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Inverse spectral problem for Hamiltonians of interacting fermionic systems in almost-periodic media

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Abstract. We consider the inverse spectral problem for two classes of Hamiltonians of interacting Fermi particles on an integer lattice of arbitrary dimension subject to a common almost-periodic potential. We show first that the inverse problem can be solved with the help of the KAM (Kolmogorov–Arnold–Moser) techniques from Craig (1983) and Pöschel (1983), initially developed for single-particle models, for a particular class of limit-periodic potentials, and then comment on the adaptations to make in the case of quasi-periodic potentials. The solution is obtained in the class of Hamiltonians with a small amplitude of the kinetic energy operator (i.e., with a low mobility of the particles) featuring a uniform exponential decay of all eigenfunctions which prove to be unimodal.

1. Introduction

1.1. Motivation

The beginning of 1980’s was marked by a wave of mathematical papers on the Anderson localization phenomenon. Since then, a majority of papers on spectral problems (mostly the direct ones) explored the properties of Hamiltonians of non-interacting quantum particles subject to an external ergodic potential $V(x, \omega)$ with strong stochastic properties. For example, a popular class of lattice models operated with IID (independent and identically distributed) values of the potential.

Nevertheless, it is worth emphasizing that the models with almost-periodic potentials, having very weak stochastic properties, made an essential contribution to shaping a new area of spectral analysis. In particular, one can hardly overestimate the role of the celebrated “Maryland model” introduced and analysed initially by physicists [18, 19]. In the wake of the papers on the direct spectral problem, the inverse problem was solved for quasi-periodic and limit-periodic Hamiltonians on the lattice $\mathbf{Z}^k$ of arbitrary dimension; see [3, 14, 22] and references therein.

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A rigorous analysis of $\mathfrak{n}$-particle Hamiltonians involving non-trivial inter-particle interactions in a disordered environment had to wait some three decades; the first papers [1, 11, 12] in this direction appeared in 2008–2009, a few years after the physical works [2, 21]. These and several subsequent mathematic papers devoted to interacting quantum particles treated only the direct spectral problem, and only in presence of a disordered media-particle potential with strong stochasticity (IID, in the lattice case). Somehow, one overlooked an opportunity to explore the inverse spectral problem with the help of a suitable adaptation of the techniques from [14, 22]. The goal of the present paper is precisely to make up for this regrettable omission.

Comparing the two main results of [14, 22], viz. [22, Theorems A and B], one can see that the hypotheses of Theorem A (cf. Theorem 3 in Section 6) for the inverse spectral problem are considerably simpler to check in a given model than for the direct problem (cf. [22, Theorem B]). Specifically, to be able to solve the inverse problem for a given diagonal matrix Λ with $\Lambda_{xx} = \lambda_x$, $\lambda\colon \mathbf{Z}^k \to \mathbf{R}$, it suffices that the function λ be "distal" in the sense of the (unnumbered) definition given in [22, Section 1]. This gives rise to a larger class of explicit examples than in the case of the direct spectral problem where λ is required to be "stable distal" (cf. [22, Section 1, the paragraph following Theorem B]).

The aforementioned simplification is particularly pronounced in the case of $\mathfrak{n}$-particle Hamiltonians; $\mathfrak{n} > 1$. We note in passing that the hypotheses of [14, 22], resulting in a uniform exponential localization of eigenfunctions (ULE; cf. [15, Appendix 3]), cannot be true for random potentials featuring a sufficiently strong stochasticity, for the latter gives rise to a semi-uniform (SULE) but not uniform localization. We addressed the direct spectral problem in a separate work [10], building on the techniques developed in earlier publications [8, 9] which are considerably more involved, and using some general ideas from the well-known work by Y. G. Sinai [24]. In the present paper, we also make use of some techniques from [8, 9], but an essential part of technical work is now more transparent.

The analytic core of the works [14, 22] is a linear variant of the KAM (Kolmogorov–Arnold–Moser) method, which is itself a particular case of the general Newton–Raphson method, albeit formulated in a specific, non-traditional way. Here, too, we would like to stress that the first strong result on spectral properties of quasi-periodic differential Schrödinger operators, viz. existence of absolutely continuous spectrum for energies not too close to some "resonant" values, was proved in [16] with the help of the KAM method. A completion of the latter work by Eliasson [17] was, too, based on the KAM-type techniques.

We do not have to *enhance* in the present work the principal analytic *techniques* of [14, 22] which are quite general and efficient.

However, a direct, formal application to fermionic Hamiltonians of the *statement* of [22, Theorem A] is far from obvious, for several reasons.

• Firstly, in order to formulate either the direct or the inverse spectral problem, in view of finding the solution by successive approximations, one has to find first a suitable, efficient initial approximation. More to the point, in the general framework of interactive disordered quantum particle systems, one can start either with the purely kinetic component of the full Hamiltonian, or with the Hamiltonian of the system of non-interacting particles. The latter option requires, of course, solving the single-particle problem first. The former one seems tempting but proves to be more complex on the technical level, as evidences a thorough analysis of the early works [1,12]. This is why in the present paper, we start with the $\mathfrak{n}$-particle system with no interaction and make use of the results and techniques on the single-particle system in an almost-periodic environment. Consequently, the full Hamiltonian $\mathbf{H}$ is decomposed in the sum $\mathbf{H}^{\text{non-int}} + \mathbf{U}$, where $\mathbf{U}$ is the inter-particle potential energy operator. Naturally, the initial orthonormal basis is the eigenbasis for $\mathbf{H}^{\text{non-int}}$, and the off-diagonal matrix elements of $\mathbf{U}$ decay exponentially fast away from the diagonal. Diagonalization of $\mathbf{H}$ is achieved with the help of a linear KAM method.

• Secondly, to overcome the usual problems due to the "small denominators" (cf. Section 2.6) invariably appearing in the KAM-type methods, one has to define a very special norm in the space of operators, since the decay of the aforementioned small denominators is strongly anisotropic in the fermionic configuration space, unlike the 1-particle case. It turns out that the lower bounds on the small denominators that one can prove decay much faster in some directions; too fast for the Craig–Pöschel technique (outlined in Section 6) to apply directly.

Nevertheless, we show that the latter, thorny technical problem can be circumvented, so that a suitable *adaptation* of the Craig–Pöschel *techniques* still applies to the fermionic Hamiltonians. To this end, we introduce in the fermionic configuration space a new metric (not a pseudo-metric like, e.g., in [1, 11, 12]), better adapted to the localization analysis of interactive Hamiltonians than those used in prior works on $\mathfrak{n}$-particle Anderson localization.

This requires a number of preliminary considerations and estimates, which explains why the statement of the main result appears only in Section 4, after a rather long Section 2 and a brief construction in Section 3 of a class of explicit examples of limit-periodic external, media-particle potentials giving rise to "distal" eigenvalue generating functions.

It is worth emphasizing that, as evidences the experience of the early years of multi-particle Anderson localization, the choice of a suitable, efficient metric (or pseudo-metric) in the $\mathfrak{n}$-particle space is not a banality, and it can be crucial for the analytic techniques to apply. It was indeed a *pseudo*-metric, not a genuine one, which proved to be the key to a quite complex technical spectral analysis of interactive Hamiltonians. Curiously, in the framework of a significantly simpler, algebraic approach, in essence going back to [3, 14, 22, 23], the pivot element of the technical analysis is a function, denoted $\mathfrak{d}(\cdot,\cdot)$ in Appendix A, verifying the triangle inequality and symmetric. As to the condition $\mathfrak{d}(a,a) \equiv 0$, it actually is almost irrelevant, because the values on the "diagonal", viz. $\mathfrak{d}(a,a)$, are of little importance. Using $\mathfrak{d}(\cdot,\cdot)$ as a new metric in the fermionic configuration space, we obtain vital bounds on small denominators compatible with the Craig–Pöschel approach.

To keep the size of the present paper within the limits it deserves, we do not include any reasonably complete survey of the prior publications on the applications of the KAM techniques to direct and inverse spectral problems for disordered operators, nor on the spectral analysis of interacting disordered quantum particle systems. Interested readers can find more information on these subjects in [10].

2. Preliminary considerations

2.1. Fermionic spaces

The spectral problems for $\mathfrak{n} > 1$ fermions in a lattice $\mathbf{Z}^k$ can be formulated in several ways. For example, once $\mathfrak{n}$ is fixed, one can consider the Cartesian power $(\mathbf{Z}^k)^{\mathfrak{n}} = \{\mathbf{x} = (x_1, \dots, x_{\mathfrak{n}})\}$ and restrict a Hamiltonian $\mathbf{H}$ to the subspace ("fermionic sector") of anti-symmetric functions of the variables x_j. Naturally, for a consistent modelling of Fermi systems, the Hamiltonian $\mathbf{H}$ has to be permutation-symmetric.

However, such a traditional representation brings up some considerable technical problems in the proofs of Anderson localization, for it gives rise to strong "phantom resonances" between the eigenstates with identical (due to the symmetry of $\mathbf{H}$) eigenvalues. In terms of the inverse spectral problem that we address here, one has to face the problem of exactly vanishing spectral spacings ("small denominators"). Clearly, this purely formal problem has nothing to do with the modelled physical system, and yet, it cannot be simply ignored. This is why the first papers [1, 12] used a specific pseudo-metric (Hausdorff metric) on the product configuration space $(\mathbf{Z}^k)^{\mathfrak{n}}$, which did not solve, alas, all the technical problems encountered on that way.

We circumvent the aforementioned formal issue in a formal way by using a different, albeit equivalent, representation of fermionic configuration spaces. Specifically, a convenient mathematical tool for representing finite systems of indistinguishable lattice particles is the so-called "symmetric power" of graphs.

Below we often denote by $\langle a, b\rangle$ a pair of distinct elements a, b of whatever nature. For the indicator functions of various sets A, we use notations like $\mathbf{1}_A$.

Let $(\mathcal{G}, \mathcal{E})$ be an at-most-countable graph without loops, with vertex set $\mathcal{G}$ and edge set $\mathcal{E}$, endowed with the canonical graph-distance $(x, y) \mapsto \mathrm{d}_{\mathcal{G}}(x, y)$ = minimal length of the paths over the edges connecting x to y. We denote by $\mathcal{B}_R(x)$ the ball $\{y \in \mathcal{G} : \mathrm{d}_{\mathcal{G}}(y, x) \le R\}$.

Fix an integer $\mathfrak{n} \geq 2$. The $\mathfrak{n}$-th symmetric power of $\mathcal{G}$ is the graph with

- the vertex set $\mathcal{G}^{(\mathfrak{n})}$ formed by all subsets $\mathbf{x} = \{x_1, \dots, x_{\mathfrak{n}}\}$ of $\mathcal{G}$ of cardinality exactly $\mathfrak{n}$, i.e., with no duplicates;
- the vertex set $\mathcal{E}^{(\mathfrak{n})}$ formed by all pairs $(\mathbf{x}, \mathbf{y})$ such that the symmetric difference (recall: $\mathbf{x}$ and $\mathbf{y}$ are *sets*!)

$$\mathbf{x} \ominus \mathbf{y} := (\mathbf{x} \setminus \mathbf{y}) \cup (\mathbf{y} \setminus \mathbf{x}) \subset \mathcal{G}$$

has cardinality 2, i.e., $\mathbf{x} \ominus \mathbf{y} = \{u, v\}$ with $u \neq v$, where $\{u, v\}$ is an edge of $\mathcal{G}$:

$$\mathbf{x} \ominus \mathbf{y} = \{u, v\} \in \mathcal{E}.$$

(We use an ad hoc notation "$\ominus$" for the symmetric difference, because a more traditional one, $\triangle$, can be easily confused with the Laplacian Δ.)

Therefore, $\mathrm{d}(u, v) = 1$ (hence $(\mathbf{x}, \mathbf{y})$ is an edge) if and only if $\mathbf{y}$ is obtained from $\mathbf{x}$ by removing exactly one point $u \in \mathbf{x}$ and replacing it with a point $v \neq u$ adjacent to, but not included in, $\mathbf{x}$:

$$\mathbf{y} = (\mathbf{x} \setminus \{u\}) \cup \{v\}, \quad \mathrm{d}(v, \mathbf{x}) = 1, \quad v \neq u.$$

An equivalent representation of fermionic configurations, helpful in the proof of an important Lemma 2.1, is given by the so-called "occupation numbers" functions

$$\mathbf{n}_{\mathbf{x}} \colon y \mapsto \sum_{x \in \mathbf{x}} \mathbf{1}_x(y), \quad y \in \mathcal{G}.$$

Clearly, the mapping $\boldsymbol{\mathcal{O}} \colon \mathbf{x} \mapsto \mathbf{n}_{\mathbf{x}}$ is an injection of $\mathcal{G}^{(n)}$ into the linear space of functions on $\mathcal{G}$, and its range $\operatorname{Ran} \boldsymbol{\mathcal{O}}$ is characterised by easily verified conditions

$$\operatorname{Ran} \mathbf{n}_{\mathbf{x}} = \{0, 1\}, \quad \operatorname{card} \operatorname{supp} \mathbf{n}_{\mathbf{x}} = \sum_{x \in \mathbf{x}} 1 = \mathfrak{n}. \tag{2.1}$$

To inverse $\boldsymbol{\mathcal{O}}$ on its range, it suffices, for any $\chi \in \operatorname{Ran} \boldsymbol{\mathcal{O}}$, to set

$$\mathbf{x} = \boldsymbol{\mathcal{O}}^{-1}(\chi) := \operatorname{supp} \chi \subset \mathcal{G}$$

noting that $\operatorname{Ran} \chi = \{0, 1\}$ and $\operatorname{card} \operatorname{supp} \chi = \mathfrak{n}$ by (2.1).

Obviously, an adaptation to bosonic particle systems requires a slightly different construction of the underlying graph. Specifically, instead of the formal linear combinations of vertices $x \in \mathcal{G}$ with coefficients $c_x \in \{0, 1\}$ obeying $\sum_{x \in \mathbf{x}} c_x = \mathfrak{n}$, one has to work with coefficients $c_x \in [\![0, \mathfrak{n}]\!] := [0, \mathfrak{n}] \cap \mathbf{Z}$ subject to the same constraint $\sum_{x \in \mathbf{x}} c_x = \mathfrak{n}$.

In the particular case where $k = 1$, there is a simpler, and well-known, realization of $\mathbf{Z}^{(\mathfrak{n})}$ as a subgraph of $\mathbf{Z}^{\mathfrak{n}}$:

$$\mathbf{Z}^{(\mathfrak{n})} \cong \{(x_1, \dots, x_{\mathfrak{n}}) \in \mathbf{Z}^{\mathfrak{n}} : x_1 < \cdots < x_{\mathfrak{n}}\}.$$

In the general case $k \geq 1$, we denote $(\mathbf{Z}^k)^{(\mathfrak{n})}$ by $\boldsymbol{\mathcal{Z}}^2$.

We focus on the case $\mathfrak{n} = 2$ revealing in fact the main techniques extendable to any $\mathfrak{n} \geq 2$, and comment where necessary on some adaptations to be made in the general case. In particular, in Section 5 we prove for $\mathfrak{n} = 2$ a simpler version of the following technical result valid for any $\mathfrak{n} > 1$. In its intended application, it leads to a conclusion that, to some extent, multi-particle configurations labelling the uniformly localised, unimodal eigenstates, can be treated as the 1-particle ones: if $\mathbf{x} \neq \mathbf{y}$, then there are some effective degrees of freedom in the "disorder" affecting much stronger the respective eigenvalue $\boldsymbol{\lambda}_{\mathbf{x}}$ than all the other eigenvalues $\boldsymbol{\lambda}_{\mathbf{y}}$. This is a weaker yet efficient analog of a trivial fact widely used in the localization analysis of 1-particle lattice Anderson models with IID external potential. However, finding the required degrees of freedom is a more complex task for the almost-periodic potentials featuring a very weak stochasticity.

Lemma 2.1. *Let be given a countable graph $\mathcal{G}$ and $\mathfrak{n} > 1$. For any $\mathbf{x} \in \mathcal{G}^{(\mathfrak{n})}$ and $\mathbf{y} \in \mathcal{G}^{(\mathfrak{n})} \setminus \{\mathbf{x}\}$, there exist $\hat{x} \in \mathbf{x} \setminus \mathbf{y}$, $\hat{y} \in \mathbf{y} \setminus \mathbf{x}$, and some subsets $\mathcal{X}, \mathcal{Y} \subset \mathcal{G}$ with* $\operatorname{card} \mathcal{X} = \operatorname{card} \mathcal{Y} = \mathfrak{n} - 1$ *such that*

$$\mathbf{x} = \{\hat{x}\} \cup \mathcal{X}, \quad \mathbf{y} = \{\hat{y}\} \cup \mathcal{Y}, \tag{2.2}$$

$$\hat{x} \notin \mathcal{X} \cup \{\hat{y}\}, \quad \hat{y} \notin \mathcal{Y} \cup \{\hat{x}\}. \tag{2.3}$$

Proof. Fix any $\mathbf{x} \neq \mathbf{y}$. The mapping $f : \mathcal{G} \ni z \mapsto \mathbf{n}_{\mathbf{x}}(z) - \mathbf{n}_{\mathbf{y}}(z) \in \{-1, 0, 1\}$ is not identically zero, for $\mathbf{n}_{\mathbf{x}} \equiv \mathbf{n}_{\mathbf{y}}$ would imply $\mathbf{x} = \mathbf{y}$. Also, $\sum_z f(z) = 0$, since $\sum_z \mathbf{n}_{\mathbf{v}}(z) = \mathfrak{n}$ for any $\mathbf{v} \in \mathcal{G}^{(\mathfrak{n})}$. Hence, there exist some lattice points $\hat{x} \neq \hat{y}$ with $f(\hat{x}) = 1$ and $f(\hat{y}) = -1$: otherwise, f would not change sign on $\mathcal{G}$ without being identically zero, in contradiction with $\sum_z f(z) = 0$. Further, $\operatorname{card} \mathbf{x} = \operatorname{card} \mathbf{y} = \mathfrak{n}$ implies that $\mathcal{X} := \mathbf{x} \setminus \{\hat{x}\}$ and $\mathcal{Y} := \mathbf{y} \setminus \{\hat{x}\}$ have cardinality $\mathfrak{n} - 1$, and $\hat{x} \notin \mathcal{X}$, $\hat{y} \notin \mathcal{Y}$. It is plain that

$$\hat{x} \in \mathbf{x} \setminus \mathbf{y}, \quad \hat{y} \in \mathbf{y} \setminus \mathbf{x}.$$

This proves (2.2). The assertion (2.3) follows from (2.2) combined with $\hat{x} \neq \hat{y}$, $\hat{x} \notin \mathcal{X}$, $\hat{y} \notin \mathcal{Y}$. ■

2.2. Fermionic Hamiltonians

In this subsection, we do not discuss dependence of Hamiltonians and their components upon any parameters. Ensembles of Hamiltonians relative to some probability spaces will be introduced later.

In the case, which we do not consider in the present paper, where the quantum particles are distinguishable, thus can be numerated in some way, the $\mathfrak{n}$-particle states form a tensor product Hilbert space $\boldsymbol{\mathcal{H}} = \bigotimes_{j=1}^{\mathfrak{n}} \mathcal{H}^j$, where $\mathcal{H}^j$ are replicas of the 1-particle quantum state space, e.g., $\mathcal{H} = \ell^2(\mathbf{Z}^k)$ for the lattice particles in dimension k. Given a 1-particle Hamiltonian H acting in $\mathcal{H}$, its counterpart for $\mathfrak{n}$ non-interacting (again, distinguishable) particles is given by

$$\mathbf{H}^{\mathrm{ni}} = \sum_{j=1}^{\mathfrak{n}} \overbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}^{j-1 \text{ factors}} \otimes H \otimes \overbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}^{\mathfrak{n}-j \text{ factors}},$$

but the case of indistinguishable particle, bosonic or fermionic, is a bit more delicate. As was explained in the previous subsection, we are compelled to avoid a standard approach with a restriction of the Hamiltonian to the subspace of symmetric/antisymmetric functions, and work in the space of square-summable functions on the $\mathfrak{n}$-th symmetric power $(\mathbf{Z}^k)^{(\mathfrak{n})}$. While the numeration of the particle positions $x_1, \dots, x_{\mathfrak{n}}$ may occasionally prove useful, it is preferable to define the Hamiltonians and their components in an invariant way, without any particle numeration.

The kinetic energy operator $\mathbf{H_0}$. It is usually assumed to be the sum of kinetic energies of all particles. In the lattice models, one often works with a finite-difference operator whose off-diagonal entries represent the "hopping" amplitudes, measuring the mobility of the particles. For example, $\mathbf{H}_0$ can be the (negative) canonical graph Laplacian $-\boldsymbol{\Delta}$ on $\boldsymbol{\mathcal{Z}}^{\mathfrak{n}}$, but it will become clear that a much richer family of "hopping" operators of finite and infinite range can be allowed without compromising the applicability of the techniques from [14,22]. Clearly, this definition operates with unordered subsets $\mathbf{x} \subset \mathbf{Z}^k$ with $\mathrm{card}(\mathbf{x}) = \mathfrak{n}$.

The media-particle interaction operator $\mathbf{V}$. Let be given a function $V\colon \mathbf{Z}^k \to \mathbf{R}$ representing the single-particle external energy potential, i.e., a media-particle interaction potential, common for all particles. Its $\mathfrak{n}$-particle counterpart $\mathbf{V}$ is the operator of multiplication by the function

$$\boldsymbol{\mathcal{Z}}^{\mathfrak{n}} \ni \mathbf{x} \mapsto \sum_{x \in \mathbf{Z}^k} \mathbf{n}_{\mathbf{x}}(x) V(x) = \sum_{x \in \mathbf{x}} V(x),$$

which, by a slight abuse of notations, we also denote by $\mathbf{V}$; a similar remark can be made about the function $\mathbf{U}$ in (2.4). Again, this construction makes no reference to any specific particle numerations, being permutation-invariant.

The inter-particle interaction operator $\mathbf{U}$. We focus on a particular but physically relevant case of a two-body interaction. It has some technical advantages making some graphical classifications and calculations easier, but this concerns mostly the direct spectral analysis. Let be given a function $\mathfrak{u}\colon \mathbf{N} \to \mathbf{R}$ called the "two-particle interaction potential". The total inter-particle interaction energy is the operator $\mathbf{U}$ of multiplication by the function

$$\mathbf{U}\colon \mathbf{x} \mapsto \sum_{\langle x,y\rangle \in (\mathbf{Z}^k)^2} \mathbf{n}_{\mathbf{x}}(x)\mathbf{n}_{\mathbf{x}}(y)\mathfrak{u}(|x-y|) = \sum_{\langle x,y\rangle \in \mathbf{x}} \mathfrak{u}(|x-y|). \tag{2.4}$$

It will not appear in the main body of the proofs of the principal results, but we shall need it in a useful example given below. To make the example simpler, let $\mathfrak{u}$ be a "fermion contact" potential vanishing for any pair of particle positions $\langle x, y\rangle$ with $|x-y| \neq 1$. A genuine on-site contact case $x = y$ is impossible for the fermions, so we allow interactions between nearest neighbors x and y in $\mathbf{Z}^k$. The reader will see below that an extension to a more general case where card supp $\mathfrak{u} < +\infty$ is not difficult. The case where $\mathfrak{u}$ decays exponentially at infinity is slightly more delicate, for the decay exponent of $\mathfrak{u}$ may interfere with other key parameters in the KAM-type induction.

Now, let us show that, if the single-particle Hamiltonian H features a uniform exponential localization of eigenfunctions (ULE, which "*. . . fails in many models*", as was observed long ago in [15, Appendix 3]), then the matrix of an interaction operator (of the above form) in the orthonormal basis of eigenstates of the Hamiltonian $\mathbf{H}^{\mathrm{ni}}$ features a strongly anisotropic decay of a specific kind. To this end, assuming that $\mathfrak{n} = 2$, $k = 1$, and the eigenbasis of $\mathbf{H}^{\mathrm{ni}}$ is formed by factorised functions $\boldsymbol{\psi}_{\mathbf{x}} = \psi_{x_1}\psi_{x_2}$, $\mathbf{x} = \{x_1, x_2\}$, where

$$|\psi_x(y)| \le \mathrm{e}^{-\mu|y-x|}, \quad \mu > 0,$$

assess the matrix elements of the operator $\mathbf{U}$ as follows:

$$\begin{aligned}
|\langle \boldsymbol{\psi}_{\mathbf{x}} \mid \mathbf{U} \mid \boldsymbol{\psi}_{\mathbf{y}}\rangle| &\le \sum_{\mathbf{Z}\ni z_1,z_2: z_1<z_2} |\psi_{x_1}(z_1)\psi_{x_2}(z_2)\mathfrak{u}(|z_2-z_1|)\psi_{y_1}(z_1)\psi_{y_2}(z_2)| \\
&\le \sum_{z\in\mathbf{Z}} |\psi_{x_1}(z)\psi_{x_2}(z+1)\psi_{y_1}(z)\psi_{y_2}(z+1)| \\
&\le \sum_{z\in\mathbf{Z}} \mathrm{e}^{-\mu f(\mathbf{x},\mathbf{y},z)}, \\
f(\mathbf{x},\mathbf{y},z) &:= |x_1 - z| + |x_2 - (z+1)| + |y_1 - z| + |y_2 - (z+1)|.
\end{aligned} \tag{2.5}$$

As shown in Appendix A, this implies the bound of the form

$$|\langle \boldsymbol{\psi}_{\mathbf{x}} \mid \mathbf{U} \mid \boldsymbol{\psi}_{\mathbf{y}}\rangle| \le C_\mu \mathrm{e}^{-\frac{\mu}{2}\mathfrak{d}(\mathbf{x},\mathbf{y})},$$

where the symmetric function

$$\mathcal{Z}^2 \times \mathcal{Z}^2 \ni (\mathbf{x},\mathbf{y}) \mapsto \mathfrak{d}(\mathbf{x},\mathbf{y}) := \|\mathbf{x}-\mathbf{y}\|_1 + \mathrm{diam}(\mathbf{x}) + \mathrm{diam}(\mathbf{y}) \in \mathbf{R}_+$$

verifies the triangle inequality, albeit it is not a metric, as $\mathrm{diam}(\mathbf{x}) + \mathrm{diam}(\mathbf{x}) > 0$ for all $\mathbf{x} = \{x_1, x_2\} \in \boldsymbol{\mathcal{Z}}^2$, hence with $x_1 \neq x_2$. Its strict positivity does not pose any problem. It merely provides stronger estimates of the diagonal matrix elements, reflecting the nature of the matrix $\mathbf{U}$ in the basis formed by the functions $\boldsymbol{\psi}_{\bullet}$.

The exponential decay of the matrix elements provided by $\mathrm{e}^{-\frac{\mu}{2}\|\mathbf{x}-\mathbf{y}\|_1}$ is to be expected, and it is isotropic in $\mathbf{R}^2 \supset \mathbf{Z}^2$ (with either $\mathbf{x}$ or $\mathbf{y}$ fixed), but $\mathrm{e}^{-\frac{\mu}{2}(\boldsymbol{\emptyset}(\mathbf{x})+\boldsymbol{\emptyset}(\mathbf{y}))}$ breaks the isotropy. For any fixed $\mathbf{y}$, the decay of $\mathbf{x} \mapsto |\mathbf{U}_{\mathbf{x}\mathbf{y}}|$ is stronger in the direction orthogonal to the diagonal $\{(x_1, x_2) : x_1 = x_2\}$ (which is actually excluded from the fermionic lattice). Of course, the roles of $\mathbf{x}$ and $\mathbf{y}$ can be reversed.

This strong anisotropy is quite fortunate for the localization analysis of our model, for our lower bounds on the spectral spacings $|\boldsymbol{\lambda}_{\mathbf{x}} - \boldsymbol{\lambda}_{\mathbf{x}}|$ deteriorate as $\mathrm{diam}(\Pi) + \mathrm{diam}(\Pi\mathbf{y}) \to +\infty$.

Remark 2.1. Observe that the rate of decay of the matrix elements of $\mathbf{U}$ is determined not by that of the two-body potential $r \mapsto \mathfrak{u}(r)$ (which is compactly supported!) but by the decay of the 1-particle eigenfunctions, provided that $\mathfrak{u}$ decays faster than exponentially. As a result, the matrix norms used in the Craig–Pöschel scheme impose the off-diagonal decay of the principal matrices which can be made as fast as one pleases, provided the 1-particle localization is strong enough. As is well known, one sufficient condition for such a scenario is a strong disorder or, equivalently, low mobility of the particles $\equiv$ small amplitude h of the 1-particle kinetic energy operator $(-h\Delta)$.

We are working with the case $\mathfrak{n} = 2$, but, speaking more generally, the configurations $\mathbf{x}$ of large diameter are characterised by a long-distance clustering of sub-systems (singletons), for $\mathfrak{n} = 2$. This is exactly the area where the lower bounds on "small denominators" $|\boldsymbol{\lambda}_{\mathbf{x}} - \boldsymbol{\lambda}_{\mathbf{y}}|$ may be much smaller, but it is also the area where localization of the sub-systems has to be treated in a more efficient way, as was the case in the works [1, 12] and subsequent papers.

A conclusion one can draw from this discussion is that, treating the matrix of $\mathbf{H}^{\mathrm{ni}}$ relative to its eigenbasis as an analog of the diagonal matrix D in Section 6, and considering the matrix of $\mathbf{U}$ in the eigenbasis for the non-interacting system, one should privilege the off-diagonal matrices (analogous to P in Section 6) featuring an anisotropic decay of the form discussed above.

It is to be stressed that we do not prove here a uniform exponential decay of the eigenfunctions of the 1-particle Hamiltonian with a given limit-periodic potential, nor do we have to, for this is the subject of the *direct* spectral problem. However, this was proved in a recent paper [13].

2.3. Matrix spaces and algebras

First, consider the vector space of all matrices labelled by the $\mathfrak{n}$-th symmetric power $\boldsymbol{\mathcal{Z}}^{\mathfrak{n}} := (\mathbf{Z}^k)^{(\mathfrak{n})}$,

$$\mathrm{Mat}(\boldsymbol{\mathcal{Z}}^{\mathfrak{n}}) = \{(\mathbf{A}_{\mathbf{x},\mathbf{y}})_{\mathbf{x},\mathbf{y}\in\boldsymbol{\mathcal{Z}}^{\mathfrak{n}}}\},$$

which can be chosen to be real or complex.

Let be given a function $\mathfrak{d}\colon \boldsymbol{\mathcal{Z}}^{\mathfrak{n}} \times \boldsymbol{\mathcal{Z}}^{\mathfrak{n}} \to \mathbf{R}_+$ satisfying the triangle inequality

$$\mathfrak{d}(x,z) \le \mathfrak{d}(x,y) + \mathfrak{d}(y,z) \quad \text{for all } x,y,z \in \boldsymbol{\mathcal{Z}}^{\mathfrak{n}}, \tag{2.6}$$

albeit not necessarily a metric. Anticipating Definition 2.2, we assume $\mathfrak{d}$ to be symmetric, but in the strict context of the present subsection, symmetry, while being welcome in general, is not necessary.

For the lack of a better word, and for the sake of terminological brevity, we call such functions $\mathfrak{d}(\cdot,\cdot)$ *orthometrics*. There is indeed some vague affinity of our main example of $\mathfrak{d}(\cdot,\cdot)$ with the so-called "orthometric height" used in cartography. Its structure can be illustrated as follows. In $\mathbf{R}^2$, one can decide to measure the length of the shortest path between $\mathbf{x} = (x_1,, x_2)$ and $\mathbf{y} = (y_1,, y_2)$ involving the segments issued from $\mathbf{x}$ and from $\mathbf{y}$ orthogonal to the line $\mathcal{L} = \{(u_1,u_2) : u_2 = 0\}$, or the length of the shortest path from $\mathbf{x}$ to $\mathbf{y}$ merely incidental to $\mathcal{L}$. In both cases, the "heights" of $\mathbf{x}$ and $\mathbf{y}$ over the baseline $\mathcal{L}$ make a considerable contribution to the Euclidean distance $\|\mathbf{x}-\mathbf{y}\|$ for $\mathbf{x}$ and/or $\mathbf{y}$ far away from $\mathcal{L}$. In the present paper, such a contribution is vital for circumventing some serious problems caused by the small denominators.

With a fixed orthometric $\mathfrak{d}(\cdot,\cdot)$ and any $t > 0$, introduce in $\mathrm{Mat}(\boldsymbol{\mathcal{Z}}^{\mathfrak{n}})$ the norm

$$\mathbf{A} \mapsto |||\mathbf{A}|||_t := \sup_{\mathbf{x}\in\boldsymbol{\mathcal{Z}}^{\mathfrak{n}}} \sum_{\mathbf{y}\in\boldsymbol{\mathcal{Z}}^{\mathfrak{n}}} \mathrm{e}^{t\,\mathfrak{d}(\mathbf{x},\mathbf{y})} |\mathbf{A}_{\mathbf{x},\mathbf{y}}|$$

and consider the vector space $\mathcal{M}_t$ of all $\mathbf{A} \in \mathrm{Mat}(\boldsymbol{\mathcal{Z}}^{\mathfrak{n}})$ with $|||\mathbf{A}|||_t < +\infty$. A simple calculation (cf. (A.4), Appendix A) shows that $\mathcal{M}_t$ with the matrix multiplication is a Banach algebra. This fact was used in prior works [3, 14, 22], where the underlying vertex set was a periodic lattice, and $\mathfrak{d}(\cdot,\cdot)$ was a bona fide metric.

Further, introduce the vector spaces of diagonal and of off-diagonal matrices:

$$\begin{aligned} \mathcal{D}\mathrm{iag} &:= \{\mathbf{A} \in \mathrm{Mat}(\boldsymbol{\mathcal{Z}}^{\mathfrak{n}}) : \text{for all } \mathbf{x} \ne \mathbf{y}\, \mathbf{A}_{\mathbf{x},\mathbf{y}} = 0\}, \\ \text{Off-diag} &:= \{\mathbf{A} \in \mathrm{Mat}(\boldsymbol{\mathcal{Z}}^{\mathfrak{n}}) : \text{for all } \mathbf{x} \in \boldsymbol{\mathcal{Z}}^{\mathfrak{n}}\, \mathbf{A}_{\mathbf{x},\mathbf{x}} = 0\}. \end{aligned}$$

Endowing the linear space $\{\mathbf{A} \in \text{Off-diag} : |||\mathbf{A}|||_\mu < \infty\}$ with the norm $|||\cdot|||_\mu$, we obtain a Banach space (but not an algebra) which we shall denote Off-diag_μ.

$\mathcal{D}\mathrm{iag}$ is a commutative Banach algebra with the norm

$$|||\mathbf{A}|||_\infty := \sup_{\mathbf{x}\in\boldsymbol{\mathcal{Z}}^{\mathfrak{n}}} |\mathbf{A}_{\mathbf{x},\mathbf{x}}|.$$

2.4. Approximation functions and distal functions

Following [22], denote by S_σ the set of all non-negative decreasing real sequences $(\sigma_l)_{\geq 1}$ with $\sum_l \sigma_l \leq \sigma$.

Definition 2.1 (cf. [14, 22, 23]). A continuous function $\mathfrak{a}\colon [0, +\infty) \to [1, +\infty)$ is called *approximation function* if, for some $\gamma > 0$, the two functions

$$0 < \sigma \mapsto \Phi(\sigma) := \sigma^{-4k} \sup_{r \geq 0} \mathfrak{a}(r) \mathrm{e}^{-\sigma r} \tag{2.7}$$

$$0 \leq \sigma \mapsto \Psi(\sigma) := \inf_{S_\sigma} \prod_{l=0}^{\infty} (\Phi(\sigma_l))^{2^{-l-1}} \tag{2.8}$$

take finite values for any $\sigma > 0$.

The factor σ^{-4k} is adapted, as in [22], to the case of the lattice $\mathbf{Z}^k$. Some minor re-calculations are required in a more general case of graphs with a polynomial rate of growth of balls of radius r as $r \to +\infty$.

Two most popular classes of approximation functions were mentioned in [14, 22]: $\mathfrak{a}(r) = r^b$ with $b > 0$, and $\mathfrak{a}(r) = \mathrm{e}^{-r^c}$, $c \in (0, 1)$. In the present paper, we need the latter form of $\mathfrak{a}$, since we are compelled to allow for the decay rate of small denominators faster than inverse polynomial.

Definition 2.2. Let be given a countable graph $(\mathcal{G}, \mathcal{E})$ endowed with some symmetric orthometric $\mathfrak{d}(\cdot, \cdot)$. A function $f\colon \mathcal{G} \to \mathbf{C}$, is called *distal* (or $\mathfrak{d}$*-distal*) if, for some approximation function $\mathfrak{a}(\cdot)$, the cocycle generated by f, of the form

$$\mathfrak{s}_f\colon (x, y) \mapsto f_x - f_y, \quad x, y \in \mathcal{G},$$

admits the bound

$$|\mathfrak{s}_f(x, y)| \geq \mathfrak{a}(\mathfrak{d}(x, y)) \quad \text{for all } \langle x, y \rangle \in \mathcal{G}^2.$$

The reader can see that omitting in the definition of orthometric the usual condition $\mathfrak{d}(x, x) = 0$ is fully warranted in the intended application to measuring the decay of spectral spacings, where $\lambda_x - \lambda_x$ is of no importance, while the symmetry of $|\mathfrak{s}_f(\cdot, \cdot)|$ makes natural the symmetry condition for $\mathfrak{d}(\cdot, \cdot)$.

For the above definition to be adapted to the Craig–Pöschel analysis, one has to assume the graph to admit a polynomial upper bound on the growth of balls, as in the periodic lattices $\mathbf{Z}^k$; this is the case with our model of fermionic space.

2.5. Ergodic ensembles of Hamiltonians

In the framework of a direct spectral problem on a periodic lattice, one would have to describe the dependence of some class of operators upon a parameter $\omega \in \Omega$, where

$(\Omega, \mathfrak{B}, \mathbf{P})$ is a probability space on which the lattice, viewed as an abelian group, acts by isomorphisms $\{T^x : \Omega \to \Omega\}$, and the dynamical system is ergodic. In the case of the so-called "diagonal disorder", the potential energy is a random field on $\mathbf{Z}^k$ relative to Ω. The off-diagonal disorder corresponds to randomness incorporated in the kinetic operator, and both kinds of disorder may be present.

We are concerned with the inverse spectral problem, so "disorder", or "randomness", is encoded in the ω-dependence of a diagonal matrix $\Lambda = \Lambda(\omega)$ which one intends to transform by a conjugation into another matrix $\mathrm{K} + \mathcal{V}(\omega)$, where the form of K is fixed and $\mathcal{V}(\omega)$ is diagonal. Therefore, it is the ω-dependence of $\Lambda(\cdot)$ that has to be defined constructively.

We consider in detail ergodic ensembles of diagonal matrices $\boldsymbol{\Lambda}(\cdot)$ related to limit-periodic functions used in [13] to generate limit-periodic external potentials. Another class of diagonal matrices, generated in a similar way by some quasi-periodic ensembles of functions in [9], can also be used. The key estimates on the small denominators, established in Section 5, were obtained in the quasi-periodic case in [9] and applied to the direct spectral analysis of fermionic operators in [10].

A diagonal matrix Λ labelled by the points of $\mathbf{Z}^k$ is uniquely defined by the sequence of its matrix elements $\lambda : \mathbf{Z}^k \ni x \mapsto \lambda_x = \Lambda_{xx}$. $\lambda_\bullet$ will be called 1*-particle eigenvalue function.* In the case of a particle system, one also needs its $\mathfrak{n}$-particle counterpart $\boldsymbol{\lambda} : (\mathbf{x}, \omega) \mapsto \boldsymbol{\lambda}_{\mathbf{x}}(\omega)$, $\mathbf{x} \in \boldsymbol{\mathcal{Z}}^{\mathfrak{n}}$, of a summatorial form,

$$\boldsymbol{\lambda}_{\mathbf{x}}(\omega) = \sum_{x \in \mathbf{x}} \lambda_x(\omega), \tag{2.9}$$

so it suffices to define a function $\lambda_\bullet(\cdot)$. The graph $\boldsymbol{\mathcal{Z}}^{\mathfrak{n}}$ is in general neither a periodic lattice nor a subgraph thereof, but $\mathbf{Z}^k$ still acts on it by the "diagonal" shifts: the mappings

$$\mathrm{S}^a : \mathbf{x} \equiv \{x_1, \ldots, x_{\mathfrak{n}}\} \mapsto \{x_1 + a, \ldots, x_{\mathfrak{n}} + a\}, \quad a \in \mathbf{Z}^k,$$

form a commutative group action.

In presence of a non-trivial inter-particle interaction $\mathbf{U}$, the diagonal elements of $\mathbf{H}(\cdot)$ incorporate the terms $\mathbf{U}(\mathbf{x})$, and the relations between the diagonal matrices $\mathrm{diag}(\mathbf{V}(\mathbf{x}, \cdot) + \mathbf{U}(\mathbf{x}))_{\mathbf{x} \in \boldsymbol{\mathcal{Z}}^{\mathfrak{n}}}$ and $\mathrm{diag}(\boldsymbol{\lambda}_{\mathbf{x}}(\cdot))_{\mathbf{x} \in \boldsymbol{\mathcal{Z}}^{\mathfrak{n}}}$ are too complex to specify the impact of $\mathbf{U}$ on the eigenvalues $\boldsymbol{\lambda}_{\mathbf{x}}(\omega)$. Conversely, it seems difficult, if at all possible, to propose a canonical form of the $\mathfrak{n}$-particle eigenvalue function $\boldsymbol{\lambda}_\bullet(\cdot)$ corresponding to Hamiltonians with a given interaction $\mathbf{U}$. The situation is somewhat similar to that encountered in the inverse spectral problem for 1-particle Hamiltonians $H = \mathrm{K} + V$ studied in [14, 22]: for an explicitly described diagonal matrix Λ, one can guarantee that it represents indeed a diagonalised form of a Hamiltonian with a prescribed kinetic (= off-diagonal) component K (subject to some conditions), but the resulting potential (= diagonal) component does not have a sufficiently explicit description. Pictorially, "it is what it is", or what one gets as an outcome of an infinite inductive construction. This is apparently an unavoidable concession to make while "dressing" a diagonal matrix Λ with a desired off-diagonal matrix K.

With these observations in mind, we propose a variant of the inverse problem where the starting point is a diagonal matrix $\mathbf{\Lambda}(\omega)$ with $\mathbf{\Lambda}_{\mathbf{x},\mathbf{x}} = \boldsymbol{\lambda}_{\mathbf{x}}(\omega)$ (cf. (2.9)), and aim to prove, for all off-diagonal matrices $\mathbf{K}$ from some class, the existence of another diagonal matrix $\mathbf{W}(\omega)$ such that $\mathbf{K} + \mathbf{W}(\omega)$ is unitarily equivalent to $\mathbf{\Lambda}(\omega)$:

$$\mathbf{\Lambda}(\omega)\boldsymbol{\mathcal{U}}(\omega) = \boldsymbol{\mathcal{U}}(\omega)(\mathbf{K} + \mathbf{W}(\omega)), \quad \boldsymbol{\mathcal{U}}^{\mathsf{T}}(\omega)\boldsymbol{\mathcal{U}}(\omega) = \mathbf{1}, \tag{2.10}$$

where $(\boldsymbol{\mathcal{U}}^{\mathsf{T}})_{\mathbf{xy}} = \boldsymbol{\mathcal{U}}_{\mathbf{yx}}$. The column-vectors of $\boldsymbol{\mathcal{U}}(\omega)$ satisfying (2.10),

$$\boldsymbol{\psi}_{\mathbf{x}}(\omega) = (\boldsymbol{\mathcal{U}}_{\mathbf{y},\mathbf{x}})_{\mathbf{y}\in\boldsymbol{\mathcal{Z}}^{\mathfrak{n}}}, \quad \mathbf{x} \in \boldsymbol{\mathcal{Z}}^{\mathfrak{n}},$$

are normalised square-summable eigenfunctions of $\mathbf{H}(\omega) = \mathbf{K} + \mathbf{W}(\omega)$, and we prove that they are uniformly exponentially decaying.

Now that the construction of suitable $\mathfrak{n}$-particle eigenvalue functions is reduced to that of parametric functions $\lambda_{\bullet}(\cdot)$ on $\mathbf{Z}^k$, it makes sense to speak of usual spaces of almost-periodic functions on $\mathbf{Z}^k$.

The dynamical systems T generating almost-periodic functions ($\lambda_{\bullet}(\cdot)$, in our case) have very weak stochastic properties, and this constitutes a significant problem in the analysis of "small denominators". Apart from a few very special, and remarkable, models which we do not discuss nor review here, the first important and quite general breakthrough was achieved in the pioneering papers [4–6] and subsequent publications where one considered analytic functions on the torus $\Omega = \mathbf{T}^{\nu}$ generating the (1-particle) potential $x \mapsto V(x,\omega) = v(T^x\omega)$. It was realised that relaxing analyticity to a finite smoothness was (and, to a large extent, still is) a formidable challenge.

For this reason, the approach proposed earlier in [7,8] operates with a richer parametric space $\Omega \times \Theta$, where $(\Theta, \mathfrak{B}^{\Theta}, \mathbf{P}^{\Theta})$ is an additional probability space unaffected by the dynamical system $T\colon \mathbf{Z}^k \times \Omega \to \Omega$. Building on some techniques from [7–9], we thus work with parametric families of Hamiltonians $\mathbf{H}(\omega,\theta)$, where $\theta \in \Theta$ is an invariant of the dynamics T. In other words, we consider parametric *families* of ergodic ensembles of Hamiltonians. Solving the inverse spectral problem for individual ergodic ensembles is very tempting, but much more difficult, while it is easier to solve it for all $(\omega,\theta) \in \Omega \times \Theta_*(\varepsilon)$, where $\mathbf{P}^{\Theta}\{\Theta_*(\varepsilon)\} \uparrow 1$ as $\varepsilon \downarrow 0$. In other words, the solvability is guaranteed for entire ergodic ensembles relative to Ω but only for typical values of the parameter θ: the smaller $\varepsilon > 0$, the smaller the $\mathbf{P}^{\Theta}$-measure of the exceptional set $\Theta \setminus \Theta_*(\varepsilon)$.

Equivalently, one can say that the spectral problem can be solved for all (not just almost all) $\omega \in \Omega$ and $\mathbf{P}^{\Theta}$-almost all $\theta \in \Theta$, but the threshold $\varepsilon_*(\theta)$ is not uniform in θ: exceptional subsets of Θ, of small $\mathbf{P}^{\Theta}$-measure, require exceptionally small values $\varepsilon(\cdot)$.

The structure of measure space on Θ is essential to our approach, while the dynamics T on Ω no longer plays a constructive role. Speaking informally, it simply must not be destructive, and to that end, we consider dynamical systems which are "sufficiently aperiodic". For example, in the quasi-periodic case, one has to avoid Liouvillian frequencies of toral rotations.

Moreover, in the case of limit-periodic eigenvalue functions $\lambda_{\bullet}(\cdot)$, the relevant phase space Ω is a Cantor group (cf., e.g., [20]), and while it is possible to refer to the parametric couples (ω, θ) and introduce the eigenvalue functions $\lambda_x(\omega, \theta)$, the reference to ω would be in fact just a fairly useless formality. For all intents and purposes, the structure of Ω is encoded here in the spatial variable x, and so we write below $\lambda_x(\theta)$ instead of $\lambda_x(\omega, \theta)$.

2.6. Jacobi rotations and small denominators

This subsection is addressed mainly to the readers not familiar with the general problematic of the so-called "small denominators" in the KAM variant of the Newton method. We also explain here why we are compelled to allow in the formulation of the main theorem some values of the spectral spacings ("small denominators") vanishing (at some controllable rate) in the limit where the key smallness parameter $\varepsilon > 0$ tends to 0.

Consider the simplest, two-dimensional example of a real symmetric matrix

$$H = \begin{pmatrix} \lambda & \varepsilon \\ \varepsilon & -\lambda \end{pmatrix}$$

whose diagonalization is achieved in a closed form, but the small denominators problem can already be illustrated. Introduce the matrices

$$\mathrm{M} := \begin{pmatrix} 0 & \phi \\ -\phi & 0 \end{pmatrix}, \quad \mathrm{U} := \mathrm{e}^{\mathrm{M}} = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix},$$

let $\lambda_1 = \lambda$, $\lambda_2 = -\lambda$, and denote for brevity $c = \cos(\phi)$, $s = \sin(\phi)$, then a straightforward, well-known calculation shows that

$$\mathrm{Ad}_{\mathrm{U}}[H] = \begin{pmatrix} *** & (c^2 - s^2)\varepsilon + sc(\lambda_1 - \lambda_2) \\ (c^2 - s^2)\varepsilon + sc(\lambda_1 - \lambda_2) & *** \end{pmatrix}$$

Therefore, diagonalization is achieved when

$$\frac{\sin(\phi)\cos(\phi)}{\cos^2(\phi) - \sin^2(\phi)} = \frac{\varepsilon}{\lambda_1 - \lambda_2}. \tag{2.11}$$

In numerical methods, a vanishing denominator $\lambda_1 - \lambda_2$ does not constitute a problem *per se*. Only the representation (2.11) is no longer valid, but a solution does exist: $\phi = \pi/4$. However, in the general context of the KAM (Kolmogorov–Arnold–Moser) approach, where a specific form of Newton's method is used, one usually seeks a transformation (here, a linear transformation) close to identity in a suitable norm, and *this* turns out to be impossible whenever ε is fixed and the denominator $\lambda_1 - \lambda_2$ is too small compared to ε.

This elementary example illustrates the reason why the differences of diagonal entries of a matrix (to be diagonalised), when they are relatively small compared to the respective off-diagonal entries, are called "small denominators".

Remark 2.2. In Section 5, the smallness parameter ε figuring in the Hamiltonians appears also in the threshold (lower bound) for the admissible spectral spacings. This is quite natural, as shows already the above two-dimensional example. Indeed, set now $\lambda = \lambda(\theta) = \theta \in [0,1]$, $\widetilde{\Lambda}(\theta) = \operatorname{diag}(\theta, -\theta)$, then a unitary matrix transforming Λ by conjugation into

$$\mathrm{K} + \widetilde{\Lambda}(\theta), \quad \mathrm{K} = \begin{pmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{pmatrix},$$

is close to $\mathbf{1}$ if $= |v(\theta) - (-v(\theta))| = |\theta - (-\theta)|$ is noticeably larger than ε. For instance, take $\theta \in [0,1] \setminus [0, \varepsilon^{1/4}]$ with $\varepsilon \ll 1$, and endow Θ with the natural structure of probability space with the Lebesgue measure. It is plain that solving this elementary instance of the inverse spectral problem for the pair of matrices $(\Lambda(\theta), \mathrm{K})$ is equivalent to solving it for a scaled pair $(\varepsilon^{-1/4}\Lambda(\theta), \varepsilon^{-1/4}\mathrm{K})$, since

$$\mathrm{U}\Lambda(\theta)\mathrm{U}^{-1} = \mathrm{K} + \widetilde{\Lambda}(\theta) \iff \mathrm{U}\varepsilon^{-1/4}\Lambda(\theta)\mathrm{U}^{-1} = \varepsilon^{-1/4}\mathrm{K} + \varepsilon^{-1/4}\widetilde{\Lambda}(\theta),$$

but, with $\varepsilon^{1/4} < \theta \le 1$, the scaled spectral matrix has the form

$$\Lambda(\theta) = \begin{pmatrix} \theta_* & 0 \\ 0 & -\theta_* \end{pmatrix}, \quad \theta_* := \varepsilon^{-1/4}\theta \ge 1,$$

so the "small" denominator $\theta_* - (-\theta_*) = 2\theta_* \ge 2$, actually, contains no small parameter. The scaled off-diagonal matrix $\varepsilon^{-1/4}\mathrm{K}$, on the contrary, still has a small amplitude $\tilde{\varepsilon} := \varepsilon^{3/4} \to 0$ as $\varepsilon \downarrow 0$. This is exactly the kind of situations where the results of [14, 22] on solvability of the inverse problem apply directly.

On the other hand, in order to make the measure (viz. probability) of the excluded parameter set small, even vanishing as $\varepsilon \downarrow 0$, one is *compelled* to allow some values of θ which are *small*, yet considerably larger than ε. Without such a compromise, our technique would merely result in an existence theorem having little to do with the ensemble point of view on the inverse spectral problem.

3. Covariant ensembles of limit-periodic functions

For simplicity, we consider limit-periodic functions $f : \mathbf{Z}^k \to \mathbf{R}$ only with diadic periods. Specifically, for each $n \geq 1$, consider the lattice cube of side length 2^n,

$$\mathcal{Q}_n := [-2^{n-1}, 2^{n-1})^k \subset \mathbf{Z}^k,$$

and introduce the product probability space $(\Theta, \mathfrak{B}^\Theta, \mathbf{P}^\Theta)$, where

$$\Theta = \bigtimes_{n=2}^{\infty} [0,1]^{\mathcal{Q}_n},$$

each factor $[0, 1]$ is endowed with the Lebesgue sigma-algebra and measure inherited from $\mathbf{R}$. The elements of Th has the form $\theta = (\theta_{n,x})_{n\geq 1, x\in\mathcal{Q}_n}$. The coordinate projections $\theta_{\bullet,\bullet}$ form a family of IID (independent and identically distributed) random variables on Θ with common uniform distribution Unif([0, 1]).

Next, for each $n \geq 1$, introduce $K_n = 2^{nk}$ lattice functions which are 2^n-periodic in all coordinate directions on $\mathbf{Z}^k$, of the form

$$\chi_{n,y}(z) := \sum_{\boldsymbol{\ell}\in(T_n\mathbf{Z})^k} \mathbf{1}_{y+\boldsymbol{\ell}}(z), \quad y \in \mathcal{Q}_n. \tag{3.1}$$

In other words, $\chi_{n,y}$ is the 2^n-periodic continuation in all coordinate directions of the indicator function $\mathbf{1}_y$, and its support is a coset (shifted sublattice) $y + (2^n\mathbf{Z})^k$.

A general case with k non-identical sequences of periods $(T_{i,n})_{n\geq 1}$, $1 \leq i \leq k$, but with uniformly bounded ratios $T_{i,n+1}/T_{i,n} \in \mathbf{N} \setminus \{0, 1\}$ would not require any major modification of the proofs. However, in absence of any restriction on the ratios, the main results would probably become false. Pictorially, an abnormal rate of growth of the periods, even in one dimension, is as harmless for the results on a specific nature of the spectral measure (pure point, absolutely continuous) as an abnormally fast rational approximation rate of the basic frequencies of quasi-periodic potentials. In such situations, the singular continuous spectral measures have an unpleasant habit to emerge.

The following simple observation will be useful in the proof of Lemma 5.1.

Remark 3.1. It follows directly from the definition (3.1) of the functions $\chi_{\bullet,\bullet}$ that the entire cube $\mathcal{Q}_n$ is covered without overlaps by the union of all sets $\mathcal{Q}_n \cap \operatorname{supp} \chi_{n,y}$ with $y \in \mathcal{Q}_n$, and the latter intersections have cardinality 1. As a result, every eigenvalue $\lambda_x(\theta)$ with $x \in \mathcal{Q}_n$ is affected by exactly one random variable among $\{a_n\theta_{n,y}, y \in \mathcal{Q}_n\}$. But this remains true for the points x of any shifted cube $z + \mathcal{Q}_n = \{y + z : y \in \mathcal{Q}_n\}$:

$$\{a_n\theta_{n,y+z}, y \in \mathcal{Q}_n\} = \{a_n\theta_{n,y}, y \in \mathcal{Q}_n\} \quad \text{for all } z \in \mathbf{Z}^k,$$

since the functions $\chi_{n,\bullet}$ are 2^n-periodic.

In this section, we work with the single-particle eigenvalue functions

$$\lambda_x(\theta) = \sum_{n=1}^{\infty} a_n \sum_{z \in \mathcal{Q}_n} \theta_{n,z} \chi_{n,z}(x) = \sum_{n \geq 1} a_n \theta_{n,[x]_n}, \tag{3.2}$$

where $[x]_n$ is the lattice point $y \in \mathcal{Q}_n$ such that

$$\{y\} = (x + (2^n \mathbf{Z})^k) \cap \mathcal{Q}_n.$$

The last equality in (3.2) follows from the definition of the functions $\chi_{\bullet,\bullet}(\cdot)$.

From this point on, by "limit-periodic" we mean a function of the above form (3.2) unless otherwise specified explicitly.

4. Main result

The direct spectral problem for fermionic particle systems subject to a limit-periodic potential was solved in [10]. The main technical arguments were proved first for $\mathfrak{n} = 2$ particles, and then it was shown how the proofs can be extended to any $\mathfrak{n} > 1$. In the present paper, which is already quite technical, we keep its size within reasonable limits by following closely the method of [14, 22], merely outlined in Section 6, so only the case of $\mathfrak{n} = 2$ particles is considered. An extension to the general case $\mathfrak{n} \geq 2$ would require a more cumbersome combinatorial and measure-theoretic analysis in the parameter space Θ.

Below, $(\Theta, \mathfrak{B}^\Theta, \mathbf{P}^\Theta)$ always stands for the probability space from Section 3.

Theorem 1. *Consider an ensemble of matrices* $\mathbf{\Lambda}(\theta) = \operatorname{diag}(\boldsymbol{\lambda}_{\mathbf{x}}(\theta), \mathbf{x} \in \mathcal{Z}^2)$, $\mathcal{Z}^2 = (\mathbf{Z}^k)^{(2)}$, *relative to the product probability space* $(\Theta, \mathfrak{B}^\Theta, \mathbf{P}^\Theta)$, *where*

$$\mathbf{\Lambda}_{\mathbf{x},\mathbf{x}}(\theta) = \boldsymbol{\lambda}_{\mathbf{x}}(\theta) = \sum_{x \in \mathbf{x}} \lambda_x(\theta), \quad \mathbf{x} \in \mathcal{Z}^2,$$

$$\lambda_x(\theta) = \sum_{n \geq 1} a_n \sum_{y \in \mathcal{Q}_n} \theta_{n,y} \chi_{n,y}(x), \quad a_n = \frac{\varepsilon^{1/4}}{\mathfrak{n}} \mathrm{e}^{-2bn^3}, b \geq 2,$$

$$\chi_{n,y}(x) := \sum_{\boldsymbol{\ell} \in (2^{n+1}\mathbf{Z})^d} \mathbf{1}_{y+\boldsymbol{\ell}}(x).$$

There exist $\varepsilon_*, c > 0$ *depending upon* k *with the following properties.*

For any $\varepsilon \in (0, \varepsilon_*)$, *there exists* $\mu(\varepsilon) \geq c \ln(\varepsilon_*/\varepsilon)$ *and a measurable subset* $\Theta_*(\varepsilon) \subset \Theta$ *with* $\mathbf{P}^\Theta\{\Theta_*(\varepsilon)\} \geq 1 - \varepsilon^{1/5}$ *such that, for any Hermitian matrix* $\mathbf{K} \in \operatorname{Off-diag}_{\mu(\varepsilon)}(\mathcal{Z}^2)$ *and* $(\theta) \in \Theta_*(\varepsilon)$, $\mathbf{\Lambda}(\theta)$ *is unitarily equivalent to a matrix of the* *form* $\varepsilon\mathbf{K} + \mathbf{W}(\theta)$, *with* $\mathbf{W}(\theta) = \mathrm{diag}(\boldsymbol{w}_{\mathbf{x}}(\theta), \mathbf{x} \in \boldsymbol{\mathcal{Z}}^2)$, *so that*

$$\begin{aligned}(\mathbf{K} + \mathbf{W}(\theta))\boldsymbol{\mathcal{U}}(\theta) &= \boldsymbol{\mathcal{U}}(\theta)\boldsymbol{\Lambda}(\theta),\\ \boldsymbol{\mathcal{U}}^{\top}(\theta)\boldsymbol{\mathcal{U}}(\theta) &= \mathbf{1},\end{aligned} \tag{4.1}$$

and one has, for some $C_1, C_2 \in (0, +\infty)$,

$$\begin{aligned}\sup_{\mathbf{x}\in\boldsymbol{\mathcal{Z}}^2} |\boldsymbol{w}(\theta)|_\infty &\le C_1\varepsilon^2,\\ \|\boldsymbol{\mathcal{U}}^{\pm 1}(\theta) - \mathbf{1}\|_\nu &\le C_2\varepsilon.\end{aligned} \tag{4.2}$$

The above theorem can be easily adapted to the case of quasi-periodic media-particle (external) potentials of the form considered in [9]. We restrict the statement to the case of Hermitian matrices due to the main intended application to the Hamiltonians of quantum systems, but, of course, the techniques of [14,22] apply also to more general matrices.

Corollary 2. *It follows from* (4.1) *that the column-vectors* $\boldsymbol{\psi}_{\mathbf{x}}(\cdot)$ *of* $\boldsymbol{\mathcal{U}}(\cdot)$ *form orthonormal eigenbases of the matrix* $\varepsilon\mathbf{K} + \mathbf{W}$ *in* $\ell^2(\boldsymbol{\mathcal{Z}}^2)$. *Moreover, owing to the definition of the norms* $\|\cdot\|_\mu$, $\mu > 0$, *the norm-bounds* (4.2) *imply a uniform exponential localization of the eigenfunctions* $\boldsymbol{\psi}_{\mathbf{x}}(\cdot)$.

It is plain that, for ε small enough, a uniform exponential decay of all eigenfunctions implies that all normalised eigenfunctions are unimodal, e.g., $|\boldsymbol{\psi}_{\mathbf{x}}(\mathbf{x})|^2 > \frac{1}{2}$.

5. Bounds on the small denominators

We work with the single-particle eigenvalue functions given by (3.2)

$$(x,\theta) \mapsto \lambda_x(\theta) = \sum_{n=1}^{\infty} a_n \sum_{z\in\mathcal{Q}_n} \theta_{n,z}\chi_{n,z}(x) = \sum_{n\ge 1} a_n\theta_{n,[x]_n}$$

and their n-particle counterparts $\boldsymbol{\lambda}_\bullet(\cdot)$ defined by

$$(\mathbf{x},\theta) \mapsto \boldsymbol{\lambda}_{\mathbf{x}}(\theta) := \sum_{x\in\mathbf{x}} \lambda_x(\theta) = \sum_{x\in\mathbf{x}}\sum_{n\ge 1} a_n\theta_{n,[x]_n}, \quad \mathbf{x} \in \boldsymbol{\mathcal{Z}}^2.$$

Fix any $n \ge 2$, consider the lattice cube $\mathcal{Q}_n = [-2^{n-1}, 2^{n-1})^k \cap \mathbf{Z}^k$, and pick any $\hat{x} \in \mathcal{Q}_n$. In the proof of Lemma 5.1, $\hat{x}$ will be chosen in a specific way.

We also need truncated 1-particle and n-particle eigenvalue functions:

$$\begin{aligned}\lambda_x^{(N)}(\theta) &:= \sum_{1\le n\le N} a_n\theta_{n,[x]_n}, \quad N \ge 1,\\ \boldsymbol{\lambda}_{\mathbf{z}}^{(N)}(\theta) &:= \sum_{z\in\mathbf{z}} \lambda_z^{(N)}(\theta).\end{aligned}$$

A simple calculation provides uniform bounds for the truncation errors:

$$\begin{aligned} \sup_{x\in\mathbf{Z}^k}\ \sup_{\theta\in\Theta} |\lambda_x(\theta)-\lambda_x^{(N)}(\theta)| &\le \frac{\varepsilon^{1/4}}{2}\beta_N a_N, & \beta_N &:= \mathrm{e}^{-2bN^2}, \\ \sup_{\mathbf{x}\in\mathcal{Z}^2}\ \sup_{\theta\in\Theta} |\boldsymbol{\lambda}_x(\theta)-\boldsymbol{\lambda}_x^{(N)}(\theta)| &\le \frac{1}{2}\eta_R, & \eta_R &:= \mathfrak{n}\varepsilon^{1/4}\beta_N a_N. \end{aligned} \tag{5.1}$$

By construction, the functions $y \mapsto \chi_{n,y}$ with $n \le N$ are 2^N-periodic in each coordinate direction in $\mathbf{Z}^k$, and so are, therefore, the truncated functions $\lambda_\bullet^{(N)}$ and $\boldsymbol{\lambda}_\bullet^{(N)}$. To be more precise, the mapping $\mathbf{x} \mapsto \boldsymbol{\lambda}_\mathbf{x}^{(N)}(\cdot)$ is invariant under the “diagonal” shifts S^a, $a \in (2^N\mathbf{Z})^k$, of the form

$$\mathrm{S}^a\colon \mathbf{x} = \{x_1,\dots,x_\mathfrak{n}\} \mapsto \{x_1+a,\dots,x_\mathfrak{n}+a\}.$$

Given any integer $R \ge 1$, set

$$\tilde{n} = \tilde{n}(R) := \min\{n\in\mathbf{N} : \mathcal{B}_R(0)\subset\mathcal{Q}_n\}, \tag{5.2}$$

so that $\mathcal{B}_R(0) \subset \mathcal{Q}_{\tilde{n}(R)}$. Clearly, $\tilde{n}(R) \asymp \ln R$. When the value of R is clear from the context, we often write for brevity $\tilde{n}$ instead of $\tilde{n}(R)$.

As was already explained, the pivot of the adaptation of the techniques from [14, 22] to our model is replacing traditional distances on the graph $\mathcal{Z}^2$ with a very special function $\langle\mathbf{x},\mathbf{y}\rangle \mapsto \mathfrak{d}(\mathbf{x},\mathbf{y})$ verifying the triangle inequality (“orthometric”), which is also symmetric. Consequently, the lower bounds on the absolute value of the cocycle $\lambda_a - \lambda_b$ in terms of the distance $|a-b|$, as would be required in the framework of the Craig–Pöschel approach, are to be replaced in our paper with lower bounds on $|\boldsymbol{\lambda}_\mathbf{x} - \boldsymbol{\lambda}_\mathbf{y}|$ for $\langle\mathbf{x},\mathbf{y}\rangle$ with $\mathfrak{d}(\mathbf{x},\mathbf{y}) \le R$, $R \ge 1$. Furthermore, some simple arguments show that $\operatorname{diam}(\mathbf{x}\cup\mathbf{y}) \le \mathfrak{d}(\mathbf{x},\mathbf{y})$ (cf. (A.3), Section A).

For these reasons, we prove below an efficient bound on the spacings $|\boldsymbol{\lambda}_\mathbf{x}(\theta) - \boldsymbol{\lambda}_\mathbf{y}(\theta)|$ for the pairs $\langle\mathbf{x},\mathbf{y}\rangle \in (\mathcal{Z}^2)^2$ with $\operatorname{diam}(\mathbf{x}\cup\mathbf{y}) \le R$, $R \ge 1$.

Lemma 5.1. *For any $R \ge 1$, there exists a subset $\bar{\Theta}(R) \subset \Theta$ of the form*

$$\bar{\Theta}(R) = \bigcup_{\substack{\langle\mathbf{x},\mathbf{y}\rangle\in(\mathcal{Z}^2)^2 \\ \mathbf{x}\cup\mathbf{y}\subset\mathcal{Q}_{\tilde{n}+1}}} \bar{\Theta}_{\mathbf{x},\mathbf{y}}(R),$$

where $\bar{\Theta}_{\mathbf{x},\mathbf{y}}(R)$ are $\mathfrak{B}^\Theta$-measurable and admit the bounds (cf. (5.1))

$$\inf_{\substack{\langle\mathbf{x},\mathbf{y}\rangle\in(\mathcal{Z}^2)^2 \\ \operatorname{diam}(\mathbf{x}\cup\mathbf{y})\le R}} |\boldsymbol{\lambda}_\mathbf{x}(\theta)-\boldsymbol{\lambda}_\mathbf{y}(\theta)| \ge \eta_R, \quad \textit{for all} \ \in \mathfrak{C} \setminus \bar{\mathfrak{C}}_{\mathbf{x}\mathbf{y}}(\mathrm{R}), \tag{5.3}$$

$$\mathbf{P}^\Theta\{\bar{\Theta}(R)\} \le 2^{(\tilde{n}+2)\mathfrak{n}d} a_{\tilde{n}}^{-1}\eta_R.$$

Proof. We proceed in three steps.

Step 1. Fix any $R \ge 1$, let $\tilde{n} = \tilde{n}(R)$ as in (5.2), and set $\mathbf{s}_{\mathbf{x},\mathbf{y}} = \boldsymbol{\lambda}_{\mathbf{x}} - \boldsymbol{\lambda}_{\mathbf{y}}$, $\mathbf{s}^{(\tilde{n})} = \boldsymbol{\lambda}_{\mathbf{x}}^{(\tilde{n})} - \boldsymbol{\lambda}_{\mathbf{y}}^{(\tilde{n})}$. By (5.1), we have for any $\theta \in \Theta$:

$$|\mathbf{s}_{\mathbf{x},\mathbf{y}}(\theta)| \ge |\mathbf{s}_{\mathbf{x},\mathbf{y}}^{(\tilde{n})}(\theta)| - \eta_R, \tag{5.4}$$

thus the lower bound $|\mathbf{s}_{\mathbf{x},\mathbf{y}}(\theta)| \ge \eta_R$ holds true for any $\theta \in \Theta$ such that

$$|\mathbf{s}_{\mathbf{x},\mathbf{y}}^{(\tilde{n})}(\theta)| \ge 2\eta_R. \tag{5.5}$$

Now, we shall construct a set $\overline{\Theta}(R)$ such that (5.5) holds for all $\theta \in \Theta \setminus \overline{\Theta}(R)$.

Step 2. *Reduction to the case where* $\mathbf{x} \cup \mathbf{y} \subset \mathcal{Q}_{\tilde{n}+1}$. As was said above, the truncated $\mathfrak{n}$-particle eigenvalue functions $\boldsymbol{\lambda}_{\bullet}^{(\tilde{n})}(\cdot)$ are invariant under the shifts S^{ℓ} with $\ell \in (2^{\tilde{n}}\mathbf{Z})^k$. Fix any $\langle \mathbf{x}, \mathbf{y} \rangle \in (\mathcal{Z}^2)^2$ with $\mathrm{diam}(\mathbf{x} \cup \mathbf{y}) \le R$ (cf. (5.3)), then there exists $a \in (2^{\tilde{n}}\mathbf{Z})^k$ such that

$$\mathrm{dist}(\mathrm{S}^a \mathbf{x}, 0) \le \mathcal{Q}_{\tilde{n}}$$

since

$$\bigcup_{a \in (2^{\tilde{n}}\mathbf{Z})^k} \mathrm{S}^a \mathcal{Q}_{\tilde{n}} = \mathbf{Z}^k.$$

Fix such a, and let $\mathbf{x}' := \mathrm{S}^a \mathbf{x}$, $\mathbf{y}' := \mathrm{S}^a \mathbf{y}$. If, in addition, $\mathrm{diam}(\mathbf{x} \cup \mathbf{y}) \le R$, hence

$$\mathrm{diam}(\mathbf{x}' \cup \mathbf{y}') \le R, \quad \mathrm{dist}(\mathrm{S}^{\ell}(\mathbf{x}' \cup \mathbf{y}'), 0) \le \mathrm{diam}\, \mathcal{Q}_{\tilde{n}},$$

then $\mathbf{x}' \cup \mathbf{y}' \subset \mathcal{Q}_{\tilde{n}+1}$, since $R \le 2^{\tilde{n}}$, and by the aforementioned periodicity of the truncated eigenvalues, we have the identity

$$\mathbf{s}_{\mathbf{x},\mathbf{y}}^{(\tilde{n})}(\theta) \equiv \boldsymbol{\lambda}_{\mathbf{x}}^{(\tilde{n})}(\theta) - \boldsymbol{\lambda}_{\mathbf{y}}^{(\tilde{n})}(\theta) = \boldsymbol{\lambda}_{\mathbf{x}'}^{(\tilde{n})}(\theta) - \boldsymbol{\lambda}_{\mathbf{y}'}^{(\tilde{n})}(\theta).$$

This brings us to the next, combinatorial reduction: it suffices to consider not all pairs $\langle \mathbf{x}, \mathbf{y} \rangle$ with $\mathrm{diam}(\mathbf{x} \cup \mathbf{y}) \le R$ but only those satisfying $\mathbf{x} \cup \mathbf{y} \subset \mathcal{Q}_{\tilde{n}+1}$. The total number of such pairs is bounded by $(2^{\tilde{n}+1})^{\mathfrak{n}k}$.

Step 3. *Analysis of the case* $\mathbf{x} \cup \mathbf{y} \subset \mathcal{Q}_{\tilde{n}+1}$. Since $\mathbf{x} \ne \mathbf{y}$, there exists $\hat{x} \in \mathbf{x} \setminus \mathbf{y}$: otherwise, one would have $\mathbf{x} \subset \mathbf{y}$, hence $\mathbf{x} = \mathbf{y}$ because $\mathrm{card}\, \mathbf{x} = \mathrm{card}\, \mathbf{y}$. (This argument is, of course, a simpler variant of Lemma 2.1.) Fix such a point $\hat{x} \in \mathbf{Z}^k$, so that $\mathbf{x} = \{\hat{x}\} \cup \mathbf{x}'$, and let $\mathfrak{B}_{\hat{x}}^{\tilde{n}} \subset \mathfrak{B}^{\Theta}$ be the sigma-algebra generated by

$$\{\theta_n([\hat{x}]_n), n \in [\![1, \tilde{n}-1]\!]; \theta_n([y]_n), y \in \mathcal{Q}_n \setminus \{\hat{x}\}, n \in [\![1, \tilde{n}]\!]\}.$$

Then we can decompose $\mathbf{s}_{\mathbf{x},\mathbf{y}}^{(\tilde{n})}(\theta)$ as follows:

$$\begin{aligned} \mathbf{s}_{\mathbf{x},\mathbf{y}}^{(\tilde{n})}(\theta) &= a_{\tilde{n}} \theta_{\tilde{n},\hat{x}} + \boldsymbol{\zeta}_{\tilde{n},\hat{x}}(\theta), \\ \boldsymbol{\zeta}_{\tilde{n},\hat{x}}(\theta) &:= \sum_{x \in \mathbf{x} \setminus \{\hat{x}\}} \lambda_x^{(\tilde{n})}(\theta) - \sum_{y \in \mathbf{y}} \lambda_y^{(\tilde{n})}(\theta), \end{aligned}$$

where $\boldsymbol{\zeta}_{\tilde{n},\hat{x}}$ is $\mathfrak{B}^{\tilde{n}}_{\hat{x}}$-measurable, since $\hat{x} \notin (\mathbf{y} \cup \{x'\})$. Conditional on $\mathfrak{B}^{\tilde{n}}_{\hat{x}}$, the Θ-random variable $a_{\tilde{n}}\theta_{\tilde{n},\hat{x}} + \boldsymbol{\zeta}_{\tilde{n},\hat{x}}(\theta)$, with $\boldsymbol{\zeta}_{\tilde{n},\hat{x}}(\theta)$ rendered constant by conditioning, has the uniform distribution on the interval $[\boldsymbol{\zeta}_{\tilde{n},\hat{x}}, \boldsymbol{\zeta}_{\tilde{n},\hat{x}} + a_{\tilde{n}}]$, thus its density is bounded by $a_{\tilde{n}}^{-1}$. Therefore, the probability of the event

$$\overline{\Theta}_{\mathbf{x},\mathbf{y}}(R) := \{\theta : |\boldsymbol{\lambda}^{(\tilde{n})}_{\mathbf{x}}(\theta) - \boldsymbol{\lambda}^{(\tilde{n})}_{\mathbf{y}}(\theta)| \le 2\eta_R\} \tag{5.6}$$

admits the bound

$$\mathbf{P}^{\Theta}\{\overline{\Theta}_{\mathbf{x},\mathbf{y}}(R)\} = \mathbf{E}^{\Theta}[\mathbf{P}^{\Theta}\{\overline{\Theta}_{\mathbf{x},\mathbf{y}}(R) \mid \mathfrak{B}^{\tilde{n}}_{\hat{x}}\}] \le 2a_{\tilde{n}}^{-1}\eta_R.$$

Now, let

$$\overline{\Theta}(R) := \bigcup_{\substack{\langle \mathbf{x},\mathbf{y}\rangle \in (\mathbf{Z}^2)^2 \\ \mathbf{x}\cup\mathbf{y}\subset \mathcal{Q}_{\tilde{n}+1}}} \overline{\Theta}_{\mathbf{x},\mathbf{y}}(R)$$

then

$$\begin{aligned}\mathbf{P}^{\Theta}\{\overline{\Theta}(R)\} &\le 2^{(\tilde{n}+1)\mathfrak{n}d} \max_{\substack{\langle \mathbf{x},\mathbf{y}\rangle \in (\mathbf{Z}^2)^2 \\ \mathbf{x}\cup\mathbf{y}\subset \mathcal{Q}_{\tilde{n}+1}}} \mathbf{P}^{\Theta}\{\overline{\Theta}_{\mathbf{x},\mathbf{y}}(R)\} \\ &\le 2^{(\tilde{n}+2)\mathfrak{n}d} a_{\tilde{n}}^{-1}\eta_R.\end{aligned}$$

By construction of the sets $\overline{\Theta}_{\mathbf{x},\mathbf{y}}(R)$ (cf. (5.6)), one has by (5.4) and (5.6):

$$|\mathbf{s}_{\mathbf{x},\mathbf{y}}(\theta)| \ge |\mathbf{s}^{(\tilde{n})}_{\mathbf{x},\mathbf{y}}(\theta)| - \eta_R \ge 2\eta_R - \eta_R = \eta_R \quad \text{for all } \theta \in \Theta \setminus \overline{\Theta}(R),$$

which proves the claim. ∎

Proposition 1. *For some $\varepsilon_* > 0$ and all $\varepsilon \in (0, \varepsilon_*)$, there exist some $c > 0$ and a measurable subset $\Theta_*(\varepsilon) \subset \Theta$ such that $\mathbf{P}^{\Theta}\{\Theta_*(\varepsilon)\} \ge 1 - \varepsilon^{1/5}$ and one has, for any $\theta \in \Theta_*(\varepsilon)$ and any $\langle \mathbf{x}, \mathbf{y}\rangle \in (\mathbf{Z}^2)^2$ with* diam $\mathbf{x} \cup \mathbf{y} \le R$*:*

$$|\boldsymbol{\lambda}_{\mathbf{x}}(\theta) - \boldsymbol{\lambda}_{\mathbf{y}}(\theta)| \ge \varepsilon^{1/3}\mathrm{e}^{-c\sqrt{R}}.$$

Proof. Let

$$\overline{\Theta}_*(\varepsilon) := \bigcup_{R\ge 1} \overline{\Theta}(R),$$

then, with $\eta_R = \mathfrak{n}\varepsilon^{1/4}\beta_{\tilde{n}(R)}a_{\tilde{n}(R)}$, we have

$$\begin{aligned}\mathbf{P}^{\Theta}\{\overline{\Theta}_*(\varepsilon)\} &\le \sum_{R\ge 1} \mathbf{P}^{\Theta}\{\overline{\Theta}(R)\} \le \sum_{R\ge 1} 2^{(\tilde{n}(R)+2)\mathfrak{n}d} a_{\tilde{n}(R)}^{-1}\eta_R \\ &\le \mathfrak{n}\varepsilon^{1/4} \sum_{R\ge 1} 2^{(\tilde{n}(R)+2)\mathfrak{n}d} \beta_{\tilde{n}(R)},\end{aligned}$$

where $\beta_{\tilde{n}} := \mathrm{e}^{-2b\tilde{n}^2}$ (cf. (5.1)) and $\tilde{n}(R) \asymp \ln(R)$ as $R \to +\infty$ (cf. (5.2)).

Now, let $\Theta_*(\varepsilon) := \Theta \setminus \overline{\Theta}(\varepsilon)$. Then by convergence of the series

$$\sum_{R\geq 1} \mathrm{e}^{-C\ln^2(R)+C'\ln(R)}$$

for any $C, C' > 0$, and taking $0 < \varepsilon \leq \varepsilon_*$ small enough to absorb $\mathfrak{n}$ and other auxiliary constants, we obtain $\mathbf{P}^{\Theta}\{\overline{\Theta}_*(\varepsilon)\} \leq \varepsilon^{1/5}$, whence

$$\mathbf{P}^{\Theta}\{\Theta_*(\varepsilon)\} \geq 1 - \varepsilon^{1/5}.$$

Proposition 1 is proved. ■

Proof of Theorem 1. For any $\theta \in \Theta_*(\varepsilon)$, the eigenvalue function $\mathbf{x} \mapsto \boldsymbol{\lambda}_{\mathbf{x}}(\theta)$ is distal with the approximation function $\mathfrak{a}(r) = \mathrm{e}^{-cr^{1/2}}$. To be more precise, we have obtained the approximation function $r \mapsto \varepsilon^{1/3}\mathrm{e}^{-cr^{1/2}}$ dependent upon ε. However, on account of Remark 2.2 in Section 2.6, the inverse spectral problem for the matrices $\Lambda(\theta)$ and K is equivalent to that for the scaled matrices $t\Lambda(\theta)$ and $t\mathrm{K}$ with any $t > 0$, so it suffices to set $t = \varepsilon^{-1/3}$ to have the aforementioned approximation function $\mathfrak{a}(r) = \mathrm{e}^{-cr^{1/2}}$.

Therefore, Theorem 1 follows by applying the inductive scheme adapting the one from the proof of [22, Theorem A] (outlined and commented in Section 6) to the fermionic model with a limit-periodic single-particle potential. ■

6. An outline of the Craig–Pöschel induction

6.1. Preliminary observations

Since our main goal in this section is merely to outline the scheme presented in [14, 22], we shall omit a number of detailed calculations, referring the reader to the original papers. However, to make the principal claims precise and notations clear, we complement Definition 2.1 of approximation functions with the following statement from [22], where some quantities useful in the sequel are introduced.

In the right-hand side of (6.1), k stands for the dimension of the lattice $\mathbf{Z}^k$ serving as the 1-particle configuration space.

Lemma 6.1 (cf. [22, Lemma A.1]). *Let $\mathfrak{a}$ be an approximation function, and Φ, Ψ are the functions associated with $\mathfrak{a}$ via the relations* (2.7)–(2.8). *For any $\sigma \in (0, +\infty)$ such that*

$$\frac{1}{\ln 2}\int_s^{+\infty} r^{-2}\ln(\mathfrak{a}(r))dr \leq \sigma$$

one has

$$\Psi(\sigma) \leq \left(\frac{4s}{\ln(\mathfrak{a}(s))}\right)^{4k} \tag{6.1}$$

and there exists a positive decreasing sequence $(\sigma_l)_{l\geq 0}$ *such that*

$$\sum_{l\geq 0}\sigma_l = \frac{\sigma}{2}, \quad \Psi\Big(\frac{\sigma}{2}\Big) = \prod_{l\geq 0}\Phi_l^{2^{-l-1}}, \quad \Phi_l := \Phi(\sigma_l).$$

Let a sequence $(\sigma_l)_{l\geq 0}$ as above be fixed, and set

$$s_n = s - 2\sum_{0\leq l\leq n-1}\sigma_l, \quad n \geq 0,$$

so that $s = s_0 > s_1 > \cdots > s_n \to s - \sigma$, and one has

$$s_{n+1} = s_n - 2\sigma_n, \quad s_{n+1} + \sigma_{n+1} \leq s_n - \sigma_n.$$

In [14, 22], one considers various matrices $X \in \mathrm{Mat}(\mathcal{X})$, where $\mathcal{X} = \mathbf{Z}^k$, $k \geq 1$. A careful reading of the original arguments and calculations evidences that only the dimension k is figuring in the calculations, making some auxiliary constants in the principal estimates k-dependent. Specifically, k enters into the estimates through [22, Lemma 4.1] (resp. [14, Lemma 4.5]) relying on the rate of growth of balls,

$$r \mapsto \sup_{x\in\mathcal{X}} \ \mathrm{card}\, \mathcal{B}_r(x),$$

which, in turn, admits a uniform upper bound on the class of countable graphs with uniformly bounded coordination numbers:

$$\mathrm{C}_{\mathcal{X}} := \sup_{x\in\mathcal{X}} \ \mathrm{card}(\mathcal{B}_r(x) \setminus \{x\}).$$

[14, Lemma 4.5] and its counterpart [22, Lemma 4.1] operate with the norm estimates of the products of matrices $X, Y \in \mathrm{Mat}(\mathcal{X})$; a similar estimate is used in [3], where the general context, examples, and techniques are slightly different. An adaptation to more general graphs with uniformly bounded coordination numbers is quite straightforward.

These observations concern *the proof* of the main *general result* from [14], reproduced in a somewhat re-arranged form in [22] and applied to additional examples compared to [14]. However, it was clear that, without constructive examples of *distal sequences* (which we call in the present paper "distal functions"), the general results would loose a significant part of their value. And it turns out that interesting and important examples of distal sequences can be naturally constructed on the periodic lattices, owing to a well-developed theory of almost-periodic (including quasi-periodic and limit-periodic) functions. The algebraic structure of the lattices $\mathbf{Z}^k$ is crucial in this respect. But once a distal sequence generating the diagonal entries of a diagonal matrix, labelled by some countable set, is given, the rest of the inductive procedure can be applied in a unified way; *this* is a strong point of the aforementioned works.

As was explained in the introductory section, the spectral problems for non-distinguishable interacting quantum particle systems give rise to more complex underlying graphs than those where individual particles evolve. In the particular case of 1-dimensional particles, one can represent the configuration space of an $\mathfrak{n}$-particle system by a subset of $(\mathbf{Z}^1)^{\mathfrak{n}}$, e.g., by $\{(x_1, x_2) : x_1 < x_2\}$ for $\mathfrak{n} = 2$, but in higher dimension, one needs a symmetric power of $\mathbf{Z}^k$ which does not admit such a simple representation. Nevertheless, almost-periodicity of some kind of the media-particle interaction potential, also known as "external potential", induces a certain form of almost-periodicity in the fermionic configuration space. Although the latter almost-periodicity is anisotropic, viz. relative to the "diagonal" shifts $\{x_1, \dots, x_{\mathfrak{n}}\} \mapsto \{x_1 + a, \dots, x_{\mathfrak{n}} + a\}$, $a \in \mathbf{Z}^k$, we prove that, under certain assumptions, it suffices for generating distal eigenvalue functions, thus making applicable the general results from [14, 22].

6.2. The inductive scheme

Throughout this section, we shall be working with a matrix $P \in \mathrm{M}_s$, with some $s > 0$, and it will be assumed that, for some $\sigma \in (0, s)$,

$$\|P\|_s \le \varepsilon\Big(\Psi\Big(\frac{\sigma}{2}\Big)\Big)^{-1}.$$

Further, once P is fixed, we define $\theta > 0$ and a positive sequence $(\theta_n)_{n\ge0}$ given by

$$\begin{aligned} \theta_n &:= c^{1-2^{-2}} \|P\|_s \prod_{0\le l\le n-1} \Phi_l^{2^{-l-1}}, \qquad c := 96b^6, \\ \theta &:= c\|P\|_s \Psi\Big(\frac{\sigma}{2}\Big), \end{aligned} \tag{6.2}$$

where $\varepsilon, b > 0$ are chosen so that $\theta \le c\varepsilon = 1/(3b)$.

Lemma 6.2. *If $V, V^{-1} \in \mathrm{M}_\sigma$ with $\|V^{\pm1} - \mathbf{1}\|_\sigma \le \sigma^k/2b$, $\sigma \in (0, 1]$, then, for any $P \in \mathrm{M}$, the equation $\mathrm{diag}(P + \mathrm{Ad}_V[A]) = 0$ has a unique solution $A \in \mathrm{M}_\infty$, with $\|A\|_\infty \le 2\|\mathrm{diag}(P)\|_\infty$.*

Proof. By the identity $A - \mathrm{diag}(V^{-1}AV) = \mathrm{diag}((V^{-1} - \mathbf{1})[\![(V - \mathbf{1}), A]\!])$, we get

$$\begin{aligned} \|A - \mathrm{diag}(V^{-1}AV)\|_\infty &\le \|(V - \mathbf{1}) \cdot [\![(V - \mathbf{1}), A]\!]\|_0 \\ &\le \frac{2b^2}{\sigma^{2d}} \|V^{-1} - \mathbf{1}\|_\sigma \|V - \mathbf{1}\|_\sigma \|A\|_\infty \le \frac{1}{2}\|A\|_\infty, \end{aligned}$$

thus for each $P \in \mathrm{M}$, the map $A \mapsto A - \mathrm{diag}(V^{-1}AV)$ is a contraction in M_∞, and its fixed point solves $\mathrm{diag}(P + \mathrm{Ad}_V[A]) = 0$. ∎

Lemma 6.3. *Let D be a diagonal matrix whose diagonal entries $d_x := D_{xx}$ are given by a distal function $d\colon x \mapsto d_x$ with an approximation function $\mathfrak{a}$ and the associated functions Φ and Ψ, as per Definition 2.1. Then for any matrix $Q \in \mathrm{M}$, the problem*

$$\begin{aligned} [\![D, \widetilde{W}]\!] + (Q - \mathrm{diag}(Q)) &= 0, \\ \mathrm{diag}(\widetilde{W}) &= 0, \end{aligned} \tag{6.3}$$

has a unique solution $\widetilde{W} \in \mathrm{M}$. If, in addition, $Q \in \mathrm{M}_s$ and $0 < \sigma \le s$, then $\widetilde{W} \in \mathrm{M}_{s-\sigma}$ with $\|\widetilde{W}\|_{s-\sigma} \le \sigma^{-4k}\Phi(\sigma)$.

Proof. Since D is diagonal, one has $[\![D, \widetilde{W}]\!]_{xy} = (d_x - d_y)\widetilde{W}_{xy}$, so the commutator has the zero diagonal, and the off-diagonal entries of $\widetilde{W}$ are (uniquely) well defined for any distal function d:

$$\widetilde{W}_{xy} = \begin{cases} \dfrac{Q_{xy}}{d_x - d_y} & \text{if } x \ne y, \\ 0 & \text{if } x = y. \end{cases}$$

The asserted norm estimates follow by straightforward calculations (cf. [22, proof of Lemma 4.4]). ∎

Remark 6.1. At each step of the inductive procedure, this statement provides the key element $W_n = \mathbf{1} + \widetilde{W}_n$, where $\widetilde{W}_n$, with $\mathrm{diag}(\widetilde{W}_n) = 0$, solves equation (6.10) of the general form (6.3), but with Q replaced with Q_n (the latter is defined in the proof of Lemma (6.4)).

Theorem 3 (cf. [22, Theorem A], [14]). *Let $d\colon \mathbf{Z}^k \to d_x \in \mathbf{R}$ be a distal function with the approximation function $\mathfrak{a}$ and the associated functions Φ and Ψ. Let $0 < s \le \infty$ and $0 < \sigma \le \min\left(1, \frac{s}{2}\right)$. There exists some $\varepsilon_* > 0$ such that for any $\varepsilon \in (0, \varepsilon_*)$ and any matrix $P \in \mathrm{M}_s$ satisfying*

$$\|P\|_s \le \frac{\varepsilon}{\Psi(\frac{\sigma}{2})}$$

there exists a diagonal matrix $\widehat{D}$ and an invertible matrix V such that

$$\mathrm{Ad}_V[\widehat{D} + P] = D.$$

Furthermore, $V, V^{-1} \in \mathrm{M}_{s-\sigma}$ and $\widehat{D} - D \in \mathrm{M}_\infty$ with

$$\begin{aligned} \|V^{\pm 1} - \mathbf{1}\|_{s-\sigma} &\le C\|P\|_s, \\ \|\widehat{D} - D - \mathrm{diag}(P)\|_\infty &\le C^2\|P\|_s^2, \quad C = \varepsilon^{-1}\Psi\left(\frac{\sigma}{2}\right). \end{aligned}$$

If P is Hermitian, then V can be chosen to be unitary.

Lemma 6.4 (Inductive estimate). *Let D be a diagonal matrix generated by a distal function $x \mapsto D_{xx}$, and $P \in \mathrm{M}_s$. Assume that, for some $n \geq 0$ and all $l \in [\![0, n]\!]$, one has*

$$V_l^{-1}(\widehat{D}_l + P)V_l = D + P_l, \tag{6.4}$$

where each $\widehat{D}_l$ is diagonal, P_l admits the bound

$$\|P_l\| \leq \theta_l^{2^l} \quad \text{(cf. (6.2))}, \tag{6.5}$$

$V_l \in \mathrm{M}_{s_l+\sigma_l}$ is invertible, with $V_l^{-1} \in \mathrm{M}_{s_l}$, and the following bounds hold:

$$\|V_l - V_{l-1}\|_{s_l+\sigma_l}, \|V_l^{-1} - V_{l-1}^{-1}\|_{s_l+\sigma_l} \leq \sigma^{2k}\theta_\infty^{2^{l-1}}, \quad l \geq 1. \tag{6.6}$$

Then there exist some matrices $\widehat{D}_{n+1} \in \mathfrak{D}\mathrm{iag}$, $P_{n+1} \in \mathrm{M}_{s_{n+1}}$, and an invertible matrix V_{n+1} such that

$$V_{n+1}^{-1}(\widehat{D}_{n+1} + P)V_{n+1} = D + P_{n+1}$$

and the bounds of the form (6.5)–(6.6) with $l = n + 1$ hold true.

Remark 6.2. It follows from the inductive argument presented in [14, 22], *upon the completion of the inductive procedure*, that there exist two sequences of diagonal matrices, $(D_n)_{n\in\mathbf{N}}$ and $(A_n)_{n\in\mathbf{N}}$, such that

$$\widehat{D} = D + \sum_{n\geq 0} A_n, \quad D_n = \sum_{l\geq n} A_l,$$

and for each $n \geq 0$, the relations (cf. [22, Section 4, p. 457])

$$V_n^{-1}(\widehat{D} + P)V_n = D + P_n + V_n^{-1}D_nV_n \tag{6.7}$$

can be re-written as in (6.4).

However, the way the inductive statement is formally presented in [14, 22], one may wonder if the "target" matrix $\widehat{D}$, unknown at any step l of the induction, can indeed be constructed without referring to the results of the induction. The aforementioned papers are somewhat sketchy in this respect. More precisely, it is not completely clear from the beginning if, in the step $l \rightsquigarrow l + 1$, the existence (and the value) of the next matrix A_{l+1} is contingent upon a specific (but unspecified) choice of the matrix $\widehat{D}$ to be made *before* the induction starts. Hopefully, the reformulation proposed in the present section can provide a possible clarification of the situation and allow one to avoid an apparent logical problem. In essence, all it takes is moving the last term from the right-hand side to the left-hand side of (6.7), while all the calculations and main arguments remain unchanged. Actually, this is precisely how the inductive relation is presented in a brief outline of the proof of the second main result, [22, Theorem B], addressing the direct spectral problem.

Proof. In the outline of the proof given below, we focus on the role and the goal of each logical component of one step of induction.

The base of induction. Let

$$V_0 = \mathbf{1}, \quad P_0 = P, \quad \widehat{D}_0 = D, \quad A_{-1} = 0,$$

then the trivial identity $D + P = D + P$ can be re-written as

$$V_n^{-1}(\widehat{D}_n + P)V_n = D + P_n, \quad \widehat{D}_n = D + \sum_{-1 \le l \le n-1} A_l, \quad n = 0.$$

Stage 0. We start each step $n \rightsquigarrow n + 1$, assuming that there are finite sequences of matrices

$$(\widehat{D}_l)_{l \in [\![0,n]\!]}, \quad (P_l)_{l \in [\![0,n]\!]}, \quad (A_l)_{l \in [\![0,n-1]\!]}, \quad (V_l)_{l \in [\![0,n]\!]},$$

satisfying

$$V_n^{-1}(\widehat{D}_n + P)V_n = D + P_n, \quad \widehat{D}_n = D + \sum_{-1 \le l \le n-1} A_l, \tag{6.8}$$

where $\widehat{D}_l$, A_l are diagonal.

Stage 1. P_n is *not* assumed to be off-diagonal, so we set $\overline{P}_n := P_n - \operatorname{diag}(P_n)$.

Stage 2. To perform the induction step, we apply Lemma 6.2 to find a bounded diagonal matrix A_n such that

$$\operatorname{diag}(V_n^{-1} A_n V_n) = -\operatorname{diag}(P_n), \quad \|A_n\|_\infty \le \|\operatorname{diag}(P_n)\|_\infty \le 2\theta_n^{2^n},$$

and so

$$\begin{aligned} \overline{P}_n &= P_n + V_n^{-1} A_n V_n, \quad \operatorname{diag}(\overline{P}_n) = 0, \\ \|\overline{P}_n\|_{s_n} &\le \|P_n\|_{s_n} + \frac{3b^2}{\sigma_n^{2m}} \|A_n\|_\infty \quad \text{(cf. [22])}. \end{aligned} \tag{6.9}$$

Stage 3. Further, by Lemma 6.3, there exists a solution $\widetilde{W}_n$ to the problem

$$[\![D, \widetilde{W}_n]\!] = -\overline{P}_n, \quad \operatorname{diag}(\widetilde{W}_n) = 0, \tag{6.10}$$

and it satisfies (cf. [22, equation (4.9)])

$$\begin{aligned} \|\widetilde{W}_n\|_{s_n} &\le \frac{\sigma_n^{3m} \theta_\infty^{2^n}}{12b} \le \frac{\sigma_n^m}{3b}, \\ \|W_n - \mathbf{1}\|_{s_n - 2\sigma_n} &\le \frac{3}{2} \|W_n - \mathbf{1}\|_{s_n} \le 12b^2 \sigma_n^{3m} \Phi_n \theta_n^{2^n}, \end{aligned}$$

so that $W_n := \mathbf{1} + \widetilde{W}_n$ is invertible in $\mathrm{M}_{s_n - 2\sigma_n}$ by Neumann series (cf. [22, Lemma 4.2]). Obviously, like $\widetilde{W}_n$, it solves $[\![D, W_n]\!] = -\overline{P}_n$.

Conclusion. Now, set

$$V_{n+1} := V_n W_n, \quad \widehat{D}_{n+1} := \widehat{D}_n + A_n,$$

and let

$$R_n := V_{n+1}^{-1}(\widehat{D} + P)V_{n+1},$$

then

$$R_n = W_n^{-1} V_n^{-1}(\widehat{D}_n + P)V_n W_n = W_n^{-1}(D + P_n)W_n.$$

By (6.10), $DW_n = W_n D - \overline{P}_n$ with $\overline{P}_n$ defined in (6.9), hence

$$\begin{aligned} W_n^{-1} D W_n &= W_n^{-1}(W_n D + [\![D, W_n]\!]) = D - W_n^{-1}\overline{P}_n \\ &= D - W_n^{-1} P_n - W_n^{-1} V_n^{-1} A_n V_n, \end{aligned}$$

whence

$$\begin{aligned} R_n &= D - W_n^{-1} P_n - W_n^{-1} V_n^{-1} A_n V_n + W_n^{-1} P_n W_n \\ &= D + W_n^{-1} P_n (W_n - \mathbf{1}) - W_n^{-1} V_n^{-1} A_n V_n \\ &= D + W_n^{-1} P_n (W_n - \mathbf{1}) \\ &\quad - W_n^{-1} V_n^{-1} A_n V_n (\mathbf{1} - W_n) - W_n^{-1} V_n^{-1} A_n V_n W_n \\ &= D + W_n^{-1} P_n (W_n - \mathbf{1}) \\ &\quad + W_n^{-1} V_n^{-1} A_n V_n (W_n - \mathbf{1}) - W_n^{-1} V_n^{-1} A_n V_n W_n. \end{aligned}$$

Next, set

$$\begin{aligned} P_{n+1} &:= W_n^{-1}(P_n + V_n^{-1} A_n V_n)(W_n - \mathbf{1}), \\ \widehat{D}_{n+1} &:= \widehat{D}_n + A_n, \end{aligned}$$

then

$$R_n = D + P_{n+1} - V_n^{-1} A_n V_{n+1},$$

and so we come to the identity of the form (6.8) with $l = n + 1$:

$$D + P_{n+1} = V_{n+1}(\widehat{D}_n + P)V_n + V_n^{-1} A_n V_{n+1} = V_{n+1}(\widehat{D}_{n+1} + P)V_n.$$

This completes the formal algebraic component of the step $n \rightsquigarrow n + 1$. As the reader can see by comparing the present reformulation of the inductive procedure with its original form in [22], all the key objects remain the same, and so the required norm estimates from [22], concluding the induction step, can be repeated almost verbatim.

Specifically, one proves as in [22] that

$$\| P_{n+1} \|_{s_{n+1}} \le \theta_{n+1}^{2^{n+1}}.$$

This completes the proof of Lemma 6.4. ∎

To derive the claim of Theorem 3 from Lemma 6.4, it remains to take the norm-limits of $V_n^{\pm 1}$, D_n, $\widehat{D}_n$, and P_n, which exist due to the inductive norm-bounds. In particular, the terms A_l of the series $\sum_\nu A_l$ have rapidly decaying norms, so we can define a diagonal matrix $\widehat{D}$ by $\widehat{D} := D + \sum_\nu A_\nu$. The sequence $(P_n)_{n\geq 0}$ is norm-convergent to 0, and the transformations $\mathrm{Ad}_{V_n}\colon X \mapsto V_n^{-1} X V_n$ also converge in norm. As noted in [14, 22], the unitarity of the limiting conjugation $\mathrm{Ad}_V[\,\cdot\,]$ follows from the fact that the column-vectors of the invertible operator V are eigenvectors of a Hermitian operator with simple spectrum, and all required norm estimates follow from the proof of Lemma 6.4.

A. "Orthometrics" and matrix norms

The main goal of this section is to reveal an important feature of the matrix $\mathbf{U}$ in the eigenbasis of the non-interactive system: a special kind of anisotropy of the decay rate of its matrix elements. To make presentation and basic calculation as explicit and clear as possible, we restrict here our analysis to the case of fermions in the one-dimensional lattice $\mathbf{Z}$, but an extension to $\mathbf{Z}^d$ with any $d \geq 1$ is quite straightforward.

As was shown in Section 2.2, the decay of the matrix elements of $\mathbf{U}$ in the eigenbasis of the non-interacting Hamiltonian admits the bound (cf. (2.5))

$$|\langle \boldsymbol{\psi}_{\mathbf{x}} \mid \mathbf{U} \mid \boldsymbol{\psi}_{\mathbf{y}} \rangle| \leq \sum_{z \in \mathbf{Z}} \mathrm{e}^{-t f(\mathbf{x},\mathbf{y},z)}, \quad t > 0,$$

where

$$f(\mathbf{x},\mathbf{y},z) := (|x_1 - z| + |x_2 - (z+1)| + |y_1 - z| + |y_2 - (z+1)|).$$

Grouping the four terms in $f(\mathbf{x},\mathbf{y},z)$ by two as below, we get

$$\begin{aligned} |x_1 - z| + |y_1 - z| &\geq |x_1 - y_1|, \\ |x_2 - (z+1)| + |y_2 - (z+1)| &\geq |x_2 - y_2|, \end{aligned}$$

hence $f(\mathbf{x},\mathbf{y},z) \geq \|\mathbf{x} - \mathbf{y}\|_1$. Now, group the same four terms in a different way and get, by virtue of the triangle inequality $|w - (z+1)| \geq |w - z| - 1$,

$$|x_1 - z| + |x_2 - (z+1)| \geq |x_1 - z| + |x_2 - z| - 1, \tag{A.1}$$

$$|y_1 - z| + |y_2 - (z+1)| \geq |y_1 - z| + |y_2 - z| - 1. \tag{A.2}$$

Furthermore, for any real numbers $u \le v$ and z,

$$|u-z|+|v-z| = \begin{cases} z-u+v-z = |v-u| & \text{if } z \in [u,v], \\ z-u+z-v = |v-u|+2|z-v| & \text{if } z > v, \\ u-z+v-z = |v-u|+2|z-u| & \text{if } z < u \end{cases}$$
$$= |v-u| + 2\,\mathrm{dist}(z,[u,v]),$$

whence, letting $\emptyset(\cdot) := \mathrm{diam}(\cdot)$ and recalling $\mathcal{Z}^2 \ni \mathbf{u} \cong \{u_1,u_2\} \subset \mathbf{Z}$,

$$\begin{aligned} |x_1-z|+|x_2-(z+1)| &= |x_1-x_2| + 2\,\mathrm{dist}(z,[x_1,x_2]) - 1 \\ &= \emptyset(\mathbf{x}) + 2\,\mathrm{dist}(z,[x_1,x_2]) - 1, \\ |y_1-z|+|y_2-(z+1)| &= |y_1-y_2| + 2\,\mathrm{dist}(z,[y_1,y_2]) - 1 \\ &= \emptyset(\mathbf{y}) + 2\,\mathrm{dist}(z,[y_1,y_2]) - 1. \end{aligned}$$

It is readily seen that

$$\begin{aligned} \mathrm{dist}(z,[x_1,x_2]) &\ge \mathrm{dist}(z,[x_1,x_2]\cup[y_1,y_2]), \\ \mathrm{dist}(z,[y_1,y_2]) &\ge \mathrm{dist}(z,[x_1,x_2]\cup[y_1,y_2]). \end{aligned}$$

Let $\widetilde{\mathcal{B}}(\mathbf{x},\mathbf{y})$ be the smallest interval containing $[x_1-1,x_2+1]\cup[y_1-1,y_2+1]$, then the sum $f(\mathbf{x},\mathbf{y},z)$ of the left-hand side terms in (A.1)–(A.2) admits the lower bound

$$\begin{aligned} f(\mathbf{x},\mathbf{y},z) &\ge \emptyset(\mathbf{x}) + \emptyset(\mathbf{y}) + 2\,\mathrm{dist}(z,[x_1,x_2]) + 2\,\mathrm{dist}(z,[y_1,y_2]) \\ &\ge \emptyset(\mathbf{x}) + \emptyset(\mathbf{y}) + 4\,\mathrm{dist}(z,\widetilde{\mathcal{B}}(\mathbf{x},\mathbf{y})) \end{aligned}$$

Here, the terms (-1) are absorbed in padding $[x_1,x_2]\cup[y_1,y_2]$ with a belt of width 1, which gives rise to a larger interval $\widetilde{\mathcal{B}}(\mathbf{x},\mathbf{y})$. Therefore,

$$2f(\mathbf{x},\mathbf{y},z) \ge \|\mathbf{x}-\mathbf{y}\|_1 + \emptyset(\mathbf{x}) + \emptyset(\mathbf{y}) + 4\,\mathrm{dist}(z,\widetilde{\mathcal{B}}(\mathbf{x},\mathbf{y}))$$

whence

$$\begin{aligned} |\langle \boldsymbol{\psi}_{\mathbf{x}} \mid \mathbf{U} \mid \boldsymbol{\psi}_{\mathbf{y}}\rangle| &\le \mathrm{e}^{-\frac{t}{2}(\|\mathbf{x}-\mathbf{y}\|_1+\emptyset(\mathbf{x})+\emptyset(\mathbf{y}))} \sum_{z\in\mathbf{Z}} \mathrm{e}^{-2t\,\mathrm{dist}(z,\widetilde{\mathcal{B}}(\mathbf{x},\mathbf{y}))} \\ &\le C_t \mathrm{e}^{-\frac{t}{2}(\|\mathbf{x}-\mathbf{y}\|_1+\tilde{\varrho}(\mathbf{x},\mathbf{y}))}, \\ \tilde{\varrho}(\mathbf{x},\mathbf{y}) &:= \emptyset(\mathbf{x}) + \emptyset(\mathbf{y}). \end{aligned}$$

Obviously, the function $\tilde{\varrho}\colon \mathcal{Z}^2 \times \mathcal{Z}^2 \to \mathbf{R}_+$ verifies the triangle inequality:

$$\begin{aligned} \tilde{\varrho}(\mathbf{x},\mathbf{z}) &= \emptyset(\mathbf{x}) + \emptyset(\mathbf{z}) \le \emptyset(\mathbf{x}) + \emptyset(\mathbf{y}) + \emptyset(\mathbf{y}) + \emptyset(\mathbf{z}) \\ &= \tilde{\varrho}(\mathbf{x},\mathbf{y}) + \tilde{\varrho}(\mathbf{y},\mathbf{z}) \end{aligned}$$

and so does, therefore, the function

$$
\begin{aligned}
\boldsymbol{\mathcal{Z}}^2 \times \boldsymbol{\mathcal{Z}}^2 \ni (\mathbf{x}, \mathbf{y}) \mapsto \mathfrak{d}(\mathbf{x}, \mathbf{y}) &:= \varrho(\mathbf{x}, \mathbf{y}) + \tilde{\varrho}(\mathbf{x}, \mathbf{y}), \\
\varrho(\mathbf{x}, \mathbf{y}) &:= \|\mathbf{x} - \mathbf{y}\|_1.
\end{aligned}
$$

Concluding, we have

$$
\begin{aligned}
|\langle \boldsymbol{\psi}_{\mathbf{x}} \mid \mathbf{U} \mid \boldsymbol{\psi}_{\mathbf{y}} \rangle| &\leq C_t \mathrm{e}^{-\frac{t}{2}(\boldsymbol{\emptyset}(\mathbf{x}) + \boldsymbol{\emptyset}(\mathbf{y}) + \|\mathbf{x}-\mathbf{y}\|_1)} \\
&= C_t \mathrm{e}^{-\frac{t}{2}\mathfrak{d}(\mathbf{x}, \mathbf{y})}, \\
\mathfrak{d} \colon (\mathbf{x}, \mathbf{y}) &\mapsto \|\mathbf{x} - \mathbf{y}\|_1 + \tilde{\varrho}(\mathbf{x}, \mathbf{y}),
\end{aligned}
$$

where $\mathfrak{d}(\cdot, \cdot)$ verifies the triangle inequality.

Furthermore,

- for all $(z, u) \in \mathbf{x} \times \mathbf{y}$,

$$
\begin{aligned}
|z - u| &\leq |z - x_1| + |y_1 - u| + |x_1 - y_1| \\
&\leq \boldsymbol{\emptyset}(\mathbf{x}) + \boldsymbol{\emptyset}(\mathbf{y}) + \|\mathbf{x} - \mathbf{y}\|_1 \\
&\leq \mathfrak{d}(\mathbf{x}, \mathbf{y}),
\end{aligned}
$$

- for all $(z, u) \in \mathbf{x} \times \mathbf{x}$

$$
|z - u| \leq \boldsymbol{\emptyset}(\mathbf{x}) \leq \mathfrak{d}(\mathbf{x}, \mathbf{y}),
$$

- for all $(z, u) \in \mathbf{y} \times \mathbf{y}$

$$
|z - u| \leq \boldsymbol{\emptyset}(\mathbf{y}) \leq \mathfrak{d}(\mathbf{x}, \mathbf{y}),
$$

yielding the bound

$$
\operatorname{diam}(\mathbf{x} \cup \mathbf{y}) \leq \mathfrak{d}(\mathbf{x}, \mathbf{y}). \tag{A.3}
$$

Incidentally, $\tilde{\varrho}$ is symmetric, and it can be re-defined on each $(\mathbf{u}, \mathbf{u})$ by setting $\tilde{\varrho}(\mathbf{u}, \mathbf{u}) := 0$, thus becoming a bona fide metric, along with $\mathfrak{d} = \varrho + \tilde{\varrho}$. However, it is just the triangle inequality that one needs for the mapping

$$
\operatorname{Mat}(\boldsymbol{\mathcal{Z}}^2) \ni A \mapsto |||A|||_t := \sup_{\mathbf{x}} \sum_{\mathbf{y}} \mathrm{e}^{-t\mathfrak{d}(\mathbf{x}, \mathbf{y})} |A_{\mathbf{x}\mathbf{y}}|
$$

to be a matrix (i.e., multiplicative) norm, as shows the calculation, well known in the case where $\mathfrak{d}$ is a genuine metric:

$$
\begin{aligned}
|||AB|||_t &= \sup_x \sum_y \mathrm{e}^{t\mathfrak{d}(x,y)} |(AB)_{xy}| \leq \sup_x \sum_y \mathrm{e}^{t\mathfrak{d}(x,y)} \sum_z |A_{xz}||B_{zy}| \\
&\leq \sup_x \sum_y \mathrm{e}^{t\mathfrak{d}(x,z)} \mathrm{e}^{t\mathfrak{d}(z,y)} \sum_z |A_{xz}||B_{zy}| \quad \text{(by (2.6))}
\end{aligned}
$$

$$
\begin{aligned}
&= \sup_x \sum_z \mathrm{e}^{t\mathfrak{d}(x,z)} |A_{xz}| \sum_y \mathrm{e}^{t\mathfrak{d}(z,y)} |B_{zy}| \\
&\le \sup_x \sum_z \mathrm{e}^{t\mathfrak{d}(x,z)} |A_{xz}| \Big(\sup_u \sum_y \mathrm{e}^{t\mathfrak{d}(u,y)} |B_{uy}| \Big) \\
&= |||A|||_t \, |||B|||_t . \qquad \text{(A.4)}
\end{aligned}
$$

Of course, the symmetry of $\mathfrak{d}$ is quite welcome, as the decay of $|\langle \boldsymbol{\psi}_{\mathbf{x}} \mid \mathbf{U} \mid \boldsymbol{\psi}_{\mathbf{y}} \rangle|$ is to be compared to that of a symmetric function $(\mathbf{x}, \mathbf{y}) \mapsto |\boldsymbol{\lambda}_{\mathbf{x}} - \boldsymbol{\lambda}_{\mathbf{y}}|$.

Remark A.1. The following geometrical observations clarify the discussion of “orthometrics” in Section 2.3. Consider for simplicity the particular case of the 1-particle configuration space $\mathbf{Z}^1$. Then, for any $\mathbf{u} = \langle u_1, u_2 \rangle \in \boldsymbol{\mathcal{Z}}^2 = \mathbf{Z}^{(2)}$, letting

$$
c_{\mathbf{u}} := \frac{u_1 + u_2}{2}, \quad \mathbf{c}_{\mathbf{u}} := (c_{\mathbf{u}}, c_{\mathbf{u}}) \in \mathbf{R}^2 \supset \boldsymbol{\mathcal{Z}}^2,
$$

we have

$$
\begin{aligned}
\emptyset(\mathbf{u}) &= |u_2 - u_1| = |u_2 - c_{\mathbf{u}}| + |c_{\mathbf{u}} - u_1| = \|\mathbf{u} - \mathbf{c}_{\mathbf{u}}\|_1 \\
&= \operatorname{dist}(\mathbf{u}, \mathcal{L}), \quad \mathcal{L} := \{(c, c) \in \mathbf{Z}^2 \subset \mathbf{R}^2, c \in \mathbf{Z}\},
\end{aligned}
$$

and therefore,

$$
\tilde{\varrho}(\mathbf{x}, \mathbf{y}) := \emptyset(\mathbf{x}) + \emptyset(\mathbf{y}) = \operatorname{dist}(\mathbf{x}, \mathcal{L}) + \operatorname{dist}(\mathbf{y}, \mathcal{L}).
$$

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