J. Spectr. Theory 16 (2026), 1143–1196
DOI 10.4171/JST/624

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Comb domains of Schrödinger operators with small quasi-periodic potentials

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Abstract. We characterize spectra of Schrödinger operators with small quasi-periodic analytic potentials in terms of their comb domains, and study action variables motivated by the KdV integrable system.

Dedicated to Michael Goldstein

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1. Introduction

For a one-dimensional Schrödinger operator

$$[H_V u](x) = -u''(x) + V(x)u(x), \quad x \in \mathbb{R}$$

with an almost periodic potential V, by gap labelling theory, the spectrum can be expressed in the form

$$\mathcal{S} = [E_0, \infty) \setminus \bigcup_{k \in \mathcal{M} \cap (0,\infty)} (E_k^-, E_k^+),$$

Mathematics Subject Classification 2020: 34L40 (primary); 30C20, 37K10 (secondary).
Keywords: quasi-periodic operator, comb domain, Marchenko–Ostrovskii map, action variable.

where $\mathcal{M}$ is the frequency module of V and $E_0 < E_k^- \le E_k^+ < E_l^-$ whenever $k, l \in \mathcal{M}$ with $0 < k < l$; compare Johnson and Moser [38] as well as [12,13,39]. The k-th gap is said to be *open* when $E_k^- < E_k^+$. If V is periodic with period T, then $\mathcal{M} = T^{-1}\mathbb{Z}$, which results in the band/gap structure of the spectrum. In this case, the connection between the sequence of gap lengths and smoothness of the potential is akin to a non-linear Fourier transform [36, 48, 54, 63]: the decay rate of gap lengths encodes the smoothness of the potential. If V is not periodic, $\mathcal{M}$ is dense in $\mathbb{R}$, and Cantor spectrum is common [24, 27, 49]; in fact, the spectrum of an almost periodic operator can have zero Lebesgue measure, and even zero Hausdorff dimension [14]. Similar results, and in fact more refined and detailed ones, are known for discrete Schrödinger operators with almost periodic potentials; compare [1–6, 29, 32, 34, 35, 56, 66].

Conversely, it is of great interest to establish strong upper bounds on gap sizes when possible, with decay with respect to an appropriate indexing of $\mathcal{M} \cap (0, \infty)$. For a class of limit-periodic potentials exponentially well approximated by periodic potentials, upper bounds were established by Pastur and Tkachenko [51, 52], see also [11] for this class of potentials. In the quasi-periodic setting, Damanik and Goldstein [15] derived exponentially decaying upper bounds for small analytic quasi-periodic operators with Diophantine frequencies, by a multiscale analysis on the dual group. This work led to thickness properties of the spectrum such as homogeneity and Craig-type conditions [17], which in turn led to a description of the isospectral torus [18, 19] and applications to the KdV equation for small quasi-periodic initial data [8, 16]. See also related work on gap sizes and homogeneity [20, 46] and on the KdV equation and the Toda flow [9, 10, 23, 47, 64, 65].

The central objects of this paper are Marchenko–Ostrovskii maps [48], originally introduced in the periodic setting in order to provide a parametrization of spectra of periodic Schrödinger operators. Let us recall a more general definition given by Johnson and Moser [38]. Denote by $m(x; z)$ the Weyl function of the half-line operator $-\partial_x^2 + V$ on $[x, \infty)$, and denote $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. If V is almost periodic, Johnson and Moser [38] proved that $m(x, z)$ is an almost periodic function of x for any $z \in \mathbb{C}_+$, and introduced its mean value

$$w(z) = \mathbb{E}(m(x; z)) = \lim_{X \to \infty} \frac{1}{X} \int_0^X m(x; z)\, dx \tag{1.1}$$

(in this paper, $\mathbb{E}$ always denotes the mean value of an almost periodic function of $x \in \mathbb{R}$, as above). The Marchenko–Ostrovskii map w is a fundamental object of interest, since $-\operatorname{Re} w$ is the Lyapunov exponent, $\operatorname{Im} w$ is the rotation number, and w is an integral transform of the density of states of H_V; the resulting connection between the Lyapunov exponent and density of states is the celebrated Thouless formula [50]. We note also that a further generalization of Marchenko–Ostrovskii maps to ergodic

operators was studied by Kotani [44], and that there is a corresponding theory even beyond the ergodic setting, which first appeared as the theory of Stahl–Totik regularity [61] in the orthogonal polynomial setting, and was extended to the Schrödinger operator setting in [22].

The formula (1.1) defines w on the set $\mathbb{C} \setminus [E_0, \infty)$, and w is shown to be injective on this set, so it is a conformal bijection; we follow a standard convention to rotate the image clockwise and call the set

$$\Omega = -i\, w(\mathbb{C} \setminus [E_0, \infty))$$

the Marchenko–Ostrovskii domain corresponding to V. This provides a powerful connection between spectral theory and geometric function theory. On the set $\mathbb{C} \setminus [E_0, \infty)$, the function obeys the symmetry $\overline{w(\bar{z})} = w(z)$ and the asymptotics

$$w(z) = -\sqrt{-z} - \frac{\mathbb{E}(V)}{2\sqrt{-z}} + o(|z|^{-1/2}), \quad z \to -\infty, \tag{1.2}$$

so the domain Ω determines the map w up to an additive constant; if w corresponds to Ω, so does $w(\cdot - c)$, for $c \in \mathbb{R}$. Since w determines the density of states measure, and the support of this measure is the spectrum $\mathcal{S}$, the domain Ω also determines $\mathcal{S}$ uniquely up to a constant. The constant corresponds to a constant shift in the potential and to a translation of the spectrum, and it can be fixed by setting $\mathbb{E}(V) = 0$ (common in direct spectral theory [18, 41]) or by setting $E_0 = 0$ (common in inverse spectral theory [23]).

In the periodic setting, the domain Ω is a comb domain of the form

$$\mathbb{C}_+ \setminus \bigcup_{n \in \mathbb{Z} \setminus \{0\}} (n/T, n/T + i h_n],$$

with $h_{-n} = h_n \geq 0$. The slits $(n/T, n/T + i h_n]$ correspond to gaps of the spectrum, and Marchenko–Ostrovskii proved that Sobolev space conditions $V \in H^k(\mathbb{T})$ are equivalent to weighted ℓ^2 conditions on the slits $\sum_n (n^{k+1} h_n)^2 < \infty$. This has inspired further work in periodic settings [43, 62] and beyond; see [25, 26, 37]. Beyond the periodic setting, comb domains have also appeared in the characterization of a class of limit periodic potentials, a class of reflectionless operators on homogeneous spectra [33, 52, 59, 60], and a class of limit periodic operators associated with quadratic iterations [58].

When the Lyapunov exponent is zero on the spectrum, the harmonic function $-\operatorname{Re} w$ on $\mathbb{C} \setminus \mathcal{S}$ is precisely the Martin function for the Denjoy domain $\mathbb{C} \setminus \mathcal{S}$ normalized by (1.2). This connection between Martin functions and reflectionless operators plays a prominent role in the papers of Yuditskii and coauthors [7, 21, 23, 42, 59, 67].

In the first part of this paper, we characterize the Marchenko–Ostrovskii domains corresponding to small quasi-periodic operators with Diophantine frequencies. These

are comb domains, and we show how the structure of the comb domain encodes the smallness and analyticity of the quasi-periodic potential. To proceed, we must precisely define the class of operators.

Definition 1.1. Let $0 < \kappa_0 \leq 1$ and $\omega \in \mathbb{R}^\nu$ for some $\nu \in \mathbb{N}$. We denote by $\mathcal{B}(\omega, \kappa_0)$ the Banach space of functions $V : \mathbb{R} \to \mathbb{R}$ of the form

$$V(x) = \sum_{n \in \mathbb{Z}^\nu} c(n) e^{2\pi i n \omega x}$$

for which the norm

$$\|V\|_{\infty,\kappa_0} = \sup_{n \in \mathbb{Z}^\nu} |c(n)| \exp(\kappa_0 |n|) \tag{1.3}$$

is finite. In particular, for $\varepsilon > 0$, we denote by $\mathcal{P}(\omega, \varepsilon, \kappa_0)$ the set of $V \in \mathcal{B}(\omega, \kappa_0)$ with $\|V\|_{\infty,\kappa_0} \leq \varepsilon$.

It has been common to view $\mathcal{P}(\omega, \varepsilon, \kappa_0)$ as a metric space with the metric inherited from $L^\infty(\mathbb{R})$, but the weighted ℓ^∞ norm (1.3) will be useful below. With this norm, the interior of $\mathcal{P}(\omega, \varepsilon, \kappa_0)$ is the set $\mathring{\mathcal{P}}(\omega, \varepsilon, \kappa_0)$ given by $\|V\|_{\infty,\kappa_0} < \varepsilon$.

We will assume that $\omega = (\omega_1, \dots, \omega_\nu) \in \mathbb{R}^\nu$ obeys the Diophantine condition

$$|n\omega| \geq a_0 |n|^{-b_0}, \quad n \in \mathbb{Z}^\nu \setminus \{0\} \tag{1.4}$$

for some

$$0 < a_0 < 1, \quad \nu < b_0 < \infty. \tag{1.5}$$

In (1.4), and everywhere else in this work, we use the ℓ^1 norm on $\mathbb{Z}^\nu$:

$$|n| = |(n_1, \dots, n_\nu)| = \sum_j |n_j|, \quad n \in \mathbb{Z}^\nu.$$

Potentials $V \in \mathcal{P}(\omega, \varepsilon, \kappa_0)$ were studied in [15] via a multiscale analysis method, which used smallness, analyticity, and Diophantine frequency to derive much more precise description of the gap sizes and their distances. An extension of this multiscale analysis to Abelian groups [19], combined with periodic approximation, was used to describe the isospectral torus of $V \in \mathcal{P}(\omega, \varepsilon, \kappa_0)$ [18], to show that the corresponding spectrum is homogeneous [17], and to study the KdV equation with this class of initial data [8].

We will apply this multiscale analysis and periodic approximation approach to describe the comb domains.

Theorem 1.2. *Assume that ω obeys the Diophantine condition (1.4), (1.5). There exists $\varepsilon_1 = \varepsilon_1(a_0, b_0, \kappa_0) > 0$ such that, if $\varepsilon < \varepsilon_1$ and $V \in \mathcal{P}(\omega, \varepsilon, \kappa_0)$, the corresponding Marchenko–Ostrovskii domain is a comb domain of the form*

$$\mathbb{C}_+ \setminus \bigcup_{m \in \mathbb{Z}^\nu \setminus \{0\}} (m\omega, m\omega + i h_m] \tag{1.6}$$

and the slit sizes h_m obey $h_{-m} = h_m \geq 0$ and

$$h_m \leq \varepsilon^{1/2} \exp\Big(-\frac{\kappa_0}{5}|m|\Big).$$

Moreover, each h_m is a continuous function of $V \in \mathcal{P}(\omega, \varepsilon, \kappa_0)$.

Recall further that the isospectral torus $\mathcal{R}(\mathcal{S})$ is defined as the set of reflectionless operators with spectrum $\mathcal{S}$, and that the torus of Dirichlet data is the product of circles obtained as double covers of the gap closures,

$$\mathcal{D}(\mathcal{S}) = \prod_{\substack{k \in \mathcal{M} \cap (0,\infty) \\ E_k^- < E_k^+}} C_k, \quad C_k = ([E_k^-, E_k^+] \times \{-1, +1\})/_{(E_k^\pm, -1) \sim (E_k^\pm, +1)}.$$

If $\mathcal{S}$ is homogeneous and has finite gap length, by Sodin and Yudiskii [59, 60], the isospectral torus is parametrized by the Dirichlet data and consists of almost periodic Schrödinger operators. A priori, the frequency module can be read off from the set of bases of the slits; in particular, quasi-periodicity with frequency ω can be read off from (1.6). However, quantitative information about the analyticity and smallness of the sampling function is more specialized, and this is the subject of the following theorem.

Theorem 1.3. *Assume that ω obeys the Diophantine condition (1.4), (1.5). There exists $\varepsilon^{(0)}(a_0, b_0, \kappa_0) > 0$ such that the following holds. Let $h_m \geq 0$, indexed by $m \in \mathbb{Z}^\nu \setminus \{0\}$, obey the conditions $h_m = h_{-m}$ and*

$$h_m \leq \varepsilon' \exp(-\kappa|m|)$$

with $\varepsilon' < \varepsilon^{(0)}$ and $\kappa \geq 5\kappa_0$. Let $\mathcal{S}$ be the corresponding spectrum with $\min \mathcal{S} = 0$. For every $\mu \in \mathcal{D}(\mathcal{S})$, there is a corresponding potential $V \in \mathcal{R}(\mathcal{S})$ with $V \in \mathcal{P}(\omega, (\varepsilon')^{1/3}, \kappa/3)$.

In particular, $\mathcal{R}(\mathcal{S}) \subset \mathcal{P}(\omega, (\varepsilon')^{1/3}, \kappa/3)$.

The proof consists of constructing, for any Dirichlet data, a quasi-periodic potential $V \in \mathcal{P}(\omega, (\varepsilon')^{1/4}, \kappa/3)$ with the desired comb domain and Dirichlet data. The explicit constants in these theorems are not optimized.

The proofs of Theorems 1.2 and 1.3 use the results of the multiscale analysis in [15, 19] to obtain direct and inverse statements for the comb domains of periodic approximants of quasi-periodic potentials. Moreover, we adapt the periodic approximation approach developed in [18] to work with comb domains rather than gap lengths, and combine it with geometric function theory to obtain direct and inverse statements for the comb domains of quasi-periodic potentials.

In the second part of the paper, we derive some differentiability results motivated by the inverse spectral theory of periodic operators. By general principles, gap edges

are 1-Lipschitz with respect to the L^∞-norm. We will consider differentiability of gap edges with respect to the potential, expressed in the language of Fréchet derivatives; for general background see [55, Appendix A] and note that $\mathring{\mathcal{P}}(\omega, \varepsilon, \kappa_0)$ is an open ball in the Banach space $\mathcal{B}(\omega, \kappa_0)$. Gap length cannot be a C^1 function on $\mathring{\mathcal{P}}(\omega, \varepsilon, \kappa_0)$; it is not differentiable even in the periodic case. However, the following related quantities are differentiable.

Theorem 1.4. *Assume that ω obeys the Diophantine condition (1.4), (1.5). There exists $\varepsilon_1 = \varepsilon_1(a_0, b_0, \kappa_0) > 0$ such that, for $m \in \mathbb{Z}^\nu$ with $m\omega > 0$, the functions*

$$\gamma_m^2 := (E_m^+ - E_m^-)^2, \quad \tau_m := \frac{E_m^- + E_m^+}{2} \tag{1.7}$$

are C^1 functions on $\mathring{\mathcal{P}}(\omega, \varepsilon_1, \kappa_0)$.

The proof will also provide explicit expressions for their derivatives, in terms of quasi-periodic eigensolutions of the Schrödinger operators at the gap edges, which mirror known formulas in the periodic setting. The periodic problem is closely related to the Dirichlet problem on an interval, and gap edges can be interpreted as isolated eigenvalues of operators with periodic/antiperiodic boundary conditions, so their differentiability as a function of the potential can be obtained using perturbation theory. In the quasi-periodic setting, there is no such interpretation; thus, the proof of Theorem 1.4 uses the results of the periodic theory, rather than its techniques.

We will also study a set of actions on the open set of quasi-periodic potentials $\mathring{\mathcal{P}}(\omega, \varepsilon, \kappa_0)$, which we define as follows: for $V \in \mathcal{P}(\omega, \varepsilon, \kappa_0)$ and $m \in \mathbb{Z}^\nu$ with $m\omega > 0$, let

$$I_m := -\frac{2}{\pi} \int_{E_m^-}^{E_m^+} \operatorname{Re} w(\lambda)\, d\lambda. \tag{1.8}$$

If $E_m^- = E_m^+$, the convention $I_m = 0$ is natural. Actions were originally introduced by Flaschka and McLaughlin [28] in the periodic case by an expression in terms of the discriminant, which can be rewritten in the form (1.8). In the periodic setting, integrability of the KdV equation motivated the construction of action-angle coordinates for periodic initial data $V \in L^2(\mathbb{T})$, whose Cartesian counterpart are a global, real analytic system of Birkhoff coordinates [40, 41]. In these coordinates, the KdV equation is represented as an integrable Hamiltonian system on $L^2(\mathbb{T})$, with the Hamiltonian a function of the actions alone. Joint analyticity of this set of coordinates has played an important role in further studies of the periodic KdV equation and has been the basis for applications of KAM techniques to perturbations of the periodic KdV equation [41].

From a spectral theorist's perspective, in the periodic case, the actions describe the isospectral torus and they stay constant along the KdV time evolution, whereas the angles are a set of coordinates on the isospectral torus and they evolve linearly with time. In the reflectionless non-periodic setting, inverse spectral theory is based on character-automorphic functions on the resolvent set of the Schrödinger operator. All results in this setting describe single solutions, and the set of angles is reinterpreted as an element of the character group [23,59,60]. It is a natural question to what extent the actions (1.8), combined with these angles/characters, can play the analogous role in the quasi-periodic setting, and how much of a Hamiltonian system structure can be found in this setting. Differentiability of the actions is a step in this direction.

Theorem 1.5. *Assume that ω obeys the Diophantine condition (1.4), (1.5). There exists $\varepsilon_1 = \varepsilon_1(a_0, b_0, \kappa_0) > 0$ such that, for $m \in \mathbb{Z}^\nu$ with $m\omega > 0$, the action I_m is well-defined for $V \in \mathcal{P}(\omega, \varepsilon_1, \kappa_0)$ and has the following properties.*

(a) *$I_m \geq 0$, with $I_m = 0$ if and only if the m-th gap is closed*

(b) *Actions obey the sum identity,*

$$\sum_{\substack{m\in\mathbb{Z}^\nu \\ m\omega>0}} 2\pi m\omega I_m(V) = \frac{1}{2}\mathbb{E}(V^2). \tag{1.9}$$

(c) *I_m is a C^1 function on $\mathring{\mathcal{P}}(\omega, \varepsilon_1, \kappa_0)$, and its Fréchet derivative is the functional*

$$(\partial_V I_m)(q) = \frac{2}{\pi}\mathbb{E}(q(x)\int_{E_m^-}^{E_m^+} G(x,x;\lambda,V)d\lambda), \tag{1.10}$$

where $G(x,x;\lambda,V)$ denotes the diagonal Green's function of the Schrödinger operator H_V.

(d) *The function I_m/γ_m^2, initially defined on the set*

$$\{V \in \mathring{\mathcal{P}}(\omega, \varepsilon_1, \kappa_0) \mid \gamma_m(V) > 0\},$$

extends to a strictly positive, C^1 function on $\mathring{\mathcal{P}}(\omega, \varepsilon_1, \kappa_0)$.

Part (d) tells us that the action I_m scales quadratically with gap size as the m-th gap closes. It is a strengthening of (c); in the periodic setting, this strengthening is a necessary step in describing the Birkhoff coordinates (compare [41, Theorem 7.3]).

The proofs cannot follow the strategy from the periodic theory. The periodic theory relies heavily on the behavior along a single period and on objects, such as the monodromy matrix, which do not exist in non-periodic settings. The monodromy matrix can be defined for complex periodic potentials, so some variables are naturally

found to be complex analytic [41]; this has no counterpart in our setting. Instead, in this paper, as in [18], we use periodic approximation. Let $\omega^{(r)} \to \omega$ be a canonical sequence of rational approximants for ω; this sequence will be fixed in what follows. Since ω has rationally independent components, each $V \in \mathcal{P}(\omega, \varepsilon, \kappa_0)$ corresponds to a unique continuous function $U: \mathbb{T}^\nu \to \mathbb{R}$ such that $V(x) = U(\omega x)$. This, in turn, allows us to unambiguously define

$$V^{(r)}(x) = U(\omega^{(r)} x) = \sum_{n \in \mathbb{Z}^\nu} c(n) e^{2\pi i n \omega^{(r)} x}. \tag{1.11}$$

We therefore have a map $\mathcal{P}(\omega, \varepsilon, \kappa_0) \to \mathcal{P}(\omega^{(r)}, \varepsilon, \kappa_0)$ given by $V \mapsto V^{(r)}$.

Our method also requires approximation of analytic sampling functions by those with subexponentially decaying Fourier coefficients, replacing the condition

$$|c(n)| \le \varepsilon \exp(-\kappa_0 |n|), \qquad \text{for all } n \in \mathbb{Z}^\nu$$

by

$$|c(n)| \le \varepsilon \exp(-\kappa_0 |n|^{\alpha_0}), \quad \text{for all } n \in \mathbb{Z}^\nu$$

for some $\alpha_0 \in (0, 1]$. The corresponding set of quasi-periodic potentials is denoted by $\mathcal{P}(\omega, \varepsilon, \kappa_0, \alpha_0)$. Of course, $\mathcal{P}(\omega, \varepsilon, \kappa_0, 1) = \mathcal{P}(\omega, \varepsilon, \kappa_0)$.

In Section 2, we consider a very general continuity property of the Marchenko–Ostrovskii maps away from the spectrum. In Section 3 we study two-sided estimates on the comb domains associated to periodic approximants $V^{(r)} \in \mathcal{P}(\omega^{(r)}, \varepsilon, \kappa_0)$. In Section 4 we combine this with geometric function theory to describe the comb domains of $V \in \mathcal{P}(\omega, \varepsilon, \kappa_0)$, proving Theorem 1.2. In Section 5 we modify a technique from [18] to prove Theorem 1.3 by periodic approximation. In Section 6 we prove Theorem 1.4 and in Section 7 we prove Theorem 1.5.

2. Continuity of Marchenko–Ostrovskii maps on $\mathbb{C}_+$

In this section, we prove very general continuity properties of Marchenko–Ostrovskii maps away from the spectrum.

For any bounded potential V, there is a unique (up to normalization) non-trivial solution $\psi(x, z)$ of $(-\partial_x^2 + V)\psi(x; z) = z\psi(x; z)$ which is square-integrable at $+\infty$. It is called the *Weyl solution;* in terms of it, the half-line Weyl functions are expressed as

$$m(x, z) = \frac{\partial_x \psi(x; z)}{\psi(x; z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

We will use the following continuity property of the Weyl function.

Lemma 2.1 ([45]). *If a sequence of potentials V_n converges to V uniformly on compacts, and m_n are m-functions corresponding to V_n, then*

$$\lim_{n\to\infty} m_n(x;z) = m(x;z)$$

uniformly on compacts in $z \in \mathbb{C} \setminus \mathbb{R}$.

In particular, this continuity property implies almost periodicity of $m(x,z)$ as a function of x for an almost periodic potential V, which justifies the definition (1.1). We will now specialize to the quasi-periodic setting

$$V_{U,\omega,\theta}(x) = U(\omega x + \theta),$$

where $U \in C(\mathbb{T}^\nu, \mathbb{R})$, $\omega \in \mathbb{R}^\nu$, $\theta \in \mathbb{T}^\nu$. We will denote the corresponding functions by $m_{U,\omega,\theta}(x;z)$ and $w_{U,\omega,\theta}(z)$. Since translation can also be written as a change of the sample point θ, we have the trivial relations

$$m_{U,\omega,\theta}(x;z) = m_{U,\omega,\omega x+\theta}(0;z),$$
$$w_{U,\omega,\theta}(z) = w_{U,\omega,\omega x+\theta}(z).$$

Lemma 2.2. *The function $m_{U,\omega,\theta}(x;z)$ is jointly continuous in the variables $(U,\omega,\theta) \in C(\mathbb{T}^\nu,\mathbb{R}) \times \mathbb{R}^\nu \times \mathbb{T}^\nu$.*

Proof. If $(U_n,\omega_n,\theta_n) \to (U,\omega,\theta)$, then the potentials $U_n(\omega_n x + \theta_n)$ converge uniformly on compacts to $U(\omega x + \theta)$, so by Lemma 2.1, we have $m_{U_n,\omega_n,\theta_n}(x;z) \to m_{U,\omega,\theta}(x;z)$. ∎

In particular, for fixed $x \in \mathbb{R}$ and $z \in \mathbb{C} \setminus \mathbb{R}$, $m_{U,\omega,\theta}(x;z)$ is a continuous function of (U,ω,θ), so it is uniformly continuous on compacts. This implies a convergence property of Marchenko–Ostrovskii maps.

Proposition 2.3. *Let $(U_n,\omega_n,\theta_n) \to (U,\omega,\theta)$. If the components of ω are linearly independent over $\mathbb{Q}$, then for all $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$\lim_{n\to\infty} w_{U_n,\omega_n,\theta_n}(z) = w_{U,\omega,\theta}(z),$$

and convergence is uniform in compacts in $\mathbb{C} \setminus \mathbb{R}$.

Note that the base space $\mathbb{T}^\nu$ is fixed in this result, but that ω_n can be rational; one of the applications is to periodic approximants of a quasi-periodic operator.

Proof. Denote $\omega = (\alpha_1,\dots,\alpha_\nu)$. Picking $t > 0$ such that $1, \alpha_1 t, \dots, \alpha_\nu t$ are linearly independent over $\mathbb{Q}$ and applying Birkhoff's ergodic theorem to the discrete rotation

with angle ωt implies

$$w_{U,\omega,\theta}(z) = \int_{\mathbb{T}^\nu} m_{U,\omega,\phi}(0;z)\, d\phi, \tag{2.1}$$

where $d\phi$ denotes normalized Lebesgue measure on $\mathbb{T}^\nu$. Since ω_n is not assumed to have linearly independent components, this is not in general true for $w_{U_n,\omega_n,\theta_n}$. Nonetheless, we will show that

$$\lim_{n\to\infty} \left| w_{U_n,\omega_n,\theta_n}(z) - \int_{\mathbb{T}^\nu} m_{U_n,\omega_n,\phi}(0;z)\, d\phi \right| = 0. \tag{2.2}$$

Since $U_n \to U$ in $C(\mathbb{T}^\nu, \mathbb{R})$, this sequence is uniformly equicontinuous on $\mathbb{T}^\nu$ and $\{U_n \mid n \in \mathbb{N}\} \cup \{U\}$ is a compact subset of $C(\mathbb{T}^\nu, \mathbb{R})$. Combining the continuity from Lemma 2.2 with compactness implies uniform continuity: for any $\epsilon > 0$ there is a $\delta > 0$ such that $|\phi - \phi'| < \delta$ implies

$$|m_{U_n,\omega_n,\phi}(0;z) - m_{U_n,\omega_n,\phi'}(0;z)| < \epsilon$$

for all n, and therefore also

$$|w_{U_n,\omega_n,\phi}(z) - w_{U_n,\omega_n,\phi'}(z)| < \epsilon.$$

Since $\omega\mathbb{R}$ is dense in $\mathbb{T}^\nu$, for each $\epsilon > 0$ there exists n_0 such that for $n > n_0$, $\omega_n\mathbb{R}$ is ϵ-dense in $\mathbb{T}^\nu$. Thus, for any $\phi \in \mathbb{T}^\nu$, we can find a ϕ' of the form $\phi' = \theta_n + \omega_n x$ such that $|\phi - \phi'| < \epsilon$. Since $w_{U_n,\omega_n,\theta_n} = w_{U_n,\omega_n,\theta_n+\omega_n x}$, this implies that for any ϕ,

$$|w_{U_n,\omega_n,\theta_n}(z) - w_{U_n,\omega_n,\phi}(z)| < \epsilon.$$

Integrating in ϕ and using

$$\int_{\mathbb{T}^\nu} w_{U_n,\omega_n,\phi}(z)\, d\phi = \int_{\mathbb{T}^\nu} m_{U_n,\omega_n,\phi}(0;z)\, d\phi,$$

we obtain (2.2).

The result now follows from (2.1) and (2.2) since, by uniform continuity,

$$\lim_{n\to\infty} \int_{\mathbb{T}^\nu} m_{U_n,\omega_n,\phi}(0;z)\, d\phi = \int_{\mathbb{T}^\nu} m_{U,\omega,\phi}(0;z)\, d\phi. \qquad \blacksquare$$

Our main focus is on small quasi-periodic potentials with analytic sampling functions, and the following lemma is useful for specializing the above results.

Lemma 2.4. *Fix $\varepsilon, \kappa_0 > 0$ and $\nu \in \mathbb{N}$. Consider sampling functions $U_n, U : \mathbb{T}^\nu \to \mathbb{R}$ with exponentially decaying Fourier coefficients:*

$$|\widehat{U}_n(m)| \le \varepsilon e^{-\kappa_0|m|}, \quad |\widehat{U}(m)| \le \varepsilon e^{-\kappa_0|m|}, \quad \text{for all } m \in \mathbb{Z}^\nu.$$

Let $Q_n(x) = U_n(\omega_n x)$, $Q(x) = U(\omega x)$, for a sequence $\omega_n \to \omega$, where $\omega \in \mathbb{R}^\nu$ has linearly independent entries over $\mathbb{Q}$. Then the following statements are equivalent:

(a) *$Q_n \to Q$ uniformly on compact subsets of $\mathbb{R}$;*

(b) *$U_n \to U$ uniformly on $\mathbb{T}^\nu$;*

(c) *for all $m \in \mathbb{Z}^\nu$, $\widehat{U}_n(m) \to \widehat{U}(m)$.*

Proof. The equivalence of (b) and (c) comes from the observation that on the set of functions of the form

$$\sum_{m \in \mathbb{Z}^\nu} c(m) e^{2\pi i m\phi}, \quad |c(m)| \le \varepsilon e^{-\kappa_0|m|}$$

pointwise convergence of Fourier coefficients is equivalent to uniform convergence on $C(\mathbb{T}^\nu)$, simply because the a priori upper bound $\varepsilon e^{-\kappa_0|m|}$ is summable in $m \in \mathbb{Z}^\nu$.

(b) $\implies$ (a) is trivial.

(a) $\implies$ (b). We first note that

$$\sum_{m \in \mathbb{Z}^\nu} (1 + |m|) e^{-\kappa_0|m|} < \infty$$

so the functions U_n, U are equicontinuous. By equicontinuity, $U_n(\omega_n x) \to U(\omega x)$ implies $U_n(\omega x) \to U(\omega x)$, and then convergence on the dense set $\{x\omega \mid x \in \mathbb{R}\} \subset \mathbb{T}^\nu$ implies uniform convergence $U_n \to U$. ∎

3. Marchenko–Ostrovskii maps of periodic approximants

There are many results about the size and location of the gaps of periodic spectra. Those results are typically derived for unit period, and when rescaled to arbitrary period, the estimates depend exponentially on the period. For the purpose of periodic approximation of almost periodic operators, such estimates are only useful when the periodic approximant is superexponentially close to the almost periodic potential, as in the Pastur–Tkachenko class of limit-periodic operators.

We will instead derive estimates for the case of analytic sampling functions, which will be uniform for a sequence of periodic approximants of a quasi-periodic operator.

With this motivation, even though the potential is periodic, we use a quasi-periodic representation

$$\widetilde{V}(x) = \sum_{m\in\mathbb{Z}^\nu} c(m) e^{2\pi i x m\tilde{\omega}}, \quad x \in \mathbb{R}, \tag{3.1}$$

where $\tilde{\omega} = (\tilde{\omega}_1, \dots, \tilde{\omega}_\nu) \neq 0$ is a vector with rational components. The potential $\widetilde{V}$ is T-periodic, where $T > 0$ is uniquely determined by the requirement that

$$\{m\tilde{\omega} \mid m \in \mathbb{Z}^\nu\} = \frac{1}{T}\mathbb{Z}. \tag{3.2}$$

To further relate the quasi-periodic representation (3.1) of the periodic potential $\widetilde{V}$ to periodic theory, we introduce the following notation. By the first isomorphism theorem for groups, the map $m \mapsto m\tilde{\omega}$ generates an isomorphism between the additive group $T^{-1}\mathbb{Z}$ and the quotient group

$$\mathfrak{Z}(\tilde{\omega}) := \mathbb{Z}^\nu/\mathfrak{N}(\tilde{\omega}), \quad \mathfrak{N}(\tilde{\omega}) := \{m \in \mathbb{Z}^\nu \mid m\tilde{\omega} = 0\}.$$

We denote the coset of $m \in \mathbb{Z}^\nu$ by $\mathfrak{m} = m + \mathfrak{N}(\tilde{\omega})$, write $\mathfrak{m}\tilde{\omega} = m\tilde{\omega}$, and equip $\mathfrak{Z}(\tilde{\omega})$ with the quotient metric

$$|\mathfrak{m} - \mathfrak{l}| := \min\{|m - l| : m \in \mathfrak{m}, l \in \mathfrak{l}\}.$$

By this bijection, we relabel gaps and comb domain slits with the label $\mathfrak{m}$, which is better suited to our analysis; throughout this section, we will always assume that $n \in \mathbb{Z}$ and $\mathfrak{m} \in \mathfrak{Z}(\tilde{\omega})$ are related via

$$\frac{n}{T} = \frac{n(\mathfrak{m})}{T} = \mathfrak{m}\tilde{\omega}. \tag{3.3}$$

For instance, the comb domain associated with the periodic potential $\widetilde{V}$ will be denoted in the form

$$\Omega = \mathbb{C}_+ \setminus \bigcup_{\mathfrak{m}\in\mathfrak{Z}(\tilde{\omega})} (\mathfrak{m}\tilde{\omega}, \mathfrak{m}\tilde{\omega} + ih_{\mathfrak{m}}]$$

with $h_0 = 0$ and $h_{-\mathfrak{m}} = h_{\mathfrak{m}}$ for all $\mathfrak{m}$.

The goal of this section is to prove two-sided estimates between exponential decay of Fourier coefficients of $\widetilde{V}$ and exponential decay of the slits of the comb domain. The estimates will be independent of the period of the potential and dependent only on a "Diophantine condition in a box" satisfied by the rational vector $\tilde{\omega} = (\ell_j/t_j)_{j=1}^\nu$, $\ell_j \in \mathbb{Z}, t_j \in \mathbb{N}$,

$$|m\tilde{\omega}| \geq a_0|m|^{-b_0}, \quad 0 < |m| \leq \bar{R}_0, \tag{3.4}$$

for some

$$0 < a_0 < 1, \quad \nu < b_0 < \infty, \quad (\bar{R}_0)^{b_0} > \prod t_j. \tag{3.5}$$

Theorem 3.1. *There exists $\varepsilon^{(0)} = \varepsilon^{(0)}(a_0, b_0, \kappa_0) > 0$ such that, if $\tilde{\omega}$ obeys (3.4), (3.5) and T is given by (3.2), then for any locally L^2, T-periodic function $\widetilde{V}$, the following holds:*

(i) *if $\widetilde{V} \in \mathcal{P}(\tilde{\omega}, \varepsilon, \kappa_0)$ for some $\varepsilon < \varepsilon^{(0)}$, then the heights $h_{\mathfrak{m}}$ obey*

$$h_{\mathfrak{m}} \leq \varepsilon^{1/2} \exp\Big(-\frac{\kappa_0}{5}|\mathfrak{m}|\Big);$$

(ii) *if, conversely, the heights obey $h_{\mathfrak{m}} \leq \varepsilon \exp(-\kappa|\mathfrak{m}|)$ with $0 < \varepsilon < \varepsilon^{(0)}$, $\kappa \geq 5\kappa_0$, then, in fact, $\widetilde{V} \in \mathcal{P}(\tilde{\omega}, \varepsilon^{1/3}, \kappa/3)$*

The proof of Theorem 3.1 consists of two parts. The first part extends the method developed in [15, 18, 19] to establish two-sided estimates between subexponential decay of Fourier coefficients and of slits, under the a priori assumption of subexponential decay of Fourier coefficients. The second part uses estimates from the theory of periodic Schrödinger operators to remove the a priori assumption and improve the statement from subexponential decay to exponential decay. The first part is the following proposition.

Proposition 3.2. *There exists $\varepsilon_1 = \varepsilon_1(a_0, b_0, \kappa_0, \nu) > 0$ such that for $1/2 \leq \alpha_0 \leq 1$, $0 < \varepsilon \leq \varepsilon_1$, if $\widetilde{V} \in \mathcal{P}(\tilde{\omega}, \varepsilon, \kappa_0, \alpha_0)$, then*

(1) *the heights $h_{\mathfrak{m}}$ obey*

$$h_{\mathfrak{m}} \leq \varepsilon^{1/2} \exp\Big(-\frac{\kappa_0}{5}|\mathfrak{m}|^{\alpha_0}\Big);$$

(2) *there exists $\varepsilon_2 = \varepsilon_2(a_0, b_0, \kappa_0) > 0$ such that if the $h_{\mathfrak{m}}$ obey the inequality $h_{\mathfrak{m}} \leq \varepsilon' \exp(-\kappa_0''|\mathfrak{m}|^{\alpha_0'})$ with $0 < \varepsilon' < \varepsilon_2$, $\kappa_0'' \geq 5\kappa_0$, $\alpha_0' \geq \alpha_0$, then, in fact,*

$$\widetilde{V} \in \mathcal{P}\Big(\tilde{\omega}, (\varepsilon')^{1/3}, \frac{\kappa_0''}{3}, \alpha_0'\Big).$$

It was shown in [18, Lemma 2.7] that (3.4) and (3.5) implies that for any $\mathfrak{m} \neq 0$,

$$|\mathfrak{m}\tilde{\omega}| \geq a_0|\mathfrak{m}|^{-b_0}.$$

Moreover, (3.1) can be rewritten by grouping all terms with the same coset,

$$\widetilde{V}(x) = \sum_{\mathfrak{m} \in \mathfrak{Z}(\tilde{\omega})} c(\mathfrak{m}) e^{2\pi i x \mathfrak{m}\tilde{\omega}}, \quad x \in \mathbb{R}. \tag{3.6}$$

By [18, Lemma 2.5], if $\widetilde{V} \in \mathcal{P}(\tilde{\omega}, \varepsilon, \kappa_0, \alpha_0)$ for some $\alpha_0 \in [1/2, 1]$, then there is a constant $D = D(\kappa_0, \nu)$, non-increasing in κ_0, such that

$$|c(\mathfrak{m})| \leq D(\kappa_0, \nu)\varepsilon \exp\Big(-\frac{\kappa_0}{2}|\mathfrak{m}|^{\alpha_0}\Big), \quad \text{for all } \mathfrak{m} \in \mathfrak{Z}(\tilde{\omega}).$$

It will sometimes be convenient to assume $D(\kappa_0, \nu) \geq 1$. The trivial, but nonetheless useful, converse is that if $\widetilde{V}$ has a representation (3.6) such that

$$|c(\mathfrak{m})| \leq \varepsilon \exp(-\kappa_0 |\mathfrak{m}|^{\alpha_0})$$

for all $\mathfrak{m} \in \mathfrak{Z}(\tilde{\omega})$, then $\widetilde{V} \in \mathcal{P}(\tilde{\omega}, \varepsilon, \kappa_0, \alpha_0)$.

We denote

$$\mathfrak{Z}(\tilde{\omega})_+ := \{\mathfrak{m} \in \mathfrak{Z}(\tilde{\omega}) : \mathfrak{m}\tilde{\omega} > 0\}$$

and denote, for $\mathfrak{m} \in \mathfrak{Z}(\tilde{\omega})_+$, the sizes of gaps by

$$\gamma_{\mathfrak{m}} = E^+_{\mathfrak{m}} - E^-_{\mathfrak{m}}.$$

These are studied in [18], where the following is proved.

Theorem 3.3 ([18, Theorem $\widetilde{B}$]). *Assume that $\widetilde{V} \in \mathcal{P}(\tilde{\omega}, \varepsilon, \kappa_0, \alpha_0)$, where $\kappa_0 \in (0, 1]$, $\alpha_0 \in [\frac{1}{2}, 1]$, and $\tilde{\omega}$ obeys (3.4), (3.5). There is an $\varepsilon_0 = \varepsilon_0(a_0, b_0, \kappa_0, \nu) > 0$ such that, if $\varepsilon \leq \varepsilon_0$, then*

(i) *the gap sizes obey*

$$\gamma_{\mathfrak{m}} \leq 2\varepsilon \exp\Big(-\frac{\kappa_0}{2}|\mathfrak{m}|^{\alpha_0}\Big), \quad \text{for all } \mathfrak{m} \in \mathfrak{Z}(\tilde{\omega})_+;$$

(ii) *there exists $\varepsilon^{(0)} = \varepsilon^{(0)}(a_0, b_0, \kappa_0, \nu) > 0$ such that if $0 < \varepsilon'' < \varepsilon^{(0)}$, $\kappa_0' \geq 4\kappa_0$, $\alpha_0' \geq \alpha_0$ and*

$$\gamma_{\mathfrak{m}} \leq \varepsilon'' \exp(-\kappa_0' |\mathfrak{m}|^{\alpha_0'}), \quad \text{for all } \mathfrak{m} \in \mathfrak{Z}(\tilde{\omega})_+,$$

then, in fact, $\widetilde{V} \in \mathcal{P}(\tilde{\omega}, \sqrt{2\varepsilon''}, \frac{\kappa_0'}{2}, \alpha_0')$.

In the study of periodic Schrödinger operators, the basic object is the discriminant Δ, which is the trace of the monodromy matrix and is related to the Marchenko–Ostrovskii map by

$$\Delta(z) = 2\cos(iTw(z)), \tag{3.7}$$

where T denotes the period. In each gap closure $[E^-_{\mathfrak{m}}, E^+_{\mathfrak{m}}]$, the derivative of the discriminant has precisely one zero $\lambda_{\mathfrak{m}}$, and it has no other zeros in $\mathbb{C}$. The derivative of the Marchenko–Ostrovskii map has the product representation

$$w'(E) = -\Big(\frac{1}{4(E_0 - E)} \prod_{\mathfrak{m}' \in \mathfrak{Z}(\tilde{\omega})_+} \frac{(\lambda_{\mathfrak{m}'} - E)^2}{(E^-_{\mathfrak{m}'} - E)(E^+_{\mathfrak{m}'} - E)}\Big)^{1/2}, \tag{3.8}$$

which follows from the product representations of $\Delta^2 - 4$ and Δ' (see, e.g., [41, Section VII.B]) after differentiating (3.7). The representation is derived on $\mathbb{C}_+$ but extends by analyticity to $\mathbb{C} \setminus \mathcal{S}$. In particular, w' is analytic on $\mathbb{C} \setminus \mathcal{S}$ and $\lambda_{\mathfrak{m}}$ are zeros

of w'. This product formula can also be concluded from a connection to Schwarz–Christoffel mappings, and can be written as an exponential Herglotz representation, see e.g., [26], [22, Section 6]. In particular, if $\sum \gamma_m < \infty$, the exponential Herglotz representation has the form

$$i w'(z) = \frac{1}{2\sqrt{E_0 - z}} e^{\int_{[E_0,\infty)\setminus S} \frac{\xi(\lambda)}{\lambda - z}\, d\lambda},$$

where

$$\xi(\lambda) = \begin{cases} 1 & \text{there exists } m : \lambda \in (E_m^-, \lambda_m), \\ -1 & \text{there exists } m : \lambda \in (\lambda_m, E_m^+), \\ 0 & \text{else,} \end{cases}$$

compare [26] and [22, Section 6].

For distinct $\mathfrak{m}, \mathfrak{m}' \in \mathfrak{Z}(\tilde{\omega})_+$, denote

$$\eta_{\mathfrak{m},\mathfrak{m}'} = \operatorname{dist}([E_{\mathfrak{m}}^-, E_{\mathfrak{m}}^+], [E_{\mathfrak{m}'}^-, E_{\mathfrak{m}'}^+]).$$

Denote also

$$\eta_{\mathfrak{m},0} = \operatorname{dist}([E_{\mathfrak{m}}^-, E_{\mathfrak{m}}^+], E_0)$$

for $\mathfrak{m} \in \mathfrak{Z}(\tilde{\omega})_+$, and define the constants

$$A(\mathfrak{m}) = \max\Big(2(\eta_{\mathfrak{m},0} + \gamma_{\mathfrak{m}})^{1/2}, \frac{1}{2\eta_{\mathfrak{m},0}^{1/2}}\Big) \prod_{\substack{\mathfrak{m}' \in \mathfrak{Z}(\tilde{\omega})_+ \\ \mathfrak{m}' \neq \mathfrak{m}}} \Big(1 + \frac{\gamma_{\mathfrak{m}'}}{\eta_{\mathfrak{m},\mathfrak{m}'}}\Big)^{1/2}. \tag{3.9}$$

Lemma 3.4. *There exists $C = C(a_0, b_0, \kappa_0, \nu) < \infty$ such that, under the assumptions of Theorem 3.3, for any $\mathfrak{m}$ with $\mathfrak{m}\omega > 0$,*

$$A(\mathfrak{m}) \leq C|\mathfrak{m}|^{C \log_2 \log_2 |\mathfrak{m}|}. \tag{3.10}$$

Proof. Versions of this lemma have already appeared in [8, 18] (compare [8, Lemma 6.1]), with the following caveats. In previous versions of this statement, the definition of $A(\mathfrak{m})$ had $1/(2\eta_{\mathfrak{m},0}^{1/2})$ instead of the more complicated prefactor in front of the product; since either of the prefactors is polynomially bounded in $|\mathfrak{m}|$, this change is irrelevant. The estimate was stated and proved on $\mathbb{Z}^\nu$, but the proofs go through without modification when terms are indexed in $\mathfrak{Z}(\tilde{\omega})$. Finally, this estimate has been proved for $\alpha_0 = 1$, but the same proof goes through uniformly in $\alpha_0 \in [1/2, 1]$, with the obvious modification of replacing some exponential terms by subexponential terms. ∎

The constants $A(\mathfrak{m})$ appear as bounds for infinite product representations such as the one for w'. Through this, they lead to two-sided estimates between gap sizes $\gamma_{\mathfrak{m}}$ and heights $h_{\mathfrak{m}}$.

Lemma 3.5. *Under the assumptions of Theorem 3.3, $\gamma_{\mathfrak{m}}$ and $h_{\mathfrak{m}}$ are related by two-sided estimates*

$$\frac{1}{2}A(\mathfrak{m})^{-1}\gamma_{\mathfrak{m}} \le h_{\mathfrak{m}} \le 2A(\mathfrak{m})\gamma_{\mathfrak{m}}. \tag{3.11}$$

Proof. For $\mathfrak{m} \in \mathfrak{Z}(\tilde{\omega})_+$, the height $h_{\mathfrak{m}}$ can be expressed as

$$h_{\mathfrak{m}} = \max_{z\in[E_{\mathfrak{m}}^-, E_{\mathfrak{m}}^+]} \mathrm{Re}(-w(z)) = \mathrm{Re}(-w(\lambda_{\mathfrak{m}}))$$

since $\mathrm{Re}(-w)$ is increasing on $[E_{\mathfrak{m}}^-, \lambda_{\mathfrak{m}}]$ and decreasing for $[\lambda_{\mathfrak{m}}, E_{\mathfrak{m}}^+]$. Since w' is real-valued on $(E_{\mathfrak{m}}^-, E_{\mathfrak{m}}^+)$ and changes sign only at $\lambda_{\mathfrak{m}}$, it follows that

$$2h_{\mathfrak{m}} = \int_{E_{\mathfrak{m}}^-}^{E_{\mathfrak{m}}^+} |w'(E)|dE.$$

For $E \in [E_{\mathfrak{m}}^-, E_{\mathfrak{m}}^+]$ and $\mathfrak{m}' \neq \mathfrak{m}$,

$$\Big(1 + \frac{\gamma_{\mathfrak{m}'}}{\eta_{\mathfrak{m},\mathfrak{m}'}}\Big)^{-1} \le \Big|\frac{(\lambda_{\mathfrak{m}'} - E)^2}{(E_{\mathfrak{m}'}^- - E)(E_{\mathfrak{m}'}^+ - E)}\Big| \le \Big(1 + \frac{\gamma_{\mathfrak{m}'}}{\eta_{\mathfrak{m},\mathfrak{m}'}}\Big).$$

Similarly, $\eta_{\mathfrak{m},0} \le |E_0 - E| \le \eta_{\mathfrak{m},0} + \gamma_{\mathfrak{m}}$, and inserting these estimates into (3.8) we get precisely products of the form (3.9), obtaining

$$A(\mathfrak{m})^{-1}\int_{E_{\mathfrak{m}}^-}^{E_{\mathfrak{m}}^+} \frac{|\lambda_{\mathfrak{m}} - E|}{\sqrt{(E_{\mathfrak{m}}^- - E)(E_{\mathfrak{m}}^+ - E)}}dE \le 2h_{\mathfrak{m}} \le A(\mathfrak{m})\int_{E_{\mathfrak{m}}^-}^{E_{\mathfrak{m}}^+} \frac{|\lambda_{\mathfrak{m}} - E|}{\sqrt{(E_{\mathfrak{m}}^- - E)(E_{\mathfrak{m}}^+ - E)}}dE.$$

We rewrite the remaining integral by the substitution

$$E = E_{\mathfrak{m}}^- + (E_{\mathfrak{m}}^+ - E_{\mathfrak{m}}^-)\sin^2\Big(\frac{t}{2}\Big),$$

and obtain

$$\int_{E_{\mathfrak{m}}^-}^{E_{\mathfrak{m}}^+} \frac{|\lambda_{\mathfrak{m}} - E|}{\sqrt{(E_{\mathfrak{m}}^- - E)(E_{\mathfrak{m}}^+ - E)}}dE = \frac{1}{2}\gamma_{\mathfrak{m}}\int_0^{\pi} |\cos t - a|dt,$$

where

$$a = \frac{2\lambda_{\mathfrak{m}} - E_{\mathfrak{m}}^+ - E_{\mathfrak{m}}^-}{E_{\mathfrak{m}}^+ - E_{\mathfrak{m}}^-} \in (-1, 1).$$

The elementary estimates $2 \le \int_0^{\pi} |\cos t - a|dt \le \pi$ complete the proof. ∎

Proof of Proposition 3.2. By (3.10), we can choose $\varepsilon_1 > 0$ small enough so that for $0 < \varepsilon < \varepsilon_1$, we have

$$A(\mathfrak{m}) \le \frac{1}{4D\varepsilon^{1/2}} \exp\Big(\frac{\kappa_0}{20}|\mathfrak{m}|^{1/2}\Big).$$

(i) Combining (3.11) and Theorem 3.3(i),

$$h_{\mathfrak{m}} \le 4A(\mathfrak{m})D\varepsilon \exp\Big(-\frac{\kappa_0}{4}|\mathfrak{m}|^{\alpha_0}\Big) \le \varepsilon^{1/2} \exp\Big(-\frac{\kappa_0}{5}|\mathfrak{m}|^{\alpha_0}\Big).$$

(ii) Using (3.11), we obtain

$$\gamma_{\mathfrak{m}} \le 2A(\mathfrak{m})\varepsilon' \exp(-\kappa_0''|\mathfrak{m}|^{\alpha_0'}) \le \varepsilon'' \exp(-\kappa_0'|\mathfrak{m}|^{\alpha_0'}),$$

where

$$\kappa_0' = \kappa_0'' - \frac{\kappa_0}{20} \ge 4\kappa_0 \quad \text{and} \quad \varepsilon'' = \frac{1}{2}\varepsilon_1^{-1/2}\varepsilon'.$$

Choosing $\varepsilon_2 = \min(\varepsilon_1^{3/2}, 2\varepsilon_1^{1/2}\varepsilon^{(0)})$, we ensure that $\varepsilon'' < \varepsilon^{(0)}$, so we can apply Theorem 3.3(ii) to obtain $\widetilde{V} \in \mathcal{P}(\tilde{\omega}, \sqrt{2\varepsilon''}, \kappa_0'/2, \alpha_0')$, which implies $\widetilde{V} \in \mathcal{P}(\tilde{\omega}, (\varepsilon')^{1/3}, \kappa_0''/3, \alpha_0')$. ∎

The second part of the proof of Theorem 3.1 consists of a continuity argument used to remove the a priori subexponential decay assumption from Proposition 3.2(2). We need to invoke the following fundamental results from the inverse spectral theory of periodic Schrödinger operators; see [30, 31, 48, 55, 63]. Let Q be a real T-periodic potential, $Q \in L^2([0, T])$, given by

$$Q(x) = \sum_{n\in\mathbb{Z}} d(n) e^{\frac{2\pi i n x}{T}}.$$

Consider the spectrum $\mathcal{S} = \sigma(H_Q)$. The isospectral torus $\mathcal{R}(\mathcal{S})$ consists of T-periodic potentials q such that $\sigma(H_q) = \mathcal{S}$. We recall the following facts.

(a) The gaps (E_n^-, E_n^+) in the spectrum obey

$$\sum_{n=1}^{\infty} (E_n^+ - E_n^-)^2 < +\infty$$

and the heights h_n obey

$$\sum_{n=1}^{\infty} n^2 h_n^2 < +\infty.$$

(b) Given a sequence $h_n \ge 0$, $n = 1, \dots$ with $\sum_n n^2 h_n^2 < +\infty$, there exists a unique up to a shift sequence of intervals (E_n^-, E_n^+) such that these intervals are the gaps in the spectrum for some $q \in L^2(0, T)$ and the h_n's are the heights of the slits in the associated comb domain.

(c) Let $\mu_n(q)$ be the Dirichlet eigenvalues and $\nu_n(q)$ be the Neumann eigenvalues for q. Then

$$E_n^- \le \mu_n(q), \nu_n(q) \le E_n^+.$$

Furthermore, q is even if and only if

$$\mu_n(q) \in \{E_n^-, E_n^+\}, \quad \text{for all } n \in \mathbb{N},$$

and if and only if

$$\nu_n(q) \in \{E_n^-, E_n^+\}, \quad \text{for all } n \in \mathbb{N}.$$

(d) Denote by $\mathcal{E}_0$ the set of all real, T-periodic even L^2 functions with zero average. Given $q \in \mathcal{E}_0$ consider the sequence $(\mu_n(q) - \nu_n(q))_{n=1}^\infty$. Since $\mu_n(q) - \nu_n(q) = \pm(E_n^+ - E_n^-)$, the values $\mu_n(q) - \nu_n(q)$ are called the *signed gap-lengths.* Similarly, the quantities

$$\rho_n(q) = \operatorname{sgn}(\mu_n(q) - \nu_n(q))h_n(q)$$

are called the *signed heights.* The map $\Upsilon : q \to (\rho_n(q))_{n=1}^\infty \in \ell_1^2$ is a real analytic diffeomorphism from $\mathcal{E}_0$ onto ℓ_1^2.

(e) Assume $E_n^- < E_n^+$ and denote $G_{n,\sigma} = \{(\lambda, \sigma) \mid E_n^- < \lambda < E_n^+\}$ for $\sigma = \pm$ and

$$\mathcal{C}_n = G_{n,-} \cup G_{n,+} \cup \{E_n^-, E_n^+\}.$$

The set $\mathcal{C}_n$ has a natural smooth structure which makes it diffeomorphic to the circle $\mathbb{T} = \{e^{i\theta} : 0 \le \theta \le 2\pi\}$. For each q one can define $\sigma_n(q) \in \{+, -\}$ so that the map $q \to ((\mu_n(q), \sigma_n(q))) \in \prod_n \mathcal{C}_n$ is a diffeomorphism.

(f) By the inequality $\gamma_n \le \max(2\pi n h_n, h_n^2)$, if

$$h_n \le A_1 \exp\Big(-\eta_1 \frac{|n|}{T}\Big), \tag{3.12}$$

then

$$\gamma_n \le A(T, A_1, \eta_1) \exp\Big(-\eta(T, A_1, \eta_1)\frac{|n|}{T}\Big).$$

(g) Assume that for all n,

$$\gamma_n \le A \exp\Big(-\eta \frac{|n|}{T}\Big).$$

Then

$$|d(n)| \le C(T, A, \eta) \exp\Big(-c(T, A, \eta)\frac{|n|}{T}\Big).$$

Besides the quotient metric introduced before, $\mathfrak{Z}(\tilde{\omega})$ can also be equipped with the metric pulled back from $\mathbb{R}$,

$$|\mathfrak{m} - \mathfrak{l}|_{\mathbb{R}} := \min\{|m\tilde{\omega} - l\tilde{\omega}| : m \in \mathfrak{m}, l \in \mathfrak{l}\}.$$

As it has been observed in [19], there are constants $c_{\tilde{\omega}}$, $C_{\tilde{\omega}}$ such that

$$c_{\tilde{\omega}}|\mathfrak{m}|_{\mathbb{R}} \le |\mathfrak{m}| \le C_{\tilde{\omega}}|\mathfrak{m}|_{\mathbb{R}}. \tag{3.13}$$

Combining (f) and (g) with (3.13), we conclude that (3.12) implies

$$|d(n(\mathfrak{m}))| \le C(T, A, \eta)\exp(-c_1(T, A, \eta)|\mathfrak{m}|).$$

Proof of Theorem 3.1. (i) follows from Prop. 3.2 with $\alpha_0 = 1$.

(ii) First we prove the statement for even potentials. Let $D\colon \ell^2_1(\mathfrak{Z}(\tilde{\omega})_+) \to \mathcal{E}_0$ be the inverse of Υ. Given $\rho = (\rho_n) \in \ell^2_1(\mathbb{Z}_+)$, denote by $d_{\mathfrak{m}}(\rho)$ the Fourier coefficients of $D(\rho)$. Let

$$\mathcal{S}(\varepsilon, \kappa) = \{\rho = (\rho_{\mathfrak{m}})_{\mathfrak{m}\in\mathfrak{Z}(\tilde{\omega})_+} : |\rho_{\mathfrak{m}}| \le \varepsilon\exp(-\kappa|\mathfrak{m}|)\}.$$

Given $1/2 \le \alpha < 1$ set

$$\mathcal{G}(\varepsilon, \kappa, \alpha) = \Big\{\rho \in \mathcal{S}(\varepsilon, \kappa) : |d_{\mathfrak{m}}(\rho)| \le \varepsilon^{1/3}\exp\Big(-\frac{\kappa}{3}|m|^{\alpha}\Big)\Big\}.$$

The set $\mathcal{G}(\varepsilon, \kappa, \alpha)$ is no-nempty, since $D(0) = 0$, and it is obviously closed since each $d_{\mathfrak{m}}$ is continuous. We claim that the set $\mathcal{G}(\varepsilon, \kappa, \alpha)$ is also relatively open in $\mathcal{S}(\varepsilon, \kappa)$ in the ℓ^2_1 metric. Indeed, note first of all that due to (f) we have

$$|d_{\mathfrak{m}}| \le C(T, \varepsilon, \kappa)\exp(-c(T, \varepsilon, \kappa)|\mathfrak{m}|)$$

for any $\rho \in \mathcal{S}(\varepsilon, \kappa)$. In particular, one can find $M = M(T, \varepsilon, \kappa, \alpha)$ large enough so that for $|\mathfrak{m}| > M$, we have

$$|d_{\mathfrak{m}}| < \varepsilon^{1/3}\exp\Big(-\frac{\kappa}{3}|\mathfrak{m}|^{\alpha}\Big).$$

Next, note that since the map D is continuous from $\ell^2(\mathfrak{Z}(\tilde{\omega})_+)$ to $L^2(0, T)$, the map $\rho \to (d_{\mathfrak{m}}(\rho))_{\mathfrak{m}\in\mathfrak{Z}(\tilde{\omega})}$ is also continuous from $\ell^2(\mathfrak{Z}(\tilde{\omega})_+)$ to $\ell^2(\mathfrak{Z}(\tilde{\omega}))$. Let $\rho^{(0)} = (\rho^{(0)}_{\mathfrak{m}}) \in \mathcal{G}(\varepsilon, \kappa, \alpha)$. Find $\delta > 0$ such that

$$\|D(\rho) - D(\rho_0)\| < \varepsilon^{1/3}\exp(-\kappa M^{\alpha})$$

for any $\|\rho - \rho_0\| < \delta$. Let $\|\rho - \rho_0\| < \delta$. Then, in particular, we have

$$|d_{\mathfrak{m}}(\rho)| < \varepsilon^{1/3}\exp\Big(-\frac{\kappa}{3}|\mathfrak{m}|^{\alpha}\Big) + \varepsilon^{1/3}\exp(-\kappa M^{\alpha}) < 2\varepsilon^{1/3}\exp\Big(-\frac{\kappa}{3}|\mathfrak{m}|^{\alpha}\Big)$$

for any $|\mathfrak{m}| \leq M$. Consider $q = D(\rho)$, i.e.,

$$q(x) = \sum_{m \in \mathfrak{Z}(\tilde{\omega})} d_{\mathfrak{m}}(\rho) e^{2\pi i x \mathfrak{m} \tilde{\omega}}.$$

Proposition 3.2 applies to q in the role of $\widetilde{V}$, as long as $2\varepsilon^{1/3} < \varepsilon_1(a_0, b_0, \kappa_0)$ and $\varepsilon < \varepsilon_2(a_0, b_0, \kappa_0)$. Thus,

$$|d_{\mathfrak{m}}(\rho)| < \varepsilon^{1/3} \exp\Big(-\frac{\kappa}{3}|\mathfrak{m}|^{\alpha}\Big),$$

i.e., $\rho \in \mathcal{G}(\varepsilon, \kappa, \alpha)$.

By connectedness, it follows that $\mathcal{G}(\varepsilon, \kappa, \alpha) = \mathcal{S}(\varepsilon, \kappa)$, i.e., for any $\rho \in \mathcal{S}(\varepsilon, \kappa)$ one has

$$|d_{\mathfrak{m}}(\rho)| < \varepsilon^{1/3} \exp\Big(-\frac{\kappa}{3}|\mathfrak{m}|^{\alpha}\Big),$$

That completes the proof for even potentials. To prove the statement for arbitrary $\widetilde{V}$, we first recall that there exists an even potential q_0 isospectral with $\widetilde{V}$. Since we have proved the statement for even potentials, we have

$$q_0(x) = \sum_{m \in \mathfrak{Z}(\tilde{\omega})} d^{(0)}_{\mathfrak{m}} e^{2\pi i x \mathfrak{m} \tilde{\omega}},$$
$$|d^{(0)}_{\mathfrak{m}}| < \varepsilon^{1/3} \exp\Big(-\frac{\kappa}{3}|\mathfrak{m}|^{\alpha}\Big).$$

Now, we employ the map $q \to ((\mu_n(q), \sigma_n(q))) \in \prod_n \mathcal{C}_n$ from (e). By another open set-closed set argument as above, it follows that the statement holds for any potential q isospectral with q_0.

We have therefore shown that

$$|c(\mathfrak{m})| \leq \varepsilon^{1/3} \exp\Big(-\frac{\kappa}{3}|\mathfrak{m}|^{\alpha}\Big)$$

for any $\alpha \in [1/2, 1)$. Taking the limit $\alpha \to 1$, we conclude

$$|c(\mathfrak{m})| \leq \varepsilon^{1/3} \exp\Big(-\frac{\kappa}{3}|\mathfrak{m}|\Big)$$

which completes the proof. ∎

4. Stability of Marchenko–Ostrovskii maps

In this section, we use geometric function theory to study Marchenko–Ostrovskii maps up to the boundary. We rely on the fact that the domains for the periodic approximants satisfy some uniform estimates, which will transfer also to the limit.

We begin by recalling some general facts [53, Chapter 4], which we rewrite with respect to the domain $\mathbb{C}_+$ and the boundary point ∞ instead of $\mathbb{D}$; these rewritings are trivially obtained by pre- and post-composing with the Cayley transform as needed. Let $g\colon \mathbb{C}_+ \to \mathbb{C}_+$ be analytic and Γ a Jordan arc in $\mathbb{C}_+ \cup \{\infty\}$ with endpoint ∞. If

$$\lim_{\substack{z\to\infty \\ z\in\Gamma}} g(z) = \infty,$$

then g has the angular limit ∞ at ∞, i.e., for any $\alpha > 0$,

$$\lim_{\substack{z\to\infty \\ \arg z\in(\alpha,\pi-\alpha)}} g(z) = \infty. \tag{4.1}$$

By the Julia–Wolff lemma [53, Proposition 4.13], (4.1) implies that g has an angular derivative at ∞, in the sense that for every $\alpha > 0$, the limit

$$g'(\infty) := \lim_{\substack{z\to\infty \\ \arg z\in(\alpha,\pi-\alpha)}} \frac{z}{g(z)}$$

exists and $g'(\infty) \in (0, \infty]$.

It is convenient to transform a Marchenko–Ostrovskii map into the form

$$\Theta(z) = -i\,w(E_0 + z^2).$$

The map Θ is a conformal bijection from $\mathbb{C}_+$ to the Marchenko–Ostrovskii domain Ω, and the asymptotics (1.2) implies that Θ has a finite angular derivative at ∞, namely,

$$\Theta'(\infty) = 1.$$

In geometric function theory it is more convenient to use an internal normalization, and our work will switch between those two. The following corollary of the Schwarz–Pick theorem is useful for comparing angular derivatives at ∞.

Lemma 4.1. *Let $g_1, g_2\colon \mathbb{C}_+ \to \mathbb{C}_+$ be analytic, injective, and have the symmetry*

$$g_j(z) = -\overline{g_j(-\bar{z})}.$$

If g_1, g_2 satisfy (4.1) and $\{iy \mid 1 \le y < \infty\} \subset g_1(\mathbb{C}_+) \subset g_2(\mathbb{C}_+)$, then

$$\frac{1}{g_2'(\infty)} \ge \frac{g_1^{-1}(i)}{g_2^{-1}(i)} \frac{1}{g_1'(\infty)}. \tag{4.2}$$

Proof. By the Schwarz–Pick theorem, analytic maps $g\colon \mathbb{C}_+ \to \mathbb{C}_+$ do not increase distances in the Poincaré metric, i.e.,

$$\left|\frac{g(z_1) - g(z_2)}{\overline{g(z_1)} - g(z_2)}\right| \le \left|\frac{z_1 - z_2}{\overline{z_1} - z_2}\right|, \quad z_1, z_2 \in \mathbb{C}_+. \tag{4.3}$$

Applying (4.3) to $g = g_2^{-1} \circ g_1 : \mathbb{C}_+ \to \mathbb{C}_+$ and taking $z_j = g_1^{-1}(w_j)$ implies

$$\left|\frac{g_2^{-1}(w_1) - g_2^{-1}(w_2)}{\overline{g_2^{-1}(w_1)} - g_2^{-1}(w_2)}\right| \le \left|\frac{g_1^{-1}(w_1) - g_1^{-1}(w_2)}{\overline{g_1^{-1}(w_1)} - g_1^{-1}(w_2)}\right|, \quad w_1, w_2 \in g_1(\mathbb{C}_+).$$

Dividing by $w_1 - w_2$ and letting $w_1 \to w_2 = iy$, we obtain

$$\frac{|(g_2^{-1})'(iy)|}{\operatorname{Im}(g_2^{-1})(iy)} \le \frac{|(g_1^{-1})'(iy)|}{\operatorname{Im}(g_1^{-1})(iy)}, \ y \ge 1.$$

Since g_j^{-1} are purely imaginary along the imaginary axis, integrating along the segment $[i, iy]$ and letting $y \to +\infty$ gives (4.2). ∎

Lemma 4.2. *Consider a simply connected domain $\Omega \subset \mathbb{C}_+$ symmetric under reflection with respect to the imaginary axis, $z \leftrightarrow -\bar{z}$. Assume that there exists $\epsilon \in (0,1)$ such that $\{z \in \mathbb{C} \mid \operatorname{Im} z > \epsilon\} \subset \Omega$. Let $f : \mathbb{C}_+ \to \Omega$ be the conformal bijection satisfying the normalization*

$$f(i) = i, \quad f'(i) > 0. \tag{4.4}$$

Then,

(i) *the map f has the symmetry*

$$f(z) = -\overline{f(-\bar{z})}; \tag{4.5}$$

(ii) *the map $iy \mapsto f(iy)$ is a strictly increasing bijection $(0, i\infty) \to (0, i\infty) \cap \Omega$;*

(iii) *the map f has an angular limit at ∞: for all $\alpha > 0$,*

$$\lim_{\substack{z \to \infty \\ \arg z \in (\alpha, \pi - \alpha)}} f(z) = \infty;$$

(iv) *the angular derivative $f'(\infty)$ exists and*

$$1 \le f'(\infty) \le \frac{1}{1-\epsilon};$$

(v) *the map Θ given by*

$$\Theta(z) = f(f'(\infty) z)$$

is the unique conformal bijection from $\mathbb{C}_+$ onto Ω obeying the normalization

$$\Theta(z) = -\overline{\Theta(-\bar{z})}, \quad \lim_{y \to \infty} \frac{\Theta(iy)}{iy} = 1. \tag{4.6}$$

Proof. (i) Since Ω is symmetric with respect to the imaginary axis, $-\overline{f(-\bar{z})}$ is also a conformal map of $\mathbb{C}_+$ to Ω, and it satisfies the same normalization (4.4), so they are equal.

(ii) By the symmetry (4.5), if $f(z) = it$ for some $t \in \mathbb{R}$, then also $f(-\bar{z}) = it$; since f is injective, this implies $z = -\bar{z}$. Conversely, $z = -\bar{z}$ implies $f(z) \in (0, i\infty)$ by the same symmetry (4.5). Thus, f maps $(0, i\infty)$ to $(0, i\infty) \cap \Omega$ bijectively. By continuity, $y \mapsto -if(iy)$ is strictly increasing or strictly decreasing; since $f'(i) > 0$, it is strictly increasing. Thus, $\lim_{y\to\infty} f(iy) = \infty$.

(iii) It follows from (ii) that $f(z)$ has a radial limit at ∞ along the imaginary axis. Thus, f has an angular limit at ∞.

(iv) The angular derivative $f'(\infty)$ exists by the Julia–Wolff lemma. Applying Lemma 4.1 to functions f and $z \mapsto z$, we obtain $f'(\infty) \geq 1$. Similarly, applying this to $z \mapsto z + i\epsilon$ and f, we obtain $f'(\infty) \leq 1/(1-\epsilon)$.

(v) For any such conformal map Θ, symmetry dictates that there is a unique $C > 0$ such that $\Theta(i/C) = i$ and that $\Theta'(i/C) > 0$. Thus, $\Theta(z/C) = f(z)$. Comparing derivatives at ∞, we see $\Theta'(\infty) = C^{-1} f'(\infty)$, so $C = f'(\infty)$. ∎

We now specialize to a family of comb domains which satisfies some uniform decay conditions on the slit heights and uniform upper bounds and Diophantine conditions for the frequency vectors.

Lemma 4.3. *Let $\{\Omega_\alpha\}_{\alpha\in\mathcal{A}}$ be a family of comb domains of the form*

$$\Omega_\alpha = \mathbb{C}_+ \setminus \bigcup_{\mathfrak{m}\in\mathfrak{Z}(\omega_\alpha)} (\mathfrak{m}\omega_\alpha, \mathfrak{m}\omega_\alpha + ih_{\mathfrak{m},\alpha}]$$

where $\omega_\alpha \in \mathbb{R}^\nu$ and $h_{\mathfrak{m},\alpha} \geq 0$ for $\mathfrak{m} \in \mathfrak{Z}(\omega_\alpha)$. Assume that the following hold, for some constants $r < \infty$, $a_0 \in (0,1)$, $b_0 \in (\nu, \infty)$, $\varepsilon > 0$, $\kappa > 0$ independent of $\alpha \in \mathcal{A}$:

(i) *$|\omega_\alpha| < r$ and ω_α obey the Diophantine condition*

$$|\mathfrak{m}\omega_\alpha| \geq a_0 |\mathfrak{m}|^{-b_0}, \quad \text{for all } \mathfrak{m} \in \mathfrak{Z}(\omega_\alpha) \setminus \{0\}; \tag{4.7}$$

(ii) *$h_{0,\alpha} = 0$, $h_{\mathfrak{m},\alpha} = h_{-\mathfrak{m},\alpha} \geq 0$, and*

$$0 \leq h_{\mathfrak{m},\alpha} \leq \varepsilon e^{-\kappa|\mathfrak{m}|}, \quad \text{for all } \mathfrak{m} \in \mathfrak{Z}(\omega_\alpha). \tag{4.8}$$

Then, for each $\alpha \in \mathcal{A}$, there is a unique conformal map $\Theta_\alpha \colon \mathbb{C}_+ \to \Omega_\alpha$ with the normalization (4.6), and the family of functions $\{\Theta_\alpha\}_{\alpha\in\mathcal{A}}$ is equicontinuous on $\overline{\mathbb{C}_+}$.

We use the spherical metric on $\widehat{\mathbb{C}}$, given by

$$d(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}, \quad z_1, z_2 \in \mathbb{C}$$

and

$$d(z_1, \infty) = d(\infty, z_1) = \frac{1}{\sqrt{1 + |z_1|^2}}.$$

Whenever we make statements about uniform convergence of sequences of functions, those statements should be understood with respect to the spherical metric for both domain and codomain.

Proof. In geometric function theory, sets $\widehat{\mathbb{C}} \setminus \Omega_\alpha$ are said to be *uniformly locally connected* if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all α and all $z, w \in \widehat{\mathbb{C}} \setminus \Omega_\alpha$ with $d(z, w) < \delta$, there exists a connected set $B \subset \widehat{\mathbb{C}} \setminus \Omega_\alpha$ such that $z, w \in B$ and $|B| < \epsilon$. We use $|\cdot|$ to denote the diameter of a set.

Consider two boundary points $z = x + iy$, $z' = x' + iy'$. Let c be the shorter of the two geodesics connecting z, z', parametrized by $[0, 1]$ so that $c(0) = z$ and $c(1) = z'$.

Let us first take care of the simple case $x = x'$ and without loss of generality let $y \le y'$. In that case, we choose $B = [z, z']$ and see that $|B| = d(z, z')$.

From now on, assume $x \ne x'$. Then, B can be chosen to be the union of three segments

$$B = [x, x + iy] \cup I \cup [x', x' + iy'],$$

where $I \subset \overline{\mathbb{R}}$ is the shorter of the two geodesics in $\widehat{\mathbb{C}}$ connecting x and x'. Since $y, y' \in [0, \varepsilon]$, it follows that

$$|c| \ge d(x + iy, x' + iy') \ge \frac{1}{1 + \varepsilon^2} d(x, x') = \frac{1}{1 + \varepsilon^2} |I|.$$

Also,

$$|\{x + it : t \in [0, y]\}| = d(x, x + iy) \le y.$$

Using c and two of the three line segments to estimate the diameter of $c \cup B$, we see that

$$|B| = |c \cup B| \le (2 + \varepsilon^2)|c| + Y, \quad Y = \min(y, y').$$

The only case when $Y \ne 0$ is when $c(0), c(1)$ lie on slits, i.e.,

$$c(0) = \mathfrak{m}\omega_\alpha + iy, \quad c(1) = \mathfrak{m}'\omega_\alpha + iy'.$$

From exponential decay of the $h_{\mathfrak{m}}$, we conclude

$$|\mathfrak{m}|, |\mathfrak{m}'| \le \frac{1}{\kappa} \ln \frac{\varepsilon}{Y}.$$

From the Diophantine condition we see

$$|(\mathfrak{m} - \mathfrak{m}')\omega_\alpha| \ge a_0 |\mathfrak{m} - \mathfrak{m}'|^{-b_0} \ge a_0 \Big(\frac{2}{\kappa} \ln \frac{\varepsilon}{Y} \Big)^{-b_0},$$

so

$$|c| \geq \frac{1}{1+\varepsilon^2} d(\mathfrak{m}\omega_\alpha, \mathfrak{m}'\omega_\alpha) \geq \frac{1}{1+\varepsilon^2} \frac{|(\mathfrak{m}-\mathfrak{m}')\omega_\alpha|}{\sqrt{1+|\mathfrak{m}\omega_\alpha|^2}\sqrt{1+|\mathfrak{m}'\omega_\alpha|^2}}$$
$$\geq \frac{1}{1+\varepsilon^2} \frac{a_0\left(\frac{2}{\kappa}\ln\frac{\varepsilon}{Y}\right)^{-b_0}}{1+\left(\frac{1}{\kappa}\ln\frac{\varepsilon}{Y} r\right)^2} =: F(Y).$$

Note that $F(Y)$ is a strictly increasing function of Y depending only on $a_0, b_0, r, \varepsilon, \kappa$, and $\lim_{Y\to 0} F(Y) = 0$. Combining this with previous estimates, we see that

$$|B| \leq (2+\varepsilon^2)|c| + F^{-1}(|c|),$$

where F^{-1} is the inverse function of F on $(0,\infty)$.

Thus, the domains Ω_α are uniformly locally connected (see [53, Chapter 2]; this reference is formulated for bounded domains, with Euclidean distance, but by conjugating our maps by the Cayley transform, it reduces to our setting of domains in $\mathbb{C}_+$). In particular, by a uniform version of Carathéodory's theorem [53, Prop. 2.3], the conformal bijections $f_\alpha\colon \mathbb{C}_+ \to \Omega_\alpha$ with

$$f_\alpha(i) = i, \quad f'_\alpha(i) > 0$$

have continuous extensions to $\overline{\mathbb{C}_+}$ and are uniformly equicontinuous in $\overline{\mathbb{C}_+}$. Thus, the maps Θ_α are uniquely determined by

$$\Theta_\alpha(z) = f_\alpha(f'_\alpha(\infty)z).$$

Since $1 \leq f'_\alpha(\infty) \leq (1-\varepsilon)^{-1}$ and f_α are uniformly equicontinuous, Θ_α are uniformly equicontinuous. ∎

The above setup includes both the case of irrational, Diophantine frequency vectors and the case of rational frequency vectors obeying a Diophantine condition in a box (3.4), (3.5). Indeed, the first application of this equicontinuity statement will be to a sequence of rational approximants of the Diophantine frequency ω; the second application will be to the constant sequence $\omega_n = \omega$.

To geometrically characterize convergence of conformal maps, Carathéodory introduced the following notion of kernel convergence [53, Section 1.4], with respect to an internal point which we take to be i. Let $\Omega_n \subset \mathbb{C}$ be domains containing i. It is said that $\Omega_n \to \Omega$ *in the sense of kernel convergence with respect to* i if

(i) either $\Omega = \{i\}$, or $\Omega \subsetneq \mathbb{C}$ is a domain with $i \in \Omega$ such that every $w \in \Omega$ has a neighborhood which lies in Ω_n for all sufficiently large n;

(ii) for every $w \in \partial\Omega$ there exist $w_n \in \partial\Omega_n$ such that $w_n \to w$ as $n \to \infty$

Proposition 4.4. *Take a sequence $\{\Omega_n\}_{n=1}^{\infty}$ from the family of comb domains $\{\Omega_\alpha\}_{\alpha\in\mathcal{A}}$ considered in Lemma 4.3. Assume that the limit*

$$\lim_{n\to\infty} \omega_n = \omega$$

exists and that ω has linearly independent entries over $\mathbb{Q}$, i.e., $m\omega \neq 0$ for all $m \in \mathbb{Z}^\nu \setminus \{0\}$. Then the following statements are equivalent.

(i) *For every $m \in \mathbb{Z}^\nu$, the limit*

$$h_m := \lim_{n\to\infty} h_{\mathfrak{m},n} \tag{4.9}$$

exists (note that we use $\mathfrak{m} = m + \mathfrak{N}(\omega_n)$ to denote the coset containing $m \in \mathbb{Z}^\nu$ in an n-dependent Abelian group).

(ii) *The maps Θ_n have a pointwise limit on $\{z \in \mathbb{C}_+ \mid \operatorname{Re} z > 0\}$.*

(iii) *The maps Θ_n converge uniformly on $\overline{\mathbb{C}_+}$.*

In this case, the limit of Θ_n is the conformal map Θ corresponding to the comb domain

$$\Omega = \mathbb{C}_+ \setminus \bigcup_{m\in\mathbb{Z}^\nu} (m\omega, m\omega + ih_m].$$

Proof. (i) $\Longrightarrow$ (ii). We begin by observing that, since $\mathfrak{N}(\omega) = \{0\}$, for each fixed $m \in \mathbb{Z}^\nu \setminus \{0\}$, $\lim_{n\to\infty} m\omega_n = m\omega \neq 0$, so $m\omega_n \neq 0$ for large enough n. Thus, for all $M \in \mathbb{N}$ there exists $N = N(M) \in \mathbb{N}$ such that

$$\{m \in \mathbb{Z}^\nu \mid |m| \leq M\} \cap \mathfrak{N}(\omega_n) = \{0\}, \quad \text{for all } n \geq N. \tag{4.10}$$

In particular, (4.10) implies for any $m \in \mathbb{Z}^\nu$, we have $|m| = |m + \mathfrak{N}(\omega_n)|$ for all large enough n, so the uniform estimates (4.7), (4.8) extend to the limit, and we can consider the comb domain Ω to be part of our family $\{\Omega_\alpha\}_{\alpha\in\mathcal{A}}$.

Assume $z \in \Omega$ and fix $L < \operatorname{Im} z$. There exists $m_0 \in \mathbb{N}$ such that $|m| > m_0$ implies $h_{\mathfrak{m},n} < L$. Of the remaining finitely many slits with $|m| \leq m_0$, each has $h_m < \operatorname{Im} z$ or $m\omega \neq \operatorname{Re} z$, so $\liminf \operatorname{dist}(z, \Omega_n) > 0$.

Now, assume $z \in \partial\Omega$. If $z = m\omega + iy$, the points $z_n = m\omega_n + i \min\{y, h_{m,n}\} \in \partial\Omega_n$ converge to z; if $z \in \mathbb{R}$, we may take $z_n = z \in \partial\Omega_n$.

Thus, pointwise convergence (4.9) implies kernel convergence of domains Ω_n to Ω with respect to the point i (see [53, Section 1.4]), so it implies that $f_n \to f$ uniformly on $\overline{\mathbb{C}_+}$, where the f_n's are the conformal bijections to Ω_n satisfying the normalization (4.4). Since $\Theta_n(z) = f_n(f_n'(\infty)z)$, it remains to prove that $f_n'(\infty) \to f'(\infty)$.

Fix $\delta > 0$ and let $M = M(\delta) \in \mathbb{N}$ be chosen so that $\varepsilon e^{-\kappa M} < \delta$. Denote by $g_{\alpha,M}$ the conformal maps $\mathbb{C}_+ \to \mathbb{C}_+ \setminus \bigcup_{|\mathfrak{m}|\leq M} \{\mathfrak{m}\omega_\alpha + i[0, h_{\mathfrak{m},\alpha}]\}$ with $g_{\alpha,M}(i) = i$,

$g'_{\alpha,M}(i) > 0$. By the same argument as above, the conformal bijections $g_{n,M}$ converge to g_M uniformly on $\overline{\mathbb{C}_+}$. Moreover, there is a finite supremum

$$L = L(M) = \sup_n g_{n,M}(M|\omega_n|) < \infty$$

such that $g_{n,M}(x) \in \mathbb{R}$ if $x \in \mathbb{R}$ with $|x| > L$. Using the reflection principle on the functions $1/g_{n,M}(1/z)$ in a neighborhood of 0 and their uniform convergence, we see that also $g'_{n,M}(\infty) \to g'_{0,M}(\infty)$.

By Lemma 4.1, since the range of $g_{\alpha,M}$ is a subset of the range of $f_{\alpha,M}$, we obtain

$$f'_\alpha(\infty) \le \frac{f_\alpha^{-1}(i)}{g_{\alpha,M}^{-1}(i)} g'_{\alpha,M}(\infty)$$

and since the range of f_α is a subset of the range of $g_{\alpha,M} - i\delta$, we obtain

$$g'_{\alpha,M}(\infty) \le \frac{g_{\alpha,M}^{-1}(i + i\delta)}{f_\alpha^{-1}(i)} f'_\alpha(\infty).$$

Thus,

$$\frac{i}{g_{\alpha,M}^{-1}(i + i\delta)} g'_{\alpha,M}(\infty) \le f'_\alpha(\infty) \le g'_{\alpha,M}(\infty). \tag{4.11}$$

Applying these estimates to the sequence f_n and taking limits as $n \to \infty$, we obtain

$$\frac{i}{g_M^{-1}(i + i\delta)} g'_M(\infty) \le \liminf_{n\to\infty} f'_n(\infty) \le \limsup_{n\to\infty} f'_n(\infty) \le g'_M(\infty).$$

Applying (4.11) also to f, we conclude

$$\frac{i}{g_M^{-1}(i + i\delta)} f'(\infty) \le \liminf_{n\to\infty} f'_n(\infty) \le \limsup_{n\to\infty} f'_n(\infty) \le \frac{g_M^{-1}(i + i\delta)}{i} f'(\infty).$$

As $\delta \to 0$, g_M converges to f, so $g_M^{-1}(i + i\delta) \to i$. This implies $f'_n(\infty) \to f'(\infty)$.

(ii) $\implies$ (iii). This follows from the equicontinuity statement in Lemma 4.3.

(iii) $\implies$ (i). If $f_n \to f$ pointwise on $\mathbb{C}_+$, then the domains Ω_n converge in kernel convergence with respect to the point i to a domain Ω [53, Theorem 1.8]. Consider a point $z = x + iy$, where $x = m\omega$ for some $m \in \mathbb{Z}^\nu$ and $y > 0$. Pick $\epsilon \in (0, y/2)$ such that the square $B_\epsilon = (x - \epsilon, x + \epsilon) + i(y - \epsilon, y + \epsilon)$ is disjoint from the boundaries of Ω_n, except possibly from their m-th slits. This can be done by finding $M > 0$ such that $|\mathfrak{m}| \ge M$ implies $h_{\mathfrak{m},n} < y/2 < y - \epsilon$, and then ensuring, by the Diophantine condition, that ϵ is small enough that $|\mathfrak{m}'| \le M$, $\mathfrak{m}' \ne \mathfrak{m}$ implies $|\mathfrak{m}\omega_n - \mathfrak{m}'\omega_n| > \epsilon$. Then,

$$B_\epsilon \setminus \Omega_n = \{m\omega_n + it \mid y - \epsilon < t < \min(y + \epsilon, h_{m,n})\}.$$

Kernel convergence implies that we cannot have $\liminf h_{m,n} < y - \epsilon$ or $\limsup h_{m,n} > y + \epsilon$. Since this is true for any $y > 0$ and any small enough $\epsilon > 0$, it implies that $\lim_{n\to\infty} h_{m,n}$ exists. ∎

5. Periodic approximation and translation flows

Applying the above ingredients to the periodic approximants $V^{(r)}$ of the small quasi-periodic potential $V \in \mathcal{P}(\omega, \varepsilon, \kappa_0)$ gives the following description:

Proposition 5.1. *Under the assumptions of Theorem 1.2, there exists*

$$\varepsilon_1 = \varepsilon_1(a_0, b_0, \kappa_0) > 0$$

such that, if $\varepsilon < \varepsilon_1$ and $V \in \mathcal{P}(\omega, \varepsilon, \kappa_0)$, then

(i) *its Marchenko–Ostrovskii domain is a comb domain of the form (1.6) with $h_m = h_{-m} \geq 0$ for all $m \in \mathbb{Z}^\nu \setminus \{0\}$, whose slit heights obey*

$$h_m \leq \varepsilon^{1/2} \exp\Big(-\frac{\kappa_0}{5}|m|\Big);$$

(ii) *the heights and gap sizes of V obey the two-sided estimates*

$$\frac{1}{2A(m)}\gamma_m \leq h_m \leq 2A(m)\gamma_m,$$

where $A(m)$ is defined by (3.9).

(iii) *the Marchenko–Ostrovskii map w has a continuous extension to $\mathbb{C}$; for each $m \in \mathbb{Z}^\nu$ with $m\omega > 0$, there is a unique $\lambda_m \in [E_m^-, E_m^+]$ with $w(\lambda_m) = m\omega + i h_m$, and for $z \in \mathbb{C}_+$, the derivative w' is given by the product formula*

$$-i w'(z) = i\Big(\frac{1}{4(E_0 - z)} \prod_{\substack{l\in\mathbb{Z}^\nu \\ l\omega>0}} \frac{(\lambda_l - z)^2}{(E_l^- - z)(E_l^+ - z)}\Big)^{1/2}; \tag{5.1}$$

(iv) *if the m-th gap is open, the derivative w' has an analytic extension through (E_m^-, E_m^+) given by the same product formula (5.1), and*

$$w'(E) = O(|E - E_m^\pm|^{-1/2}), \quad E \to E_m^\pm, \quad E \in (E_m^-, E_m^+).$$

The slit height is given by the convergent integral

$$h_m = \frac{1}{2}\int_{E_m^-}^{E_m^+} |w'(E)|dE.$$

We note that some qualitative parts of this statement can also be obtained in other ways by using previously proved properties of these potentials, including homogeneity of its spectrum [17]; for instance, existence of a product representation (5.1) is a general consequence of the fact that the Lyapunov exponent is zero on the spectrum. We give a self-contained proof which also prepares for the more specialized quantitative estimates.

Proof. We use the sequence of periodic approximants (1.11), denoting their associated objects with the superscript (r). We transform their Marchenko–Ostrovskii maps $w^{(r)}$ into the form

$$\Theta^{(r)}(z) = -i\, w^{(r)}(z^2 + E_0^{(r)}).$$

The bottom of the spectrum converges, $E_0^{(r)} \to E_0$, see [18, Theorem 7.1]. By Proposition 2.3, $w^{(r)}$ converge pointwise on $\mathbb{C}_+$ to w, so the functions $\Theta^{(r)}$ converge uniformly on $\overline{\mathbb{C}_+}$ to the function

$$\Theta(z) = -i\, w(z^2 + E_0).$$

The derivatives $(\Theta^{(r)})'$ also converge to Θ' on $\mathbb{C}_+$.

Next, let us establish convergence of gap edges and critical points. On the real line, $\operatorname{Re}\Theta^{(r)}$ are continuous increasing functions converging to $\operatorname{Re}\Theta$ and they all map 0 to 0. Note

$$E_m^{\pm} = \pm \max\{\pm E \in \mathbb{R} \mid \operatorname{Re}\Theta(E) = m\omega\}$$

and analogous characterizations of $E_m^{\pm,(r)}$.

If $E > E_m^+$, then $\Theta(E) > m\omega$, so $\Theta^{(r)}(E) > m\omega$ for all large enough r, so $E > E_m^{+,(r)}$ for all large enough r. Thus, $\limsup_{r\to\infty} E_m^{+,(r)} \le E$. Together with analogous arguments with opposite inequalities, this gives

$$E_m^- \le \liminf_{r\to\infty} E_m^{-,(r)} \le \limsup_{r\to\infty} E_m^{+,(r)} \le E_m^+.$$

If the m-th gap is closed, this implies

$$E_m^- = \lim_{r\to\infty} E_m^{-,(r)} = \lim_{r\to\infty} \lambda_m^{(r)} = \lim_{r\to\infty} E_m^{+,(r)} = E_m^+.$$

If the m-th gap is open, convergence of the exponential Herglotz representations

$$i(w^{(r)})'(z) = \frac{1}{2\sqrt{E_0 - z}} e^{\int_{[E_0^{(r)},\infty)\setminus S^{(r)}} \frac{\xi^{(r)}(\lambda)}{\lambda - z}\, d\lambda}$$

implies weak convergence of $\xi^{(r)}(\lambda)\, d\lambda$ against $C_c(\mathbb{R})$ functions, so it implies convergence $E_m^{\pm,(r)} \to E_m^{\pm}$ and $\lambda_m^{(r)}$ to λ_m for some $\lambda_m \in [E_m^-, E_m^+]$. Due to $w(\lambda_m) = \lim_{r\to\infty} w^{(r)}(\lambda_m^{(r)}) = m\omega + ih_m$, we have $E_m^- < \lambda_m < E_m^+$.

For $z \in \mathbb{C} \setminus \mathcal{S}$, the estimates

$$\left| \frac{\lambda_m - z}{E_m^{\pm} - z} - 1 \right| = \left| \frac{\lambda_m - E_m^{\pm}}{E_m^{\pm} - z} \right| \leq \frac{\gamma_m}{\operatorname{dist}(z, \mathcal{S})}$$

give estimates uniform on compacts, and uniform in r, on the terms in the product representations of w'. Thus, convergence of each term of the product implies that the product formulas converge. The remaining claims now follow from the estimates in Lemma 3.5. ∎

In particular, this contains Theorem 1.2.

Proof of Theorem 1.2. The comb domain structure is described in Prop. 5.1, and continuity of h_m in V follows from Proposition 4.4. ∎

We now turn to the converse direction: given a comb domain with exponentially decaying slit heights h_m, does it correspond to quasi-periodic Schrödinger operators in our regime? The frequency module could be read off from the locations of the bases (x-coordinates) of the slits [26, 59, 60], but the quantitative information about the analyticity and smallness of the potential is more specialized. We will study this by periodic approximation, building upon the result of Marchenko that in the periodic case, any ℓ_1^2 set of slits is possible.

The proof is an adaptation of methods of [18]. The work [18] starts from the isospectral tori of a small quasi-periodic potential V and its periodic approximants $V^{(r)}$, and shows convergence of the trajectories of the corresponding Dubrovin flows. This is used to conclude that the trajectory on the isospectral torus of V gives a potential of the desired form. Here, instead, we will start from a desired set of slits h_m. We will take periodic isospectral tori with frequencies $\omega^{(r)}$ and with essentially the same set of slits. We will show that the trajectories of the Dubrovin flows converge, that the limits are quasi-periodic operators in our regime, and that their slits are precisely h_m. Another application of this method is a continuity statement: within our class of potentials, we wish to know that convergence of slit heights and Dirichlet data implies convergence of potentials. We therefore point out that the approach in [18] is robust enough to yield both of those applications. Assume that we have a sequence of sets

$$\mathcal{S}^{(r)} = [E_0^{(r)}, \infty) \setminus \bigcup_{\substack{\mathfrak{m} \in \mathfrak{Z}(\omega^{(r)}) \\ \mathfrak{m}\omega^{(r)} > 0}} (E_{\mathfrak{m}}^{-,(r)}, E_{\mathfrak{m}}^{+,(r)})$$

such that, uniformly in r, we have the following gap-band estimates:

(i) for any $\mathfrak{m}$,

$$E_{\mathfrak{m}}^{+,(r)} - E_{\mathfrak{m}}^{-,(r)} \leq \varepsilon_1 \exp(-\kappa|\mathfrak{m}|);$$

(ii) for any $\mathfrak{m}' \neq \mathfrak{m}$ with $|\mathfrak{m}'| \geq |\mathfrak{m}|$,

$$\operatorname{dist}([E_{\mathfrak{m}}^{-,(r)}, E_{\mathfrak{m}}^{+,(r)}], [E_{\mathfrak{m}'}^{-,(r)}, E_{\mathfrak{m}'}^{+,(r)}]) \geq a|\mathfrak{m}'|^{-b} \geq (\varepsilon_1 \exp(-\kappa|\mathfrak{m}'|))^{1/4};$$

(iii) for any $\mathfrak{m}$ with $\mathfrak{m}\omega > 0$,

$$a|\mathfrak{m}|^{-b} \leq E_{\mathfrak{m}}^{-,(r)} - E_0^{(r)} \leq C|\mathfrak{m}|^2.$$

Assume that the sets $\mathcal{S}^{(r)}$ converge to

$$\mathcal{S} = [E_0, \infty) \setminus \bigcup_{\substack{m \in \mathbb{Z}^\nu \\ m\omega > 0}} (E_m^-, E_m^+)$$

in the sense that for each m, $E_m^{\pm,(r)} \to E_m^{\pm}$, and that $\mathfrak{N}(\omega) = \{0\}$. Clearly, then, the set $\mathcal{S}$ obeys the same gap-band estimates listed above.

We will then consider Dubrovin flows corresponding to the sets $\mathcal{S}^{(r)}$. Recall that the Dirichlet data $(\mu_{\mathfrak{m}}^{(r)}, \sigma_{\mathfrak{m}}^{(r)})_{\mathfrak{m}} \in \mathcal{D}(\mathcal{S}^{(r)})$ can be reparametrized in terms of angular variables $\varphi_{\mathfrak{m}}^{(r)}$ so that

$$\begin{aligned} \mu_{\mathfrak{m}}^{(r)} &= E_{\mathfrak{m}}^{-,(r)} + (E_{\mathfrak{m}}^{+,(r)} - E_{\mathfrak{m}}^{-,(r)}) \sin^2 \varphi_{\mathfrak{m}}^{(r)}, \\ \sigma_{\mathfrak{m}}^{(r)} &= \operatorname{sgn} \sin \varphi_{\mathfrak{m}}^{(r)} \end{aligned}$$

and that the Dubrovin flow corresponding to translation is then expressed as

$$\partial_x \varphi_{\mathfrak{m}}^{(r)} = \sqrt{(\mu_{\mathfrak{m}}^{(r)} - E_0^{(r)}) \prod_{\mathfrak{m}' \neq \mathfrak{m}} \frac{(E_{\mathfrak{m}'}^{-,(r)} - \mu_{\mathfrak{m}}^{(r)})(E_{\mathfrak{m}'}^{+,(r)} - \mu_{\mathfrak{m}}^{(r)})}{(\mu_{\mathfrak{m}'}^{(r)} - \mu_{\mathfrak{m}}^{(r)})^2}}.$$

The corresponding potential $Q^{(r)}(x, \varphi_{\mathfrak{m}}^{(r)})$ is obtained by the trace formula,

$$Q^{(r)}(x, \varphi_{\mathfrak{m}}^{(r)}) = E_0^{(r)} + \sum_{\mathfrak{m}} E_{\mathfrak{m}}^{+,(r)} + E_{\mathfrak{m}}^{-,(r)} - 2\mu_{\mathfrak{m}}^{(r)}.$$

The following lemma follows from the arguments in [18, Sections 8–9].

Lemma 5.2. *If $\mathcal{S}^{(r)}$, $\mathcal{S}$ are as above, and the initial data converge, i.e., $\varphi_{\mathfrak{m}}^{(r)} \to \varphi_m$ for all m, then the functions $Q^{(r)}(t)$ converge uniformly on compacts, i.e., there is a function $Q(t)$ such that for all $T < \infty$,*

$$\lim_{r \to \infty} \max_{-T \leq t \leq T} |Q(t) - Q^{(r)}(t)| = 0.$$

Further, we characterize isospectrality in terms of comb domains and averages.

Lemma 5.3. *Let $V, Q \in \mathcal{P}(\omega, \varepsilon, \kappa_0)$. The operators H_V and H_Q are isospectral if and only if*

$$\mathbb{E}(Q) = \mathbb{E}(V) \tag{5.2}$$

and

$$h_m(Q) = h_m(V), \quad \text{for all } m \in \mathbb{Z}^\nu. \tag{5.3}$$

Proof. If H_V, H_Q are isospectral, they have the same map w. From (1.2), we obtain equality of averages, and from the comb domain $w(\mathbb{C}_+)$ we obtain equality of slits.

For the converse, note that (5.3) implies that V and Q have the same Marchenko–Ostrovskii map Θ. The spectrum is determined by Θ up to an additive constant; therefore, $\mathcal{S} := \sigma(H_V) = \sigma(H_{Q-c})$ for some $c \in \mathbb{R}$. This implies that $Q - c \in \mathcal{R}(\mathcal{S})$ and, by the previous proposition, that $\mathbb{E}(Q - c) = \mathbb{E}(V)$. Comparing with (5.2), we conclude that $c = 0$. ∎

Further, we can conclude the following:

Proposition 5.4. *Let $V \in \mathcal{P}(\omega, \varepsilon, \kappa_0)$ and denote $\mathcal{S} = \sigma(H_V)$. Let $Q \in \mathcal{R}(\mathcal{S})$. Then $Q \in \mathcal{P}(\omega, \sqrt{2\varepsilon}, \kappa_0/2)$ and (5.2), (5.3) hold.*

Proof. This proposition follows from the arguments in the proof of Theorem I in [18]. The proof given there starts from the periodic approximants $V^{(r)}(x) = U(\omega^{(r)}x)$ and constructs certain periodic potentials $Q^{(r)}(x)$ (denoted by $Q^{(r)}(x, \gamma^{(r)})$ there) which are isospectral with $V^{(r)}(x)$. This implies that $Q^{(r)}$ and $V^{(r)}$ have the same Marchenko–Ostrovskii slits $h_m^{(r)}$, since the slits are determined by the spectrum. Moreover, $Q^{(r)}$ are of the form

$$Q^{(r)}(x) = \sum_{m \in \mathbb{Z}^\nu} d^{(r)}(m) e^{2\pi i m \omega^{(r)} x}$$

and it is shown that

$$|d^{(r)}(m)| \le \sqrt{2\varepsilon} \exp\Big(-\frac{\kappa_0}{2}|m|\Big), \quad m \in \mathbb{Z}^\nu$$

and that $Q^{(r)}$ converge to Q in the sense that $\lim_{r\to\infty} d^{(r)}(m) \to d(m)$ for each m and

$$Q(x) = \sum_{m \in \mathbb{Z}^\nu} d(m) e^{2\pi i m \omega x}.$$

By Proposition 4.4, this suffices to conclude

$$h_m(Q) = \lim_{r\to\infty} h_m^{(r)} = h_m(V).$$

Isospectrality of $Q^{(r)}$ with $V^{(r)}$ implies that $\mathbb{E}(Q^{(r)}) = \mathbb{E}(V^{(r)})$, so Lemma 2.4 implies (5.2). ∎

Proposition 5.5. *Let $\varepsilon^{(0)}(a_0, b_0, \kappa_0)$ be as in Theorem 3.1. Let h_m, $m \in \mathbb{Z}^\nu$, obey the conditions*

$$h_m \le \varepsilon' \exp(-\kappa|m|)$$

with $\varepsilon' < \varepsilon^{(0)}$ and $\kappa \ge 5\kappa_0$. Assume that the angular Dirichlet data φ_m are provided.

Then there exists a quasi-periodic operator $V \in \mathcal{P}(\omega, (\varepsilon')^{1/4}, \kappa/3)$ with slits $h_m(V) = h_m$ and angular Dirichlet data $\varphi_m(V) = \varphi_m$.

Proof. For the rational approximants $\omega^{(r)}$, we know that this set of slits h_m is allowed, i.e., there are periodic potentials $V^{(r)}$ which have the set of slits equal to h_m and angular Dirichlet data φ_m. Moreover, by Theorem 3.1, these potentials are of the form $V^{(r)}(x) = U^{(r)}(\omega^{(r)}x)$, with

$$U^{(r)}(\alpha) = \sum_{m\in\mathbb{Z}^\nu} c^{(r)}(m)\exp(2\pi i m\alpha), \quad |c^{(r)}(m)| \le (\varepsilon')^{1/3}\exp\Big(-\frac{\kappa}{5}|m|\Big).$$

By Lemma 5.2, these potentials converge uniformly on compacts to a potential V.

On the other hand, if a subsequence of $U^{(r)}$ converges uniformly to a function U, then $V(x) = U(\omega x)$, which determines the function U uniquely. Thus, every convergent subsequence of $U^{(r)}$ converges to the same limit, and by precompactness of the set of functions $U^{(r)}$ in $C(\mathbb{T}^\nu, \mathbb{R})$, we conclude that the sequence $U^{(r)}$ does in fact converge uniformly to a limit U.

Clearly, $U \in C(\mathbb{T}^\nu, \mathbb{R})$ is of the form

$$U(\alpha) = \sum_{m\in\mathbb{Z}^\nu} c(m)\exp(2\pi i m\alpha), \quad |c(m)| \le (\varepsilon')^{1/3}\exp\Big(-\frac{\kappa}{3}|m|\Big).$$

This provides us with the desired quasi-periodic operator $V(x) = U(\omega x)$ which, by Proposition 4.4, obeys $h_m(V) = \lim_{r\to\infty} h_m(V^{(r)}) = h_m$.

Moreover, V is given by the Dubrovin flow corresponding to the spectrum $\mathcal{S}$ determined by the slits h_m, with initial conditions φ_m, so we conclude that $\varphi_m(V) = \varphi_m$. ∎

Proof of Theorem 1.3. Since the spectrum is homogeneous, the isospectral torus is parametrized by the Dirichlet data [59]; by the previous proposition, any $Q \in \mathcal{R}(\mathcal{S})$ is in $\mathcal{P}(\omega, (\varepsilon')^{1/3}, \kappa/3)$. ∎

We now know that for each set of exponentially decaying h_m and each set of angular Dirichlet data φ_m, there is a unique potential V. In fact, the above methods also imply continuity of this construction.

Proposition 5.6. *Consider $V \in \mathcal{P}(\omega, \varepsilon, \kappa_0)$ and a sequence of $V_n \in \mathcal{P}(\omega, \varepsilon, \kappa_0)$. If $\mathbb{E}(V_n) \to \mathbb{E}(V)$, $h_m(V_n) \to h_m(V)$ and $\varphi_m(V_n) \to \varphi_m(V)$ for all m, then $V_n \to V$.*

Proof. By the same arguments as above, if $\mathbb{E}(V_n)$, $h_m(V_n)$ converge, then the spectra $\mathcal{S}_n$ converge; moreover, if the $\mathcal{S}_n$ converge and the φ_m converge, then the potentials converge. ∎

6. Gap edges and differentiability

It is clear from general principles that gap edges are 1-Lipschitz functions of V. However, for what follows, we need to establish stronger smoothness properties.

We begin by recalling the gap edge behavior of Weyl solutions [8, Proposition 2.1]. Fix $m \in \mathbb{Z}^\nu$, $m\omega > 0$, such that the m-th gap of H_V is open. For $z \in (E_m^-, E_m^+)$, denote by $\psi_\pm(x;z)$ the Weyl solutions of H_V at $\pm\infty$. Let $E \in \{E_m^-, E_m^+\}$ be one of the gap edges. There exists a non-trivial formal eigensolution $\psi(x;E)$ of $H_V\psi(x;E) = E\psi(x;E)$ which is a limit of Weyl solutions from the gap, in the following sense: there exist $c_\pm(z) \in \mathbb{C}$ such that

$$\lim_{\substack{z\to E \\ z\in(E_m^-,E_m^+)}} c_\pm(z)\psi_\pm(x;z) = \psi(x;E) \tag{6.1}$$

uniformly in x on compact subsets of $\mathbb{R}$.

Let $V(x) = U(\omega x)$ be our quasi-periodic potential, with $U\colon \mathbb{T}^\nu \to \mathbb{R}$ its sampling function. For $\beta \in \mathbb{T}^\nu$, let us denote by $\mu_l(\beta)$ the Dirichlet data corresponding to the potential $V_\beta(x) = U(\beta + \omega x)$; at a closed gap we will set $\mu_l = E_l^\pm$.

The following product will be repeatedly useful, for $m \in \mathbb{Z}^\nu$ and $z \in [E_m^-, E_m^+]$:

$$\Phi_m(\beta;z) = \frac{1}{2}\sqrt{\frac{1}{E_0 - z}\prod_{\substack{l\neq m \\ l\omega>0}} \frac{(\mu_l(\beta)-z)^2}{(E_l^- - z)(E_l^+ - z)}}.$$

This combination of factors occurs in Craig's product formula for the Green's function,

$$G(x,x;z,V) = \frac{1}{2}\sqrt{\frac{1}{E_0 - z}\prod_{\substack{l\in\mathbb{Z}^\nu \\ l\omega>0}} \frac{(\mu_l(\omega x)-z)^2}{(E_l^- - z)(E_l^+ - z)}} \tag{6.2}$$

but there the product includes the term with index $l = m$. Therefore, the formula (6.2) can be written as

$$G(x,x;z,V) = \frac{z - \mu_m(\omega x)}{\sqrt{(E_m^+ - z)(z - E_m^-)}}\Phi_m(\omega x;z). \tag{6.3}$$

We also define a similar product involving the critical values of w,

$$\chi_m(z) = \frac{1}{2\sqrt{z-E_0}}\Big(\prod_{l\neq m}\frac{(\lambda_l - z)^2}{(E_l^- - z)(E_l^+ - z)}\Big)^{1/2},$$

so that (5.1) can be rewritten as

$$w'(z) = \frac{z-\lambda_m}{\sqrt{(E_m^+ - z)(z-E_m^-)}}\chi_m(z).$$

Lemma 6.1. *Fix $m \in \mathbb{Z}^\nu$, $m\omega > 0$, and assume that the m-th gap is open. Let $E \in \{E_m^-, E_m^+\}$ be one of the gap edges. The squared eigensolution $\psi^2(x;E)$ is quasi-periodic in x, with frequency vector ω. When normalized so that*

$$\mathbb{E}(\psi^2(x;E)) = 1, \tag{6.4}$$

it is represented by the product formula

$$\psi^2(x;E) = \frac{|\mu_m(\omega x) - E|\Phi_m(\omega x;E)}{|\lambda_m - E|\chi_m(E)}. \tag{6.5}$$

Proof. Recall that

$$G(x,x;z) = \frac{\psi_-(x;z)\psi_+(x;z)}{W(\psi_+(\cdot;z),\psi_-(\cdot;z))}.$$

It follows from (6.1) that

$$\lim_{\substack{z\to E\\ z\in(E_m^-,E_m^+)}} \frac{W(\psi_+(\cdot,z),\psi_-(\cdot,z))}{c_-(z)c_+(z)}G(x,x;z) = \psi^2(x;E)$$

uniformly in x on compact subsets of $\mathbb{R}$. Meanwhile, (6.2) implies that

$$\lim_{\substack{z\to E\\ z\in(E_m^-,E_m^+)}} \sqrt{(E_m^+ - z)(z-E_m^-)}G(x,x;z) = (\mu_m(\omega x) - E)\Phi_m(\omega x;E)$$

converges uniformly on compacts, so the two limits must be equal up to a multiplicative constant. Thus, the squared eigensolution is of the form

$$\psi^2(x;E) = \frac{1}{C}|\mu_m(\omega x) - E|\Phi_m(\omega x;E) \tag{6.6}$$

for some $C \in (0,\infty)$. The right-hand side of (6.6) is a continuous function of $\mu(\omega x) \in \mathcal{D}(\mathcal{S})$, and the map $\mu\colon \mathbb{T}^\nu \to \mathcal{D}(\mathcal{S})$ is continuous, so $\psi(x;\lambda)^2$ is a quasi-periodic function with frequency ω.

As for normalization, for any $z \in (E_m^-, E_m^+)$, we have

$$\mathbb{E}(G(x,x;z)) = w'(z)$$

(see [38]) which can be rewritten as

$$\int_{\mathbb{T}^\nu} \frac{z-\mu_m(\beta)}{\sqrt{(E_m^+ - z)(z-E_m^-)}} \Phi_m(\beta;z)\, d\beta = \frac{z-\lambda_m}{\sqrt{(E_m^+ - z)(z-E_m^-)}} \chi_m(z).$$

After cancelling the square roots in the denominator, we obtain

$$\int_{\mathbb{T}^\nu} |E-\mu_m(\beta)| \Phi_m(\beta;E) d\beta = |E-\lambda_m| \chi_m(E).$$

Both sides of this equality converge as $z \to E$, so the equality holds also for $z = E$, and the choice

$$C = |\lambda_m - E| \chi_m(E)$$

results in the normalization (6.4). ∎

From now on, we will fix the normalization (6.5), so that (6.4) holds. Note that (6.5) can be rewritten as

$$\psi^2(x;E) = \prod_{\substack{l\in\mathbb{Z}^\nu \\ l\omega>0}} \frac{\mu_l(\omega x) - E}{\lambda_l - E}, \quad x \in \mathbb{R}. \tag{6.7}$$

We will also denote this by $\psi^2(x;E,V)$ when needed. We will also denote

$$\Psi^2(\beta; E_m^\pm, V) = \prod_{\substack{l\in\mathbb{Z}^\nu \\ l\omega>0}} \frac{\mu_l(\beta) - E_m^\pm}{\lambda_l - E_m^\pm}, \quad \beta \in \mathbb{T}^\nu,$$

noting that $\Psi^2(\omega x; E, V) = \psi^2(x; E, V)$. This encodes the quasi-periodic nature of ψ^2, and it will be helpful to do some estimates on the compact torus.

Moreover, we define the set

$$D_m = \{V \in \mathring{\mathcal{P}}(\omega, \varepsilon, \kappa_0) \mid \gamma_m(V) = 0\}.$$

This set is closed, since γ_m is a continuous function.

It should be kept in mind that $\psi^2(\beta; E_m^\pm, V)$ depends on V through the gap edge $E_m^\pm$, through all the critical points λ_l, and through all the Dirichlet data μ_l. Nonetheless, uniform control of the products implies a continuity statement.

Lemma 6.2. *If $V \in \mathring{\mathcal{P}}(\omega, \varepsilon, \kappa_0) \setminus D_m$, then*

$$\lim_{\substack{q \to 0 \\ q \in \mathcal{B}(\omega,\kappa_0)}} \mathbb{E}(|\psi^2(x; E_m^{\pm}, V+q) - \psi^2(x; E_m^{\pm}, V)|) = 0. \tag{6.8}$$

Proof. For $l \neq m$, the terms of the product are estimated uniformly by

$$\left| \frac{\mu_l(\beta) - E_m^{\pm}}{\lambda_l - E_m^{\pm}} - 1 \right| \leq \frac{\gamma_l}{\eta_{l,m}},$$

so pointwise convergence of Dirichlet data implies, for each $\beta \in \mathbb{T}^\nu$,

$$\lim_{\substack{q \to 0 \\ q \in \mathcal{B}(\omega,\kappa_0)}} \Psi^2(\beta; E_m^{\pm}, V+q) = \Psi^2(\beta; E_m^{\pm}, V).$$

The same upper bounds give a (constant) dominating function, so by dominated convergence,

$$\lim_{\substack{q \to 0 \\ q \in \mathcal{B}(\omega,\kappa_0)}} \int_{\mathbb{T}^\nu} |\Psi^2(\beta; E_m^{\pm}, V+q) - \Psi^2(\beta; E_m^{\pm}, V)| \, d\beta = 0,$$

but this is precisely (6.8), rewritten as an integral over the hull. ∎

Our next result is that $E_m^{\pm}$ is differentiable where the m-th gap is open.

Lemma 6.3. *For $m \in \mathbb{Z}^\nu$ with $m\omega > 0$ and any choice of $\pm$ sign, the gap edge $E_m^{\pm}(V)\colon \mathring{\mathcal{P}}(\omega, \varepsilon, \kappa_0) \setminus D_m \to \mathbb{R}$ is a C^1 function with Fréchet derivative*

$$(\partial_V E_m^{\pm})(q) = \mathbb{E}(\psi^2(x; E_m^{\pm}) q(x)).$$

Proof. By the estimates

$$\begin{aligned} &|\mathbb{E}(\psi^2(x; E_m^{\pm}, V_1) q(x)) - \mathbb{E}(\psi^2(x; E_m^{\pm}, V_2) q(x))| \\ &\quad \leq \mathbb{E}(|\psi^2(x; E_m^{\pm}, V_1) - \psi^2(x; E_m^{\pm}, V_2)|) \|q\|_\infty \end{aligned}$$

and

$$\|q\|_\infty \leq \sum_{n \in \mathbb{Z}^\nu} \|q\|_{\infty,\kappa_0} e^{-\kappa_0 |n|} = C \|q\|_{\infty,\kappa_0} \tag{6.9}$$

with an explicit constant $C = C(\kappa_0) < \infty$, the proposed derivative is, as a function of V, a continuous family of bounded linear functionals.

For the periodic approximants, it is known that the gap edges $E_m^{\pm}$ are C^1 on the set where the gap is open [41, 55], so in a neighborhood of some V with $\gamma_m(V) > 0$, the integral identities

$$E_m^{\pm}(V+q) - E_m^{\pm}(V) = \int_0^1 \mathbb{E}(\psi^2(x; E_m^{\pm}(V+tq), V+tq) q) \, dt$$

follow by periodic approximation, since we have uniform control of products of the form (6.7). By continuity of $\psi^2(x;E)$, this implies the C^1 property. ∎

We note that the derivative is precisely what one would expect from perturbation theory, and in the periodic case this can be made rigorous by interpreting $E_m^{\pm}$ as an isolated eigenvalue of an operator with (anti)periodic boundary conditions; this approach does not extend to the quasi-periodic case.

Proposition 6.4. *The function $\gamma_m^2 \colon \mathring{\mathcal{P}}(\omega,\varepsilon,\kappa_0) \to \mathbb{R}$ is C^1, with Fréchet derivative*

$$(\partial_V \gamma_m^2)(q) = \begin{cases} 2\gamma_m \mathbb{E}((\psi^2(x;E_m^+) - \psi^2(x;E_m^-))q(x)), & V \notin D_m, \\ 0, & V \in D_m. \end{cases}$$

Proof. At points where $\gamma_m(V) > 0$, this follows from $\gamma_m(V) = E_m^+(V) - E_m^-(V)$ and Lemma 6.3.

The Lipschitz estimate $|\gamma_m(V+q) - \gamma_m(V)| \le \|q\|_\infty$ implies that $\partial_V \gamma_m^2 = 0$ at points V where $\gamma_m(V) = 0$, which completes the proof. ∎

To prove differentiability of the gap midpoint, we need more precise estimates of the behavior of products as the gap length goes to zero.

Lemma 6.5. *For $V \in \mathcal{P}(\omega,\varepsilon,\kappa_0)$, fix $m \in \mathbb{Z}^\nu$, $m\omega > 0$, and let*

$$f(E;V) \in [E_m^-(V), E_m^+(V)] \to \mathbb{R}$$

be given by the product formula

$$f(E;V) = \sqrt{\frac{1}{E-E_0} \prod_{l \neq m} \frac{(E-a_l)^2}{(E-E_l^-)(E-E_l^+)}} \tag{6.10}$$

with arbitrary choices of $a_l(V) \in [E_l^-(V), E_l^+(V)]$.

(1) *There is a universal constant C such that $C^{-1} \le f(E;V) \le C$ for all V and all $E \in [E_m^-(V), E_m^+(V)]$. Moreover,*

$$f(E;V) = f(\tau_m) + O(\gamma_m), \quad \gamma_m \to 0,$$

uniformly in $V \in \mathcal{P}(\omega,\varepsilon,\kappa_0)$ and $E \in [E_m^-(V), E_m^+(V)]$.

(2) *If the functions $a_l(V)$ are continuous for all l, and $E = E(V)$ is a continuous function such that $E(V) \in [E_m^-(V), E_m^+(V)]$, then $f(E(V);V)$ is continuous in V.*

Proof. We can bound

$$\Big|\frac{(E-a_l)^2}{(E-E_l^-)(E-E_l^+)}\Big|^{\pm 1} \le \frac{\max(|E-E_l^-|,|E-E_l^+|)}{\min(|E-E_l^-|,|E-E_l^+|)} = 1+\frac{\gamma_l}{\eta_{m,l}}$$

so (3.9), (3.10) gives uniform bounds on $f(E)$. The logarithmic derivative of $f(E;V)$ is

$$\frac{\partial_E f(E;V)}{f(E;V)} = \frac{1}{2}\Big(-\frac{1}{E-E_0}+\sum_{l\neq m}\Big(\frac{2}{E-a_l}-\frac{1}{E-E_l^-}-\frac{1}{E-E_l^+}\Big)\Big)$$

and terms of this series are bounded by

$$\Big|\frac{2}{E-a_l}-\frac{1}{E-E_l^-}-\frac{1}{E-E_l^+}\Big| \le \Big|\frac{1}{E-E_l^-}-\frac{1}{E-E_l^+}\Big| \le \frac{\gamma_l}{\eta_{m,l}^2}.$$

This implies uniform convergence of the series, due to exponential bounds on the gaps and polynomial lower bounds on the gap distances. This completes the proof. ∎

We now consider the function

$$b_m = \begin{cases} 2\frac{\lambda_m-\tau_m}{\gamma_m}, & \gamma_m > 0, \\ 0, & \gamma_m = 0, \end{cases}$$

introduced to quantify how much the critical point λ_m deviates from the gap midpoint τ_m defined in (1.7). If the m-th gap is small but non-zero, the rest of the infinite product for w' is roughly constant along the m-th gap, so the critical point can be expected to be roughly in the middle of the gap; the following lemma is a rigorous version of this intuition.

Lemma 6.6. *The function b_m on $\mathcal{P}(\omega,\varepsilon,\kappa_0)\setminus D_m$ satisfies $b_m = O(\gamma_m)$ as $\gamma_m \to 0$. In particular, when extended by $b_m(V)=0$ for $\gamma_m(V)=0$, it is a continuous function on $\mathcal{P}(\omega,\varepsilon,\kappa_0)$.*

Proof. Lemma 6.5 implies the product asymptotics

$$\chi_m(E) = \chi_m(\tau_m) + O(\gamma_m), \quad \gamma_m \to 0.$$

Observe that

$$0 = w(E_m^+) - w(E_m^-) = \int_{E_m^-}^{E_m^+} w'(E)\,dE$$

and use the change of variables $E = \tau_m + \frac{\gamma_m}{2} s$ to rewrite this as

$$0 = \frac{\gamma_m}{2} \int_{-1}^{1} \frac{b_m - s}{\sqrt{1-s^2}} (\chi_m(\tau_m) + O(\gamma_m)) ds = \frac{\gamma_m}{2} \chi_m(\tau_m) \int_{-1}^{1} \frac{b_m - s}{\sqrt{1-s^2}} ds + O(\gamma_m^2).$$

Computing the remaining explicit integral gives

$$0 = \frac{\gamma_m}{2} \chi_m(\tau_m) \pi b_m + O(\gamma_m^2)$$

which implies $b_m = O(\gamma_m)$. ∎

The product formula for $\psi^2(x; E_m^{\pm})$ contains the factor

$$\frac{\mu_m(\omega x) - E_m^{\pm}}{\lambda_m - E_m^{\pm}},$$

whose denominator goes to zero as the gap closes. The previous lemma helps us to control this term in order to prove differentiability of gap midpoints.

Proposition 6.7. *The function $\tau_m : \mathring{\mathcal{P}}(\omega, \varepsilon, \kappa_0) \to \mathbb{R}$ is C^1, with Fréchet derivative*

$$(\partial_V \tau_m)(q) = \begin{cases} \mathbb{E}\Big(\frac{1}{2}(\psi^2(x; E_m^+) + \psi^2(x; E_m^-))q(x)\Big), & \gamma_m > 0, \\ \mathbb{E}(\chi_m(\tau_m)^{-1} \Phi_m(\omega x; \tau_m) q(x)), & \gamma_m = 0. \end{cases}$$

Proof. The main step is to verify that the quasi-periodic function

$$g(\beta; V) = \begin{cases} \frac{1}{2}(\Psi^2(\beta; E_m^+) + \Psi^2(\beta; E_m^-)), & \gamma_m > 0, \\ \chi_m(\tau_m)^{-1} \Phi_m(\beta; \tau_m), & \gamma_m = 0, \end{cases}$$

is continuous in V. It is clearly continuous on the closed set D_m and on its open complement, so the only non-trivial case is continuity at points $V \in D_m$ when approached from its complement. Using

$$\frac{2}{\gamma_m} |\mu_m(\beta) - E^{\pm}| \Phi_m(\beta; E^{\pm}) = (1 \mp \cos \varphi_m(\beta))(\Phi_m(\beta; \tau_m) + O(\gamma_m))$$

and

$$\frac{2}{\gamma_m} |\lambda_m - E_m^{\pm}| \chi_m(E_m^{\pm}) = (1 \mp b_m)(\chi_m(\tau_m) + O(\gamma_m)) = \chi_m(\tau_m) + O(\gamma_m),$$

we conclude that

$$\Psi^2(\beta; E_m^{\pm}) = \frac{|\mu_m(\beta) - E^{\pm}| \Phi_m(\beta; E^{\pm})}{|\lambda_m - E^{\pm}| \chi_m(E^{\pm})} = \frac{(1 \mp \cos \varphi_m(\beta)) \Phi_m(\beta; \tau_m)}{\chi_m(\tau_m)} + O(\gamma_m)$$

and therefore

$$\frac{1}{2}(\Psi^2(\beta; E_m^-) + \Psi^2(\beta; E_m^+)) = \frac{\Phi_m(\beta; \tau_m)}{\chi_m(\tau_m)} + O(\gamma_m)$$

which shows continuity of g at a point $V \in D_m$.

Now, the rest of the proof follows by periodic approximation, analogously to the proof of Lemma 6.3, since in the periodic case this is known to be the formula for the derivative. ∎

Proof of Theorem 1.4. This result is contained in Propositions 6.4 and 6.7. ∎

7. Actions

In this section, we study the actions I_m defined by (1.8).

For $V \in \mathcal{P}(\omega, \varepsilon, \kappa_0)$, by the previous sections, w' has square-root singularities at the gap edges, and $w(\lambda) \to im\omega$ as $\lambda \to E_m^{\pm}$, so by an integration by parts, (1.8) can be rewritten in the form

$$I_m = \frac{2}{\pi} \int_{E_m^-}^{E_m^+} \lambda w'(\lambda)\, d\lambda$$

or the form

$$I_m(V) = \frac{2}{\pi} \int_{E_m^-}^{E_m^+} (\lambda - \lambda_m) w'(\lambda)\, d\lambda. \tag{7.1}$$

From either (1.8) or (7.1), we see that $I_m \geq 0$, and $I_m(V) = 0$ if and only if the m-th gap is closed. The first goal of this section is to prove the differentiability of actions.

Proposition 7.1. *The function $I_m : \mathring{\mathcal{P}}(\omega, \varepsilon, \kappa_0) \to \mathbb{R}$ is a C^1 function, with Fréchet derivative* (1.10).

In the previous section, we have already used the diagonal Green's function. By Scharf [57] (see also Johnson and Moser [38]), for $z \in \mathbb{C} \setminus \mathcal{S}$, $G(x, x; z, V)$ is an almost periodic function of x with the same frequency module as V (or a proper subset), and the product formula (6.2) is well suited to be rewritten with respect to the hull: for each $z \in \mathbb{C} \setminus \mathcal{S}$, if we define

$$\Gamma(\beta; z, V) = \frac{1}{2} \sqrt{\frac{1}{E_0 - z} \prod_{\substack{l \in \mathbb{Z}^\nu \\ l\omega > 0}} \frac{(\mu_l(\beta) - z)^2}{(E_l^- - z)(E_l^+ - z)}},$$

then $\Gamma(\cdot\,;\lambda,V)\in C(\mathbb{T}^\nu)$ and

$$G(x,x;\lambda,V)=\Gamma(\omega x;\lambda,V),\quad \text{for all } x\in\mathbb{R}.$$

This agrees with changes of the potential in the hull in the sense that if $V(x)=U(\omega x)$ with $U\colon\mathbb{T}^\nu\to\mathbb{R}$, and if $V_\beta(x)=U(\omega x+\beta)$, then

$$G(x,x;\lambda,V_\beta)=\Gamma(\omega x+\beta;\lambda,V),\quad \text{for all } x\in\mathbb{R}.$$

We will use this to rewrite the candidate derivative (the right-hand side of (1.10)) as an integral over the hull and to prove that it is a continuous functional on $\mathcal{P}(\omega,\varepsilon,\kappa_0)$.

Lemma 7.2. *For any $V\in\mathcal{P}(\omega,\varepsilon,\kappa_0)$, the expression*

$$\Lambda_m(q;V)=\mathbb{E}\left(q(x)\int_{E_m^-}^{E_m^+}G(x,x;\lambda,V)\,d\lambda\right)$$

defines a bounded linear functional $\Lambda_m(\cdot\,;V):\mathcal{B}(\omega,\kappa_0)\to\mathbb{R}$ and

$$|\Lambda_m(q;V)|\le A(m)\gamma_m\pi\|q\|_\infty. \tag{7.2}$$

Moreover, the functional $\Lambda_m(\cdot\,;V)$ depends continuously on $V\in\mathcal{P}(\omega,\varepsilon,\kappa_0)$.

Proof. It is beneficial to rewrite the functional as an integral over $\beta\in\mathbb{T}^\nu$, instead of a spatial average over $x\in\mathbb{R}$. For this, we denote the sampling map $p\in C(\mathbb{T}^\nu)$ such that $q(x)=p(\omega x)$ for all $x\in\mathbb{R}$, and write

$$\Lambda_m(q;V)=\int_{\mathbb{T}^\nu}\left(\int_{E_m^-}^{E_m^+}\Gamma(\beta;\lambda,V)d\lambda\right)p(\beta)\,d\beta.$$

Bounding the product formula (6.2) as in the proof of Lemma 3.5, we conclude

$$|\Gamma(\beta;\lambda,V)|\le A(m)\frac{\gamma_m}{\sqrt{(E_m^+-\lambda)(\lambda-E_m^-)}}. \tag{7.3}$$

Integrating this estimate along the gap gives

$$\int_{E_m^-}^{E_m^+}\sup_{\beta\in\mathbb{T}^\nu}|\Gamma(\beta;\lambda,V)|d\lambda\le A(m)\gamma_m\pi \tag{7.4}$$

which implies that $\Lambda_m(q;V)$ is well defined and bounded by (7.2).

Fix a sequence $\{V_n\}_{n=1}^{\infty} \in \mathcal{P}(\omega, \varepsilon, \kappa_0)$ such that $V_n \to V$ uniformly on $\mathbb{R}$. Then $V_n = U_n(\omega x)$ and $U_n \to U$ uniformly on $\mathbb{T}^\nu$. To prove continuity, we separate cases.

If the m-th gap of V is closed, apply (7.2) to the V_n and the universal estimate on $A(m)$ to conclude

$$|\Lambda_m(q; V_n) - \Lambda_m(q; V)| = |\Lambda_m(q; V_n)| \le \pi C |m|^{C \log_2 \log_2 |m|} \gamma_m(V_n) \|q\|_\infty.$$

This shows the desired continuity at V, since $\gamma_m(V_n) \to \gamma_m(V) = 0$ as $n \to \infty$.

We assume from now on that $\gamma_m(V) > 0$. Since uniform convergence of potentials on compacts implies convergence of Green's functions,

$$\lim_{n\to\infty} \Gamma(\beta; \lambda_n, V_n) = \Gamma(\beta; \lambda, V)$$

for any $\beta \in \mathbb{T}^\nu$ and any sequence $\lambda_n \to \lambda \in (E_m^-, E_m^+)$. If we make the choice of sequence

$$\lambda_n = E_m^-(V_n) + \frac{\gamma_m(V_n)}{\gamma_m(V)}(\lambda - E_m^-(V)),$$

then (7.3) provides the bound for dominated convergence to conclude

$$\lim_{n\to\infty} \int_{E_m^-(V_n)}^{E_m^+(V_n)} \Gamma(\beta; \lambda, V_n) d\lambda = \int_{E_m^-(V)}^{E_m^+(V)} \Gamma(\beta; \lambda, V) d\lambda, \quad \text{for all } \beta \in \mathbb{T}^\nu.$$

Another application of dominated convergence using the bound (7.4) implies that

$$\lim_{n\to\infty} \int_{\mathbb{T}^\nu} \left(\int_{E_m^-(V_n)}^{E_m^+(V_n)} \Gamma(\beta; \lambda, V_n) d\lambda - \int_{E_m^-(V)}^{E_m^+(V)} \Gamma(\beta; \lambda, V) d\lambda \right) d\beta = 0,$$

which immediately implies

$$\lim_{n\to\infty} \sup_{q \in \mathcal{P}(\omega,\varepsilon,\kappa_0)\setminus\{0\}} \frac{|\Lambda_m(q; V_n) - \Lambda_m(q; V)|}{\|q\|_\infty} = 0.$$

By (6.9), $\|q\|_\infty$ in the denominator can also be replaced by $\|q\|_{\infty,\kappa_0}$. ∎

To prove the other properties, we will use periodic approximation. For a periodic potential $\widetilde{V}$, the action $I_{\mathfrak{m}}$ is defined by a contour integral in the Riemann surface which can be rewritten as an integral over the gap, see [41, Proof of Theorem 7.1],

$$I_{\mathfrak{m}}(\widetilde{V}) = \frac{2}{\pi} \int_{E_{\mathfrak{m}}^-}^{E_{\mathfrak{m}}^+} (-1)^{n(\mathfrak{m})-1} (\lambda - \lambda_{\mathfrak{m}}) \frac{\Delta'(\lambda)}{\sqrt{\Delta^2(\lambda) - 4}} d\lambda,$$

where $n = n(\mathfrak{m})$ is given by (3.3), Δ is the discriminant of $\widetilde{V}$, and the branch of square root is such that the integrand is non-negative. Using $\Delta(z) = 2\cos(iTw(z))$ and $w(z) = -y + in(\mathfrak{m})\pi$, $y > 0$, for $z \in [E_{\mathfrak{m}}^-, E_{\mathfrak{m}}^+]$, we see that $\sin(iw(z)) = i(-1)^{n(\mathfrak{m})}\sinh y$, so

$$\frac{2}{\pi}\int_{E_{\mathfrak{m}}^-}^{E_{\mathfrak{m}}^+}(-1)^{n(\mathfrak{m})-1}(\lambda-\lambda_{\mathfrak{m}})\frac{\Delta'(\lambda)}{\sqrt{\Delta^2(\lambda)-4}}d\lambda = \frac{2}{\pi}\int_{E_{\mathfrak{m}}^-}^{E_{\mathfrak{m}}^+}(\lambda-\lambda_{\mathfrak{m}})w'(\lambda)d\lambda$$

and hence the standard definition in the periodic setting is equivalent to ours.

In the periodic case, $I_{\mathfrak{m}}$ is known to be analytic, with Fréchet derivative

$$\partial_{\widetilde{V}} I_{\mathfrak{m}} = \frac{2}{\pi}\int_{E_{\mathfrak{m}}^-}^{E_{\mathfrak{m}}^+}(-1)^{n(\mathfrak{m})}\frac{\partial_{\widetilde{V}}\Delta(\lambda)}{\sqrt{\Delta^2(\lambda)-4}}d\lambda = \frac{2}{\pi}\int_{E_{\mathfrak{m}}^-}^{E_{\mathfrak{m}}^+}\partial_{\widetilde{V}} w(\lambda)d\lambda.$$

It is known that $\partial_{\widetilde{V}} w(\lambda) = -\mathbb{E}(G(x,x;\lambda)q(x))$ (see, e.g., [38, Theorem 6.4]), so this implies that Proposition 7.1 holds for the periodic approximants.

Proof of Prop. 7.1. The identity

$$I_m(V+q) - I_m(V) = \frac{2}{\pi}\int_0^1 \Lambda_m(q; V+tq)\,dt \tag{7.5}$$

holds for periodic approximants by the discussion above. Applying it to the sequence of periodic approximants $V^{(r)}$ and letting $r \to \infty$, the actions converge since the Marchenko–Ostrovskii maps are continuous, and the functionals Λ_m converge by the same proof used to prove their continuity (Lemma 7.2). Thus, the identity (7.5) holds also for $V \in \mathring{\mathcal{P}}(\omega, \varepsilon, \kappa_0)$ and small enough $q \in \mathcal{B}(\omega, \kappa_0)$. Since by Lemma 7.2, the functional Λ_m is continuous, the identity (7.5) shows that I_m is C^1 with Fréchet derivative Λ_m. ∎

As an interlude, let us note that a trace formula can also be obtained by periodic approximation.

Lemma 7.3. *For any $V \in \mathcal{P}(\omega, \varepsilon, \kappa_0)$, (1.9) holds.*

Proof. The T-periodic approximants $V^{(r)}$ obey the trace formula [41, Theorem E.1]

$$\sum_{n=1}^{\infty} 2\pi\frac{n}{T} I_n(V^{(r)}) = \frac{1}{2}\mathbb{E}((V^{(r)})^2). \tag{7.6}$$

The identity (7.1) implies $I_m \le 2\gamma_m h_m$. Given existing uniform estimates on the gap sizes and slit heights of V and its periodic approximants $V^{(r)}$, this provides a dominating function for the series, so from pointwise convergence $I_{\mathfrak{m}}(V^{(r)}) \to I_m(V)$ we obtain convergence of the series. Thus, taking $r \to \infty$ in the trace formula (7.6) proves (1.9). ∎

Our next goal is to characterize more precisely the behavior of the actions as a gap closes. We will show that the action goes to zero quadratically in the gap size, by showing that the expression $8I_m/\gamma_m^2$ extends to a continuous function on $\mathring{\mathcal{P}}(\omega, \varepsilon, \kappa_0)$.

Proposition 7.4. *The function*

$$\rho_m = \begin{cases} \sqrt{8I_m/\gamma_m^2}, & \gamma_m > 0, \\ \sqrt{2\chi_m(\tau_m)}, & \gamma_m = 0, \end{cases}$$

is a strictly positive, C^1 function on $\mathring{\mathcal{P}}(\omega, \varepsilon, \kappa_0)$.

This requires a strengthening of Lemma 6.5:

Lemma 7.5. *For $V \in \mathcal{P}(\omega, \varepsilon, \kappa_0)$, fix $m \in \mathbb{Z}^\nu \setminus \{0\}$ and let*

$$f(E; V) \in [E_m^-(V), E_m^+(V)] \to \mathbb{R}$$

be given by the product formula (6.10) with arbitrary choices of $a_l(V) \in [E_l^-(V), E_l^+(V)]$.

(i) *There is a universal constant C such that $C^{-1} \le f(E; V) \le C$ and $-C \le \partial_E f(E; V) \le C$ for all V and all $E \in [E_m^-(V), E_m^+(V)]$. Moreover,*

$$f(E; V) = f(\tau_m; V) + (E - \tau_m)\partial_E f(\tau_m) + O(\gamma_m^2), \quad \gamma_m \to 0,$$

uniformly in $V \in \mathcal{P}(\omega, \varepsilon, \kappa_0)$ and $E \in [E_m^-(V), E_m^+(V)]$.

(ii) *If the functions $a_l(V)$ are continuous for all l, and $E = E(V)$ is a continuous function such that $E(V) \in [E_m^-(V), E_m^+(V)]$, then $f(E(V); V)$ and $\partial_E f(E(V); V)$ are continuous in V.*

Proof. We build upon the proof of Lemma 6.5 and the bounds on $f(E; V)$ and its logarithmic derivative given there. By analogous estimates, we obtain a universal bound on the second logarithmic derivative of $f(E)$,

$$\partial_E^2 \log f(E; V) = \frac{\partial_E^2 f(E; V)}{f(E; V)} - \Big(\frac{\partial_E f(E; V)}{f(E; V)}\Big)^2$$

which is sufficient to complete the proof. ∎

We also need a strengthening of Lemma 6.6, which shows not just that the critical point approaches the midpoint as the m-th gap shrinks, but also gives the first correction term to this approach.

Lemma 7.6. *The function b_m has asymptotic behavior*

$$b_m = 2\frac{\lambda_m - \tau_m}{\gamma_m} = \frac{\gamma_m}{4}\frac{\partial_E \chi_m(\tau_m)}{\chi_m(\tau_m)} + O(\gamma_m^2), \quad \gamma_m \to 0.$$

Proof. Using Lemma 7.5, we write

$$\chi_m(E) = \chi_m(\tau_m) + (E - \tau_m)\partial_E \chi_m(\tau_m) + O(\gamma_m^2).$$

Observe that

$$0 = w(E_m^+) - w(E_m^-) = \int_{E_m^-}^{E_m^+} w'(E)dE$$

and use the change of variables $E = \tau_m + \frac{\gamma_m}{2}s$ to rewrite this as

$$\begin{aligned} 0 &= \frac{\gamma_m}{2}\int_{-1}^{1} \frac{b_m - s}{\sqrt{1-s^2}}(\chi_m(\tau_m) + \frac{\gamma_m}{2}s\partial_E \chi_m(\tau_m) + O(\gamma_m^2))ds \\ &= \frac{\gamma_m}{2}\chi_m(\tau_m)\int_{-1}^{1} \frac{b_m - s}{\sqrt{1-s^2}}ds + \frac{\gamma_m^2}{4}\partial_E \chi_m(\tau_m)\int_{-1}^{1} \frac{b_m - s}{\sqrt{1-s^2}}sds + O(\gamma_m^3) \\ &= \frac{\gamma_m}{2}\chi_m(\tau_m)\pi b_m - \frac{\gamma_m^2}{8}\pi\partial_E \chi_m(\tau_m) + O(\gamma_m^3) \end{aligned}$$

which implies the lemma. ∎

Proof of Proposition 7.4. To prove continuity of ρ_m^2, the only non-trivial case is to prove convergence of ρ_m^2 along a sequence V_n with $\gamma_m(V_n) > 0$ but $\gamma_m(V_n) \to 0$ as $n \to \infty$. We write

$$I_m = \frac{2}{\pi}\int_{E_m^-}^{E_m^+} (E - \lambda_m)\frac{dw}{dE}dE = \frac{2}{\pi}\int_{E_m^-}^{E_m^+} \frac{(E - \lambda_m)^2}{\sqrt{(E_m^+ - E)(E - E_m^-)}}\chi_m(E)\,dE.$$

We use the change of variables

$$E = \tau_m + s\gamma_m/2, \quad s \in [-1, 1] \tag{7.7}$$

to write

$$I_m = \frac{1}{2\pi}\gamma_m^2\int_{-1}^{1} \frac{(s - b_m)^2}{\sqrt{1-s^2}}\chi_m(\tau_m + s\gamma_m/2)\,ds$$

and therefore, on the complement of D_m,

$$\rho_m^2 = 8\frac{I_m}{\gamma_m^2} = \frac{4}{\pi}\int_{-1}^{1} \frac{(s-b_m)^2}{\sqrt{1-s^2}}\chi_m(\tau_m + s\gamma_m/2)\,ds. \tag{7.8}$$

Using $\chi_m(\tau_m + s\gamma_m/2) = \chi_m(\tau_m) + O(\gamma_m)$ and Lemma 6.6 proves that

$$\rho_m^2 = \frac{4}{\pi}\chi_m(\tau_m)\int_{-1}^{1} \frac{s^2}{\sqrt{1-s^2}}ds + O(\gamma_m) = 2\chi_m(\tau_m) + O(\gamma_m),$$

which proves continuity at points in D_m. Thus, ρ_m^2 is continuous.

For differentiability, we need a higher order version of the above analysis: starting from (7.8) but using Lemma 7.5,

$$\chi_m\Big(\tau_m + s\frac{\gamma_m}{2}\Big) = \chi_m(\tau_m) + \frac{\gamma_m}{2}s\partial_E\chi_m(\tau_m) + O(\gamma_m^2)$$

gives

$$\begin{aligned}
\rho_m^2 &= \frac{4}{\pi}\int_{-1}^{1} \frac{(s-b_m)^2}{\sqrt{1-s^2}}\Big(\chi_m(\tau_m) + \frac{\gamma_m}{2}s\partial_E\chi_m(\tau_m) + O(\gamma_m^2)\Big)\,ds \\
&= \frac{4}{\pi}\int_{-1}^{1}\Big(\frac{s^2}{\sqrt{1-s^2}}\chi_m(\tau_m) - 2b_m\frac{s}{\sqrt{1-s^2}}\chi_m(\tau_m) \\
&\qquad + \frac{\gamma_m}{2}\frac{s^3}{\sqrt{1-s^2}}\partial_E\chi_m(\tau_m) + O(\gamma_m^2)\Big)\,ds \\
&= 2\chi_m(\tau_m) + O(\gamma_m^2),
\end{aligned}$$

where we used

$$\begin{aligned}
\int_{-1}^{1}\frac{1}{\sqrt{1-s^2}}ds &= \pi, \quad \int_{-1}^{1}\frac{s}{\sqrt{1-s^2}}ds = 0, \\
\int_{-1}^{1}\frac{s^2}{\sqrt{1-s^2}}ds &= \frac{\pi}{2}, \quad \int_{-1}^{1}\frac{s^3}{\sqrt{1-s^2}}ds = 0.
\end{aligned}$$

Using this, let us prove that the derivative

$$\partial_V\rho_m^2 = \frac{1}{\gamma_m^2}(8\partial_V I_m - \rho_m^2\partial_V\gamma_m^2) \tag{7.9}$$

extends continuously to $\mathcal{P}(\omega, \varepsilon, \kappa_0)$. To write the derivative as an integral over the hull, we will once again denote $p \in C(\mathbb{T}^\nu)$ so that $q(x) = p(\omega x)$. Using (6.3) and the change of variables (7.7) and applying Lemma 7.5 to $\Phi_m(\omega x; E)$ for all x we conclude

$$(\partial_V I_m)(q) = \int_{\mathbb{T}^\nu} G_m(\beta) p(\beta)\, d\beta,$$

where G_m is the function on the hull representing the derivative, given by

$$\begin{aligned} G_m(\beta) &= \frac{2}{\pi} \int_{E_m^-}^{E_m^+} \frac{E - \mu_m(\beta)}{\sqrt{(E_m^+ - E)(E - E_m^-)}} \Phi_m(\beta; E)\, dE \\ &= \frac{\gamma_m}{\pi} \int_{-1}^{1} \frac{s - \cos\varphi_m(\beta)}{\sqrt{1 - s^2}} \Phi_m(\beta; \tau_m + \frac{\gamma_m}{2} s)\, ds \\ &= \frac{\gamma_m}{\pi} \int_{-1}^{1} \frac{s - \cos\varphi_m(\beta)}{\sqrt{1 - s^2}} \Phi_m(\beta; \tau_m)\, ds \\ &\quad + \frac{\gamma_m^2}{2\pi} \int_{-1}^{1} \frac{s - \cos\varphi_m(\beta)}{\sqrt{1 - s^2}} \partial_E \Phi_m(\beta; \tau_m) s\, ds + O(\gamma_m^3) \\ &= -\gamma_m \cos\varphi_m(\beta) \Phi_m(\beta; \tau_m) + \frac{\gamma_m^2}{4} \partial_E \Phi_m(\beta; \tau_m) + O(\gamma_m^3). \end{aligned} \tag{7.10}$$

Note that the estimate in Lemma 7.5 persisted through the integration since the integrand obeys a uniform bound times a square root singularity at the edges. Now, write

$$\begin{aligned} \frac{2}{\gamma_m} |E_m^\pm - \lambda_m| \chi_m(E_m^\pm) &= (1 \mp b_m)(\chi_m(\tau_m) \pm \frac{\gamma_m}{2} \partial_E \chi_m(\tau_m) + O(\gamma_m^2)) \\ &= \chi_m(\tau_m) \mp b_m \chi_m(\tau_m) \pm \frac{\gamma_m}{2} \partial_E \chi_m(\tau_m) + O(\gamma_m^2) \\ &= \chi_m(\tau_m) \pm \frac{\gamma_m}{4} \partial_E \chi_m(\tau_m) + O(\gamma_m^2) \end{aligned}$$

which implies that

$$\frac{1}{\frac{2}{\gamma_m} |E_m^\pm - \lambda_m| \chi_m(E_m^\pm)} = \frac{1}{\chi_m(\tau_m)} \Big(1 \mp \frac{\gamma_m}{4} \frac{\partial_E \chi_m(\tau_m)}{\chi_m(\tau_m)}\Big) + O(\gamma_m^2). \tag{7.11}$$

Similarly,

$$\begin{aligned} &\frac{2}{\gamma_m} |E_m^\pm - \mu_m(\beta)| \Phi_m(\beta; E_m^\pm) \\ &= (1 \mp \cos\varphi_m(\beta)) \Big(\Phi_m(\beta; \tau_m) \pm \frac{\gamma_m}{2} \partial_E \Phi_m(\beta; \tau_m) + O(\gamma_m^2) \Big). \end{aligned} \tag{7.12}$$

Multiplying (7.11) and (7.12),

$$\begin{aligned}
&\frac{|E_m^{\pm} - \mu_m(\beta)|\Phi_m(\beta; E_m^{\pm})}{|E_m^{\pm} - \lambda_m|\chi_m(E_m^{\pm})} \\
&= \frac{1}{\chi_m(\tau_m)}\Big((1 \mp \cos\varphi_m(\beta))(\Phi_m(\beta;\tau_m) \pm \frac{\gamma_m}{2}\partial_E\Phi_m(\beta;\tau_m)) \\
&\qquad \mp \frac{\gamma_m}{4}\frac{\partial_E\chi_m(\tau_m)}{\chi_m(\tau_m)}(1 \mp \cos\varphi_m(\beta))\Phi_m(\beta;\tau_m)\Big) + O(\gamma_m^2).
\end{aligned}$$

Subtracting this with + minus with − gives

$$\begin{aligned}
&\frac{|E_m^{+} - \mu_m(\beta)|\Phi_m(\beta; E_m^{+})}{|E_m^{+} - \lambda_m|\chi_m(E_m^{+})} - \frac{|E_m^{-} - \mu_m(\beta)|\Phi_m(\beta; E_m^{-})}{|E_m^{-} - \lambda_m|\chi_m(E_m^{-})} \\
&= \frac{2}{\chi_m(\tau_m)}\Big(-\cos\varphi_m(\beta)\Phi_m(\beta;\tau_m) + \frac{\gamma_m}{2}\partial_E\Phi_m(\beta;\tau_m) \\
&\qquad - \frac{\gamma_m}{4}\frac{\partial_E\chi_m(\tau_m)}{\chi_m(\tau_m)}\Phi_m(\beta;\tau_m)\Big) + O(\gamma_m^2).
\end{aligned}$$

Therefore, the following derivative is of the form

$$\rho_m^2 2\gamma_m(\partial_V E_m^{+} - \partial_V E_m^{-})(q) = \int_{\mathbb{T}^\nu} H_m(\beta)\, p(\beta)\, d\beta$$

where the function on the hull representing this derivative is

$$\begin{aligned}
H_m(\beta) = 8\gamma_m\Big(&-\cos\varphi_m(\beta)\Phi_m(\beta;\tau_m) + \frac{\gamma_m}{2}\partial_E\Phi_m(\beta;\tau_m) \\
&- \frac{\gamma_m}{4}\frac{\partial_E\chi_m(\tau_m)}{\chi_m(\tau_m)}\Phi_m(\beta;\tau_m)\Big) + O(\gamma_m^3). \qquad (7.13)
\end{aligned}$$

Subtracting (7.13) from 8 times (7.10), dividing by γ_m^2, and using (7.9), we conclude

$$(\partial_V\rho_m^2)(q) = \int_{\mathbb{T}^\nu} L_m(\beta)\, p(\beta)\, d\beta,$$

where

$$L_m(\beta) = -2\partial_E\Phi_m(\beta;\tau_m) + 2\frac{\partial_E\chi_m(\tau_m)}{\chi_m(\tau_m)}\Phi_m(\beta;\tau_m) + O(\gamma_m),$$

so $\partial_V\rho_m^2$ extends continuously to all of $\mathcal{P}(\omega, \varepsilon, \kappa_0)$. ∎

Proof of Theorem 1.5. This is contained in the results proved in this section. ∎

Funding. I. B. was supported in part by an NSERC Discovery Grant. D. D. was supported in part by NSF grants DMS–2054752 and DMS–2349919. M. L. was supported in part by NSF grant DMS–2154563/2626207 and DMS–2453758/2626193.

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Received 15 April 2025; revised 5 June 2026.

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