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# Generic simplicity of ellipses

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**Abstract.** We study the Laplace operator in an ellipse with either Dirichlet or Neumann boundary. We prove that except for a countable set of eccentricities, the spectrum is simple.

*Steve Zelditch's impressive work in a wide variety of subjects has been very influential on ours. We cannot acknowledge all of the discussions with him that eventually led to something, whether a result, a proof, an example… Our use of his note (2004) in this paper is representative of his influence on us. We hope that he would have liked the result and this text is dedicated to his memory.*

## 1. Introduction

In this article, we adapt the methods of [11] to prove that the Laplace spectrum of the generic ellipse is simple, both with Dirichlet or Neumann boundary condition. Multiplicities are unchanged if one applies an isometry or a homothety, and hence it suffices to consider ellipses of the form

$$\mathcal{E}_h := \{(x, y) : (hx)^2 + y^2 < 1\},$$

where $h \in ]0, 1]$.

**Theorem 1.1.** *There exists a countable subset $\mathcal{C} \subset ]0, 1]$ so that if $h \notin \mathcal{C}$, then each eigenspace of the Dirichlet (resp. Neumann) Laplace operator of the ellipse $\mathcal{E}_h$ is one-dimensional.*

The spectral geometry of ellipses has a long history, starting with the work of Émile Mathieu (see [13]) and it has been recently the subject of a number of striking results, both on its dynamical and spectral sides. In [10], H. Hezari and S. Zelditch, use (among many other ingredients) dynamical results initiated in [2] to prove that ellipses of small eccentricity are spectrally determined. In a previous paper ([9]) the same

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authors conjecture that eigenvalues of a generic ellipse have multiplicity at most 2. The latter result is a consequence of analyticity of the spectrum and the well-known fact that eigenvalues of the disk have multiplicity at most 2. In this paper, we refine this result and prove that the spectrum of a generic ellipse is actually simple. It turns out that the method of proof also gives that there do exist ellipses (besides disks) that have at least one multiple eigenvalue.

In [11], we designed a general approach to prove generic simplicity in settings that depend only on a finite number of geometric parameters. This should be contrasted with the well-known results of Albert [1] and Uhlenbeck [15] that require infinitely many parameters. This approach relies on two main ingredients: analytic perturbation theory and (semiclassical) concentration of eigenfunctions. In [11], we used this approach for triangles and here we use it for ellipses. Although the general philosophy is the same, the exact method described in [11] does not apply directly to ellipses. For the ellipse $\mathcal{E}_h$ above, the natural profile curve is $y = L(x) = \sqrt{1-(hx)^2}$ and the derivative $L'$ does not exist at the endpoints $x = \pm\frac{1}{h}$. The method of [11] requires that $L'$ be finite at each endpoint. To apply our approach in the case of ellipses, we need an extra new ingredient that is a careful control of the mass of eigenfunctions near $x = \pm\frac{1}{h}$. We also use the structure of the spectrum of the unit disk while the method of [11] does not require a knowledge of the spectrum of a particular member of the family of domains.

We outline the contents of the paper. We treat the case of the Dirichlet Laplacian in the main part of the text and provide details of the Neumann case in Appendix B. In Section 2, we state and prove a non-concentration estimate for a 1-dimensional semiclassical family $P_h u = -h^2 u'' + Vu$ of Schrödinger operators with an unbounded, single-well potential $V$. In Section 3, we apply this non-concentration estimate to prove an estimate for a family of semiclassical Sturm–Liouville operators $A_h u = -h^2 L^{-1}(Lu')' + L^{-2}u$ on a finite interval where $L$ is a non-negative function that vanishes on the endpoints and has exactly one critical point. These operators arise from naïvely applying separation of variables to a domain $\Omega$ that lies between the $x$-axis and the graph of the function $L$. The estimate is a non-concentration estimate for $H^1$ functions supported away from the endpoints.

In Section 4, we make sense of the formal sum $\mathcal{A}_h = \sum_k -h^2 L^{-1}(Lu')' + \left(\pi\frac{k}{L}\right)^2 u$, and we obtain a non-concentration estimate in this context. In Section 5 we show that the quadratic form $a_h$ associated to $\mathcal{A}_h$ is comparable and in fact 'asymptotic' to the quadratic form $q_h(u) = \int_\Omega h^2 \cdot |u_x|^2 + |u_y|^2$ on $H^1_0(\Omega)$. This allows us to use the non-concentration estimate for $\mathcal{A}_h$ to identify the semiclassical limits of real-analytic eigenvalue branches $h \mapsto E_h$ of $q_h$ under Dirichlet and certain mixed conditions.

The quadratic form $q_h$ is the pull-back of the Dirichlet energy on a domain $\Omega_h$ obtained by stretching $\Omega$ horizontally. In Section 6 we consider the domain $\widetilde{\Omega}_h$ obtained by reflecting $\Omega$ across the $x$-axis. For example, $\widetilde{\Omega}_h$ is the ellipse $\mathcal{E}_h$ if $L(x) = \sqrt{1-x^2}$. The symmetry of $\widetilde{\Omega}_h$ allows one to decompose into 'even' and 'odd' functions. The results of Section 5 immediately imply that limits of 'odd' eigenvalue branches and the limits of 'even' eigenvalue branches are distinct. By specializing this result to the family of ellipses and by using Bourget's hypothesis we are able to prove Theorem 1.1.

In Section 7 we use the same method to prove that the generic ellipsoid in $\mathbf{R}^3$ has simple spectrum. In [6], this was proven for ellipsoids near the round ball.

## 2. A one-dimensional non-concentration estimate

For each $u \in \mathcal{C}_0^\infty(\mathbf{R})$ and $h \in ]0,\infty[$, define $P_h u = -h^2 \cdot u'' + V \cdot u$. That is, $P_h$ is a one-dimensional semiclassical Schrödinger operator. For basic results on Schrödinger operators, see for example [3, Section 7.1].

We will assume that the 'potential' $V$ satisfies the following conditions:

V1. $V$ is positive and smooth and $\lim_{\pm\infty} |V(x)| = +\infty$;

V2. if $x \neq 0$, then $x \cdot V'(x) > 0$.

Condition V2 implies that $V$ attains its global minimum at $x = 0$ and hence $P_h$ is semi-bounded from below. We will use $P_h$ to denote the Friedrichs extension.[1] Condition V1 implies that $P_h$ has compact resolvent, and so its spectrum, $\mathrm{spec}(P_h)$, is discrete and consists of eigenvalues. Condition V1 also implies that each eigenspace of $P_h$ is 1-dimensional. For each eigenvalue $\lambda$, choose an $L^2$-normalised eigenfunction $\psi_\lambda$. Then $(\psi_\lambda)_{\lambda \in \mathrm{spec}(P_h)}$ is a Hilbertian basis for $L^2(\mathbf{R})$: Each $u \in L^2(\mathbf{R})$ may be written as

$$u = \sum_{\lambda \in \mathrm{spec}(P_h)} \langle \psi_\lambda, u \rangle \cdot \psi_\lambda,$$

where $\langle v, u \rangle = \int_{\mathbb{R}} \bar{v} \cdot u \, dx$.

It is customary to set $H^1_{P_h} := \mathrm{dom}((P_h+1)^{1/2})$, and

$$\begin{aligned}\|u\|^2_{H^1_{P_h}} &:= \|(P_h+1)^{1/2} u\|^2_{L^2} \\ &= h^2 \int_{\mathbf{R}} |u'(x)|^2 \, dx + \int_{\mathbf{R}} V(x) \cdot |u(x)|^2 \, dx + \int_{\mathbf{R}} |u(x)|^2 \, dx. \\ &= \sum_{\lambda \in \mathrm{spec}(P_h)} (\lambda+1) \cdot |\langle \psi_\lambda, u \rangle|^2.\end{aligned}$$

[1] In fact, $P_h$ is essentially self-adjoint, but this is not important for us.

We denote by $H^{-1}_{P_h}$ the dual space to $H^1_{P_h}$ equipped with the dual norm. If $u \in H^1_{P_h}$ and if $E \in \mathbf{R}$, then $(P_h - E)u$ belongs to $H^{-1}_{P_h}$. We have

$$\begin{aligned}
&\|(P_h - E)u\|_{H^{-1}_{P_h}} \\
&\quad = \Big( \sum_{\lambda \in \operatorname{spec} P_h} \frac{(\lambda - E)^2}{\lambda + 1} \cdot |\langle \psi_\lambda, u\rangle|^2 \Big)^{1/2} \\
&\quad = \sup \Big\{ \frac{|h^2 \int \bar{u}'(x) v'(x)\, dx + \int (V(x) - E)\bar{u}(x) v(x)\, dx|}{\|v\|_{H^1_{P_h}}}, v \in H^1_{P_h} \Big\}.
\end{aligned} \tag{1}$$

The following proposition implies a non-concentration result for $O(h)$ quasi-modes.

**Proposition 2.1.** *Let $K$ be a compact set such that $K \subset ]V(0), +\infty[$. For any $\varepsilon > 0$ there exists a constant $C$ and $h_0$ such that, if $h \in ]0, h_0]$, $E \in K$ and $u \in \mathcal{C}_0^\infty(\mathbf{R})$, then*

$$\|u\|_{H^1_{P_h}} \leqslant \frac{\varepsilon}{h} \cdot \|(P_h - E)u\|_{H^{-1}_{P_h}} + C \cdot \|h \cdot u'\|_{L^2}.$$

*Proof.* By assumption, there exists an interval $[a, b]$ such that $K \subset ]a, b[$ and $[a, b] \subset ]V(0), +\infty[$. We argue by contradiction. Suppose that the statement is not true, and let $\mathbf{N}^*$ denote the set of positive integers. Then there exists $E_0 \in K$, a constant $M$, a subset $\mathbb{H} \subset ]0, \infty[$ with zero as an accumulation point, and functions $E\colon \mathbb{H} \to \mathbf{R}$ and $\varepsilon\colon \mathbb{H} \to \mathbf{R}$, such that as $h \in \mathbb{H}$ tends to zero, we have $E_h \to E_0$, $\varepsilon_h \to 0$,

$$\|(P_h - E_h)u_h\|_{H^{-1}_{P_h}} \leqslant M \cdot h \cdot \|u_h\|_{H^1_{P_h}}, \tag{2}$$

and

$$\|h \cdot u_h'\|_{L^2} \leqslant \varepsilon_h \cdot \|u_h\|_{H^1_{P_h}}.$$

In the language of semiclassical analysis, equation (2) means that $u_h$ is a quasimode for $P_h$ of order $O(h)$.

The assumptions on $V$ imply that the compact interval $[a, b]$ consists of non-critical energies. Using semiclassical analysis, the distribution of the eigenvalues of $P_h$ in $K$ is thus well understood.[2] In particular, using Bohr–Sommerfeld rules for instance, we know that the spectrum in the interval $[a, b]$ is given by an (ordered) sequence $\lambda_{i,h}$ and that there exist two positive constants $c_1$ and $c_2$ such that if $h$ is small enough, then, for any eigenvalue $\lambda_{i,h}$ of $P_h$ in $[a, b]$ we have

$$c_1 \cdot h \leqslant \lambda_{i+1,h} - \lambda_{i,h} \leqslant c_2 \cdot h. \tag{3}$$

[2]See [4, 5, 8, 18], and the references therein.

The number $E_h$ lies in the closure of some component of $\mathbf{R} \setminus \operatorname{spec} P_h$, thus, there exists a unique integer $i_h$ so that

$$\lambda_{i_h,h} \leqslant E_h < \lambda_{i_h+1,h} \leqslant \lambda_{i_h,h} + c_2 \cdot h.$$

It follows that, for each sufficiently large integer $N$ and $h$ small enough,

$$a < \lambda_{i_h-N,h} \leqslant \lambda_{i_h,h} - N \cdot c_1 \cdot h \leqslant \lambda_{i_h,h} \leqslant \lambda_{i_h,h} + N \cdot c_1 \cdot h \leqslant \lambda_{i_h+N,h} < b.$$

Using $N$, we decompose $u_h$ onto those modes that are close to $E_h$ and those that are far from $E_h$.

Concretely, we define the projection of $u_h$ onto the modes close to $E_h$ to be

$$v_h := \sum_{j=-N}^{N} \langle \psi_{i_h+j,h}, u_h \rangle \cdot \psi_{i_h+j,h}.$$

We now show that $v_h$ approximates $u_h$ as $h$ goes to zero. Define $r_h := u_h - v_h$ to be the remainder. Using the definitions of the $H^1_{P_h}$ and $H^{-1}_{P_h}$ norms we have

$$\|r_h\|^2_{H^1_{P_h}} \leqslant \max\left\{ \frac{(\lambda_{i_h-N-1,h} + 1)^2}{(\lambda_{i_h-N-1,h} - E_h)^2}, \frac{(\lambda_{i_h+N+1,h} + 1)^2}{(\lambda_{i_h+N+1,h} - E_h)^2} \right\} \cdot \|(P_h - E_h)u_h\|^2_{H^{-1}_{P_h}}. \tag{4}$$

The prefactor on the right of (4) is bounded above by

$$\frac{(b+1)^2}{(Nc_1 - c_2)^2 \cdot h^2} \leqslant \frac{c_3}{N^2 \cdot h^2}$$

for $N$ large. By combining this with (2)—the assumption that $u_h$ is a quasimode—we obtain the following estimate for $h$ small and $N$ large:

$$\|r_h\|^2_{H^1_{P_h}} \leqslant \frac{c_3}{N^2} \cdot \|u_h\|^2_{H^1_{P_h}}.$$

Here $c_3$ is a constant that depends neither on $h$ nor on $N$.

We next proceed to estimate $\|v_h\|^2$ by comparing it to $\|h \cdot v_h'\|^2_{L^2}$. Set $V_{h,N}$ to be the span of $\{\psi_{i,h} : |i - i_h| < N\}$. By construction $v_h$ belongs to each $V_{h,N}$. The space $V_{h,N}$ is a $2N+1$-dimensional vector space.

Let $\mathcal{B}_{h,N} \colon V_{h,N} \to \mathbf{R}$ denote the quadratic form defined by $\mathcal{B}_{h,N}(v) = \|h \cdot v'\|^2_{L^2}$. This quadratic form is represented by a hermitian matrix $B_{h,N}$ in the basis of eigenfunctions.

The small $h$ behavior of the diagonal entries of $B_{h,N}$ is described by semiclassical (defect) measures (see for example [18]). In dimension 1, there is only one semiclassical measure associated to a non-critical energy. Indeed, this measure, denoted $\mu_{E_0}$, equals the Liouville measure on $T^*\mathbf{R}$ restricted to the energy shell[3]

$$\Sigma_{E_0} = \{(x,\xi) : \xi^2 + V(x) = E_0\}.$$

From this, we find that each diagonal entry of $B_h$ converges to the positive number

$$\beta = \int_{\Sigma_{E_0}} \xi^2 d\mu_{E_0}.$$

A nice argument of Steve Zelditch in [17] when semiclassically reformulated, shows that, for any $j,k$ between $-N$ and $N$, there exists a (complex) measure $\mu_{jk}$ such that

$$\langle \mathrm{Op}_h(a)\psi_{i_h+j}, \psi_{i_h+k}\rangle \xrightarrow[h\to 0]{} \int a d\mu_{jk} \quad \text{for all } a,$$

and that, moreover, the measure $\mu_{jk}$ is absolutely continuous with respect to $\mu_{E_0}$. We can thus write

$$\mu_{jk} = f_{jk}\mu_{E_0}$$

for some integrable $f_{jk}$. It follows that the matrix $\mathcal{B}_{h,N}$ converge to the matrix

$$B_0 = \left[\int_{\Sigma_{E_0}} \xi^2 f_{jk}\, d\mu_{E_0}\right]_{\substack{-N\leqslant j\leqslant N\\ -N\leqslant k\leqslant N}}.$$

By construction, this matrix is Hermitian and we claim that it is also positive. For, otherwise, there would be a normalised eigenvector $(w_j)_{-N\leqslant j\leqslant N}$ such that

$$\int_{\Sigma_{E_0}} \xi^2 \sum_{j,k} \overline{w_j} f_{jk} w_k d\mu_{E_0} \leqslant 0.$$

The latter implies that the integrable function $\sum_{j,k} \overline{w_j} f_{jk} w_k$ vanishes. This is in contradiction with the fact that the measure $\sum_{j,k} \overline{w_j} f_{jk} w_k d\mu_{E_0}$ is the semiclassical measure associated with the sequence

$$w_h = \sum_{j=-N}^{N} w_j \psi_{i_h+j,h},$$

and, as such, is a probability measure.

[3] See [8, Appendix] for precise statements.

Denote by $\beta$ the smallest (positive) eigenvalue of $B_0$. For $h$ small enough, we have

$$\begin{aligned}\|h \cdot v_h'\|^2 \geqslant \frac{\beta}{2} \cdot \|v_h\|_{L^2}^2 \geqslant \frac{\beta}{2(b+1)} \cdot \|v_h\|_{H^1_{P_h}}^2 \\ \geqslant \frac{\beta}{2(b+1)} \cdot (\|u_h\|_{H^1_{P_h}}^2 - \|r_h\|_{H^1_{P_h}}^2) \\ \geqslant \frac{\beta}{2(b+1)} \cdot \Big(1 - \frac{c_3}{N^2}\Big) \cdot \|u_h\|_{H^1_{P_h}}^2 .\end{aligned}$$

This yields a contradiction with the definition of the sequence $\varepsilon_h$ provided that $N$ is sufficiently large. ■

**Remark 2.2.** Note that, by using a semiclassical quantization, the controlling term $\|h \cdot v_h'\|_{L^2}^2$ may be thought of as $\langle \mathrm{Op}_h(\xi^2)u_h, u_h\rangle$. In the proof we can replace $\xi^2$ by any non-negative symbol $a$ of order 0 provided that $\int a \, d\mu_{E_0} > 0$.

**Remark 2.3.** Because $\int \xi^2 \, d\mu_{E_0} > 0$, the probability measure $\mu_{E_0}$ is not concentrated at the turning points for non-critical $E$. This can also be observed by using an Airy function analysis near a turning point (see [11, Proposition 9.1 and Remark 9.3]). Proposition 2.1 can thus be rephrased as *any $O(h)$ quasimode cannot concentrate on the turning points*. The corresponding statement for an $o(h)$ quasimode is a straightforward implication of the invariance of semiclassical measures under the hamiltonian flow of the symbol.

## 3. An application to a singular Schrödinger operator

To study the eigenvalue problem for an ellipse, we will use singular Schrödinger operators on $]-1,1[$. In this section we derive a non-concentration estimate of the same form as above.

Let $L$ be a smooth positive function on $]-1,1[$, that satisfies the following assumptions:

L1. for any $x \in \,]-1,1[$, we have $x \cdot L'(x) < 0$ when $x \neq 0$;

L2. $L$ extends continuously to $[-1,1]$ by setting $L(\pm 1) = 0$.

The function $L^{-2}$ will be used to construct a potential that satisfies conditions V1 and V2 of Section 2.

**Remark 3.1.** For our application to the disk, we will have that $L$ vanishes like the square root $\sqrt{(1-x)}$ at $x=1$ (resp. $\sqrt{1+x}$ at $x=1$). We note here that for functions $L$ that either do not vanish at $x=\pm 1$ or that vanish to order 1, then the results below would follow directly from [11].

Let $\mathscr{H} = L^2(]-1,1[, L(x)\,dx)$ and let $A_h$ be the symmetric operator defined on $\mathscr{C}_0^\infty(]-1,1[)$ by

$$A_h u := -h^2 \cdot \frac{1}{L}(Lu')' + \frac{1}{L^2}u.$$

The study of the essential self-adjointness of $A_h$ is a standard procedure that we do not pursue here. We simply observe that the quadratic form $\langle A_h u, u\rangle_{\mathscr{H}}$ is non-negative, so that we may consider the Friedrichs extension that we still denote by $A_h$.

The operator $A_h$ is non-negative and has compact resolvent so that its spectrum consists of eigenvalues and we may define spaces $H^1_{A_h}$ and $H^{-1}_{A_h}$ in the same way that we defined $H^1_{P_h}$ and $H^{-1}_{P_h}$ as in the previous section.

For $\delta \in ]0,1[$, we denote by $I_\delta$ the interval $[-1+\delta, 1-\delta]$. The following proposition is a consequence of Proposition 2.1.

**Proposition 3.2.** *Let $b > a > \frac{1}{L(0)^2}$. There exists $\delta_0 \in ]0,1[$ such that the following holds. For any $\delta \in ]0,\delta_0[$ and any $\varepsilon > 0$ there exist constants $C$ and $h_0$ such that if $h \in ]0,h_0[$, if $E \in [a,b]$ and if $u \in \mathscr{C}_0^\infty(I_\delta)$ then*

$$\|u\|_{H^1_{A_h}} \leqslant \frac{\varepsilon}{h} \cdot \|(A_h - E)u\|_{H^{-1}_{A_h}} + C \cdot \|h \cdot u'\|_{\mathscr{H}}.$$

**Remark 3.3.** We warn the reader that the interval $[a,b]$ in the statement of Proposition 3.2 corresponds to the compact set $K$ in the statement of Proposition 2.1 and not to the interval used in the proof of Proposition 2.1.

*Proof.* Because $L$ is continuous and $L(-1) = 0 = L(1)$, there exists $\delta_0 \in ]0,1[$ so that $x \notin I_{\delta_0}$ implies that $L(x)^{-2} \geqslant b+1$. Let $\delta \in ]0,\delta_0[$. Choose a potential $V_\delta$ that coincides with $\frac{1}{L^2}$ on $I_{\frac{\delta}{2}}$ and that satisfies assumptions V1 and V2 of Section 2. See Figure 1. Define $P_h u := -h^2 \cdot u'' + V_\delta \cdot u$.

Let $\varepsilon > 0$. Proposition 2.1 applies to give a constant $C$ so that for any $u \in \mathscr{C}_0^\infty(I_\delta)$

$$\|u\|_{H^1_{P_h}} \leqslant \frac{\varepsilon}{h} \cdot \|(P_h - E)u\|_{H^{-1}_{P_h}} + C \cdot \|h \cdot u'\|_{L^2(\mathbf{R})}.$$

The claim now follows from 3.4 below. ∎

**Lemma 3.4.** *Let $a, b, \delta$ and $V_\delta$ be as in the proof of Proposition 3.2. Then, there exist constants $m_i$, $i = 1,2,3$, such that for any $u \in \mathscr{C}_0^\infty(I_\delta)$,*

$$m_1 \cdot \|u\|_{L^2(\mathbf{R})} \leqslant \|u\|_{\mathscr{H}} \leqslant \frac{1}{m_1} \cdot \|u\|_{L^2(\mathbf{R})}, \tag{5}$$

$$m_2 \cdot \|u\|_{H^1_{P_h}} \leqslant \|u\|_{H^1_{A_h}} \leqslant \frac{1}{m_2} \cdot \|u\|_{H^1_{P_h}}, \tag{6}$$

$$\|(P_h - E)u\|_{H^{-1}_{P_h}} \leqslant \frac{1}{m_3} \cdot \|(A_h - E)u\|_{H^{-1}_{A_h}} + m_3 \cdot h \cdot \|h \cdot u'\|_{\mathscr{H}}. \tag{7}$$

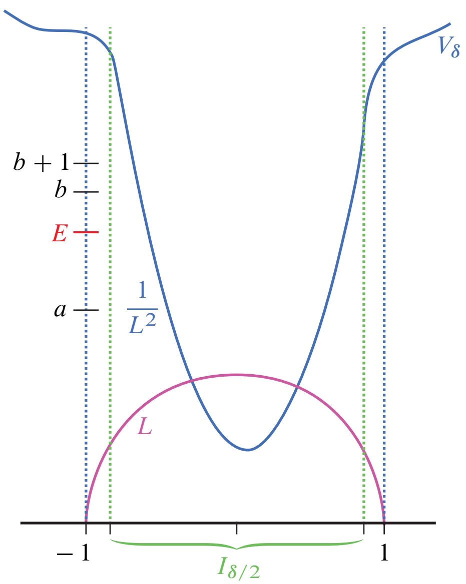


**Figure 1.** The construction of the potential $V_\delta$.

*Proof.* Everywhere in the proof of the lemma, we will use the fact that $u$ has support in $I_\delta = [-1+\delta, 1-\delta]$.

Estimate (5) follows from the fact that $L$ is uniformly bounded above and below on $I_\delta$. For (6), we observe that

$$\begin{aligned}\|u\|^2_{H^1_{P_h}} &= \int_{I_\delta} |h \cdot u'(x)|^2 \, dx + \int_{I_\delta} (V(x)+1) \cdot |u(x)|^2 \, dx,\\ \|u\|^2_{H^1_{A_h}} &= \int_{I_\delta} |h \cdot u'(x)|^2 L(x) \, dx + \int_{I_\delta} \Big(\frac{1}{L(x)^2}+1\Big) \cdot |u(x)|^2 L(x) \, dx.\end{aligned}$$

and we obtain (6) using (5) and the fact that the functions $V$ and $\frac{1}{L^2}$ coincide on $I_\delta$.

To prove estimate (7), we use the definitions of the $H^{-1}$ norms. In particular, if we let

$$I(\phi) := h^2 \int_{\mathbf{R}} \bar{u}'(x)\phi'(x) \, dx + \int_{\mathbf{R}} (V(x)-E)\bar{u}(x)\phi(x) \, dx.$$

then $\|(P_h - E)u\|_{H^{-1}_{P_h}}$ equals the supremum of $|I(\phi)|/\|\phi\|_{H^1_{P_h}}$ over $\phi \in \mathscr{C}^\infty_0(\mathbf{R})$. We now proceed to compute $I(\phi)$. Because $u$ has support in $I_\delta$, we can insert a cutoff function $\chi$ that is 1 on $I_\delta$ and 0 outside $I_{\delta/2}$. Since $L > 0$ on $I_{\delta/2}$, there exists

$\psi \in \mathcal{C}_0^\infty(\mathbf{R})$ such that $L \cdot \psi = \phi \cdot \chi$. We have

$$\begin{aligned}
I(\phi) &= h^2 \int_{\mathbf{R}} \bar{u}'(x)(\chi \cdot \phi)'(x)\, dx + \int_{\mathbf{R}} (V(x) - E)\bar{u}(x)(\chi \cdot \phi)(x)\, dx \\
&= h^2 \int_{\mathbf{R}} \bar{u}'(x)(\psi \cdot L)'(x)\, dx + \int_{\mathbf{R}} (V(x) - E)\bar{u}(x)(\psi \cdot L)(x)\, dx \\
&= h^2 \int_{\mathbf{R}} \bar{u}'(x)\psi'(x)L(x)\, dx + \int_{\mathbf{R}} \Big(\frac{1}{L(x)^2} - E\Big)\bar{u}(x)\psi(x)L(x)\, dx \\
&\quad + h^2 \int \bar{u}'(x)\psi(x)L'(x)\, dx. \\
&= \langle (A_h - E)u, \psi\rangle_{\mathcal{H}} + h^2 \int \bar{u}'(x)\psi(x)L'(x)\, dx.
\end{aligned}$$

Using the definition of the $H^{-1}_{A_h}$ norm and the fact that $L'$ is bounded on $I_\delta$, we obtain

$$|I(\phi)| \leqslant \|(A_h - E)u\|_{H^{-1}_{A_h}} \cdot \|\psi\|_{H^1_{A_h}} + c \cdot h \cdot \|hu'\|_{L^2(\mathbf{R})} \cdot \|\psi\|_{L^2(\mathbf{R})}.$$

Since $\psi$ has support in $I_{\delta/2}$ we have the estimate $\|\psi\|_{H^1_{A_h}} \leqslant C\|\psi\|_{H^1_{P_h}}$.

Since $\psi = \frac{\chi \cdot \phi}{L}$ it is straightforward to prove that

$$\|\psi\|_{H^1_{P_h}} \leqslant C\|\phi\|_{H^1_{P_h}}.$$

Finally, we obtain

$$|I(\phi)| \leqslant (C \cdot \|(A - E)u\|_{H^{-1}_{A_h}} + ch\|h \cdot u'\|_{L^2(\mathbf{R})})\|\phi\|_{H^1_{P_h}}.$$

This yields (7). ∎

## 4. Summing non-concentration estimates

Let $L$ satisfy conditions L1 and L2 of the previous section and as before let $\mathcal{H} = L^2(]-1,1[, L(x)\,dx)$. For each $k \in \mathbf{N}^*$, $h \in ]0,\infty[$, and $v \in \mathcal{C}_0^\infty(]-1,1[)$ define

$$A_{k,h}v = -h^2 \frac{1}{L}(Lv') + \frac{k^2\pi^2}{L^2(x)}v.$$

We will let $A_{k,h}\colon D_{k,h} \to \mathcal{H}$ denote the Friedrichs extension. Note that the domain $D_{k,h}$ does not depend on $k$ or $h$ and hence we will simply write $D$. The analysis of Section 3 applies to each of the operators $A_{k,h}$. We will consider a 'sum' of the $A_{k,h}$ over $k \in \mathbf{N}^*$.

Define the Hilbert space $\widetilde{\mathcal{H}} := \ell^2(\mathbf{N}^*; \mathcal{H})$. An element in $\widetilde{\mathcal{H}}$ can be uniquely written as

$$u = \sum_{k \geqslant 1} u_k \otimes e_k,$$

where $(e_k)_{k \geqslant 1}$ is the canonical Hilbert basis of $\ell^2(\mathbf{N}^*)$. Note that $\|u\|^2_{\widetilde{\mathcal{H}}} = \sum_k \|u_k\|^2_{\mathcal{H}}$.

Let $\widetilde{\mathcal{D}}_f$ denote the subspace of $\widetilde{\mathcal{H}}$ consisting of $u$ such that each $u_k \in D$ and only finitely many of the $u_k$ are non-zero. (Here the subscript $f$ refers to the finiteness of each sum.) Define the operator $\mathcal{A}_{f,h} \colon \widetilde{\mathcal{D}}_f \to \widetilde{\mathcal{H}}$ by

$$\mathcal{A}_{f,h} u = \sum_{k \geqslant 1} A_{k,h} u \otimes e_k = \sum_{k \geqslant 1} \Big( -h^2 \frac{1}{L} (L u_k')' + \frac{k^2 \pi^2}{L^2(x)} u_k \Big) \otimes e_k.$$

The quadratic form $a_h$ associated with $\mathcal{A}_{f,h}$ is given by[4]

$$\begin{aligned} a_h(u) &= \sum_{k \geqslant 1} h^2 \int_{-1}^{1} |u_k'(x)|^2 L(x)\, dx + \int_{-1}^{1} \frac{k^2 \pi^2}{L^2(x)} |u_k(x)|^2 L(x)\, dx \\ &= \sum_{k \geqslant 1} a_{k,h}(u_k). \end{aligned}$$

The following theorem summarises the spectral theory of $\mathcal{A}_{f,h}$. Its proof is left to the reader. Let $\widetilde{\mathcal{D}}$ consist of $u \in \widetilde{\mathcal{H}}$ such that $u_k \in D$ for each $k$ and $\sum_k \|A_{k,h} u_k\|^2 < +\infty$.

**Theorem 4.1.** *The operator $\mathcal{A}_h \colon \widetilde{\mathcal{D}} \to \widetilde{\mathcal{H}}$ defined by*

$$\mathcal{A}_h \Big( \sum_{k \geqslant 1} u_k \otimes e_k \Big) = \sum_{k \geqslant 1} A_{k,h} u_k \otimes e_k$$

*is the Friedrichs extension of $\mathcal{A}_{f,h}$. It has compact resolvent. Its spectrum is given by*

$$\operatorname{spec} \mathcal{A}_h = \bigcup_{k \geqslant 1} \operatorname{spec} A_{k,h}.$$

*More precisely, if, for any $k$, $(\psi_{k,\ell})_{\ell \geqslant 0}$ is a Hilbert basis of $\mathcal{H}$ consisting in eigenvectors for $A_{k,h}$ then $(\psi_{k,\ell} \otimes e_k)_{k,\ell \geqslant 0}$ is a Hilbert basis of eigenvectors of $\mathcal{A}_h$.*

[4] A similar quadratic form was introduced in [11, Section 11] as part of the so-called method of asymptotic separation of variables. Here we include the otherwise unimportant factor of $\pi^2$ in the definition of $a_h$ in order to be consistent with the approach in [11].

The $H^1_{\mathcal{A}_h}$ norm is easily computed:

$$\|u\|^2_{H^1_{\mathcal{A}_h}} = \sum_{k\geqslant 1} \|u_k\|^2_{H^1_{A_{k,h}}},$$

and $u$ belongs to $H^1_{\mathcal{A}_h}$ if and only if the sum on the right-hand side is finite. By duality,

$$\|(\mathcal{A}_h - E)u\|^2_{H^{-1}_{\mathcal{A}_h}} = \sum_{k\geqslant 1} \|(A_{k,h} - E)u_k\|^2_{H^{-1}_{A_{k,h}}}.$$

The analysis in Section 3 leads to the following.

**Proposition 4.2.** *Let $k_0 \in \mathbf{N}^*$ and suppose that*

$$[a,b] \subset \Big]\Big(\frac{k_0\pi}{L(0)}\Big)^2, \Big(\frac{(k_0+1)\pi}{L(0)}\Big)^2\Big[.$$

*There exists $\delta_0 \in ]0,1[$ such that, for any $\varepsilon > 0$, for any $\delta \in ]0,\delta_0[$, there exist constants $C$ and $h_0$ such that, for any $h \in ]0,h_0[$, for any $E \in [a,b]$ and any $u \in H^1_{\mathcal{A}_h} \cap \ell^2(\mathbf{N}^*, \mathcal{C}^\infty_0(I_\delta))$ the following estimate holds:*

$$\|u\|_{H^1_{\mathcal{A}_h}} \leqslant \frac{\varepsilon}{h}\|(\mathcal{A}_h - E)u\|_{H^{-1}_{\mathcal{A}_h}} + C\|h \cdot D'u\|_{\widetilde{\mathcal{H}}},$$

*where $D'$ is the operator defined by $D'(\sum u_k \otimes e_k) = \sum u'_k \otimes e_k$.*

*Proof.* Write $u = \sum_{k\geqslant 1} u_k \otimes e_k$. For $k \in \mathbf{N}^*$, let $T_k$ denote the 'threshold' $\big(\frac{k\pi}{L(0)}\big)^2$. Note that $a_{k,h}(v) \geqslant T_k\|v\|_{\mathcal{H}}$ and hence spec $A_{k,h} \subset [T_k, +\infty[$ for each $k$ and $h$. In particular, if $k \geqslant k_0 + 1$ and $\lambda \in$ spec $A_{k,h}$, then by hypothesis

$$\lambda - E \geqslant T_{k_0+1} - b > 0.$$

Thus, by using the analogue of equation (1), we find that if $k \geqslant k_0 + 1$, then

$$\|u_k\|_{H^1_{A_k}} \leqslant c_+\|(A_{k,h} - E)u\|_{H^{-1}_{A_k}},$$

where

$$c_+ = \sup\Big\{\frac{|\lambda+1|}{|\lambda - E|} : \lambda \geqslant T_{k_0+1} \text{ and } E \in [a,b]\Big\} = 1 + \frac{b+1}{T_{k_0+1} - b}.$$

For $k \leqslant k_0$, we may use the estimate in Proposition 3.2:

$$\|u_k\|_{H^1_{A_k}} \leqslant \frac{\varepsilon}{h}\|(A_{k,h} - E)u\|_{H^{-1}_{A_k}} + C\|h \cdot u'_k\|_{\mathcal{H}}.$$

By squaring and summing all of these estimates, we obtain

$$\|u\|^2_{H^1_{\mathcal{A}_h}} \leqslant \Big(\max\Big\{\frac{2\varepsilon}{h}, c_+\Big\}\Big)^2 \cdot \|(\mathcal{A} - E)u\|^2_{H^{-1}_{\mathcal{A}_h}} + C\|h \cdot D'u\|^2_{\widetilde{\mathcal{H}}}.$$

We choose $h_0 < \frac{2\varepsilon}{c_+}$, and the claim follows. ∎

## 5. Compressing half-ovals with Dirichlet and mixed conditions

We apply the preceding estimates to study the analytic eigenvalue branches of the family $q_h$ of quadratic forms defined on certain subspaces of $H^1(\Omega)$ where

$$\Omega = \{(x, y) \in \mathbf{R}^2, |x| < 1, 0 < y < L(x)\}$$

and $L$ satisfies the conditions L1 and L2 of Section 3. For each $u \in H^1(\Omega)$, we define

$$q_h(u) := h^2 \int_\Omega |\partial_x u(x, y)|^2\, dx\, dy + \int_\Omega |\partial_y u(x, y)|^2\, dx\, dy. \tag{8}$$

We consider individually the restriction of $q_h$ to three distinct subspaces of $H^1(\Omega)$, and thus obtain three distinct spectral problems.

- *Full Dirichlet.* Restrict $q_h$ to $H^1_0(\Omega)$.
- *Dirichlet on curved part.* Restrict $q_h$ to the subspace $H^1_{0c}(\Omega)$ consisting of $u \in H^1(\Omega)$ whose trace vanishes on the graph of $L$. In this case, eigenfunctions satisfy Dirichlet conditions on the graph of $L$ and Neumann conditions on the $x$-axis.
- *Dirichlet on straight part.* We restrict $q_h$ to the subspace $H^1_{0s}(\Omega)$ consisting of $u \in H^1(\Omega)$ that vanish on the $x$-axis. In this case, eigenfunctions satisfy Neumann conditions on the graph of $L$ and Dirichlet conditions on the $x$-axis.

Here we study the spectrum of $q_h$ relative to $\|\cdot\|^2$ where $\|\cdot\|$ denotes the standard norm on $L^2(\Omega, dx\, dy)$. Recall that $u$ is an eigenfunction of $q_h$ relative to $\|u\|^2$ with eigenvalue $E$ if and only if $u$ belongs to the chosen subspace of $H^1(\Omega)$ and $q_h(u, v) = E\langle u, v\rangle$ for each $v$ belonging to the same subspace.

**Remark 5.1.** The quadratic form $q_h$ arises from deforming the domain $\Omega$. Indeed, suppose that $\Omega_h$ is the image of $\Omega$ under the map $(x, y) \mapsto \left(\frac{x}{h}, y\right)$. By making the change variables $\bar{x} = \frac{x}{h}$, we have

$$\frac{1}{h} q_h(u) = \int_{\Omega_h} |\partial_{\bar{x}} u(\bar{x}, y)|^2\, d\bar{x}\, dy + \int_{\Omega_h} |\partial_y u(\bar{x}, y)|^2\, d\bar{x}\, dy.$$

Note that the map $(x, y) \mapsto \left(\frac{x}{h}, y\right)$ 'stretches' $\Omega$ in the horizontal direction by a factor of $\frac{1}{h}$.

On the other hand, the map $(x, y) \mapsto (x, hy)$ compresses the domain $\Omega$ in the vertical direction. If $\Omega_h$ is the image of this map and we set $\bar{y} = h \cdot y$, then we find that

$$\frac{1}{h} q_h(u) = \int_{\Omega_h} |\partial_x u(x, \bar{y})|^2\, dx\, d\bar{y} + \int_{\Omega_h} |\partial_{\bar{y}} u(x, \bar{y})|^2\, dx\, d\bar{y}.$$

In either case, the study of the spectrum of Dirichlet and mixed Laplace operators on $\Omega_h$ may be reduced to the study of the spectrum of $q_h$ restricted to the appropriate subspace of $H^1(\Omega)$ and relative to $\|\cdot\|^2$. Observe that the latter norm also is simply related to the $L^2$ norm in $\Omega_h$.

Kato–Rellich analytic perturbation theory [12] applies to the family $q_h$, and so in each of the three cases above, the spectrum of $q_h$ relative to $\|\cdot\|^2$ may be organised into analytic eigenbranches $(E_h, u_h)$. The derivative of the eigenvalue branch $E_h$ is given by the Feynman–Hellmann formula:

$$\frac{dE_h}{dh} = \frac{(\partial_h q_h)(u_h)}{\|u_h\|^2} = \frac{2h}{\|u_h\|^2} \int_\Omega |\partial_x u_h(x,y)|^2 \, dx \, dy = \frac{1}{h} \frac{\|h \cdot \partial_x u_h(x,y)\|^2}{\|u_h\|^2}. \tag{9}$$

In particular, $\partial_h E_h \geqslant 0$ and so each eigenvalue branch $E_h$ is an increasing function of $h$. Since $E_h \geqslant 0$ for all $h$, we deduce that $E_h$ tends to a limit as $h$ tends to 0.

The following theorem restricts the possible limits of eigenvalue branches in the full Dirichlet case. After the proof we will state the analogous result for the mixed cases and we will discuss how to modify the proof.

**Theorem 5.2** (Full Dirichlet). *For any analytic eigenvalue branch $E_h$ of $q_h$ on $H_0^1(\Omega)$ relative to $\|\cdot\|^2$, there exists $k \in \mathbf{N}^*$ such that*

$$\lim_{h \to 0} E_h = \left(\frac{k\pi}{L(0)}\right)^2.$$

We will use two lemmas to prove Theorem 5.2.

The first lemma is a corollary of the Poincaré inequality on the segment $[0, L(x)]$. For $\delta \in ]0,1[$, define $J_\delta := ]-1,1[ \setminus I_\delta$ where $I_\delta$ is defined as in Section 3.

**Lemma 5.3** (Poincaré estimate). *There exists $f : ]0,1[ \to \mathbf{R}^*$ with $\lim_{\delta \to 0} f(\delta) = 0$ so that if $\delta \in ]0,1[$ and $u \in H_0^1(\Omega)$, then*

$$\int_\Omega \mathbb{1}_{J_\delta}(x) \cdot |u(x,y)|^2 \, dx \, dy \leqslant f(\delta) \int_\Omega \mathbb{1}_{J_\delta}(x) \cdot |\partial_y u(x,y)|^2 \, dx \, dy.$$

*Proof.* For each fixed $x \in ]-1,1[$, we perform a Fourier sine decomposition in $y$:

$$u(x,y) = \sum_{k \geqslant 1} u_k(x) \cdot \sqrt{2} \sin\left(\frac{k\pi}{L(x)} y\right). \tag{10}$$

A computation gives

$$\int_\Omega \mathbb{1}_{J_\delta}(x) \cdot |u(x,y)|^2 \, dx \, dy = \sum_{k \geqslant 1} \int_{J_\delta} |u_k(x)|^2 L(x) \, dx,$$

$$\int_{\Omega} \mathbb{1}_{J_\delta}(x) \cdot |\partial_y u(x,y)|^2 \, dx \, dy = \sum_{k \geqslant 1} \int_{J_\delta} \left( \frac{k\pi}{L(x)} \right)^2 |u_k(x)|^2 L(x) \, dx.$$

Thus, the desired inequality holds with

$$f(\delta) := \sup_{x \in J_\delta} \left( \frac{L(x)}{\pi} \right)^2.$$

Since $L(x) \to 0$ as $x \to \pm 1$, we have $\lim_{\delta \to 0} f(\delta) = 0$. ∎

The second lemma will show that the quadratic form $q_h$ may be approximated by the quadratic form $a_h$ of Section 4.[5] To be precise, we identify the space $L^2(\Omega, dx\,dy)$ with the space $\ell^2(\mathbf{N}^*, \mathcal{H})$ using the unitary map $u \mapsto \sum_k u_k \otimes e_k$, where $u_k$ is as in equation (10). Under this identification, for $u \in \mathcal{C}_0^\infty(\Omega)$, we set

$$\begin{aligned} a_h(u) &= \sum_{k \geqslant 1} h^2 \int_{-1}^{1} |u_k'(x)|^2 L(x) \, dx + \int_{-1}^{1} \left( \frac{k\pi}{L(x)} \right)^2 |u_k(x)|^2 L(x) \, dx \\ &= \|h \cdot D'u\|^2 + \|\partial_y u\|^2, \end{aligned}$$

where $D'$ is the operator defined in Proposition 4.2. As in Section 4, we will let $\mathcal{A}_h$ denote the self-adjoint operator associated to $a_h$.

For any $\delta \in ]0,1[$, we define

$$\Omega_\delta = \{(x,y) \in \Omega : x \in I_\delta\},$$

where we recall that $I_\delta = ]-1+\delta, 1-\delta[$. On the domain $\Omega_\delta$, the function $\frac{L'}{L}$ is uniformly bounded.

**Lemma 5.4** (Asymptotic at first order). *For any $\delta \in ]0,1[$, there exists a constant $C_\delta$ such that, for any $u \in H_0^1(\Omega_\delta)$ and $v \in H_0^1(\Omega)$, we have*

$$|q_h(u,v) - a_h(u,v)| \leqslant C_\delta h a_h(u)^{1/2} a_h(v)^{1/2}. \tag{11}$$

*Proof.* We first assume that $v \in H_0^1(\Omega_\delta)$ and then later show how to extend the inequality to $v \in H_0^1(\Omega)$.

Since $\frac{L'}{L}$ is uniformly bounded on $I_\delta$, if $w \in H_0^1(\Omega_\delta)$, then the function $w^*$ defined by

$$w^*(x,y) := -y \frac{L'(x)}{L(x)} \cdot w(x,y)$$

[5] This approximation is the basis for the method of asymptotic separation of variables described in [11].

also belongs to $H_0^1(\Omega_\delta)$. A computation shows that

$$\partial_x u = D'u + \partial_y u^*. \tag{12}$$

Using this, a further computation shows that

$$q(u,v) - a(u,v) = h^2 \cdot (\langle D'u, \partial_y v^* \rangle + \langle \partial_y u^*, D'v \rangle + \langle \partial_y u^*, \partial_y v^* \rangle),$$

where $\langle \cdot, \cdot \rangle$ is the inner product for $L^2(\Omega, dx\, dy)$. Let $C = \sup\{ \left| \frac{L'(x)}{L(x)} \right| : x \in I_\delta \}$. By using the Cauchy–Schwarz inequality we find that

$$|q(u,v) - a(u,v)| \leqslant h^2 \cdot (C\|D'u\|\|\partial_y v\| + C\|\partial_y u\|\|D'v\| + C^2\|\partial_y u\|\|\partial_y v\|).$$

In general, $h\|D'w\| \leqslant a(w)^{1/2}$ and $\|\partial_y w\| \leqslant a(w)^{1/2}$. Thus, we obtain (11) when $v \in H_0^1(\Omega_\delta)$.

If $v \in H_0^1(\Omega)$, then define $\chi(x,y) = \rho(x)$ where $\rho$ is smooth with compact support in $I_{\delta/2}$ and identically one on $I_\delta$. Because $\mathrm{supp}(u) \subset \Omega_\delta$, we have $a_h(u, \chi v) = a_h(u, \chi v)$ and $q_h(u, \chi v) = q_h(u, \chi v)$. Since $\chi \cdot v \in H_1(\Omega_{\delta/2})$ the argument above gives

$$|a_h(u,v) - q_h(u,v)| = |a_h(u, \chi v) - q_h(u, \chi v)| \leqslant C' h a_h(u)^{1/2} a_h(\chi v)^{1/2}.$$

A direct computation shows that there exists a constant $C''$ so that $a_h(\chi w) \leqslant C'' a(w)$ for each $w \in H_0^1(\Omega)$ when $h$ is small. The claim follows. ∎

*Proof of Theorem* 5.2. The proof is by contradiction. Assume that the conclusion is not true. Then there exists an eigenbranch $(E_h, u_h)$ such that $E_h$ tends to $E_0$ and $\sqrt{E_0} \cdot \frac{L(0)}{\pi}$ is not an integer. Thus, there exists $a < b$, $h_0 > 0$, and a non-negative integer $k_0$, so that if $h \leqslant h_0$, then $E_h \in [a,b]$ and

$$[a,b] \subset \left] \left( \frac{k_0 \pi}{L(0)} \right)^2, \left( \frac{(k_0+1)\pi}{L(0)} \right)^2 \right[. \tag{13}$$

Because $E_h$ has a limit as $h$ tends to zero, its derivative $\partial_h E_h$ is integrable with respect to $dh$ near $h = 0$. Thus, it follows from (9) that there exists a subset $\mathbb{H} \subset ]0, \infty[$ with accumulation point 0 and a sequence $(\varepsilon_h)_{h \in \mathbb{H}}$ such that $\varepsilon_h \to 0$ and

$$\|h \cdot \partial_x u_h\|^2 \leqslant \varepsilon_h \|u_h\|^2 \tag{14}$$

where $\|\cdot\|$ is the Euclidean $L^2$-norm on $\Omega$. Our goal is to contradict (14), the *integrability condition.*

Suppose $h \leqslant h_0$. Then $E_h < b$ and so since $\||\nabla u_h|\|^2 = E_h \|u_h\|^2$ we find that $\|\partial_y u_h\|^2 \leqslant b\|u_h\|^2$. Therefore, it follows from Lemma 5.3 that if $\delta > 0$ is sufficiently small and $h \leqslant h_0$, then

$$\|u_h \cdot \mathbb{1}_{J_\delta}\|^2 \leqslant \frac{1}{4}\|u_h\|^2.$$

We fix $\delta > 0$ so that this estimate holds and so that the conclusion of Proposition 4.2 holds. Choose a cut-off function $\chi$ with support in $I_{\delta/2}$ and that is identically 1 on $I_\delta$ and set

$$w_h := \chi \cdot u_h \quad \text{and} \quad r_h := u_h - w_h.$$

Since $|r_h| \leqslant |u_h| \cdot \mathbb{1}_{J_\delta}$, the preceding estimate gives $\|r_h\| \leqslant \frac{1}{2}\|u_h\|$ and hence $\|w_h\| \geqslant \frac{1}{2}\|u_h\|$.

Let $Q_h$ be the self-adjoint operator associated with $q_h$. We compute

$$Q_h(\chi \cdot u_h) = E_h \chi u_h + [Q_h, \chi]u_h \quad \text{and} \quad [Q_h, \chi]u_h = -h^2(2\chi' \partial_x u_h + \chi'' u_h).$$

Hence, we obtain

$$\|(Q_h - E)w_h\| \leqslant Ch\|u_h\|.$$

Thus, if $\phi$ belongs to the form domain of $q_h$, then

$$|q_h(w_h, \phi) - E\langle w_h, \phi_h\rangle| \leqslant Ch\|u_h\|\|\phi\|.$$

Using Lemma 5.4, we then obtain

$$|a_h(w_h, \phi) - E_h\langle w_h, \phi\rangle| \leqslant Ch(a_h(w_h)^{1/2} + \|u_h\|)a_h(\phi)^{1/2}.$$

The quantity $a_h(w_h)$ is majorised by a multiple of $q_h(u_h) = E_h\|u_h\|^2$, and so

$$|a_h(w_h, \phi) - E_h\langle w_h, \phi\rangle| \leqslant Ch\|u_h\|a_h(\phi)^{1/2}. \tag{15}$$

We have $a_h(w_h, \phi) = \langle \mathcal{A}_h w_h, \phi\rangle$ where we regard $\mathcal{A}_h w_h$ as an element of $H^{-1}_{\mathcal{A}_h}$, and moreover, $\|\phi\|^2_{H^1_{\mathcal{A}_h}} = a_h(\phi) + \|\phi\|^2$. Therefore, from (15) we find that if $h \leqslant h_0$, then

$$\|(\mathcal{A}_h - E)w_h\|_{H^{-1}_{\mathcal{A}_h}} \leqslant Ch\|u_h\|. \tag{16}$$

Because of (13) and because each Fourier coefficient of $w_h$ lies in $\mathcal{C}_0^\infty(I_\delta)$, we may apply Proposition 4.2 to estimate $\|w_h\|$. By combining the resulting estimate with (16) we find that for $h \in \mathbb{H}$ with $h \leqslant h_0$

$$\|w_h\| \leqslant \varepsilon\|u_h\| + C\|h \cdot D' w_h\|,$$

where $D'$ is the operator defined in Proposition 4.2.

Using (12), we find that

$$D' w_h = \chi' u_h + \chi \partial_x u_h - \chi \frac{L'}{L} \partial_y u_h.$$

Thus, because $\chi$ has support in $I_\delta$, we get the estimate

$$\|h \cdot D' w_h\| \leqslant C(\|h \cdot \partial_x u_h\| + h\|u_h\|).$$

We finally obtain

$$\frac{1}{2}\|u_h\| \leqslant \|w_h\| \leqslant (C\varepsilon\|u_h\| + C\|h\cdot\partial_x u\| + Ch\|u_h\|).$$

We now can choose $\varepsilon$ and $h_0$ small enough to absorb the first and last terms of the RHS into the LHS, yielding

$$\frac{1}{4}\|u_h\| \leqslant C\|h\cdot\partial_x u_h\|.$$

This contradicts the integrability condition (14). ∎

**Theorem 5.5** (Mixed conditions). *Let $E_h$ be an analytic eigenvalue branch of $q_h$ restricted $H^1_{0c}(\Omega)$ (or restricted to $H^1_{0s}(\Omega)$) relative to $\|\cdot\|^2$. Then there exists $k\in\mathbf{N}^*$ such that*

$$\lim_{h\to 0} E_h = \left(\frac{(k-\frac{1}{2})\pi}{L(0)}\right)^2.$$

*Proof.* The proofs are nearly identical to the proof of Theorem 5.2. In the case where $q_h$ is restricted to $H^1_{0c}(\Omega)$, we use the Fourier decomposition

$$u(x,y) = \sum_{k\geqslant 1} u_k(x)\cdot\sqrt{2}\cos\left(\frac{(k-\frac{1}{2})\pi y}{L(x)}\right).$$

In the case where $q_h$ is restricted to $H^1_{0s}$, the identification of $L^2(\Omega)$ with $\ell^2(\mathbf{N}^*,\mathscr{H})$ is given by

$$u(x,y) = \sum_{k\geqslant 1} u_k(x)\cdot\sqrt{2}\sin\left(\frac{(k-\frac{1}{2})\pi y}{L(x)}\right).$$

In each case, an analogue of the Poincaré estimate—Lemma 5.3—still holds as does an analogue of Lemma 5.4. Using the same method used to prove Theorem 5.2, we obtain the theorem. ∎

## 6. Eigenvalue limits for symmetric domains with Dirichlet conditions

In this section, we consider domains $\Omega\subset\mathbf{R}^2$ of the form

$$\Omega = \{(x,y)\in\mathbf{R}^2, |x|<1, -L(x)<y<L(x)\},$$

where $L$ satisfies conditions L1 and L2 of Section 3. We consider the family of quadratic forms $q_h$ defined as in (8) and restricted to $H^1_0(\Omega)$.

Note that $\Omega$ is invariant under the reflection symmetry $(x, y) \to (x, -y)$. This symmetry defines an orthogonal decomposition,

$$L^2(\Omega, dx\, dy) = L^2_{\text{odd}}(\Omega) \oplus L^2\text{even}(\Omega),$$

into the sum of the space of 'odd' functions and the space of 'even' functions. If $u, v \in H^1_0(\Omega)$ and $u$ and $v$ have opposite parity, then $q_h(u, v) = 0$. It follows that each eigenspace $V$ of $q_h$ on $H^1_0(\Omega)$ with respect to $\|\cdot\|^2_{L^2}$ equals $(V \cap L^2_{\text{odd}}(\Omega)) \oplus (V \cap L^2\text{even}(\Omega))$. In particular, the spectral problem for $q_h$ on $H^1_0(\Omega)$ reduces to the study of $q_h$ on $H^1_0(\Omega) \cap L^2\text{odd}(\Omega)$ and the study of $q_h$ on $H^1_0(\Omega) \cap L^2\text{even}(\Omega)$.

The Kato–Rellich theory [12] applies separately to the family $q_h$ restricted to 'odd' functions and the family $q_h$ restricted to 'even' functions. The theory provides analytic paths $h \mapsto u_{h,\ell} \in L^2\text{odd}(\Omega)$ (resp. $L^2\text{even}(\Omega)$) and $h \mapsto E_{h,\ell}$ so that for each $h$ the set $\{u_{h,\ell} : \ell \in \mathbf{N}^*\}$ is a Hilbertian basis for $L^2\text{odd}(\Omega)$ (resp. $L^2\text{even}(\Omega)$) and $q_h(u_{h,\ell}, v) = E_{h,\ell}\langle u_{h,\ell}, v\rangle$ for $v \in H^1_0(\Omega)$.

**Theorem 6.1.** *If $E_h$ is an 'odd' analytic eigenvalue branch of $q_h$ restricted to the intersection $H^1_0(\Omega) \cap L^2\text{odd}(\Omega)$, then*

$$\lim_{h\to 0} E_h = \left(\frac{k\pi}{L(0)}\right)^2.$$

*If $E_h$ is an 'even' analytic eigenvalue branch of $q_h$ restricted to the intersection $H^1_0(\Omega) \cap L^2\text{even}(\Omega)$, then*

$$\lim_{h\to 0} E_h = \left(\frac{(k - \frac{1}{2})\pi}{L(0)}\right)^2.$$

*Proof.* This follows from the results of Section 5. Indeed, let $\Omega'$ denote the intersection of $\Omega$ with the upper half plane $\{y > 0\}$. Restriction $u \mapsto u|_{\Omega'}$ defines an isomorphism from the space of 'odd' $H^1_0(\Omega)$ functions onto the space $H^1_0(\Omega')$. This restriction also defines an isomorphism from the space of 'even' $H^1_0(\Omega)$ functions onto the space $H^1_{0c}(\Omega')$ of functions whose trace along the graph of $L$ vanishes identically. Moreover, the ratio $q_h(u)/\|u\|^2$ is unchanged by restriction. The claim then follows from Theorems 5.2 and 5.5. ∎

We next combine 6.1 and Bourget's hypothesis (Appendix A) to prove the generic simplicity of ellipses. First note that, up to isometry, each ellipse has the form

$$\mathcal{E}_{a,b} := \left\{(x, y) : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leqslant 1\right\}.$$

Thus, we may naturally identify the set of isometry classes of ellipses with the set of $\{(a, b) : 0 < b \leqslant a < \infty\}$.

**Theorem 6.2** (Theorem 1.1). *There exists a countable subset $\mathcal{C} \subset ]0, 1]$ so that if $\frac{b}{a} \notin \mathcal{C}$, then each eigenvalue of the Dirichlet Laplace operator of the ellipse $\mathcal{E}_{a,b}$ is simple.*

*Proof.* The multiplicities of the Laplace spectrum of a domain are unchanged by a homothety of the domain. In particular, the spectrum of $\mathcal{E}_{a,b}$ is simple if and only if the spectrum of $\mathcal{E}_{a/b,1}$ is simple. Let $\Omega$ be the unit disk. If we let $\Omega_h$ denote the image of $\Omega$ under $(x, y) \mapsto \left(\frac{x}{h}, y\right)$, then $\mathcal{E}_{1/h,1} = \Omega_h$. As discussed in Remark 5.1, the study of the Dirichlet Laplace spectrum of $\Omega_h$ is equivalent to the study of the quadratic form $q_h$ of (8) restricted to $H_0^1(\Omega)$ relative to $\|\cdot\|^2$. Thus, to prove the claim, it suffices to show that there exists a countable set $\mathcal{C} \subset ]0, 1]$ such that $q_h$ has simple spectrum.

To 'construct' the set $\mathcal{C}$, it suffices to show that each of the real-analytic eigenvalue branches of $q_h$ is distinct. Indeed, let $\{h \mapsto E_{h,\ell} : \ell \in \mathbf{N}^*\}$ denote the collection of analytic eigenvalue branches associated to the family $q_h$. If for some $h$ we have $E_{h,\ell} \neq E_{h,\ell'}$ then analyticity implies that the set $\mathcal{C}_{\ell,\ell'} := \{h : E_{h,\ell} = E_{h,\ell'}\}$ is countable. Thus, $\mathcal{C} = \bigcup_{\ell\neq\ell'} \mathcal{C}_{\ell,\ell'}$ is the desired set.

To prove that the various analytic eigenvalue branches are distinct, we argue by contradiction. Suppose to the contrary, that $\ell \neq \ell'$ but $E_{h,\ell} = E_{h,\ell'}$ for each $h \in ]0, 1]$. Let $u_{h,\ell}$ and $u_{h,\ell'}$ denote the corresponding analytic eigenfunction branches. For each $h$, we have $u_{h,\ell} \perp u_{h,\ell'}$ and in particular $u_{1,\ell} \perp u_{1,\ell'}$. It follows from the discussion before Theorem 6.1 that either $u_{h,\ell}$ (resp. $u_{h,\ell'}$) is 'odd' for each $h$ or $u_{h,\ell}$ (resp. $u_{h,\ell'}$) is 'even' for each $h$.

It is well known that each eigenspace $V$ of $q_1$ is at most two-dimensional, and moreover, if $\dim(V) = 2$, then $\dim(V \cap L^2\text{odd}(\Omega)) = 1 = \dim(V \cap L^2\text{even}(\Omega))$ (see Corollary A.2 in Appendix A). Therefore, $u_{1,\ell}$ and $u_{1,\ell'}$ span an eigenspace of $q_1$ and they have opposite parity. Hence, they have opposite parity for each $h$. Theorem 6.1 then implies $\lim_{h\to 0} E_{h,\ell} \neq \lim_{h\to 0} E_{h,\ell'}$, a contradiction. ∎

We now remark that this proof also yields a way to construct ellipses that have some multiplicity in the spectrum. We obtain the following proposition.

**Proposition 6.3.** *There exists a sequence $(h_n)_{n\geq 0}$ going to zero such that, for any $n$, if $\frac{b}{a} = h_n$ then the ellipse $\mathcal{E}_{a,b}$ has at least one multiple eigenvalue.*

*Proof.* Fix some $h_0$, it suffices to show that there exists $h < h_0$ so that the spectrum of $\mathcal{E}_{1/h,1}$ has (at least) one eigenvalue of multiplicity at least 2. Let $h \mapsto E_{h,\text{odd}}$ be the smallest odd eigenvalue of $\mathcal{E}_{h,1}$. This map is continuous, piecewise analytic, non-decreasing and it tends to $\pi^2$. It is fairly standard that for any $N$ the $N$-th eigenvalue of $\mathcal{E}_h$ converges to $\frac{\pi^2}{4}$. It follows from min-max theory, asymptotic separation of variables and semiclassical analysis near the bottom of a well (see e.g., [7] for a related problem). Let now $N$ be such that the $N$-th eigenvalue of $\mathcal{E}_{h_0,1}$ is greater than $E_{h_0,\mathrm{odd}}$. By continuity, there must be some $h < h_0$ such that the $N$-th eigenvalue of $\mathcal{E}_{1/h,1}$ is equal to $E_{h,\mathrm{odd}}$. ∎

## 7. Ellipsoids

In this section, we prove that the generic ellipsoid has simple spectrum. As for ellipses, this result relies on knowing precisely how the multiplicities in the spectrum of the unit ball are distributed and on the limiting behaviour of analytic eigenbranches upon the stretching/compressing degeneration.

We denote by $\mathbb{E}_{a,b,c}$ the ellipsoid:

$$\mathbb{E}_{a,b,c} = \left\{(x,y,z) \in \mathbf{R}^3 : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 < 1\right\}.$$

We first consider the ellipsoids with 'circular cross-sections':

$$\mathbb{E}_{1/h,1,1} = \{(x,y,z) \in \mathbf{R}^3 : (hx)^2 + y^2 + z^2 < 1\}.$$

**Proposition 7.1.** *There exists a countable subset $\mathcal{C} \in ]0,1]$ such that if $h \subset \mathcal{C}$, then each eigenspace of the Dirichlet Laplacian on $\mathbb{E}_{1/h,1,1}$ is at most 2-dimensional. Moreover, if $V$ is a 2-dimensional eigenspace of $\mathbb{E}_{1/h,1,1}$, then there exist $u_\pm \in V$ so that $V = \mathrm{span}\{u_+, u_-\}$ and $u_\pm(x,y,-z) = \pm u_\pm(x,y,z)$.*

The proof of Proposition 7.1 will use the same ideas as the proof of Theorem 6.2 in a slightly different setting. First, by a change of variables, we are led to consider the following quadratic form

$$q_h(u) := \int_B (h^2 \cdot |\partial_x u|^2 + |\partial_y u|^2 + |\partial_z u|^2)\, dx\, dy \tag{17}$$

defined on $H^1(B)$ where $B$ is the unit ball in $\mathbf{R}^3$.

Instead of using a reflection symmetry to decompose the spectrum, we will use the natural rotational symmetry. To be concrete, it will prove convenient to use cylindrical coordinates $(x, r, \theta)$ where $(y,z) = (x, r\cos(\theta), r\sin(\theta))$. Then the rotation about the $x$-axis through angle $\alpha$ is given by

$$R_\alpha(x,r,\theta) = (x, r, \theta + \alpha).$$

The map $u \mapsto u \circ R_\alpha$ is an isometry of both $L^2(B)$ and $H^1_0(B)$. Note that $q_h(u \circ R_\alpha) = q_h(u)$ for each $u \in H^1(B)$. For $m \in \mathbf{Z}$, define

$$L^2_m(B) = \{u \in L^2(B) : u \circ R_\alpha = e^{im\alpha} \cdot u\}$$

and

$$H^1_{0,m}(B) = \{u \in H^1_0(B) : u \circ R_\alpha = e^{im\alpha} \cdot u\}. \tag{18}$$

Then $L^2_0(B) = \bigoplus_m L^2_m(B)$ and $H^1_0(B) = \bigoplus_m H^1_{0,m}(B)$.

We fix $m \in \mathbf{Z}$, and let $q_{h,m}$ denote the restriction $q_h$ to $H^1_{0,m}(B)$. Let $L(x) = \sqrt{1-x^2}$, and let $D_x$ denote the disk $\{(r,\theta) : r < L(x)\}$. The rotation $R_\alpha$ preserves $D_x$, and we define the spaces $L^2_m(D_x)$ and $H^1_{0,m}(D_x)$ as above. Let $\{\phi^m_\lambda\}$ be a Hilbertian basis of $H^1_{0,m}(D_0)$ consisting of eigenfunctions of the Dirichlet Laplacian on the unit disk $D_0$.[6] Let $\mathrm{spec}_m(D_0)$ denote the set of eigenvalues associated with $\{\phi^m_\lambda\}$.

As with the case of ellipses, the proof of Proposition 7.1 is based on a determination of the limits of analytic eigenbranches.

**Proposition 7.2.** *For any analytic eigenvalue branch $E_h$ of $q_{h,m}$ on $H^1_{0,m}(\Omega)$ relative to $\|\cdot\|^2$, there exists $\lambda \in \mathrm{spec}_m(D_0)$ such that*

$$\lim_{h\to 0} E_h = \frac{\lambda}{L(0)^2}.$$

*Proof.* The proof follows the same outline as the proof of Theorem 5.2. We provide a sketch and leave the details to the reader. For $u \in H^1_0(B)$ with compact support we have

$$u(x,r,\theta) = \sum_{\lambda\in \mathrm{spec}_m(D_0)} u_\lambda(x)\cdot \phi^m_\lambda\Big(\frac{r}{L(x)},\theta\Big). \tag{19}$$

Given $\delta \in ]0,1[$, let $B_\delta = \{(x,y,z) \in B : -1+\delta < x < 1-\delta\}$. By using (19) in place of the Fourier sine decomposition, one proves a Poincaré estimate on $B \setminus B_\delta$ analogous to Lemma 5.3 with $\partial_y$ replaced by $\nabla = \partial_r u \partial_r + r^{-2}\partial_\theta u \partial_\theta$.

For $u \in H^1_{0,m}(B)$ define

$$(D'u)(x,r,\theta) = \sum_{\lambda\in \mathrm{spec}_m(D_0)} u'_\lambda(x)\cdot \phi^m_\lambda\Big(\frac{r}{L(x)},\theta\Big)$$

and

$$a_{h,m}(u) = \|h\cdot D'u\|^2 + \|\nabla u\|^2,$$

where $\nabla u = \partial_r u \partial_r + r^{-2}\partial_\theta u \partial_\theta$. By using the method of proof of Lemma 5.4, one finds that $q_{h,m}$ and $a_{h,m}$ are asymptotic at first order in the sense of (11) where $\Omega_\delta$ is replaced by $B_\delta$.

Following Section 4, one now defines an operator $\mathcal{A}_{h,m}$ associated with the quadratic form $a_{h,m}$ by identifying $e_k$ with $\phi^m_{\lambda_k}$ where $\lambda_k$ is the $k$-th element of $\mathrm{spec}_m(D_0)$.

[6]Each $\phi^m_\lambda$ is a multiple of $J_m(\sqrt{\lambda} r)e^{im\theta}$ where $J_m$ is the Bessel function of order $m$.

The Hilbert space $\mathcal{H}$ is now $\mathcal{H} = L^2(]-1,1[, L^2(x)dx)$ and the operators $A_{h,m,k}$ are defined as

$$A_{h,m,k}v = -\frac{1}{L^2}(L^2 v')' + \frac{\lambda_k}{L^2}v.$$

One proves the analogue of Proposition 4.2 with $(k_0\pi)^2$ and $((k_0+1)\pi)^2$ replaced by successive elements $\lambda_{k_0} < \lambda_{k_0+1}$ in $\mathrm{spec}_m(D_0)$.

Given the analogues of Proposition 4.2, Lemma 5.3, and Lemma 5.4, the proof of 7.2 follows the same lines as does the proof of Theorem 5.2. ■

*Proof of Proposition* 7.1. Given Proposition 7.2 and well-known facts about the Dirichlet spectrum of the unit ball, much of the proof follows the same outline as the proof of Theorem 6.2. As described there, the 'construction' of the set $\mathcal{C}$ will follow from proving that the various analytic eigenvalue branches of $q_h$ are all distinct.

In the present case, Kato–Rellich theory implies that there exist analytic branches $h \mapsto E^m_{h,\ell} \in \mathbf{R}$ and $h \mapsto u^m_{h,\ell} \in H^1_{0,m}$ so that, for each $h \in ]0,1]$, the set $\{u^m_{h,\ell} : \ell \in \mathbf{N}^*\}$ is a Hilbertian basis for $L^2_m(B)$ and so that for each $\ell \in \mathbf{N}^*$ the function $u^m_{h,\ell}$ is an eigenfunction of $q_{h,m}$ relative to $\|\cdot\|^2$ with eigenvalue $E^m_{h,\ell}$. Bourget's hypothesis implies that if $|m| \neq |m'|$ then $\mathrm{spec}_m(D_0) \cap \mathrm{spec}_{m'}(D_0) = \emptyset$ (see Appendix A). Thus, Lemma 7.2 implies that if $|m| \neq |m'|$, then for every $\ell, \ell' \in \mathbf{N}^*$, the branch $E^m_{h,\ell}$ is distinct from the branch $E^{m'}_{h,\ell'}$.

Suppose for contradiction that $q_h$ has two distinct analytic eigenfunction branches $u_h$ and $u'_h$ whose corresponding eigenvalue branches coincide for all $h$. Then the preceding paragraph implies that there exist $m, m', \ell$, and $\ell'$ so that $|m| = |m'|$, $u_h = u^m_{h,\ell}$ and $u'_h = u^{m'}_{h,\ell'}$.

The functions $u_1$ and $u'_1$ lie in the same eigenspace $V_1$ of the Dirichlet Laplace operator on the unit ball. The intersection of $H^1_{0,m}(B)$ with any such eigenspace is one-dimensional (see Appendix A). Hence, we cannot have $m' = m$. Therefore, $m' = -m$, the function $u_1$ spans $V_1 \cap H^1_{0,m}(B)$, and $u'_1$ spans $V_1 \cap H^1_{0,-m}(B)$.

From (18), one sees that complex conjugation defines a unitary isomorphism from $H^1_{0,m}(B)$ onto $H^1_{0,-m}(B)$. Because $q_h(\bar{v}) = q_h(v)$, if $(E_h, u_h)$ is an analytic eigenbranch for $q_{h,m}$, then $(E_h, \bar{u}_h)$ is an analytic eigenbranch for $q_{h,-m}$. In particular, we may assume that $(E^{-m}_{h,\ell}, u^{-m}_{h,\ell}) = (E^m_{h,\ell}, \bar{u}^m_{h,\ell})$ for each $\ell$, $h$, and $m$.

Since $u'_1$ spans $V_1 \cap H^1_{0,-m}(B)$, it follows that $u'_1 = c\bar{u}_1$ for some constant $c$. Hence, without loss of generality, $u'_h = \bar{u}_h$ for each $h$. It follows that $\ell = \ell'$ and one can find real eigenfunctions $u_+$ and $u_- \in V_1 \cap (H^1_{0,m}(B) \oplus H^1_{0,-m}(B))$ that satisfy $u_\pm(x,y,-z) = \pm u_\pm(x,y,z)$. ■

**Theorem 7.3.** *There exists a set $\mathcal{C} \subset ]0,1] \times ]0,1]$ of Lebesgue measure zero so that if $(h,h') \notin \mathcal{C}$, then each eigenspace of the Dirichlet Laplacian on $\mathbb{E}_{1/h,1/h',1}$ is one-dimensional.*

*Proof.* Let $\mathcal{E}_h$ denote the ellipse $\{(y,z) : (hy)^2 + z^2 < 1\}$. By Theorem 6.2 and Proposition 7.1, there exists $h_0 \in ]0,1]$ so that the Dirichlet spectrum of $\mathcal{E}_{h_0}$ is simple and so that each Dirichlet eigenspace $V$ of the ellipsoid $\mathbb{E}_{1,1/h_0,1}$ is at most 2-dimensional (exchanging the $x$ and $y$ coordinates is harmless), and moreover, if $\dim(V) = 2$, then $V = \mathrm{span}\{u_+, u_-\}$ where $u_\pm(x,y,-z) = \pm u(x,y,z)$. We fix $h_0$ and consider the family of domains $\Omega_h := \mathbb{E}_{1/h,1/h_0,1}$. It suffices to show that there exists some $h \in ]0,1]$ so that $\Omega_h$ has simple Dirichlet Laplace spectrum. Indeed, analyticity of the spectrum would then imply that for each line $\ell$ in $]0,1] \times ]0,1]$ that passes through $(h, h_0)$, there exists a countable subset $\mathcal{C}_\ell \subset \ell$ such that if $(g, g') \in \ell \setminus \mathcal{C}_\ell$, then $\mathbb{E}_{1/g,1/g',1}$ has simple spectrum. Then $\mathcal{C} = \cup_\ell \mathcal{C}_\ell$ would be the desired set of measure zero.

Let $\Omega := \Omega_1 = \mathbb{E}_{1,1/h_0,1}$ and consider the quadratic form $q_h$ of (17) on $H^1_0(\Omega)$. Let $q_{h,\pm}$ denote the restriction of $q_h$ to the space

$$H^1_{0,\pm}(\Omega) = \{u \in H^1_0(\Omega) : u(x,y,-z) = \pm u(x,y,z)\}.$$

Let $\{\phi^\pm_\lambda\}$ be a Hilbertian basis of Dirichlet eigenfunctions of the ellipse $\mathcal{E}_{h_0}$ such that $\phi^\pm_\lambda(y,-z) = \pm\phi^\pm_\lambda(y,z)$. Let $\mathrm{spec}_\pm(\mathcal{E}_{h_0})$ denote the set of (simple) eigenvalues associated to the eigenfunctions $\phi^\pm_\lambda$. The now familiar method that was used to prove Theorem 5.2 and Proposition 7.2 may be used to prove that each real-analytic eigenvalue branch $E_h$ of $q_{h,\pm}$ limits to an element of $\mathrm{spec}_\pm(\mathcal{E}_{h_0})$.

The idea used to prove Theorem 6.2 can now be used to prove that there exists a countable set $\mathcal{C}' \subset ]0,1]$ so that if $h \notin \mathcal{C}'$, then the spectrum of $q_h$ (and hence $\Omega_h$) is simple. Indeed, each analytic eigenvalue branch of $q_h$ on $H^1_0(\Omega)$ is either an eigenvalue branch of $q_{h,-}$ or an eigenvalue branch of $q_{h,+}$. Because $\mathrm{spec}(\mathcal{E}_{h_0})$ is simple, we have $\mathrm{spec}_-(\mathcal{E}_{h_0}) \cap \mathrm{spec}_+(\mathcal{E}_{h_0}) = \emptyset$. Therefore, since the $q_{h,\pm}$ branches limit to an element of $\mathrm{spec}_\pm(\mathcal{E}_{h_0})$, the $q_{h,-}$ branches are distinct from the $q_{h,+}$ branches. In particular, if two distinct eigenfunction branches $u_h$ and $u_{h'}$ of $q_h$ were to have coincident eigenvalue branches, then either each branch is a branch of $q_{h,-}$ or each is a branch of $q_{h,+}$. Thus, $u_1$ and $u_1'$ would either be two independent 'odd' Dirichlet eigenfunctions of $\mathbb{E}_{1,1/h_0,1}$ or would be two independent 'even' Dirichlet eigenfunctions of $\mathbb{E}_{1,1/h_0,1}$. This would contradict the choice of $h_0$ made above.

Therefore, no two analytic eigenvalue branches of $q_h$ coincide, and so analyticity provides the existence of the desired set $\mathcal{C}'$. The spectrum of $q_h$ on $\Omega$ is the same as the spectrum of the Dirichlet Laplacian on $\mathbb{E}_{1/h,1/h_0,1}$, and thus the latter is simple for some $h \in ]0,1]$. ∎

**Remark 7.4.** We observe that the strategy of the present paper could apply to study the quadratic form $q_h$ on a domain $\Omega \subset \mathbf{R}^N$ that satisfies the following property for a profile function $L$:

$$(x_1, \dots, x_N) \in \Omega \iff \left(0, \frac{x_2}{L(x)}, \dots, \frac{x_N}{L(x)}\right) \in \Omega.$$

## A. The spectrum of the unit disk and the unit ball

The Dirichlet and Neumann spectra of the unit disk $D$ may be computed using separation of variables and Bessel functions. A Hilbertian basis of eigenfunctions of $q_1$ restricted to $H_0^1(D) \cap L^2\mathrm{odd}(D)$ is given by $J_k(\sqrt{E}r)\sin(k\theta)$ where $J_k$ is the standard Bessel and $\sqrt{E}$ is a zero of $J_k$. A Hilbertian basis of eigenfunctions of $q_1$ on $H^1(\Omega)(D) \cap L^2\mathrm{even}(D)$ is given by $J_k(\sqrt{E}r)\cos(k\theta)$. It follows that the part of the spectrum of $q_1^N$ and $q_1^D$ that correspond to positive $k$ coincide, so that the multiplicity of an eigenfunction that has a non-radial corresponding eigenfunction is at least 2.

The following theorem is a well-known corollary of Siegel's work [14].

**Theorem A.1** (Bourget's hypothesis). *If $k$ and $k'$ are distinct integers, then the Bessel functions $J_k$ and $J_{k'}$ have no common zeros other than* 0.

*Proof.* See [16, Section 15.28].[7] If $k' > k$, then a recurrence relation implies that $J_{k'}(z) = p\left(\frac{1}{z}\right)J_k(z) + q\left(\frac{1}{z}\right)J_{k-1}(z)$ where $p, q$ are polynomials[8] with rational coefficients. It is known that $J_k$ and $J_{k-1}$ have no common zeros other than zero [16]. Thus, if $J_k$ and $J_{k'}$ were to have a common zero $z_0 \neq 0$, then $q\left(\frac{1}{z_0}\right) = 0$. Hence, $z_0$ is an algebraic number and Siegel's theorem [14] would imply that $J_k(z_0)$ is not algebraic. But $J_k(z_0) = 0$, a contradiction. ∎

Bourget's hypothesis has the following corollary. Recall from Section 6, the notion of 'even' and 'odd' function with respect to the symmetry $(x, y) \mapsto (x, -y)$.

**Corollary A.2.** *The dimension of each eigenspace $V$ of $q_1$ on $H^1(D)$ is at most two. If $\dim(V) = 1$, then each $u \in V$ is a radial function (hence 'even'). If $\dim(V) = 2$, then $V$ is spanned by an 'even' function and an 'odd' function.*

*Proof.* For each $k$, let $Z_k$ be the set of $E > 0$ so that $J_k(\sqrt{E}) = 0$. Theorem A.1 implies that the sets $Z_k$ are disjoint. By the discussion before Theorem A.1, the spectrum of $q_1$ equals the union $\bigcup_{k \geqslant 1} Z_k$. Moreover, if $k = 0$, then for each $E \in Z_k$, the space is spanned by $J_0(\sqrt{E} \cdot r)$ which is radial. If $k > 0$ and $E \in Z_k$, the space is spanned by $J_k(\sqrt{E} \cdot r)\sin(\theta)$ which is 'odd' and $J_k(\sqrt{E} \cdot r)\cos(\theta)$ which is 'even'. ∎

[7] Bourget's hypothesis was presented as a conjecture in the first edition of [16], and later was presented as a theorem in the second edition.

[8] The polynomials $p$ and $q$ are called *Lommel polynomials.*

To describe the eigenspaces of the Dirichlet Laplace operator on the unit ball $B \subset \mathbb{R}^3$, we use the harmonic polynomials $P_\ell$ of degree $\ell$ on $\mathbb{R}^3$. To describe a useful basis, for $P_\ell$ we use the differential operators associated to rotating about the $x$, $y$, and $z$-axes:

$$\begin{aligned} R_x &= -z\partial_y + y\partial_z, \\ R_y &= -x\partial_z + z\partial_x, \\ R_z &= -y\partial_x + x\partial_y. \end{aligned}$$

The 'ladder operators' $L := R_y + iR_z$ and $\overline{L} := R_y - iR_z$ preserve $P_\ell$. A computation shows that the polynomial $(y - iz)^\ell$ is harmonic. If we define

$$Y_{\ell,k} := L^k (y - iz)^\ell,$$

then for each $k$, we have

$$R_x Y_{\ell,k} = -i(\ell - k) \cdot Y_{\ell,k}. \tag{20}$$

Moreover, the set

$$\{Y_{\ell,k} : k = 0, 1, \ldots, 2\ell\}$$

is a basis for $P_\ell$.

In terms of spherical coordinates $(r, \omega)$ for $B$, a Hilbertian basis of Dirichlet eigenfunctions is given by

$$\phi_{\lambda,\ell,k}(r,\omega) := r^{-1/2} J_{\ell+\frac{1}{2}}(\sqrt{\lambda} r) \cdot Y_{\ell,k}(\omega)$$

where $\ell \in \mathbf{N}$, $k = 0, \ldots, 2\ell$, and $J_{\ell+\frac{1}{2}}(\sqrt{\lambda}) = 0$. Bourget's hypothesis also holds for Bessel functions of half-integer order, and so the non-zero zeros of $J_{\ell+\frac{1}{2}}$ are distinct from the non-zero zeros of $J_{\ell'+\frac{1}{2}}$ if $\ell \neq \ell'$. The eigenspace $V$ associated to the eigenvalue $\lambda$ where $J_{\ell+\frac{1}{2}}(\sqrt{\lambda}) = 0$ has basis $\{\phi_{\lambda,\ell,k} : k = 0, \ldots, 2\ell\}$.

Recall the subspace $H^1_{0,m}(B)$ defined in (18).

**Lemma A.3.** *Let $V$ be an eigenspace of the Dirichlet Laplace operator on the unit ball associated to a zero of $J_{\ell+\frac{1}{2}}$. For $|m| \leqslant \ell$, the dimension of $V \cap H^1_{0,m}(B)$ equals one.*

*Proof.* It follows from (20) that the function $Y_{\ell,\ell+m}$ belongs to $H^1_{0,m}(B)$. ∎

## B. The Neumann case

The purpose of this appendix is to prove the analogues of Theorems 5.2 and 5.5 in the full Neumann case. Namely, we prove the following theorem, in which we use the notation of Section 5.

**Theorem B.1** (Full Neumann). *For any analytic eigenvalue branch $E_h$ of $q_h$ on $H^1(\Omega)$ relative to $\|\cdot\|^2$, there exists $k \in \mathbf{N} = \mathbf{N}^* \cup \{0\}$ such that*

$$\lim_{h\to 0} E_h = \Big(\frac{k\pi}{L(0)}\Big)^2.$$

As is obvious from the statement, the main difference between the full Neumann case and the boundary conditions considered in Section 5 is the presence of a zeroth mode corresponding to $k = 0$. Of course, the presence of this mode follows from the fact that constant functions are in the kernel of the Neumann Laplace operator in $\Omega$. Given $v \in L^2(\Omega)$, we perform the transversal Fourier decomposition:

$$v = v_0 \otimes \mathbb{1} + \sum_{k\geqslant 1} v_k \otimes e_k,$$

where

$$e_k(x, y) = \sqrt{2}\cos\frac{k\pi y}{L(x)},$$

$$v_k(x) = \frac{1}{L(x)}\int_0^{L(x)} v(x, y)e_k(x, y)\, dy,$$

$$v_0(x) = \frac{1}{L(x)}\int_0^{L(x)} v(x, y)\, dy.$$

To prove Theorem B.1, we apply the same overall strategy as in the proof of Theorem 5.2. That is, we suppose for contradiction that a (normalised) analytic eigenbranch $(E_h, u_h)$ does not limit to one of the thresholds $(\pi k)^2/L(0)^2$, and we seek to contradict the resulting integrability condition (14). As we will see, the arguments that we used in the central region $\Omega_\delta$ still work so that the integrability condition gives that, for any $\delta$ and any 'horizontal' cutoff function $\chi_\delta$, the quantity $\|\chi_\delta u_h\|$ is eventually, for $h$ small, arbitrarily small. This will imply that $u_h$ concentrates away from $\Omega_\delta$. The Poincaré inequality is also still valid for $k \geqslant 1$ so that, for any $\delta$, the quantity $\|(1-\chi_\delta)(u_h - u_{h,0})\|$ also tends to 0. As a consequence, it will follow that $u_h$ eventually has all of its mass supported on its zeroth mode near the points $x = \pm 1$. We will obtain a contradiction by making a careful study of the second order inhomogeneous ODE satisfied by $u_{h,0}$.

### B.1. In the central region $\Omega_\delta$

Recall the region $\Omega_\delta$ defined in Section 5. In this section, we fix $\delta \in \left(0, \frac{1}{2}\right)$ and define $H^1_v(\Omega_\delta)$ to be the subspace of functions in $H^1$ that vanish on the vertical sides $x = 1-\delta$ and $x = -1+\delta$.

For each $v \in H^1_v(\Omega_\delta)$, we define

$$D'(v) = v_0' \otimes \mathbb{1} + \sum_{k \geqslant 1} v_k' \otimes e_k,$$

and the quadratic form

$$a_h(v) = \sum_{k \geqslant 0} \int_{-1+\delta}^{1-\delta} [h^2 |v_k'(x)|^2 + \frac{k^2\pi^2}{L^2(x)} |u(x)|^2] L(x)\, dx.$$

For $h$ small enough, using that $L'$ and $\frac{1}{L}$ are uniformly bounded on $\Omega_\delta$, we obtain the following lemma that implies that $q_h$ and $a_h$ are asymptotic at first order on $H^1_v(\Omega_\delta)$

**Lemma B.2.** *For any $\delta$, there exists $C > 0$ and $h_0$ such that, for any $h < h_0$ and any $v, w \in H^1_v(\Omega_\delta)$*

$$|q_h(v,w) - a_h(v,w)| \leqslant C \cdot h \cdot a_h(v)^{1/2} a_h(w)^{1/2}.$$

The proof is the same as the proof of Lemma 5.4. Details are left to the reader.

Following the proof in the Dirichlet case, we introduce the abstract Hilbert space $\ell^2(\mathbf{N}, \mathcal{H}_\delta)$ by formally identifying $e_k$ with the canonical Hilbertian basis of $\ell^2(\mathbf{N})$. This enables us to define an operator $\mathcal{A}_h$ that coincides with the previous one on the modes $k \geqslant 1$. On the zeroth mode, we have

$$A_{0,h} v_0 = -h^2 \frac{1}{L} (L v')'.$$

In order to prove the analogue of Proposition 2.1, we thus only need to study the concentration estimate for this operator. Actually, using that

$$A_{0,h} v_0 = -h^2 v'' - h^2 \frac{L'}{L} v',$$

and the fact that we work in $I_\delta$, the following lemma is enough.

**Lemma B.3.** *Let $P_h$ be the Dirichlet realization of $-h^2 v''$ in $L^2(I_\delta)$. For any $\varepsilon$, and any compact set $K \subset ]0, +\infty[$, there exist constants $C$ and $h_0$ such that if $h \in ]0, h_0]$, $E \in K$ and $v \in H^1_0(I_\delta)$*

$$\|v\|_{H^1_{P_h}} \leqslant \frac{\varepsilon}{h} \|(P_h - E)v\|_{H^{-1}_{P_h}} + C\|hv'\|_{L^2(I_\delta)}.$$

*Proof.* We follow the proof of Proposition 2.1. The spectrum of $P_h$ is given by $\{\frac{k^2\pi^2}{\ell^2}, k \geqslant 1\}$ where $\ell = 2 - 2\delta$ is the length of the interval $I_\delta$. We observe that if $k$ is such that $\frac{h^2k^2\pi^2}{\ell^2} \in [a,b] \subset ]0,+\infty[$ then $k$ is of order $\frac{1}{h}$ so that the distance between two consecutive eigenvalues is of order $h$ as desired (see equation (3)). Next, we need a description of the quadratic form $\mathcal{B}_h$. The matrix element of the latter can be explicitly described since the eigenfunctions of $P_h$ are known: They are given by $x \mapsto \sin\big(\frac{k\pi}{\ell}(x+1-\delta)\big)$. In the interval $[a,b]$, the relevant $k$ is of order $\frac{1}{h}$ so that the asymptotics of $\mathcal{B}$ are easily computed. All of the arguments in the proof of Proposition 2.1 are then seen to hold in the present case. ■

By following the proof of Theorem 5.2, we are able to prove the following proposition. We recall that a 'threshold' is an element of the set

$$\left\{\frac{k^2\pi^2}{L^2(0)}, k \in \mathbf{N}\right\}.$$

**Proposition B.4.** *For any $\delta$, any $\varepsilon$, and any interval $[a,b]$ that does not contain any threshold, there exists a constant $C$ and $h_0$ such that, for any $h < h_0$ and for any eigenfunction $u_h$ of $q_h$ whose eigenvalue $E_h$ is in $[a,b]$, we have*

$$\|u_h\|_{L^2(\Omega_\delta)} \leqslant \varepsilon\|u_h\|_{L^2(\Omega)} + C\|h\partial_x u_h\|_{L^2(\Omega)}.$$

*Proof.* We fix a cutoff function $\chi$ that is identically 1 in $I_\delta$ and identically 0 outside $I_{\frac{\delta}{2}}$. The previous lemma implies that the analogue of Proposition 4.2 holds true:

$$\|\chi u\|_{H^1_{\mathcal{A}_h}} \leqslant \frac{\varepsilon}{h}\|(\mathcal{A}_h - E_h)(\chi u_h)\|_{H^{-1}_{\mathcal{A}_h}} + C\|hD'(\chi u_h)\|_{L^2(\Omega_{\delta/2})},$$

where all the norms are taken on $I_{\delta/2}$. Using the same arguments as in the proof of Theorem 5.2, we show first that there exists some $M$ (that depends only on $\delta$, $a$, and $b$) such that

$$\|(\mathcal{A}_h - E_h)(\chi u_h)\|_{H^{-1}_{\mathcal{A}_h}} \leqslant M\|u_h\|_{L^2(\Omega)},$$

and then that there exists another constant $M$ such that

$$\|hD'(\chi u_h)\|_{L^2(\Omega_{\delta/2})} \leqslant M(\|h\partial_x u_h\|_{L^2(\Omega)} + h\|u_h\|).$$

The claim follows. ■

As before, we will use the preceding proposition to prove that if an eigenvalue branch converges to a limit that is not a threshold, then its mass must concentrate away from $\Omega_\delta$. Our aim will then be to obtain a contradiction.

So for the remainder of this appendix, we make the following assumption.

**Assumption B.5.** We assume that $(u_h, E_h)$ is an analytic eigenbranch of $q_h$ such that

$$\lim_{h\to 0} E_h \notin \left\{\frac{k^2\pi^2}{L^2(0)}, k \in \mathbf{N}\right\}.$$

Using the Feynman–Hellmann formula (9) as in the proof of Theorem 5.2, we find that there exists a subset $\mathbb{H} \subset \mathbf{R}^*$ with accumulation point 0 such that for each $\varepsilon > 0$ there exists $h_0$ so that if $h \in ]0, h_0[ \cap \mathbb{H}$, then

$$\|h \cdot \partial_x u_h\|_{L^2(\Omega)} \leqslant \varepsilon \|u_h\|_{L^2(\Omega)}. \tag{21}$$

Assumption B.5 implies that there exists an interval $[a, b]$ not containing a threshold so that if $h$ is sufficiently small, then $E_h \in [a, b]$. Therefore, we may apply Proposition B.4 to the sequence $(u_h)_{h\in\mathbb{H}}$ and obtain the following result.

**Proposition B.6.** *Under assumption B.5, there exists a subsequence $\mathbb{H}$ going to 0 such that, for any $\varepsilon, \delta > 0$, there exists $h_0$ such that if $h \in ]0, h_0[ \cap \mathbb{H}$, then*

$$\|u_h\|_{L^2(\Omega_\delta)} \leqslant \varepsilon \|u_h\|_{L^2(\Omega)}. \tag{22}$$

### B.2. Outside of the central region

In this section, we assume that $(u_h)_{h\in\mathbb{H}}$ is a sequence provided by Proposition B.6. The proof of Theorem B.1 will be completed by showing that estimate (22) cannot hold true. That is, Assumption B.5 will be contradicted.

We want to control $u_h$ near $x = \pm 1$. These two problems are completely equivalent so that we will only treat the case where $x$ is near $-1$. In order to simplify the notation, we make the change of variable $x \leftarrow x + 1$ so that $x$ is now near 0, and $L(x) = \sqrt{x(2-x)}$. We set

$$U_\delta = \{(x, y) : x \in ]0, \delta[, 0 < y < L(x)\}.$$

In other words, $U_\delta$ is the connected component of $\Omega \setminus \Omega_\delta$ that lies to the left of the central region $\Omega_\delta$.

Since $u_h$ is an eigenvalue, we will use several times the following estimate that we call the $H_h^1$ bound:

$$\|h\partial_x u_h\|^2_{L^2(\Omega)} + \|\partial_y u_h\|^2 \leqslant C \|u_h\|_{L^2(\Omega)}.$$

We recall the transversal Fourier decomposition of $u_h$:

$$u_h = v_h \otimes \mathbb{1} + \sum_{k\geqslant 1} u_{h,k} \otimes e_k.$$

The function $v_h \otimes \mathbb{1}$ is the (transversal) zeroth mode of $u_h$, and we define $u_h^\perp = u_h - v_h \otimes \mathbb{1}$. Our aim is to prove there exists some $\delta$ so that, for $h$ small enough, all of the mass of $u_h$ cannot lie in $U_\delta$. We will use three different estimates to achieve our goal:

- an estimate for $u_h^\perp$ in $U_\delta$ that follows from a Poincaré inequality (see Lemma B.7);
- an estimate for $v_h$ in the interval $[0, \varepsilon h]$ that follows from the $H_h^1$ bound (see Corollary B.9);
- an estimate for $v_h$ in the interval $[\varepsilon h, \delta]$ that is obtained using standard methods in the study of one-dimensional Schrödinger (or Sturm–Liouville) equations (see Proposition B.11).

It will be possible to choose the parameters $\varepsilon$ and $\delta$, and we will always assume that $h$ is chosen so that $\varepsilon h < \delta$. We also fix some (small) upper bound $\delta_0$ so as to have estimates that depend on $\delta_0$ and not on $\delta < \delta_0$.

### B.2.1. The estimate for $u_h^\perp$

**Lemma B.7.** *There exists a function $f$ that goes to 0 when $\delta$ goes to zero so that if $h \leqslant h_0$, then*

$$\|u_h^\perp\|_{L^2(U_\delta)} \leqslant f(\delta)\|u_h\|_{L^2(\Omega)}.$$

*Proof.* For any $k \geqslant 1$, we have

$$\int_0^\delta |u_{h,k}(x)|^2 L(x)\,dx \leqslant L^2(\delta)\int_0^\delta \frac{k^2\pi^2}{L^2(x)}|u_{h,k}(x)|^2 L(x)\,dx.$$

It follows that

$$\|u_h^\perp\|^2_{L^2(U_\delta)} \leqslant L^2(\delta)\|\partial_y u_h\|^2_{L^2(U_\delta)} \leqslant C\cdot L^2(\delta)\|\partial_y u_h\|^2_{L^2(\Omega)}. \qquad \blacksquare$$

**B.2.2. The estimate for $v_h$ in $[0, \varepsilon h]$.** In this regime, we only need the fact that $u_h$ is in $H^1(\Omega)$. Thus, we will supress the dependence of $u$ on $h$ here.

**Lemma B.8.** *There exists a constant $C$ such that, for any $u \in H^1(\Omega)$, for any $\alpha, \beta$ such that $\alpha < \beta < \frac{1}{2}$, we have*

$$\int_0^\alpha |v(x)|^2 L(x)dx \leqslant C\Big[\Big(\frac{\alpha}{\beta}\Big)^{3/2}\|u\|^2_{L^2(\Omega)} + \alpha^{\frac{3}{2}}\beta^{1/2}\|\partial_x u\|^2_{L^2(\Omega)} + \alpha\|\partial_y u\|^2_{L^2(\Omega)}\Big],$$

*where $v \otimes \mathbb{1}$ is the zeroth-mode of $u$.*

*Proof.* By density, we may assume that $u \in C^\infty(\overline{\Omega})$. Using the transversal Fourier decomposition, we have

$$\partial_x u = v' \otimes \mathbb{1} + \sum_{k \geqslant 1} u_k' \otimes e_k - \frac{yL'}{L}\partial_y u.$$

Since $\frac{yL'}{L} \in L^2(\Omega)$ and $|\nabla u|$ is bounded, it follows that

$$\partial_x u + \frac{yL'}{L}\partial_y u \in L^2(\Omega),$$

so that

$$v'(x) = \frac{1}{L(x)} \int_0^{L(x)} \Big(\partial_x u(x,y) + \frac{yL'(x)}{L(x)}\partial_y u(x,y)\Big)\, dy.$$

For a function $w \in L^2(\Omega)$, we introduce the notation

$$N_w(x) \ = \Big( \int_0^{L(x)} |w(x,y)|^2 \, dy\Big)^{1/2}.$$

By construction, for any $\delta > 0$,

$$\int_0^\delta |N_w(x)|^2 \, dx \leqslant \|w\|^2_{L^2(\Omega)}.$$

Using the Cauchy–Schwarz inequality on the definition of $v'$ and the behaviour at 0 of $L$, we get that there is a constant $C$ such that

$$|v'(x)| \leqslant C[x^{-\frac{1}{4}}N_{\partial_x u}(x) + x^{-\frac{3}{4}}N_{\partial_y u}(x)] \quad \text{for all } x \leqslant \frac{1}{2}.$$

Integrating this inequality on an interval $[x, y]$, and using Cauchy–Schwarz again, we obtain

$$|v(y) - v(x)| \leqslant C[(y^{1/2} - x^{1/2})^{1/2}\|\partial_x u\|_{L^2(\Omega)} + (x^{-1/2} - y^{-1/2})^{1/2}\partial_y u\|_{L^2(\Omega)}].$$

We square, use Young's inequality, and remove the obviously negative terms to obtain that for each $x < y$,

$$|v(x)|^2 \leqslant C[|v(y)|^2 + y^{1/2}\|\partial_x u\|^2 + x^{-1/2}\|\partial_y u\|^2].$$

We now fix $\alpha < \beta$, multiply by $L(y)$ and integrate the preceding over $y \in [\beta, 2\beta]$. Using that

$$\int_\beta^{2\beta} L(y)\, dy \asymp \int_\beta^{2\beta} y^{1/2}\, dy \asymp \beta^{3/2},$$

we obtain

$$|v(x)|^2 \leqslant C[\beta^{-3/2}\|u\|^2_{L^2(\Omega)} + \beta^{1/2}\|\partial_x u\|^2_{L^2(\Omega)} + x^{-1/2}\|\partial_y u\|^2_{L^2(\Omega)}]$$

for all $x \in (0, \alpha)$. We finally multiply by $L(x)$ and integrate over $x \in ]0, \alpha[$. ■

We obtain the following corollary for $v_h$.

**Corollary B.9.** *There exists a constant $C$ such that, for any $\varepsilon \in ]0, 1[$ and $h$ small enough, we have*

$$\int_0^{\varepsilon h} |v(x)|^2 L(x)\, dx \leqslant C\varepsilon \|u_h\|^2.$$

*Proof.* We let $\beta = h$ and $\alpha = \varepsilon h$ to obtain

$$\int_0^{\varepsilon h} |v(x)|^2 L(x)\, dx \leqslant \varepsilon^{3/2}\|u_h\|^2_{L^2(\Omega)} + \varepsilon^{3/2}\|h\partial_x u_h\|^2_{L^2(\Omega)} + \varepsilon h\|\partial_y u_h\|^2_{L^2(\Omega)}.$$

The claim follows using the eigenvalue $H^1_h$ bound on $u_h$. ■

We now move on to study $v_h$ on $[\varepsilon h, \delta]$.

**B.2.3. The estimate for $v_h$ in $[\varepsilon h, \delta]$.** Fix some $\delta_0 < 2$, and let $\phi \in \mathcal{C}_0^\infty(]0, \delta[)$. Integration against the eigenequation gives

$$\begin{aligned} h^2 \int_\Omega \partial_x u_h(x, y)\phi'(x)\, dx\, dy &= E_h \int_\Omega u_h(x, y)\phi(x)\, dx\, dy \\ &= E_h \int_0^{\delta_0} v_h(x)\phi(x)L(x)\, dx. \end{aligned}$$

Now, recall that

$$\partial_x u_h = v_h' \otimes \mathbb{1} + \sum_{k \geqslant 1} u_{h,k}' \otimes e_k - \frac{yL'}{L}\partial_y u_h,$$

and integrate this equation against $\phi'$ to find

$$h^2 \int_0^{\delta_0} v_h'(x)\phi'(x)L(x)dx - E_h \int_0^{\delta_0} v_h(x)\phi(x)L(x)\, dx = h^2 \int_0^{\delta_0} r_h(x)\phi'(x)\, dx,$$

where we have set

$$r_h(x) = \int_0^{L(x)} \frac{yL'(x)}{L(x)} \partial_y u_h \, dy.$$

We divide by $h^2$ and make an integration by parts on the RHS to find that

$$\int_0^{\delta_0} v_h'(x)\phi'(x)L(x)\,dx - \frac{E_h}{h^2}\int_0^{\delta_0} v_h(x)\phi(x)L(x)\,dx = -\int_0^{\delta_0} r_h'(x)\phi(x)\,dx$$

It follows that, on the interval $(0, \delta_0)$, $v_h$ satisfies the Sturm–Liouville equation

$$-\frac{1}{L}(Lv_h')' - \frac{E}{h^2}v_h = -\frac{1}{L}r_h'.$$

We first make the change of dependent variable $w = L^{1/2}v$. The equation becomes

$$-w'' - \frac{E}{h^2}w + \frac{1}{2}L^{-1/2}(L'L^{-1/2})'w = -L^{-1/2}r_h'. \tag{23}$$

In order to solve it, we first study the associated homogeneous equation:

$$-w'' - \frac{E}{h^2}w + \frac{1}{2}L^{-1/2}(L'L^{-1/2})'w = 0.$$

We can write $L$ as

$$L(x) = \sqrt{2x}(1 + \tilde{\ell}(x)),$$

where $\tilde{\ell}$ is smooth on $[0, \delta_0]$. It follows that we can write

$$\frac{1}{2}L^{-1/2}(L'L^{-1/2})'(x) = -\frac{3}{16x^2}(1 + x\ell(x)), \tag{24}$$

where $\ell$ is smooth on $[0, \delta_0]$.

We now perform the change of independent variable $x = \frac{\sqrt{E}}{h}z$. Setting $W(z) = w(x)$, we are led to the following homogeneous equation on $\left[\varepsilon\sqrt{E_h}, \frac{\delta_0\sqrt{E}}{h}\right]$:

$$W'' + W + \frac{3}{16z^2}W = f_h(z)W.$$

We set $J_h = \left[\varepsilon\sqrt{E_h}, \frac{\delta_0\sqrt{E}}{h}\right]$ and observe that, for any $h \in \mathbb{H}$, we have $J_h \subset \overline{J}_h = [\varepsilon\sqrt{a}, Z_h]$ with $Z_h = \frac{\delta_0\sqrt{b}}{h}$. The function $f_h$ satisfies the following uniform bound:

$$\text{there exists } M \text{ such that}$$
$$|f_h(x)| \leqslant M\frac{h}{z} \quad \text{for all } h \leqslant h_0, E \in [a,b], z \in \overline{J}_h. \tag{25}$$

We solve the preceding equation as a perturbation of the equation

$$W'' + W + \frac{3}{16z^2} W = 0. \tag{26}$$

It is standard that any solution to equation (26) on the half-line is asymptotic to a linear combination of $\cos x$ and $\sin x$ near $+\infty$. We define $\psi_c$ to be the solution that is asymptotic to cosine and $\psi_s$ to be the solution that is asymptotic to sine. We now apply the variation of constants method: We define an operator $K$ on $C^0(\bar{J}_h)$ by

$$K[W](z) = \frac{1}{2} \int_z^{Z_h} f_h(\zeta) W(\zeta) [\psi_c(z)\psi_s(\zeta) - \psi_s(z)\psi_c(\zeta)]\, d\zeta,$$

and observe that the function $W + K[W]$ is a solution to the unperturbed equation (26). It follows that a basis of solutions is given by $\{\phi_s, \phi_c\}$ where

$$\phi_s = [\mathrm{id} + K]^{-1} \psi_s \quad \text{and} \quad \phi_c = [\mathrm{id} + K]^{-1} \psi_c,$$

provided that $\mathrm{id} + K$ is invertible. Since $\psi_c$ and $\psi_s$ are uniformly bounded on $[\varepsilon\sqrt{a}, +\infty)$, the uniform bound on $f_h$ in (25) implies that there exists some constant $C$ that depends on $a, b, \varepsilon, \delta_0$ such that for each $h \in \mathbb{H}$ the operator norm of $K$ is at most $Ch|\ln h|$. This ensures the invertibility of $\mathrm{id} + K$ for small $h$, and so $\phi_s$ and $\phi_c$ are well defined.

By unwinding the change of independent variable, we find that

$$\left\{x \mapsto \phi_s\left(\frac{x\sqrt{E_h}}{h}\right), x \mapsto \phi_c\left(\frac{x\sqrt{E_h}}{h}\right)\right\}$$

is a basis of solutions to the homogeneous equation (24). Since the Wronskian of these two solutions is $\frac{2\sqrt{E_h}}{h}$ we find that the following function $w_p$ solves (23) on the interval $[\varepsilon h, \delta]$ for any $\delta$ such that $\delta \leqslant \delta_0$.

$$\begin{aligned} w_p(x) = {} & \frac{h}{2\sqrt{E}} \phi_c(x) \int_x^{\delta} L^{-1/2}(\xi) r'(\xi) \phi_s\left(\frac{\sqrt{E}\xi}{h}\right) d\xi \\ & - \frac{h}{2\sqrt{E}} \phi_s(x) \int_x^{\delta} L^{-1/2}(\xi) r'(\xi) \phi_c\left(\frac{\sqrt{E}\xi}{h}\right) d\xi. \end{aligned}$$

We perform an integration by parts in the integrals. The boundary terms at $x$ cancel out and the boundary terms coming from $\delta$ contribute to a solution of the homogeneous equation. We deduce that the sum of the following four terms is a solution to the equation:

$$
\begin{aligned}
w_p(x) = &- \frac{h}{2\sqrt{E}} \phi_c(x) \int_x^{2\delta} (L^{-1/2})'(\xi) r(\xi) \phi_s\Big(\frac{\sqrt{E}\xi}{h}\Big)\, d\xi \\
&- \phi_c(x) \int_x^{2\delta} L^{-1/2}(\xi) r(\xi) \phi_s'(\sqrt{E}\xi h)\, d\xi \\
&+ \frac{h}{2\sqrt{E}} \phi_s(x) \int_x^{2\delta} (L^{-1/2})'(\xi) r(\xi) \phi_c\Big(\frac{\sqrt{E}\xi}{h}\Big)\, d\xi \\
&+ \phi_s(x) \int_x^{2\delta} L^{-1/2}(\xi) r(\xi) \phi_c'\Big(\frac{\sqrt{E}\xi}{h}\Big)\, d\xi.
\end{aligned}
$$

We denote these four terms by $w_{p,i}$ for $i = 1, 2, 3, 4$. In order to bound these terms, we use the following bound, obtained by applying the Cauchy–Schwarz inequality to the definition of $r$. There exists $C$ so that

$$
|r(\xi)| \leqslant C\xi^{-1/4} N_{\partial_y u}(\xi).
$$

We also use that the functions $\phi_s$, $\phi_c$, and their derivatives are uniformly bounded on $[\varepsilon\sqrt{a}, Z_h)$.

For $i = 1$ and 3, we find that

$$
|w_{p,i}(x)| \leqslant Ch\|\partial_y u\| \Big( \int_x^{\delta_0} y^{-3} \Big)^{1/2} \leqslant Ch\|\partial_y u\| x^{-1}.
$$

It follows that, for any $\delta \leqslant \delta_0$,

$$
\int_{\varepsilon h}^{\delta} |w_{p,i}(x)|^2\, dx \leqslant Ch^2 \|\partial_y u\|^2 \int_{\varepsilon h}^{\delta} x^{-2}\, dx \leqslant Ch\|\partial_y u\|^2.
$$

We now proceed to estimate $w_{p,i}$ for $i = 2$ or 4. We proceed as above, using that $\phi'_{c,s}$ is uniformly bounded on $[\varepsilon, \infty)$. For $i = 2, 4$, we obtain that

$$
|w_{p,i}(x)| \leqslant C\|\partial_y u\| \Big( \int_x^{\delta_0} \xi^{-1} d\xi \Big)^{1/2} \leqslant C\|\partial_y u\| |\ln x|^{1/2}.
$$

It follows that, for any $\delta \leqslant \delta_0$,

$$\int_{\varepsilon h}^{\delta} |w_{p,i}(x)|^2\, dx \leqslant C\|\partial_y u\|^2 \int_{\varepsilon h}^{\delta} |\ln \xi|\, d\xi \leqslant C\|\partial_y u\|^2 \delta \ln|\delta|.$$

Finally, the $L^2$ norm of $w_p$ on $[\varepsilon h, \delta]$ satisfies the following. For any $\varepsilon > 0$, there exists a constant $C$ such that, for any $\delta$ and $h$ small enough,

$$\|w_p\|_{L^2([\varepsilon h,\delta])} \leqslant C(h + \delta|\ln\delta|))^{1/2}\|u\|_{L^2(\Omega)}. \tag{27}$$

Now, since $w_p$ is a solution to the same equation as $w_h$ there exists a solution $\phi_h$ to the homogeneous equation such that $w_h = w_p + \phi_h$.

We now claim the following statement.

**Lemma B.10.** *For any $\delta \in ]0,1[$, there exist constants $C$ and $h_0$ such that, for any $h \leqslant h_0$ and any $\phi$ solution of the homogeneous equation*

$$\int_{\varepsilon h}^{\delta} \Big|\phi\Big(\frac{\sqrt{E}}{h}x\Big)\Big|^2\, dx \leqslant C\int_{\delta}^{2\delta} \Big|\phi\Big(\frac{\sqrt{E}}{h}x\Big)\Big|^2\, dx.$$

*Proof.* Changing variables, it suffices to study the behaviour of

$$\int_{\varepsilon}^{X/h} |\phi(z)|^2\, dz,$$

as $h$ tends to 0 and $X \leqslant X_0 = 2\sqrt{b}\delta_0$. There exists $\alpha, \beta$ such that $\phi = \alpha\phi_c + \beta\phi_s$. By construction, on $\big[\varepsilon, \frac{X_0}{h}\big]$, we have

$$\begin{aligned}\|\phi_c - \psi_c\|_{C^0} &\leqslant Mh|\ln h|\|\psi_c\|_{C^0},\\ \|\phi_s - \psi_s\|_{C^0} &\leqslant Mh|\ln h|\|\psi_s\|_{C^0}.\end{aligned}$$

and, for $z$ going to infinity, we have

$$\begin{aligned}\psi_c(z) &= \cos z + O(z^{-1}),\\ \psi_s(z) &= \sin z + O(z^{-1}).\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\int_{\varepsilon}^{X/h} |\phi(z)|^2\, dz = &\int_{\varepsilon}^{X/h} (\alpha\cos z + \beta\sin z)^2\, dz\\ &+ (|\alpha|^2 + |\beta|^2) O\Big(\ln\Big|\frac{X}{h}\Big|\Big)\\ &+ (|\alpha|^2 + |\beta|^2)\Big|\frac{X}{h}\Big| O(h^2\ln|h|^2).\end{aligned}$$

A direct computation gives that

$$\int_{\varepsilon}^{X/h} (\alpha \cos z + \beta \sin z)^2 \, dz = \frac{\alpha^2 + \beta^2}{2} \frac{X}{h} + |ab| O(1).$$

It follows that there exist constants $C$ and $h_0$ so that, for any $X$, $\alpha$, $\beta$, and $h$ small enough we have

$$C^{-1} \cdot \frac{X(\alpha^2 + \beta^2)}{h} \leqslant \int_{\varepsilon}^{X/h} |\phi(z)|^2 \, dz \leqslant C \cdot \frac{X(\alpha^2 + \beta^2)}{h}.$$

The claim now follows. ■

All of these estimates may be combined to give the following result.

**Proposition B.11.** *For any $\varepsilon$ and $\delta_0$, there exists a constant $C$ such that for each $\delta < \delta_0$ there exists $h_0$ such that if $h \leqslant h_0$, then*

$$\int_{\varepsilon h}^{\delta} |v_h(x)|^2 L(x) \, dx \leqslant C (\|u\|^2_{L^2(\Omega_\delta)} + (h + \delta |\ln \delta|) \cdot \|u\|^2_{L^2(\Omega)}). \tag{28}$$

*Proof.* By definition of $w_h$, we have

$$\int_{\varepsilon h}^{\delta} |v_h(x)|^2 L(x) \, dx = \int_{\varepsilon h}^{\delta} |w_h(x)|^2 \, dx.$$

Since $w_h = w_p + \phi$, we obtain

$$\int_{\varepsilon h}^{\delta} |w_h(x)|^2 \, dx \leqslant 2\Big( \int_{\varepsilon h}^{\delta} |\phi_h(x)|^2 \, dx + \int_{\varepsilon h}^{\delta} |w_p(x)|^2 \, dx \Big).$$

The second integral may be estimated using Lemma B.10:

$$\int_{\varepsilon h}^{\delta} |\phi_h(x)|^2 \, dx \leqslant \int_{\delta}^{2\delta} |\phi_h(x)|^2 \, dx \leqslant 2\Big( \int_{\delta}^{2\delta} |w_h(x)|^2 \, dx + \int_{\delta}^{2\delta} |w_p(x)|^2 \, dx \Big).$$

Estimate (27) can be used to estimate each of the integrals involving $w_p$, and so we obtain

$$\int_{\varepsilon h}^{\delta} |w_h(x)|^2 \, dx \leqslant C \Big( \int_{\delta}^{2\delta} |w_h(x)|^2 \, dx + (h + \delta \ln \delta) \|u\|^2_{L^2(\Omega)} \Big).$$

The claim follows since

$$\int_{\delta}^{2\delta} |w_h(x)|^2 \, dx \leqslant \int_{\delta}^{2\delta} |v(x)|^2 L(x) \, dx \leqslant \|u_h\|^2_{L^2(\Omega_\delta)}. \qquad \blacksquare$$

**B.2.4. Obtaining the contradiction.** We now finish the proof of Theorem B.1. Recall that we are assuming for contradiction that the statement of the theorem is false, and hence Proposition B.6 provides us with a sequence $(u_h)_{h\in\mathbb{H}}$ so that given $\varepsilon, \delta > 0$, there exists $h_0$ such that if $h \in ]0, h_0[ \cap \mathbb{H}$, then

$$\|u_h\|_{L^2(\Omega_\delta)} \leqslant \varepsilon \|u_h\|_{L^2(\Omega)}.$$

Fix $\eta > 0$. By Proposition B.6, Lemma B.7, Corollary B.9, there exists $\varepsilon, \delta_0, h_0 > 0$ so that if $\delta \leqslant \delta_0$ and $h \in ]0, h_0[ \cap \mathbb{H}$, then

$$\|u_h\|_{L^2(\Omega_\delta)} \leqslant \eta \|u_h\|_{L^2(\Omega)}, \tag{29}$$

$$\|u_h^{\perp}\|^2_{L^2(U_\delta)} \leqslant \eta \|u_h\|^2_{L^2(\Omega)},$$

and

$$\int_0^{\varepsilon h} |v_h(x)|^2 L(x) \, dx \leqslant \eta \, \|u_h\|^2_{L^2(\Omega)}.$$

With these choices of $\varepsilon$ and $\delta_0$, we apply Proposition B.11 to obtain a constant $C$ for which (28) holds if $h$ is sufficiently small. Choose $\delta < \delta_0$ so that $C\delta|\ln \delta| \leqslant \eta$. With this choice of $\delta$ we have

$$\begin{aligned}
\|u_h\|^2_{L^2(U_\delta)} &= \|u_h^{\perp}\|^2_{L^2(U_\delta)} + \int_0^{\delta} |v_h(x)|^2 L(x) \, dx \\
&\leqslant \eta \|u_h\|^2_{L^2(\Omega)} + \eta \|u_h\|^2_{L^2(\Omega)} + \int_{\varepsilon h}^{\delta} |v_h(x)|^2 L(x) \, dx \\
&\leqslant 2\eta \|u_h\|^2_{L^2(\Omega)} + C \|u_h\|^2_{L^2(\Omega_\delta)} + (C \cdot h + \eta) \|u_h\|^2_{L^2(\Omega)} \\
&\leqslant 3\eta \|u_h\|^2_{L^2(\Omega)} + C \|u_h\|^2_{L^2(\Omega_\delta)} + C \cdot h \|u_h\|^2_{L^2(\Omega)}.
\end{aligned}$$

The last term is less than $\eta \|u_h\|^2_{L^2(\Omega)}$ if $h < \frac{\eta}{C}$. To bound the second term, we apply Proposition B.4: There exists $C'$ so that if $h$ is sufficiently small, then

$$\|u_h\|^2_{L^2(\Omega_\delta)} \leqslant \frac{\eta}{C} \|u\|^2_{L^2(\Omega)} + C' \|h\partial_x u\|^2.$$

Using these estimates, we find that for $h$ small enough we have

$$\|u_h\|^2_{L^2(U_\delta)} \leqslant 5\eta \|u_h\|^2_{L^2(\Omega)} + CC' \|h\partial_x u\|^2.$$

Therefore, the integrability condition (21) implies that for $h$ small enough

$$\|u_h\|^2_{L^2(U_\delta)} \leqslant 6\eta \|u_h\|^2_{L^2(\Omega)}.$$

Recall that $U_\delta$ is the component of $\Omega \setminus \Omega_\delta$ on the left of the central region $\Omega_\delta$. The same argument applies to the component of $\Omega \setminus \Omega_\delta$ on the right of the central region. Thus, in sum we have

$$\|u_h\|^2_{L^2(\Omega \setminus \Omega_\delta)} \leqslant 12\eta \|u_h\|^2_{L^2(\Omega)}.$$

By combining this with (29) we find that

$$\|u_h\|^2_{L^2(\Omega)} \leqslant 13\eta \|u_h\|^2_{L^2(\Omega)},$$

which is absurd if $\eta < \frac{1}{13}$.

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