$ by a dynamical system approach; while if $\epsilon = -1$ and $\beta_q > \beta_S$, the existence is proved in [20] by a suitable adaptation of Tolksdorf's construction. Notice that no functional can be associated to (1.8), except in the case $q = q^* = Np/(N-p) - 1$. In that case, (1.8) is the Euler–Lagrange equation for the functional

$$J_q(\phi) := \int_S \left(\frac{1}{p} (\beta_{q^*}^2 \phi^2 + |\nabla' \phi|^2)^{p/2} - \frac{\epsilon}{q^* + 1} |\phi|^{q^*+1} \right) dv_g, \tag{1.11}$$

and existence of a nontrivial solution of (1.8) with $\epsilon = 1$ is derived from the mountain pass theorem. In all the other cases variational techniques cannot be used and have to be replaced by topological methods based upon Leray–Schauder degree. Define q_c by

$$q_c = q_{c,p} = \begin{cases} \dfrac{(N-1)p}{N-1-p} - 1 & \text{if } p < N-1, \\ \infty & \text{if } p \geq N-1. \end{cases}$$

Then we prove the following results:

I. *Let $\epsilon = 1$. Assume $p > 1$, $q < q_c$ and $\beta_q < \beta_S$. Then (1.8) admits a positive solution in S vanishing on ∂S.*

II. *Let $\epsilon = -1$. Assume $p > 1$ and $\beta_q > \beta_S$. Then (1.8) admits a unique positive solution in S vanishing on ∂S.*

The result **I** is based upon sharp Liouville theorems for solutions of (1.5) in $\mathbb{R}^N$ or $\mathbb{R}^N_+$ respectively due to Serrin–Zou [17] and Zou [23]. In the case of **II**, the existence part is already known, but we give here a simpler form than the one in [20], using a topological deformation acting on the exponent p. In the case $\epsilon = 1$, the result is optimal in the case $q = q_c$; indeed, using an integral identity, we also prove

III. *Let $\epsilon = 1$, $S \subsetneq S_+^{N-1}$ be a starshaped domain and $1 < p < N-1$. If $q = q_c$, then (1.8) admits no positive solution in S vanishing on ∂S.*

Notice that when $p = 2$ an integral identity was used in [3] to prove nonexistence for all $q \geq q_{c,2}$. The form which is derived in the case $p \neq 2$ is much more complicated and we prove nonexistence only in the case $q = q_{c,p}$.

Finally, the constraint $\beta_q < \beta_S$ in **I** (respectively, $\beta_q > \beta_S$ in **II**) is sharp. When $\epsilon = 1$, the nonexistence of positive solutions of (1.8) when $\beta_q \geq \beta_S$ has been proved in [2]. The method is based upon the strong maximum principle. When $\epsilon = -1$ a somewhat similar method is used in [22] and yields nonexistence results when $\beta_q \leq \beta_S$. Notice that obtaining such results when $p = 2$ is straightforward.

2. Nonexistence for the reaction problem

Let S be a bounded C^2 subdomain of S^{N-1}. We consider positive solutions in S of

$$-\operatorname{div}((\beta^2\omega^2 + |\nabla'\omega|^2)^{(p-2)/2}\nabla'\omega) - \beta\lambda(\beta)(\beta^2\omega^2 + |\nabla'\omega|^2)^{(p-2)/2}\omega = \omega^q \tag{2.1}$$

vanishing on ∂S. Recall that $\lambda(\beta)$ is given by (1.9) and that, in connection with problem (1.5), we are interested in the special case where $\beta = \beta_q$ is given by (1.7). The following Pohozaev type identity, which is valid for any β, is the key to nonexistence. We denote by S_+^{N-1} the half-sphere.

Proposition 2.1. *Let $S \subsetneq S^{N-1}$ be a C^2 domain and ϕ the first eigenfunction of $-\Delta'$ in $W_0^{1,2}(S_+^{N-1})$. If $\omega \in W_0^{1,p}(S) \cap C(\overline{S})$ is a positive solution of (2.1) in S, and if we set $\Omega = (\beta^2\omega^2 + |\nabla'\omega|^2)^{1/2}$, then*

$$\left(1 - \frac{1}{p}\right)\int_{\partial S} |\omega_\nu|^p \phi_\nu \, dS = A\int_S \omega^{q+1}\phi \, d\sigma + B\int_S \Omega^{p-2}|\nabla'\omega|^2\phi \, d\sigma + C\int_S \Omega^{p-2}\omega^2\phi \, d\sigma \tag{2.2}$$

with

$$A = A(\beta) := -\frac{N-1}{q+1} - \beta(p\beta + p - N), \tag{2.3}$$

$$B = B(\beta) := \frac{N-1-p}{p} + \beta(p\beta + p - N), \tag{2.4}$$

$$C = C(\beta) := \beta^2\left(\frac{N-1}{p} - (p\beta + p - N)\lambda(\beta)\right). \tag{2.5}$$

In order to prove Proposition 2.1, we start with the following lemma.

Lemma 2.1. *Let $S \subset S^{N-1}$ be a C^2 domain and $\phi \in C^2(\overline{S})$. If $\omega \in W_0^{1,p}(S) \cap C(\overline{S})$ is a positive solution of (2.1) in S, then*

$$\begin{aligned}\left(1 - \frac{1}{p}\right)\int_{\partial S} |\omega_\nu|^p \phi_\nu \, dS &= \int_S \left(\frac{\Delta'\phi}{q+1} - \beta(p\beta + p - N)\phi\right)\omega^{q+1} \, d\sigma - \frac{1}{p}\int_S \Omega^p \Delta'\phi \, d\sigma \\ &\quad + \int_S \Omega^{p-2} D^2\phi(\nabla'\omega, \nabla'\omega) \, d\sigma + \beta(p\beta + p - N)\int_S \Omega^{p-2}|\nabla'\omega|^2\phi \, d\sigma \\ &\quad - \beta^2(p\beta + p - N)\lambda(\beta)\int_S \Omega^{p-2}\omega^2\phi \, d\sigma. \end{aligned} \tag{2.6}$$

Proof. By the regularity theory of p-Laplace type equations (see e.g. [6], [19] and Appendix in [13]) it turns out that $\omega \in C^{1,\gamma}(\overline{S})$ for some $\gamma \in (0,1)$, and since $\beta^2\omega^2 + |\nabla'\omega|^2 > 0$ in the interior, by elliptic regularity we have $\omega \in C^2(S)$. Let $\phi \in C^2(S)$ be a given function and $\zeta \in C_c^1(S)$; since ζ is compactly supported we can multiply

(2.1) by the test function $\langle \nabla' \omega, \nabla' \phi \rangle \zeta$. Integrating by parts we get (using the notation $\Omega := (\beta^2 \omega^2 + |\nabla' \omega|^2)^{1/2}$)

$$\int_S \Omega^{p-2} \left(\frac{1}{2} \langle \nabla' |\nabla' \omega|^2, \nabla' \phi \rangle + D^2 \phi (\nabla' \omega, \nabla' \omega) \right) \zeta \, d\sigma + \int_S \Omega^{p-2} \langle \nabla' \omega, \nabla \zeta \rangle \, \langle \nabla' \omega, \nabla' \phi \rangle \, d\sigma$$
$$= \beta \lambda(\beta) \int_S \Omega^{p-2} \omega \langle \nabla' \omega, \nabla' \phi \rangle \, \zeta \, d\sigma + \frac{1}{q+1} \int_S \langle \nabla' \omega^{q+1}, \nabla' \phi \rangle \, \zeta \, d\sigma.$$

Since

$$\Omega^{p-2} \frac{1}{2} \langle \nabla' |\nabla' \omega|^2, \nabla' \phi \rangle = \frac{1}{p} \langle \nabla' \Omega^p, \nabla \phi \rangle - \beta^2 \Omega^{p-2} \omega \langle \nabla' \omega, \nabla' \phi \rangle$$

we obtain, due to (1.9),

$$\frac{1}{p} \int_S \langle \nabla' \Omega^p, \nabla' \phi \rangle \zeta \, d\sigma + \int_S \Omega^{p-2} \, D^2 \phi (\nabla' \omega, \nabla' \omega) \, \zeta \, d\sigma + \int_S \Omega^{p-2} \langle \nabla' \omega, \nabla' \zeta \rangle \langle \nabla' \omega, \nabla' \phi \rangle \, d\sigma$$
$$= \beta (p\beta + p - N) \int_S \Omega^{p-2} \omega \langle \nabla' \omega, \nabla' \phi \rangle \, \zeta \, d\sigma + \frac{1}{q+1} \int_S \langle \nabla' \omega^{q+1}, \nabla' \phi \rangle \, \zeta \, d\sigma.$$

Integrating by parts the first and the last terms we get

$$- \frac{1}{p} \int_S \Omega^p \langle \nabla' \phi, \nabla' \zeta \rangle \, d\sigma + \frac{1}{q+1} \int_S \omega^{q+1} \langle \nabla' \phi, \nabla' \zeta \rangle \, d\sigma + \int_S \left(\frac{\omega^{q+1}}{q+1} - \frac{\Omega^p}{p} \right) \Delta' \phi \, \zeta \, d\sigma$$
$$+ \int_S \Omega^{p-2} \, D^2 \phi (\nabla' \omega, \nabla' \omega) \, \zeta \, d\sigma + \int_S \Omega^{p-2} \langle \nabla' \omega, \nabla' \zeta \rangle \, \langle \nabla' \omega, \nabla' \phi \rangle \, d\sigma$$
$$= \beta (p\beta + p - N) \int_S \Omega^{p-2} \omega \langle \nabla' \omega, \nabla' \phi \rangle \, \zeta \, d\sigma. \tag{2.7}$$

Now we choose $\zeta = \zeta_\delta$, where ζ_δ is a sequence of compactly supported C^1 functions such that $\zeta_\delta(\sigma) \to 1$ for every $\sigma \in S$ and $|\nabla' \zeta_\delta|$ is bounded in $L^1(S)$. It is easy to see by integration by parts that for every continuous vector field $F \in C(\overline{S})$ we have

$$\int_S \langle F, \nabla' \zeta_\delta \rangle \, d\sigma \to - \int_{\partial S} \langle F, \nu(\sigma) \rangle \, d\sigma$$

where ν is the outward unit normal on ∂S. We take $\zeta = \zeta_\delta$ in (2.7) and we let $\delta \to 0$. Using that $\omega \in C^1(\overline{S})$ and that, by the Hopf lemma, $\omega_\nu := \langle \nabla' \omega, \nu(\sigma) \rangle < 0$ we can actually pass to the limit in the integrals containing $\nabla' \zeta_\delta$. Recalling that $\omega = 0$ and $\nabla' \omega = -|\omega_\nu| \nu$ on ∂S we obtain

$$\left(1 - \frac{1}{p} \right) \int_{\partial S} |\omega_\nu|^p \phi_\nu \, dS = \int_S \left(\frac{\omega^{q+1}}{q+1} - \frac{\Omega^p}{p} \right) \Delta' \phi \, d\sigma + \int_S \Omega^{p-2} D^2 \phi (\nabla' \omega, \nabla' \omega) \, d\sigma$$
$$- \beta (p\beta + p - N) \int_S \Omega^{p-2} \omega \langle \nabla' \omega, \nabla' \phi \rangle \, d\sigma. \tag{2.8}$$

Multiplying (2.1) by $\omega\phi$ we derive

$$\int_S \Omega^{p-2}\omega\langle\nabla'\omega, \nabla'\phi\rangle\, d\sigma$$
$$= -\int_S \Omega^{p-2}|\nabla'\omega|^2\phi\; d\sigma + \beta\lambda(\beta)\int_S \Omega^{p-2}\omega^2\phi\; d\sigma + \int_S \omega^{q+1}\phi\; d\sigma,$$

so that (2.8) becomes, replacing its last term,

$$\begin{aligned}\left(1-\frac{1}{p}\right)\int_{\partial S}|\omega_\nu|^p\phi_\nu\, dS &= \int_S\left(\frac{\omega^{q+1}}{q+1}-\frac{\Omega^p}{p}\right)\Delta'\phi\; d\sigma + \int_S \Omega^{p-2}D^2\phi(\nabla'\omega,\nabla'\omega)\, d\sigma\\ &\quad - \beta(p\beta+p-N)\int_S \omega^{q+1}\,\phi\; d\sigma + \beta(p\beta+p-N)\int_S \Omega^{p-2}|\nabla'\omega|^2\phi\, d\sigma\\ &\quad - \beta^2(p\beta+p-N)\lambda(\beta)\int_S \Omega^{p-2}\omega^2\phi\, d\sigma,\end{aligned}$$

which is (2.6). □

Proof of Proposition 2.1. We use Lemma 2.1 choosing in (2.6) ϕ to be the first eigenfunction of $-\Delta'$ in $W_0^{1,2}(S_+^{N-1})$. Since $\Delta'\phi = (1-N)\phi$ and $D^2\phi = -\phi g_0$, we get

$$\begin{aligned}\left(1-\frac{1}{p}\right)\int_{\partial S}|\omega_\nu|^p\phi_\nu\, dS = &-\int_S\left(\frac{N-1}{q+1}+\beta(p\beta+p-N)\right)\omega^{q+1}\phi\; d\sigma\\ &+\frac{N-1}{p}\int_S \Omega^p\phi\; d\sigma - \int_S \Omega^{p-2}|\nabla'\omega|^2\,\phi\; d\sigma\\ &+\beta(p\beta+p-N)\int_S \Omega^{p-2}|\nabla'\omega|^2\phi\; d\sigma\\ &-\beta^2(p\beta+p-N)\lambda(\beta)\int_S \Omega^{p-2}\omega^2\phi\, d\sigma.\end{aligned} \tag{2.9}$$

Then, using also the definition of Ω, (2.2) follows, with A, B and C given by (2.3)-(2.5). □

We shall say that a C^2 domain $S \subset S_+^{N-1}$ is *starshaped* if there exists a spherical harmonic ϕ of degree 1 such that $\phi > 0$ on S and for any $a \in \partial S$,

$$\langle\nabla\phi, \nu_a\rangle \leq 0 \tag{2.10}$$

where ν_a is the unit outward normal vector to ∂S at a in the tangent plane T_a to S^{N-1}. It also means that there exists some $x_0 \in S$ such that the geodesic connecting x_0 and a remains inside S.

Theorem 2.1. *Assume that* $1 < p < N-1$, $q = q_c$ *and* $S \subset S_+^{N-1}$ *is starshaped. Then* (2.1) *admits no positive solution in* S *vanishing on* ∂S.

Proof. Recall that in (1.8) we have $\beta_q = p/(q-(p-1))$, hence different values of q are in one-to-one correspondence with different values of β. We first notice that if $q = q_c$ then the corresponding critical β is given by

$$\beta_c := \frac{p}{q_c - (p-1)} = \frac{N-1-p}{p}. \tag{2.11}$$

We now use Proposition 2.1 with $\beta = \beta_q$ and we analyze the values of the coefficients A, B, C given by (2.3)–(2.5) as functions of β. First of all, since $q+1 = p(1+\beta)/\beta$, we have

$$A = -\frac{(N-1)\beta}{p(1+\beta)} - \beta(p\beta + p - N) = -\frac{\beta}{\beta+1}\left(\frac{N-1}{p} + p(\beta+1)^2 - N(\beta+1)\right),$$

and since from (2.11) we have $\beta_c + 1 = \frac{N-1}{p}$, we deduce

$$A = -\frac{\beta}{\beta+1}\, p\left(\beta + 1 - \frac{1}{p}\right)(\beta - \beta_c).$$

Still using (2.11), we also get

$$B = \beta_c + \beta(p(\beta - \beta_c) - 1) = (\beta - \beta_c)(\beta p - 1).$$

Finally, using (1.9) and (2.11) we have

$$\begin{aligned} C &= \beta^2\left(\frac{N-1}{p} - (p\beta + p - N)((p-1)\beta + p - N)\right) \\ &= \beta^2(\beta_c + 1 - (p(\beta - \beta_c) - 1)(p(\beta - \beta_c) - (\beta + 1))) \\ &= \beta^2(\beta - \beta_c)(1-p)\left(p\beta - 1 - \frac{p(N-p)}{p-1}\right). \end{aligned} \tag{2.12}$$

Therefore $A \geq 0$, $B \geq 0$ and $C \geq 0$ can be obtained only if $q = q_c$, i.e. $\beta = \beta_c$, in which case $A = B = C = 0$. Since $\phi_\nu \leq 0$ because S is starshaped, we deduce from (2.2) that $|\omega_\nu|^p \phi_\nu = 0$ on ∂S. Unless ω is identically zero, we have $\omega_\nu < 0$ by the Hopf lemma. Then $\phi_\nu \equiv 0$, and using the equation satisfied by ϕ and the Gauss formula, we derive

$$\lambda_S \int_S \phi \, d\sigma = 0, \quad \text{so} \quad \phi \equiv 0 \text{ in } S,$$

which is impossible since $\phi > 0$ in S_+^{N-1}. This proves the first assertion. □

Remark. If $p = 2$, it is proved in [3] that the nonexistence result of Theorem 2.1 holds for every $q \geq q_c$, which suggests that our result above is not optimal. The proof in [3] cannot be applied here since the term $\int_S \Omega^{p-2}\omega\langle\nabla'\omega, \nabla'\phi\rangle \, d\sigma$ is completely integrable only if $p = 2$. However, we conjecture that, even when $p \neq 2$, the conclusion of Theorem 2.1 holds under the more general condition $q \geq q_c$.

Remark. If we assume that $p \neq 2$, the proof of Theorem 2.1 relies on the existence of a positive function ϕ in S, satisfying (2.10) on ∂S and

$$\frac{\Delta'\phi}{(q+1)\phi} - \beta(p\beta + p - N) \geq 0, \tag{2.13}$$

$$\frac{pD^2\phi(\xi,\xi) - \Delta'\phi}{p\phi} + \beta(p\beta + p - N) \geq 0 \quad \forall \xi \in S^{N-1}, \tag{2.14}$$

$$-\frac{\Delta'\phi}{p\phi} - (p\beta + p - N)((p-1)\beta + p - N) \geq 0. \tag{2.15}$$

Remark 2.1. For completeness, we recall the nonexistence result obtained in [2, Th. 1]:

Let $\epsilon = 1$ and $0 < p - 1 < q$. If $\beta_q \geq \beta_S$, then there exists no positive solution of (1.8) *in S which vanishes on ∂S.*

3. Existence for the reaction problem

Concerning the problem with reaction we consider a more general statement than Theorem I, replacing the sphere by a complete d-dimensional Riemannian manifold (M, g) and supposing that S is a relatively compact smooth open domain of M. We denote by $\nabla := \nabla_g$ the gradient of a function identified with its covariant derivatives, and by $\mathrm{div} := \mathrm{div}_g$ the intrinsic divergence operator acting on vector fields. The following result is proved in [13].

Theorem 3.1. *For any $\beta > 0$ there exists a unique $\Lambda_\beta > 0$ and a unique (up to homothety) positive function $\omega_\beta \in C^2(S) \cap C^1(\overline{S})$ satisfying*

$$\begin{cases} -\mathrm{div}((\beta^2\omega_\beta^2 + |\nabla\omega_\beta|^2)^{(p-2)/2}\nabla\omega_\beta) = \beta\Lambda_\beta(\beta^2\omega_\beta^2 + |\nabla\omega_\beta|^2)^{(p-2)/2}\omega_\beta & \text{in } S, \\ \omega_\beta = 0 & \text{on } \partial S. \end{cases} \tag{3.1}$$

The mapping $\beta \mapsto \Lambda_\beta$ is continuous and decreasing, and the spectral exponent β_S is the unique $\beta > 0$ such that $\Lambda_{\beta_S} = \beta_S(p-1) + p - d - 1$.

Remark 3.1. Let us notice that the monotonicity character of $\beta \mapsto \Lambda_\beta$ implies that

$$0 < \beta < \beta_S \iff \Lambda_\beta - \beta(p-1) > \Lambda_{\beta_S} - \beta_S(p-1) = p - d - 1.$$

Therefore, if we set $\lambda(\beta) = \beta(p-1) + p - d - 1$, we deduce that

$$0 < \beta < \beta_S \iff \Lambda_\beta > \lambda(\beta). \tag{3.2}$$

Let us now prove the existence of solutions for the reaction problem.

Theorem 3.2. *Assume $1 < p < d$ and $p - 1 < q < q_c := pd/(d-p) - 1$. Then for any $0 < \beta < \beta_S$, there exists a positive function $\omega \in C(\overline{S}) \cap C^2(S)$ satisfying*

$$\begin{cases} -\mathrm{div}((\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\nabla\omega) = \beta\lambda(\beta)(\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\omega + \omega^q & \text{in } S, \\ \omega = 0 & \text{on } \partial S, \end{cases} \tag{3.3}$$

where $\lambda(\beta) = \beta(p-1) + p - d - 1$.

In order to prove Theorem 3.2, we use topological arguments as is often needed in a nonvariational setting. In particular, following a strategy similar to [15], our proof is based upon the following fixed point theorem which is a consequence of the Leray–Schauder degree theory to compute the fixed point index of compact mappings. Such results were developed mostly by Krasnosel'skiĭ [9]; we refer to Proposition 2.1 and Remark 2.1 in [5] for the statement below.

Theorem 3.3. *Let X be a Banach space and $K \subset X$ a closed cone with non-empty interior. Let $F : K \times \mathbb{R}_+ \to K$ be a compact mapping, and let $\Phi(u) = F(u, 0)$ (a compact mapping from K into K). Assume that there exist $R_1 < R_2$ and $T > 0$ such that*

(i) *$u \neq s\Phi(u)$ for every $s \in [0, 1]$ and every u with $\|u\| = R_1$.*
(ii) *$F(u, t) \neq u$ for every (u, t) with $\|u\| \leq R_2$ and $t \geq T$.*
(iii) *$F(u, t) \neq u$ for every u with $\|u\| = R_2$ and every $t \geq 0$.*

Then the mapping Φ has a fixed point u such that $R_1 < \|u\| < R_2$.

We also recall the following nonexistence results respectively due to Serrin and Zou [17] and Zou [23].

Theorem 3.4. *Assume $1 < p < d$ and $p - 1 < q < q_c$. Then there exists no positive C^1 solution of*

$$-\Delta_p u = u^q \tag{3.4}$$

in $\mathbb{R}^d$.

Theorem 3.5. *Assume $1 < p < d$ and $p - 1 < q < q_c$. Then there exists no positive C^1 solution of*

$$-\Delta_p u = u^q \tag{3.5}$$

in $\mathbb{R}^d_+ := \{x = (x_1, \ldots, x_d) : x_d > 0\}$ vanishing on $\partial\mathbb{R}^d_+ := \{x = (x_1, \ldots, x_d) : x_d = 0\}$.

Proof of Theorem 3.2. Define the operator $\mathcal{A}$ in $W_0^{1,p}(S)$ as

$$\mathcal{A}(\omega) := -\operatorname{div}_g((\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\nabla\omega) + \beta^2\,\omega(\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1}.$$

Note that $\mathcal{A}$ is the derivative of the functional

$$J(w) = \frac{1}{p}\int_S (\beta^2\omega^2 + |\nabla\omega|^2)^{p/2}\,dv_g.$$

Since J is strictly convex, $\mathcal{A}$ is a strictly monotone operator from $W_0^{1,p}(S)$ into $W^{-1,p'}(S)$, hence its inverse is well defined and continuous [12]. In order to apply Theorem 3.3, we denote by $X = C_0^1(\overline{S})$ the closure of $C_0^1(S)$ in $C^1(\overline{S})$. Clearly $X \subset W_0^{1,p}(S)$, with continuous imbedding, if it is endowed with its natural norm $\|\cdot\|_X := \|\cdot\|_{C^1(\overline{S})}$. Furthermore, since ∂S is C^2, $C^1(\overline{S}) \cap W_0^{1,p}(S) = C_0^1(\overline{S})$. If K is the cone of nonnegative functions in S, it has a nonempty interior. For $t > 0$, we set

$$F(\omega, t) := \mathcal{A}^{-1}\big(\beta(\lambda(\beta) + \beta + t)\omega(\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1} + (\omega + t)^q\big).$$

Note that

$$\Phi(\omega) := F(\omega, 0) = \mathcal{A}^{-1}\big(\beta(\lambda(\beta)+\beta)\omega(\beta^2\omega^2+|\nabla\omega|^2)^{p/2-1}+\omega^q\big);$$

hence any nontrivial fixed point for Φ would solve problem (3.3).

We have to verify the assumptions of Theorem 3.3. First of all, the compactness of $F(\omega,t)$: If we set $F(\omega,t)=\phi$, then it means that $\phi\in W_0^{1,p}(S)$ satisfies

$$\begin{aligned}-\operatorname{div}_g((\beta^2\phi^2+|\nabla\phi|^2)^{p/2-1}\nabla\phi)+\beta^2\,\phi(\beta^2\phi^2+|\nabla\phi|^2)^{p/2-1}\\ =\big(\beta(\lambda(\beta)+\beta+t)\omega(\beta^2\omega^2+|\nabla\omega|^2)^{p/2-1}+(\omega+t)^q\big).\end{aligned}\tag{3.6}$$

Thus, if we assume that ω belongs to a bounded set in $K\cap X$, the right-hand side of (3.6) is bounded in $C(\overline{S})$. Hence, by standard regularity estimates up to the boundary for p-Laplace type operators (see [13, Appendix] and [6], [19]), ϕ remains bounded in $C^{1,\alpha}(\overline{S})$ and therefore relatively compact in $C^1(\overline{S})$. It remains to show that conditions (i)–(iii) of Theorem 3.3 hold.

Step 1: Condition (i) holds. Suppose for contradiction that there exists a sequence $\{s_n\}$ in $[0,1]$ such that for any $n\in\mathbb{N}$ the problem

$$\begin{cases}-\operatorname{div}_g((\beta^2\omega^2+|\nabla\omega|^2)^{p/2-1}\nabla\omega)+\beta^2(\beta^2\omega^2+|\nabla\omega|^2)^{p/2-1}\omega\\ \quad =s^{p-1}\beta(\lambda(\beta)+\beta)(\beta^2\omega^2+|\nabla\omega|^2)^{p/2-1}\omega+s_n^{p-1}\omega^q \quad \text{in } S,\\ \omega=0 \quad \text{on } \partial S,\end{cases}\tag{3.7}$$

admits a positive solution ω_n, and that

$$\|\omega_n\|_X\to 0 \qquad \text{as } n\to\infty.$$

Set $w_n=\omega_n/\|\omega_n\|$; then w_n solves

$$\begin{cases}-\operatorname{div}_g((\beta^2w_n^2+|\nabla w_n|^2)^{p/2-1}\nabla w_n)+\beta^2 w_n(\beta^2w_n^2+|\nabla w_n|^2)^{p/2-1}\\ \quad =s_n^{p-1}\beta(\lambda(\beta)+\beta)(\beta^2w_n^2+|\nabla w_n|^2)^{p/2-1}w_n+s_n^{p-1}w_n^q\,\|w_n\|_X^{q-(p-1)} \quad \text{in } S,\\ w_n=0 \quad \text{on } \partial S.\end{cases}$$

Up to subsequences, we assume that $s_n\to s$ for some $s\in[0,1]$. Using compactness arguments we deduce that w_n will converge strongly in $C^1(\overline{S})$ to some positive function w such that $\|w\|_X=1$ and

$$\begin{cases}-\operatorname{div}_g((\beta^2w^2+|\nabla w|^2)^{p/2-1}\nabla w)\\ \quad =\beta(s^{p-1}\lambda(\beta)+(s^{p-1}-1)\beta)(\beta^2w^2+|\nabla w|^2)^{p/2-1}w \quad \text{in } S,\\ w=0 \quad \text{on } \partial S.\end{cases}\tag{3.8}$$

Using Theorem 3.1, we derive $\Lambda_\beta=s^{p-1}\lambda(\beta)+(s^{p-1}-1)\beta$. Since $\beta<\beta_S$, we have $\lambda(\beta)<\Lambda_\beta$ by (3.2). Therefore, as $s\le 1$, we get

$$s^{p-1}\lambda(\beta)+(s^{p-1}-1)\beta\le s^{p-1}\lambda(\beta)<\Lambda_\beta,$$

which is a contradiction. Consequently, there exists $R_1 > 0$ such that for any $s \in [0,1]$, we have $\omega \neq s\Phi(\omega)$ for any ω such that $\|\omega\|_X = R_1$.

Step 2: Condition (ii) holds. Consider the first eigenvalue $\lambda_{1,\beta}$ associated with the operator $\mathcal{A}$, i.e.

$$\lambda_{1,\beta} = \min\left\{\int_S (\beta^2\omega^2 + |\nabla\omega|^2)^{p/2}\,dv_g : \omega \in W_0^{1,p}(S),\ \int_S |\omega|^p\,dv_g = 1\right\}. \tag{3.9}$$

Note that for t large enough, we have $\lambda(\beta) + \beta + t \geq 0$, hence, using that $q > p-1$, we can find $T > 0$ such that

$$\beta(\lambda(\beta)+\beta+t)\omega(\beta^2\omega^2+|\nabla\omega|^2)^{p/2-1} + (\omega+t)^q \geq (\lambda_1+\delta)\omega^{p-1} \quad \forall t \geq T,\ \forall\omega \geq 0.$$

Therefore, if $t \geq T$ and $F(\omega,t) = \omega$ we deduce that $\omega \neq 0$ and satisfies

$$\begin{cases} -\operatorname{div}_g((\beta^2\omega^2+|\nabla\omega|^2)^{p/2-1}\nabla\omega)+\beta^2\omega(\beta^2\omega^2+|\nabla\omega|^2)^{p/2-1} \geq (\lambda_{1,\beta}+\delta)\omega^{p-1} \text{ in } S,\\ \omega = 0 \quad \text{on } \partial S. \end{cases}$$

The existence of a positive supersolution with $\lambda_{1,\beta}+\delta$ would make it possible to construct a positive solution as well. But since $\lambda_{1,\beta}$ is an isolated eigenvalue (see Appendix) this yields a contradiction. Therefore, for $t \geq T$ the equation $F(\omega,t) = \omega$ has no solution at all. Note that T only depends on λ_1, β.

Step 3: Condition (iii) holds. Since we proved that (ii) holds independently of the choice of R_2, it is enough to show that (iii) holds for every $t \leq T$.

This is done if we have the existence of universal a priori estimates, i.e. if we can prove the existence of a constant R_2 such that for any $t \leq T$ every positive solution of

$$\begin{cases} -\operatorname{div}_g((\beta^2\omega^2+|\nabla\omega|^2)^{p/2-1}\nabla\omega)+\beta^2\omega(\beta^2\omega^2+|\nabla\omega|^2)^{p/2-1}\\ \quad = \beta(\lambda(\beta)+\beta+t)(\beta^2\omega^2+|\nabla\omega|^2)^{p/2-1}\omega + (\omega+t)^q \quad \text{in } S,\\ \omega = 0 \quad \text{on } \partial S, \end{cases}$$

satisfies $\|\omega\| < R_2$.

The crucial step is to prove that there exist universal a priori estimates for the L^∞-norm (a bound for the $W_0^{1,p}$-norm would follow immediately, and then a bound in X from regularity theory). A standard procedure is to reach this result reasoning by contradiction and using a blow-up argument. Indeed, if a universal bound does not exist, there exist a sequence of solutions ω_n and $t_n \leq T$ such that

$$\|\omega_n\|_\infty \to \infty.$$

Let σ_n be the (local coordinates of) maximum points of ω_n; up to subsequences, we have $\sigma_n \to \sigma_0 \in \overline{S}$. Setting $M_n = \|\omega_n\|_\infty^{-(q-(p-1))/p}$, define

$$v_n(y) = \frac{\omega_n(\sigma_n + M_n y)}{\|\omega_n\|_\infty} = M_n^{p/(q-(p-1))}\,\omega_n(\sigma_n + M_n y).$$

Then v_n is a sequence of uniformly bounded solutions, which will be locally compact in the C^1-topology. Rescaling the equation and passing to the limit in n we find that the limit function v is positive and satisfies the equation

$$-\Delta_p v = c_0 v^q$$

for some constant c_0 (coming from the local expression of the Laplace–Beltrami operator). Depending on whether $\sigma_0 \in S$ or $\sigma_0 \in \partial S$, the equation holds either in $\mathbb{R}^d$ or in the half-space $\mathbb{R}^d_+$, where $d = N-1$, in which case v vanishes on $\partial\mathbb{R}^d_+$. Since $p-1 < q < q_c$, this contradicts either Theorem 3.4, or Theorem 3.5, because, by construction, we have $v(0) = 1$. □

Remark. In the case $p = 2$, existence is proved in [3] using a standard variational method. It is also proved that, if $(M, g) = (S^d, g_0)$ (the standard sphere), and if S is a spherical cap with center a, then any positive solution of

$$\begin{cases} \Delta'\omega + \beta(\beta+1-d))\omega + \omega^q = 0 & \text{in } S, \\ \omega = 0 & \text{on } \partial S, \end{cases} \tag{3.10}$$

depends only on the angle θ from a. Furthermore, uniqueness is proved by a delicate analysis of the nonautonomous second order O.D.E. satisfied by ω. In the case $p \neq 2$ and assuming always that S is a spherical cap of (S^d, g_0), it is still possible to construct a radial (i.e. depending only on θ) positive solution of (3.3): it suffices to restrict the functional analysis framework to radial functions. However, there are two interesting open questions the answer to which would be important:

(i) *Are all positive solutions of* (3.3) *radial?*
(ii) *Is there uniqueness of positive radial solutions of* (3.3)*?*

4. Existence for the absorption problem

Let us now consider the absorption problem, i.e. (1.8) with $\epsilon = -1$. We give an existence result which extends the previous ones obtained in [20], with a simpler proof.

Theorem 4.1. *Assume* $0 < p-1 < q$. *Then for any* $\beta > \beta_S$, *there exists a unique positive function* $\omega \in C(\overline{S}) \cap C^2(S)$ *satisfying*

$$\begin{cases} -\operatorname{div}_g((\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\nabla\omega) = \beta\lambda(\beta)(\beta^2\omega^2+|\nabla\omega|^2)^{p/2-1}\omega - \omega^q & \text{in } S, \\ \omega = 0 & \text{on } \partial S, \end{cases} \tag{4.1}$$

where $\lambda(\beta) = \beta(p-1) + p - d - 1$.

To prove Theorem 4.1, we will need the following lemma.

Lemma 4.1. *For* $\beta > 0$ *and* $p > 1$, *let* Λ_β *and* β_S *be defined by Theorem* 3.1. *Then both* Λ_β *and* β_S *are continuous functions of* p, *varying in* $(1, \infty)$.

Proof. By Theorem 3.1, Λ_β is uniquely defined for any fixed $p > 1$. To emphasize the dependence of Λ_β on p, let us denote it now by $\Lambda_{\beta,p}$. The continuity of $\Lambda_{\beta,p}$ with respect to p can be proved in the same way as we proved (see Proposition 2.4 in [13]) the continuity of $\Lambda_{\beta,p}$ with respect to β. Thus, we only sketch the argument, which relies on the construction itself of $\Lambda_{\beta,p}$. Indeed, we proved in [13] that $\Lambda_{\beta,p}$ is the unique constant such that there exists a function $v \in C^2(S)$ satisfying

$$\begin{cases} -\Delta_g v - (p-2)\dfrac{D^2 v \nabla v.\nabla v}{1+|\nabla v|^2} + \beta(p-1)|\nabla v|^2 = -\Lambda_{\beta,p} & \text{in } S, \\ \lim\limits_{\sigma\to\partial S} v(\sigma) = \infty. \end{cases} \tag{4.2}$$

If we normalize v by setting, for example, $v(\sigma_0) = 0$ for some $\sigma_0 \in S$, then v is unique. Moreover $v \in C^2(S)$ and v satisfies estimates in $W^{1,\infty}_{\text{loc}}(S)$ which are uniform as $\beta \in (0,\infty)$ and $p \in (1,\infty)$ vary in compact sets. It is also easy to check (see [13]) that $\Lambda_{\beta,p}$ remains bounded whenever β varies in a compact subset of $(0,\infty)$ and p vary in a compact subset of $(1,\infty)$. The estimates obtained on v and ∇v imply that, whenever β_n or p_n are convergent sequences, the sequence of the corresponding solutions v_n of (4.2) (such that $v_n(\sigma_0) = 0$) is relatively compact (locally uniformly in C^1). The equation (4.2) turns out then to be stable (including the boundary estimates); finally, the uniqueness property of $\Lambda_{\beta,p}$, and of the associated (normalized) solution v, implies the continuity of $\Lambda_{\beta,p}$ with respect to both β and p.

Let now $\beta_{S,p}$ be the spectral exponent defined by the equation

$$\Lambda_{\beta,p} = \beta(p-1) + p - d - 1. \tag{4.3}$$

First of all note that when p lies in a compact set in $(1,\infty)$, then necessarily $\beta_{S,p}$ is bounded. Indeed, since $\Lambda_{\beta,p} \le \Lambda_{1,p}$ whenever $\beta \ge 1$, we have

$$\beta_S(p-1) + p - d - 1 \le \Lambda_{1,p} \quad \text{if } \beta_S \ge 1,$$

so that

$$\beta_S \le 1 + \frac{1}{p-1}(\Lambda_{1,p} - (p-d-1)).$$

Therefore, if p belongs to a compact set in $(1,\infty)$, then β_S remains also in a bounded set. Now, if $p_n \to p_0$, setting $\beta_n = \beta_{S,p_n}$, we see that β_n is bounded and, up to subsequences, it is convergent to some β_0. From (4.3), we deduce that Λ_{β_n,p_n} is bounded, which implies that β_n cannot converge to zero, hence $\beta_0 > 0$. Then, using the continuity of $\Lambda_{\beta,p}$, we can pass to the limit in (4.3) and we deduce that β_0 is the spectral exponent with $p = p_0$, i.e. $\beta_0 = \beta_{S,p_0}$. This proves that $\beta_{S,p}$ is continuous with respect to p. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1

Step 1: construction of a solution. We use similar ideas to the proof of Theorem 3.2, i.e. a topological degree argument. On the Banach space $X = C^1_0(\overline{S})$ (endowed with its natural norm) with positive cone K, we set

$$\mathcal{B}(\omega) = -\operatorname{div}_g((\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\nabla\omega) + \beta^2(\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\omega + |\omega|^{q-1}\omega,$$
$$\Psi(\omega) = \mathcal{B}^{-1}\big(\beta(\lambda(\beta) + \beta)(\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\omega_+\big).$$

Clearly, $\Psi(w) = w$ implies that $w \geq 0$ and solves (4.1). Then, it is enough to prove the existence of a nontrivial fixed point for Ψ. Observe that, as in Theorem 3.2, Ψ is a continuous compact operator in X thanks to the $C^{1,\alpha}$ estimates for p-Laplace operators, and $\Psi(K) \subset K$.

We now wish to compute the degree of $I - \Psi$. First of all we consider, if R is sufficiently large, $\deg(I - \Psi, B_R^+, 0)$ where $B_R^+ = B_R \cap K$ for $t \in [0, 1]$. To this end, define $\Psi^*(\omega, t) = t\Psi(\omega)$. Then Ψ^* is a compact map on $X \times [0, 1]$ and if $\Psi^*(\omega, t) = \omega$, we have

$$-\operatorname{div}_g((\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\nabla\omega) + \beta^2(\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\omega + \frac{1}{t^{q-(p-1)}}\omega^q = t^{p-1}\beta(\lambda(\beta) + \beta)(\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\omega. \tag{4.4}$$

We get, by the maximum principle,

$$\left\|\frac{\omega}{t}\right\|_\infty^{q-(p-1)} \leq t^{p-1}\beta^{p-1}(\lambda(\beta) + \beta) \leq \beta^{p-1}(\lambda(\beta) + \beta).$$

Since $t \leq 1$, we deduce in particular that $\|\omega\|_\infty$ is bounded independently of t. Then, we have

$$\frac{1}{t^{q-(p-1)}}\omega^q \leq \left\|\frac{\omega}{t}\right\|_\infty^{q-(p-1)} \|\omega\|_\infty^{p-1} \leq C\|\omega\|_\infty^{p-1} \leq C.$$

Multiplying by ω we obtain a similar bound for $\|\omega\|_{W_0^{1,p}(S)}$, and the regularity theory for p-Laplace type equations yields a further estimate on $\|\nabla\omega\|_\infty$. Therefore, we conclude that there exists a constant M, independent of $t \in [0, 1]$, such that $t\Psi(\omega) = \omega$ implies $\|\omega\|_X \leq M$. As a consequence, if R is sufficiently large we have $t\Psi(\omega) \neq \omega$ on ∂B_R. We deduce that $\deg(I - t\Psi, B_R^+, 0)$ is constant. Therefore

$$\deg(I - \Psi, B_R^+, 0) = \deg(I - t\Psi, B_R^+, 0) = \deg(I, B_R^+, 0) = 1.$$

Next, we compute $\deg(I - \Psi, B_r^+, 0)$ for small r. We set

$$\mathcal{B}_t(\omega) = -\operatorname{div}_g((\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\nabla\omega) + \beta^2(\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\omega + t|\omega|^{q-1}\omega,$$
$$F(\omega, t) = \mathcal{B}_t{}^{-1}\big(\beta(\lambda(\beta) + \beta)\omega_+(\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\big).$$

Again, we have $\Psi(\cdot) = F(\cdot, 1)$. We claim that there exists a small $r > 0$ such that $F(\omega, t) \neq \omega$ for every $t \in [0, 1]$ and $\omega \in \partial B_r$. Indeed, if this were not true there would exist a nonnegative sequence ω_n such that $0 \neq \|\omega_n\| \to 0$, and $t_n \in [0,1]$ such that $F(\omega_n, t_n) = \omega_n$, which means that

$$
\begin{aligned}
-\operatorname{div}_g((\beta^2\omega_n^2 + |\nabla\omega_n|^2)^{p/2-1}\nabla\omega_n) + \beta^2(\beta^2\omega_n^2 + |\nabla\omega_n|^2)^{p/2-1}\omega_n + t_n\omega_n^q \\
= \beta(\lambda(\beta)+\beta)\omega_n(\beta^2\omega_n^2 + |\nabla\omega_n|^2)^{p/2-1}.
\end{aligned}
$$

Dividing by $\|\omega_n\|^{p-1}$ and letting $n \to \infty$, we find that $\omega_n/\|\omega_n\|$ would converge to some function $\hat{\omega}$ such that $\hat{\omega} \geq 0$, $\|\hat{\omega}\| = 1$ and

$$
\begin{aligned}
-\operatorname{div}_g((\beta^2\hat{\omega}^2 + |\nabla\hat{\omega}|^2)^{p/2-1}\nabla\hat{\omega}) + \beta^2(\beta^2\hat{\omega}^2 + |\nabla\hat{\omega}|^2)^{p/2-1}\hat{\omega} \\
= \beta(\lambda(\beta)+\beta)\hat{\omega}(\beta^2\hat{\omega}^2 + |\nabla\hat{\omega}|^2)^{p/2-1}.
\end{aligned}
$$

By Theorem 3.1 this means that $\lambda(\beta) = \Lambda_\beta$, which is not possible since $\lambda(\beta) > \Lambda_\beta$ because $\beta > \beta_S$ (see Remark 3.1). We conclude that $F(\omega,t) \neq \omega$ for every $t \in [0,1]$ and $\omega \in \partial B_r$ provided r is sufficiently small. We deduce that $\deg(I - F(\cdot,t), B_r, 0)$ is constant and in particular

$$
\deg(I - \Psi, B_r^+, 0) = \deg(I - F(\cdot,0), B_r^+, 0).
$$

In order to compute this degree, we perform a homotopy acting on p and β by setting $p_t = 2t + (1-t)p$ and by taking β_t so that $t \mapsto \beta_t$ is continuous on $[0,1]$, $\beta_0 = \beta$, $\beta_t > \beta_{S,p_t}$ for every $t \in [0,1]$ (where β_{S,p_t} is the spectral exponent for S with $p = p_t$) and $\beta_1 > 0$ is large enough. It follows from Lemma 4.1 that β_{S,p_t} is a continuous function of t and remains bounded as $t \in [0,1]$. Therefore, a similar choice of a function β_t is possible. In the space $C_0^1(\overline{S})$ we define the mapping $\mathcal{C}_t$ by

$$
\mathcal{C}_t(\omega) = -\operatorname{div}_g((\beta_t^2\omega^2 + |\nabla\omega|^2)^{p_t/2-1}\nabla\omega) + \beta_t^2(\beta_t^2\omega^2 + |\nabla\omega|^2)^{p_t/2-1}\omega.
$$

We set

$$
\tilde{F}(\omega,t) = \mathcal{C}_t^{-1}\big(\beta_t(\lambda(\beta_t)+\beta_t)(\beta_t^2\omega^2 + |\nabla\omega|^2)^{p_t/2-1}\omega\big).
$$

Combining Tolksdorf's construction [19] which shows the uniformity with respect to p_t of the $C^{1,\alpha}$ estimates (with $\alpha = \alpha_t \in (0,1)$), with the perturbation method of [13, Th. A1], we deduce that $(\omega,t) \mapsto \tilde{F}(\omega,t)$ is compact in $C_0^1(\overline{S}) \times [0,1]$. Since $\beta_t > \beta_{S,p_t}$, clearly $I - \tilde{F}(\cdot,t)$ does not vanish on $\|\omega\|_X = r$ for any $r > 0$, which implies that

$$
\deg(I - \Psi, B_r^+, 0) = \deg(I - \tilde{F}(\cdot,0), B_r^+, 0) = \deg(I - \tilde{F}(\cdot,1), B_r^+, 0).
$$

But

$$
I - \tilde{F}(\cdot,1) = I - \beta_1(\lambda(\beta_1)+\beta_1)(-\Delta_g + \beta_1^2)^{-1}.
$$

Since $-\Delta_g$ has only one eigenvalue in S with positive eigenfunction and multiplicity one, choosing β_1 so large that $\lambda(\beta_1)\beta_1 > \lambda_1(S)$ it follows that

$$
\deg(I - \tilde{F}(\cdot,1), B_r^+, 0) = -1 = \deg(I - \Psi, B_r^+, 0).
$$

To conclude, since

$$\deg(I-\Psi, B_R^+ \setminus \overline{B}_r^+, 0) = \deg(I-\Psi, B_R^+, 0) - \deg(I-\Psi, B_r^+, 0) \neq 0$$

we deduce the existence of some ω such that $r < \|\omega\| < R$ which is a solution of (4.1).

Step 2: uniqueness. If ω is any positive solution, then $\beta^2\omega^2 + |\nabla\omega^2|$ is positive in $\overline{S}$. This is obvious in S and it is a consequence of the Hopf boundary lemma on ∂S. Let $\overline{\omega}$ and ω be two positive solutions. Either the two functions are ordered or their graphs intersect. Since all the solutions are positive in S and satisfy the Hopf boundary lemma, we can define

$$\theta := \inf\{s \geq 1 : s\omega \geq \overline{\omega}\},$$

and denote $\omega^* := \theta\omega$. Either the graphs of $\overline{\omega}$ and $\omega^* := \theta\omega$ are tangent at some interior point $\alpha \in S$, or $\omega^* > \overline{\omega}$ in S and there exists $\alpha \in \partial S$ such that $\overline{\omega}_\nu(\alpha) = \omega^*_\nu(\alpha) < 0$. We put $w = \overline{\omega} - \omega^*$ and use local coordinates $(\sigma_1, \ldots, \sigma_d)$ on M near α. We denote by $g = (g_{ij})$ the metric tensor on M and g^{jk} its contravariant components. Then, for any $\varphi \in C^1(S)$,

$$|\nabla\varphi|^2 = \sum_{j,k} g^{jk} \frac{\partial\varphi}{\partial\sigma_j}\frac{\partial\varphi}{\partial\sigma_k} = \langle\nabla\varphi, \nabla\varphi\rangle_g.$$

If $X = (X^1, \ldots, X^d) \in C^1(TM)$ is a vector field, if we lower indices by setting $X^\ell = \sum_i g^{\ell i} X_i$, then

$$\mathrm{div}_g\, X = \frac{1}{\sqrt{|g|}} \sum_\ell \frac{\partial}{\partial\sigma_\ell}(\sqrt{|g|}X^\ell) = \frac{1}{\sqrt{|g|}} \sum_{\ell,i} \frac{\partial}{\partial\sigma_\ell}(\sqrt{|g|}g^{\ell i}X_i).$$

By the mean value theorem applied to

$$t \mapsto \Phi(t) = (\beta^2(\omega^* + tw)^2 + |\nabla(\omega^* + tw)|^2)^{p/2-1}(\omega^* + tw), \qquad t \in [0,1],$$

we have, for some $t \in (0,1)$,

$$(\beta^2\overline{\omega}^2 + |\nabla\overline{\omega}|^2)^{p/2-1}\overline{\omega} - (\beta^2\omega^{*2} + |\nabla\omega^*|^2)^{p/2-1}\omega^* = \sum_j a_j \frac{\partial w}{\partial\sigma_j} + bw,$$

where

$$b = \left(\beta^2(\omega^* + tw)^2 + |\nabla(\omega^* + tw)|^2\right)^{p/2-2}\left((p-1)\beta^2(\omega^* + tw)^2 + |\nabla(\omega^* + tw)|^2\right)$$

and

$$a_j = (p-2)(\beta^2(\omega^* + tw)^2 + |\nabla(\omega^* + tw)|^2)^{p/2-2}(\omega^* + tw) \sum_k g^{jk} \frac{\partial(\omega^* + tw)}{\partial\sigma_k}.$$

Considering now

$$t \mapsto \Phi_i(t) = (\beta^2(\omega^* + tw)^2 + |\nabla(\omega^* + tw)|^2)^{p/2-1} \frac{\partial(\omega^* + tw)}{\partial\sigma_i}, \qquad t \in [0,1],$$

we see that there exists some $t_i \in (0, 1)$ such that

$$(\beta^2\overline{\omega}^2 + |\nabla\overline{\omega}|^2)^{p/2-1}\frac{\partial\overline{\omega}}{\partial\sigma_i} - (\beta^2\omega^{*2} + |\nabla\omega^*|^2)^{p/2-1}\frac{\partial\omega^*}{\partial\sigma_i} = \sum_j a_{ij}\frac{\partial w}{\partial\sigma_j} + b_i w,$$

where

$$b_i = (p-2)(\beta^2(\omega^* + t_i w)^2 + |\nabla(\omega^* + t_i w)|^2)^{p/2-2}\beta^2(\omega^* + t_i w)\frac{\partial(\omega^* + t_i w)}{\partial\sigma_i}$$

and

$$a_{ij} = (p-2)(\beta^2(\omega^* + t_i w)^2 + |\nabla(\omega^* + t_i w)|^2)^{p/2-2}\frac{\partial(\omega^* + t_i w)}{\partial\sigma_i}\sum_k g^{jk}\frac{\partial(\omega^* + t_i w)}{\partial\sigma_k}$$
$$+ \delta_i^j(\beta^2(\omega^* + t_i w)^2 + |\nabla(\omega^* + t_i w)|^2)^{p/2-1}.$$

Set $P = \omega^*(\alpha) = \overline{\omega}(\alpha)$ and $Q = \nabla\omega^*(\alpha) = \nabla\overline{\omega}(\alpha)$. Then $P^2 + |Q|^2 > 0$ and

$$b_i(\alpha) = (p-2)(\beta^2 P^2 + |Q|^2)^{p/2-2}\beta^2 P Q_i,$$

and

$$a_{ij}(\alpha) = (\beta^2 P^2 + |Q|^2)^{p/2-2}\Big(\delta_i^j(\beta^2 P^2 + |Q|^2) + (p-2)Q_i\sum_k g^{jk}Q_k\Big).$$

Because ω^* is a supersolution for (4.1), the function w satisfies

$$-\frac{1}{\sqrt{|g|}}\sum_{\ell,j}\frac{\partial}{\partial\sigma_\ell}\left(A_{j\ell}\frac{\partial w}{\partial\sigma_j}\right) + \sum_i C_i\frac{\partial w}{\partial\sigma_i} + Dw \le 0 \qquad (4.5)$$

where the C_i and D are continuous functions and

$$A_{j\ell} = \sqrt{|g|}\sum_i g^{\ell i}a_{ij}.$$

The matrix $(a_{ij})(\alpha)$ is symmetric and positive definite since it is the Hessian of

$$x = (x_1, \ldots, x_d) = \frac{1}{p}(P^2 + |x|^2)^{p/2} = \frac{1}{p}\Big(P^2 + \sum_{j,k} g^{jk}x_j x_k\Big)^{p/2}.$$

Therefore the matrix $(A_{j\ell})$ keeps the same property in a neighborhood of a. Since w is nonpositive and vanishes at some $a \in S$, or $w < 0$ and $w_\nu = 0$ at some boundary point, it follows from the strong maximum principle or the Hopf boundary lemma (see [14]) that $w \equiv 0$, i.e. $\theta\omega = \overline{\omega}$. This implies that actually $\theta = 1$ and $\omega = \overline{\omega}$. □

5. Appendix

Here we prove the following result:

Theorem 5.1. *Let S be a subdomain of a complete d-dimensional Riemannian manifold (M, g). If $\beta > 0$ and $p > 1$, then the first eigenvalue $\lambda_{1,\beta}$ of the operator $\omega \mapsto -\operatorname{div}((\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\nabla\omega) + \beta^2\omega(\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1}$ in $W_0^{1,p}(S)$ is isolated. Furthermore any corresponding eigenfunction has constant sign.*

Proof. The proof is an adaptation of the original one due to Anane [1] and Lindqvist [10, 11] when $\beta = 0$. We recall that

$$\lambda_{1,\beta} = \inf\left\{\int_S (\beta^2\omega^2 + |\nabla\omega|^2)^{p/2}\, dv_g : \omega \in W_0^{1,p}(S),\ \int |\omega|^p\, dv_g = 1\right\}, \tag{5.1}$$

and that there exists $\omega \in W_0^{1,p}(S) \cap C^{1,\alpha}(S)$ such that

$$-\operatorname{div}((\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\nabla\omega) + \beta^2\omega(\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1} = \lambda_{1,\beta}|\omega|^{p-2}\omega \quad \text{in } S. \tag{5.2}$$

The function $|\omega|$ is also a minimizer for $\lambda_{1,\beta}$, thus it is a positive solution of (5.2). By the Harnack inequality [16], for any compact subset K of S, there exists C_K such that

$$\frac{|\omega|(\sigma_1)}{|\omega|(\sigma_2)} \leq C_K \quad \forall \sigma_i \in K,\ i = 1, 2.$$

Thus any minimizer ω must keep a constant sign in S. If $\lambda_{1,\beta}$ is not isolated, there exists a decreasing sequence $\{\mu_n\}$ of real numbers converging to $\lambda_{1,\beta}$ and a sequence of functions $\omega_n \in W_0^{1,p}(S)$ satisfying

$$-\operatorname{div}(\beta^2\omega_n^2 + |\nabla\omega_n|^2)^{p/2-1}\nabla\omega_n) + \beta^2\omega_n(\beta^2\omega_n^2 + |\nabla\omega_n|^2)^{p/2-1} = \mu_n|\omega_n|^{p-2}\omega_n \quad \text{in } S \tag{5.3}$$

such that $\|\omega_n\|_{L^p(S)} = 1$. By standard compactness and regularity results, we can assume that $\omega_n \to \overline{\omega}$ weakly in $W_0^{1,p}(S)$ and strongly in $L^p(S)$. Thus

$$\int_S (\beta^2\overline{\omega}^2 + |\nabla\overline{\omega}|^2)^{p/2}\, dv_g \leq \liminf_{n\to\infty} \int_S (\beta^2\omega_n^2 + |\nabla\omega_n|^2)^{p/2}\, dv_g = \lambda_{1,\beta},$$

which implies that $\overline{\omega}$ is an eigenfunction associated with $\lambda_{1,\beta}$.

We observe that ω_n cannot have constant sign. Indeed, if ω_n were positive in Ω, we could proceed as in the proof of Theorem 4.1, Step 2; up to rescaling ω_n, we could assume that $w = \omega - \omega_n$ is nonpositive, is not zero, and the graphs of ω and ω_n are tangent. In that case, using (5.2) and (5.3), we see that w satisfies a nondegenerate elliptic equation (as in (4.5)), and we obtain a contradiction either by the strict maximum principle or by the Hopf lemma. Thus, any eigenfunction ω_n must change sign in Ω. Set $S_n^+ = \{\sigma \in S : \omega_n(\sigma) > 0\}$ and $S_n^- = \{\sigma \in S : \omega_n(\sigma) < 0\}$. Clearly, for $0 < \theta < 1$,

$$\int_{S_n^\pm} (\beta^2\omega_n^2 + |\nabla\omega_n|^2)^{p/2}\, dv_g \geq (1-\theta)\beta^p \int_{S_n^\pm} |\omega_n|^p\, dv_g + \theta \int_{S_n^\pm} |\nabla\omega_n|^p\, dv_g.$$

It follows from (5.3), multiplying by ω_n^+, that

$$\int_{S_n^+} (\beta^2 \omega_n^2 + |\nabla \omega_n|^2)^{p/2} \, dv_g = \mu_n \int_{S_n^+} |\omega_n|^p \, dv_g,$$

hence

$$\mu_n \int_{S_n^+} |\omega_n|^p \, dv_g \geq (1-\theta)\beta^p \int_{S_n^+} |\omega_n|^p \, dv_g + \theta \int_{S_n^+} |\nabla \omega_n|^p \, dv_g.$$

Since for some suitable $q > p$ (for example $q = p^*$ if $p < d$, or any $p < q < \infty$ if $p \geq d$)

$$\int_{S_n^+} |\nabla \omega_n|^p \, dv_g \geq c(p,q) \left(\int_{S_n^+} |\omega_n|^q \, dv_g \right)^{p/q} \geq c(p,q) |S_n^+|^{(p-q)/q} \int_{S_n^+} |\omega_n|^p \, dv_g$$

we obtain

$$\mu_n \geq (1-\theta)\beta^p + \theta c(p,q) |S_n^+|^{(p-q)/q}.$$

Similarly we get, multiplying (5.3) by ω_n^-,

$$\mu_n \geq (1-\theta)\beta^p + \theta c(p,q) |S_n^-|^{(p-q)/q}.$$

It follows that the two sets

$$S^{\pm} = \limsup_{n \to \infty} S_n^{\pm}$$

have positive measure. Since $\overline{\omega} \geq 0$ on S^+ and $\overline{\omega} \leq 0$ on S^-, we derive a contradiction with the fact that any eigenfunction corresponding to $\lambda_{1,\beta}$ has constant sign. □

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