

On the zeros of partition functions with multi-spin interactions

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Abstract. Let $X_1, \dots, X_n$ be probability spaces, let X be their direct product, let $\phi_1, \dots, \phi_m : X \rightarrow \mathbb{C}$ be random variables, each depending only on a few coordinates of $x = (x_1, \dots, x_n)$, and let $f = \phi_1 + \dots + \phi_m$. The expectation $\mathbf{E} e^{\lambda f}$, where $\lambda \in \mathbb{C}$, appears in statistical physics as the partition function of a system with multi-spin interactions, and also in combinatorics and computer science, where it is known as the partition function of edge-coloring models, tensor network contractions, or a Holant polynomial. Assuming that each ϕ_i is 1-Lipschitz in the Hamming metric of X , that each $\phi_i(x)$ depends on at most $r \geq 2$ coordinates $x_1, \dots, x_n$ of $x \in X$, and that for each j there are at most $c \geq 1$ functions ϕ_i that depend on the coordinate x_j , we prove that $\mathbf{E} e^{\lambda f} \neq 0$ provided $|\lambda| \leq (3c\sqrt{r-1})^{-1}$ and that the bound is sharp up to a constant factor. Taking a scaling limit, we prove a similar result for functions $\phi_1, \dots, \phi_m : \mathbb{R}^n \rightarrow \mathbb{C}$ that are 1-Lipschitz in the ℓ^1 metric of $\mathbb{R}^n$ and where the expectation is taken with respect to the standard Gaussian measure in $\mathbb{R}^n$. As a corollary, the value of the expectation can be efficiently approximated, provided λ lies in a slightly smaller disc.

1. Introduction and the main results

We investigate the complex zeros and computational complexity of functionals of a particular type, which appear under different names in statistical physics, computer science, and combinatorics. The functionals can be viewed as the partition function of a spin system with multiple spin interactions (statistical physics) or as the partition function of a hypergraph edge-coloring model, also known as a tensor network contraction, or as a Holant polynomial (combinatorics and computer science).

In what follows, we consider functions on the direct product $X_1 \times \dots \times X_n$ of probability spaces. For a point $x \in X$, $x = (x_1, \dots, x_n)$, we refer to $x_j \in X_j$ as the j th coordinate of x . The *Hamming distance* between two points $x, y \in X$ is the number of coordinates where they differ:

$$\text{dist}(x, y) = |j : x_j \neq y_j|, \quad \text{where } x = (x_1, \dots, x_n) \text{ and } y = (y_1, \dots, y_n).$$

Mathematics Subject Classification 2020: 82B20 (primary); 30C15, 68R05, 68W25 (secondary).

Keywords: partition function, spin systems, hypergraphs, edge-coloring models, zeros, algorithm.

We prove the following main result.

Theorem 1.1. *Let $X_1, \dots, X_n$ be probability spaces, let $X = X_1 \times \dots \times X_n$ be the product space and let $\phi_1, \dots, \phi_m : X \rightarrow \mathbb{C}$ be measurable functions. Assume that*

- (1) *each function ϕ_i is 1-Lipschitz in the Hamming metric, that is,*

$$|\phi_i(x) - \phi_i(y)| \leq 1$$

provided $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ differ in one coordinate;

- (2) *each function ϕ_i depends only on at most $r \geq 2$ coordinates, that is, for each $i = 1, \dots, m$ there is a subset $J_i \subset \{1, \dots, n\}$ with $|J_i| \leq r$ such that*

$$\phi_i(x_1, \dots, x_n) = \phi_i(y_1, \dots, y_n) \quad \text{whenever } x_j = y_j \text{ for all } j \in J_i;$$

- (3) *for every $j = 1, \dots, n$, there are at most $c \geq 1$ functions ϕ_i that depend on the coordinate x_j , that is, $|i : j \in J_i| \leq c$ for all j .*

Let

$$f = \sum_{i=1}^m \phi_i,$$

and suppose that λ is a complex number such that

$$|\lambda| \leq \frac{1}{3c\sqrt{r-1}}.$$

Then,

$$\mathbf{E} e^{\lambda f} \neq 0.$$

Moreover, if, additionally,

$$|\phi_i(x)| \leq L \quad \text{for all } x \in X \text{ and } i = 1, \dots, m \tag{1.1}$$

and some $L > 0$, then

$$e^{m|\lambda|L} \geq |\mathbf{E} e^{\lambda f}| \geq e^{-m|\lambda|L} \left(\cos \frac{\pi}{4\sqrt{r-1}} \right)^n. \tag{1.2}$$

We prove Theorem 1.1 in Section 2, and in Section 4, we show that the bound for $|\lambda|$ is optimal, up to a constant factor. The method of proof extends those of [4, 6]. The dependence on r is worth noting. One approach frequently used for problems of this type is the *cluster expansion* method; see [23] for a recent survey. In the situation of Theorem 1.1, it apparently gives only $|\lambda| = \Omega(1/rc)$ as a bound for the zero-free region, while also requiring $|\phi_i|$ to remain uniformly bounded [10].

The dependence on r allows us to obtain a zero-free region for a partition function which can be considered as a scaling limit of that of Theorem 1.1. We consider the standard Gaussian probability measure in $\mathbb{R}^n$ with density

$$\frac{1}{(2\pi)^{n/2}} \exp \left\{ -\frac{1}{2}(x_1^2 + \dots + x_n^2) \right\} \quad \text{for } x = (x_1, \dots, x_n)$$

and prove the following result.

Theorem 1.2. *Let $\phi_1, \dots, \phi_m : \mathbb{R}^n \rightarrow \mathbb{C}$ be functions on Euclidean space $\mathbb{R}^n$, endowed with the standard Gaussian probability measure. Assume that*

- (1) *each function ϕ_i is 1-Lipschitz in the ℓ^1 metric of $\mathbb{R}^n$, that is,*

$$|\phi_i(x_1, \dots, x_n) - \phi_i(y_1, \dots, y_n)| \leq \sum_{i=1}^n |x_i - y_i|;$$

- (2) *each function ϕ_i depends only on at most $r \geq 2$ coordinates;*

- (3) *for every $j = 1, \dots, n$, there are at most $c \geq 1$ functions ϕ_i that depend on the coordinate x_j .*

Let

$$f = \sum_{i=1}^m \phi_i$$

and suppose that λ is a complex number such that

$$|\lambda| < \frac{1}{6c\sqrt{r}}.$$

Then,

$$\mathbf{E} e^{\lambda f} \neq 0.$$

Moreover, if, additionally,

$$|\phi_i(x)| \leq L \quad \text{for all } x \in \mathbb{R}^n \text{ and } i = 1, \dots, m \quad (1.3)$$

and some $L > 0$, then

$$e^{m|\lambda|L} \geq |\mathbf{E} e^{\lambda f}| \geq e^{-m|\lambda|L} e^{-\pi^2 n/32r}. \quad (1.4)$$

We prove Theorem 1.2 in Section 3.

Suppose we want to approximate the value of $\mathbf{E} e^{\lambda f}$ efficiently. As we are dealing with complex numbers, it is convenient to adopt the following definition: we say that a complex number $z_1 \neq 0$ approximates another complex number $z_2 \neq 0$ within relative error ε if $z_1 = e^{w_1}$ and $z_2 = e^{w_2}$ for some w_1 and w_2 such that $|w_1 - w_2| \leq \varepsilon$.

Let us fix some $0 < \delta < 1$. From by now standard method of polynomial interpolation [3, 31], it follows that for any λ such that

$$|\lambda| \leq \frac{1 - \delta}{3c\sqrt{r-1}} \quad (\text{in Theorem 1.1}) \quad \text{or} \quad |\lambda| \leq \frac{1 - \delta}{6c\sqrt{r}} \quad (\text{in Theorem 1.2})$$

approximating the value of $\mathbf{E} e^{\lambda f}$ within relative error $0 < \varepsilon < 1$ reduces to the computation of the m^k expectations

$$\mathbf{E}(\phi_{i_1} \cdots \phi_{i_k}) \quad \text{for all } 1 \leq i_1, i_2, \dots, i_k \leq m,$$

where

$$k = O_\delta(\ln(m+n) - \ln \varepsilon).$$

We sketch the algorithm in Section 5. In many applications of Theorem 1.1, the probability spaces X_j are finite. Assuming that each X_j contains at most q elements, computing each expectation $\mathbf{E}(\phi_{i_1} \cdots \phi_{i_k})$ by the direct enumeration takes $O(q^{rk})$ time, provided we have an access to the values of $\phi_i(x)$ for a given $x \in X$. If the parameter r is fixed in advance, we obtain a quasi-polynomial algorithm of $(q(m+n))^{O(\ln(m+n) - \ln \varepsilon)}$ complexity to approximate $\mathbf{E} e^{\lambda f}$ within relative error $0 < \varepsilon < 1$. Moreover, the general technique of Patel and Regts [31] allows one to speed it up to a genuinely polynomial time algorithm of $(q(m+n)\varepsilon^{-1})^{O(1)}$ complexity, provided the parameter c is also fixed in advance. In the context of Theorem 1.2, the expectations $\mathbf{E}(\phi_{i_1} \cdots \phi_{i_k})$ are represented by integrals in the space of dimension at most kr and often can be efficiently computed or approximated with high accuracy, assuming again that r is fixed in advance. In that case, we also obtain an algorithm of quasi-polynomial complexity to approximate $\mathbf{E} e^{\lambda f}$.

As we mentioned, the expectation $\mathbf{E} e^{\lambda f}$ appears in several different, though closely related contexts.

1.1. Statistical physics: partition functions of systems with multi-spin interactions

Here, we describe the statistical physics context of Theorem 1.1. Suppose that we have a system of n particles, where the j th particle can be in a state described by a point $x_j \in X_j$, also called the *spin* of the particle. The vector $x = (x_1, \dots, x_n)$ of all spins is called a *spin configuration*. The particles interact in various ways, and the energy of a spin configuration is given by a function $-f(x_1, \dots, x_n)$. Then, for a real $\lambda > 0$, interpreted as the *inverse temperature*, the value of $\mathbf{E} e^{\lambda f}$ is the *partition function* of the system; see, for example, [14] for the general setup. If the function f is written as a sum of ϕ_i , as in Theorem 1.1, then the energy of the system is a sum

over subsystems, each containing at most r particles, and each particle participating in at most c subsystems.

The idea to relate complex zeros of the partition function $\lambda \mapsto \mathbf{E} e^{\lambda f}$ to the physical phenomenon of *phase transition* apparently goes back to Mayer [29], when it became known as “Mayer’s conjecture”; see Section 4.12.3 of [14] for a discussion. It then took off with the classical works of Lee and Yang [25, 39], who showed that zero-free regions like the one described by Theorem 1.1 correspond to regimes with no phase transition.

In terms of Theorem 1.1, the most studied case is that of $r = 2$, which includes the classical Ising, Potts, and Heisenberg models; see [14]. In that case, we have a (finite) graph $G = (V, E)$ with set V of vertices and set E of edges. The particles are identified with the vertices of G , the spins are ± 1 in the case of the Ising model, elements of some finite set in the case of the Potts model, or vectors in Euclidean space, in the case of the Heisenberg model. The interactions are pairwise and described by functions attached to the edges of E . Hence, in the context of Theorem 1.1, we have $r = 2$ and c is the largest degree of a vertex of G . Starting with [25], the complex zeros of the partition function of systems with pairwise interactions were actively studied in many papers. We refer to [14] for earlier, and to [8, 16, 32, 33] for more recent works.

Zero-free regions for systems with multiple spin interactions were considered by Suzuki and Fisher [36], again in connection with phase transition; see also [26, 28] for recent developments. In this case, the particles are identified with the vertices of a hypergraph, while functions attached to the edges (sometimes called hyperedges) of the hypergraph describe interactions. In terms of Theorem 1.1, the maximum number of vertices of an edge of the hypergraph is r , while c is the largest degree of vertex. The papers [36] and [26], respectively, [28], consider rather special *ferromagnetic*, respectively, *antiferromagnetic*, types of interaction, so their results are not directly comparable to ours. While [26, 28, 36] say more about those specific models, our Theorem 1.1 allows one to handle a wider class of interactions.

1.2. Statistical physics: partition functions of interacting particles with an external field

Here, we give an example of how Theorem 1.2 applies to systems of pairwise interacting particles, cf. [14]. For an integer $N > 1$, let us consider N particles represented as vectors $x^{(1)}, \dots, x^{(N)}$ in $\mathbb{R}^d$, chosen independently at random from the standard Gaussian distribution in $\mathbb{R}^d$. Suppose that there are pairwise repulsive constant forces, so that the total energy of the system is

$$-f(x^{(1)}, \dots, x^{(N)}) = -\sum_{i \neq j} \|x^{(i)} - x^{(j)}\|,$$

where $\|\cdot\|$ is the standard Euclidean norm in $\mathbb{R}^d$. The energy is minimized when the particles are far away from each other. However, the probability of a configuration with large pairwise distances is small, so that the Gaussian density can be interpreted as an external force pushing the particles towards the origin. In terms of Theorem 1.2, we have $n = dN$, $m = \binom{N}{2}$, $r = 2d$, and $c = N$. Theorem 1.2 establishes a zero-free region of the order

$$|\lambda| = \Omega\left(\frac{1}{N\sqrt{d}}\right)$$

for the partition function $\mathbf{E} e^{\lambda f}$. This model is related to the old problem of finding a configuration of points on the unit sphere in $\mathbb{R}^d$ that maximizes the sum of pairwise distances between points, see [2] (we note that for large d the standard Gaussian measure in $\mathbb{R}^d$ is concentrated around the sphere $\|x\| = \sqrt{d}$).

Essentially identical models described by Theorem 1.1 are studied in connection with questions in combinatorics and theoretical computer science.

1.3. Combinatorics and computer science: edge-coloring models and tensor network contractions

Let $G = (V, E)$ be a graph. Suppose now that every edge of G can be in one of the q states $\{1, \dots, q\}$, typically called “colors”. In terms of Theorem 1.1, we have $n = |E|$ and $X_1 = \dots = X_n = \{1, \dots, q\}$, so that $X = \{1, \dots, q\}^E$. Suppose further, that to each vertex v of G a function $\psi_v : X_1 \times \dots \times X_n \rightarrow \mathbb{C}$ is attached, which depends only on the colors of the edges of G that contain v . The expression

$$\sum_{x \in X} \prod_{v \in V} \psi_v(x) \tag{1.5}$$

is known as the *tensor network contraction*, or the partition function of the *edge coloring model*, or as a *Holant polynomial*; see [34] for the relations between different models and references therein. For relations with spin systems in statistical physics and also knot invariants, see [12]. Assuming that $\psi_v(x) \neq 0$ for all x and v , we can write $\psi_v(x) = \exp\{\phi_v(x)\}$ and (1.5) can be written as the scaled expectation $\mathbf{E} e^{\lambda f}$ of Theorem 1.1, assuming the uniform probability measure in each space X_j . In the context of Theorem 1.1, we have $c = 2$, while r is equal to the maximum degree of a vertex of G . Note that, compared to the graph interpretation of the Ising and Potts models from Section 1.1, the parameters r and c switch places.

An expression similar to (1.5) expression can be built for hypergraphs, in which case r is still the maximum degree of vertex, while c is the maximum size of an edge; see, for example, Chapter 6 of [38] for the discussion of the partition function of a k -satisfiability problem.

In combinatorics and computer science, there is a lot of interest in efficient approximation of (1.5), also in connection with zero-free regions [1, 10, 15, 20, 34], since many interesting counting problems on graphs and hypergraphs can be stated as a problem of computing (1.5). To illustrate, in Section 6, we consider the example of graph coloring in some detail; see [38] for further motivation.

1.4. Numerical integration in higher dimensions

Computationally efficient numerical integration of multivariate functions is an old problem that is often associated with “the curse of dimensionality”. The most spectacular success is achieved for integration of log-concave functions on $\mathbb{R}^n$ via the Monte Carlo Markov chain method; see [27] for a survey. Deterministic methods enjoyed much less success. In [18], the authors, using the deterministic *decay of correlations* approach, considered integration of functions over the unit cube $[0, 1]^n$. The model of [18] fits the setup of our Theorem 1.1, if we choose $X_1 = \dots = X_n = [0, 1]$ endowed with the Lebesgue probability measure. The results of [18] appear to be weaker than our Theorem 1.1, in the dependence on both parameters r and c , as well as in the class of allowed functions ϕ_i .

Theorem 1.2 allows us to integrate efficiently some functions that are decidedly not log-concave (or log-convex), for example, if we choose $\phi_i(x) = ||x_i| - 1|$ for some i , thus reaching beyond the realm of functions efficiently integrated by randomized methods.

1.5. Other applications

Zero-free regions of partition functions of the type covered by Theorem 1.1 turn out to be relevant to the decay of correlations [17, 35], to the mixing time of Markov chains [11], to the validity of the Central Limit Theorem for combinatorial structures [30] and to other related algorithmic applications [22].

2. Proof of Theorem 1.1

We will be proving a more general version of the theorem.

Theorem 2.1. *Let $X_1, \dots, X_n$ be probability spaces, and let $X = X_1 \times \dots \times X_n$ be the product space and let $\phi_1, \dots, \phi_m : X \rightarrow \mathbb{C}$ be measurable functions.*

For $i = 1, \dots, m$, let $J_i \subset \{1, \dots, n\}$ be the set of the coordinates that ϕ_i depends on, that is,

$$\phi_i(x_1, \dots, x_n) = \phi_i(y_1, \dots, y_n) \quad \text{whenever } x_j = y_j \text{ for all } j \in J_i,$$

and where J_i is the minimal set under inclusion with this property.

Assume that each function ϕ_i is 1-Lipschitz in the Hamming metric, that is,

$$|\phi_i(x) - \phi_i(y)| \leq 1$$

provided $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ differ in one coordinate and that for some $r \geq 2$, we have

$$|J_i| \leq r \quad \text{for } i = 1, \dots, m$$

so that every function ϕ_i depends on at most r coordinates.

Let

$$f = \sum_{i=1}^m \lambda_i \phi_i$$

for some $\lambda_i \in \mathbb{C}$ such that

$$\sum_{i: j \in J_i} |\lambda_i| \leq \frac{1}{3\sqrt{r-1}} \quad \text{for } j = 1, \dots, n.$$

Then,

$$\mathbf{E} e^f \neq 0.$$

Moreover, if, additionally,

$$|\phi_i(x)| \leq L \quad \text{for all } x \in X \text{ and } i = 1, \dots, m \quad (2.1)$$

and some $L > 0$, then

$$e^{\alpha m L} \geq |\mathbf{E} e^f| \geq e^{-\alpha m L} \left(\cos \frac{\pi}{4\sqrt{r-1}} \right)^n, \quad \text{where } \alpha = \max_{i=1, \dots, m} |\lambda_i|. \quad (2.2)$$

Choosing in Theorem 2.1,

$$\lambda_i = \lambda \quad \text{for } i = 1, \dots, m,$$

we deduce Theorem 1.1.

We start with the following lemma.

Lemma 2.2. *Let X be a probability space and let $f : X \rightarrow \mathbb{C}$ be an integrable function. Suppose that $f(x) \neq 0$ for all $x \in X$ and, moreover, for any two points $x, y \in X$, the angle between $f(x)$ and $f(y)$, considered as non-zero vectors in $\mathbb{R}^2 = \mathbb{C}$ does not exceed θ for some $0 \leq \theta < 2\pi/3$. Then,*

$$|\mathbf{E} f| \geq \left(\cos \frac{\theta}{2} \right) \mathbf{E} |f|.$$

Proof. This is Lemma 3.3 from [6], see also Lemma 3.6.3 of [3]. The idea of the proof is to compare both sides of the inequality to the expected length of the orthogonal projection of $f(x)$ onto the bisector of the angle enclosing the range of f . ■

We will need some technical inequalities.

Lemma 2.3. *The following inequalities hold:*

$$\begin{aligned} \left(\cos \frac{\alpha}{\sqrt{k}} \right)^k &\geq \cos \alpha \quad \text{for } 0 \leq \alpha \leq \frac{\pi}{2} \text{ and } k \geq 1, \\ \sin(\tau \alpha) &\geq \tau \sin \alpha \quad \text{for } 0 \leq \alpha \leq \frac{\pi}{2} \text{ and } 0 \leq \tau \leq 1, \\ |e^z - 1| &\leq 2|z| \quad \text{for } z \in \mathbb{C} \text{ such that } |z| \leq 1. \end{aligned}$$

Proof. The inequality of Part (1) is proved, for example, in [5], see Lemma 6.1 therein. To prove the inequality of Part (2), let

$$f(\alpha) = \sin(\tau \alpha) \quad \text{and} \quad g(\alpha) = \tau \sin \alpha.$$

Then, $f(0) = g(0) = 0$ and

$$f'(\alpha) = \tau \cos(\tau \alpha) \geq \tau \cos \alpha = g'(\alpha),$$

from which the proof follows. To prove the inequality of Part (3), we note that

$$|e^z - 1| = \left| \sum_{k=1}^{\infty} \frac{z^k}{k!} \right| \leq \sum_{k=1}^{\infty} \frac{|z|^k}{k!} = e^{|z|} - 1,$$

and hence, it suffices to check the inequality, assuming that z is non-negative real. Since the function

$$\frac{e^z - 1}{z} = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!}$$

is increasing for $z \geq 0$, it suffices to check the inequality at $z = 1$, where it states that

$$|e - 1| \leq 2. \quad \blacksquare$$

Let $X_1, \dots, X_n$ be probability spaces and let $f : X_1 \times \dots \times X_n \rightarrow \mathbb{C}$ be an integrable function. For a subset $S \subset \{1, \dots, n\}$, by $\mathbf{E}_S f$ we denote the conditional expectation obtained by integrating f over the coordinates $x_j \in X_j$ with $j \in S$. Thus, $\mathbf{E}_S f$ is a function of the coordinates x_j with $j \notin S$. In particular, for $S = \emptyset$, we have $\mathbf{E}_S f = f$ and for $S = \{1, \dots, n\}$, we have $\mathbf{E}_S = \mathbf{E} f$.

Now, we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let

$$\theta = \frac{\pi}{2\sqrt{r-1}}. \quad (2.3)$$

For a subset $S \subset \{1, \dots, n\}$, we prove by induction on $|S|$ the following.

Statement 2.4. Let functions ϕ_i , f and complex numbers λ_i be as in Theorem 2.1. For every $S \subset \{1, \dots, n\}$, we have $\mathbf{E}_S e^f \neq 0$, by which we mean that $\mathbf{E}_S e^f \neq 0$ for any choice of the coordinates $x_j \in X_j$ with $j \notin S$. Moreover, for every $j \notin S$, the value of $\mathbf{E}_S e^f \neq 0$, considered as a vector in $\mathbb{R}^2 = \mathbb{C}$, rotates by not more than an angle of θ when only the x_j coordinate of $x = (x_1, \dots, x_n)$ changes, while all other coordinates stay the same.

Once we have Statement 2.4 for $S = \{1, \dots, n\}$, we conclude that $\mathbf{E} e^f \neq 0$, which is what we want to prove.

Suppose that $|S| = 0$, so that $S = \emptyset$ and

$$\mathbf{E}_S e^f = e^f = \exp \left\{ \sum_{i=1}^m \lambda_i \phi_i \right\}.$$

Let $x', x'' \in X$ be two points that differ only in the x_j coordinate. Then,

$$|f(x') - f(x'')| = \left| \sum_{i: j \in J_i} \lambda_i (\phi_i(x') - \phi_i(x'')) \right| \leq \sum_{i: j \in J_i} |\lambda_i| \leq \frac{1}{3\sqrt{r-1}} < \theta,$$

which establishes the base of the induction.

Suppose now that Statement 2.4 holds for every S with $|S| \leq k$, where $0 \leq k < n$. Let us choose an arbitrary $S \subset \{1, \dots, n\}$ such that $|S| = k + 1$ and an index $j \in S$. Applying Lemma 2.2 and the induction hypothesis, we conclude that

$$|\mathbf{E}_S e^f| = |\mathbf{E}_{\{j\}} \mathbf{E}_{S \setminus \{j\}} e^f| \geq \left(\cos \frac{\theta}{2} \right) \mathbf{E}_{\{j\}} |\mathbf{E}_{S \setminus \{j\}} e^f|. \quad (2.4)$$

It follows that

$$\mathbf{E}_S e^f \neq 0.$$

Assuming that $k + 1 < n$, let us pick some index not in S , without loss of generality index n . We need to prove that as the coordinate x_n changes from some value x'_n to some value x''_n , while other coordinates remain the same, the value of $\mathbf{E}_S e^f$ rotates through an angle of at most θ . Let

$$I = \{i : n \in J_i\}$$

be the set of indices i such that ϕ_i depends on x_n .

Without loss of generality, $I \neq \emptyset$. For each $i \in I$, we define two functions ϕ'_i and ϕ''_i , obtained from ϕ_i by fixing the coordinate x_n to x'_n and x''_n , respectively. Although ϕ'_i and ϕ''_i are functions of the first $n - 1$ coordinates $x_1, \dots, x_{n-1}$ of $x \in X$, we formally consider them as functions on X , by ignoring the last coordinate x_n .

Since each function ϕ_i is 1-Lipschitz in the Hamming metric, we have

$$|\phi'_i(x) - \phi''_i(x)| \leq 1 \quad \text{for all } x \in X \text{ and } i \in I. \quad (2.5)$$

Thus, the value of $\mathbf{E}_S e^f$, where we fix $x_n = x'_n$, is

$$\mathbf{E}_S \exp \left\{ \sum_{i \in I} \lambda_i \phi'_i + \sum_{i \notin I} \lambda_i \phi_i \right\}, \quad (2.6)$$

while the value of $\mathbf{E}_S e^{f'}$, where we fix $x_n = x''_n$, is

$$\mathbf{E}_S \exp \left\{ \sum_{i \in I} \lambda_i \phi''_i + \sum_{i \notin I} \lambda_i \phi_i \right\}. \quad (2.7)$$

We pass from (2.6) to (2.7) step by step, replacing one ϕ'_i by ϕ''_i at each step. Our goal is to prove that at each step, the expectation rotates by at most $3|\lambda_i|\pi/2$. Once we prove that, it would follow that the expectation $\mathbf{E}_S e^f$ rotates by at most

$$\frac{3\pi}{2} \sum_{i \in I} |\lambda_i| \leq \frac{\pi}{2\sqrt{r-1}} = \theta,$$

when we replace the value of $x_n = x'_n$ by $x_n = x''_n$.

Let us pick an index in I , without loss of generality index m . We define

$$f' = \lambda_m \phi'_m + \psi \quad \text{and} \quad f'' = \lambda_m \phi''_m + \psi, \quad (2.8)$$

where ψ is the sum of some functions $\lambda_i \phi'_i$, $\lambda_i \phi''_i$, and $\lambda_i \phi_i$, where for each $i \neq m$, we select exactly one of the three functions $\lambda_i \phi'_i$, $\lambda_i \phi''_i$, or $\lambda_i \phi_i$ into the sum for ψ . Hence, our goal is to show that the angle between $\mathbf{E}_S e^{f'} \neq 0$ and $\mathbf{E}_S e^{f''} \neq 0$ does not exceed $3|\lambda_m|\pi/2$.

Since each of the functions ϕ'_i and ϕ''_i is obtained from ϕ_i by specifying some value of x_n , we can apply the induction hypothesis both to f' and to f'' . Let $S_0 = S \cap J_m$ be the set of indices $j \in S$ such that ϕ'_m and ϕ''_m depend on x_j . In particular, $|S_0| \leq r-1$.

If $S_0 = \emptyset$, then

$$\mathbf{E}_S e^{f'} = e^{\lambda_m \phi'_m} \mathbf{E}_S e^{\psi} \quad \text{and} \quad \mathbf{E}_S e^{f''} = e^{\lambda_m \phi''_m} \mathbf{E}_S e^{\psi}.$$

Using (2.5), we conclude the angle between $\mathbf{E}_S e^{f'} \neq 0$ and $\mathbf{E}_S e^{f''} \neq 0$ does not exceed

$$|\lambda_m \phi''_m(x) - \lambda_m \phi'_m(x)| \leq |\lambda_m| \leq \frac{3|\lambda_m|\pi}{2}.$$

Suppose now that $S_0 \neq \emptyset$. Since ϕ'_m and ϕ''_m do not depend on the coordinates x_j with $j \in S \setminus S_0$, from (2.8), we have

$$\begin{aligned} \mathbf{E}_S e^{f''} &= \mathbf{E}_S (e^{\lambda_m \phi''_m} e^{\psi}) = \mathbf{E}_S (e^{\lambda_m (\phi''_m - \phi'_m)} e^{\lambda_m \phi'_m + \psi}) = \mathbf{E} (e^{\lambda_m (\phi''_m - \phi'_m)} e^{f'}) \\ &= \mathbf{E}_{S_0} (e^{\lambda_m (\phi''_m - \phi'_m)} \mathbf{E}_{S \setminus S_0} e^{f'}), \end{aligned}$$

and hence,

$$\begin{aligned}\mathbf{E}_S e^{f''} - \mathbf{E}_S e^{f'} &= \mathbf{E}_{S_0}(e^{\lambda_m(\phi_m'' - \phi_m')} \mathbf{E}_{S \setminus S_0} e^{f'}) - \mathbf{E}_{S_0}(\mathbf{E}_{S \setminus S_0} e^{f'}) \\ &= \mathbf{E}_{S_0}((e^{\lambda_m(\phi_m'' - \phi_m')} - 1) \mathbf{E}_{S \setminus S_0} e^{f'}).\end{aligned}$$

By (2.5) and by Part (3) of Lemma 2.3, we have

$$|e^{\lambda_m(\phi_m'' - \phi_m')} - 1| \leq 2|\lambda_m|.$$

Therefore,

$$|\mathbf{E}_S e^{f''} - \mathbf{E}_S e^{f'}| \leq 2|\lambda_m| \mathbf{E}_{S_0} |\mathbf{E}_{S \setminus S_0} e^{f'}|. \quad (2.9)$$

Iterating (2.4) with f replaced by f' , we obtain that

$$\begin{aligned}|\mathbf{E}_S e^{f'}| &= |\mathbf{E}_{S_0} \mathbf{E}_{S \setminus S_0} e^{f'}| \geq \left(\cos \frac{\theta}{2}\right)^{|S_0|} \mathbf{E}_{S_0} |\mathbf{E}_{S \setminus S_0} e^{f'}| \\ &\geq \left(\cos \frac{\theta}{2}\right)^{r-1} \mathbf{E}_{S_0} |\mathbf{E}_{S \setminus S_0} e^{f'}|.\end{aligned} \quad (2.10)$$

Combining (2.9) and (2.10), we conclude that

$$\frac{|\mathbf{E}_S e^{f''} - \mathbf{E}_S e^{f'}|}{|\mathbf{E}_S e^{f'}|} \leq \frac{2|\lambda_m|}{\cos^{r-1}(\theta/2)}.$$

Recalling formula (2.3) for θ and using Part (1) of Lemma 2.3, we obtain

$$\begin{aligned}\frac{|\mathbf{E}_S e^{f''} - \mathbf{E}_S e^{f'}|}{|\mathbf{E}_S e^{f'}|} &\leq 2|\lambda_m| \left(\cos \frac{\pi}{4\sqrt{r-1}}\right)^{-(r-1)} \\ &\leq 2|\lambda_m| \frac{1}{\cos(\pi/4)} = 2\sqrt{2}|\lambda_m| < 1.\end{aligned}$$

It follows now that the angle, call it β , between $\mathbf{E}_S e^{f'}$ and $\mathbf{E}_S e^{f''}$ is acute and that

$$\sin \beta \leq 2\sqrt{2}|\lambda_m| \leq 3|\lambda_m|.$$

From Part (2) of Lemma 2.3, we have

$$\sin \frac{3|\lambda_m|\pi}{2} \geq 3|\lambda_m| \sin \frac{\pi}{2} = 3|\lambda_m|.$$

Hence, the angle between $\mathbf{E}_S e^{f'}$ and $\mathbf{E}_S e^{f''}$ indeed does not exceed $3|\lambda_m|\pi/2$ and $\mathbf{E}_S e^f$ rotates by not more than an angle of θ when the value of one of the coordinates x_j with $j \notin S$ changes, while the others remain the same. This completes the induction step in proving Statement 2.4.

It remains to prove (2.2) assuming (2.1). Clearly, we have

$$|f(x)| \leq L \sum_{i=1}^m |\lambda_i| \leq \alpha m L \quad \text{for all } x \in X,$$

and the upper bound in (2.2) follows. To prove the lower bound, iterating (2.4), we get

$$|\mathbf{E} e^f| \geq \left(\cos \frac{\theta}{2} \right)^n \mathbf{E} |e^f| \geq e^{-\alpha m L} \left(\cos \frac{\pi}{4\sqrt{r-1}} \right)^n,$$

as required. ■

3. Proof of Theorem 1.2

We will be proving a more general version of the theorem.

Theorem 3.1. *Let $\phi_1, \dots, \phi_m : \mathbb{R}^n \rightarrow \mathbb{C}$ be functions on Euclidean space $\mathbb{R}^n$, endowed with the standard Gaussian probability measure.*

For $i = 1, \dots, m$, let $J_i \subset \{1, \dots, n\}$ be the set of the coordinates that ϕ_i depends on, that is,

$$\phi_i(x_1, \dots, x_n) = \phi_i(y_1, \dots, y_n) \quad \text{whenever } x_j = y_j \text{ for all } j \in J_i.$$

Assume that each function ϕ_i is 1-Lipschitz in the ℓ^1 metric of $\mathbb{R}^n$, that is,

$$|\phi_i(x_1, \dots, x_n) - \phi_i(y_1, \dots, y_n)| \leq \sum_{i=1}^n |x_i - y_i|$$

and that for some $r \geq 2$, we have

$$|J_i| \leq r \quad \text{for } i = 1, \dots, m,$$

so that every function ϕ_i depends on at most $r \geq 2$ coordinates.

Let

$$f = \sum_{i=1}^m \lambda_i \phi_i$$

for some $\lambda_i \in \mathbb{C}$ such that

$$\sum_{i: j \in J_i} |\lambda_i| < \frac{1}{6\sqrt{r}} \quad \text{for } j = 1, \dots, n.$$

Then,

$$\mathbf{E} e^f \neq 0.$$

Moreover, if, additionally,

$$|\phi_i(x)| \leq L \quad \text{for all } x \in \mathbb{R}^n \text{ and } i = 1, \dots, m \quad (3.1)$$

and some $L > 0$, then

$$e^{\alpha mL} \geq |\mathbf{E} e^f| \geq e^{-\alpha mL} e^{-\pi^2 n/32r}, \quad \text{where } \alpha = \max_{i=1, \dots, m} |\lambda_i|. \quad (3.2)$$

Choosing in Theorem 3.1,

$$\lambda_i = \lambda \quad \text{for } i = 1, \dots, m,$$

we deduce Theorem 1.2.

We obtain Theorem 3.1 as a scaling limit of Theorem 2.1.

Proof of Theorem 3.1. First, we consider the case of bounded functions ϕ_i , so that the condition (3.1) is satisfied.

We are going to use Theorem 2.1. For an integer $N \geq 1$, we consider nN probability spaces $X_1 = \dots = X_{nN} = \{-1, 1\}$ and their direct product $X = \{-1, 1\}^{nN}$, all endowed with the uniform probability measure. We write a point $x \in X$ as $x = (x_{jk})$, where $j = 1, \dots, n$ and $k = 1, \dots, N$. For $i = 1, \dots, m$, we define functions $\psi_{i,N} : \{-1, 1\}^{nN} \rightarrow \mathbb{C}$ by

$$\psi_{i,N}(x) = \frac{\sqrt{N}}{2} \phi_i \left(\frac{x_{11} + \dots + x_{1N}}{\sqrt{N}}, \dots, \frac{x_{j1} + \dots + x_{jN}}{\sqrt{N}}, \dots, \frac{x_{n1} + \dots + x_{nN}}{\sqrt{N}} \right)$$

and

$$g_N = \sum_{i=1}^m \lambda_{i,N} \psi_{i,N}, \quad \text{where } \lambda_{i,N} = \frac{2\lambda_i}{\sqrt{N}} \quad \text{for } i = 1, \dots, m.$$

Let $J_{i,N}$ be the set of coordinates jk such that $\psi_{i,N}$ depends on x_{jk} .

Since the functions ϕ_i are 1-Lipschitz in the ℓ^1 metric of $\mathbb{R}^n$, the functions $\psi_{i,N}$ are 1-Lipschitz in the Hamming metric of X . Moreover, each function $\psi_{i,N}$ depends on not more than $r_N = rN$ coordinates and

$$\sum_{i: jk \in J_{i,N}} |\lambda_{i,N}| = \frac{2}{\sqrt{N}} \sum_{i: jk \in J_i} |\lambda_i| < \frac{1}{3\sqrt{rN}} \quad \text{for } j = 1, \dots, n \text{ and } k = 1, \dots, N.$$

We note that

$$\max_{i=1, \dots, n} |\lambda_{i,N}| = \frac{2}{\sqrt{N}} \max_{i=1, \dots, n} |\lambda_i| = \frac{2\alpha}{\sqrt{N}}$$

and that by (3.1), we have

$$|\psi_i(x)| \leq \frac{\sqrt{N}}{2} L \quad \text{for } i = 1, \dots, m.$$

Therefore, from formulas (2.1)–(2.2) of Theorem 2.1, we conclude that

$$|\mathbf{E} e^{g_N}| \geq e^{-\alpha m L} \left(\cos \frac{\pi}{4\sqrt{rN-1}} \right)^{nN}. \quad (3.3)$$

By the Central Limit Theorem, as $N \rightarrow \infty$, the random vector

$$\left(\frac{x_{11} + \dots + x_{1N}}{\sqrt{N}}, \dots, \frac{x_{j1} + \dots + x_{jN}}{\sqrt{N}}, \dots, \frac{x_{n1} + \dots + x_{nN}}{\sqrt{N}} \right)$$

converges in distribution to the standard Gaussian measure in $\mathbb{R}^n$. Since the functions ϕ_i are continuous and bounded, we have

$$\lim_{N \rightarrow \infty} \mathbf{E} e^{g_N} = \mathbf{E} e^f,$$

where the expectation on the right-hand side is taken with respect to the standard Gaussian measure in $\mathbb{R}^n$; see, for example, Section 7.2 of [19]. Since

$$\lim_{N \rightarrow \infty} \left(\cos \frac{\pi}{4\sqrt{rN-1}} \right)^{nN} = \lim_{N \rightarrow \infty} \left(1 - \frac{\pi^2}{32(rN-1)} \right)^{nN} = \exp \left\{ -\frac{\pi^2 n}{32r} \right\},$$

from (3.3), we obtain the lower bound in the inequality (3.2). The upper bound in (3.2) is trivial.

It remains to consider the general case of not necessarily bounded functions ϕ_i . Shifting, if necessary,

$$\phi_i := \phi_i - \phi_i(0) \quad \text{for } i = 1, \dots, m,$$

without loss of generality, we assume that

$$\phi_i(0) = 0 \quad \text{for } i = 1, \dots, m.$$

Then,

$$|\phi_i(x_1, \dots, x_n)| \leq \sum_{j=1}^n |x_j| \quad \text{for } i = 1, \dots, m. \quad (3.4)$$

For $L > 0$, we define the truncation of ϕ_i by

$$\phi_{i,L}(x) = \begin{cases} \phi_i(x) & \text{if } |\phi_i(x)| \leq L, \\ L\phi_i(x)/|\phi_i(x)| & \text{if } |\phi_i(x)| > L. \end{cases}$$

Then,

$$|\phi_{i,L}(x) - \phi_{i,L}(y)| \leq |\phi_i(x) - \phi_i(y)|,$$

so $\phi_{i,L}$ satisfy the conditions of Theorem 3.1 and are bounded. Hence, for

$$f_L = \sum_{i=1}^m \lambda_i \phi_{i,L},$$

we have

$$\mathbf{E} e^{f_L} \neq 0.$$

Let us pick a $\rho > 1$ such that

$$\rho \sum_{i=1}^m |\lambda_i| < \frac{1}{6\sqrt{r}}.$$

Replacing f by μf , we conclude that $\mathbf{E} e^{\mu f_L} \neq 0$ provided $|\mu| < \rho$. From (3.4), it follows that

$$\lim_{L \rightarrow +\infty} \mathbf{E} e^{\mu f_L} = \mathbf{E} e^{\mu f}$$

and that the convergence is uniform on any compact set of μ in the disc $|\mu| < \rho$. Then, by the Hurwitz theorem, see, for example, Section 1 of [24], we have two options: either $\mathbf{E} e^{\mu f} = 0$ for all μ in the disc $|\mu| < \rho$ or $\mathbf{E} e^{\mu f} \neq 0$ for all μ in the open disc $|\mu| < \rho$. Since for $\mu = 0$, we clearly have $\mathbf{E} e^{\mu f} = 1$, the first option is not realized. Choosing $\mu = 1$, we conclude that $\mathbf{E} e^f \neq 0$. ■

4. Optimality

Our goal is to show that the bound

$$|\lambda| \leq \frac{1}{3c\sqrt{r-1}}$$

in Theorem 1.1 is optimal, up to a constant factor, replacing $1/3$. We do it in two steps. First, we show that for $c = 1$ the dependence on r is optimal, up to a constant factor, and then that the dependence on both c and r is optimal.

4.1. Dependence on r

Let $X_1 = \dots = X_n = \{-1, 1\}$ and let $X = \{-1, 1\}^n$, all endowed with the uniform probability measure. We intend to show that for all sufficiently large n there is a function $\phi : X \rightarrow \mathbb{C}$ which is 1-Lipschitz in the Hamming metric of X and such that $\mathbf{E} e^{\lambda \phi} = 0$ for some λ satisfying

$$|\lambda| \leq \frac{\gamma}{\sqrt{n}},$$

where $\gamma > 0$ is an absolute constant.

We reverse engineer an example from our proof of Theorem 3.1 in Section 3.

The functions

$$z \mapsto \int_0^{+\infty} e^{zx} e^{-x^2/2} dx \quad \text{and} \quad z \mapsto \int_{-\infty}^0 e^{zx} e^{-x^2/2} dx,$$

where we integrate over a real variable x , are non-constant entire functions of $z \in \mathbb{C}$, and hence, the range of each, by the Picard theorem, is the whole plane $\mathbb{C}$, except perhaps one point. Therefore, there are two points $u, v \in \mathbb{C}$ such that

$$\int_{-\infty}^0 e^{ux} e^{-x^2/2} dx + \int_0^{+\infty} e^{vx} e^{-x^2/2} dx = 0.$$

Let us define $\psi : \mathbb{R} \rightarrow \mathbb{C}$:

$$\psi(x) = \begin{cases} vx & \text{if } x \geq 0, \\ ux & \text{if } x < 0. \end{cases}$$

Then,

$$\int_{-\infty}^{+\infty} e^{\psi(x)} e^{-x^2/2} dx = 0. \quad (4.1)$$

It is easy to check that ψ is Lipschitz; more precisely,

$$|\psi(x) - \psi(y)| \leq \tau |x - y| \quad \text{where } \tau = \max\{|u|, |v|\}.$$

For an $L > 0$, we consider the truncation

$$\psi_L(x) = \begin{cases} \psi(x) & \text{if } |\psi(x)| \leq L, \\ L\psi(x)/|\psi(x)| & \text{if } |\psi(x)| > L. \end{cases}$$

Then,

$$|\psi_L(x) - \psi_L(y)| \leq |\psi(x) - \psi(y)| \leq \tau |x - y|, \quad (4.2)$$

so ψ_L is also Lipschitz and, in addition, bounded.

Let us fix some $\rho > 1$. We have

$$\lim_{L \rightarrow \infty} \int_{-\infty}^{+\infty} e^{z\psi_L(x)} e^{-x^2/2} dx = \int_{-\infty}^{+\infty} e^{z\psi(x)} e^{-x^2/2} dx, \quad (4.3)$$

and from (4.2) it follows that the convergence is uniform in z on all compact sets in $\mathbb{C}$. Since the right-hand side of (4.3) for $z = 0$ is equal to 1, while for $z = 1$ is equal to 0 by (4.1), by the Hurwitz theorem, see, for example, Section 1 of [24], we conclude that for all sufficiently large L , we must have

$$\int_{-\infty}^{+\infty} e^{z\psi_L(x)} e^{-x^2/2} dx = 0 \quad \text{for some } z \text{ with } |z| < \rho. \quad (4.4)$$

Let us pick some L such that (4.4) holds. For an integer n , we consider n probability spaces $X_1 = \dots = X_n = \{-1, 1\}^n$ and their product $X = \{-1, 1\}^n$, all endowed with the uniform probability measure. We define $\phi : X \rightarrow \mathbb{C}$ by

$$\phi(x) = \frac{\sqrt{n}}{2\tau} \psi_L\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n x_i\right), \quad \text{where } x = (x_1, \dots, x_n) \quad (4.5)$$

(recall that τ is the Lipschitz constant of ψ_L). It follows from (4.2) that ϕ is 1-Lipschitz in the Hamming metric of $\{-1, 1\}^n$. Hence, ϕ satisfies the conditions of Theorem 1.1 with $c = 1$ and $r = n$.

Since by the Central Limit Theorem the normalized sum

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n x_i$$

converges in distribution to the standard Gaussian measure as n grows, and the function ψ_L is continuous and bounded, we have

$$\mathbf{E} \exp\left\{z \psi_L\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n x_i\right)\right\} \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{z \psi_L(x)} e^{-x^2/2} dx \quad (4.6)$$

and that the convergence in (4.6) is uniform in z on all compact subsets of $\mathbb{C}$; see, for example, Section 7.2 of [19]. The right-hand side of (4.6) is equal to 1 for $z = 0$. Then, by (4.4) and the Hurwitz theorem, for all sufficiently large n , we must have

$$\mathbf{E} \exp\left\{z \psi_L\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n x_i\right)\right\} = 0 \quad \text{for some } z \text{ with } |z| < \rho.$$

Therefore, from (4.5), we conclude that for all sufficiently large n , there is λ satisfying

$$|\lambda| < \frac{2\tau\rho}{\sqrt{n}} \quad \text{and} \quad \mathbf{E} e^{\lambda\phi} = 0,$$

which proves that the bound in Theorem 1.1 is indeed optimal in terms of r , up to an absolute constant factor.

4.2. Dependence on r and c

To prove that the dependence on both r and c is optimal, for an integer $k \geq 1$, we introduce

$$f_k(x) = \underbrace{\phi(x) + \dots + \phi(x)}_{k \text{ times}}, \quad (4.7)$$

where ϕ is defined by (4.5). While the parameter r remains the same for all f_k defined by (4.7), the parameter c changes, $c = k$. Furthermore,

$$\mathbf{E} e^{\lambda f_k} = \mathbf{E} e^{(k\lambda)f},$$

and hence, the zero-free region for λ scales

$$|\lambda| = O\left(\frac{1}{k\sqrt{r}}\right) = O\left(\frac{1}{c\sqrt{r}}\right).$$

4.3. Question: computational complexity

It would be interesting to find out whether the dependence on r is optimal from the computational complexity point of view, that is, whether for real-valued f and $\lambda \gg 1/c\sqrt{r}$, the approximation of $\mathbf{E} e^{\lambda f}$ becomes computationally difficult, possibly conditioned on $P \neq NP$ or other commonly believed hypothesis. The argument with the “cloning” of f as in (4.7) shows that the dependence on c is indeed optimal.

5. Approximations

5.1. The goal

Here, we sketch how Theorem 1.1, respectively, Theorem 1.2, allow us to approximate $\mathbf{E} e^{\lambda f}$ provided

$$|\lambda| \leq \frac{1-\delta}{3c\sqrt{r-1}}, \text{ respectively, } |\lambda| \leq \frac{1-\delta}{6c\sqrt{r}} \quad (5.1)$$

for some $0 < \delta < 1$, fixed in advance. The approach was used many times before, in particular, in [4] in the context closest to ours.

The algorithm is based on the following result.

Lemma 5.1. *Let $p(z)$ be a univariate polynomial of degree N in a complex variable z . Suppose that for some $\beta > 1$, we have $p(z) \neq 0$ for all z satisfying $|z| < \beta$ and let us choose a branch of $g(z) = \ln p(z)$ in the disc $|z| < \beta$. For an integer $k \geq 1$, let $T_k(z)$ be the Taylor polynomial of $g(z)$ degree k computed at $z = 0$, that is,*

$$T_k(z) = g(0) + \sum_{s=1}^k \frac{g^{(s)}(0)}{s!} z^s.$$

Then,

$$|g(z) - T_k(z)| \leq \frac{N}{(k+1)(\beta-1)\beta^k} \text{ for all } |z| \leq 1.$$

Proof. This is Lemma 2.2.1 from [3]. ■

As follows from Lemma 5.1, to approximate $g(1)$ by $T_k(1)$ within an additive error $0 < \varepsilon < 1$, and hence, to approximate $p(1)$ by $\exp\{T_k(1)\}$ within a relative error $0 < \varepsilon < 1$, it suffices to choose $k = O_\beta(\ln N - \ln \varepsilon)$. Moreover, the derivatives $g^{(s)}(0)$ can be computed from $p(0)$ and the derivatives $p^{(s)}(0)$ for $s = 1, \dots, k$ in $O(k^2)$ time; see Section 2.2.2 of [3].

5.2. Bounds

In the context of Theorem 1.1, let us pick an arbitrary $x_0 \in X$ and let $\gamma = f(x_0)$. In the context of Theorem 1.2, let $\gamma = f(0)$. Since

$$\mathbf{E} e^{\lambda(f-\gamma)} = e^{-\lambda\gamma} \mathbf{E} e^{\lambda f},$$

without loss of generality, we assume that $f(x_0) = 0$ (Theorem 1.1) or $f(0) = 0$ (Theorem 1.2).

Then, in the context of Theorem 1.1, respectively, Theorem 1.2, we have

$$\begin{aligned} |f(x)| &\leq nm \quad \text{for all } x \in X, \quad \text{respectively,} \\ |f(x_1, \dots, x_n)| &\leq m \sum_{j=1}^n |x_j| \quad \text{for all } (x_1, \dots, x_n) \in \mathbb{R}^n. \end{aligned} \quad (5.2)$$

Given a complex λ satisfying (5.1), we consider a function

$$z \mapsto \mathbf{E} e^{\lambda z f} = \mathbf{E} \exp \left\{ \lambda z \sum_{i=1}^m \phi_i \right\}. \quad (5.3)$$

While (5.3) is not a polynomial, we can approximate it close enough by its Taylor polynomial $p_N(z)$ and then use Lemma 5.1 to approximate $\ln p_N(z)$ by a polynomial of a low degree. To accomplish the first step, one can use, for example, the following standard estimate.

Lemma 5.2. *Let $\rho > 0$ be a real number and let $N \geq 5\rho$ be an integer. Then,*

$$\left| e^z - \sum_{k=0}^N \frac{z^k}{k!} \right| \leq e^{-2\rho} \quad \text{for all } z \in \mathbb{C} \text{ such that } |z| \leq \rho.$$

Proof. This is Lemma 3.2 in [4]. The proof is a standard estimate of the truncation error. ■

Using (5.2), Lemma 5.2, and the bounds (1.1)–(1.2) and (1.3)–(1.4), for a given $0 < \varepsilon < 1$, one can compute

$$N = \left((m+n) \ln \frac{1}{\varepsilon} \right)^{O(1)}$$

such that the Taylor polynomial of (5.3),

$$p_N(z) = \sum_{s=0}^N \frac{\lambda^s z^s}{s!} \mathbf{E} f^s = \sum_{s=0}^N \frac{\lambda^s z^s}{s!} \left(\sum_{i=1}^m \phi_i \right)^s,$$

satisfies

$$p_N(z) \neq 0 \text{ provided } |z| \leq \frac{1}{1-\delta}$$

and $p_N(1)$ approximates $\mathbf{E} e^{\lambda f}$ within a relative error of $\varepsilon/3$ (in the context of Theorem 1.2, we replace ϕ_i by their appropriate truncations).

Using Lemma 5.1 with $\beta = (1-\delta)^{-1}$, we further approximate $p_N(1)$ within a relative error of $\varepsilon/3$ using only $k = O_\delta(\ln(m+n) - \ln \varepsilon)$ first derivatives,

$$p_N^{(s)}(0) = \lambda^s \mathbf{E} \left(\sum_{i=1}^m \phi_i \right)^s \text{ for } s = 1, \dots, k,$$

which in turn reduces to computing $m^{O_\delta(\ln(m+n) - \ln \varepsilon)}$ different expectations

$$\mathbf{E}(\phi_{i_1} \cdots \phi_{i_k});$$

see, for example, [4] for a similar computation.

This results in a quasi-polynomial algorithm for approximating $\mathbf{E} e^{\lambda f}$.

As we remarked, Patel and Regts [31], see also [26], developed methods for faster computation of the relevant derivatives of $g(z) = \ln p(z)$, which in some cases allows one to obtain a genuinely polynomial algorithm of $((m+n)/\varepsilon)^{O(1)}$ complexity, provided the parameters r and c are fixed in advance.

6. Example: vertex and edge colorings in graphs

In this section, following a suggestion by an anonymous referee, we illustrate the algorithmic consequences of Theorem 1.1 by an example of graph colorings.

In what follows, γ denotes some positive absolute constant, whose value may change from line to line.

6.1. Vertex coloring

Let $G = (V, E)$ be an undirected graph with set V of vertices, set E of edges, and without multiple edges or loops, and let $q \geq 2$ be an integer. A map $V \rightarrow \{1, \dots, q\}$ is called a *vertex q -coloring*. A vertex q -coloring is *proper* if the endpoints of every edge of G have different colors. It is one of the oldest NP-complete problems to determine

whether a proper vertex q -coloring exists for a given graph G and given $q \geq 3$. The smallest q for which a proper vertex q -coloring of G exists is called the *chromatic number* of G . It turns out that even approximating the chromatic number within a factor of $|V|^{1-\varepsilon}$ for any $0 < \varepsilon < 1$, fixed in advance, is an NP-hard problem [13, 40].

The set $X = \{1, \dots, q\}^V$ of all maps $V \rightarrow \{1, \dots, q\}$ is naturally interpreted as the set of all vertex colorings, proper or not, of G into one of the q colors. For $x \in X$ and $v \in V$, let x_v denote the corresponding coordinate of x , that is, the color assigned to vertex v in x . For $0 \leq \delta \leq 1$, we consider the sum

$$\sum_{x \in X} \prod_{\{u, v\} \in E} \eta_\delta(x_u, x_v), \quad \text{where } \eta_\delta(i, j) = \begin{cases} 1 & \text{if } i \neq j, \\ 1 - \delta & \text{if } i = j. \end{cases} \quad (6.1)$$

If $\delta = 1$, then (6.1) is the number of proper vertex q -colorings of G and if $\delta = 0$ then (6.1) is the number $q^{|V|}$ of all vertex q -colorings. For $0 < \delta < 1$, the sum (6.1) represents a weighted sum over all vertex q -colorings, where the proper colorings are counted with weight 1 and improper colorings are discounted by a factor, exponentially small in the number of edges that are spanned by identically colored vertices. As δ increases towards 1, the discount of improper vertex colorings increases. We note that the sum (6.1) is an example of an antiferromagnetic Potts partition function, cf. Section 1.1.

Let $\deg v$ be the degree of a vertex v , that is, the number of edges of G incident to v and let $\Delta(G)$ be the maximum degree of a vertex in G . It is shown in [7], see also Section 7.1 of [3], that for some absolute constant $\gamma > 0$ and

$$\delta = \frac{\gamma}{\Delta(G)}, \quad (6.2)$$

the sum (6.1) can be approximated within relative error $0 < \varepsilon < 1$ in quasi-polynomial $|V|^{O(\ln|V| - \ln \varepsilon)}$ time. In [31], it is further shown that if $\Delta(G)$ is fixed in advance, then the algorithm can be made to run in genuinely polynomial $(|V|/\varepsilon)^{O(1)}$ time. Moreover, unless $P = NP$, the value of δ in (6.2) cannot be increased to $\delta = \gamma \Delta^{-\alpha}(G)$ for any fixed $0 < \alpha < 1$ [7].

Finally, we observe that the fact that for δ defined by (6.2) for some absolute constant $\gamma > 0$, the sum (6.1) can be approximated efficiently can be deduced from Theorem 1.1. Indeed, in (6.1) each function η_δ depends on two coordinates of $x \in X$, and every coordinate x_v enters at most $\Delta(G)$ different functions η_δ , so we can choose $r = 2$ and $c = \Delta(G)$ in Theorem 1.1.

6.2. Edge coloring

Let $G = (V, E)$ be a graph as in Section 6.1. A map $E \rightarrow \{1, \dots, q\}$ is called an *edge q -coloring*. An edge q -coloring is *proper* if every two edges sharing a vertex

have different colors. While by Vizing's theorem the smallest number q of colors needed for a proper edge q -coloring of a graph, called the *chromatic index* of G , is either $\Delta(G)$ or $\Delta(G) + 1$, deciding between these two numbers is an NP-complete problem [21]. The problem of counting proper edge q -colorings is #P-complete for any values of q and $\Delta(G)$ satisfying $q \geq \Delta(G) \geq 3$ (see [9]).

The set $Y = \{1, \dots, q\}^E$ of all maps $E \rightarrow \{1, \dots, q\}$ is naturally interpreted as the set of all edge q -colorings, proper or not, of G into one of the q colors. For $y \in Y$ and $e \in E$, let y_e denote the corresponding coordinate of y , that is, the color assigned to edge e in y . For a vertex $v \in V$, let $E_v \subset E$ be the set of edges with one endpoint at v . For $0 \leq \delta \leq 1$, we consider the sum

$$\sum_{y \in Y} \prod_{v \in V} \psi_\delta(y_e : e \in E_v), \quad \text{where} \quad (6.3)$$

$$\psi_\delta(y_e : e \in E_v) = \begin{cases} 1 & \text{if } y_e : e \in E_v \text{ are distinct,} \\ 1 - \delta & \text{otherwise.} \end{cases}$$

If $\delta = 1$, then (6.3) counts proper edge q -colorings of G and if $\delta = 0$ then (6.3) is the number $q^{|E|}$ of all edge q -colorings.

From Theorem 1.1, it follows that for some absolute constant $\gamma > 0$ and

$$\delta = \frac{\gamma}{\sqrt{\Delta(G)}}, \quad (6.4)$$

the sum (6.3) can be approximated within relative error $0 < \varepsilon < 1$ in quasi-polynomial (and if $\Delta(G)$ is fixed in advance, in polynomial) time. Indeed, each function ψ_δ depends on at most $\Delta(G)$ coordinates of $y \in Y$, and every coordinate y_e enters at most two different functions $\psi_\delta(y_e : e \in E_v)$, so we can choose $r = \Delta(G)$ and $c = 2$ in Theorem 1.1. The dependence in (6.4) on $\Delta(G)$ appears to be new, as the results of [10, 34] allow only for a weaker dependence of

$$\delta = \gamma / \Delta(G).$$

At the moment, it is not clear whether the bound in (6.4) is indeed optimal, assuming some standard computational complexity hypothesis, such as $P \neq NP$.

One can impose a more severe penalty for the deviation of an edge colorings from being proper as follows. For a vertex $v \in V$ and an edge q -coloring $y \in Y$, let

$$\phi_v(y_e : e \in E_v) = \deg v - \text{the number of distinct colors in } y_e : e \in E_v. \quad (6.5)$$

Hence, $\phi_v(y) = 0$ if and only if the edges incident to v have distinct colors, and $\phi_v(y) > 0$ is the number of “missing colors” otherwise. Importantly, ϕ_v is 1-Lipschitz

in the Hamming metric of Y . Applying Theorem 1.1, we conclude that for some absolute constant $\gamma > 0$, the sum

$$\sum_{y \in Y} \prod_{v \in V} \Psi_v(y_e : e \in E_v), \quad \text{where} \quad (6.6)$$

$$\Psi_v(y_e : e \in E_v) = \exp \left\{ - \frac{\gamma}{\sqrt{\Delta(G)}} \phi_v(y_e : e \in E_v) \right\}$$

can be efficiently approximated.

6.3. Vertex colorings of the line graph

Given a graph $G = (V, E)$ as in Sections 6.1 and 6.2, one can construct the *line graph* $L(G)$ of G as follows. The vertices of $L(G)$ are the edges of G , and two vertices of $L(G)$ span an edge of $L(G)$ if the corresponding edges in G share an endpoint.

Note that each vertex v of G gives rise to a clique in $L(G)$ with $\deg v$ vertices, corresponding to the edges $E_v \subset E$, incident to v .

A proper edge q -coloring of G corresponds to a proper vertex q -coloring of $L(G)$ and vice versa. We intend to compare the Potts partition function (6.1)–(6.2) of vertex q -colorings of $L(G)$ with the partition function (6.5)–(6.6) of edge colorings of G . While in general the partition functions are incomparable, it turns out that (6.5)–(6.6) discounts *random* colorings more efficiently under some general regimes.

We sketch the argument below, assuming that G is an r -regular graph, that is, the degree of every vertex of G is $r \geq 2$. Then, $|E| = r|V|/2$ and the line graph $L(G)$ is $2(r-1)$ regular, with $r|V|/2$ vertices and $\binom{r}{2}|V|$ edges. We further assume that $|V| \gg r$.

We see first what penalty a random vertex q -coloring of $L(G)$ incurs under the Potts partition function (6.1)–(6.2). The probability that the endpoints of a given edge are colored with the same color is $1/q$, so the expected number of edges that have the endpoints of the same color is $q^{-1} \binom{r}{2} |V|$. Standard concentration results, see [37], imply that with high probability the number of miscolored edges is close to its expectation, so with high probability the weight of a random coloring under the (6.1)–(6.2) partition function is roughly

$$\exp \left\{ - \frac{\gamma}{2(r-1)} q^{-1} \binom{r}{2} |V| \right\} = \exp \left\{ - \gamma \frac{r|V|}{4q} \right\}. \quad (6.7)$$

Next, we see what penalty a random vertex q -coloring of $L(G)$, that is a random edge q -coloring of G , incurs under the partition function (6.5)–(6.6). Pick a vertex v of G and any particular color. The probability that the color is used to color some of the edges incident to v is $1 - (1 - q^{-1})^r$. Hence, the expected number of colors used to color the edges in E_v is $q - q(1 - q^{-1})^r$ and the expected value of ϕ_v in

(6.5) is $r - q + q(1 - q^{-1})^r$. Again, standard concentration results [37] imply that with high probability the value of ϕ_v is close to its expectation, and hence, with high probability the weight of a random coloring in the partition function (6.5)–(6.6) is roughly

$$\exp \left\{ -\frac{\gamma}{\sqrt{r}} |V| \left(r - q + q \left(1 - \frac{1}{q} \right)^r \right) \right\}. \quad (6.8)$$

Assuming that $q \geq r$, we have

$$\begin{aligned} \left(1 - \frac{1}{q} \right)^r &= \exp \left\{ r \ln \left(1 - \frac{1}{q} \right) \right\} = \exp \left\{ -\frac{r}{q} - \frac{r}{2q^2} + O\left(\frac{r}{q^3} \right) \right\} \\ &= \exp \left\{ -\frac{r}{q} \right\} \left(1 + O\left(\frac{r}{q^2} \right) \right) = \exp \left\{ -\frac{r}{q} \right\} + O\left(\frac{r}{q^2} \right) \\ &\geq \left(1 - \frac{r}{q} + \frac{r^2}{3q^2} \right) + O\left(\frac{r}{q^2} \right). \end{aligned}$$

Hence, we conclude that (6.8) does not exceed

$$\exp \left\{ -\gamma |V| \left(\frac{r^{3/2}}{3q} + O\left(\frac{\sqrt{r}}{q} \right) \right) \right\}. \quad (6.9)$$

Comparing (6.9) and (6.7), we conclude that the partition function (6.5)–(6.6) provides an exponentially better discount of random colorings, compared to the partition function (6.3)–(6.4), as r grows.

Acknowledgments. The author is thankful to the anonymous referee for suggesting graph coloring partition functions to illustrate the main results of the paper.

Funding. This research was partially supported by NSF Grant DMS 1855428.

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Communicated by Joanna Ellis-Monaghan

Received 27 June 2024; revised 1 February 2026.

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