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On the dynamics of quasi-periodic Schrödinger cocycles for positive measure sets of frequencies

Kristian Bjerklöv

Abstract. We consider the family of one-frequency quasi-periodic Schrödinger coycles $G_{\omega,E}$, parametrized by the energy E. For potential functions $v(x) = \lambda v_0(x)$, where $v_0 \in C^2(\mathbb{T}, \mathbb{R})$ is a Morse function with finitely many critical points and $\lambda > 0$ is large, we show that, for any value of $E \in \mathbb{R}$ and for any phase $x^* \in \mathbb{T}$ such that $|v_0(x^*) - E/\lambda|$ is not too small, there exists a set of frequencies $\Omega = \Omega(E, x^*)$ of positive measure such that the following hold: (1) for every $\omega \in \Omega$, the upper Lyapunov exponent of the cocycle $G_{\omega,E}$ is $\gtrsim \log \lambda$ and x^* is (essentially) a typical point in Oseledets' theorem; (2) either $G_{\omega,E}$ is uniformly hyperbolic, or there exists a phase $x_0 \in \mathbb{T}$ such that E is an eigenvalue of the corresponding discrete Schrödinger operator $H_{x_0,\omega}$.

In memory of Misha Goldstein

1. Introduction

For several decades, the quasi-periodic Schrödinger cocycle has attracted much attention in the literature. This is because its close connection to the one-dimensional discrete Schrödinger operator with a quasi-periodic potential; but also because it has shown to be a natural model-problem in the theory of dynamical systems. We refer to the excellent articles [8, 18] for an overview of this rich subject.

1.1. The cocycle

Assume that $v_0 \in C(\mathbb{T}, \mathbb{R})$ and let

$$A_{E,\lambda}(x) = \begin{pmatrix} 0 & 1 \\ -1 & \lambda v_0(x) - E \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}), \quad E, \lambda \in \mathbb{R}.$$

Given an irrational $\omega \in \mathbb{T}$, let the Schrödinger cocycle $G_{\omega,E,\lambda}\colon \mathbb{T} \times \mathbb{R}^2 \to \mathbb{T} \times \mathbb{R}^2$ be given by

$$G_{\omega,E,\lambda}(x, u) = (x + \omega, A_{E,\lambda}(x)u). \tag{1.1}$$

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In this paper, we shall focus on the dynamics of $G_{\omega,\lambda,E}$ for large *coupling constants* λ. We thus assume that $\lambda > 0$ is fixed and sufficiently large (depending on v_0), and we use E, ω as the parameters. For this reason, we do not explicitly write out the dependence on λ, but write only A_E and $G_{\omega,E}$.

We define the transfer matrix (or fundamental solution) $A^n_{\omega,E} \in SL(2,\mathbb{R})$ $(n \in \mathbb{Z})$ by

$$A^n_{\omega,E}(x) = \begin{cases} A_E(x + (n-1)\omega) \cdots A_E(x), & n \geq 1 \\ I, & n = 0, \\ A_E(x + n\omega)^{-1} \cdots A_E(x - \omega)^{-1}, & n \leq -1. \end{cases}$$

Then, we can write

$$G^n_{\omega,E}(x,u) = (x + n\omega, A^n_{\omega,E}(x)u), \quad n \in \mathbb{Z}.$$

The growth of the norm $\|A^n_{\omega,E}(x)\|$ is measured by the (upper) Lyapunov exponent $L(\omega, E)$, defined by

$$L(\omega, E) = \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}} \log \|A^n_{\omega,E}(x)\| \, dx \geq 0,$$

where the limit exists by sub-additivity. This is one of the main objects in the present paper.

In general, it is difficult to obtain non-trivial lower bounds of $L(\omega, E)$ (for E in the range of λv_0), even showing positivity. The main reason is essentially the following: if λ is large, then the matrix $A_E(x)$ is strongly hyperbolic for all x outside a tiny region where $\lambda v_0(x) - E$ is close to zero; close to these zeroes $A_E(x)$ acts a rotation (which can interchange expanding and contracting directions). This causes major problems in estimating the growth of $A^n(x)$. In fact, this is a central problem in dynamics.

If $L(\omega, E) > 0$, it follows from Oseledets' theorem (note that the transformation $x \mapsto x + \omega$ on $\mathbb{T}$ is (uniquely) ergodic with respect to the Lebesgue measure) that there exists a measurable splitting $W^+(x) \oplus W^-(x) = \mathbb{R}^2$ such that for a.e. $x \in \mathbb{T}$ we have

$$\lim_{n\to\pm\infty} \frac{1}{n} \log |A^n_{\omega,E}(x)u| = L(\omega) \qquad \text{for all } 0 \neq u \in W^+(x) \tag{1.2a}$$

and

$$\lim_{n\to\pm\infty} \frac{1}{n} \log |A^n_{\omega,E}(x)u| = -L(\omega) \quad \text{for all } 0 \neq u \in W^-(x). \tag{1.2b}$$

The cocycle $G_{\omega,E}$ is said to be *uniformly hyperbolic* if there exist a continuous splitting $W^+(x) \oplus W^-(x) = \mathbb{R}^2$ and constants $C, \alpha > 0$ such that for every $x \in \mathbb{T}$

and $n \geq 1$ there holds:

$$|A^n_{\omega,E}(x)u| \leq Ce^{-\alpha n}|u| \quad \text{for } u \in W^-(x),$$
$$|A^{-n}_{\omega,E}(x)u| \leq Ce^{-\alpha n}|u| \quad \text{for } u \in W^+(x).$$

It is well known that an equivalent definition of uniform hyperbolicity is that there exist constants $C, \beta > 0$ such that

$$\|A^n_{\omega,E}(x)\| \geq Ce^{\beta n} \quad \text{for all } x \in \mathbb{T} \text{ and } n \geq 1. \tag{1.3}$$

One of the connections between the dynamics of the cocycle $G_{\omega,E}$ and the corresponding Schrödinger operator $H_{x,\omega}\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, given by

$$(H_{x,\omega}a)_n = -(a_{n+1} + a_{n-1}) + \lambda v_0(x + (n-1)\omega)a_n, \tag{1.4}$$

is Johnson's theorem [15] which states that $G_{\omega,E}$ is uniformly hyperbolic if and only if E is not in the spectrum of the operator $H_{x,\omega}$ (it is well known that the spectrum of $H_{x,\omega}$, as a set, is independent of the phase x).

We say that $G_{\omega,E}$ is *non-uniformly hyperbolic* if $L(\omega, E) > 0$ and $G_{\omega,E}$ is not uniformly hyperbolic.

1.2. Related results

We now summarize some of the main results concerning, in particular, positivity of the Lyapunov exponent for the one-frequency Schrödinger cocycle $G_{\omega,E}$. As mentioned above, we focus on the case when the coupling constant λ is large (large disorder); there is also a rich theory (many times based on KAM theory) for the case when λ is small; and there are also more global results [1,2]. (There is a huge literature on quasi-periodic Schrödinger operators and cocycle. We refer to the overview articles [8, 18] for a more complete discussion.)

In the case when the function v_0 is real-analytic, it is possible to complexify the problem and use powerful tools from, among others, harmonic analysis. Herman [13] showed that if v_0 is a non-constant trigonometric polynomial, then, for any irrational $\omega \in \mathbb{T}$, we have $L(\omega, E) \gtrsim \log \lambda$ uniformly for all $E \in \mathbb{R}$; and Sorets and Spencer [21] extended this result to the case of v_0 being non-constant and real-analytic.

A control on the Lyapunov exponents (for example, positivity on some interval) is many times the starting point for a finer analysis of more delicate properties of the cocyle (1.1) and the operator (1.4). This includes: regularity of the Lyapunov exponent and the integrated density of states, existence of eigenvalues, presence of Anderson localization, structure of the spectrum. Remarkable results have been obtained via such an approach, in particular in the real-analytic case. We mention the seminal works [6, 9, 11, 12].

Under a Gevrey assumption on v_0 (and a Diophantine-type assumption on ω), it is possible to extend the techniques from the real-analytic case and show (for large enough λ) uniform (in E) positivity of the Lyapunov exponent and Anderson localization [16]. Very related results (pure-point spectrum) are also obtained in [9] by other methods.

To investigate the cases when v_0 is of lower regularity, there are so-far two direction: consider "simple" v_0, or start excluding parameters. In the first direction, most results concerning uniform (in E) non-trivial lower bounds of the Lyapunov exponent are for $v_0 \in C^2(\mathbb{T}, \mathbb{R})$ with exactly two critical points [5, 24] (closely related to the seminal works [10, 20] on the Schrödinger operator (1.4); see also [17] and references therein for recent developments for this class of v_0).

Concerning the parameter exclusion, one way is to fix either ω (irrational, or satisfying some Diophantine condition), i.e., the base dynamics is fixed, and excludes parameters E from some interval, or E and excludes (resonant) frequencies ω. We also mention the technique in [7, Theorem 1.2], where one, for a large class of $\tilde{v}_0 \in C^3(\mathbb{T}, \mathbb{R})$ and $\lambda \gg 1$, can perform small changes in $\tilde{v}_0$ and obtain a v_0 such that for a large set of $\omega \in \mathbb{T}$ one has $L(E) \gtrsim \log \lambda$ for all $E \in \mathbb{R}$.

For a fixed irrational ω, and under very mild assumptions on v_0 (bounded and measurable), one has $L(E) \gtrsim \log \lambda$ for all $E \in \mathbb{R}$ outside an exceptional set of exponentially small (in λ) measure [19, 22] (see also [23] for related results).

In the present paper, we will show that if $v_0 \in C^2(\mathbb{T}, \mathbb{R})$ is a Morse function with finitely many critical points and if $\lambda \gg 1$, then for any $E \in \mathbb{R}$ we can construct a positive measure set of ω (depending on E) such that for all such ω we have a very good control on the dynamics of the cocycle $G_{\omega,E}$; see Theorem 1 below. This includes that a chosen $x_0 \in \mathbb{T}$ (such that $|v_0(x) - E/\lambda|$ is not too small) can be shown to be "typical" in the sense of the Oseledets' theorem (1.2) (the excluded set of ω also depends on x_0).

This extends the results in [4] (see also [7, Theorem 1.1], [26], and [25]), in that every $E \in \mathbb{R}$ can be considered. In the latter paper(s), one considers $v_0 \in C^1(\mathbb{T}, \mathbb{R})$ and restricts to energies E for which $E/\lambda \in \mathcal{E}$, where $\mathcal{E} \subset \mathbb{R}$ is such that $|v_0'(x)| > s$ on $v_0^{-1}(\mathcal{E})$ for some $s > 0$.

The general approach used in the proof is based on the seminal paper [3] by Benedicks and Carleson. This kind of approach was first used in the cocycle case in [25] (not for Schrödinger cocyles; but it is possible to adapt to this class, see [26]). We will build on the techniques from [4, 5].

We end by mentioning that the methods in [4] have been used to analyze more general quasi-periodically forced circle maps [14]. Similarly, the techniques in the present paper could be extended to more general systems.

1.3. Main results

We are now ready to formulate the main results in this paper. First, we shall make precise the assumptions on v_0; and we shall introduce the first critical set I_0, as well as the set Ω of frequencies.

1.3.1. Assumption on v_0. Assume that $v_0 \in C^2(\mathbb{T}, \mathbb{R})$ has a finite number of critical points $c_1, \dots, c_d \in \mathbb{T}$ and that

$$v_0''(c_j) \neq 0 \quad \text{for all } 1 \leq j \leq d. \tag{1.5}$$

1.3.2. Definition of Ω and the definition of the critical set I_0. Let $\omega_{\max}$ be the minimum distance (on $\mathbb{T}$) between any two distinct critical points c_i, c_j and let

$$\Omega = \{\omega : |\omega| \leq \omega_{\max}\}. \tag{1.6}$$

Remark 1. The reason for this set is that $v'(x - \omega)$ and $v'(x + \omega)$ have different signs if x is close to a critical point and $\varepsilon < |\omega| < \omega_{\max} - \varepsilon$ (see Figure 1). This will guarantee that certain "bad sets" move with ω (like a "twist condition"). In fact, this is only needed in the cases when the parameter E is close to $\lambda v(c_i)$ $(1 \leq i \leq d)$. However, to get an easier formulation of the main theorem we focus only on $\omega \in \Omega$.

We also introduce the critical set I_0 defined by

$$I_0 = I_0(E, \lambda) = \{x \in \mathbb{T} : |\lambda v_0(x) - E| \leq 3\lambda^{3/4}\}. \tag{1.7}$$

Moreover, if $S \subset \mathbb{T}$ is a Lebesgue measurable set, we denote by $|S|$ the Lebesgue measure of S.

With these assumptions and definitions, we can now formulate the main theorem.

Theorem 1. *There is a $\lambda_0 = \lambda_0(v_0) > 0$ such that, for all $\lambda \geq \lambda_0$, the following hold. For every $E \in \mathbb{R}$ and every $x^* \in \mathbb{T} \setminus I_0(E)$, there exists a set $\Omega_\infty = \Omega_\infty(E, x^*) \subset \Omega$ ($|\Omega_\infty| > |\Omega| - \lambda^{-1/60}$) such that, for every $\omega \in \Omega_\infty$, the conditions below are met.*

(1) $L(\omega, E) > (2 \log \lambda)/3$.

(2) *There exist two one-dimensional subspaces $W^\pm \subset \mathbb{R}^2$ such that*

$$W^+ \oplus W^- = \mathbb{R}^2,$$

and, for all $n \geq 1$, we have

$$|A^n(x^*)u| \leq \lambda^{-n/3}|u| \quad \textit{for all } u \in W^-,$$
$$|A^{-n}(x^*)u| \leq \lambda^{-n/3}|u| \quad \textit{for all } u \in W^+.$$

(3) *Either the cocycle $G_{\omega,E}$ is uniformly hyperbolic, or there exist an $x_0 \in \mathbb{T}$ and a unit vector $u_0 \in \mathbb{R}^2$ such that $|A^n(x_0)u_0| < 2\lambda^{-2|n-1|/3}$ for all $n \neq 0$. Moreover, the latter case holds if $E \in [\lambda \min v_0 + 4\lambda^{3/4}, \lambda \max v_0 - 4\lambda^{3/4}]$.*

Remark 2. (a) From (1), it follows that (1.2) hold for a.e. $x \in \mathbb{T}$. However, if we pick an $x \in \mathbb{T}$, we do not know if this point is typical in the sense of (1.2). Condition (2) says that the chosen point x^* is (close of being) a typical point in Oseledets' theorem. (Another situation when a prescribed point is shown to be typical for the dynamics can be found in [3, Theorem 5].)

(b) In (3), if the cocycle $G_{\omega,E}$ is not uniformly hyperbolic, it follows from the definition of the cocycle that if $\binom{a_0}{a_1} = u_0$, then $\binom{a_n}{a_{n+1}} = A^n(x_0)\binom{a_0}{a_1}$ $(n \in \mathbb{Z})$ satisfies $-(a_{n+1} + a_{n-1}) + \lambda v_0(x_0 + (n-1)\omega)a_n = Ea_n$. Thus, $(a_n) \in \ell^2(\mathbb{Z})$ is an exponentially decaying (at $\pm\infty$) eigenfunction of the Schrödinger operator $H_{x_0,\omega}$ defined in (1.4).

(c) In (3), if the cocycle $G_{\omega,E}$ is not uniformly hyperbolic, then the corresponding projective cocycle (which is obtained by considering the action of $A(x)$ on the projective space $\mathbb{P}^1$, see also the map (2.1) below) is minimal. The proof of this follows the same lines as in [4] (see also [14]); however, in order to keep this article of a reasonable length, we have omitted the proof.

(d) It should be mentioned that we have not tried to optimize any exponents in the estimates.

1.4. Outline of the paper

As mentioned above, the overall strategy of the proof of Theorem 1 is inspired by the seminal work by Benedicks and Carleson in [3]. We follow, in particular, the approach in [4]. In more detail, the structure is as detailed below. In Section 2 we introduce the projectivization F of the cocycle G, which will be the main object we work with. Elementary properties of the map F are derived, and a classification of the components in the critical set I_0 is introduced. Furthermore, a first parameter exclusion (of the frequencies in Ω) is performed. In Section 3, we introduce the critical sets I_n $(n \geq 1)$ of higher order and introduce arithmetic conditions (on ω) with respect to these critical sets. Under a suitable (abstract) assumption on ω, we prove the main inductive step (Proposition 3.1) for controlling iterates of F and F^{-1}, up to a fixed scale. We also derive an important inclusion property (Lemma 3.3).

In Section 4, we apply the estimates from Section 3, in order to control the geometry of the iterates of certain curves (which are used to define the critical sets). This then allows to inductively obtain (in Proposition 4.4) information about the geometry of the critical sets (small intervals) and how they move with the parameter ω.

With the information about the critical sets I_n (for a fixed scale), it is possible to estimate the measure and geometry of the set of ω which do not satisfy certain arithmetic conditions and which will be removed. This is the content of Section 5, and the main result is Proposition 5.1.

Finally, in Section 6, we can put the three main inductive steps (control on the iterates, control of the geometry and the behavior of the critical sets, and the frequency exclusion) together to obtain a set $\Omega_\infty \subset \Omega$ such that, for all $\omega \in \Omega_\infty$, for a large set of initial conditions, we have control on the iterates of F and F^{-1} for all time.

In Appendix A, we have collected some results of a more computational nature which are used in the analysis.

2. Preliminaries

In this section, we shall introduce some of the main objects needed for the analysis and derive some elementary properties about them.

2.1. Some standing assumptions

We assume that the function v_0 is fixed, satisfying the assumptions in Section 1.3.1. For simplicity, we also assume that

$$\|v_0\|_{C^0} \leq \frac{1}{3}.$$

This only scales λ. Moreover, to simplify notation, we let

$$v(x) = \lambda v_0(x) - E.$$

(Recall that, for each fixed E, we want to construct a good set of ω; so E is fixed, and ω acts as the parameter.)

We always assume that λ is large enough, depending only on v_0 (and, in particular, not on E); we stress that there are only finitely many largeness assumptions on λ, so indeed there exists $\lambda_0 > 0$ such that everything below holds for all $\lambda > \lambda_0$. Note that this (implicit) standing assumption on the largeness of λ also applies in the statements of the lemmas and propositions below, that is, we do not explicitly write out that we assume that λ is large enough, depending on v_0.

2.2. The projective cocyle

Let $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. We note that

$$A(x)\binom{1}{r} = r\binom{1}{v(x) - \frac{1}{r}}.$$

Thus, the skew-product map $F_\omega: \mathbb{T} \times \widehat{\mathbb{R}} \to \mathbb{T} \times \widehat{\mathbb{R}}$ defined by

$$F_\omega(x, r) = \Big(x + \omega, v(x) - \frac{1}{r}\Big) \tag{2.1}$$

is the projective action of the cocycle G_ω. Its inverse is given by

$$F_\omega^{-1}(x, r) = \Big(x - \omega, \frac{1}{v(x-\omega) - r}\Big).$$

If $\omega \in \mathbb{T}$ is fixed, then we use the following notation: if $x \in \mathbb{T}$, we let

$$x_n = x + n\omega, \quad n \in \mathbb{Z};$$

if $(x, r) \in \mathbb{T} \times \widehat{\mathbb{R}}$, then we let

$$r_n = \pi_2(F_\omega^n(x, r)), \quad n \geq 0; \tag{2.2}$$

and, if $(x, s) \in \mathbb{T} \times \widehat{\mathbb{R}}$, then we let

$$s_n = \pi_2(F_\omega^{-n}(x, s)), \quad n \geq 0. \tag{2.3}$$

With this notation, we have

$$F_\omega^n(x, r) = (x_n, r_n)$$

and

$$F_\omega^{-n}(x, s) = (x_{-n}, s_n), \quad n \geq 0.$$

Here (and in the rest of the paper),

$$\pi_1: \mathbb{T} \times \widehat{\mathbb{R}} \to \mathbb{T}, \quad \pi_1(x, r) = x,$$
$$\pi_2: \mathbb{T} \times \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}, \quad \pi_2(x, r) = r.$$

Remark 3. We shall always use this convention: given r (and x, ω), we assume that r_n are the forward iterates of r under F, as define above; and given s (and x, ω), we assume that s_n are the backward iterates of s under F (i.e., the forward iterates of s under F^{-1}), as above.

With this definition, we see that if $(x, r) \in \mathbb{T} \times \mathbb{R}$, then

$$A^n(x)\begin{pmatrix}1\\ r\end{pmatrix} = r_{n-1}\cdots r\begin{pmatrix}1\\ r_n\end{pmatrix}, \quad n \geq 1. \tag{2.4}$$

By recalling (1.2), we see that the above relation gives the following result.

Lemma 2.1. *If $\omega \in \mathbb{T}$ is irrational and if there is a measurable set $X \subset \mathbb{T}$, $|X| > 0$, such that there for each $x \in X$ exists $r \in \mathbb{R}$ such that*

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log |r_{n-1} \cdots r| > \gamma > 0,$$

then $L(\omega) > \gamma$.

Similarly, we have that if $(x, s) \in \mathbb{T} \times \mathbb{R}$, then

$$A^{-n}(x)\begin{pmatrix}1\\ s\end{pmatrix} = \frac{1}{s_n \cdots s_1}\begin{pmatrix}1\\ s_n\end{pmatrix}, \quad n \geq 1. \tag{2.5}$$

2.3. The cones $P_\pm$ and $\widehat{P}_\pm$

The following regions (cones) in $\widehat{\mathbb{R}}$ will be used frequently in the analysis.

Let

$$P_+ = \{r \in \widehat{\mathbb{R}} : |r| \geq \lambda^{3/4}\} \quad \text{and} \quad P_- = \{s \in \widehat{\mathbb{R}} : |s| \leq \lambda^{-3/4}\}.$$

We also let

$$\widehat{P}_+ = \{r \in \widehat{\mathbb{R}} : |r| \geq 2\lambda^{3/4}\} \quad \text{and} \quad \widehat{P}_- = \left\{s \in \widehat{\mathbb{R}} : |s| \leq \frac{1}{2}\lambda^{-3/4}\right\}.$$

By definition, we thus have

$$\widehat{P}_+ \subset P_+ \quad \text{and} \quad \widehat{P}_- \subset P_-.$$

2.4. The first entry times $T_\omega^\pm$

Given $x \in \mathbb{T}$ and $S \subset \mathbb{T}$, let

$$T_\omega^+(x, S) = \min\{m \geq 0 : x + m\omega \in S\}.$$

Here, we assume that $T_\omega^+(x, S) = \infty$ if $x + m\omega \notin S$ for all $m \geq 0$. Similarly, let

$$T_\omega^-(x, S) = \min\{m \geq 0 : x - m\omega \in S\},$$

where $T_\omega^-(x, S) = \infty$ if $x - m\omega \notin S$ for all $m \geq 0$.

2.5. Basic lemmas

Here, we collect some elementary facts about iterations of $F = F_\omega$. Recall the definition of the critical set $I_0 \subset \mathbb{T}$ in (1.7).

Lemma 2.2. *We have that*

(1) *if* $(x, r) \in (\mathbb{T} \setminus I_0) \times (\widehat{\mathbb{R}} \setminus P_-)$ *then* $r_1 \in \widehat{P}_+$*;*

(2) *if* $(x, s) \in (\mathbb{T} \setminus (I_0 + \omega)) \times (\widehat{\mathbb{R}} \setminus P_+)$ *then* $s_1 \in \widehat{P}_-$*.*

Proof. (1) Since, $x \notin I_0$ we have $|v(x)| > 3\lambda^{3/4}$, and $|r| > \lambda^{-3/4}$. Hence, $|r_1| = |v(x) - 1/r| > 3\lambda^{3/4} - \lambda^{3/4} = 2\lambda^{3/4}$.

(2) Since $x \notin I_0 + \omega$, we have $|v(x - \omega)| > 3\lambda^{3/4}$, and $|s| < \lambda^{3/4}$. Therefore, $|s_1| = 1/|v(x - \omega) - s| < 1/(3\lambda^{3/4} - \lambda^{3/4}) = (1/2)\lambda^{-3/4}$. ∎

We clearly have $\widehat{P}_+ \subset (\widehat{\mathbb{R}} \setminus P_-)$ and $\widehat{P}_- \subset (\widehat{\mathbb{R}} \setminus P_+)$. Thus, from the previous lemma, we immediately get the following result (which is the base case of Proposition 3.1 below).

Lemma 2.3. *Assume that* $\omega \in \mathbb{T}$ *is fixed.*

(1) *If* $(x, r) \in \mathbb{T} \times (\widehat{\mathbb{R}} \setminus P_-)$ *and* $N = T^+(x, I_0)$*, then* $r_k \in \widehat{P}_+$ *for all* $k = 1, \dots, N$*.*

(2) *If* $(x, s) \in \mathbb{T} \times (\widehat{\mathbb{R}} \setminus P_+)$ *and* $N = T^-(x, I_0 + \omega)$*, then* $s_k \in \widehat{P}_-$ *for all* $k = 1, \dots, N$*.*

2.6. Proof of Theorem 1 for $|E| > \lambda/2$

We can now prove the statement in Theorem 1 in the (trivial) case $|E| > \lambda/2$. The proof consists of two parts.

Lemma 2.4. *If* $|E| > \lambda/2$*, then* $I_0 = \emptyset$*.*

Proof. This follows immediately from the definition of I_0 in (1.7), recalling that $\|v_0\|_{C^0} \leq 1/3$. ∎

Lemma 2.5. *Assume that* $\omega \in \mathbb{T}$ *is irrational and that* E *is such that* $I_0 = \emptyset$*, and assume that* $x^* \in \mathbb{T}$*. Then, both Theorem* 1 (1) *and Theorem* 1 (2) *hold. Moreover,* $G_{\omega,E}$ *is uniformly hyperbolic.*

Proof. We have $I_0 = \emptyset$. Thus, from Lemma 2.3, we obtain that, for all $(x, r) \in \mathbb{T} \times P_+$, we get $|r_k| \geq 2\lambda^{3/4}$, for all $k \geq 1$; and, for all $(x, s) \in \mathbb{T} \times P_-$, we get $|s_k| \leq \lambda^{-3/4}/2$, for all $k \geq 1$.

Recalling Lemma 2.1, we see that $L(\omega, E) > 3(\log \lambda)/4$. And clearly (1.3) is satisfied, so G_ω is indeed uniformly hyperbolic.

It remains to verify condition (2). Although this is quite obvious, we include a proof, since we shall make a similar approach in the proof of the general case in Section 6.

For each $k \geq 1$, let

$$U_k = \pi_2(F^{-k}(x^* + k\omega, P_-)).$$

From the above estimates, we get $U_{k+1} \subset U_k$ for all $k \geq 1$; and, by Lemma A.2, we get that $|U_k| \to 0$ as $k \to \infty$. Thus, there is a unique $r^- \in P_-$ such that $(x^*, r^-) \in F^{-k}(x^* + k\omega, P_-)$ for all $k \geq 1$. We conclude that $r_k^- \in P_-$ for all $k \geq 1$, so $|r \cdots r_k| \leq \lambda^{-3/4(k+1)}$ for all $k \geq 1$. Let $W^- \subset \mathbb{R}^2$ be the line passing through the origin which corresponds to r^-. Recalling (2.4) gives one half of Theorem 1 (2).

Let $V_k = \pi_2(F^k(x^* - k\omega, P_+))$. Proceeding similarly (we have $V_{k+1} \subset V_k$ for all k), using Lemma A.1, we get a unique $r^+ \in P_+$ such that $(x^*, r^+) \in F^k(x^* - k\omega, P_+)$ for all $k \geq 1$. Let W^+ be the line corresponding to r^+. recalling (2.5) finishes the proof of Theorem 1 (2). ∎

2.7. A uniform lower bound for $|E| \leq \lambda/2$

If $I_0 = \emptyset$, then there is nothing to do, as we have seen in Lemma 2.5. From now on, we shall always assume that $I_0 \neq \emptyset$. In particular, we assume that

$$|E| \leq \frac{\lambda}{2}.$$

By this assumption on E, we have

$$|v(x)| \leq \frac{5\lambda}{6} \quad \text{for all } x \in \mathbb{T}.$$

We get the uniform bounds below.

Lemma 2.6. *For any $\omega \in \mathbb{T}$, $(x, r), (x, s) \in \mathbb{T} \times \widehat{\mathbb{R}}$ and $N \geq 1$ the following holds:*

(1) *if $|r_N| > 1/\lambda$, then $|r \cdot r_1 \cdots r_N| > 1/\lambda^{N+1}$;*

(2) *if $|s_N| < \lambda$, then $|s \cdot s_1 \cdots s_N| < \lambda^{N+1}$.*

Proof. To prove (1) we only need to note that if $|r_j| \leq 1/\lambda$, then one has $|r_j r_{j+1}| = |v(x_j) r_j - 1| \geq 1 - 5/6 = 1/6 \gg 1/\lambda^2$. The proof of (2) is similar. ∎

2.8. Classification of critical intervals

We introduce some notations and derive several estimates which will be used in later sections. The estimates are of elementary "calculus-type," but, since they are crucial for the proof of Theorem 1, we have included some details.

Recalling assumption (1.5), we let $\kappa > 0$ be such that

$$2\kappa = \min_{j=1,\ldots,d} \{|v_0''(c_j)|\} > 0.$$

For each j, there is an open interval $(c_j - \delta_j, c_j + \delta_j)$, $\delta_j > 0$, such that $|v_0''(x)| > \kappa$ for all $x \in U_j$. If $\ell_0 > 0$ is sufficiently small, then we have

$$\{x : |v_0'(x)| \le \ell_0\} \subset \bigcup_{j=1}^{d} \Big(c_j - \frac{\delta_j}{2}, c_j + \frac{\delta_j}{2}\Big). \tag{2.6}$$

The set I_0 defined in (1.7) can be written

$$I_0 = I_0(E, \lambda) = \Big\{x : \Big|v_0(x) - \frac{E}{\lambda}\Big| \le 3\lambda^{-1/4}\Big\}.$$

Since v_0 has d critical points, the set I_0 can consist of at most d intervals. Furthermore, we have the following.

Lemma 2.7. *Assume that I_0^j is a component in I_0.*

(1) *If $c_k \in I_0^j$ for some k, then $|v_0''(x)| > \kappa$ on I_0^j.*

(2) *We have $|I_0^j| < C\lambda^{-1/8}$, where $C = C(v_0)$.*

Proof. (1) Since $c_k \in I_0^j$, we have $|v_0(c_k) - E/\lambda| \le 3\lambda^{-1/4}$. Let $t^{\pm} = c_k \pm \lambda^{-1/9}$. Then, $[t^-, t^+] \subset (c_k - \delta_k, c_k + \delta_k)$, so we have $|v_0''(x)| > \kappa$ on $[t^-, t^+]$. This implies that $|v_0(t^{\pm}) - v_0(c_k)| \ge \kappa(t^{\pm} - c_k)^2/2 = \kappa\lambda^{-2/9}/2 \gg 6\lambda^{-1/4}$, so $t^{\pm} \notin I_0$. We conclude that $I_0^j \subset [t^-, t^+]$.

(2) If $I_0^j \cap (c_k - \delta_k/2, c_k + \delta_k/2) = \emptyset$ for each k, it follows from (2.6) that $|v_0'| > \ell_0$ on I_0^j, so $|I_0^j| < 6\lambda^{-1/4}/\ell_0$. If $I_0^j \cap (c_k - \delta_k/2, c_k + \delta_k/2) \neq \emptyset$, then we must have $I_0^j \subset (c_k - \delta_k, c_k + \delta_k)$ since $|v_0''| > \kappa$ on this interval. From this, it follows that $|I_0^j| < 2\sqrt{12\lambda^{-1/4}/\kappa} = C\lambda^{-1/8}$. ■

By the previous lemma, we can make the following classification of the components in I_0. Let I_0^j be an interval in I_0. We say that I_0^j is of type

(ii)$^+$ if $c_k \in I_0^j$ for some k and $v_0'' > \kappa$ on I_0^j;

(ii)$^-$ if $c_k \in I_0^j$ for some k and $v_0'' < -\kappa$ on I_0^j;

(i)$^+$ if $v_0' > 0$ on I_0^j;

(i)$^-$ if $v_0' < 0$ on I_0^j.

Lemma 2.8. *If I_0 contains k intervals of type* (ii)*, then I_0 consists of at most $d - k$ intervals.*

Proof. Note that between any two intervals in I_0 there has to be a critical point; and there is one critical point in each interval of type (ii). ■

Lemma 2.9. *If $E \in [\lambda \min v_0 + 4\lambda^{3/4}, \lambda \max v_0 - 4\lambda^{3/4}]$, then $I_0 = I_0(E)$ consists of at least one (in fact two) intervals of type* (i).

Proof. To get an easier notation, we assume that $v_0(c_1) = \min v_0$ and $v_0(c_k) = \max v_0$, $k \geq 2$; and we assume that the critical points in between (if there are any) are ordered as $c_1 < c_2 < c_3 < \cdots < c_k$. Since each critical point c_j is non-degenerate (assumption (1.5)), it follows that $v_0''(c_j)$ and $v_0''(c_{j+1})$ must have different signs for each $j = 1, 2, \ldots, k-1$; and we have $|v_0(c_j) - v_0(c_{j+1})| > 0$ for all $j = 1, \ldots, k-1$. Since $v_0''(c_1) > 0$ and $v_0''(c_k) < 0$, we see that k must be even. Moreover, v_0 is increasing on $[c_{2i-1}, c_{2i}] (i = 1, 2, \ldots, k/2)$ and decreasing on $[c_{2i}, c_{2i+1}] (i = 1, 2, \ldots, k/2 - 1)$. From this, it follows that $\bigcup_{i=1}^{k/2} (v_0(c_{2i-1}), v_0(c_{2i}))$ is a finite open cover of $(v_0(c_1), v_0(c_k)) = (\min v_0, \max v_0)$. Hence, there exists $\tau > 0$ such that, for any interval $J \subset (\min v_0, \max v_0)$ of length $< \tau$, there is an $i \in [1, k/2]$ such that $J \subset (v_0(c_{2i-1}), v_0(c_{2i}))$.

Assume now that $E \in [\lambda \min v_0 + 4\lambda^{3/4}, \lambda \min v_0 - 4\lambda^{3/4}]$. Then $J := [E/\lambda - 3\lambda^{-1/4}, E/\lambda + 3\lambda^{-1/4}] \subset (\min v_0, \max v_0)$ (and clearly $|J| < \tau$); so, there exists $i \in [1, k/2]$ such that $J \subset (v_0(c_{2i-1}), v_0(c_{2i}))$. By definition, we have $I_0 = v_0^{-1}(J)$. Thus, there is a component of I_0 in (c_{2i-1}, c_{2i}).

By doing the same analysis on the arc $[c_k, c_1]$, we would get one more component in I_0 of type (i). ■

Lemma 2.10. *If I_0^j is of type* (i)$^\pm$*, then $|v'(x)| > 2\lambda^{3/4}$ if $x \in I_0^j$ and $|v(x)| < 1$.*

Proof. We write $I_0^j = [a, b]$. We have seen in the proof of Lemma 2.7 that either we have $|v'| = |\lambda v_0'| > \lambda \ell_0$ on I_0^j, or $|v''| = |\lambda v_0''| > \lambda\kappa$ on I_0^j. In the first case, there is nothing to prove. We thus assume that $|v''| > \lambda\kappa$ on I_0^j. By assumption, we have $|v'| > 0$ on I_0^j. We assume that $v' > 0$ and $v'' > \lambda\kappa$; the other cases are treated similarly. By the definition of $I_0^j = [a, b]$ we must have $v(a) = -3\lambda^{3/4}$ and $v(b) = 3\lambda^{3/4}$.

If $x \in (a, b)$ is such that $|v(x)| < 1$, then $v(x) - v(a) > 3\lambda^{3/4} - 1 > \lambda^{3/4}$. Since $v'(a) \geq 0$ and $v'' > \lambda\kappa$, it is easy to verify that $\lambda\kappa(x - a) \leq v'(x)$, and $v'(t) \leq v'(x) - \lambda\kappa(x - t)$ for $t \in [a, x]$; so, $\lambda^{3/4} < v(x) - v(a) \leq v'(x)(x - a) - \lambda\kappa(x - a)^2/2 < v'(x)(x - a)$. Combining these two estimates yields $v'(x) > \sqrt{\lambda\kappa\lambda^{3/4}} > 2\lambda^{3/4}$. ■

2.9. Frequency assumptions

Recall the definition of $\omega_{\max}$ and the set Ω in (1.6). We shall now make the first frequency exclusion in Ω.

For simplicity, we shall focus only on "one half" of the set Ω, i.e., we shall only consider $0 < \omega < \omega_{\max}$. The "second half", $-\omega_{\max} < \omega < 0$, is treated in the same way. In some of the estimates, it slightly simplifies to know the "sign" of ω and not have to consider two cases.

We let

$$\Omega' = \{\omega : \lambda^{-1/10} \leq \omega \leq \omega_{\max} - \lambda^{-1/10}\} \subset \Omega. \tag{2.7}$$

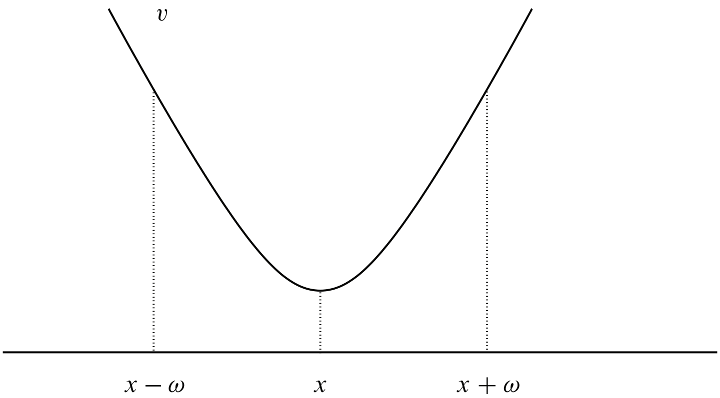

Figure 1. Different signs of $v'(x\pm\omega)$ for x close to a critical point.

The next lemma will be needed later (for example in Lemma 2.12) in the process of parameter exclusion in the cases when E is such that I_0 contains an interval of type (ii). It will guarantee that we can control how things move when ω changes.

Lemma 2.11. *If I_0^j is of type* (ii)$^\pm$ *and $x\in I_0^j$, $\omega\in\Omega'$, then $|v'(x\pm\omega)|>\lambda^{4/5}$, and $v'(x+\omega)$ and $v'(x-\omega)$ have different signs. Moreover, we have $|v(x\pm\omega)|>3\lambda^{3/4}$.*

Proof. We know that I_0^j contains a critical point, which we denote by c. Let a and b be the closest critical points to c such that $a<c<b$ (a and b can be the same point on $\mathbb{T}$). We have $|c-a|,|c-b|\geq\omega_{\max}$. Since $|v''(c)|>0$, we must have that v' has different sign on (a,c) and (c,b); see Figure 1.

From (2.6), we see that $|v_0'(t)|\geq\min\{\kappa|t-a|,\kappa|t-b|,\kappa|t-c|,\ell_0\}$ for all $t\in(a,b)$. Since $x\in I_0^j$, we know that $|x-c|<\lambda^{-1/9}$ (recall Lemma 2.7 (2)). Thus, if $\omega\in\Omega'$, we have $|x\pm\omega-c|,|x\pm\omega-a|,|x\pm\omega-b|>\lambda^{-1/10}-\lambda^{-1/9}$. We conclude that $|v'(x\pm\omega)|>\kappa\lambda^{9/10}/2>\lambda^{4/5}$.

The last statement follows since clearly $x\pm\omega\notin I_0$. ■

We now introduce the set $\widetilde{\Omega}$. Let

$$g(x,\omega)=v(x)-\frac{1}{v(x-\omega)}-\frac{1}{v(x+\omega)}$$

and let

$$\begin{aligned}\widetilde{\Omega}=\{\omega\in\Omega':|\min_{x\in I_0^j}g(x,\omega)|>2\lambda^{-7/5}\text{ for each }I_0^j\text{ of type (ii)}^+\}\\ \cap\{\omega\in\Omega':|\max_{x\in I_0^j}g(x,\omega)|>2\lambda^{-7/5}\text{ for each }I_0^j\text{ of type (ii)}^-\}.\end{aligned}\tag{2.8}$$

Remark 4. By introducing this set $\widetilde{\Omega}$, it will in fact be enough to keep control on the first derivatives (with respect to x and ω) of the iterates. It would be possible to work without making this parameter exclusion, but then one would need to keep control on the second derivatives of the iterates; and the parameter exclusion in Section 5 would be more involved (since the endpoints of the critical intervals defined in Section 3 could move fast with ω, like $\sqrt{\omega_1 - \omega}$ for $\omega < \omega_1$ close to ω_1). However, since we anyway will make larger parameter exclusions later, we make this simplification.

The assumption on $\widetilde{\Omega}$ will be used in the proof of Lemma 4.3, which is the base case for Proposition 4.4.

Lemma 2.12. *The set $\Omega' \setminus \widetilde{\Omega}$ is either $\emptyset$ or consists of at most $d/2$ intervals, each of length $< 2\lambda^{-1/5}$. Moreover, for each fixed $\omega \in \widetilde{\Omega}$, we have, writing $g(x) = g(x,\omega)$, $|g'(x)| > 2\lambda^{-1/4}$ if $|g(x)| \le \lambda^{-7/5}$ and $x \in I_0^j$ where I_0^j is of type* (ii).

Proof. There can be at most $d/2$ intervals of type (ii) in I_0.

We consider the case when $I_0^j := [a_0, b_0]$ is of type (ii)$^+$, and we denote by c_j the critical point of v in I_0^j. Let $\widetilde{\Omega}(j) = \{\omega \in \Omega' : |\min_{x\in I_0^j} g(x,\omega)| > 2\lambda^{-7/5}\}$, where $g(x,\omega)$ is as above. By the estimates in Lemma 2.11, together with the fact that $|v(t)| < 5\lambda/6$ for all $t \in \mathbb{T}$ and $v''(x) > \lambda\kappa$ on I_0^j, we have

$$\partial_\omega g(x,\omega) > 2\lambda^{4/5-2} \quad \text{and} \quad \partial^2_{xx} g(x,\omega) > \frac{\kappa\lambda}{2} \quad \text{for all } (x,\omega) \in I_0^j \times \Omega'. \tag{2.9}$$

If $v(c_j) \ge 1$, then $v(x) \ge 1$ on I_0^j; thus, it follows from the estimates in Lemma 2.11 that $g(x,\omega) > 1/2$ for all $x \in I_0^j$ and $\omega \in \Omega'$. Therefore, $\widetilde{\Omega}(j) = \Omega'$.

We assume that $v(c_j) < 1$. Since $v(a_0) = v(b_0) = 3\lambda^{3/4}$, and, thus, $g(a_0,\omega)$, $g(b_0,\omega) > \lambda^{3/4}$ for all $\omega \in \Omega'$, it follows from the second estimate in (2.9) that there is a unique $a(\omega) \in I_0^j$ such that $\partial_x g(a(\omega),\omega) = 0$ for $\omega \in \Omega'$. Letting $h(\omega) = g(a(\omega),\omega)$, we, get from the definition of $a(\omega)$ (the point in I_0^j where $g(\cdot,\omega)$ has its minimum) and the first estimate in (2.9), that $h'(\omega) > 2\lambda^{4/5-2}$. From this, it follows that $\Omega' \setminus \widetilde{\Omega}(j)$ either is empty, or is an interval of length $< 2\lambda^{-1/5}$ (since we need to remove the ω for which $|h(\omega)| \le 2\lambda^{-7/5}$).

We turn to the second statement. Assume that $\omega \in \widetilde{\Omega}(j)$ is fixed. If $\min_{x\in I_0^j} g(x) > 2\lambda^{-7/5}$, then there is nothing to prove. We therefore assume that $\min_{x\in I_0^j} g(x) < -2\lambda^{-7/5}$. We assume that the unique minimum of g in I_0^j is attained at $x = a$. Assume that $p \in I_0^j$ is such that $p > a$ and $g(p) \ge -\lambda^{-7/5}$ (the case then $p < a$ is treated similarly). Since the second statement in (2.9) holds, we have $g'(x) \le g'(p) - (\kappa\lambda/2)(p-x)$ on $[a,p]$. Thus, we can conclude

$$\lambda^{-7/5} < g(p) - g(a) \le g'(p)(p-a) - \left(\frac{\kappa\lambda}{4}\right)(p-a)^2 \le \frac{g'(p)^2}{\kappa\lambda}.$$

From this, the statement follows. ∎

3. Arithmetic

We recall that we have assumed that $\|v_0\|_{C^0} \leq 1/3$ and that $v(x) := \lambda v_0(x) - E$. We have also assumed that $I_0 \neq \emptyset$; in particular, we thus have $|E| \leq \lambda/2$ (recall Lemma 2.4). Note that $\|v\|_{C^0} \leq 5\lambda/6$.

By this assumption, we have the following trivial fact which we will use many times:

$$\text{if } x \in \mathbb{T} \text{ and } |r| \leq 1/\lambda, \text{ then } |r_1| > \frac{1}{\lambda} \tag{3.1}$$

and

$$\text{if } x, \omega \in \mathbb{T} \text{ and } |s| \geq \lambda, \text{ then } |s_1| < \lambda.$$

Indeed, we have $|r_1| = |v(x) - 1/r| \geq \lambda/6 \gg 1/\lambda$; and $|s_1| = |1/(v(x-\omega) - s)| \leq 6/\lambda \ll \lambda$.

3.1. Basic definitions

We shall now define some of the central objects for the analysis. This includes the sequence of integers (K_n), the "critical sets" (I_n), and the "good starting locations" $(X_n^{\pm})$. Let

$$K_0 = [\lambda^{1/20}] \quad \text{and} \quad K_{n+1} = [\lambda^{K_n/20}] \quad \text{for } n \geq 1, \tag{3.2}$$

where $[\cdot]$ denotes the integer part. Note that the sequence (K_n) grows very fast (since λ is large). In particular, we have

$$K_n^{1/2} \gg 12^{n+2} \quad \text{for all } n \geq 0 \tag{3.3}$$

provided that $\lambda > 0$ is large enough. This fact will be used below.

Recall that the set I_0 is defined in (1.7). Given $\omega \in \mathbb{T}$ and a sequence $0 < M_0 < M_1 < \cdots$ of integers, we inductively define

$$I_{n+1} = \pi_1(F_\omega^{M_n}(A_n) \cap F_\omega^{-M_n}(B_n)), \quad n \geq 0, \tag{3.4}$$

where

$$A_n = (I_n - M_n\omega) \times P_+ \quad \text{and} \quad B_n = (I_n + M_n\omega) \times P_-. \tag{3.5}$$

Thus, $I_{n+1} = I_{n+1}(M_0, \ldots, M_n, \omega)$. (Of course, it also depends on E, but we have assumed that E is fixed.)

By definition, we have

$$I_0 \supset I_1 \supset I_2 \supset \cdots.$$

Moreover, for any $n \geq 0$, we have

$$(x, r) \in A_n \text{ and } x \notin I_{n+1} - M_n\omega \implies r_{2M_n} \notin P_-. \tag{3.6}$$

and

$$(x,s) \in B_n \text{ and } x \notin I_{n+1} + M_n\omega \implies s_{2M_n} \notin P_+.$$

Remark 5. In our application, we will have $M_n \approx K_n$. It would have been easier if we could let $M_n = K_n$, but in fact we will have to vary them slightly. But we can, an will, always take $M_0 = K_0$.

We also let $X_{-1}^{\pm} = \mathbb{T}$ and

$$X_n^+ = \mathbb{T} \setminus \bigcup_{j=0}^{n} \bigcup_{m=1}^{[K_j^{3/2}]} (I_j + m\omega) \quad \text{and} \quad X_n^- = \mathbb{T} \setminus \bigcup_{j=0}^{n} \bigcup_{m=0}^{[K_j^{3/2}]} (I_j - m\omega), \quad n \geq 0. \tag{3.7}$$

Note that, if $x \in X_n^-$, then $T_\omega^+(x, I_j) > [K_j^{3/2}]$ for $j \in [0,n]$; and if $x \in X_n^+$, then $T_\omega^-(x, I_j + \omega) \geq [K_j^{3/2}]$ for $j \in [0,n]$.

Furthermore, for each $n \geq 0$, we let

$$\widehat{X}_n = \mathbb{T} \setminus \bigcup_{j=0}^{n} \bigcup_{|m| \leq K_j^2} (I_j + m\omega). \tag{3.8}$$

Clearly, we have $\widehat{X}_n \subset X_n^{\pm}$. These sets $\widehat{X}_n$ will first be used in Section 5 (and we need them for proving that Theorem 1 (2) hold).

3.2. Arithmetic conditions

The following two conditions will be central for the analysis. The first condition is a non-resonance condition (stronger than a Diophantine condition, since the set I_n need not be a single interval):

$$I_n \cap \bigcup_{m=1}^{K_n^2} (I_n + m\omega) = \emptyset. \tag{NR$_n$}$$

It is easy to check that (NR)$_n$ is equivalent with

$$I_n \cap \bigcup_{1 \leq |m| \leq K_n^2} (I_n + m\omega) = \emptyset.$$

The second one is a condition on position:

$$I_n \pm M_n\omega, I_n \pm (M_n + 1)\omega \subset X_{n-1}^+ \cap X_{n-1}^-. \tag{Pos$_n$}$$

Note that (Pos)$_0$ trivially holds. (It is for this condition to hold, that we need to vary M_n, instead of fixing it to be K_n, when $n \geq 1$.)

3.3. Key arithmetic result

We are now ready to formulate and prove the main inductive step in the arithmetic part of the proof of Theorem 1. Recall the definitions of the cones $P_\pm$, $\widehat{P}_\pm$ in Section 2.3. Note that we clearly have $P_\pm \subset \widehat{\mathbb{R}} \setminus P_\mp$.

The following proposition consists of two parts, $(\mathrm{S})_n$ and $(\mathrm{R})_n$. The fist one concerns forward iterations of F; the second one concerns backward iterations. Recall the convention of the notation of the iterates of F in Remark 3. (In particular, recall the notation $F_\omega^{-n}(x,s) = (x_{-n}, s_n)$ for $n \geq 0$.)

Proposition 3.1. *For all $n \geq 0$, we have the following. Assume that $\omega \in \mathbb{T}$ and $M_k \in [K_k, 2K_k]$ $(k = 0, 1, \ldots, n-1)$ are fixed. Assume further that the following statements holds.*

$(\mathrm{S})_n$. *For each $(x,r) \in X_{n-1}^- \times (\widehat{\mathbb{R}} \setminus P_-)$ and $N = T^+(x, I_n)$, we have the followig:*

- $(1)_n$ *if $r_k \notin \widehat{P}_+$ for some $1 \leq k \leq N$, then $x_k \in \mathbb{T} \setminus X_{n-1}^+$;*
- $(2)_n$ $|r_1 \cdots r_k| \geq \lambda^{k(2/3+1/12^{n+1})}$ *for all $1 \leq k \leq N$ such that $|r_k| > 1/\lambda$.*

If $N < \infty$, then we also have the following:

- $(3)_n$ $|r_k \cdots r_{N-1}| \geq \lambda^{(N-k)(2/3+1/12^{n+1})}$ *for all $1 \leq k \leq N-1$.*

$(\mathrm{R})_n$. *For each $(x,s) \in X_{n-1}^+ \times (\widehat{\mathbb{R}} \setminus P_+)$ and $N = T^-(x, I_n + \omega)$ we have the following:*

- $(1)_n$ *if $s_k \notin \widehat{P}_-$ for some $1 \leq k \leq N$, then $x_{-k} \in \mathbb{T} \setminus X_{n-1}^-$;*
- $(2)_n$ $|s_1 \cdots s_k| \leq \lambda^{-k(2/3+1/12^{n+1})}$ *for all $1 \leq k \leq N$ such that $|s_k| < \lambda$.*

If $N < \infty$, then we also have the following:

- $(3)_n$ $|s_k \cdots s_{N-1}| \leq \lambda^{-(N-k)(2/3+1/12^{n+1})}$ *for all $1 \leq k \leq N-1$.*

Then, if $M_n \in [K_n, 2K_n]$ and if $(\mathrm{NR})_n$ and $(\mathrm{Pos})_n$ hold, we have that $(S,R)_{n+1}$ hold.

Proof. We assume that $M_n \in [K_n, 2K_n]$ and that $\omega \in \mathbb{T}$ is such that $(\mathrm{NR})_n$ and $(\mathrm{Pos})_n$ hold. Furthermore, we assume that $(\mathrm{S})_n$ and $(\mathrm{R})_n$ hold. We shall verify that $(\mathrm{S})_{n+1}$ hold; the case of $(\mathrm{R})_{n+1}$ is analogous.

Take $(\xi, \rho) \in X_n^- \times (\widehat{\mathbb{R}} \setminus P_-)$ and let $N = T^+(\xi, I_{n+1})$. We shall assume that $N < \infty$. (The case $N = \infty$ is handled in the same way. In that case, only $(1-2)_{n+1}$ need to be verified.) We use the notation

$$(\xi_k, \rho_k) = F_\omega^k(\xi, \rho).$$

Let $0 < k_1 < k_2 < \cdots < k_\ell = N$ be the indices $k \in [0, N]$ such that $x_k \in I_n$. Since $\xi \in X_n^-$ and since $(\mathrm{NR})_n$ holds, we have

$$k_1 > [K_n^{3/2}] \quad \text{and} \quad k_{i+1} - k_i > K_n^2, \quad i \geq 1. \tag{3.9}$$

If $\ell = 1$, the weaker statements $(1-4)_{n+1}$ follow from $(1-4)_n$. We therefore assume that $\ell \geq 2$.

We first prove that

$$\rho_{k_i+M_n+1} \in \widehat{P}_+ \quad \text{for all } 1 \leq i \leq \ell - 1. \tag{3.10}$$

Then, since $\xi_{k_i+M_n+1} \in I_n + (M_n + 1)\omega$, it follows from $(\mathrm{Pos})_n$ that

$$(\xi_{k_i+M_n+1}, \rho_{k_i+M_n+1}) \in X_{n-1}^- \times \widehat{P}_+ \quad \text{for all } 1 \leq i \leq \ell - 1. \tag{3.11}$$

To show that (3.10) holds, we first note that, by applying $(\mathrm{S})_n$ to the point (ξ, ρ), we get from $(1)_n$, combined with $(\mathrm{Pos})_n$, that $\rho_{k_1-M_n} \in \widehat{P}_+$. Thus, $(\xi_{k_1-M_n}, \rho_{k_1-M_n}) \in A_n$. Since $\xi_{k_1-M_n} \notin I_{n+1} - M_n\omega$, we get from (3.6) that $\rho_{k_1+M_n} \notin P_-$; and since $\xi_{k_1+M_n} \in I_n + M_n\omega$, it follows from that $(\mathrm{Pos})_n$ that $\xi_{k_1+M_n} \notin I_0$. It hence follows from Lemma 2.2 that $\rho_{k_1+M_n+1} \in \widehat{P}_+$. If $\ell = 2$, we are done. If $\ell > 2$, we note that $\xi_{k_1+M_n+1} \in X_{n-1}^-$ by $(\mathrm{Pos})_n$; thus, we can apply $(\mathrm{S})_n$ to $(\xi_{k_1+M_n+1}, \rho_{k_1+M_n+1}) \in X_{n-1}^- \times \widehat{P}_+$ and proceed inductively.

Verifying $(1)_{n+1}$. As we noted above, by applying $(\mathrm{S})_n$ to the point (ξ, ρ), we get from $(1)_n$ that

$$\text{if } \rho_k \notin \widehat{P}_+ \text{ for some } 1 \leq k \leq k_1, \text{ then } \xi_k \in \mathbb{T} \setminus X_{n-1}^+.$$

Furthermore, from (3.11) we have $(\xi_{k_i+M_n+1}, \rho_{k_i+M_n+1}) \in X_{n-1}^- \times \widehat{P}_+$ for $1 \leq i \leq \ell - 1$. By applying $(\mathrm{S})_n$ to each of these points, we get from condition $(1)_n$ that, for each $1 \leq i \leq \ell - 1$, we have that

$$\text{if } \rho_k \notin \widehat{P}_+ \text{ for some } k_i + M_n + 1 \leq k \leq k_{i+1}, \text{ then } \xi_k \in \mathbb{T} \setminus X_{n-1}^+.$$

Thus, we have control on the intervals $[1, k_1], [k_1 + M_n + 1, k_2], \ldots, [k_{l-1} + M_n + 1, k_l]$. But note that, for each $k \in [k_i + 1, k_i + M_n]$, $1 \leq i \leq \ell - 1$, we have $\xi_k \in \bigcup_{m=1}^{M_n} (I_n + M_n\omega) \subset \mathbb{T} \setminus X_n^+$. Thus, we conclude that $(1)_{n+1}$ holds.

Verifying $(2)_{n+1}$. Again, we apply $(\mathrm{S})_n$ to the points (ξ, ρ), $(\xi_{k_i+M_n+1}, \rho_{k_i+M_n+1})$, $i = 1, \ldots, \ell - 1$ (recalling (3.11)). We then get from $(2)_n$ that

$$|\rho_1 \cdots \rho_k| \geq \lambda^{k(2/3+1/12^{n+1})} \quad \text{for all } k \in [1, k_1] \text{ such that } |\rho_k| > \frac{1}{\lambda}, \tag{3.12}$$

and, for each $1 \leq i \leq \ell - 1$, we have

$$|\rho_{k_i+M_n+2} \cdots \rho_k| \geq \lambda^{(k-(k_i+M_n+1))(2/3+1/12^{n+1})} \tag{3.13}$$

for all $k \in [k_i + M_n + 2, k_{i+1}]$ such that $|\rho_k| > 1/\lambda$.

We shall first prove that

$$|\rho_1 \cdots \rho_k| \geq \lambda^{k(2/3+1/12^{n+2})} \tag{3.14}$$

for all $k \in [1, k_1 + M_n + 1]$ such that $|\rho_k| > 1/\lambda$. For $k \in [1, k_1]$, this is just a weaker version of (3.13). We therefore assume that $k \in [k_1 + 1, k_1 + M_n + 1]$ and that $|\rho_k| > 1/\lambda$. By Lemma 2.6, we have $|\rho_{k_1} \cdots \rho_k| > \lambda^{-(k-k_1+1)}$. Combining this with (3.12) (with $k = k_1 - 1$; recall that $\max\{|\rho_{k_1-1}|, |\rho_{k_1}|\} > 1/\lambda$; we also know from $(3)_n$ that $\rho_{k_1-1} > \lambda^{2/3}$), we see that (3.14) follows if we have

$$(k_1 - 1)\Big(\frac{2}{3} + \frac{1}{12^{n+1}}\Big) - (k - k_1 + 1) > k\Big(\frac{2}{3} + \frac{1}{12^{n+2}}\Big).$$

It is easy to check that this is satisfied if $k_1 > 12^{n+2} K_n$ (consider the worst case $k = k_1 + M_n + 1$, and recall that $M_n \leq 2K_n$). By recalling (3.9) and (3.3), we see that this indeed is satisfied.

In the same way, for $1 \leq i \leq \ell - 2$, we use (3.13) to prove

$$|\rho_{k_i+M_n+2} \cdots \rho_k| \geq \lambda^{(k-(k_i+M_n+1))(2/3+1/12^{n+2})}$$

for all $k \in [k_i + M_n + 2, k_{i+1} + M_n + 1]$ such that $|\rho_k| > 1/\lambda$. (In this situation, it is even better, since $k_{i+1} - k_i > K_n^2$; recall (3.9)). Combing these estimates (recalling that $|r_{M_n+k_i+1}| > 1/\lambda$ for all $1 \leq i \leq \ell - 1$, by (3.11)) finishes the proof.

Verifying $(3)_{n+1}$ is similar to what we just did. ■

The base case for the previous proposition is the following.

Lemma 3.2. *Conditions* $(\mathrm{S})_0$ *and* $(\mathrm{R})_0$ *hold for all* $\omega \in \mathbb{T}$.

Proof. This follows immediate from Lemma 2.3. ■

3.4. Some remarks on the inductive scheme

From Proposition 3.1, combined with Lemma 3.2, we can make the following conclusion. Suppose that $\omega \in \mathbb{T}$ is irrational and the sequence (M_n) – where $M_n \in [K_n, 2K_n]$, for all $n \geq 0$, are such that $X_\infty := \bigcap_{n\geq 0}(X_n^+ \cap X_n^-)$ – has positive measure and such that $(\mathrm{NR})_n$ and $(\mathrm{Pos})_n$ hold for all $n \geq 0$. Then, by induction, we have that $(\mathrm{S})_n$ holds for all $n \geq 0$. For every $x \in X_\infty$ and $r \in P_+$, we have that, in particular, $(x, r) \in X_{n-1}^- \times P_+$, $n \geq 0$; and, by definition of the set X_∞, we have that $T^+(x, I_n) > M_n^{3/2}$, $n \geq 0$. Since $(\mathrm{S})_n(2)_n$ holds for all $n \geq 0$, we conclude that

$$\overline{\lim_{k\to\infty}} \frac{1}{k} \log |r_k \cdots r_1| \geq \frac{2}{3} \log \lambda.$$

(Recall that if $|r_k| < 1/\lambda$ for some k, then $|r_{k+1}| \gg 1/\lambda$.) Thus, from Lemma 2.1, we get that $L(\omega) \geq (2/3)\log\lambda$.

Another consequence is the following. We have seen (Lemma 2.5) that if $I_0 = \emptyset$, then the cocycle G_ω is uniformly hyperbolic for each irrational ω. In fact, this also holds if some $I_n = \emptyset$ $(n \geq 1)$. More precisely, assume that there is an irrational number $\omega \in \mathbb{T}$, an $n \geq 1$, and integers $M_j \in [K_j, 2K_j]$, $j = 0, 1, \ldots, n-1$, such that $I_n = \emptyset$, the set $X := (X^+_{n-1} \cap X^-_{n-1})$ contains an interval J of positive length, and such that $(\mathrm{NR})_j$, $(\mathrm{Pos})_j$ hold for each $j \in [0, n-1]$. Then, G_ω is uniformly hyperbolic. Indeed, to see this, we do as follows. As we did above, we note that $(\mathrm{S})_n$ holds. Moreover, since ω is irrational and J is an (non-degenerate) interval, there exists a number $m > 0$ such that, for any $x \in \mathbb{T}$, there is an $0 \leq j \leq m$ such that $x + j\omega \in J$. We now take any $x \in \mathbb{T}$, and we take an $r \in \mathbb{R}$ such that $\lambda \leq r_j < \infty$ (thus $r_j \in P_+$), where $0 \leq j \leq m$ is such that $x_j \in J \subset X^-_{n-1}$. By Lemma 2.6, we have

$$|r \cdots r_j| \geq \lambda^{-(j+1)} \geq \lambda^{-(m+1)} \quad \text{and} \quad |r_1 \cdots r_j| \geq \lambda^{-j} \geq \lambda^{-m};$$

hence

$$\frac{|r|}{\sqrt{1+r^2}}|r_1 \cdots r_j| \geq \frac{\lambda^{-(m+1)}}{\sqrt{2}}$$

(consider the two cases $|r| < 1$ and $|r| \geq 1$). And, by applying $(\mathrm{S})_n$ to the point (x_j, r_j), recalling that $T^+(x_j, I_n) = \infty$, since $I_n = \emptyset$, we get from $(\mathrm{S})_n(2)_n$ that

$$\max\{|r_{j+1} \cdots r_k|, |r_{j+1} \cdots r_{k+1}|\} \geq \lambda^{(2/3)(k-j)} \quad \text{for all } k \geq j+1.$$

(again using that if $|r_k| < 1/\lambda$ for some k, then $|r_{k+1}| \gg 1/\lambda$). Recalling (2.4), we thus get

$$\begin{aligned}\|A^{k+1}\| &\geq \left|\frac{1}{\sqrt{1+r^2}}A^{k+1}(x)\begin{pmatrix}1\\ r\end{pmatrix}\right| \geq \max\{|r \cdots r_k|, |r \cdots r_{k+1}|\} \\ &\geq \frac{\lambda^{-(m+1)}}{\sqrt{2}}\lambda^{(2/3)(k-m)} \quad \text{for all } k \geq m+1.\end{aligned}$$

Thus, we have a uniform growth, and, therefore, by (1.3), we can conclude that the cocycle G_ω is uniformly hyperbolic.

3.5. An important inclusion

We are now ready to derive an important inclusion property which will be needed to control the geometry in the next section. Recall that $\widehat{P}_+$ is strictly included in P_+ and that $\widehat{P}_-$ is strictly included in P_-.

Lemma 3.3. *The following holds for all $n \geq 0$. Assume that $\omega \in \mathbb{T}$ and that $M_k \in [K_k, 2K_k]$, $k = 0, 1, \ldots, n+1$, are such that* $(\mathrm{NR})_i$, $(\mathrm{Pos})_i$, $(\mathrm{S})_i(1)$, *and* $(\mathrm{R})_i(1)$ *hold for $i \in \{n, n+1\}$. Then,*

$$F^{M_{n+1}-M_n}(A_{n+1}) \subset (I_n - M_n\omega) \times \widehat{P}_+ \subset (I_n - M_n\omega) \times P_+ = A_n$$

and

$$F^{-M_{n+1}+M_n}(B_{n+1}) \subset (I_n + M_n\omega) \times \widehat{P}_- \subset (I_n + M_n\omega) \times P_- = B_n.$$

Proof. We prove the first statement. The proof of the second one is similar.

Take $(x, r) \in A_{n+1}$, i.e., $x \in I_{n+1} - M_{n+1}\omega$ and $r \in P_+$. Then, $x \in X_n^+ \cap X_n^-$ by $(\mathrm{Pos})_{n+1}$, and $T(x, I_{n+1}) = M_{n+1}$ by $(\mathrm{NR})_{n+1}$. Hence, by $(\mathrm{S})_{n+1}(1)$, we get that $r_k \in \widehat{P}_+$ for all $k \in [1, M_{n+1}]$ such that $x_k \in X_n^+$. If we can show that $I_n - M_n\omega \subset X_n^+$, we are thus done (since $x_{M_{n+1}-M_n} \in I_{n+1} - M_n\omega \subset I_n - M_n\omega$). To see that this indeed holds, we first note that, by $(\mathrm{Pos})_n$, we have $[]I_n - M_n\omega \subset X_{n-1}^+ \cap X_{n-1}^-$. Secondly, by $(\mathrm{NR})_n$, we have $(I_n - M_n\omega) \cap \bigcup_{m=1}^{[K_n^{3/2}]}(I_n + m\omega) = \emptyset$ (since clearly $M_n, K_n^{3/2} \ll K_n^2$). By combining these two facts, and recalling the definition of X_n^+, we see that indeed $I_n - M_n\omega \subset X_n^+$. ■

4. Geometry

Recall the definition of the sets A_n, B_n in (3.5), and the definition of the critical set I_{n+1} in (3.4). To control the critical set I_{n+1}, we need to control the geometry of the set $F_\omega^{M_n}(A_n) \cap F_\omega^{-M_n}(B_n)$. In fact, for convenience, we will focus on the set $F_\omega^{M_n+1}(A_n) \cap F_\omega^{-M_n+1}(B_n)$.

In the proofs of the statements, in this section we will apply some results from Appendix A.

Given $n \geq 0$ and an $M_n > 0$, we define

$$\varphi_n^\pm(x, \omega) = \pi_2(F_\omega^{M_n+1}(x - M_n\omega, \mp\lambda^{3/4}))$$

and

$$\psi_n^\pm(x, \omega) = \pi_2(F_\omega^{-M_n+1}(x + M_n\omega, \pm\lambda^{-3/4})).$$

(The reason for why there is an $\mp$ in the definition of $\varphi_n^\pm$ is due to orientation. Recall that $P_+ = \{r \in \widehat{\mathbb{R}} : |r| \geq \lambda^{3/4}\}$ and note that $\pi_2(F_\omega(x, \lambda^{3/4}) = v(x) - 1/\lambda^{3/4} < v(x) + 1/\lambda^{3/4} = \pi_2(F_\omega(x, -\lambda^{3/4}))$.

If ω is fixed, we also write $\varphi_n^\pm(x) = \varphi_n^\pm(x, \omega)$ and $\psi_n^\pm(x) = \psi_n^\pm(x, \omega)$.

In the next lemma, we connect the arithmetic information from Proposition 3.1 with the geometry of $\varphi_n^\pm, \psi_n^\pm$.

Lemma 4.1. *For each $n \geq 0$, the following holds. Assume that ω and $M_k \in [K_k, 2K_k]$ ($k = 0, \dots, n$) are fixed and that* (NR)$_0$, (NR)$_n$, (Pos)$_n$, *as well as* (S)$_n$, (R)$_n$ *in Proposition 3.1, hold. Then,*

$$\varphi_n^{\pm}(x, \omega) = v(x) - \frac{1}{v(x-\omega) + e_n^{\pm}(x, \omega)}$$

and

$$\psi_n^{\pm}(x, \omega) = \frac{1}{v(x+\omega) - w_n^{\pm}(x, \omega)},$$

where, for all $x \in I_n$, we have the bounds

$$|e_n^{\pm}(x, \omega)|, |w_n^{\pm}(x, \omega)| < \lambda^{-1/2}; \tag{4.1}$$

$$0 < e_n^{+}(x, \omega) - e_n^{-}(x, \omega) < \lambda^{-4(M_n-1)/3}, \tag{4.2a}$$

$$0 < w_n^{+}(x, \omega) - w_n^{-}(x, \omega) < \lambda^{-4(M_n-2)/3}; \tag{4.2b}$$

$$|\partial_\alpha e_n^{\pm}(x, \omega)|, |\partial_\alpha w_n^{\pm}(x, \omega)| < \lambda^{-1/4} \quad \textit{for } \alpha = x, \omega. \tag{4.3}$$

Furthermore, we have

$$F^{M_n+1}(A_n) = \{(x+\omega, r) : x \in I_n, \varphi_n^{-}(x) \leq r \leq \varphi_n^{+}(x)\} \tag{4.4a}$$

$$F^{-M_n+1}(B_n) = \{(x+\omega, r) : x \in I_n, \psi_n^{-}(x) \leq r \leq \psi_n^{+}(x)\}. \tag{4.4b}$$

Proof. We assume that $I_n \neq \emptyset$, since otherwise there is nothing to prove.

Let

$$r_k^{\pm}(x, \omega) = \pi_2(F_\omega^k(x - M_n\omega, \mp\lambda^{3/4})).$$

Then, by definition, we have $\varphi_n^{\pm}(x, \omega) = r_{M_n+1}^{\pm}(x, \omega)$. Moreover, we can write

$$r_{M_n+1}^{\pm}(x, \omega) = v(x) - \frac{1}{v(x-\omega) - \frac{1}{r_{M_n-1}^{\pm}(x,\omega)}}.$$

Thus, we let

$$e_n^{\pm}(x, \omega) = -\frac{1}{r_{M_n-1}^{\pm}(x, \omega)}. \tag{4.5}$$

Since (NR)$_n$ holds, we have $T^+(x - M_n\omega, I_n) = M_n$ for $x \in I_n$. Moreover, since also (Pos)$_n$ holds, and $r_0^{\pm} = \mp\lambda^{3/4} \in P_+$, we can apply (S)$_n$ (to each point $(x - M_n\omega, r_0^{\pm})$, $x \in I_n$) to conclude that the estimates $S_n(1-3)$ hold for the $r_j(x, \omega)$, $x \in I_n$, $j = 1, \dots, N$, with $N = M_n$.

By recalling the definition of the set A_n in (3.5), we see that the first statement in (4.4) follows from Lemma A.1.

From $S_n(2)$ (with $k = N - 1 = M_n - 1$), we get $|r_{M_n-1}^{\pm}(x)| \geq \lambda^{2/3}$ for $x \in I_n$. Thus, the first part in (4.1) follows from the definition (4.5).

From Lemma A.1, we get

$$0 < r^+_{M_n-1} - r^-_{M_n-1} \le 2\lambda^{-4(M_n-2)/3-3/4}.$$

Since $|r^\pm_{M_n-1}(x)| \ge \lambda^{2/3}$, we conclude that $r^\pm_{M_n-1}$ must have the same sign. Thus, it follows that

$$0 < e^+_n - e^-_n = \frac{r^+_{M_n-1} - r^-_{M_n-1}}{r^+_{M_n-1} r^-_{M_n-1}} \le 2\lambda^{-4(M_n-1)/3-3/4} < \lambda^{-4(M_n-1)/3},$$

which is the first statement in (4.2).

Finally, (4.3) follows from Lemma A.3.

The estimates of $\psi^\pm_n$ and $w^\pm_n$ follow similarly by letting

$$s^\pm_k(x,\omega) = \pi_2(F^{-k}_\omega(x + M_n\omega, \pm\lambda^{-3/4})) \quad \text{and} \quad w^\pm_n(x,\omega) = -s_{M_n-2}(x,\omega)$$

and using the estimates from $(\mathrm{R})_n$, combined with Lemma A.2 and Lemma A.4. ∎

Before formulating the main geometric inductive step, we will make a certain classification. Recall that, by definition (3.4), the set $I_1 = I_1(M_0)$ depends on M_0 (and ω). Below, we use $M_0 = K_0$ and write $I_1(K_0)$ to emphasis this choice.

Let

$$\widetilde{\Omega}_0 = \{\omega \in \widetilde{\Omega} : (\mathrm{NR})_0 \text{ is satisfied}\}. \tag{4.6}$$

Lemma 4.2. *Assume that I^j_0 is of type* (ii) *and that $\widehat{\Omega}_0$ is a component in $\widetilde{\Omega}_0$. Then, either $I^j_0 \cap I_1(K_0) = \emptyset$ for all $\omega \in \widetilde{\Omega}_0$, or $I^j_0 \cap I_1(K_0) \ne \emptyset$ for all $\omega \in \widetilde{\Omega}_0$.*

The proof of this lemma is included in the proof of Lemma 4.3 below.

We now use the previous lemma to classify the components in $\widetilde{\Omega}_0$ with respect to intervals in I_0 of type (ii). If I^j_0 is of type (ii), we say that an interval $\widehat{\Omega} \subset \widetilde{\Omega}_0$ is *relevant for* I^j_0 if $I^j_0 \cap I_1(K_0) \ne \emptyset$ for all $\omega \in \widehat{\Omega}$.

Remark 6. By Lemma 4.2, we thus have that if I^j_0 is of type (ii) and if an interval $\widehat{\Omega} \subset \widetilde{\Omega}_0$ is not revlevant for I^j_0, then $I^j_0 \cap I_1(K_0) = \emptyset$ for all $\omega \in \widehat{\Omega}$.

For an interval $\widehat{\Omega} \subset \widetilde{\Omega}_0$, we let

$$\begin{aligned} d_0 = d_0(\widehat{\Omega}) &= \#(\text{components of type (i) in } I_0) \\ &\quad + 2 \cdot \#(\text{components of type (ii) in } I_0 \text{ which are relevant for } \widehat{\Omega}) \le d. \end{aligned} \tag{4.7}$$

Note that the last inequality follows from Lemma 2.8. The number d_0 will in fact be the number of components in the critical sets I_n $(n \ge 1)$ for "good" ω in relevant sets of frequencies.

The next lemma shows that the base case for Proposition 4.4, which is the main geometric inductive step, hold under a condition on ω.

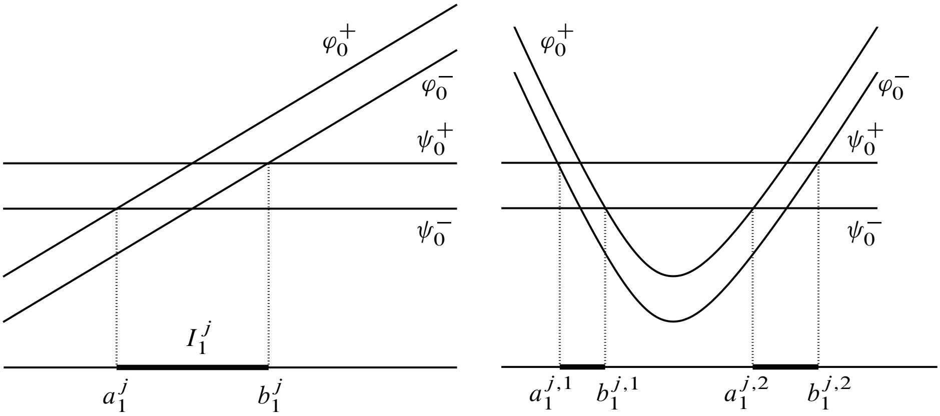

Figure 2. The intersections.

Lemma 4.3. *Let $M_0 = K_0$ and assume that $\widehat{\Omega}_0 \subset \widetilde{\Omega}_0$ is an interval. Then condition* $(\mathrm{Geo})_0$ *in proposition* (4.4) *holds.*

Proof. Fix any $\omega \in \widehat{\Omega}_0$. We recall that $(\mathrm{Pos})_0$, defined in Section 3.2, trivially holds; and, from Lemma 3.2, we know that conditions $(\mathrm{S})_0$ and $(\mathrm{R})_0$ hold. Thus, we can apply Lemma 4.1 (with $n = 0$) to get

$$\varphi_0^\pm(x) = v(x) - \frac{1}{v(x-\omega) + e_0^\pm(x,\omega)}$$

and

$$\psi_0^\pm(x) = \frac{1}{v(x+\omega) - w_0^\pm(x,\omega)},$$

where $e_0^\pm, w_0^\pm$ satisfy the estimates $(4.1)_0$, $(4.2)_0$, and $(4.3)_0$ on I_0, and $(4.4)_0$ holds. Moreover, from $(\mathrm{NR})_0$, we have $(I_0 \pm \omega) \cap I_0 = \emptyset$, so $|v(x \pm \omega)| > 3\lambda^{3/4}$ for $x \in I_0$.

As usual, we let

$$g(x) = g(x,\omega) = v(x) - \frac{1}{v(x-\omega)} - \frac{1}{v(x+\omega)}.$$

We also let

$$\theta^+(x) = \varphi_0^+(x) - \psi_0^-(x) \quad \text{and} \quad \theta^-(x) = \varphi_0^-(x) - \psi_0^+(x).$$

From the above estimates, we get that, for all $x \in I_0$, there holds

$$\begin{gathered} \varphi_0^+(x) - \varphi_0^-(x) > 0 \quad \text{and} \quad \psi_0^+(x) - \psi_0^-(x) > 0, \\ |g(x) - \theta^\pm(x)| < \lambda^{-2}, \end{gathered} \tag{4.8}$$

$$|g'(x) - (\theta^{\pm})'(x)| < \lambda^{-1/3}, \tag{4.9}$$
$$|g(x) - v(x)| < \lambda^{-2}, \quad |g'(x) - v'(x)| < \lambda^{-1/3}.$$

Since $(4.4)_0$ holds, we have

$$I_1 \cap I_0^j = \{x \in I_0^j : \theta_0^-(x) \leq 0 \leq \theta_0^+(x)\}. \tag{4.10}$$

We shall now verify that condition $(\text{Geo})_0(1)$ holds. Assume that I_0^j is of type $(\text{i})^+$. Then, we have $v(I_0^j) = [-3\lambda^{3/4}, 3\lambda^{3/4}]$ and $v'(x) > 0$ on I_0^j. Let $J_0^j = \{x \in I_0^j : |v(x)| \leq 1\}$. Writing $J_0^j = [a, b]$, we have $v(a) = -1$ and $v(b) = 1$. Furthermore, by Lemma 2.10, we have $v'(x) > 2\lambda^{3/4}$ on J_0^j. Using (4.8)–(4.10), we conclude that $I_1 \cap I_0^j \subset J_0^j$, $(\theta_0^{\pm})'(x) > \lambda^{3/4}$ on J_0^j, and $\theta_0^+(a) < -1/2, \theta_0^-(b) > 1/2$. Thus, there are unique points $a_1^j < b_1^j$ in J_0^j such that $\theta_0^+(a_1^j) = \theta_0^-(b_1^j) = 0$. From this, it is easy to see that $I_1 \cap I_0^j = [a_1^j, b_1^j]$; see Figure 2. Recalling the definition of $\theta_0^{\pm}$ finishes the verification of $(\text{Geo})_0(1)$.

We turn to condition $(\text{Geo})_0(2)$ and the proof of Lemma 4.2. Assume that $I_0^j := [a_0^j, b_0^j]$ is of type $(\text{ii})^+$ (the case of type $(\text{ii})^-$ is treated analogously). We therefore have

$$v(a_0^j) = v(b_0^j) = 3\lambda^{3/4} \quad \text{and} \quad v''(x) > \kappa \quad \text{on } I_0^j. \tag{4.11}$$

Since $\widehat{\Omega}_0 \subset \widetilde{\Omega}_0 \subset \widetilde{\Omega}$, we have (by definition (2.8) and Lemma 2.12) that either $\min_{x \in I_0^j} g(x, \omega) > 2\lambda^{-7/5}$ for all $\omega \in \widehat{\Omega}_0$, or $\min_{x \in I_0^j} g(x, \omega) < -2\lambda^{-7/5}$ for all $\omega \in \widehat{\Omega}_0$. In the first case, it follows from (4.8) (which holds for all $\omega \in \widehat{\Omega}_0$) that $\theta_0^-(x) > \lambda^{-7/5}/2$ for all $x \in I_0^j$. Recalling (4.10), we conclude that $I_1 \cap I_0^j = \emptyset$. Assume now that the latter holds, i.e., that $\min_{x \in I_0^j} g(x) < -2\lambda^{-7/5}$.

Let $J_0^j = \{x \in I_0^j : |g(x)| \leq \lambda^{-7/5}\}$. By Lemma 2.12, we have $|g'(x)| > 2\lambda^{-1/4}$ on J_0^j. Since (4.11) holds, J_0^j must consist of two intervals, $J_0^{j,i}$ $(i = 1, 2)$, and $g'(x) > 2\lambda^{-1/4}$ on $J_0^{j,1}$ and $g'(x) < -2\lambda^{-1/4}$ on $J_0^{j,2}$. By using the estimates (4.8) and (4.9), it is easy to see that there exists unique points $a_1^{j,i} < b_1^{j,i}$ in $J_0^{j,i}$ such that $\theta_0^+(a_1^{j,i}) = \theta_0^-(b_1^{j,i}) = 0$ $(i = 1, 2)$. By recalling (4.10), we get $I_1 \cap I_0^j = [a_1^{j,1}, b_1^{j,1}] \cup [a_1^{j,2}, b_1^{j,2}]$; see Figure 2. Thus, $(\text{Geo})_0(2b)$ holds; and the proof of Lemma 4.2 is finished. ■

We now formulate the inductive step. In the statement, we assume that

$$g(x) = g(x, \omega) = v(x) - \frac{1}{v(x - \omega)} - \frac{1}{v(x + \omega)}.$$

Proposition 4.4. *For all $n \geq 0$, the following holds. Assume that $M_0 = K_0$ and $M_k \in [K_k, 2K_k]$ $(k = 1, \ldots, n)$ are fixed and that there exists an interval $\widehat{\Omega}_n \subset \widetilde{\Omega}_0$ such that, for each $\omega \in \widehat{\Omega}_n$, the conditions* $(\text{NR})_n$, $(\text{Pos})_n$, $(\text{S})_n$, *and* $(\text{R})_n$ *(in Proposition 3.1) hold. Moreover, assume the following.*

(Geo)$_n$. *Each component I_0^j in I_0 satisfies the conditions below.*

$(1)_n$ *If I_0^j is of type* (i)*, then, for each $\omega \in \widehat{\Omega}_n$, we have $I_0^j \cap I_{n+1} = [a_{n+1}^j, b_{n+1}^j]$ and*

- *if I_0^j is of type* (i)$^-$*, then*

$$\varphi_n^-(a_{n+1}^j) = \psi_n^+(a_{n+1}^j) \quad \textit{and} \quad \varphi_n^+(b_{n+1}^j) = \psi_n^-(b_{n+1}^j);$$

- *if I_0^j is of type* (i)$^+$*, then*

$$\varphi_n^+(a_{n+1}^j) = \psi_n^-(a_{n+1}^j) \quad \textit{and} \quad \varphi_n^-(b_{n+1}^j) = \psi_n^+(b_{n+1}^j).$$

Moreover, $g'(x) > 0$ on $[a_{n+1}^j, b_{n+1}^j]$ if I_0^j is of type (i)$^+$*, and $g'(x) < 0$ on $[a_{n+1}^j, b_{n+1}^j]$ if I_0^j is of type* (i)$^-$*.*

$(2)_n$ *If I_0^j is of type* (ii) *and if $\widehat{\Omega}_n$ is relevant for I_0^j, then, for each $\omega \in \widehat{\Omega}_n$, we have that $I_0^j \cap I_{n+1}$ consists of two disjoint components $[a_{n+1}^{j,i}, b_{n+1}^{j,i}]$ ($i = 1, 2$) and*

$$\varphi_n^-(a_{n+1}^{j,1}) = \psi_n^+(a_{n+1}^{j,1}) \quad \textit{and} \quad \varphi_n^+(b_{n+1}^{j,1}) = \psi_n^-(b_{n+1}^{j,1}),$$
$$\varphi_n^+(a_{n+1}^{j,2}) = \psi_n^-(a_{n+1}^{j,2}) \quad \textit{and} \quad \varphi_n^-(b_{n+1}^{j,2}) = \psi_n^+(b_{n+1}^{j,2}).$$

Moreover, $g'(x) > 0$ on $[a_{n+1}^{j,1}, b_{n+1}^{j,1}]$ and $g'(x) < 0$ on $[a_{n+1}^{j,2}, b_{n+1}^{j,2}]$.

If I_0^j is of type (ii) *and if $\widehat{\Omega}_n$ is not relevant for I_0^j, then, for each $\omega \in \widehat{\Omega}_n$, we have that $I_0^j \cap I_{n+1} = \emptyset$.*

Then, the following holds with $\widehat{\Omega} = \widehat{\Omega}_n$.

(CS)$_n(\widehat{\Omega})$. *Either $I_{n+1}(\omega) = \emptyset$ for all $\omega \in \widehat{\Omega}$, or the set $I_{n+1}(\omega)$ (for each $\omega \in \widehat{\Omega}$) consist of $d_0(\widehat{\Omega}) \leq d$ intervals, and, for each of these intervals in $I(\omega)$, we have*

$$|I(\omega)| < \lambda^{-K_n}.$$

Furthermore, if $a(\omega)$ is an endpoint of the component $I(\omega)$, then $|a'(\omega)| < 1$ for all $\omega \in \widehat{\Omega}$.

Moreover, if $M_{n+1} \in [K_{n+1}, 2K_{n+1}]$ and $\widehat{\Omega}_{n+1} \subset \widehat{\Omega}_n$ are such that bath (NR)$_{n+1}$ *and* (Pos)$_{n+1}$ *are satisfied for each $\omega \in \widehat{\Omega}_{n+1}$, then* (Geo)$_{n+1}$ *holds.*

Proof. For each $k \geq 0$, we let

$$\theta_k^+(x,\omega) = \varphi_k^+(x,\omega) - \psi_k^-(x,\omega) \quad \text{and} \quad \theta_k^-(x,\omega) = \varphi_k^-(x,\omega) - \psi_k^+(x,\omega).$$

We shall first verify that $(\mathrm{CS})_n(\widehat{\Omega}_n)$ holds. To begin, we note that, from the statements about $I_0 \cap I_{n+1}$ in assumption $(\mathrm{Geo})_n$, it follows from the definition of d_0 in (4.7) that I_{n+1} consists of d_0 components (note that we can have $d_0 = 0$).

By applying Lemma 4.1, we get that

$$\varphi_n^{\pm}(x,\omega) = v(x) - \frac{1}{v(x-\omega) + e_n^{\pm}(x,\omega)}$$

and

$$\psi_n^{\pm} = \frac{1}{v(x+\omega) - w_n^{\pm}(x,\omega)}$$

satisfy $(4.1)_n$–$(4.3)_n$ and $(4.4)_n$ on $I_n \supset I_{n+1}$ for each $\omega \in \widehat{\Omega}_n$.

Note that, from the assumptions in $(1)_n$, we have $\theta_n^+(a_{n+1}^j) = \theta_n^-(b_{n+1}^j) = 0$ and $g'(x) > 0$ on $[a_{n+1}^j, b_{n+1}^j]$ if I_0^j is of type $(\mathrm{i})^+$; and $\theta_n^-(a_{n+1}^j) = \theta_n^+(b_{n+1}^j) = 0$ and $g'(x) < 0$ on $[a_{n+1}^j, b_{n+1}^j]$ if I_0^j is of type $(\mathrm{i})^-$. And, from assumption $(2)_n$, we have that if I_0^j is of type (ii) and $\widehat{\Omega}_n$ is relevant for I_0^j, then $\theta_n^+(a_{n+1}^{j,1}) = \theta_n^-(b_{n+1}^{j,1}) = 0$ and $g'(x) > 0$ on $[a_{n+1}^{j,1}, b_{n+1}^{j,1}]$; and $\theta_n^-(a_{n+1}^{j,2}) = \theta_n^+(b_{n+1}^{j,2}) = 0$ and $g'(x) < 0$ on $[a_{n+1}^{j,2}, b_{n+1}^{j,2}]$.

We can now apply Lemma A.5 with $\varphi^{\pm} = \varphi_n^{\pm}$, $\psi^{\pm} = \psi_n^{\pm}$, $I = [a_{n+1}^j, b_{n+1}^j]$ and $\varepsilon = \lambda^{-4(K_n-2)/3}$. In each of the cases above, we then get the bound $|I| < \varepsilon\lambda^{-5/4} \ll \lambda^{-K_n}$. We also get the estimate of the ω-derivative of the endpoint of the intervals. Thus, we can conclude that $(\mathrm{CS})_n(\widehat{\Omega}_n)$ indeed holds.

We now turn to the second statement in the proposition. We therefore assume that $M_{n+1} \in [K_{n+1}, 2K_{n+1}]$ and $\widehat{\Omega}_{n+1} \subset \widehat{\Omega}_n$ are such that $(\mathrm{NR})_{n+1}$, $(\mathrm{Pos})_{n+1}$ are satisfied for each $\omega \in \widehat{\Omega}_{n+1}$. We shall check that $(\mathrm{Geo})_{n+1}$ holds.

From Lemma 4.1 and Lemma 3.3 (which implies both that $F^{M_n+1}(A_n)$ is strictly included in $F^{M_{n-1}+1}(A_{n-1})$ and that $F^{-M_n+1}(B_n)$ in strictly included in $F^{-M_{n-1}+1}(B_{n-1})$), we get

$$\varphi_n^-(x) < \varphi_{n+1}^-(x) < \varphi_{n+1}^+(x) < \varphi_n^+(x) \tag{4.12a}$$

and

$$\psi_n^-(x) < \psi_{n+1}^-(x) < \psi_{n+1}^+(x) < \psi_n^+(x), \quad x \in I_{n+1}. \tag{4.12b}$$

Moreover,

$$\varphi_{n+1}^{\pm}(x,\omega) = v(x) - \frac{1}{v(x-\omega) + e_{n+1}^{\pm}(x,\omega)}$$

and

$$\psi_{n+1}^{\pm} = \frac{1}{v(x+\omega) - w_{n+1}^{\pm}(x,\omega)}$$

satisfy $(4.1)_{n+1}$–$(4.4)_{n+1}$ on I_{n+1}.

Verifying $(1)_{n+1}$. We assume that I_0^j is of type $(\mathrm{i})^+$. By assumption $(1)_n$, we have $\varphi_n^+(a_{n+1}^j) = \psi_n^-(a_{n+1}^j)$ and $\varphi_n^-(b_{n+1}^j) = \psi_n^+(b_{n+1}^j)$. Thus, by the inclusion in (4.12), we have

$$\varphi_{n+1}^+(a_{n+1}^j) < \psi_{n+1}^-(a_{n+1}^j) \quad \text{and} \quad \varphi_{n+1}^-(b_{n+1}^j) > \psi_{n+1}^+(b_{n+1}^j),$$

that is $\theta_{n+1}^+(a_{n+1}^j) < 0$ and $\theta_{n+1}^-(b_{n+1}^j) > 0$. By applying Lemma A.5 with $\varphi^\pm = \varphi_n^\pm$, $\psi_n^\pm = \psi_n^\pm$, and $I = [a_{n+1}^j, b_{n+1}^j]$, we get that the statement in $(1)_{n+1}$ holds.

Verifying $(2)_{n+1}$. We assume that I_0^j is of type $(\mathrm{ii})^+$. If I_0^j is not relevant for $\widehat{\Omega}_{n+1}$, then, by $(2)_n$, we have that $I_n \cap I_0^j = \emptyset$ for all $\omega \in \widehat{\Omega}_{n+1} \subset \widetilde{\Omega}_n$. Thus, by definition, we get that $I_{n+1} \cap I_0^j \subset I_n \cap I_0^j = \emptyset$.

Assume now that I_0^j is relevant for $\widehat{\Omega}_{n+1}$. By proceeding as we did in the previous case on each of the intervals $[a_{n+1}^{j,i}, b_{n+1}^{j,i}]$ $(i = 1, 2)$, using the inclusion (4.12) and Lemma A.5, we see that the statement in $(2)_{n+1}$ holds. ■

5. Excluding frequencies

Let

$$\varepsilon_0 = C\lambda^{-1/8} \quad \text{and} \quad \varepsilon_j = \lambda^{-K_{j-1}}, \quad j \ge 1,$$

where $C > 0$ is the constant in Lemma 2.7 (so the length of each component in I_0 is less than ε_0).

In the proposition, below we shall use the following notation. If we have $I_n^k = I_n^k(\omega) = [a_n^k(\omega), b_n^k(\omega)]$, we let

$$J_n^k(\varepsilon, \omega) = [a_n^k(\omega) - \varepsilon, b_n^k(\omega) + \varepsilon] \quad \text{for } \varepsilon > 0;$$

and if $I_n(\omega) = \bigcup_{k=1}^{d_0} I_n^k(\omega)$, we let $J_n(\varepsilon, \omega) = \bigcup_{k=1}^{d_0} J_n^k(\varepsilon, \omega)$. Thus, we make the intervals in I_n a bit wider.

Recall the definition of the sets $\widehat{X}_n$ in (3.8). Moreover, let x^* be as in Theorem 1, i.e., $x^* \in \mathbb{T} \setminus I_0$.

Proposition 5.1. *For all, $n \ge 0$ the following holds. Assume that $\widehat{\Omega}_n \subset \mathbb{T}$ is an interval and that $M_0 = K_0$ and $M_j \in [K_j, 2K_j]$, $j = 1, \dots, n$, are such that* $(\mathrm{CS})_j(\widehat{\Omega}_n)$ *(in Proposition 4.4) hold for $j \in [0, n]$, and such that* $(\mathrm{Pos})_n$*,*

$$x^* \in \widehat{X}_n, \tag{5.1}$$

and

$$J_j(2\varepsilon_j) \cap \bigcup_{m=1}^{K_j^2} (J_j(2\varepsilon_j) + m\omega) = \emptyset \quad \textit{for each } j \in [0, n] \tag{5.2}$$

hold for all $\omega \in \widehat{\Omega}_n$.

Then, there exists a set $R_{n+1} \subset \widehat{\Omega}_n$*, such that*

$$|R_{n+1}| < \frac{1}{K_{n+1}^{1/3}} \quad \textit{and} \quad \#R_{n+1} < K_{n+1}^4$$

and, for every interval $\widehat{\Omega}_{n+1}$ *in* $\widehat{\Omega}_n \setminus R_{n+1}$*, there is an* $M_{n+1} \in [K_{n+1}, 2K_{n+1}]$ *such that, for every* $\omega \in \widehat{\Omega}_{n+1}$*, we have that* $(\mathrm{Pos})_{n+1}$*,* $(5.1)_{n+1}$*, and* $(5.2)_{n+1}$ *hold.*

Proof. From the assumption that $(\mathrm{CS})_j(\widehat{\Omega}_n)$ holds for $0 \le j \le n$, we know that the sets $I_{j+1}(\omega)$ consist of $d_0 \le d$ intervals, each of length $< \lambda^{-K_j} = \varepsilon_{j+1}$ for each $j \in [0, n]$ and each $\omega \in \widehat{\Omega}_n$. If $I_{n+1} = \emptyset$ for all $\omega \in \widehat{\Omega}_n$, we can just let $R_{n+1} = \emptyset$. If this does not hold, we have (by the $(\mathrm{CS})_j$) $I_{n+1} \neq \emptyset$ for all $\omega \in \widehat{\Omega}_n$. From now on, we assume that this is the case. Letting $I_j^k(\omega) = [a_j^k(\omega), b_j^k(\omega)]$ denote the intervals in I_j, there holds (by the $(\mathrm{CS})_j$)

$$|(a_j^k)'(\omega)| < 1 \quad \text{for all } \omega \in \widehat{\Omega}_n, 1 \le k \le d_0, \text{ and } j \in [1, n+1]. \tag{5.3}$$

Recalling the definition of J_j above, we note that each interval in $J_j(2\varepsilon_j)$ $(0 \le j \le n+1)$ has length $< 5\varepsilon_j$.

Since $(5.1)_n$ holds and $I_{n+1} \subset I_n$, we have that $(5.1)_{n+1}$ holds if

$$\{x^*\} \cap \bigcup_{K_n^2+1 \le |m| \le K_{n+1}^2} (I_{n+1} + m\omega) = \emptyset;$$

and, since $(5.2)_n$ holds and $J_{n+1}(2\varepsilon_{n+1}) \subset J_n(2\varepsilon_n)$, we see that $(5.2)_{n+1}$ holds if

$$J_{n+1}(2\varepsilon_{n+1}) \cap \bigcup_{m=K_n^2+1}^{K_{n+1}^2} (J_{n+1}(2\varepsilon_{n+1}) + m\omega) = \emptyset. \tag{5.4}$$

Constructing the set $R_{n+1} \subset \widehat{\Omega}_n$. We denote $[\omega_0, \omega_1] = \widehat{\Omega}_n$. Take $m \in [K_n^2 + 1, K_{n+1}^2]$. For each $i, k \in [1, d_0]$, we want to construct a set $R_{n+1}(i, k, m) \subset \widehat{\Omega}_n$ such that

$$J_{n+1}^i(2\varepsilon_{n+1}) \cap (J_{n+1}^k(2\varepsilon_{n+1}) + m\omega) = \emptyset \quad \text{for every } \omega \in \widehat{\Omega}_n \setminus R_{n+1}(i, k, m). \tag{5.5}$$

For simplicity, we also denote the corresponding lifts by a_{n+1}^k, a_{n+1}^i. Let

$$h(\omega) = a_{n+1}^k(\omega) + m\omega - a_{n+1}^i(\omega).$$

Since (5.3), hold we have that $|h([\omega_0, \omega_1])| < (m+2)(\omega_1 - \omega_0)$ and $h'(\omega) > m - 2$. Thus, $h([\omega_0, \omega_1])$ can intersect at most $[(m+2)(\omega_1 - \omega_0)] + 2$ different intervals of the form $[p, p+1]$, $p \in \mathbb{Z}$. We conclude that the set

$$R_{n+1}(i, k, m) := \{\omega \in [\omega_0, \omega_1] : \inf_{q \in \mathbb{Z}} |h(\omega) - q| \le 5\varepsilon_{n+1}\}$$

consists of at most $[(m+2)(\omega_1-\omega_0)]+2$ intervals, each of which of length $<10\varepsilon_{n+1}/(m-2)$. We see that, by definition, (5.5) holds for all $\omega\in\widehat{\Omega}_n\setminus R_{n+1}(i,k,m)$. Note that this estimate also holds in the case $i=k$ (and then $h(\omega)=m\omega$).

Similarly, for $K_n^2+1\le|m|\le K_{n+1}^2$ and $k\in[1,d_0]$, we want a set

$$R_{n+1}(x^*,k,m)\subset\widehat{\Omega}_n$$

such that

$$\{x^*\}\cap(I_{n+1}^k+m\omega)=\emptyset\quad\text{for every } \omega\in\widehat{\Omega}_n\setminus R_{n+1}(x^*,k,m).$$

We let $h(\omega)=a_{n+1}^k+m\omega-x^*$ and define

$$R_{n+1}(x^*,k,m):=\{\omega\in[\omega_0,\omega_1]:\inf_{q\in\mathbb{Z}}|h(\omega)-q|\le\varepsilon_{n+1}\}.$$

In the same way as above, $R_{n+1}(x^*,k,m)$ consists of at most $[(|m|+2)(\omega_1-\omega_0)]+2$ intervals, each which of length $<2\varepsilon_{n+1}/(|m|-2)$.

We now let

$$R_{n+1}=\Big(\bigcup_{i=1}^{d_0}\bigcup_{k=1}^{d_0}\bigcup_{m=K_n^2+1}^{K_{n+1}^2}R_{n+1}(i,k,m)\Big)\cup\bigcup_{k=1}^{d_0}\bigcup_{K_n^2+1\le|m|\le K_{n+1}^2}R_{n+1}(x^*,k,m).$$

From the estimates above, we get (recall the definitions of ε_{n+1} and K_{n+1})

$$|R_{n+1}|<100d^2K_{n+1}^2\varepsilon_{n+1}|\widehat{\Omega}_n|<K_{n+1}^{-1/3}$$

and

$$\#R_{n+1}<10d^2\sum_{m=1}^{K_{n+1}^2}m|\widehat{\Omega}_n|<K_{n+1}^4.$$

Since, by $(5.2)_n$, $J_n(2\varepsilon_n)\cap(J_n(2\varepsilon_n)+K_n^2\omega)=\emptyset$ for all $\omega\in\widehat{\Omega}_n$, we must have $|\widehat{\Omega}_n|<1/K_n^2$.

Choosing M_{n+1}. Let $\widehat{\Omega}_{n+1}$ be an interval in $\widehat{\Omega}_n\setminus R_{n+1}$. By the above observation, we get that (since (5.4) hold)

$$|\widehat{\Omega}_{n+1}|<\frac{1}{K_{n+1}^2}. \tag{5.6}$$

Claim. *For each $\omega\in\widehat{\Omega}_{n+1}$ and any interval $I\subset\mathbb{T}$ of length $<\varepsilon_n$, there are at least $K_n^2(1-10K_0^{-1/2})$ integers M in $[K_{n+1},K_{n+1}+K_n^2]$ such that*

$$I\pm M\omega, I\pm(M+1)\omega\subset\mathbb{T}\setminus\bigcup_{j=0}^{n}\bigcup_{m=-[K_j^{3/2}]}^{[K_j^{3/2}]}(J_j(\varepsilon_j)+m\omega). \tag{5.7}$$

To prove this claim, we do as follows. Fix $j \in [0, n]$. Since $|I| < \varepsilon_n \le \varepsilon_j$, we note that if $I \cap (J(\varepsilon_j) + m\omega)$ for some m, then $I \subset J(2\varepsilon_j) + m\omega$. Hence, we get from $(5.2)_j$ that in any interval $[q, q + K_j^2]$, $q \in \mathbb{Z}$, there are at most $2[K_j^{3/2}] + 3$ indices ℓ such that

$$(I + \ell\omega) \cap \bigcup_{m=-[K_j^{3/2}]-1}^{[K_j^{3/2}]+1} (J_j(\varepsilon_j) + m\omega) \neq \emptyset$$

and, in any interval $[q, q + K_j^2]$, $q \in \mathbb{Z}$, there are at most $2[K_j^{3/2}] + 3$ indices ℓ such that

$$(I - \ell\omega) \cap \bigcup_{m=-[K_j^{3/2}]-1}^{[K_j^{3/2}]+1} (J_j(\varepsilon_j) + m\omega) \neq \emptyset.$$

From this, we get that, in any interval $[q, q + K_n^2]$, there are at most

$$2([K_n^2/K_j^2] + 1)(2[K_j^{3/2}] + 3) < 9K_n^2 K_j^{-1/2}$$

indices ℓ such that

$$((I - \ell\omega) \cup (I + \ell\omega)) \cap \bigcup_{m=-[K_j^{3/2}]-1}^{[K_j^{3/2}]+1} (J_j(\varepsilon_j) + m\omega) \neq \emptyset.$$

We conclude that, in any interval $[q, q + K_n^2]$, there are at most

$$9K_n^2 \sum_{j=0}^{n} K_j^{-1/2} < 9K_n^2 \sum_{j=0}^{\infty} K_j^{-1/2} < 10K_n^2 K_0^{-1/2}$$

incides M such that (5.7) does not hold. Thus, the claim holds.

Let $\overline{\omega} \in \widehat{\Omega}_{n+1}$ be the midpoint of the interval. Since $I_{n+1} = I_{n+1}(\overline{\omega})$ consists of at most d intervals, each of a length $\ll \varepsilon_n$, it follows from the above claim that we can find an $M_{n+1} \in [K_{n+1}, K_{n+1} + K_n^2] \subset [K_{n+1}, 2K_{n+1}]$ such that

$$\begin{aligned} I_{n+1}(\overline{\omega}) \pm M_{n+1}\overline{\omega}, I_{n+1}(\overline{\omega}) \pm (M_{n+1} + 1)\overline{\omega} \\ \subset \mathbb{T} \setminus \bigcup_{j=0}^{n} \bigcup_{m=-[K_j^{3/2}]}^{[K_j^{3/2}]} (J_j(\varepsilon_j, \overline{\omega}) + m\overline{\omega}). \end{aligned} \tag{5.8}$$

Since (5.3) and (5.6) hold, we have, for all $\omega \in \widehat{\Omega}_{n+1}$ and all $1 \le j \le n + 1$,

$$|a_j^i(\omega) - a_j^i(\omega_{\text{mid}})| < |\omega - \overline{\omega}| < \frac{1}{2K_{n+1}^2} \ll \varepsilon_n$$

and

$$M_{n+1}|\omega - \overline{\omega}| < \frac{1}{K_{n+1}} \ll \varepsilon_n.$$

Thus, recalling the definitions of $X_n^{\pm}$ in (3.7), we see that $(\mathrm{Pos})_{n+1}$ follows from (5.8), combined with the last observation. ■

Let $M_0 = K_0$. Note that condition $(\mathrm{Pos})_0$ is void (since $X_{-1}^{\pm} = \emptyset$), and I_0 consists of at most d intervals. Thus, we can use the next lemma as the base step for Proposition 5.1.

Lemma 5.2. *The set*

$$\Omega_0 = \{\omega \in \widetilde{\Omega} : \ (5.1)_0 \textit{ and } (5.2)_0 \textit{ hold}\} \subset \widetilde{\Omega}_0$$

satisfies $\#\Omega_0 < 2d^2K_0^4$ *and* $|\Omega_0| > |\widetilde{\Omega}| - K_0^{-1/3}/10$.

Proof. We first observe the following trivial fact. If $p \in \mathbb{T}$ is a given point and $|m| \neq 1$ is an integer, then the set $\{\omega \in \mathbb{T} : |p + m\omega| \leq \varepsilon\}$ consists of $|m|$ intervals of length $\varepsilon/|m|$. And if $U \subset \mathbb{T}$ is an interval, the set $\{\omega \in U : |p + m\omega| \leq \varepsilon\}$ consists of at most $|m| + 1$ intervals, and the total measure is $\leq \varepsilon$. The last estimate is of course very rough if the length of U is small.

Recall the definition of the set Ω' in (2.7). Let $d_1 \leq d$ be the number of intervals in $J_0 = J_0(2\varepsilon_0)$. We have

$$\Omega' \setminus \Omega_0 = \Big(\bigcup_{i=1}^{d_1}\bigcup_{j=1}^{d_1}\bigcup_{m=1}^{K_0^2}\{\omega \in \Omega' : J_0^j \cap (J_0^i + m\omega) \neq \emptyset\}\Big)$$
$$\cup \Big(\bigcup_{i=1}^{d_1}\bigcup_{1\leq|m|\leq K_0^2}\{\omega \in \Omega' : x^* \in (I_0^i + m\omega)\}\Big).$$

Recall that we have assumed $x^* \notin I_0$. By Lemma 2.7 and the definition of ε_0, we have $|J_0^j| < \varepsilon_0$. Denoting $J_0^j = [p_0^j, q_0^j]$, we have

$$\{\omega \in \Omega' : J_0^j \cap (J_0^i + m\omega) \neq \emptyset\} \subset \{\omega \in \Omega' : |(p_0^i - p_0^j) + m\omega| \leq \varepsilon_0\}$$

and

$$\{\omega \in \Omega' : x^* \in (I_0^i + m\omega)\} \subset \{\omega \in \Omega' : |(p_0^i - x^*) + m\omega| < \varepsilon_0\}.$$

Thus,

$$|\Omega' \setminus \Omega_0| \leq d^2K_0^2\varepsilon_0 + 2dK_0^2\varepsilon_0 \ll K_0^{-1/3}/20$$

and

$$\#(\Omega' \setminus \Omega_0) \leq d^2 \sum_{m=1}^{K_0^2}(m+1) + 2d \sum_{m=1}^{K_0^2}(m+1) < \frac{3}{2} d^2 K_0^4.$$

To finish, recall the definition of $\widetilde{\Omega} \subset \Omega'$ in (2.8). By Lemma 2.12, we know that $\Omega' \setminus \widetilde{\Omega}$ consists of at most $d/2$ intervals, each of length $< 2\lambda^{-1/5}$. From this, combined with the estimates above, the statement of the lemma follows. ∎

6. Putting everything together

We now have everything needed to prove Theorem 1. First, we will combine the inductive steps from Sections 3–5.

6.1. Putting together the inductive construction

Recall that we have assumed that the first critical set $I_0 \neq \emptyset$. (Otherwise, we just use Lemma 2.5.)

Let $\Omega_0 \subset \widetilde{\Omega}_0$ be the set in Lemma 5.2. We thus have the estimates

$$|\Omega_0| > |\widetilde{\Omega}| - \frac{\lambda^{-1/3}}{10} \quad \text{and} \quad \#\Omega_0 < 2d^2 K_0^4. \tag{6.1}$$

If I_0 consist only of intervals of typ (ii), let Ω_0^{nt} be the union of the components of Ω_0 which are relevant for a least one I_0^j; if I_0 consists of a least one interval of type (i), let $\Omega_0^{\mathrm{nt}} = \Omega_0$. Recall that, by Lemma 2.9, the latter case holds if $E \in [\lambda \min v_0 + 4\lambda^{3/4}, \lambda \max v_0 - 4\lambda^{3/4}]$.

Lemma 6.1. *If $\omega \in \Omega_0 \setminus \Omega_0^{\mathrm{nt}}$ is irrational, then both Theorem 1 (1) and Theorem 1 (2) hold. Moreover, $G_{\omega,E}$ is uniformly hyperbolic.*

Proof. By definition, we have that $I_1(K_0) = I_1(K_0, \omega) = \emptyset$ for all $\omega \in \Omega_0 \setminus \Omega_0^{\mathrm{nt}}$. Moreover, since condition $(\mathrm{NR})_0$ is satisfied for all $\omega \in \Omega_0$ it follows from Proposition 3.1 (applied with $n = 0$ and $M_0 = K_0$; recall Lemma 3.2) that conditions $(\mathrm{S})_1$ and $(\mathrm{R})_1$ hold for all $\omega \in \Omega_0$.

Recall the definition of the sets $X_n^{\pm}$ in (3.7). Note that we have (recall Lemma 2.7)

$$\begin{aligned} |X_0^+ \cap X_0^-| \geq 1 - (2K_0^{3/2} + 1)|I_0| &> 1 - \text{const.}\, \lambda^{3/40} \lambda^{-1/8} \\ &= 1 - \text{const.}\, \lambda^{-1/20} > 0. \end{aligned}$$

The statement of the lemma now follows from the discussion in Section 3.4. ∎

From now on, we focus only on Ω_0^{nt}, which we assume is non-empty (otherwise there is nothing to do). We will consider each component in Ω_0^{nt} separately. Thus, we fix a component $\Omega_0^q \subset \Omega_0^{\mathrm{nt}}$. We know that I_0 contains at least one component of type (i), or I_0 contains at least one interval of type (ii) which is relevant for Ω_0^q. Let $d_0 = d_0(\Omega_0^q) \geq 1$ be as in (4.7).

By combining the inductive steps from the previous sections, we obtain the following statement.

Proposition 6.2. *For all $n \geq 0$, we have the following. Assume that the set $\Omega_n^q \subset \Omega_0^q$ has been constructed with the properties that*

(a)$_n$ $|\Omega_n^q| > |\Omega_0^q| - \sum_{k=1}^n K_k^{-1/10}$ *and* $\#\Omega_n^q \leq K_1^4 \cdots K_n^4$ *if $n \geq 1q$;*

(b)$_n$ *for each component $\Omega_n^{q,\ell}$ in Ω_n^q, there are integers $M_j(\ell) \in [K_j, 2K_j]$, $j = 0, \ldots, n$, $M_0 = K_0$, such that* (5.1)$_n$, (5.2)$_n$, *and* (Pos)$_k$, (S)$_k$, (R)$_k$, *and* (Geo)$_k$ *hold for all $\omega \in \Omega_n^{q,\ell}$ and all $0 \leq k \leq n$.*

Then, the set $I_{n+1} = I_{n+1}(M_0(\ell), \ldots, M_n(\ell), \omega)$ consists of d_0 components, each of length $< \lambda^{-K_n}$, for all $\omega \in \Omega_n^{q,\ell}$. Moreover, there is a set $\Omega_{n+1}^q \subset \Omega_n^q$ such that (a)$_{n+1}$ *holds and such that, for each component $\Omega_{n+1}^{q,\ell'} \subset \Omega_n^{q,\ell}$, there is an $M_{n+1}(\ell')$ such that, for $M_j = M_j(\ell)$, $j = 0, 1, \ldots, n$, and $M_{n+1} = M_{n+1}(\ell')$, we have that* (5.1)$_{n+1}$, (5.2)$_{n+1}$, *and* (Pos)$_k$, (S)$_k$, (R)$_k$, *and* (Geo)$_k$ *hold for all $\omega \in \Omega_{n+1}^{q,\ell'}$ and all $0 \leq k \leq n+1$. Thus, in particular,* (b)$_{n+1}$ *holds.*

Before proving this statement, we note that the base case ($n = 0$) indeed holds (recall that (a)$_0$ is void),

Lemma 6.3. *Condition* (b)$_0$ *holds.*

Proof. As always, we take $M_0 = K_0$. Since Ω_0^q is a component of the set Ω_0 in Lemma 5.2, we have that (5.1)$_0$ and (5.2)$_0$ hold for all $\omega \in \Omega_0$; and (Pos)$_0$ is trivially satisfied. Moreover, by Lemma 3.2, we know that (S)$_0$ and (R)$_0$ hold. Finally, (Geo)$_0$ hold by Lemma 4.3. ∎

Proof of Proposition 6.2. We assume that (a)$_n$ and (b)$_n$ hold. Let $\Omega_n^{q,\ell}$ be one of the components in Ω_n^q and assume that $M_j = M_j(\ell)$ for $j = 0, \ldots, n$ are as in (b)$_n$. From Proposition 4.4, we get that (CS)$_k(\Omega_n^\ell)$ holds for $k \in [0, n]$ (recall that condition (5.2)$_k$ is stronger than (NR)$_k$ for all k). Applying Proposition 5.1 gives a set $R_{n+1}^{q,\ell} \subset \Omega_n^{q,\ell}$ such that

$$|R_{n+1}^{q,\ell}| < K_{n+1}^{-1/3} \quad \text{and} \quad \#R_{n+1}^{q,\ell} < K_{n+1}^4, \tag{6.2}$$

with the property that, for each component $\Omega_{n+1}^{q,\ell'}$ in $\Omega_n^{q,\ell} \setminus R_{n+1}^{q,\ell}$, there is $M_{n+1}(\ell') \in [K_{n+1}, 2K_{n+1}]$ such that (5.1)$_{n+1}$, (5.2)$_{n+1}$, and (Pos)$_{n+1}$ hold for all $\omega \in \Omega_{n+1}^{q,\ell'}$.

Thus, it follows from Proposition 3.1 that also $(\mathrm{S})_{n-1}$ and $(\mathrm{R})_{n-1}$ hold for each $\omega \in \Omega_{n+1}^{q,\ell'}$; and, by Proposition 4.4, we get that $(\mathrm{Geo})_{n+1}$ holds for each $\omega \in \Omega_{n+1}^{\ell'}$.

To finish, we let

$$\Omega_{n+1}^{q} = \bigcup_{\ell=1}^{\#\Omega_n^q} \Omega_n^{q,\ell} \setminus R_{n+1}^{q,\ell}.$$

Then, by $(\mathrm{a})_n$ and (6.2), we have

$$\#\Omega_{n+1}^{q} \leq \#\Omega_n^q \cdot K_{n+1}^4 \leq K_1^4 \cdots K_n^4$$

and

$$|\Omega_{n+1}^q| > |\Omega_0^q| - \sum_{k=1}^{n} K_k^{-1/10} - \#\Omega_n^q \cdot K_{n+1}^{-1/3}.$$

Noticing that

$$\#\Omega_n \cdot K_{n+1}^{-1/3} \leq K_1^4 \cdots K_n^4 \cdot K_{n+1}^{-1/3} \leq \lambda^{(K_0+\cdots+K_{n-1})/5 - K_n/60} < K_{n+1}^{-1/10}$$

provided that λ is large enough, independently of n, finishes the proof. ∎

6.2. Proof of Theorem 1

From Lemma 6.1, we know that Theorem 1 (1)–(3) hold for all irrational number $\omega \in \Omega_0 \setminus \Omega_0^{\mathrm{nt}}$.

For each component Ω_0^q in Ω_0^{nt} we get, by combining Lemma 6.3 and Proposition 6.2, a nested sequence $\Omega_0^q \supset \Omega_1^q \supset \Omega_2^q \supset \ldots$ such that $(\mathrm{a})_n$ and $(\mathrm{b})_n$ hold for all $n \geq 0$. Let

$$\Omega_\infty^q = \bigcap_{n\geq 0} \Omega_n^q.$$

By the estimates in $(\mathrm{a})_n$, we have

$$|\Omega_\infty^q| > |\Omega_0^q| - 2K_1^{-1/10}.$$

We also let

$$\Omega_\infty^{\mathrm{nt}} = \bigcup_{q=1}^{\#\Omega_0^{\mathrm{nt}}} \Omega_\infty^q.$$

Since (6.1) holds, we have $\#\Omega_0^{\mathrm{nt}} \leq \#\Omega_0 < 2d^2 K_0^4$. Thus, the above estimate gives

$$|\Omega_\infty^{\mathrm{nt}}| > |\Omega_0^{\mathrm{nt}}| - 4d^2 K_0^4 \cdot K_1^{-1/10} > |\Omega_0^{\mathrm{nt}}| - K_1^{-1/9}. \tag{6.3}$$

Finally, we let

$$\Omega_\infty = (\Omega_0 \setminus (\Omega_0^{\mathrm{nt}} \cup \mathbb{Q}) \cup \Omega_\infty^{\mathrm{nt}}.$$

This is half of the set Ω_∞ in the statement of Theorem 1 (recall the discussion in the beginning of Section 2.9). By the estimates in (6.1) and (6.3), we get

$$|\Omega_\infty| > |\widetilde{\Omega}| - \frac{K_0^{-1/3}}{5} > \frac{|\Omega|}{2} - \frac{\lambda^{-1/60}}{2}.$$

It remains to show that statements (1)–(3) holds for all $\omega \in \Omega_\infty^{\mathrm{nt}}$. As we will see, the cocycle $G_{\omega,E}$ is not uniformly hyperbolic for such ω.

We now fix $\omega \in \Omega_\infty^{\mathrm{nt}}$. Thus, $\omega \in \Omega_\infty^q$ for some q. Let $d_0 = d_0(\Omega_0^q) \geq 1$. By Proposition 6.2 and Lemma 6.3, there is thus a nested sequence of intervals $\ldots \subset \Omega_2^{\ell_2} \subset \Omega_1^{\ell_1} \subset \Omega_0^q$ such that $\omega \in \bigcap_{n=1}^\infty \Omega_n^{\ell_n}$. Furthermore, there is a sequence $(M_n(\ell_n))_{n=0}^\infty$ such that $M_0 = K_0$ and $M_n \in [K_n, 2K_n]$ for each $n \geq 1$, and $(5.1)_n$, $(\mathrm{NR})_n$, $(\mathrm{Pos})_n$, $(\mathrm{S})_n$, $(\mathrm{R})_n$, and $(\mathrm{Geo})_n$ hold for all $n \geq 0$; and for each $n \geq 1$ we have that I_n consists of d_0 intervals, each of a length $< \lambda^{-K_{n-1}}$.

Proof of (1). Let

$$X_\infty = \bigcap_{n\geq 0}(X_n^+ \cap X_n^-) = \mathbb{T} \setminus \bigcup_{j=0}^{\infty} \bigcup_{|m|\leq [K_j^{3/2}]}(I_j + m\omega).$$

(Recall the definition of $X_n^\pm$ in (3.7).) By the estimates above, and recalling the definition of (K_n) in (3.2), and that $|I_0| < C\lambda^{-1/8}$, we have

$$|X_\infty| \geq 1 - \sum_{n=0}^{\infty}(2[K_n^{3/2}] + 1)|I_n| > 1 - 3C\lambda^{-1/20} - \sum_{n=1}^{\infty}\lambda^{-K_{n-1}/2}$$

which is close to 1 since λ is large.

If $(x, r) \in X_\infty \times (\widehat{\mathbb{R}} \setminus P_-)$, we can apply $(\mathrm{S})_n(2)$ for each $n \geq 0$ (recall the statement after equation 3.7) and get

$$|r_1 \cdots r_k| \geq \lambda^{2k/3} \quad \text{for every } k \geq 1 \text{ such that } |r_k| > 1/\lambda.$$

(Recall (3.1)). Since $|X_\infty| > 0$, it thus follows from Lemma 2.1 that $L(E, \omega) \geq (2/3)\log\lambda$.

Proof of (3). By Lemma 3.3, we have

$$F^{M_{n+1}}(A_{n+1}) \cap F^{-M_{n+1}}(B_{n+1}) \subset F^{M_n}(A_n) \cap F^{-M_n}(B_n) \quad \text{for all } n \geq 0.$$

Since these sets are closed, and since $\emptyset \neq I_{n+1} = \pi_1(F^{M_n}(A_n) \cap F^{-M_n}(B_n))$ for all $n \geq 0$, we can take

$$(x_0, r_0) \in \bigcap_{n=0}^{\infty}(F^{M_n}(A_n) \cap F^{-M_n}(B_n)) \neq \emptyset.$$

Let $s_0 = r_0$. By definition, we have

$$(x_{-M_n}, s_{M_n}) \in A_n \quad \text{and} \quad (x_{M_n}, r_{M_n}) \in B_n \quad \text{for all } n \geq 0.$$

Thus, for each fixed $n \geq 0$ we have, by the definition of A_n, B_n and the fact that $(\mathrm{Pos})_n$ holds, that $(x_{-M_n}, s_{M_n}) \in X^-_{n-1} \times P_+$ and $(x_{M_n}, r_{M_n}) \in X^+_{n-1} \times P_-$; and, since $(\mathrm{NR})_n$ holds, we have that $T^+(x_{-M_n}, I_n) = M_n$ and $T^-(x_{M_n}, I_n + \omega) = M_n - 1$.

For each $n \geq 0$, we can therefore apply $(\mathrm{S})_n(3)$ to the point (x_{-M_n}, s_{M_n}), and $(\mathrm{R})_n(3)$ to the point (x_{M_n}, r_{M_n}). Thus, we can conclude that

$$|s_1 \cdots s_k| \geq \lambda^{2k/3} \qquad \text{for all } k \geq 1$$

and

$$|r_2 \cdots r_k| \leq \lambda^{-2(k-1)/3} \quad \text{for all } k \geq 2.$$

In particular, we have $|s_1| \geq \lambda^{2/3}$ and $|r_2| \leq \lambda^{-2/3}$. Since $(I_0 - \omega) \cap I_0 = \emptyset$ and $(I_0 + 2\omega) \cap I_0 = \emptyset$ (by $(\mathrm{NR})_0$), and since $x_{-1} \in I_0 - \omega$ and $x_2 \in I_0 + 2\omega$, we get $|s_0| = |v(x_{-1}) - 1/s_1| \in (\lambda^{2/3}, \lambda)$ and $|r_1| = |v(x_1) - r_2|^{-1} < \lambda^{-2/3}$. We hence have

$$|s_0 s_1 \cdots s_k| \geq \lambda^{2(k+1)/3} \quad \text{for all } k \geq 0$$

and

$$|r_1 \cdots r_k| \leq \lambda^{-2k/3} \qquad \text{for all } k \geq 1.$$

Now, let

$$u_0 = \frac{1}{\sqrt{1 + s_0^2}} \begin{pmatrix} 1 \\ s_0 \end{pmatrix}.$$

Since $r_0 = s_0$ and $\lambda^{2/3} < |s_0| < \lambda$, it follows from the above estimates, combined with the formulae (2.4) and (2.5), that

$$|A^{-k}(x_0) u_0| < 2\lambda^{-2k/3} \qquad \text{for all } k \geq 1$$

and

$$|A^k(x_0) u_0| < 2\lambda^{-2(k-1)/3} \quad \text{for all } k \geq 1.$$

Proof of (2). We turn to Theorem 1 (2). We need to construct the subspaces $W^{\pm}$. Since (5.1) holds for each $n \geq 1$, we have that

$$x^* \in \bigcap_{n \geq 0} \widehat{X}_n = \mathbb{T} \setminus \bigcup_{j=0}^{\infty} \bigcup_{|m| \leq K_j^2} (I_j + m\omega) =: \widehat{X}_\infty.$$

Note that $\widehat{X}_\infty \subset X_\infty$.

To simplify notation, we let $X_{-1} = \mathbb{T}$ and

$$X_n := X_n^- \cap X_n^+ = \mathbb{T} \setminus \bigcup_{j=0}^{n} \bigcup_{|m| \le [K_j^{3/2}]} (I_j + m\omega).$$

Sublemma 6.4. *For every $n \ge 0$, if $|k| < K_n^2 - [K_n^{3/2}]$ and $x^* + k\omega \in X_{n-1}$, then $x^* + k\omega \in X_\infty$.*

Proof. Since $x^* \in \widehat{X}_\infty$, we must have $x^* + k\omega \in \mathbb{T} \setminus \bigcup_{j=n}^{\infty} \bigcup_{|m| \le [K_j^{3/2}]} (I_j + m\omega)$. Since $X_\infty = \big(\mathbb{T} \setminus \bigcup_{j=n}^{\infty} \bigcup_{|m| \le [K_j^{3/2}]} (I_j + m\omega)\big) \cap X_{n-1}$, the statement follows. ∎

This lemma implies, in particular, that $x^* + k\omega \in X_\infty$ for all $|k| < K_0^2/2$ (recall that $X_{-1} = \mathbb{T}$).

Sublemma 6.5. *For every $n \ge 0$ we have that if $K_n^2/2 \le k < K_{n+1}^2/2$, then there exists $\ell \in [k - 3K_n^{3/2}, k]$ such that $x^* + \ell\omega \in X_\infty$, and, if $-K_{n+1}^2/2 < k \le -K_n^2/2$, then there exists $\ell \in [k, k + 3K_n^{3/2}]$ such that $x^* + \ell\omega \in X_\infty$.*

Proof. We prove the first statement. By the previous sublemma, we are done if we can find an $\ell \in [k, k - 3K_n^{3/2}]$ such that $x^* + \ell\omega \in X_n$.

Since $(\mathrm{NR})_j$ holds for all $j \ge 0$, we know that in any interval of length K_j^2, there are at most $(2[K_j^{3/2}] + 1)$ indices i such that $x^* + i\omega \in \bigcup_{|m| \le [K_j^{3/2}]} (I_j + m\omega)$. Thus, in an interval of length $3[K_n^{3/2}]$ there are at most

$$(2[K_n^{3/2}] + 1) + \sum_{j=0}^{n-1} \Big(\frac{3[K_n^{3/2}]}{K_j^2} + 1 \Big) (2[K_j^{3/2}] + 1) < \frac{5}{2} K_n^{3/2}$$

indices i such that $x^* + i\omega \in \mathbb{T} \setminus X_n$. Therefore, we can find an ℓ as in the statement of the lemma. ∎

We shall now construct the subspace $W^- \subset \mathbb{R}^2$ and show that the first half of Theorem 1 (2) holds.

From the previous sublemma, we get an increasing sequence $0 < k_1 < k_2 < \cdots$ such that $x^* + k_i\omega \in X_\infty$ for all $i \ge 1$.

Sublemma 6.6. *We have*

$$\pi_2(F^{-k_{i+1}}(x^* + k_{i+1}\omega, P_-)) \subset \pi_2(F^{-k_i}(x^* + k_i\omega, P_-)) \subset P_- \quad \textit{for all } i \ge 1$$

and

$$|\pi_2(F^{-k_i}(x^* + k_i\omega, P_-))| \to 0 \quad \textit{as } i \to \infty.$$

Proof. If we take any $(x,s) \in X_\infty \times P_-$, it follows from $(\mathrm{R})_n(1)$, which we can apply to (x,s) for each $n \geq 0$, that $s_k \in \widehat{P}_- \subset P_-$ for each $k \geq 1$ such that $x_{-k} \in X_\infty$. Since $x^* \in X_\infty$ and $x^* + k_i\omega \in X_\infty$, for each $i \geq 1$, and since $k_{i+1} > k_i$, we can conclude that the first statement of the lemma holds.

That the second statement holds is a consequence of the following observation. Take $x = x^* + k_i\omega$, $i \geq 1$ and $s \in P_-$. Then, we have $s_{k_i} \in P_-$ (in particular, $|s_{k_i}| < \lambda$), as we saw above. By $(\mathrm{R})_n(2)$ (take sufficiently large n), we get $|s_1 \cdots s_{k_i}| \leq \lambda^{-(2/3)k_i}$. Applying Lemma A.2 now gives the result. ∎

From this last sublemma, it follows that there is a unique $r^- \in P_-$ such that

$$(x^*, r^-) \in F^{-k_i}(x^* + k_i\omega, P_-) \quad \text{for all } i \geq 1. \tag{6.4}$$

Sublemma 6.7. *If $r_j^- = \pi_2(F^j(x^*, r^-))$, then*

$$|r^- r_1^- \cdots r_k^-| \leq \lambda^{-(k+1)/2} \quad \textit{for all } k \geq 0.$$

Proof. We first note that $r_j^- \in P_-$ for all $j \geq 0$ such that $x_j^* \in X_\infty$. Indeed, if we would have $r_j^- \in \widehat{\mathbb{R}} \setminus P_-$ and $x_j^* \in X_\infty$ for some $j \geq 0$, it would follow from $(\mathrm{S})_n(1)$ (which holds for each n), applied to the point (x_j^*, r_j^-), that $r_k^- \in \widehat{\mathbb{R}} \setminus P_-$ for all $k > j$ such that $x_k^* \in X_\infty$. But this would contradict (6.4).

Next, take any $k \geq 1$. If $k < K_0^2/2$, then (as we saw above) $x_k^* \in X_\infty$. Thus, we have $r_k^- \in P_-$. From $(\mathrm{R})_n(2)$ (with large enough n) applied to (x_k^*, r_k^-), we get that $|r^- \cdots r_k^-| < \lambda^{-(2/3)(k+1)}$ (recall that $r^-, r_k^- \in P_-$). Assume now that $K_n^2/2 \leq k < K_{n+1}^2/2$ for some $n \geq 0$. From Sublemma 6.5, we get an $\ell \in [k - 3K_n^{3/2}, k]$ such that $x_\ell^* \in X_\infty$ (and thus $r_\ell^- \in P_-$). Applying $(\mathrm{R})_m(2)$ (with large enough m) to (x_l^*, r_l^-) yields $|r^- \cdots r_{\ell-1}^-| < \lambda^{-(2/3)\ell}$; and, from Lemma 2.6(2), we get $|r_\ell^- \cdots r_k^-| < \lambda^{k-\ell+1}$. Hence,

$$|r^- \cdots r_k^-| < \lambda^{-(2/3)\ell}\lambda^{k-\ell+1} < \lambda^{-(k+1)/2}.$$

For the last inequality to hold, we need that $(5/3)\ell > (3/2)k + 3/2$. But, recalling the estimates on k and ℓ above, we see that this inequality clearly is satisfied. ∎

Finally, by letting $W^- \subset \mathbb{R}^2$ be the line passing through the origin corresponding to $r^- \in \widehat{\mathbb{R}}$, and noticing that (by using Sublemma 6.7 and recalling (2.4))

$$\left| A^k(x^*)\begin{pmatrix} 1 \\ r^- \end{pmatrix} \right| = \left| \begin{pmatrix} r^- \cdots r_{k-1}^- \\ r^- \cdots r_k^- \end{pmatrix} \right| < 2\lambda^{-k/2} < \lambda^{-k/3}, \quad k \geq 1,$$

we see that the first half of Theorem 1 (2) holds.

In a similar way, we can construct the subspace W^+. From Sublemma 6.5, we get an increasing sequence $0 < k_1 < k_2 < \cdots$ such that $x^* - k_i\omega \in X_\infty$ for all $i \geq 0$.

By using $(\mathrm{S})_n$, which holds for all n, we can prove the analogue of Sublemma 6.6, namely,

$$\begin{aligned}\pi_2(F^{k_{i+1}}(x^* - k_{i+1}\omega, P_+)) &\subset \pi_2(F^{k_i}(x^* - k_i\omega, P_+)) \\ &\subset (-\lambda, -\lambda^{3/4}] \cup [\lambda^{3/4}, \lambda) \quad \text{for all } i \geq 1\end{aligned}$$

and

$$|\pi_2(F^{k_{i+1}}(x^* - k_{i+1}\omega, P_+))| \to 0 \quad \text{as } i \to \infty.$$

To see that the last inclusion above holds, recall that, in particular, $x^* - \omega \in X_\infty$. Therefore, $\pi_2(F^{k_1-1}(x^* - k_1\omega, P_+)) \subset P_+$. Since $I_0 \cap X_\infty = \emptyset$ (so $x^* - \omega \notin I_0$), we get $\pi_2(F^{k_i}(x^* - k_i\omega, P_+)) \subset (-\lambda, -\lambda^{3/4}] \cup [\lambda^{3/4}, \lambda)$.

It follows that there is a unique $r^+ \in P^+$ such that $(x^*, r^+) \in F^{k_i}(x^* - k_i\omega, P_+)$ for all $i \geq 1$. And, letting $s_j^+ = \pi_2(F^{-j}(x^*, r^+)$, we get, similarly to Sublemma 6.7, that $|s_k^+ \cdots s^+| > \lambda^{(k+1)/2}$ for all $k \geq 0$.

A. Contraction and derivative estimates

In this appendix, we have collected some elementary results of a computational nature.

A.1. Contraction

We use the notation from Section 2.2.

Lemma A.1. *Assume that $n \geq 1$ and that there is an $x \in \mathbb{T}$ and $\omega \in \mathbb{T}$ such that, for every $r \in P_+$, there holds $|r_n \cdots r_1| \geq \lambda^{2n/3}$. Then,*

$$|\pi_2(F^{n+1}(\{x\} \times P_+))| \leq 2\lambda^{-3/4}\lambda^{-4n/3}.$$

Moreover, if we let $r^- = \lambda^{3/4}$ and $r^+ = -\lambda^{3/4}$, then

$$\pi_2(F^{n+1}(\{x\} \times P_+) = [r_{n+1}^-, r_{n+1}^+].$$

Proof. Since $r_1 = v(x) - 1/r$, it is easy to check that

$$\pi_2(F(\{x\} \times P_+) = [v(x) - \lambda^{-3/4}, v(x) + \lambda^{-3/4}] = [r_1^-, r_1^+];$$

and, by induction, we have the formula

$$r_{n+1} - \rho_{n+1} = \frac{r_1 - \rho_1}{r_n \cdots r_1 \rho_n \cdots \rho_1}.$$

Thus, for any $r, \rho \in P_+$, we have, by using the assumption,

$$|r_{n+1} - \rho_{n+1}| \leq |r_1 - \rho_1|\lambda^{-4n/3} \leq 2\lambda^{-3/4}\lambda^{-4n/3}.$$

Since $r_{n+1}^{\pm}$ must be boundary points of $\pi_2(F^{n+1}(\{x\} \times P_+))$, we can therefore conclude that $\pi_2(F^{n+1}(\{x\} \times P_+)) = [r_{n+1}^-, r_{n+1}^+]$, and thus

$$|\pi_2(F^{n+1}(\{x\} \times P_+))| \leq 2\lambda^{-3/4}\lambda^{-4n/3}. \qquad \blacksquare$$

We have the corresponding result for backward iterations.

Lemma A.2. *Assume that $n \geq 1$ and that there is an $x \in \mathbb{T}$ and $\omega \in \mathbb{T}$ such that, for every $s \in P_-$, there holds $|s_n \cdots s_1| \leq \lambda^{-2n/3}$. Then,*

$$|\pi_2(F^{-n}(\{x\} \times P_-))| \leq 2\lambda^{-3/4}\lambda^{-4n/3}$$

Moreover, if we let $s^- = -\lambda^{-3/4}$ and $s^+ = \lambda^{-3/4}$, then

$$\pi_2(F^{-n}(\{x\} \times P_-) = [s_n^-, s_n^+].$$

Proof. We use the formula

$$s_n - \sigma_n = (s_0 - \sigma_0)(s_1 \cdots s_n)(\sigma_1 \cdots \sigma_n). \qquad \blacksquare$$

A.2. Derivative

In this section, we assume that $|E| \leq \lambda/2$. Thus, we have $\|v\|_{C^0} \leq 5\lambda/6$.

A.2.1. Forward. We assume that r_0 is a constant and that $n \geq 2$ is fixed. For our applications, we define

$$r_k = r_k(x, \omega) = \pi_2(F_\omega^k(x - n\omega, r_0)).$$

Then, we have

$$r_{k+1} = v(x - (n-k)\omega) - \frac{1}{r_k} \quad \text{for } k \geq 0. \tag{A.1}$$

Lemma A.3. *Assume that, for some fixed $(x, \omega) \in \mathbb{T}$, the following holds for some $n \geq 2$: $|r_{n-1} \cdots r_j| \geq \lambda^{2(n-j)/3}$ for all $1 \leq j \leq n-1$. Then $|\partial_\alpha(1/r_{n-1})(x, \omega)| < \lambda^{-1/4}$ for $\alpha = x, \omega$.*

Proof. We estimate $\partial_\omega(1/r_{n-1})$. The estimate of $\partial_x(1/r_{n-1})$ is similar.

Differentiating (A.1) with respect to ω gives

$$\partial_\omega r_{k+1} = -(n-k)v'(x - (n-k)\omega) + \frac{\partial_\omega r_k}{r_k^2}. \tag{A.2}$$

For the estimate, it does not matter at which points we evaluate v', so we do not write out explicitly at which points they are evaluated.

By inductively using the formula (A.2), and recalling that r_0 is constant, we get

$$\partial_\omega\Big(\frac{1}{r_{n-1}}\Big) = -\frac{\partial_\omega r_{n-1}}{r_{n-1}^2} = \frac{2v'}{r_{n-1}^2} + \cdots + \frac{(n-1)v'}{(r_{n-1}\cdots r_1)^2}.$$

Using the assumption yields

$$\Big|\partial_\omega\Big(\frac{1}{r_{n-1}}\Big)\Big| < C\lambda^{-1/3}.$$

■

A.2.2. Backward. We consider backward iterations analogously. We assume that s_0 is a constant and that $n \geq 3$ is fixed. Define

$$s_k = s_k(x,\omega) = \pi_2(T_\omega^{-k}(x+n\omega, s_0)).$$

Then, we have

$$s_{k+1} = \frac{1}{v(x+(n-k-1)) - s_k} \quad \text{for } k \geq 0.$$

Lemma A.4. *Assume that, for some fixed $(x,\omega) \in \mathbb{T}$, the following holds for some $n \geq 3$: $|s_{n-2}\cdots s_j| \leq \lambda^{-2(n-j)/3}$ for all $1 \leq j \leq n-2$. Then, $|\partial_\alpha s_{n-2}(x,\omega)| < \lambda^{-1/4}$ for $\alpha = x, \omega$.*

A.3. Geometry

The lemma in this section is adapted for the proof of Proposition 4.4.

As before, we let

$$g(x) = g(x,\omega) = v(x) - \frac{1}{v(x-\omega)} - \frac{1}{v(x+\omega)}.$$

We also recall the definition of the set $\widetilde{\Omega}_0$ in (4.6).

Lemma A.5. *Let $\omega \in \widetilde{\Omega}_0$ and assume that $0 < \varepsilon < \lambda^{-1}$. Assume that $I = [p,q]$ is an interval in I_0^j (some j), and that the functions $e^\pm, w^\pm$ are C^2 in a neighborhood of $I \times \{\omega\}$ and such that, for each $x \in I$, there holds*

$$|e^\pm(x,\omega)|, |w^\pm(x,\omega)| < \lambda^{-1/2}, \tag{A.3}$$

$$0 < e^+(x,\omega) - e^-(x,\omega) < \varepsilon, \quad 0 < w^+(x,\omega) - w^-(x,\omega) < \varepsilon, \tag{A.4}$$

$$|\partial_\alpha e^\pm(x,\omega)|, |\partial_\alpha w^\pm(x,\omega)| < \lambda^{-1/4} \quad \textit{for } \alpha = x,\omega. \tag{A.5}$$

Letting

$$\varphi^\pm(x,\omega) = v(x) - \frac{1}{v(x-\omega) + e^\pm(x,\omega)}$$

and

$$\psi^{\pm}(x,\omega) = \frac{1}{v(x+\omega) - w^{\pm}(x,\omega)},$$

and

$$\theta^{+}(x) = \varphi^{+}(x) - \psi^{-}(x)$$

and

$$\theta^{-}(x) = \varphi^{-}(x) - \psi^{+}(x),$$

we further assume that $g'(x) \neq 0$ *on* I *and*

$$\theta^{+}(p) \leq 0 \leq \theta^{-}(q) \quad \text{if } g'(x) > 0 \text{ on } [p,q], \tag{A.6a}$$

$$\theta^{-}(p) \geq 0 \geq \theta^{+}(q) \quad \text{if } g'(x) < 0 \text{ on } [p,q]. \tag{A.6b}$$

If

$$\begin{aligned} A' &= \{(x,r) : x \in I, \varphi^{-}(x) \leq r \leq \varphi^{+}(x)\} \\ B' &= \{(x,r) : x \in I, \psi^{-}(x) \leq r \leq \psi^{+}(x)\}, \end{aligned}$$

then there exist $p \leq p_1 < q_1 \leq q$, $q_1 - p_1 < \varepsilon\lambda^{-5/4}$, *such that* $\pi_1(A' \cap B') = [p_1, q_1]$, *and* $\theta^{+}(p_1) = \theta^{-}(q_1) = 0$ *if* $g'(x) > 0$ *on* $[p,q]$, *and* $\theta^{-}(p_1) = \theta^{+}(q_1) = 0$ *if* $g'(x) < 0$ *on* $[p,q]$. *Moreover, if* $\theta = \theta^{+}$ *or* θ^{-} *and if* $a = a(\omega) \in I$ *is such that* $\theta(a(\omega),\omega) = 0$, *then* $|a'(\omega)| < 1$.

Proof. By the definition of the sets A', B' and the functions $\theta^{\pm}$, we note that

$$\pi_1(A' \cap B') = \{x \in I : \theta^{-}(x) \leq 0 \leq \theta^{+}(x)\}. \tag{A.7}$$

From the assumption on ω, we have that $I_0 \cap (I_0 \pm \omega) = \emptyset$. Combining this with (A.3), we have the bounds

$$|v(x-\omega) + e^{\pm}(x,\omega)|, |v(x+\omega) + w^{\pm}(x,\omega)| > 2\lambda^{3/4} \quad \text{for all } x \in I. \tag{A.8}$$

Thus, we have

$$|\theta^{\pm}(x) - g(x)| < \lambda^{-2} \quad \text{for all } x \in I \tag{A.9}$$

and, by using (A.5), we have for $\alpha = x, \omega$:

$$|\partial_\alpha g(x,\omega) - \partial_\alpha \theta^{\pm}(x,\omega)| < \lambda^{-1} \quad \text{for all } x \in I. \tag{A.10}$$

Moreover, from assumption (A.4) it follows that

$$0 < \theta^{+}(x) - \theta^{-}(x) < \varepsilon\lambda^{-3/2}, \quad x \in I. \tag{A.11}$$

We consider the case when $g'(x) > 0$ on I. Note that we have

$$|\theta^{\pm}(x)| < \lambda^{-2} \implies (\theta^{\pm})'(x) > \lambda^{-1/4}. \tag{A.12}$$

Indeed, if $|\theta^{+}(x)| < \lambda^{-2}$, then by (A.9) we have $|g(x)| < 2\lambda^{-2}$; and since (A.8) holds, we must then have $v(x) < 1/2$. If I_0^j is of type (i), it follows from Lemma 2.10 that $|v'(x)| > 2\lambda^{3/4}$; we conclude that $g'(x) > \lambda^{3/4}$. If I_0^j is of type (ii), it follows from Lemma 2.12 that $g'(x) > 2\lambda^{-1/4}$. In both cases, the statement now follows from (A.10).

By assumption (A.6), we have $\theta^{+}(p) \leq 0$ and $\theta^{+}(q) > 0$; and $\theta^{-}(p) < 0$ and $\theta^{-}(q) \geq 0$ Since (A.10) and (A.11), hold we conclude that there are unique $p \leq p_1 < q_1 \leq q$ such that $\theta^{+}(p_1) = 0$ and $\theta^{-}(q_1) = 0$. Recalling (A.7), we see that $\pi_1(A' \cap B') = [p_1, q_1]$. Furthermore, by (A.11) we have $\theta^{-}(p_1) > -\varepsilon\lambda^{-3/2} > -\lambda^{-2}$. Since $\theta^{-}(q_1) = 0$ and since (A.12) holds, we obtain the bound $q_1 - p_1 < \varepsilon\lambda^{-5/4}$.

To verify the last statement, assume that $\theta = \theta^{+}$ and that $\theta(a(\omega), \omega) = 0$. By differentiating with respect to ω, we get

$$a'(\omega) = -\frac{\partial_\omega \theta(a(\omega), \omega)}{\partial_x \theta(a(\omega), \omega)} \tag{A.13}$$

Since $\theta(a(\omega), \omega) = 0$, it follows from (A.12) that $\partial_x \theta(a(\omega), \omega) > \lambda^{-1/4}$. Moreover, we have $|\partial_\omega g(x, \omega)| < \lambda^{-1/3}/2$ on I (recall (A.8)); combined with (A.10) this gives $|\partial_\omega \theta(a(\omega), \omega)| < \lambda^{-1/3}$. Plugging these two estimates into (A.13) yields the result. ■

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Kristian Bjerklöv
Department of Mathematics, KTH Royal Institute of Technology, Lindstedtsvägen 25, 100 44 Stockholm, Sweden; bjerklov@kth.se