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Hölder continuity of the integrated density of states for Liouville frequencies

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Abstract. We prove Hölder continuity of the Lyapunov exponent $L(\omega, E)$ and of the integrated density of states in neighborhoods of energies that satisfy $L(\omega, E) > 4\kappa(\omega, E) \cdot \beta(\omega) \geq 0$ for general analytic potentials, with $\kappa(\omega, E)$ being Avila's acceleration, and $\beta(\omega)$ measures the closeness of ω to rationals on a logarithmic scale.

In memory of Michael Goldstein

1. Introduction

We study one-dimensional quasi-periodic Schrödinger operators with analytic potentials:

$$(H_{\omega,\theta}\phi)_n = \phi_{n+1} + \phi_{n-1} + v(\theta + n\omega)\phi_n, \tag{1.1}$$

in which $\theta \in \mathbb{T}$ is called the *phase*, $\omega \in \mathbb{T} \setminus \mathbb{Q}$ is called the *frequency*. Throughout the paper, for $\theta \in \mathbb{T}$, $\|\theta\|_{\mathbb{T}} := \operatorname{dist}(\theta, \mathbb{Z})$, and we shall simply write it as $\|\theta\|$.

Our result concerns the regularity of the integrated density of states in energy for quasi-periodic Schrödinger operators. For integers $a < b$ and interval $\Lambda = [a, b] \cap \mathbb{Z}$, let $H_{\omega,\theta}|_\Lambda$ be the restriction of $H_{\omega,\theta}$ to the interval Λ with Dirichlet boundary condition. Let $\{E_{\Lambda,j}(\omega, \theta)\}_{j=1}^{|\Lambda|}$ be the eigenvalues of $H_{\omega,\theta}|_\Lambda$. Consider

$$N_\Lambda(\omega, E, \theta) := \frac{1}{|\Lambda|}\sum_{j=1}^{|\Lambda|} \chi_{(-\infty,E)}(E_{\Lambda,j}(\omega, \theta)).$$

It is well known that the weak limit

$$\lim_{\substack{a\to-\infty\\ b\to\infty}} \mathrm{d}N_\Lambda(\omega, E, \theta) =: \mathrm{d}\mathcal{N}(\omega, E).$$

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exists and is independent of θ. The distribution $\mathcal{N}(\omega,\cdot)$ is called the *integrated density of states* (IDS). It is connected to the Lyapunov exponent $L(\omega, E)$, see (2.7), through the Thouless formula:

$$L(\omega, E) = \int \log|E - E'| \, \mathrm{d}\mathcal{N}(\omega, E').$$

Concerning the regularity of $\mathcal{N}(\omega, E)$ in E, Craig and Simon [9] established log-Hölder continuity for general ergodic operators. Goldstein and the second name author [13] established the first Hölder continuity result for general analytic potentials on $\mathbb{T}$ throughout the positive Lyapunov exponent regime for Diophantine frequencies, as well as a weaker regularity for general $\mathbb{T}^d$, $d \geq 2$. The Diophantine condition was relaxed to cover some weak Liouville ω's in [21] and then further strengthened in [20] to cover all the ω's such that $0 \leq \beta(\omega) < c_v L(\omega, E)$, where $c_v > 0$ is an implicit constant and the exponent $\beta(\omega)$ measures the exponential convergence rate of the continued fraction approximants to ω, see the definition in (2.1). The results of [20,21] are both for the positive Lyapunov exponent regime, and follow the scheme of Goldstein and the second named author [13]: combining two important tools, the large deviation estimates (LDT) and the Avalanche Principle (AP). The former was first introduced in the seminal paper of Bourgain and Goldstein [6] with significant further developments by Bourgain, Goldstein, and the second named author in [5, 7, 8, 14, 15], and the latter was introduced in [13] and has been being greatly expanded by Duarte and Klein [10, 11].

In the regime of small coupling constants (which is a sub-regime of the zero Lyapunov exponent regime), $1/2$ Hölder continuity was established using almost reducibility by Avila and Jitomirskaya [2] for Diophantine frequencies. For a special family, the almost Mathieu operator with $v(\theta) = 2\lambda\cos(2\pi\theta)$ and Diophantine frequencies, $1/2$-Hölder was established in the same work for any non-critical coupling constant $\lambda \neq 1$. Recently, Avila, Last, Shamis, and Zhou [3] proved that the regularity of the IDS for the almost Mathieu operator is only log-Hölder in the regime $\beta(\omega) > 1.5\log\lambda \geq 0$. Recall that $\log\lambda = L(\omega, E)$ on the spectrum for $\lambda \geq 1$.

The main result of this paper deals with the opposite regime of [3], and it applies to general analytic potentials assuming positive Lyapunov exponents.

Theorem 1.1. *Let v be an analytic function on $\mathbb{T}$. Let $\beta(\omega)$ be as in (2.1), $L(\omega, E)$ be the Lyapunov exponent, and $\kappa(\omega, E)$ be the acceleration as in (2.8). Then for any E such that*

$$L(\omega, E) > 4\kappa(\omega, E) \cdot \beta(\omega), \tag{1.2}$$

the integrated density of states $\mathcal{N}(\omega, E)$ is Hölder continuous in a neighborhood of E.

Remark 1.2. Note that (1.2) holds on an open set of E due to continuity of $L(\omega, \cdot)$ and upper semi-continuity of $\kappa(\omega, \cdot)$, see [1]. By the Thouless formula, the same Hölder continuity holds for $L(\omega, E)$ on the same set of energies.

Theorem 1.1 strengthens the result of [20] by replacing the implicit condition $L(\omega, E) > C_v \beta(\omega)$ there with an effective condition. The main reason for such an improvement is the recently established connection between the LDT technology and Avila's acceleration in [17] and its developments in various settings [16, 18, 19]. In addition to these ingredients, the technical core of this paper is Theorem 3.1, an effective version of the LDT for Liouville frequencies. The mechanism for bounding the IDS proceeds as in [14] by Goldstein and the second named author: one controls the number of eigenvalues falling into an interval I by estimating the trace of the finite volume resolvent for energies $E = E_0 + i\epsilon$, where $\epsilon = |I|$.

The following notations will be used throughout. For any $R > 1$, let $\mathcal{A}_R := \{z \in \mathbb{C} : 1/R < |z| < R\}$ be the annulus. Let $\mathcal{C}_r := \{z \in \mathbb{C} : |z| = r\}$ be the circle with radius $r > 0$. For $z \in \mathbb{C}$ and $r > 0$, let $B_r(z)$ be the open ball centered at z with radius r. For $x \in \mathbb{R}$, let $[x]$ be the largest integer such that $[x] \le x$. Throughout this paper, for an analytic function $g(z)$ on $\mathcal{A}_R$, let

$$\hat{g}(k) := \int_{\mathbb{T}} g(e^{2\pi i\theta}) e^{-2\pi i k\theta}\, \mathrm{d}\theta \quad \text{for } k \in \mathbb{Z}$$

be the Fourier coefficients.

We organize the rest of the paper as follows. Section 2 covers preliminary material. We present a quick proof of Theorem 1.1 in Section 3, assuming the large deviation Theorem 3.1, whose proof is postponed to Sections 4 and 5.

2. Preliminaries

2.1. Continued fraction expansion

Given $\omega \in (0,1)$, let $[a_1, a_2, \dots]$ be the continued fraction expansion of ω. For $n \ge 1$, let $p_n/q_n = [a_1, a_2, \dots, a_n]$ be the continued fraction approximants. The following property is well known for $n \ge 1$,

$$\|q_n\omega\|_{\mathbb{T}} = \min_{1 \le k < q_{n+1}} \|k\omega\|_{\mathbb{T}}.$$

The $\beta(\omega)$ exponent measures the exponential closeness of ω to rational numbers:

$$\beta(\omega) := \limsup_{n\to\infty} \frac{\log q_{n+1}}{q_n} = \limsup_{n\to\infty} \Big(-\frac{\log \|n\omega\|_{\mathbb{T}}}{n}\Big). \tag{2.1}$$

Let

$$\beta_n(\omega) := \frac{\log q_{n+1}}{q_n}.$$

It is well known that

$$\|q_n\omega\| \in \Big(\frac{1}{2q_{n+1}}, \frac{1}{q_{n+1}}\Big) = \Big(\frac{1}{2}e^{-\beta_n(\omega)q_n}, e^{-\beta_n(\omega)q_n}\Big).$$

The following relations are standard:

$$q_{n+1} = a_{n+1}q_n + q_{n-1}, \quad p_{n+1} = a_{n+1}p_n + p_{n-1}, \tag{2.2}$$

and

$$\|q_{n-1}\omega\| = a_{n+1}\|q_n\omega\| + \|q_{n+1}\omega\|. \tag{2.3}$$

To see the latter is true, note that by (2.2),

$$q_{n+1}\omega - p_{n+1} = a_{n+1}(q_n\omega - p_n) + (q_{n-1}\omega - p_{n-1}),$$

which yields (2.3) since $\|q_n\omega\| = |q_n\omega - p_n|$ and

$$(q_{n+1}\omega - p_{n+1}) \cdot (q_n\omega - p_n) < 0,$$

for any $n \geq 1$.

2.2. Green's function for the annulus

We state the precise Green's kernel, which can be derived by the method of images.

Lemma 2.1 ([17, Lemma 3.1]). *The Green's function on the annulus $\mathcal{A}_R$ is given by*

$$G_R(z,w) = \frac{1}{2\pi}\log|z-w| + \Gamma_R(z,w), \tag{2.4}$$

where

$$\Gamma_R(z,w) = \frac{\log(\frac{|z|}{R})\log(\frac{|w|}{R})}{4\pi\log R} + \frac{1}{2\pi}\log\Big(\frac{\prod_{k=1}^{\infty}\big|1-\frac{1}{R^{4k}}\frac{z}{w}\big|\cdot\big|1-\frac{1}{R^{4k}}\frac{w}{z}\big|}{R\cdot\prod_{k=1}^{\infty}\big|1-\frac{1}{R^{4k-2}}w\bar{z}\big|\cdot\big|1-\frac{1}{R^{4k-2}}\frac{1}{\bar{z}w}\big|}\Big).$$

The Green's function is symmetric and invariant under rotations:

$$G_R(z,w) = G_R(w,z) \quad \text{and} \quad G_R(z,w) = G_R(e^{i\phi}z, e^{i\phi}w).$$

It is also easy to check that

$$
\begin{aligned}
2\pi G_R\Big(\frac{1}{\bar z},\frac{1}{\bar w}\Big) &= \log\Big|\frac{1}{\bar z}-\frac{1}{\bar w}\Big| + \frac{\log\big(\frac{1}{R|z|}\big)\log\big(\frac{1}{R|w|}\big)}{2\log R} \\
&\quad + \log\Big(\frac{\prod_{k=1}^{\infty}\big|1-\frac{1}{R^{4k}}\frac{\bar w}{\bar z}\big|\cdot\big|1-\frac{1}{R^{4k}}\frac{\bar z}{\bar w}\big|}{R\cdot\prod_{k=1}^{\infty}\big|1-\frac{1}{R^{4k-2}}\frac{1}{\bar w z}\big|\cdot\big|1-\frac{1}{R^{4k-2}}z\bar w\big|}\Big) \\
&= 2\pi G_R(z,w).
\end{aligned} \tag{2.5}
$$

The following integral is useful, see [17, Lemma 3.2]:

$$
\int_0^1 \Gamma_R(re^{2\pi i\theta},w)\,\mathrm{d}\theta = \frac{\log\big(\frac{r}{R}\big)}{4\pi\log R}\log\Big(\frac{|w|}{R}\Big) - \frac{\log R}{2\pi}.
$$

Note that Γ_R here is H_R in loc. cit.

2.3. Cocycles and Lyapunov exponents

Let $(\omega, A)\in(\mathbb{T}, C^{\omega}(\mathbb{T},\mathrm{SL}(2,\mathbb{R})))$. Let

$$
A_n(\omega,\theta) = A(\theta+(n-1)\omega)\cdots A(\theta).
$$

Let the finite-scale Lyapunov exponents be defined as

$$
L_n(\omega,A) := \frac{1}{n}\int_{\mathbb{T}}\log\|A_n(\omega,\theta)\|\,\mathrm{d}\theta,
$$

and the infinite-scale Lyapunov exponents as

$$
L(\omega,A) = \lim_{n\to\infty} L_n(\omega,A).
$$

We denote the phase-complexified Lyapunov exponents as

$$
L_n(\omega,A(\cdot+i\varepsilon)) =: L_n(\omega,A,\varepsilon), \quad \text{and} \quad L(\omega,A(\cdot+i\varepsilon)) =: L(\omega,A,\varepsilon),
$$

respectively. One can rewrite the eigenvalue equation $H_{\omega,\theta}\phi = E\phi$ of (1.1) as follows:

$$
\begin{pmatrix}\phi_{n+1}\\ \phi_n\end{pmatrix} = M_E(\theta+n\omega)\begin{pmatrix}\phi_n\\ \phi_{n-1}\end{pmatrix},
$$

in which

$$
M_E(\theta) = \begin{pmatrix}E-v(\theta) & -1\\ 1 & 0\end{pmatrix}
$$

is the one-step transfer matrix. Let $M_{m,E}(\theta) := M_E(\theta + (m-1)\omega) \cdots M_E(\theta)$ be the m-step transfer matrix. The following relation between $M_{m,E}$ and the Dirichlet determinants is well known:

$$M_{m,E}(\theta) = \begin{pmatrix} P_{m,E}(\theta) & -P_{m-1,E}(\theta+\omega) \\ P_{m-1,E}(\theta) & -P_{m-2,E}(\theta+\omega) \end{pmatrix}, \tag{2.6}$$

in which $P_{m,E}(\theta) = \det(H_{\omega,\theta}|_{[0,m-1]} - E)$ and $H_{\omega,\theta}|_{[0,m-1]}$ is the restriction of operator $H_{\omega,\theta}$ to the interval $[0, m-1]$ with Dirichlet boundary condition. For the given transfer matrix M_E, we denote

$$L(\omega, M_E) =: L(\omega, E), \tag{2.7}$$

for simplicity.

2.4. Avila's acceleration

Let $(\omega, A) \in (\mathbb{T}, C^\omega(\mathbb{T}, \mathrm{SL}(2,\mathbb{R})))$. The Lyapunov exponent $L(\omega, A, \varepsilon)$ is a convex and even function in ε. Avila defined the acceleration to be the right-derivative as follows:

$$\kappa(\omega, A, \varepsilon) := \lim_{\varepsilon' \to 0+} \frac{L(\omega, A, \varepsilon + \varepsilon') - L(\omega, A, \varepsilon)}{2\pi\varepsilon'}. \tag{2.8}$$

As a cornerstone of his global theory [1], he showed that for any analytic A with values in $\mathrm{SL}(2,\mathbb{R})$ and irrational ω, $\kappa(\omega, A, \varepsilon) \in \mathbb{Z}$ is always quantized.

Recall that v is an analytic functions on $\mathbb{T}_{\varepsilon_0}$ for some $\varepsilon_0 > 0$. We may shrink ε_0 when necessary such that

$$L(\omega, E, \varepsilon) = L(\omega, E, 0) + 2\pi\kappa(\omega, E, 0) \cdot |\varepsilon|$$

holds for any $|\varepsilon| \le \varepsilon_0$. For the rest of the paper, when $\varepsilon = 0$, we shall omit ε from various notations of Lyapunov exponents and accelerations.

2.5. Upper semi-continuity

The following upper bound is standard.

Lemma 2.2 ([12]). *For any irrational ω and continuous cocycle $(\omega, A) \in (\mathbb{T}, C(\mathbb{T}, \mathrm{SL}(2,\mathbb{R})))$, for any small $\delta > 0$, and n large enough, there holds*

$$\frac{1}{n} \log \|A_n(\omega, \theta)\| \le L(\omega, A) + \delta,$$

uniformly in $\theta \in \mathbb{T}$.

2.6. Basic estimates in large deviation estimates

Let

$$u_{m,E}(\theta) := \frac{1}{m}\log\|M_{m,E}(\theta)\|.$$

It is easy to see that the following holds for some constant $C_{v,1} = C(\|v\|_{\mathbb{T}_{\varepsilon_0}}) > 0$:

$$|u_{m,E}(\theta) - u_{m,E}(\theta+\omega)| \le \frac{C_{v,1}}{m}. \tag{2.9}$$

Consider the Cesàro average $u_{m,E}^{(Q)}$ of $u_{m,E}$ along the dynamics $\theta \to \theta + \omega$:

$$\begin{aligned} u_{m,E}^{(Q)}(\theta) &:= \sum_{|j|<Q} \frac{Q-|j|}{Q^2} u_{m,E}(\theta + j\omega) \\ &= \sum_{|j|<Q} \frac{Q-|j|}{Q^2} \sum_{k\in\mathbb{Z}} \hat{u}_{m,E}(k) e^{2\pi i k(\theta+j\omega)} \\ &= \sum_{k\in\mathbb{Z}} \hat{u}_{m,E}(k) F_Q(k) e^{2\pi i k\theta}, \end{aligned} \tag{2.10}$$

in which $F_Q(k)$ is the Fourier multiplier of the Fejér kernel

$$F_Q(k) = \sum_{|j|<Q} \frac{Q-|j|}{Q^2} e^{2\pi i k j\omega}.$$

The reason for introducing the Fejér kernel lies with its better averaging properties as compared to the arithmetic mean. This feature was already exploited by Bourgain for similar reasons, see [4]. The following estimates concerning the multiplier F_Q can be found in [20], with their proofs in [20, Appendix E]. Below, p/q is an arbitrary continued fractional approximant of ω, as in (2.1). One has

$$0 \le F_Q(k) \le \min\Big(1, \frac{2}{1+Q^2\|k\omega\|^2}\Big), \tag{2.11}$$

$$\sum_{1\le|k|<q/4} \frac{1}{1+Q^2\|k\omega\|^2} \le 2\pi\frac{q}{Q}, \tag{2.12}$$

and

$$\sum_{\ell q/4\le k<(\ell+1)q/4} \frac{1}{1+Q^2\|k\omega\|^2} \le 2+2\pi\frac{q}{Q}. \tag{2.13}$$

We will always use (2.11) to bound the Fejér kernel without explicitly referring to it in Section 5. Furthermore, we will rely on the following two basic estimates for the Fourier coefficients of $u_{m,E}$. First,

$$|\hat{u}_{m,E}(k)| \leq \frac{C_{v,2}}{|k|}, \quad k \neq 0, \tag{2.14}$$

for some constant $C_{v,2} = C(\|v\|_{\mathbb{T}_{\varepsilon_0}}, \varepsilon_0) > 0$. And, second, we have the following lemma which was first proved in [20]. It leads to gains for k which are small relative to m.

Lemma 2.3 ([20, Lemma 2.4]). *One has*

$$|\hat{u}_{m,E}(k)| \leq \frac{C_{v,1}}{4m\|k\omega\|}, \quad k \neq 0. \tag{2.15}$$

3. Hölder continuity of the IDS

Our proof of Hölder continuity is built on a large deviation estimate. The setup is as follows. Let $E \in \mathbb{R}$ be an energy such that $L(\omega, E) > 4\kappa(\omega, E) \cdot \beta(\omega)$. Let $\varepsilon_0 > 0$ be such that

$$L(\omega, E, \varepsilon) = L(\omega, E) + 2\pi\kappa(\omega, E)|\varepsilon| \quad \text{for } |\varepsilon| \leq \varepsilon_0. \tag{3.1}$$

Let $\eta_0 > 0$ be a small constant such that for any $|\eta| \leq \eta_0$ (recall that the acceleration κ is upper semi-continuous with respect to the cocycle [1])

$$\begin{cases} \kappa(\omega, E + i\eta, \varepsilon) \leq \kappa(\omega, E, \varepsilon) = \kappa(\omega, E) & \text{for } |\varepsilon| \leq \varepsilon_0, \\ L(\omega, E + i\eta) > 4\kappa(\omega, E) \cdot \beta(\omega). \end{cases}$$

We now state the large deviation estimates.

Theorem 3.1. *Let $E \in \mathbb{R}$ be such that $L(\omega, E) > 4\kappa(\omega, E) \cdot \beta(\omega)$. Then for any small $\delta \in (0, 1/(\beta(\omega) + 1))$, and for m large enough,*

$$\mathrm{mes}(\mathcal{B}_m) := \mathrm{mes}\{\theta : |u_{m,E}(\theta) - L_m(\omega, E)| > 2\kappa(\omega, E)\beta(\omega) + C_{\omega,v}\delta\} \leq e^{-\delta^4 m},$$

for some constant $C_{\omega,v} > 1$.

As a corollary of Theorem 3.1, we have the following.

Corollary 3.2. *Let E, δ be as in Theorem 3.1, for large enough m, and η such that*

$$|\eta| \leq \exp(-2m(\kappa(\omega, E)\beta(\omega) + 2C_{\omega,v}\delta)). \tag{3.2}$$

We have

$$\mathrm{mes}(\mathcal{B}_m^\eta) := \mathrm{mes}\{\theta : u_{m,E+i\eta}(\theta) < L(\omega, E) - 2\kappa(\omega, E)\beta(\omega) - 2C_{\omega,v}\delta\} \le e^{-\delta^4 m}.$$

Remark 3.3. Since $\kappa(\omega, \cdot)$ is upper semi-continuous, $\kappa(\omega, \widetilde{E}) \le \kappa(\omega, E)$ in a neighborhood of E. The same proofs of Theorem 3.1 and Corollary 3.2 yield the following large deviation estimates: for any

$$|\eta| \le \exp(-2m(\kappa(\omega, E)\beta(\omega) + 2C_{\omega,v}\delta)), \tag{3.3}$$

we have

$$\mathrm{mes}(\mathcal{B}_{m,\widetilde{E}}^\eta) := \mathrm{mes}\{\theta : |u_{m,\widetilde{E}}(\theta) - L_m(\omega, \widetilde{E})| > 2\kappa(\omega, E)\beta(\omega) + C_{\omega,v}\delta\} \le e^{-\delta^4 m}, \tag{3.4}$$

for all $\widetilde{E}$ in a compact interval $I_E := [E - \tau, E + \tau]$ and $m \ge m_0(I_E, \delta, \omega)$.

We give a quick proof of this corollary. Within the proof, we write $\kappa(\omega, E) = \kappa$ and $\beta(\omega) = \beta$.

Proof. It suffices to prove $(\mathcal{B}_m)^c \subseteq (\mathcal{B}_m^\eta)^c$. Taking any $\theta \in (\mathcal{B}_m)^c$, we have

$$\begin{aligned} \|M_{m,E}(\theta)\| &\ge \exp(m(L_m(\omega, E) - 2\kappa\beta - C_{\omega,v}\delta)) \\ &\ge \exp(m(L(\omega, E) - 2\kappa\beta - C_{\omega,v}\delta)). \end{aligned} \tag{3.5}$$

By telescoping and upper-semi-continuity (Lemma 2.2), we have for η satisfying (3.2) that

$$\begin{aligned} \big|\|M_{m,E}(\theta)\| - \|M_{m,E+i\eta}(\theta)\|\big| &\le \|M_{m,E}(\theta) - M_{m,E+i\eta}(\theta)\| \\ &\le \exp(m(L(\omega, E) + \delta)) \cdot |\eta| \\ &\le \exp(m(L(\omega, E) - 2\kappa\beta - 3C_{\omega,v}\delta)). \end{aligned} \tag{3.6}$$

Combining (3.5) with (3.6), we have

$$\begin{aligned} u_{m,E+i\eta}(\theta) &= u_{m,E}(\theta) + \frac{1}{m}\log\frac{\|M_{m,E+i\eta}(\theta)\|}{\|M_{m,E}(\theta)\|} \\ &= u_{m,E}(\theta) + \frac{1}{m}\log\Big(1 + \frac{\|M_{m,E+i\eta}(\theta)\| - \|M_{m,E}(\theta)\|}{\|M_{m,E}(\theta)\|}\Big) \\ &\ge u_{m,E}(\theta) - \frac{2}{m}\frac{\big|\|M_{m,E}(\theta)\| - \|M_{m,E+i\eta}(\theta)\|\big|}{\|M_{m,E}(\theta)\|} \\ &\ge u_{m,E}(\theta) - \delta, \end{aligned}$$

which yields $\theta \in (\mathcal{B}_m^\eta)^c$. ∎

We postpone the proof of Theorem 3.1 to Section 5. Below we show how to utilize it to obtain Theorem 1.1.

Proof of Theorem 1.1

Fix any E_0 such that $L(\omega, E_0) > 4\kappa(\omega, E_0) \cdot \beta(\omega)$. Let I_{E_0} be the compact interval for which $\kappa(\omega, E) \le \kappa(\omega, E_0)$, and

$$\inf_{E \in I_{E_0}} \min_{|\eta| \le \eta_0} L(\omega, E + i\eta) > 4\kappa(\omega, E_0)\beta(\omega),$$

for some $\eta_0 > 0$. And furthermore, assume $m \ge m_0(I_{E_0}, \delta, \omega)$ so that the large deviation estimate in (3.4) holds for all $E \in I_{E_0}$ and $m \ge m_0(I_{E_0}, \delta, \omega)$.

Our goal is to analyze the following quantity for an arbitrary fixed θ and $E \in I_{E_0}$ as $N \to \infty$:

$$\begin{aligned} d_{\omega,N}(\theta, E-\eta, E+\eta) :&= N_{[0,N-1]}(\omega, E+\eta, \theta) - N_{[0,N-1]}(\omega, E-\eta, \theta) \\ &= \frac{1}{N} \cdot \#\{\sigma(H_{\omega,\theta}|_{[0,N-1]}) \cap [E-\eta, E+\eta)\} \\ &= \frac{1}{N} \mathrm{tr}(P_{[E-\eta, E+\eta)}(H_{\omega,\theta}|_{[0,N-1]})), \end{aligned} \tag{3.7}$$

in which $H_{\omega,\theta}|_{[0,N-1]}$ is $H_{\omega,\theta}$ restricted to the interval $[0, N-1]$ with Dirichlet boundary conditions. Throughout the rest of this section, we shall omit ω in various notations. Let $\delta > 0$ be small such that

$$0 < \delta < \min\Big(\frac{1}{\beta(\omega)+1}, \frac{\min_{|\eta| \le \eta_0} L(\omega, E+i\eta) - 4\kappa(\omega, E_0)\beta(\omega)}{10 C_{\omega,v}}\Big), \tag{3.8}$$

in which $C_{\omega,v}$ is the constant in Theorem 3.1. We proceed as in [14], bounding (3.7) via the trace of the Green's function using the well-known relation

$$\chi_{[E-\eta, E+\eta)}(x) \le \frac{2\eta^2}{\eta^2 + (x-E)^2} = 2\eta \cdot \mathrm{Im}\Big(\frac{1}{x - (E+i\eta)}\Big).$$

This implies that

$$\begin{aligned} d_N(\theta, E-\eta, E+\eta) &\le \frac{2\eta}{N} \mathrm{Im}(\mathrm{tr}(G^{E+i\eta}_{[0,N-1]}(\theta))) \\ &= \frac{2\eta}{N} \sum_{k=0}^{N-1} \mathrm{Im}(G^{E+i\eta}_{[0,N-1]}(\theta; k, k)), \end{aligned} \tag{3.9}$$

in which $G^{E+i\eta}_{[0,N-1]}(\theta) = (H_\theta|_{[0,N-1]} - (E+i\eta))^{-1}$ is the Green's function.

For $\eta > 0$ small, let $m \in \mathbb{N}$ be chosen such that

$$\begin{aligned} \eta &\simeq_{\kappa(E_0),\beta,v,\omega,\delta} \exp(-m(4\kappa(E_0)\beta + 12 C_{\omega,v}\delta + 2\delta^4)) \\ &\le \exp(-4m(\kappa(E_0)\beta(\omega) + 2C_{\omega,v}\delta)). \end{aligned} \tag{3.10}$$

We say that k is *good,* denoted by $k \in \mathcal{Q}_m$, if

$$\theta \notin (\mathcal{B}^{\eta}_{2m,E} + m\omega - k\omega). \tag{3.11}$$

Otherwise, we say that k is *bad.* Since η satisfies (3.3) for $2m$ in place of m, see (3.10) above, we conclude from Corollary 3.2 that

$$\operatorname{mes}(\mathcal{B}^{\eta}_{2m,E}) \leq e^{-2\delta^4 m}. \tag{3.12}$$

Note that (2.6) together with (3.11) implies that for each good $k \in \mathcal{Q}_m$, there exists $m_k \in \{2m, 2m-1, 2m-2\}$ and $s_k \in \{0, 1\}$ such that

$$\begin{aligned}\frac{1}{2m} \log |P^{E+i\eta}_{m_k}(\theta + k\omega + s_k\omega - m\omega)| &\geq u_{2m,E+i\eta}(\theta + k\omega - m\omega) - \frac{\log \sqrt{2}}{2m} \\ &\geq L(E) - 2\kappa(E_0)\beta - 3C_{\omega,v}\delta.\end{aligned} \tag{3.13}$$

For each good $k \in \mathcal{Q}_m$, we let $[k + s_k - m, k + s_k - m + m_k - 1] =: [a_k, b_k]$. Distinguishing the good from the bad k's in (3.9), and bounding the bad terms using the trivial bound

$$|G^{E+i\eta}_{[0,N-1]}(\theta; k, k)| \leq \|G^{E+i\eta}_{[0,N-1]}(\theta)\| \leq \eta^{-1},$$

we obtain the upper bound

$$\begin{aligned}&d_N(\theta, E-\eta, E+\eta) \\ &\leq \frac{2\eta}{N}\Big(\sum_{k=0}^{m-1} + \sum_{k=N-m}^{N-1} + \sum_{\substack{k=m \\ k\notin\mathcal{Q}_m}}^{N-1-m} + \sum_{\substack{k=m \\ k\in\mathcal{Q}_m}}^{N-1-m}\Big)\operatorname{Im}(G^{E+i\eta}_{[0,N-1]}(\theta; k, k)) \\ &\leq \frac{4m}{N} + \frac{2}{N}\sum_{k=m}^{N-1-m} \chi_{\mathcal{B}^{\eta}_{2m,E}+m\omega}(\theta + k\omega) + \frac{2\eta}{N}\sum_{\substack{k=m \\ k\in\mathcal{Q}_m}}^{N-1-m} \operatorname{Im}(G^{E+i\eta}_{[0,N-1]}(\theta; k, k)).\end{aligned} \tag{3.14}$$

For N large enough, we can invoke the ergodic theorem and (3.12) to see that

$$\begin{aligned}\frac{4m}{N} + \frac{2}{N}\sum_{k=m}^{N-1-m} \chi_{\mathcal{B}^{\eta}_{2m,E}+m\omega}(\theta + k\omega) &\leq 2 \cdot \operatorname{mes}(\mathcal{B}^{\eta}_{2m,E} + m\omega) + \eta \\ &\leq 2e^{-2\delta^4 m} + \eta.\end{aligned}$$

Hence, (3.14) turns into

$$d_N(\theta, E-\eta, E+\eta) \leq 2e^{-2\delta^4 m} + \eta + 2\eta \cdot \sup_{k\in\mathcal{Q}_m} \operatorname{Im}(G^{E+i\eta}_{[0,N-1]}(\theta; k, k)). \tag{3.15}$$

Next, we study $G^{E+i\eta}_{[0,N-1]}(\theta;k,k)$ for good k's. Let $\widetilde{H}_\theta|_{[0,N-1],k} = H_\theta|_{[0,N-1]} - \Gamma_k$, where Γ_k is a $N\times N$ matrix such that

$$\Gamma_k(j,\ell) = \begin{cases} 1 & \text{if } (j,\ell) = (a_k, a_k-1), (a_k-1, a_k), (b_k, b_k+1), (b_k+1, b_k), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\widetilde{H}_\theta|_{[0,N-1],k}$ decouples into

$$H_\theta|_{[a_k,b_k]} \quad \text{and} \quad H_\theta|_{[0,N-1]\setminus[a_k,b_k]}.$$

Let $\widetilde{G}^{E+i\eta}_{[0,N-1],k}(\theta) = (\widetilde{H}_\theta|_{[0,N-1],k} - (E+i\eta))^{-1}$. By the resolvent identity,

$$G^{E+i\eta}_{[0,N-1]}(\theta) - \widetilde{G}^{E+i\eta}_{[0,N-1],k}(\theta) = -G^{E+i\eta}_{[0,N-1]}(\theta)\cdot\Gamma_k\cdot\widetilde{G}^{E+i\eta}_{[0,N-1],k}(\theta).$$

This implies that

$$\begin{aligned} &G^{E+i\eta}_{[0,N-1]}(\theta;k,k) \\ &\quad= \widetilde{G}^{E+i\eta}_{[0,N-1],k}(\theta;k,k) \\ &\qquad - \sum_{j,\ell} G^{E+i\eta}_{[0,N-1]}(\theta;k,j)\cdot\Gamma_k(j,\ell)\cdot\widetilde{G}^{E+i\eta}_{[0,N-1],k}(\theta;\ell,k) \\ &\quad= \widetilde{G}^{E+i\eta}_{[a_k,b_k]}(\theta;k,k) \\ &\qquad - G^{E+i\eta}_{[0,N-1]}(\theta;k,a_k-1)\cdot\widetilde{G}^{E+i\eta}_{[a_k,b_k]}(\theta;a_k,k) \\ &\qquad - G^{E+i\eta}_{[0,N-1]}(\theta;k,b_k+1)\cdot\widetilde{G}^{E+i\eta}_{[a_k,b_k]}(\theta;b_k,k) \\ &\quad= \widetilde{G}^{E+i\eta}_{[a_k,b_k]-k}(\theta+k\omega;0,0) \\ &\qquad - G^{E+i\eta}_{[0,N-1]}(\theta;k,a_k-1)\cdot\widetilde{G}^{E+i\eta}_{[a_k,b_k]-k}(\theta+k\omega;a_k-k,0) \\ &\qquad - G^{E+i\eta}_{[0,N-1]}(\theta;k,b_k+1)\cdot\widetilde{G}^{E+i\eta}_{[a_k,b_k]-k}(\theta+k\omega;b_k-k,0). \end{aligned} \tag{3.16}$$

We now bound the last two product terms above as follows:

$$\max(|G^{E+i\eta}_{[0,N-1]}(\theta;k,a_k-1)|, |G^{E+i\eta}_{[0,N-1]}(\theta;k,b_k+1)|) \le \|G^{E+i\eta}_{[0,N-1]}(\theta)\| \le \eta^{-1}. \tag{3.17}$$

Using (3.13) and Cramer's rule, we obtain that for each good k (recall $a_k = k + s_k - m$),

$$\begin{aligned} |\widetilde{G}^{E+i\eta}_{[a_k,b_k]-k}(\theta+k\omega;a_k-k,0)| &= \frac{|P^{E+i\eta}_{b_k-k}(\theta+(k+1)\omega)|}{|P^{E+i\eta}_{m_k}(\theta+a_k\omega)|} \\ &\le e^{m(L(E+i\eta)+\delta)}\cdot e^{-2m(L(E)-2\kappa(E_0)\beta-3C_{\omega,v}\delta)} \\ &\le e^{-m(L(E)-4\kappa(E_0)\beta-8C_{\omega,v}\delta)}. \end{aligned} \tag{3.18}$$

A similar estimate holds for $|\widetilde{G}^{E+i\eta}_{[a_k,b_k]-k}(\theta+k\omega;b_k-k,0)|$ as well. Combining (3.17) and (3.18) with (3.16), we infer that

$$\begin{aligned}&|G^{E+i\eta}_{[0,N-1]}(\theta;k,k)|\\&\quad\le|\widetilde{G}^{E+i\eta}_{[a_k,b_k]-k}(\theta+k\omega;0,0)|+2\eta^{-1}e^{-m(L(E)-4\kappa(E_0)\beta-8C_{\omega,v}\delta)}.\end{aligned}\tag{3.19}$$

Once again, by Cramer's rule,

$$\begin{aligned}|\widetilde{G}^{E+i\eta}_{[a_k,b_k]-k}(\theta+k\omega;0,0)|&=\frac{|P^{E+i\eta}_{k-a_k}(\theta+a_k\omega)|\cdot|P^{E+i\eta}_{b_k-k}(\theta+(k+1)\omega)|}{|P^{E+i\eta}_{m_k}(\theta+a_k\omega)|}\\&\le e^{2m(L(E+i\eta)+\delta)}\cdot e^{-2m(L(E)-2\kappa(E_0)\beta-3C_{\omega,v}\delta)}\\&\le e^{4m(\kappa(E_0)\beta+3C_{\omega,v}\delta)},\end{aligned}\tag{3.20}$$

where we used (3.13) as before to estimate the denominator. Combining (3.19) with (3.20) yields the following bound:

$$\begin{aligned}&\sup_{k\in\mathcal{Q}_m}\ \mathrm{Im}(G^{E+i\eta}_{[0,N-1]}(\theta;k,k))\\&\quad\le e^{4m(\kappa(E_0)\beta+3C_{\omega,v}\delta)}+2\eta^{-1}e^{-m(L(E)-4\kappa(E_0)\beta-8C_{\omega,v}\delta)}.\end{aligned}$$

Plugging this into (3.15), and noting that we use the smallness of δ as in (3.8), we arrive at the following conclusion:

$$\begin{aligned}&d_N(\theta,E-\eta,E+\eta)\\&\quad\le 2e^{-2\delta^4m}+\eta+2\eta e^{4m(\kappa(E_0)\beta+3C_{\omega,v}\delta)}+4e^{-m(L(E)-4\kappa(E_0)\beta-8C_{\omega,v}\delta)}\\&\quad\le 3e^{-2\delta^4m}+3\eta e^{4m(\kappa(E_0)\beta+3C_{\omega,v}\delta)}\\&\quad\lesssim_{\kappa(E_0),\beta,v,\omega,\delta}\ \eta^{\frac{\delta^4}{2\kappa(E_0)\beta+7C_{\omega,v}\delta}},\end{aligned}$$

provided that $\eta\simeq_{\kappa(E_0),\beta,v,\omega,\delta}\exp(-m(4\kappa(E_0)\beta+12C_{\omega,v}\delta+2\delta^4))$, as $N\to\infty$. This implies

$$\mathcal{N}(E-\eta,E+\eta)\lesssim_{\kappa(E_0),\beta,v,\omega,\delta}\ \eta^{\frac{\delta^4}{2\kappa(E_0)\beta+7C_{\omega,v}\delta}},$$

due to weak convergence, thus completing the proof of Theorem 1.1.

4. Fourier coefficients of $u_{m,E}$

Clearly, (3.1) implies for any $0\le\varepsilon_1<\varepsilon_2\le\varepsilon_0$,

$$L(\omega,E,\varepsilon_2)-L(\omega,E,\varepsilon_1)=2\pi\kappa(\omega,E)(\varepsilon_2-\varepsilon_1).\tag{4.1}$$

Hence, for any $\delta > 0$, for m large enough, we have

$$L_m(\omega, E, \varepsilon_0) - L_m\Big(\omega, E, \frac{\varepsilon_0}{2}\Big) \le \pi\varepsilon_0\kappa(\omega, E) + \delta. \tag{4.2}$$

One key new ingredient in this paper, in addition to the recent developments in [16, 17], is the following effective bound on $C_{v,2}$, see (2.14).

Lemma 4.1. *For any $\delta > 0$, for m large, we have*

$$|\hat{u}_{m,E}(k)| \le \frac{\kappa(\omega, E) + \delta}{|k|} + C_{\omega,v,3}R^{-|k|/2} \quad \textit{for any } k \neq 0,$$

in which $C_{\omega,v,3} = C(\omega, v, \varepsilon_0) > 0$ and $R = e^{2\pi\varepsilon_0}$.

Proof. As a preparation, we start off by discussing the Riesz representation of $u_{m,E}$.

4.1. Control of Riesz mass of $u_{m,E}$

The first subsection focuses on the control of Riesz mass of $u_{m,E}$. This part is analogous to [17, Section 5], with an improvement due to Lemma 4.4 below.

Lemma 4.2 ([17, Lemma 3.2]). *For $1/R \le r \le R$ and w in the annulus $\mathcal{A}_R$, we have*

$$2\pi\int_0^1 G_R(re^{2\pi i\theta}, w)\,\mathrm{d}\theta = (2\log R)^{-1}\begin{cases}\log(rR)\log\Big|\dfrac{w}{R}\Big| & \textit{if } |w| \ge r,\\ \log\Big(\dfrac{r}{R}\Big)\log|wR| & \textit{if } |w| < r.\end{cases}$$

Theorem 4.3. *The Riesz representation of $u_{m,E}$ yields*

$$u_{m,E}(z) = \int_{\mathcal{A}_R} 2\pi G_R(z, w)\,\mu_{m,E}(\mathrm{d}w) + h_{R,m,E}(z), \tag{4.3}$$

G_R is as in (2.4) and the harmonic part satisfies $h_{R,m,E} = u_{m,E}$ on $\partial\mathcal{A}_R$. We have for any $\delta > 0$, for m large enough,

$$\mu_{m,E}(\mathcal{A}_{\sqrt{R}}) \le 2\kappa(\omega, E) + \delta.$$

Proof. Within the proof, we shall omit ω in various notations. We first show an analogue of [16, Lemma 4.7].

Lemma 4.4. *There exists $b \in \mathbb{R}$ such that*

$$\int_{\mathbb{T}} h_{R,m,E}(e^{2\pi i(\theta+i\varepsilon)})\,\mathrm{d}\theta \equiv b.$$

Proof. Let

$$I_m^G(E,\varepsilon) := \int_{\mathbb{T}} \int_{\mathcal{A}_R} 2\pi G_R(e^{2\pi i(\theta+i\varepsilon)}, w)\, \mu_{m,E}(\mathrm{d}w)\, \mathrm{d}\theta$$

and

$$I_m^h(E,\varepsilon) := \int_{\mathbb{T}} h_{R,m,E}(e^{2\pi i(\theta+i\varepsilon)})\, \mathrm{d}\theta.$$

Since $h_{R,m,E}$ is a harmonic function, its integral along an arbitrary circle centered at 0 is a radial harmonic function, which implies

$$I_m^h(E,\varepsilon) = a\varepsilon + b \quad \text{for some } a, b \in \mathbb{R}. \tag{4.4}$$

Next, we show $I_m^h(E,\varepsilon)$ is an even function in ε. This follows from the evenness of $L_m(E,\varepsilon)$, see (4.6), and $I_m^G(E,\varepsilon)$, see (4.10) below.

Since the potential function $v(e^{2\pi i\theta})$ is real-valued for $\theta \in \mathbb{T}$, we have

$$v(z) = \overline{v\Big(\frac{1}{\bar{z}}\Big)} \quad \text{for } z \in \mathcal{A}_R.$$

This implies, since $E \in \mathbb{R}$,

$$u_{m,E}(e^{2\pi i(\theta+i\varepsilon)}) = u_{m,E}(e^{2\pi i(\theta-i\varepsilon)}) \quad \text{for } |\varepsilon| \le \varepsilon_0, \tag{4.5}$$

and hence

$$L_m(E,\varepsilon) = L_m(E,-\varepsilon), \tag{4.6}$$

as well as

$$\int_{\mathbb{T}} u_{m,E}(e^{2\pi i(\theta+i\varepsilon)}) e^{-2\pi ik\theta}\, \mathrm{d}\theta = \int_{\mathbb{T}} u_{m,E}(e^{2\pi i(\theta-i\varepsilon)}) e^{-2\pi ik\theta}\, \mathrm{d}\theta. \tag{4.7}$$

We will use (4.7) and (4.11) below in Section 4.2.

Since $u_{m,E}(z) = u_{m,E}(1/\bar{z})$, and $\Delta u_{m,E} = \mu_{m,E}$, the measure $\mu_{m,E}$ exhibits the same reflection symmetry as $u_{m,E}$ in (4.5):

$$\mu_{m,E}(\mathrm{d}z) = \mu_{m,E}\Big(\mathrm{d}\Big(\frac{1}{\bar{z}}\Big)\Big). \tag{4.8}$$

This implies that

$$\int_{\mathcal{A}_R} 2\pi G_R(e^{2\pi i(\theta+i\varepsilon)}, w)\, \mu_{m,E}(\mathrm{d}w) = \int_{\mathcal{A}_R} 2\pi G_R(e^{2\pi i(\theta+i\varepsilon)}, w)\, \mu_{m,E}\Big(\mathrm{d}\Big(\frac{1}{\bar{w}}\Big)\Big)$$

$$= \int_{\mathcal{A}_R} 2\pi G_R\Big(e^{2\pi i(\theta+i\varepsilon)}, \frac{1}{\overline{w}}\Big)\, \mu_{m,E}(\mathrm{d}w)$$
$$= \int_{\mathcal{A}_R} 2\pi G_R(e^{2\pi i(\theta-i\varepsilon)}, w)\, \mu_{m,E}(\mathrm{d}w), \quad (4.9)$$

in which we used $G_R(1/\bar{z}, 1/\overline{w}) = G_R(z, w)$ as in (2.5), in the last inequality. Clearly, (4.9) implies

$$I_m^G(E, \varepsilon) = I_m^G(E, -\varepsilon), \tag{4.10}$$

as well as

$$\int_{\mathbb{T}} \Big(\int_{\mathcal{A}_R} 2\pi G_R(e^{2\pi i(\theta+i\varepsilon)}, w)\, \mu_{m,E}(\mathrm{d}w)\Big) e^{-2\pi i k\theta}\, \mathrm{d}\theta$$
$$= \int_{\mathbb{T}} \Big(\int_{\mathcal{A}_R} 2\pi G_R(e^{2\pi i(\theta-i\varepsilon)}, w)\, \mu_{m,E}(\mathrm{d}w)\Big) e^{-2\pi i k\theta}\, \mathrm{d}\theta. \tag{4.11}$$

Combining (4.6) with (4.10), we have $I_m^h(E, \varepsilon) = I_m^h(E, -\varepsilon)$ which further implies

$$I_m^h(E, \varepsilon) = b,$$

when combined with (4.4). This is the claimed result. ■

Integrating (4.3) along $w \in \mathcal{C}_{r_1}$ and $\mathcal{C}_{r_2}$, with $1 < r_1 < r_2 \leq R$, and subtracting one from the other yields that

$$\begin{aligned}
&L_m\Big(E, \frac{\log r_2}{2\pi}\Big) - L_m\Big(E, \frac{\log r_1}{2\pi}\Big)\\
&= \int_0^1 u_{m,E}(r_2 e^{2\pi i\theta})\, \mathrm{d}\theta - \int_0^1 u_{m,E}(r_1 e^{2\pi i\theta})\, \mathrm{d}\theta\\
&= \int_{\mathcal{A}_R} \Big(\int_0^1 2\pi G_R(r_2 e^{2\pi i\theta}, w)\, \mathrm{d}\theta - \int_0^1 2\pi G_R(r_1 e^{2\pi i\theta}, w)\, \mathrm{d}\theta \Big) \mu_{m,E}(\mathrm{d}w)\\
&\quad + \int_0^1 h_{R,m,E}(r_2 e^{2\pi i\theta})\, \mathrm{d}\theta - \int_0^1 h_{R,m,E}(r_1 e^{2\pi i\theta})\, \mathrm{d}\theta\\
&= \int_{\mathcal{A}_R} \Big(\int_0^1 2\pi G_R(r_2 e^{2\pi i\theta}, w)\, \mathrm{d}\theta - \int_0^1 2\pi G_R(r_1 e^{2\pi i\theta}, w)\, \mathrm{d}\theta \Big) \mu_{m,E}(\mathrm{d}w),
\end{aligned} \tag{4.12}$$

where we used Lemma 4.4 in the last equality.

The rest of the proof is completely analogous to that of [17, Theorem 5.1]. By (4.2), for m large, for $r_1 = \sqrt{R} = e^{\pi\varepsilon_0}$ and $r_2 = R = e^{2\pi\varepsilon_0}$,

$$L_m(E, \varepsilon_0) - L_m\left(E, \frac{\varepsilon_0}{2}\right) \leq \pi\varepsilon_0\kappa(\omega, E) + \varepsilon_0\delta. \tag{4.13}$$

By Lemma 4.2, we have for any $1 < r_1 < r_2 \leq R$,

$$\begin{aligned}
&\int_0^1 2\pi G_R(r_2 e^{2\pi i\theta}, w)\,\mathrm{d}\theta - \int_0^1 2\pi G_R(r_1 e^{2\pi i\theta}, w)\,\mathrm{d}\theta \\
&= \begin{cases}
\dfrac{\log r_2 - \log r_1}{2\log R}\log\dfrac{|w|}{R} & \text{if } |w| \geq r_2, \\
\dfrac{\log(\frac{r_2}{R})}{2\log R}\log|wR| - \dfrac{\log(r_1 R)}{2\log R}\log\dfrac{|w|}{R} & \text{if } r_1 \leq |w| \leq r_2, \\
\dfrac{\log r_2 - \log r_1}{2\log R}\log(|w|R) & \text{if } |z| < r_1.
\end{cases}
\end{aligned} \tag{4.14}$$

By (4.8) and (4.14), we conclude that

$$\begin{aligned}
&\int_{|w|\geq r_2} \frac{\log r_2 - \log r_1}{2\log R}\log\frac{|w|}{R}\,\mu_{m,E}(\mathrm{d}w) + \int_{|w|\leq r_2^{-1}} \frac{\log r_2 - \log r_1}{2\log R}\log(|w|R)\,\mu_{m,E}(\mathrm{d}w) \\
&= 0,
\end{aligned} \tag{4.15}$$

as well as

$$\begin{aligned}
&\int_{r_1\leq|w|<r_2} \left(\frac{\log(\frac{r_2}{R})}{2\log R}\log|wR| - \frac{\log(r_1 R)}{2\log R}\log\frac{|w|}{R}\right)\mu_{m,E}(\mathrm{d}w) \\
&\quad + \int_{r_2^{-1}<|w|\leq r_1^{-1}} \frac{\log r_2 - \log r_1}{2\log R}\log(|w|R)\mu_{m,E}(\mathrm{d}w) \\
&= \int_{r_1\leq|w|<r_2} \left(\frac{\log(\frac{r_2}{R})}{2\log R}\log|wR| - \frac{\log(r_1 R)}{2\log R}\log\frac{|w|}{R}\right. \\
&\qquad \left. + \frac{\log r_2 - \log r_1}{2\log R}\log\frac{R}{|w|}\right)\mu_{m,E}(\mathrm{d}w) \\
&= \int_{r_1\leq|w|<r_2} \log\frac{r_2}{|w|}\mu_{m,E}(\mathrm{d}w),
\end{aligned} \tag{4.16}$$

and

$$\int\limits_{1<|w|<r_1} \frac{\log r_2 - \log r_1}{2\log R} \log(|w|R)\mu_{m,E}(\mathrm{d}w)$$
$$+ \int\limits_{r_1^{-1}<|w|<1} \frac{\log r_2 - \log r_1}{2\log R} \log(|w|R)\,\mu_{m,E}(\mathrm{d}w)$$
$$+ \int\limits_{|w|=1} \frac{\log r_2 - \log r_1}{2}\,\mu_{m,E}(\mathrm{d}w)$$
$$= \frac{\log r_2 - \log r_1}{2} \cdot \mu_{m,E}(\mathcal{A}_{r_1}). \tag{4.17}$$

Combining (4.13), (4.15), (4.16), and (4.17) with (4.12), one obtains

$$\begin{aligned}\pi\varepsilon_0\kappa(\omega, E) + \varepsilon_0\delta \geq & \frac{\pi\varepsilon_0}{2}\mu_{m,E}(\mathcal{A}_{\sqrt{R}}) + \int\limits_{\sqrt{R}\leq|w|<R} \log\frac{R}{|w|}\,\mu_{m,E}(\mathrm{d}w) \\ \geq & \frac{\pi\varepsilon_0}{2}\mu_{m,E}(\mathcal{A}_{\sqrt{R}}).\end{aligned}$$

This yields

$$\mu_{m,E}(\mathcal{A}_{\sqrt{R}}) \leq 2\kappa(\omega, E) + \delta,$$

as claimed. ■

4.2. Proof of Lemma 4.1

In this subsection we use the Riesz representation of $u_{m,E}$ on $\mathcal{A}_{\sqrt{R}}$, whose Riesz mass is controlled by Theorem 4.3. Throughout the rest of the paper, for simplicity, denote $\sqrt{R} =: R_1$. We have

$$u_{m,E}(z) = G_{R_1,m,E}(z) + h_{R_1,m,E}(z),$$

where

$$G_{R_1,m,E}(z) = \int\limits_{\mathcal{A}_{R_1}} 2\pi G_{R_1}(z,w)\,\mu_{m,E}(\mathrm{d}w).$$

We will prove the Fourier decay of $G_{R_1,m,E}$ and $h_{R_1,m,E}$ separately.

Lemma 4.5. *For $k \in \mathbb{Z} \setminus \{0\}$,*

$$|\widehat{G}_{R_1,m,E}(k)| \leq \frac{\mu_{m,E}(\mathcal{A}_{R_1})}{2|k|}(1 + R_1^{-|k|}).$$

Proof. It suffices to prove

$$|(2\pi G_{R_1}(\cdot, w))^\wedge(k)| \le \frac{1}{2|k|}(1 + R_1^{-|k|}).$$

This follows from explicit computations: if $w = re^{2\pi i\phi}$, then

$$\begin{aligned}\int_{\mathbb{T}} \log|e^{2\pi i\theta} - w| e^{-2\pi i k\theta}\, \mathrm{d}\theta &= \frac{1}{2\pi i k}\int_{\mathbb{T}} \frac{d}{d\theta}(\log|e^{2\pi i\theta} - w|) e^{-2\pi i k\theta}\, \mathrm{d}\theta \\ &= \frac{1}{ik}\int_{\mathbb{T}} \frac{r\sin(2\pi(\theta-\phi))}{1 - 2r\cos(2\pi(\theta-\phi)) + r^2} e^{-2\pi i k\theta}\, \mathrm{d}\theta \\ &= \frac{1}{ik} e^{-2\pi i k\phi}\int_{\mathbb{T}} \frac{r\sin(2\pi\theta)}{1 - 2r\cos(2\pi\theta) + r^2} e^{-2\pi i k\theta}\, \mathrm{d}\theta \\ &= \frac{\mathrm{sgn}(k)}{-2k} e^{-2\pi i k\phi} \begin{cases} r^{|k|} & \text{if } r \le 1, \\ r^{-|k|} & \text{if } r > 1. \end{cases}\end{aligned} \tag{4.18}$$

Recall the definition of $G_{R_1}(z, w)$ in (2.4); we have

$$\begin{aligned}(2\pi G_{R_1}(\cdot, w))^\wedge(k) &= \int_{\mathbb{T}} \log|e^{2\pi i\theta} - w| e^{-2\pi i k\theta}\, \mathrm{d}\theta \\ &+ \sum_{\ell=1}^{\infty}\int_{\mathbb{T}} \log\left|e^{2\pi i\theta} - \frac{1}{R_1^{4\ell}\overline{w}}\right| e^{-2\pi i k\theta}\, \mathrm{d}\theta \\ &+ \sum_{\ell=1}^{\infty}\int_{\mathbb{T}} \log\left|e^{2\pi i\theta} - \frac{w}{R_1^{4\ell}}\right| e^{-2\pi i k\theta}\, \mathrm{d}\theta \\ &- \sum_{\ell=1}^{\infty}\int_{\mathbb{T}} \log\left|e^{2\pi i\theta} - \frac{w}{R_1^{4\ell-2}}\right| e^{-2\pi i k\theta}\, \mathrm{d}\theta \\ &- \sum_{\ell=1}^{\infty}\int_{\mathbb{T}} \log\left|e^{2\pi i\theta} - \frac{1}{R_1^{4\ell-2}\overline{w}}\right| e^{-2\pi i k\theta}\, \mathrm{d}\theta.\end{aligned}$$

Note that we have converted each of the integrals above into the form of (4.18). Then directly applying (4.18), with $w = re^{2\pi i\phi}$, yields

$$\begin{aligned}(2\pi G_{R_1}(\cdot, w))^\wedge(k) &= \frac{\mathrm{sgn}(k)}{-2k} e^{-2\pi i k\phi} \begin{cases} r^{|k|} & \text{if } r \le 1, \\ r^{-|k|} & \text{if } r > 1 \end{cases} \\ &+ \frac{\mathrm{sgn}(k)}{-2k} e^{-2\pi i k\phi}\sum_{\ell=1}^{\infty}\Big(\frac{1}{R_1^{4\ell|k|}} - \frac{1}{R_1^{(4\ell-2)|k|}}\Big)(r^{|k|} + r^{-|k|})\end{aligned}$$

$$= \frac{\operatorname{sgn}(k)}{-2k} e^{-2\pi i k\phi} \begin{cases} r^{|k|} & \text{if } r \le 1, \\ r^{-|k|} & \text{if } r > 1 \end{cases}$$

$$+ \frac{\operatorname{sgn}(k)}{2kR_1^{|k|}} e^{-2\pi i k\phi} \frac{r^{|k|} + r^{-|k|}}{R_1^{|k|} + R_1^{-|k|}}.$$

Here we used that $R_1^{4\ell-2} \min(|w|, |w^{-1}|) > 1$ for $w \in \mathcal{A}_{R_1}$. This implies

$$|(2\pi G_{R_1}(\cdot, w))^\wedge(k)| \le \frac{1}{2|k|}(1 + R_1^{-|k|}),$$

as claimed. ■

As an immediate corollary of Theorem 4.3 and Lemma 4.5, one has the following result.

Corollary 4.6. *For any $\delta > 0$, for m large, uniformly in $k \neq 0$, we have*

$$|2\pi \widehat{G}_{R_1,m,E}(k)| \le \frac{\kappa(\omega, E) + \delta}{|k|}(1 + R_1^{-|k|}).$$

Lemma 4.7. *For any $\delta > 0$, for m large enough, uniformly in $k \neq 0$, we have*

$$|\hat{h}_{R_1,m,E}(k)| \le \frac{2L(\omega, E) + 2\pi\varepsilon_0\kappa(\omega, E) + \delta}{R_1^{|k|}}.$$

Proof. In the proof, we shall write $h_{R_1,m,E}$ as h_E for simplicity. We compute for $k \neq 0$ that, via integration by parts twice,

$$\begin{aligned} I_k(E, r) :&= \int_0^{2\pi} h_E(re^{i\theta}) e^{-ik\theta}\, d\theta \\ &= \frac{1}{ik} \int_{\mathbb{T}} (\partial_\theta h_E)(re^{i\theta}) e^{-ik\theta}\, d\theta \\ &= -\frac{1}{k^2} \int_{\mathbb{T}} (\partial_{\theta\theta} h_E)(re^{i\theta}) e^{-ik\theta}\, d\theta \\ &= \frac{1}{k^2} \int_{\mathbb{T}} [r^2(\partial_{rr} h_E)(re^{i\theta}) + r(\partial_r h_E)(re^{i\theta})] e^{-2\pi i k\theta}\, d\theta \\ &= \frac{1}{k^2}(r^2 I_k''(E, r) + r I_k'(E, r)). \end{aligned}$$

The solution to this ODE is $I_k(E, r) = a_k(E) r^k + b_k(E) r^{-k}$, where $a_k(E), b_k(E)$ are constants. By (4.7) and (4.11), we have $I_k(E, r) = I_k(E, 1/r)$. Hence, $a_k(E) =$

$b_k(E)$, and

$$I_k(E,r) = a_k(E)(r^k + r^{-k}). \tag{4.19}$$

Since the harmonic part is equal to $u_{m,E}$ on the boundary $\partial\mathcal{A}_{R_1}$, for $\delta > 0$, we have

$$0 \le h(R_1 e^{2\pi i\theta}) = u_{m,E}(R_1 e^{2\pi i\theta}) \le L\Big(\omega, E, -\frac{\varepsilon_0}{2}\Big) + \delta, \tag{4.20}$$

for m large enough, where we used the upper semi-continuity Lemma 2.2 in the last inequality. Combining (4.19), (4.20) with (4.1), we obtain that

$$|a_k(E)|(R_1^{|k|} + R_1^{-|k|}) \le 2\pi \sup_{\theta\in\mathbb{T}} |h(R_1 e^{2\pi i\theta})| \le 2\pi(L(\omega,E) + \pi\varepsilon_0\kappa(\omega,E) + \delta).$$

This implies

$$\begin{aligned}|\hat{h}_{R_1,m,E}(k)| &= (2\pi)^{-1}|I_k(E,1)| = \pi^{-1}|a_k(E)| \\ &\le \frac{2L(\omega,E) + 2\pi\varepsilon_0\kappa(\omega,E) + 2\delta}{R_1^{|k|}},\end{aligned}$$

as claimed. ■

Combining Corollary 4.6 with Lemma 4.7, we have proved Lemma 4.1. ■

5. Large deviation estimate

In this section we prove Theorem 3.1. Within the proof, let $\beta = \beta(\omega)$, $\beta_n = \beta_n(\omega)$ for simplicity. Let n be such that

$$\delta^{-2}(\beta+1)q_n \le m < \delta^{-2}(\beta+1)q_{n+1}. \tag{5.1}$$

Let $Q = [\delta m]$ and let

$$u_{m,E}^{(Q)}(\theta) = \sum_{|j|<Q} \frac{Q-|j|}{Q^2} u_{m,E}(\theta + j\omega) = \sum_{k\in\mathbb{Z}} \hat{u}_{m,E}(k) F_Q(k) e^{2\pi i k\theta}$$

be the Cesàro sum of $u_{m,E}$, see (2.10).

Note that $\hat{u}_{m,E}(0) = L_m(\omega,E)$. Fourier expansion yields (note that the zeroth coefficient cancels):

$$\begin{aligned}u_{m,E}(\theta) - L_m(\omega,E) = u_{m,E}(\theta) - u_{m,E}^{(Q)}(\theta) &=: U_1(\theta) \\ + \sum_{1\le|k|\le\delta^{-2}} \hat{u}_{m,E}(k) F_Q(k) e^{2\pi i k\theta} &=: U_2(\theta)\end{aligned}$$

$$+\sum_{\delta^{-2}<|k|<q_n} \hat{u}_{m,E}(k)F_Q(k)e^{2\pi ik\theta} =: U_3(\theta)$$

$$+\sum_{|\ell|=1}^{q_{n+1}/(4q_n)} \hat{u}_{m,E}(\ell q_n)F_Q(\ell q_n)e^{2\pi i\ell q_n\theta} =: U_4(\theta)$$

$$+\sum_{\ell=1}^{q_{n+1}/(4q_n)} \sum_{\ell q_n<|k|<(\ell+1)q_n} \hat{u}_{m,E}(k)F_Q(k)e^{2\pi ik\theta} =: U_5(\theta)$$

$$+\sum_{q_{n+1}/4\le|k|<e^{4\delta^4 m}} \hat{u}_{m,E}(k)F_Q(k)e^{2\pi ik\theta} =: U_6(\theta)$$

$$+\sum_{|k|\ge e^{4\delta^4 m}} \hat{u}_{m,E}(k)F_Q(k)e^{2\pi ik\theta} =: U_7(\theta).$$

Our key new estimates are for U_4.

By (2.9), we have

$$\|U_1\|_{L^\infty(\mathbb{T})} = \|u_{m,E}-u_{m,E}^{(Q)}\|_{L^\infty(\mathbb{T})} \le C_{v,1}\sum_{|j|<Q}\frac{(Q-|j|)|j|}{Q^2 m} \le C_{v,1}\frac{Q}{m} \le C_{v,1}\delta. \tag{5.2}$$

Regarding U_2, we infer from (2.15) that

$$\|U_2\|_{L^\infty(\mathbb{T})} \le \sum_{1\le|k|\le\delta^{-2}} \frac{C_{v,1}}{4m\|k\omega\|} \le \frac{C_{v,1}\delta^{-2}}{2m}\cdot \max_{1\le|k|\le\delta^{-2}}\frac{1}{\|k\omega\|} \le \delta, \tag{5.3}$$

provided m is large enough. By (2.12) and (2.14),

$$\|U_3\|_{L^\infty(\mathbb{T})} \le C_{v,2}\delta^2 \sum_{1\le|k|<q_n/4} \frac{1}{1+Q^2\|k\omega\|_{\mathbb{T}}^2} \le 2\pi C_{v,2}\delta^2\frac{q_n}{Q} \le 2\pi C_{v,2}\delta^3, \tag{5.4}$$

where we used $Q=\delta m \ge \delta^{-1}q_n$, due to (5.1). Regarding U_5, we have by (2.13) and (2.14) that

$$\|U_5\|_{L^\infty(\mathbb{T})} \le 2C_{v,2}\sum_{\ell=1}^{q_{n+1}/(4q_n)} \sum_{k\in(\ell q_n,(\ell+1)q_n)} \frac{1}{\ell q_n}\frac{1}{1+Q^2\|k\omega\|^2}. \tag{5.5}$$

Since $\ell \le [q_{n+1}/(4q_n)] = a_{n+1}/4$,

$$\|\ell q_n\omega\| \le \frac{a_{n+1}}{4}\|q_n\omega\| \le \frac{1}{4}\|q_{n-1}\omega\|, \tag{5.6}$$

where we used (2.3) in the last inequality. This implies for any $k \in (\ell q_n, (\ell+1)q_n)$ that

$$\|k\omega\| \geq \|(k-\ell q_n)\omega\| - \|\ell q_n \omega\| \geq \|q_{n-1}\omega\| - \frac{1}{4}\|q_{n-1}\omega\| \geq \frac{3}{4}\|q_{n-1}\omega\|, \quad (5.7)$$

in which we used $0 < |k-\ell q_n| < q_n$ and hence $\|(k-\ell q_n)\omega\| \geq \|q_{n-1}\omega\|$.

Next, we divide $(\ell q_n, (\ell+1)q_n)$ into the following two subsets:

$$\begin{cases} K_1 := \left\{k \in (\ell q_n, (\ell+1)q_n) : k\omega - [k\omega] \in \left(0, \frac{1}{2}\right)\right\}, \\ K_2 := \left\{k \in (\ell q_n, (\ell+1)q_n) : k\omega - [k\omega] \in \left(\frac{1}{2}, 1\right)\right\}. \end{cases}$$

For $\{k_1 \neq k_2\} \subset K_1$, one has

$$|\|k_1\omega\| - \|k_2\omega\|| = \|(k_1-k_2)\omega\| \geq \|q_{n-1}\omega\|. \quad (5.8)$$

Similarly for $\{k_1 \neq k_2\} \subset K_2$, we also have $|\|k_1\omega\| - \|k_2\omega\|| \geq \|q_{n-1}\omega\|$. Combining (5.7) with (5.8), we conclude that $\{\|k\omega\|\}_{k\in K_1}$ (same for K_2) are at least $\|q_{n-1}\omega\|$ spaced with the smallest term being at least $3\|q_{n-1}\omega\|/4$. This implies in view of $\|q_{n-1}\omega\| \geq 1/(2q_n)$, that

$$\begin{aligned} \sum_{s=1}^{2}\sum_{k\in K_s}\frac{1}{1+Q^2\|k\omega\|^2} &\leq 2\sum_{j=1}^{q_n}\frac{1}{1+Q^2j^2\left(\frac{\|q_{n-1}\omega\|}{2}\right)^2} \\ &\leq 2\sum_{j=1}^{q_n}\frac{1}{1+Q^2\frac{j^2}{(4q_n)^2}} \\ &\leq 2\int_0^\infty \frac{1}{1+Q^2\frac{x^2}{(4q_n)^2}}\,\mathrm{d}x \\ &= \frac{8q_n}{Q}\int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x \\ &= \frac{4\pi q_n}{Q}. \end{aligned} \quad (5.9)$$

Combining (5.5) with (5.9) yields that

$$\|U_5\|_{L^\infty(\mathbb{T})} \leq \frac{8\pi C_{v,2}}{Q}\sum_{\ell=1}^{q_{n+1}}\frac{1}{\ell} \leq 8\pi C_{v,2}\frac{\log q_{n+1}}{Q} \leq 8\pi C_{v,2}\beta_n\frac{q_n}{Q} \leq 8\pi C_{v,2}\delta,$$

where we used that $Q \geq \delta m \geq \delta^{-1}(1+\beta)q_n > \delta^{-1}\beta_n q_n$, for n large.

We also have by (2.13) and (2.14) that

$$\begin{aligned}
\|U_6\|_{L^\infty(\mathbb{T})} &\leq 8C_{v,2} \sum_{\ell=1}^{4e^{4\delta^4 m}/q_{n+1}} \frac{1}{\ell q_{n+1}} \sum_{k\in[\ell q_{n+1}/4,(\ell+1)q_{n+1}/4)} \frac{1}{1+Q^2\|k\omega\|^2} \\
&\leq 8C_{v,2} \sum_{\ell=1}^{4e^{4\delta^4 m}/q_{n+1}} \frac{1}{\ell q_{n+1}} \Big(2+2\pi\frac{q_{n+1}}{Q}\Big) \\
&\leq 8C_{v,2}\Big(\frac{10\delta^4 m}{q_{n+1}}+\frac{10\pi\delta^4 m}{Q}\Big) \\
&\leq 80C_{v,2}((\beta+1)\delta^2+\pi\delta^3) \\
&\leq 400C_{v,2}\delta, \qquad (5.10)
\end{aligned}$$

where we used $m \leq \delta^{-2}(\beta+1)q_{n+1}$, $Q=[\delta m]$, and $\delta < 1/(\beta+1)$.

Regarding U_7, the following L^2 estimate holds by (2.14):

$$\|U_7\|_{L^2(\mathbb{T})}^2 = \sum_{|k|>e^{4\delta^4 m}} |\hat{u}_{m,E}(k)|^2|F_Q(k)|^2 \leq C_{v,2}^2 \sum_{|k|>e^{4\delta^4 m}} \frac{1}{|k|^2} \leq 2C_{v,2}^2 e^{-4\delta^4 m}.$$

The estimate of U_4 is new and is the key to Theorem 3.1. This lemma is the only place that requires the improved explicit bound in Lemma 4.1.

Lemma 5.1. *For any small $\delta>0$, for m large enough, we have*

$$\|U_4\|_{L^\infty(\mathbb{T})} \leq 2\kappa(\omega,E)\beta(\omega)+C_{\omega,v,4}\delta.$$

Proof. We write $\kappa(\omega,E)$ as κ for simplicity. We begin by decomposing

$$\begin{aligned}
U_4(\theta) = &\sum_{|\ell|=1}^{[q_{n+1}^{1-\delta}/Q]} \hat{u}_{m,E}(\ell q_n)F_Q(\ell q_n)e^{2\pi i\ell q_n} \ (=: U_{4,1}(\theta)) \\
&+ \sum_{|\ell|=[q_{n+1}^{1-\delta}/Q]+1}^{q_{n+1}/(4q_n)} \hat{u}_{m,E}(\ell q_n)F_Q(\ell q_n)e^{2\pi i\ell q_n} \ (=: U_{4,2}(\theta)).
\end{aligned}$$

Note that by (5.6), $\|\ell q_n\omega\| = |\ell|\|q_n\omega\| \geq |\ell|/(2q_{n+1})$. This implies, by Lemma 4.1, that

$$\begin{aligned}
&\|U_{4,2}\|_{L^\infty(\mathbb{T})} \\
&\leq 2\sum_{\ell=[q_{n+1}^{1-\delta}/Q]+1}^{q_{n+1}/(4q_n)} \Big(\frac{\kappa+\delta}{\ell q_n}\frac{2}{1+Q^2\|\ell q_n\omega\|^2}+C_{\omega,v,3}e^{-\pi\varepsilon_0\ell q_n}\Big)
\end{aligned}$$

$$\leq \frac{4(\kappa+\delta)}{q_n}\Big(\sum_{\ell=[q_{n+1}^{1-\delta}/Q]+1}^{q_{n+1}/(4q_n)} \frac{1}{\ell\big(1+Q^2\frac{\ell^2}{(2q_{n+1})^2}\big)}\Big) + 3C_{\omega,v,3}e^{-\pi\varepsilon_0 q_n}$$

$$\leq \frac{4(\kappa+\delta)}{q_n}\left(1+\int_{[q_{n+1}^{1-\delta}/Q]+1}^{\infty} \frac{1}{x(1+Q^2x^2/(2q_{n+1})^2)}\,\mathrm{d}x\right) + 3C_{\omega,v,3}e^{-\pi\varepsilon_0 q_n}$$

$$\leq \frac{4(\kappa+\delta)}{q_n}\left(1+\int_{q_{n+1}^{-\delta}/2}^{\infty} \frac{1}{x(1+x^2)}\,\mathrm{d}x\right) + 3C_{\omega,v,3}e^{-\pi\varepsilon_0 q_n}$$

$$\leq \frac{4(\kappa+\delta)}{q_n}\left(1+\int_{q_{n+1}^{-\delta}/2}^{1} \frac{1}{x}\,\mathrm{d}x + \int_{1}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x\right) + 3C_{\omega,v,3}e^{-\pi\varepsilon_0 q_n}$$

$$\leq 4(\kappa+\delta)(\delta\beta_n + 4q_n^{-1}) + 3C_{\omega,v,3}e^{-\pi\varepsilon_0 q_n}$$

$$\leq 4(\kappa+\delta)\delta\beta_n + \delta, \tag{5.11}$$

for m large enough (when m is large, q_n and n are also large).

For $1 \leq \ell \leq q_{n+1}^{1-\delta}/Q$, we give a more precise estimate of $F_Q(\ell q_n)$ than (2.11). Note that for such ℓ's, and any $|j| < Q$, we have

$$\|\ell j q_n\omega\| = \ell|j|\|q_n\omega\| \leq \ell Q\|q_n\omega\| \leq q_{n+1}^{-\delta}.$$

Hence, $e^{2\pi i\ell j q_n\omega} = 1 + O(q_{n+1}^{-\delta})$. Going back to its definition of F_Q, we have

$$\begin{aligned} F_Q(\ell q_n) &= \sum_{|j|<Q} \frac{Q-|j|}{Q^2} e^{2\pi i\ell j q_n\omega} \\ &= \sum_{|j|<Q} \frac{Q-|j|}{Q^2}(1+O(q_{n+1}^{-\delta})) \\ &= 1 + O(q_{n+1}^{-\delta}). \end{aligned} \tag{5.12}$$

Combining Lemma 4.1 with (5.12), we have

$$\begin{aligned} \|U_{4,1}\|_{L^\infty(\mathbb{T})} &\leq 2\sum_{\ell=1}^{[q_{n+1}^{1-\delta}/Q]} \Big(\frac{\kappa+\delta}{\ell q_n}(1+O(q_{n+1}^{-\delta})) + C_{\omega,v,4}e^{-\pi\ell q_n\varepsilon_0}\Big) \\ &\leq 2(\kappa+\delta)\beta_n(1-\delta)(1+O(q_{n+1}^{-\delta})) + 3C_{\omega,v,4}e^{-\pi q_n\varepsilon_0} \\ &\leq 2(\kappa+\delta)\beta_n\Big(1-\frac{\delta}{2}\Big) + \delta. \end{aligned} \tag{5.13}$$

Therefore, combining (5.11) with (5.13), we have, using $\delta < (1+\beta)^{-1}$,

$$\begin{aligned}\|U_4\|_{L^\infty(\mathbb{T})} &\le 2(\kappa+\delta)\beta_n(1+2\delta)+2\delta\\ &\le 2(\kappa+\delta)\beta(1+3\delta)+2\delta\\ &\le 2\kappa\beta+2(\beta(1+3(1+\beta)^{-1})+1)\delta,\end{aligned}$$

for m large enough (which implies n is large), as claimed. ■

Combining (5.2)–(5.5) and (5.10) with Lemma 5.1, we have for m large enough,

$$\|u_{m,E}-L_m-U_7\|_{L^\infty(\mathbb{T})} \le \sum_{j=1}^{6}\|U_j\|_{L^\infty(\mathbb{T})} \le 2\kappa\beta+C_{\omega,v,5}\delta,$$

where $C_{\omega,v,5}$ depends on $C_{v,1}$, $C_{v,2}$ and $C_{\omega,v,4}$. Combining this with (5.10) and the Chebyshev's inequality, we have

$$\begin{aligned}&\operatorname{mes}\{\theta : |u_{m,E}(\theta)-L_m(\omega,E)|>2\kappa\beta+2C_{\omega,v,5}\delta\}\\ &\le \operatorname{mes}\{\theta : |U_7(\theta)|>C_{\omega,v,5}\delta\} \le \frac{1}{C_{\omega,v,5}^2\delta^2}\|U_7\|_{L^2(\mathbb{T})}^2\\ &\le \frac{2\delta^{-2}C_{v,2}}{C_{\omega,v,5}^2}e^{-4\delta^4 m} \le e^{-\delta^4 m},\end{aligned}$$

for m large enough, as claimed.

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