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Generalized cycles on spectral curves

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Abstract. Generalized cycles can be thought of as the extension of form-cycle duality between holomorphic forms and cycles to meromorphic forms and generalized cycles.

They appeared as a ubiquitous tool in the study of spectral curves and integrable systems in the topological recursion approach. They parametrize deformations, implementing the special geometry, where moduli are periods, and derivatives with respect to moduli are other periods, or more generally "integrals", whence the name "generalized cycles". They appeared over the years in various works, each time in specific applied frameworks, and here, we provide a comprehensive self-contained corpus of definitions and properties for a very general setting. The geometry of generalized cycles is also fascinating in itself.

1. Introduction

The idea of generalized cycles is to extend to meromorphic 1-forms the *form-cycle duality* that exists for *holomorphic* 1-forms with *cycles*. Generalized cycles could thus be called "*meromorphic cycles*". They were introduced because of their ubiquitous role in the theory of topological recursion [7]; in particular, they parametrize deformations of spectral curves, by implementing some "special geometry" relations, i.e., the fact that derivatives with respect to some periods of the spectral curve are also periods (integrals on the curve).

Motivation example 1: Seiberg–Witten theory

In Seiberg–Witten theory, there is a compact Riemann surface Σ of some genus $\mathfrak{g}$ and a 1-form y. The curve Σ is endowed with a choice of $2\mathfrak{g}$ cycles (in the usual sense) $\mathcal{A}_1, \dots, \mathcal{A}_{\mathfrak{g}}, \mathcal{B}_1, \dots, \mathcal{B}_{\mathfrak{g}}$, with symplectic intersections:

$$\mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j}, \quad \mathcal{A}_i \cap \mathcal{A}_j = 0, \quad \mathcal{B}_i \cap \mathcal{B}_j = 0.$$

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Such a choice, not unique, is called a "Torelli marking of cycles". The $\mathcal{A}$-periods of y,

$$a_i = \frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{A}_i} \mathrm{y},$$

are moduli of the Seiberg–Witten curve. The Seiberg–Witten prepotential is defined as a function F_0 of the moduli that satisfies "special geometry":

$$\frac{\partial}{\partial a_i} F_0 = \oint_{\mathcal{B}_i} \mathrm{y}.$$

In other words, derivatives with respect to $\mathcal{A}$ periods are $\mathcal{B}$ periods.

Example 2: Random matrices

Consider a random Hermitian matrix $M \in H_N$ with probability law of the form

$$\frac{1}{Z} e^{-N \operatorname{Tr} V(M)} D_{\text{Lebesgue}}(M), \quad Z = \int_{H_N} e^{-N \operatorname{Tr} V(M)} D_{\text{Lebesgue}}(M), \tag{1.1}$$

where D_{Lebesgue} is the Lebesgue measure of H_N and Z is the partition function (that makes the total probability equal to 1), and $V(M) = \sum_{k=1}^{d} \frac{t_k}{k} M^k$ is a polynomial called the potential. The expectation value of the resolvent is written as

$$W_1(x) = \mathbb{E}\left(\frac{1}{N} \operatorname{Tr}(x - M)^{-1}\right) = \sum_{k=0}^{\infty} \frac{1}{x^{k+1}} \mathbb{E}\left(\frac{1}{N} \operatorname{Tr} M^k\right).$$

Let us denote the 1-form

$$\mathrm{y} = V'(x)dx - W_1(x)dx,$$

which is a 1-form analytic outside of the support of eigenvalues distribution. Since $\mathrm{y} = V'(x)dx + O(1/x)dx$ at large x, the coefficients of $V(x)$, called here the "moduli", are worth

$$t_k = -\operatorname{Res}_\infty x^{-k} \mathrm{y},$$

and taking the derivative with respect to t_k in (1.1) gives

$$\frac{\partial}{\partial t_k} \frac{1}{N^2} \ln Z = \mathbb{E}\left(\frac{-1}{kN} \operatorname{Tr} M^k\right) = \frac{1}{k} \operatorname{Res}_\infty x^k W_1(x)dx = \frac{-1}{k} \operatorname{Res}_\infty x^k \mathrm{y}.$$

This is a special case of the Miwa–Jimbo relation for Tau functions in isomonodromic integrable systems. But this is another example of special geometry, where the derivative with respect to an integral of y (a residue) is another integral of y (another residue).

Moreover, it is known in random matrix theory that, in the large N limit, $W_1(x)$ tends to an algebraic function of x and thus y tends to a meromorphic 1-form on a Riemann surface Σ. Topological recursion was invented to compute the asymptotic expansion at large N and to satisfy the Miwa–Jimbo relation to all orders in the expansion.

The same kind of relations appear in many other integrable systems, except that instead of integrals over closed cycles, one can have other types of integrals: integrals on open chains, or residues... on some Riemann surface Σ equipped with a certain differential 1-form y. This is what led, in the topological recursion framework to the notion of generalized cycles associated to a spectral curve.

2. Spectral curve

Generalized cycles are associated to spectral curves.

2.1. Definitions

Definition 2.1 (Spectral curve). A spectral curve $\mathcal{S} = (\Sigma, \mathrm{x}, \mathrm{y}, B)$ is

- Σ is an open Riemann surface, not necessarily compact nor connected.
- $\mathrm{x} : \Sigma \to \underline{\Sigma}$ is a ramified covering of a Riemann surface $\underline{\Sigma}$ called the "base", and most of the time, we will choose $\underline{\Sigma} = \mathbb{C}P^1$.
- y is a meromorphic 1-form on Σ (in any chart $\mathrm{y} = f(\zeta)d\zeta$, where $f(\zeta)$ is a meromorphic function of a local coordinate ζ).
- B is a symmetric $1 \boxtimes 1$-form with a double pole on the diagonal with bi-residue 1:

$$B \in H^0(\Sigma \times \Sigma, K_\Sigma \overset{\mathrm{sym}}{\boxtimes} K_\Sigma(2\mathrm{diag}))_1,$$

which means that in any pair of charts $U_i \times U_j$, and local coordinates ζ_i, ζ_j (and we use the same charts and coordinates for neighborhoods of the diagonal), B behaves as follows:

$$B(z_1, z_2) \sim \frac{\delta_{i,j}}{(\zeta_i(z_1) - \zeta_j(z_2))^2} d\zeta_i(z_1) \otimes d\zeta_j(z_2) + \text{bi-holomorphic}.$$

We recall the usual examples of B in Section 2.2.

Remark 2.1. If $\underline{\Sigma} = \mathbb{C}P^1$, it is customary to write

$$\mathrm{y} = y\, d\mathrm{x},$$

where y is a meromorphic function on Σ. In other words, one can identify $T^*\mathbb{C}P^1$ with $\mathbb{C} \times \mathbb{C}P^1$.

The following is an immediate consequence of the definition.

Proposition 2.1 (Lagrangian immersion). *The map* $\mathrm{i} : \Sigma \to T^*\underline{\Sigma}$ *given by* $\mathrm{i}(p) = (\mathrm{x}(p), \mathrm{y}(p))$ *is a Lagrangian immersion of* Σ *into the cotangent space of the base curve:*

$$\begin{tikzcd}
\Sigma \arrow[r, hook, "\mathrm{i}=(\mathrm{x},\mathrm{y})"] \arrow[dr, "x"'] & T^*\underline{\Sigma} \arrow[d] \\
& \Sigma
\end{tikzcd}$$

y *is the pullback by* i *of the tautological 1-form of* $T^*\underline{\Sigma}$*, i.e., the Liouville form.*

Definition 2.2 (Meromorphic forms). Here, we define a meromorphic 1-form on Σ to be a 1-form holomorphic on the interior of $\Sigma \setminus$ isolated points, and bounded by a rational function in a local coordinate in any neighborhood, and in fact a meromorphic 1-form is the equivalence class of such 1-forms that differ on at most a set of isolated points:

$$\omega_1 \equiv \omega_2 \quad \text{iff they are equal on } \Sigma \setminus \{\text{isolated points}\}.$$

We will call pole, any point where a 1-form has no holomorphic representative.

Let $\mathfrak{M}^1(\Sigma)$ be the sheaf of meromorphic 1-forms. Notice that it is of infinite uncountable dimension.

We needed to adapt the residue definition in case we would choose two representatives that differ exactly at the point where we compute the residue, and with the following definition, the residue depends only on the equivalence class.

Definition 2.3 (Residue). The residue of a 1-form is defined as follows:

$$\mathrm{Res}_p\, \omega = \frac{1}{2\pi \mathrm{i}} \liminf_{r\to 0} \oint_{\mathcal{C}_{p,r}} \omega,$$

where, in a given chart around p, $\mathcal{C}_{p,r}$ is a circle of radius $r > 0$. The projective limit $r \to 0$ is independent of a choice of representative of ω and is well defined because for any representative the integral is constant as soon as r is smaller than the distance of the closest non-holomorphic isolated point to p.

2.2. The bidifferential B

The bidifferential B is here quite arbitrary. The reader can skip this subsection whose role is only to illustrate the zoology of possible B in some specific examples and relies on a prior knowledge of *Theta functions* on Riemann surfaces. For this subsection, we refer to classical textbooks on compact Riemann surfaces; see [8, 9, 12].

Notice that one can add to B any symmetric tensor product of holomorphic forms, so that the space of possible B is an affine space

$$B + \Omega^1(\Sigma) \overset{\text{sym}}{\boxtimes} \Omega^1(\Sigma).$$

Let us consider the particularly interesting case where Σ is a connected compact Riemann surface of some genus $\mathfrak{g}$.

- *Case genus* $= 0$, i.e., $\Sigma \sim \mathbb{C}P^1$, the Riemann sphere. In that case, there is no non-zero holomorphic 1-form, and B is unique:

  $$B(z_1, z_2) = \frac{1}{(z_1 - z_2)^2} dz_1 \otimes dz_2 = d_{z_1} \otimes d_{z_2}(\ln(z_1 - z_2)),$$

  where z is the canonical coordinate of $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$.
- *Case genus* $= 1$, i.e., $\Sigma \sim T_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, the torus of modulus τ, with its canonical coordinate $z \equiv z + 1 \equiv z + \tau$. In that case, B is of the form

  $$B(z_1, z_2) = (\wp(z_1 - z_2) + C) dz_1 \otimes dz_2,$$

  where $\wp$ is the Weierstrass function and C is an arbitrary constant that parametrizes the one-dimensional affine space of possible B.
  The following choices of C are particularly interesting.
  - $C = G_2(\tau)$, the second Eisenstein series. In that case, B is called the *Bergman kernel* (cf. [3, 10]). It is normalized on the cycle $\mathcal{A} = [\frac{i}{2}, 1 + \frac{i}{2}]$:

    $$\forall z_2, \quad \int_{z_1 \in \mathcal{A}} B(z_1, z_2) = 0.$$

    It is the second derivative of the log of the Theta function

    $$B(z_1, z_2) = d_{z_1} \otimes d_{z_2} \ln \Theta\left(z_1 - z_2 + \frac{1}{2} + \frac{1}{2}\tau\right),$$

    where $\Theta = \vartheta_3$ is the Riemann Theta-function, or also the 3rd Jacobi Theta function ϑ_3.
  - $C = G_2(\tau) + \pi / \operatorname{Im} \tau$. In that case, B is called the *Schiffer kernel* (cf. [3, 10]). It is related to the *Green function*

    $$B(z_1, z_2) = d_{z_1} \otimes d_{z_2} \mathcal{G}(z_1, z_2),$$

    where $\mathcal{G}(z_1, z_2) : \Sigma \times \Sigma \to \mathbb{R}$ has a logarithmic singularity on the diagonal, and satisfies the Poisson equation $\Delta_{z_1} \mathcal{G}(z_1, z_2) = 4\pi \delta^{(2)}_{z_2}(z_1)$.
  - Let $\zeta \in \mathbb{C}$, and choose $C = G_2(\tau) - \frac{d^2}{d\zeta^2} \ln \Theta(\zeta + \frac{1}{2} + \frac{1}{2}\tau)$. In that case, B is called the *Klein kernel* (cf. [10]).

- *Higher genus*: we have

$$B(z_1, z_2) = d_{z_1} \otimes d_{z_2} \ln \Theta\Big(\mathfrak{a}(z_1) - \mathfrak{a}(z_2) + \frac{1}{2}(\mathbf{n} + \tau \mathbf{m})\Big) + 2\pi \mathrm{i} \sum_{i=1}^{\mathfrak{g}} \sum_{j=1}^{\mathfrak{g}} \kappa_{i,j} \omega_i(z_1) \otimes \omega_j(z_2),$$

  where $\Theta : \mathbb{C}^{\mathfrak{g}} \to \mathbb{C}$ is the higher genus Riemann Theta function, $\mathfrak{a} : \Sigma \to \mathbb{C}^{\mathfrak{g}}$ is the Abel map, and $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^{\mathfrak{g}} \times \mathbb{Z}^{\mathfrak{g}}$ is an integer non-singular characteristic, odd $(\mathbf{n}, \mathbf{m}) \in 2\mathbb{Z} + 1$, and $\omega_{i=1,\dots,\mathfrak{g}}$ is a basis of holomorphic 1-forms such that $d\mathfrak{a} = (\omega_1, \dots, \omega_{\mathfrak{g}})$.
  The symmetric $\mathfrak{g} \times \mathfrak{g}$ matrix κ parametrizes the affine space of possible B.
  $\kappa = 0$ is the *Bergman kernel.*
  $\kappa = (\overline{\tau} - \tau)^{-1}$ is the *Schiffer kernel.*
- There also exist Klein kernel and many others; see [4, 9].
- *Non-compact curves*: in that case, we have a much larger choice of possible B, in an infinite-dimensional affine space. In particular, if Σ is an open subdomain of a compact curve, one can choose the restriction of the Bergman kernel. But many other choices have been considered.

3. Generalized cycles

Since we want to define a duality pairing between meromorphic forms and "generalized cycles", we have to start by taking "generalized cycles" in the algebraic dual of $\mathfrak{M}^1(\Sigma)$. The algebraic dual is a very nasty space, we will consider a much nicer subspace.

3.1. Definitions

Let $\mathfrak{M}^1(\Sigma)^*$ be the algebraic dual of $\mathfrak{M}^1(\Sigma)$, i.e., the complex vector space of all linear forms on $\mathfrak{M}^1(\Sigma)$.

Definition 3.1 (Notation: Poincaré pairing). If $\gamma \in \mathfrak{M}^1(\Sigma)^*$ and $\omega \in \mathfrak{M}^1(\Sigma)$, we will often denote the duality pairing with the same symbol as an integral

$$\gamma(\omega) \overset{\text{notation}}{=} \langle \gamma, \omega \rangle \overset{\text{notation}}{=} \int_\gamma \omega.$$

The bidifferential B is, in each chart×chart, a meromorphic form valued in meromorphic forms. In other words, for $p \in \Sigma$, in particular, in a chart U with local coordinate ξ, then $B(\cdot, p)/d\xi(p)$ is a meromorphic 1-form on Σ, which has a double pole at p and no other pole. We will define $\langle\gamma, B\rangle$ as follows:

$$p \mapsto \langle\gamma, B(\cdot, p)/d\xi(p)\rangle d\xi(p).$$

This defines, in each chart, a function of p multiplied by $d\xi(p)$, i.e., locally a 1-form. However, nothing guarantees that it is meromorphic (even outside of isolated points), and even just continuous on some open sets. It is easy to find some γ such that $\langle\gamma, B\rangle$ is nowhere holomorphic, as we will see in Section 3.2.1. However, if γ is such that $\langle\gamma, B\rangle$ is locally meromorphic in each chart, then it is independent of the choice of local coordinate ξ and is globally a meromorphic 1-form on Σ.

Definition 3.2 (Generalized cycles). We define the space of generalized cycles as the following subspace of the algebraic dual

$$\mathfrak{M}_1(\mathcal{S}) := \{\gamma \in \mathfrak{M}^1(\Sigma)^* \mid \langle\gamma, B\rangle_{\pi_1} \in \mathfrak{M}^1(\Sigma)_{\pi_2}\},$$

where π_1 (resp., π_2) means the projection to the first (resp., second) factor of $\Sigma \times \Sigma$ for B. Since we pair the first factor of B, the result is a 1-form in the second factor. What we require here is that $\langle\gamma, B\rangle_{\pi_1}$ is meromorphic in the sense of Definition 2.2.

$\mathfrak{M}_1(\mathcal{S})$ is a complex vector space, and we will see below that it is of infinite uncountable dimension.

Remark 3.1. The space of generalized cycles is denoted by $\mathfrak{M}_1(\mathcal{S})$ and not $\mathfrak{M}_1(\Sigma)$ because it depends also on B, so it depends on the full spectral curve, not only the underlying curve.

Definition 3.3 (Map $\hat{B}$: cycles to forms). We define

$$\begin{aligned} \hat{B} : \mathfrak{M}_1(\mathcal{S}) &\to \mathfrak{M}^1(\Sigma), \\ \gamma &\mapsto \hat{B}(\gamma) = \int_\gamma B = \langle\gamma, B\rangle_{\pi_1}. \end{aligned}$$

It is a linear map.

As we will see below, this map is not injective; it has a very large kernel.

Definition 3.4 (Symplectic structure). We define the symplectic form on $\mathfrak{M}_1(\mathcal{S}) \times \mathfrak{M}_1(\mathcal{S})$:

$$\gamma_1 \cap \gamma_2 := \frac{1}{2\pi \mathrm{i}}\left(\int_{\gamma_1} \hat{B}(\gamma_2) - \int_{\gamma_2} \hat{B}(\gamma_1)\right).$$

It obviously satisfies $\gamma_1 \cap \gamma_2 = -\gamma_2 \cap \gamma_1$. We will show below that $\cap$ is a nondegenerate symplectic form and thus $\mathfrak{M}_1(\mathcal{S})$ is a symplectic space.

Lemma 3.1. *If $\Sigma = \Sigma_1 \sqcup \Sigma_2$ is a disjoint union, if we denote the restricted spectral curves $\mathcal{S}_i = (\Sigma_i, \mathrm{x}|_{\Sigma_i}, \mathrm{y}|_{\Sigma_i}, B|_{\Sigma_i \times \Sigma_i})$ for $i = 1, 2$, we have*

$$\mathfrak{M}_1(\mathcal{S}) = \mathfrak{M}_1(\mathcal{S}_1) \oplus \mathfrak{M}_1(\mathcal{S}_2),$$

which means that from now on we may consider only connected spectral curves.

Proof. By the bijection $\omega \mapsto (\omega|_{\Sigma_1}, \omega|_{\Sigma_2})$, we have

$$\mathfrak{M}^1(\Sigma) = \mathfrak{M}^1(\Sigma_1) \oplus \mathfrak{M}^1(\Sigma_2), \quad \mathfrak{M}^1(\Sigma)^* = \mathfrak{M}^1(\Sigma_1)^* \oplus \mathfrak{M}^1(\Sigma_2)^*.$$

Let $\gamma = \gamma_1 + \gamma_2 \in \mathfrak{M}^1(\Sigma)^*$. We have

$$\hat{B}(\gamma) = \omega = \omega_1 + \omega_2,$$

where

$$\omega_1 = \hat{B}_1(\gamma_1) + \hat{B}(\gamma_2)\big|_{\Sigma_1}.$$

Remark that the restriction $B_{2,1}$ of B to $\Sigma_2 \times \Sigma_1$ is holomorphic; it has no pole; therefore, $\hat{B}_{2,1}(\gamma_2)$ is always holomorphic on Σ_1. This implies that $\hat{B}(\gamma)$ is meromorphic if and only if each $\hat{B}_i(\gamma_i)$ is meromorphic. This implies that

$$\gamma = \gamma_1 + \gamma_2 \in \mathfrak{M}_1(\mathcal{S}) \quad \text{iff } \gamma_i \in \mathfrak{M}_1(\mathcal{S}_i) \text{ for } i = 1, 2. \qquad \blacksquare$$

3.2. Explicit construction of $\mathfrak{M}_1(\mathcal{S})$

In fact, we can build explicitly generating families of $\mathfrak{M}_1(\mathcal{S})$. This is done in [6] and in fact was already in [7]. Let us assume that Σ is connected.

3.2.1. What is not a generalized cycle. First, observe that the dual $\mathfrak{M}^1(\mathcal{S})^*$ is a very wild space, and in most cases, for an arbitrary $\gamma \in \mathfrak{M}^1(\mathcal{S})^*$, $\hat{B}(\gamma)$ would *not* be meromorphic, and in fact not even continuous, it can even be analytic nowhere on Σ. Let us show it on an example. Let γ be a Jordan arc and f a complex function analytic in a tubular neighborhood of γ. The map

$$\begin{aligned} \tilde{\gamma} : \mathfrak{M}^1(\Sigma) &\to \mathbb{C}, \\ \omega &\mapsto \int_\gamma f\omega \end{aligned}$$

is called a current. One finds that $\Omega = \hat{B}(\tilde{\gamma})$ is a 1-form, which is discontinuous across γ, with discontinuity

$$\Omega(z^{\text{left}}) - \Omega(z^{\text{right}}) = 2\pi \mathrm{i}\, df(z).$$

Therefore, a current cannot be a generalized cycle unless f is constant.

More generally, if μ is a distribution on a compact closed surface Σ, then the following 2-dimensional integral

$$\hat{\mu} : \omega \mapsto \int_\Sigma \omega \wedge \bar{d}\mu$$

is an element of the dual $\mathfrak{M}^1(\Sigma)^*$, but we have the Poisson relation

$$\bar{\partial}\hat{B}(\hat{\mu}) = 2\pi\mathrm{i}\Delta\mu.$$

In other words, unless μ is harmonic outside isolated points, $\hat{\mu}$ cannot be a generalized cycle.

So, now let us focus on what are generalized cycles.

3.2.2. Canonical local coordinates. The ramified cover $\mathrm{x} : \Sigma \to \underline{\Sigma}$ provides canonical coordinates on Σ.

Definition 3.5 (Canonical local coordinate on Σ). Let $p \in \Sigma$.

- Case $\underline{\Sigma} = \mathbb{C}P^1$.
  Let us define:
  - if $\mathrm{x}(p) \neq \infty$:
    $$d_p := \mathrm{order}_p(\mathrm{x} - \mathrm{x}(p)), \quad x_p := \mathrm{x}(p),$$
    we have $d_p > 0$.
  - if $\mathrm{x}(p) = \infty$:
    $$d_p := \mathrm{order}_p\, \mathrm{x}, \quad x_p := 0,$$
    we have $d_p < 0$.

  Then, we define in a neighborhood of p,
  $$\zeta_p(z) := (\mathrm{x}(z) - x_p)^{1/d_p}.$$
  It is a local coordinate in a neighborhood of p, vanishing at p, called the "canonical local coordinate at p".
  If p is ramified, then $|d_p| > 1$, and the canonical local coordinate at p is defined only up to a d_pth root of unity, we also have the rotated canonical local coordinates
  $$\zeta_p(z)e^{2\pi\mathrm{i}\frac{k}{d_p}}, \quad k = 0, \dots, |d_p| - 1.$$
- Case $\underline{\Sigma}$ a Riemann surface equipped with charts and local coordinates $\underline{\Sigma} \to \mathbb{C}$ chosen once for all, i.e., $x \in \underline{\Sigma}$ can be considered in a given chart as a fixed once for all complex coordinate. We then use for each p, as above, the local coordinate chart associated to $\mathrm{x}(p)$.

3.2.3. Second-kind cycles.

Definition 3.6 (Generalized A-cycles). Let $p \in \Sigma$ and an integer $k \geq 0$. We define $\mathcal{A}_{p,k} \in \mathfrak{M}^1(\Sigma)^*$ by its pairing with any meromorphic 1-form ω:

$$\langle \mathcal{A}_{p,k}, \omega \rangle := 2\pi \mathrm{i} \operatorname{Res}_p {\zeta_p}^k \omega.$$

$\mathcal{A}_{p,k}$ is linear in ω and is thus a well-defined element of $\mathfrak{M}^1(\Sigma)^*$. For $q \neq p$, $B(z,q)$ has no pole at $z = p$, and if $q = p$, it has a double pole at $z = p$, which gives

$$\hat{B}(\mathcal{A}_{p,k})(q) = 2\pi \mathrm{i} \delta_{p,q} \delta_{k,1} d\zeta_p(q).$$

This is a 1-form equal to 0 except at p and bounded by a rational function (in fact a constant) at $q = p$; therefore, it belongs to $\mathfrak{M}^1(\Sigma)$ in the sense of Definition 2.2, and is in the equivalence class of 0, i.e., $\hat{B}(\mathcal{A}_{p,k}) \equiv 0$. This shows that $\mathcal{A}_{p,k} \in \mathfrak{M}_1(\mathcal{S})$. Moreover,

$$\mathcal{A}_{p,k} \in \operatorname{Ker} \hat{B} \subset \mathfrak{M}_1(\mathcal{S}).$$

This shows that $\operatorname{Ker} \hat{B}$ is of infinite uncountable dimension.

Definition 3.7 (Generalized B-cycles). Let $p \in \Sigma$ and an integer $k > 0$. We define $\mathcal{B}_{p,k} \in \mathfrak{M}^1(\Sigma)^*$ by its pairing with any meromorphic 1-form ω:

$$\langle \mathcal{B}_{p,k}, \omega \rangle = \frac{1}{k} \operatorname{Res}_p \zeta_p^{-k} \omega.$$

It is a well-defined element of $\mathfrak{M}^1(\Sigma)^*$. Then,

$$\hat{B}(\mathcal{B}_{p,k}) = \Omega_{p,k} = \text{meromorphic 1-form that has a pole at } p \text{ of degree } k+1,$$

and thus belongs to $\mathfrak{M}^1(\Sigma)$; more precisely,

$$\Omega_{p,k} \sim \zeta_p^{-k-1} d\zeta_p + \text{holomorphic}. \tag{3.1}$$

This shows that

$$\mathcal{B}_{p,k} \in \mathfrak{M}_1(\mathcal{S}).$$

This shows that a complement of $\operatorname{Ker} \hat{B}$ is of infinite uncountable dimension.

Definition 3.8 (2nd kind times (generalized periods)). Let $p \in \Sigma$ and an integer $k \geq 0$. We define for any $\omega \in \mathfrak{M}^1(\Sigma)$:

$$t_{p,k}(\omega) = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{A}_{p,k}} \omega = \operatorname{Res}_p \zeta_p^k \omega.$$

Notice that

$$t_{p,0}(\omega) = \operatorname{Res}_p \omega.$$

Under a change of root of unity for the canonical local coordinate $\zeta_p \to \zeta_p e^{2\pi \mathrm{i}\frac{j}{d_p}}$, we have

$$t_{p,k}(\omega) \to t_{p,k}(\omega) e^{2\pi \mathrm{i}\frac{kj}{d_p}}.$$

Lemma 3.2. *For any $\omega \in \mathfrak{M}^1(\Sigma)$, for any $p \in \Sigma$, in a neighborhood of p, the following 1-form:*

$$\omega - \sum_{k=0}^{\deg_p \omega - 1} t_{p,k}(\omega) \zeta_p^{-k-1} d\zeta_p$$

is holomorphic in a neighborhood of p (coincides with a holomorphic 1-form except maybe at isolated points).

Proof. It is obvious. ∎

Theorem 3.1 (Symplectic normalization of 2nd kind $\mathcal{A}$ and $\mathcal{B}$ cycles). *We have*

$$\mathcal{A}_{p,k} \cap \mathcal{B}_{q,j} = \delta_{p,q} \delta_{k,j}.$$

Proof. It can be easily computed using (3.1). ∎

Definition 3.9 (Evaluation cycle). Let us define

$$\mathrm{ev}_z = d\zeta_z(z) \otimes \mathcal{B}_{z,1}.$$

If $\omega \in \mathfrak{M}^1(\Sigma)$ is holomorphic at z, we have

$$\langle \mathrm{ev}_z, \omega \rangle = \omega(z),$$

whence the name "evaluation" cycle.

If ω is not holomorphic at z, then $\langle \mathrm{ev}_z, \omega \rangle$ is the finite part of $\omega(z)$, i.e., obtained after subtracting the polar part

$$\langle \mathrm{ev}_z, \omega \rangle = \lim_{z' \to z, z' \neq z} \left(\omega(z') - \sum_{k=0}^{\deg_z \omega - 1} t_{z,k}(\omega) \zeta_z(z')^{-k-1} d\zeta_z(z') \right).$$

We have

$$\mathrm{ev} \in H_0(\Sigma, K_\Sigma \otimes \mathfrak{M}_1(\mathcal{S})).$$

3.2.4. Third-kind cycles, Jordan arcs. We will immerse Jordan arcs as generalized cycles.

Let $\mathcal{J}$ be the set of all Jordan arcs on Σ, piecewise C^1 and with compact support and nowhere stationary (having a non-zero left and right tangent vector at every point). Let $\gamma \in \mathcal{J}$ be a Jordan arc, i.e., a continuous piecewise C^1 function with compact image and $\gamma : I \to \Sigma$, where $I = [a, b] \subset \mathbb{R}$ is an interval of $\mathbb{R}$ with $b > a$ (and possibly $a = -\infty$ and/or $b = +\infty$). We denote the endpoints $p = \gamma(a)$ and $q = \gamma(b)$. (if $p = q$ then γ is called a Jordan loop). The image $\operatorname{supp}\gamma = \gamma(I) \subset \Sigma$ is called the support of γ. We also allow the point, i.e., the case $a = b$ and $\operatorname{supp}\gamma = \{p\}$.

Definition 3.10 (Regularized integrals). Let $\gamma \in \mathcal{J}$ be a Jordan arc. We define a map $\hat{\gamma} : \mathfrak{M}^1(\Sigma) \to \mathbb{C}$ as follows.

For any $\omega \in \mathfrak{M}^1(\Sigma)$, we define the following:

- if $\operatorname{supp}\gamma$ is a point, we define $\hat{\gamma} = 0$, i.e.,

$$\langle \hat{\gamma}, \omega \rangle = 0, \tag{3.2}$$

- if ω has no pole on the support of γ, we define

$$\langle \hat{\gamma}, \omega \rangle = \int_\gamma \omega, \tag{3.3}$$

- if ω has a pole at the starting point $p = \gamma(a)$ and at no other $\gamma(t)$ for $t \in]a, b]$, we let $c \in]a, b]$ close enough to a such that $\gamma|_{[a,c]}$ is in a neighborhood of p in which the canonical local coordinate ζ_p is well defined. Then, define $o = \gamma(c)$ and

$$\begin{aligned}\langle \hat{\gamma}, \omega \rangle = &\int_{\gamma|_{]c,b]}} \omega + \int_{\gamma|_{[a,c]}} (\omega - \sum_k t_{p,k}(\omega)\zeta_p^{-k-1} d\zeta_p) \\ &- \sum_{k=1}^{\deg_p \omega - 1} \frac{t_{p,k}(\omega)}{k} \zeta_p(o)^{-k} + t_{p,0}(\omega)(-\mathrm{i}\theta_p + \ln \zeta_p(o)),\end{aligned} \tag{3.4}$$

where the branch-cut of the log is defined so that $\operatorname{Im}\ln\zeta_p(o) = \operatorname{Arg}\zeta_p(o) \in [\theta_p - \pi, \theta_p + \pi[$ for o close enough to p, and where $\theta_p = \frac{1}{d_p}\operatorname{Arg}\gamma'(a)$ is the angle of the tangent of γ at $p = \gamma(a)$ in the coordinate ζ_p.

This definition is independent of the choice of $c \in]a, b]$, i.e., of $o \in \operatorname{supp}\gamma$,

- if ω has a pole at the endpoint $q = \gamma(b)$ and at no other $\gamma(t)$ for $t \in [a, b[$, we define

$$\langle \hat{\gamma}, \omega \rangle = -\langle \widehat{-\gamma}, \omega \rangle, \tag{3.5}$$

where $-\gamma(t) := \gamma(b + a - t)$, and we use (3.4) for $\widehat{-\gamma}$,

- if ω has a pole at $p = \gamma(a)$ and $q = \gamma(b)$ and at no other $\gamma(t)$ for $t \in]a, b[$, we let any $c \in]a, b[$ and define

$$\langle \hat{\gamma}, \omega \rangle = \langle \hat{\gamma}|_{[a,c]}, \omega \rangle + \langle \hat{\gamma}|_{[c,b]}, \omega \rangle \tag{3.6}$$

(this is independent of the choice of $c \in]a, b[$),

- since $\operatorname{supp}\gamma$ is compact and ω meromorphic, ω can have only finitely many poles on $\operatorname{supp}\gamma$. If ω has poles at $p_1 = \gamma(t_1), \dots, p_\ell = \gamma(t_\ell)$ with $t_1 < t_2 < \cdots < t_\ell$ on the support of γ, we let $t_0 = a$ and $t_{\ell+1} = b$, and define

$$\langle \hat{\gamma}, \omega \rangle = \sum_{i=0}^{\ell} \langle \hat{\gamma}|_{[t_i, t_{i+1}]}, \omega \rangle. \tag{3.7}$$

Lemma 3.3 (Avoiding poles from left or right). *Let U be a connected compact lens tubular neighborhood of γ such that $U \setminus \gamma$ is a union of 2 connected components $U \setminus \gamma = U_{\text{left}} \cup U_{\text{right}}$. Let γ_{left} (resp., γ_{right}) denote a Jordan arc homotopic deformation of γ in U_{left} (resp., in U_{right}), and crepant to γ at the extremities (same tangents at the extremities). We define*

$$\langle \hat{\gamma}_{\text{left}}, \omega \rangle = \lim_{\text{width of } U \to 0} \langle \widehat{\gamma_{\text{left}}}, \omega \rangle, \quad (\textit{resp.}, \langle \hat{\gamma}_{\text{right}}, \omega \rangle = \lim_{\text{width of } U \to 0} \langle \widehat{\gamma_{\text{right}}}, \omega \rangle).$$

The limit is a well-defined projective limit because $\langle \widehat{\gamma_{\text{left}}}, \omega \rangle$ (resp., $\langle \widehat{\gamma_{\text{right}}}, \omega \rangle$) is constant when width of $U < \inf_{r = \text{poles of } \omega \text{ in } U} D(r, \gamma)$ for any distance D in U.

We have

$$\begin{aligned} \hat{\gamma} &= \hat{\gamma}_{\text{left}} + \sum_{p \in \overset{\circ}{\operatorname{supp}}\,\gamma} \frac{\pi + \theta_{p_-} - \theta_{p_+}}{2\pi} \mathcal{A}_{p,0} \\ &= \hat{\gamma}_{\text{right}} - \sum_{p \in \overset{\circ}{\operatorname{supp}}\,\gamma} \frac{\pi - \theta_{p_-} + \theta_{p_+}}{2\pi} \mathcal{A}_{p,0} = \frac{1}{2}(\hat{\gamma}_{\text{left}} + \hat{\gamma}_{\text{right}}), \end{aligned}$$

where, if $p = \gamma(t)$, we define p_- (resp., p_+)$=\lim_{s \to t+0} \gamma(s)$ (resp., $= \lim_{s \to t-0} \gamma(s)$) and θ_{p_-} (resp., θ_{p_+}) is the angle (in the coordinate ζ_p) of the tangent vector of γ to the left (resp., right) of p.

Notice that if p is a point where γ is C^1, the 2 angles coincide $\theta_{p_-} = \theta_{p_+}$.

Proof. First, remark that the sum $\sum_{p \in \overset{\circ}{\operatorname{supp}}\,\gamma} c_p \mathcal{A}_{p,0}$ is a well-defined generalized cycle because for any meromorphic ω, there can be only finitely many poles on $\operatorname{supp}\gamma$, and

$$\left\langle \sum_{p \in \overset{\circ}{\operatorname{supp}}\,\gamma} c_p \mathcal{A}_{p,0}, \omega \right\rangle = 2\pi \mathrm{i} \sum_{p = \text{poles of } \omega \text{ on } \overset{\circ}{\operatorname{supp}}\,\gamma} c_p \operatorname{Res}_p \omega$$

is well defined.

If ω has no pole between p and q, the lemma is obvious.

It suffices to prove it in the case (other cases follow by concatenation) where ω has a unique pole at r on γ and no other pole, and assuming that γ is contained in a chart where coordinate ζ_r is well defined and where ω has no other pole than r.

In that chart, we have

$$\begin{aligned}\langle\hat{\gamma},\omega\rangle &= \int_p^r (\omega - \sum_{k=0}^{\deg_r\omega-1} t_{r,k}(\omega)\zeta_r^{-k-1}d\zeta_r) \\ &\quad + \sum_{k=1}^{\deg_r\omega-1} \frac{t_{r,k}(\omega)}{k}\zeta_r(p)^{-k} - t_{r,0}(\omega)(\ln_-\zeta_r(p) - \pi\mathrm{i} - \mathrm{i}\theta_{r_-}) \\ &\quad + \int_r^q (\omega - \sum_{k=0}^{\deg_r\omega-1} t_{r,k}(\omega)\zeta_r^{-k-1}d\zeta_r) \\ &\quad - \sum_{k=1}^{\deg_r\omega-1} \frac{t_{r,k}(\omega)}{k}\zeta_r(q)^{-k} + t_{r,0}(\omega)(\ln_+\zeta_r(q) - \mathrm{i}\theta_{r_+}),\end{aligned}$$

where $\ln_-$ (resp., $\ln_+$) is the determination of the logarithm with cut in the opposite direction of the left (resp., right) tangent at r, i.e.,

$$\operatorname{Im}\ln_-\zeta_r(z) \in [\theta_{r_-}, \theta_{r_-} + 2\pi[, \quad \operatorname{Im}\ln_+\zeta_r(z) \in [\theta_{r_+} - \pi, \theta_{r_+} + \pi[.$$

Since $\omega - \sum_{k=0}^{\deg_r\omega-1} t_{r,k}(\omega)\zeta_r^{-k-1}d\zeta_r$ has no pole on the chart, we can deform the integration contour to γ_{left}:

$$\begin{aligned}\langle\hat{\gamma},\omega\rangle &= \int_{\gamma_{\mathrm{left}}} (\omega - \sum_{k=0}^{\deg_r\omega-1} t_{r,k}(\omega)\zeta_r^{-k-1}d\zeta_r) \\ &\quad + \sum_{k=1}^{\deg_r\omega-1} \frac{t_{r,k}(\omega)}{k}\zeta_r(p)^{-k} - t_{r,0}(\omega)\ln_-\zeta_r(p) \\ &\quad - \sum_{k=1}^{\deg_r\omega-1} \frac{t_{r,k}(\omega)}{k}\zeta_r(q)^{-k} + t_{r,0}(\omega)\ln_+\zeta_r(q) + \mathrm{i}t_{r,0}(\omega)(\pi + \theta_{r_-} - \theta_{r_+}) \\ &= \int_{\gamma_{\mathrm{left}}} (\omega - \sum_{k=0}^{\deg_r\omega-1} t_{r,k}(\omega)\int_{\gamma_{\mathrm{left}}} \zeta_r^{-k-1}d\zeta_r) \\ &\quad + \sum_{k=1}^{\deg_r\omega-1} \frac{t_{r,k}(\omega)}{k}\zeta_r(p)^{-k} - t_{r,0}(\omega)\ln_-\zeta_r(p) \\ &\quad - \sum_{k=1}^{\deg_r\omega-1} \frac{t_{r,k}(\omega)}{k}\zeta_r(q)^{-k} + t_{r,0}(\omega)\ln_+\zeta_r(q) + \mathrm{i}t_{r,0}(\omega)(\pi + \theta_{r_-} - \theta_{r_+}) \\ &= \int_{\gamma_{\mathrm{left}}} \omega - t_{r,0}(\omega)\Big(\int_{\gamma_{\mathrm{left}}} \zeta_r^{-1}d\zeta_r + \ln_-\zeta_r(p) - \ln_+\zeta_r(q)\Big) \\ &\quad + \mathrm{i}t_{r,0}(\omega)(\pi + \theta_{r_-} - \theta_{r_+}).\end{aligned}$$

The integral of $\zeta_r^{-1} d\zeta_r$ on γ_{left} (resp., γ_{right}) goes above (resp., below) zero and gives $\ln_+ \zeta_r(q) - \ln_- \zeta_r(p)$ modulo $2\pi\mathrm{i}$, and eventually we get the result. ∎

Theorem 3.2 (Map Jordan arcs → generalized cycles). *We have the following.*

- *The map $\gamma \mapsto \hat{\gamma}$ is well defined.*
- *$\langle \hat{\gamma}, \omega \rangle$ is a linear in ω; i.e., $\hat{\gamma} \in \mathfrak{M}^1(\Sigma)^*$.*
- *$\hat{\gamma}$ is invariant by reparametrization of γ; i.e., if $f : [a', b'] \to [a, b]$ is a monotonic piecewise C^1 bijection, we have*

$$\widehat{\gamma \circ f} = \hat{\gamma}.$$

- *The map $\gamma \mapsto \hat{\gamma}$ is additive under concatenation and satisfies*

$$\widehat{\gamma_1 + \gamma_2} = \hat{\gamma}_1 + \hat{\gamma}_2, \quad \widehat{-\gamma} = -\hat{\gamma}.$$

- *$\hat{\gamma}$ is a generalized cycle, $\hat{\gamma} \in \mathfrak{M}_1(\mathcal{S})$. This defines a map that immerses Jordan arcs into generalized cycles*

$$\begin{aligned} \mathcal{J} &\to \mathfrak{M}_1(\mathcal{S}), \\ \gamma &\mapsto \hat{\gamma}, \end{aligned}$$

  which is such that for any meromorphic 1-form $\omega \in \mathfrak{M}^1(\Sigma)$ not having pole on the support of γ we have

$$\langle \hat{\gamma}, \omega \rangle = \int_{\hat{\gamma}} \omega = \int_{\gamma} \omega.$$

  (This justifies the Poincaré pairing notation and the name "generalized cycles".)
- *$\hat{B}(\hat{\gamma})$ is a meromorphic 1- form that has poles at the extremities of γ, a simple pole at p of residue -1, a simple pole at q of residue $+1$ and no other pole on Σ (whence the name 3rd-kind cycle):*

$$-\operatorname{Res}_p \hat{B}(\hat{\gamma}) = \operatorname{Res}_q \hat{B}(\hat{\gamma}) = 1.$$

- *It can be extended to a linear map on an $\mathbb{A}$-module subring of $\mathbb{C}$ (like $\mathbb{A} = \mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$) of linear combinations of Jordan arcs*

$$\widehat{\sum_{i=1}^{k} c_i \gamma_i} = \sum_{i=1}^{k} c_i \hat{\gamma}_i.$$

Remark 3.2 (Identifying γ with $\hat{\gamma}$). From now on, thanks to this map, and for the sake of simplifying notations, we will identify a Jordan arc γ with its generalized cycle $\hat{\gamma}$. This is what is done most of the time in the literature of topological recursion.

Proof. • First, the map $\hat{\gamma}$ is well defined. One can check that it is independent of o in (3.4) (take the derivative with respect to o), independent of c in (3.6). Each summand in (3.7) is exclusively defined by (3.4) or (3.6) or (3.5) or (3.2) or (3.3), so is well defined.

- It is easy to see that each term is linear in ω, so that $\hat{\gamma} \in \mathfrak{M}^1(\Sigma)^*$.
- $\hat{\gamma}$ is invariant by reparametrization because all terms are.
- It is easy to see that $\hat{\gamma}$ is additive by concatenation, nearly from construction (it is defined by adding concatenations of pieces).
- To prove that $\hat{\gamma}$ is a generalized cycle, we need to compute $\Omega_\gamma = \hat{B}(\hat{\gamma})$.
  Let $z \in \Sigma$; then, we have the following:
  - if $z \notin \operatorname{supp}\gamma$, then $B(.,z)$ has no pole on $\operatorname{supp}\gamma$, and $\Omega_\gamma(z) = \int_\gamma B(.,z)$ is a convergent integral and locally analytic of z,
  - if $z \in \overset{\circ}{\operatorname{supp}}\,\gamma$ such that $z \neq p$ and $z \neq q$, Lemma 3.3 gives

$$\begin{aligned}\Omega_\gamma(z) &= \langle \hat{\gamma}_{\text{left}}, B(\cdot,z)\rangle + \pi \mathrm{i} \sum_{r \in \overset{\circ}{\operatorname{supp}}\,\gamma} \operatorname{Res}_r B(\cdot,z) \\ &= \langle \hat{\gamma}_{\text{left}}, B(\cdot,z)\rangle + \pi \mathrm{i} \operatorname{Res}_z B(\cdot,z).\end{aligned}$$

    Since B is symmetric, it has a double pole at z without residue, so that we have $\operatorname{Res}_z B(\cdot,z) = 0$.
    Moreover, since $z \notin \gamma_{\text{left}}$, we are back to the 1st situation, and Ω_γ is locally analytic of z,
  - if z is in a neighborhood of p and outside of $\operatorname{supp}\gamma$, we have

$$\begin{aligned}\Omega_\gamma(z) &= \int_{z'=p}^{o} B(z',z) + \int_{z'=o}^{q} B(z',z) = \int_{z'=p}^{o} \frac{d\zeta_p(z') \otimes d\zeta_p(z)}{(\zeta_p(z)-\zeta_p(z'))^2} \\ &\quad + \int_{z'=p}^{o} \left(B(z',z) - \frac{d\zeta_p(z') \otimes d\zeta_p(z)}{(\zeta_p(z)-\zeta_p(z'))^2} \right) + \int_{z'=o}^{q} B(z',z) \\ &= -\frac{d\zeta_p(z)}{\zeta_p(z)} + \frac{d\zeta_p(z)}{\zeta_p(z)-\zeta_p(o)} \\ &\quad + \int_{z'=p}^{o} \left(B(z',z) - \frac{d\zeta_p(z') \otimes d\zeta_p(z)}{(\zeta_p(z)-\zeta_p(z'))^2} \right) + \int_{z'=o}^{q} B(z',z) \\ &= -\frac{d\zeta_p(z)}{\zeta_p(z)} + \text{holomorphic at } p,\end{aligned}$$

  - similarly, if z is in a neighborhood of q, we have

$$\Omega_\gamma(z) = \frac{d\zeta_q(z)}{\zeta_p(z)} + \text{holomorphic at } q.$$

Putting all together, we obtain that $\Omega_\gamma = \hat{B}(\hat{\gamma})$ is a meromorphic 1-form with a simple pole at $p = \gamma(a)$ with residue -1, a simple pole at $q = \gamma(b)$ with residue $+1$, and no other pole. And if $p = q$, it is a holomorphic form without any poles on Σ. This implies that $\hat{\gamma} \in \mathfrak{M}_1(\mathcal{S})$. ∎

Theorem 3.3 (Intersections of Jordan arcs). *Let γ_1 and γ_2 be two Jordan arcs.*

- *If γ_1 and γ_2 do not intersect, we have*
$$\hat{\gamma}_1 \cap \hat{\gamma}_2 = 0 = \gamma_1 \cap \gamma_2,$$
where $\gamma_1 \cap \gamma_2$ is the usual intersection form.
- *If γ_1 and γ_2 intersect transversally at a point p different from the extremities, and the tangent of γ_i is in the direction of angle θ_i, and $\theta_2 - \theta_1 \in]-\pi, \pi[$, we have*
$$\hat{\gamma}_1 \cap \hat{\gamma}_2 = \operatorname{sign}(\theta_2 - \theta_1) = \gamma_1 \cap \gamma_2.$$
- *If the intersection p is the starting point of γ_1 and a middle point of γ_2, we have*
$$\hat{\gamma}_1 \cap \hat{\gamma}_2 = \frac{1}{2} \operatorname{sign}(\theta_2 - \theta_1).$$
- *If the intersection is the starting point of γ_1 and the starting point of γ_2, we have*
$$\hat{\gamma}_1 \cap \hat{\gamma}_2 = \frac{1}{2} \operatorname{sign}(\theta_2 - \theta_1) - \frac{1}{2\pi}(\theta_2 - \theta_1) \in \Big[-\frac{1}{2}, \frac{1}{2}\Big].$$
- *If p is an ending point, revert $\gamma \to -\gamma$ and use the fact that*
$$(-\hat{\gamma}_1) \cap \hat{\gamma}_2 = -\hat{\gamma}_1 \cap \hat{\gamma}_2, \quad \hat{\gamma}_2 \cap \hat{\gamma}_1 = -\hat{\gamma}_1 \cap \hat{\gamma}_2.$$
- *If there are several intersection points, the intersection is the algebraic sum of contributions of each intersection point.*

Proof. Let $\gamma_1 : [a, b] \to \Sigma$ and $\gamma_2 : [\tilde{a}, \tilde{b}] \to \Sigma$. We have to compute
$$\int_{z \in \gamma_1} \Big(\int_{z' \in \gamma_2} B(z', z) \Big) - \int_{z' \in \gamma_2} \Big(\int_{z \in \gamma_2} B(z', z) \Big).$$

• If γ_1 and γ_2 do not intersect, the pole of B at $z = z'$ is never on the integration paths, and the order of integration commutes, and the difference gives 0.

• Now, consider the case where γ_1 and γ_2 have an intersection point $\gamma_1 \cap \gamma_2 = \{p\}$, which we assume transverse. Let $r > 0$, small enough such that inside the disc $D(p, r)$ of center p and radius r, $\Gamma_1 = \gamma_1 \cap D(p, r)$ is a connected Jordan arc and $\Gamma_2 = \gamma_2 \cap D(p, r)$ is a connected Jordan arc. Let us decompose the paths γ_i with the part outside the disc $D(p, r)$ and the part inside the disc
$$\gamma_1 = \overline{\gamma}_1 + \Gamma_1, \quad \gamma_2 = \overline{\gamma}_2 + \Gamma_2.$$

We thus have

$$\int_{z\in\gamma_1}\Big(\int_{z'\in\gamma_2} B(z',z)\Big) = \int_{z\in\overline{\gamma}_1}\Big(\int_{z'\in\overline{\gamma}_2} B(z',z)\Big) + \int_{z\in\overline{\gamma}_1}\Big(\int_{z'\in\Gamma_2} B(z',z)\Big)$$
$$+ \int_{z\in\Gamma_1}\Big(\int_{z'\in\overline{\gamma}_2} B(z',z)\Big) + \int_{z\in\Gamma_1}\Big(\int_{z'\in\Gamma_2} B(z',z)\Big).$$

In the first 3 integrals, the pole of $B(z,z')$ is not on the integration paths and thus the order of integration commutes. This implies that we can reduce the computation only for the parts inside the disc

$$2\pi\mathrm{i}\hat{\gamma}_1 \cap \hat{\gamma}_2 = 2\pi\mathrm{i}\hat{\Gamma}_1 \cap \hat{\Gamma}_2.$$

Inside the disc $D(p,r)$, we use the local coordinate $z=\zeta_p$ and write

$$B(z',z) = \frac{dz'\otimes dz}{(z'-z)^2} + f(z',z)dz'\otimes dz,$$

where $f(z',z)$ is analytic in $D(p,r)\times D(p,r)$. In particular, the double integral with f behaves as $O(r^2)$ as $r\to 0$.

Let us assume that the tangents of γ_1 and γ_2 at p are in direction θ_1 and θ_2, and assume $\theta_2-\theta_1\in\,]-\pi,\pi[$. Let us define the following integral, that is, the integral from p on the half-Jordan arcs starting from p of γ_1 and γ_2:

$$\begin{aligned}
I_{\theta_1,\theta_2} &= \mathrm{Im}\int_{\mathrm{e}^{\mathrm{i}\theta_1}0_+}^{\mathrm{e}^{\mathrm{i}\theta_1}r} dx \int_{\mathrm{e}^{\mathrm{i}\theta_2}0_+}^{\mathrm{e}^{\mathrm{i}\theta_2}r} \frac{dx'}{(x-x')^2}\\
&= \mathrm{Im}\int_{0_+}^{1} dx\Big(\frac{1}{x-e^{\mathrm{i}(\theta_2-\theta_1)}} - \frac{1}{x}\Big)\\
&= \mathrm{Arg}\,\frac{1-e^{\mathrm{i}(\theta_2-\theta_1)}}{-e^{\mathrm{i}(\theta_2-\theta_1)}}\\
&= \mathrm{Arg}(1-e^{-\mathrm{i}(\theta_2-\theta_1)})\\
&= \frac{1}{2}(\pi\,\mathrm{sign}(\theta_2-\theta_1)-(\theta_2-\theta_1)) \in [-\pi/2,\pi/2].
\end{aligned}$$

The integrals computing the intersection are then sums, taking into account all ways to glue the half-Jordan arcs to make the full arcs.

- If p is not an endpoint for both paths, we have

$$\begin{aligned}
&(I_{\theta_1,\theta_2} - I_{\theta_1,\theta_2\pm\pi}) - (I_{\theta_1\pm\pi,\theta_2} - I_{\theta_1\pm\pi,\theta_2\pm\pi})\\
&\quad= \Big(\frac{1}{2}(\pi+\theta_1-\theta_2) - \frac{1}{2}(\theta_1-\theta_2)\Big) - \Big(\frac{1}{2}(\theta_1-\theta_2) - \frac{1}{2}(\pi+\theta_1-\theta_2)\Big)\\
&\quad= \pi\,\mathrm{sign}(\theta_2-\theta_1).
\end{aligned}$$

This gives

$$\begin{aligned}2\pi\hat{\gamma}_1 \cap \hat{\gamma}_2 &= ((I_{\theta_1,\theta_2} - I_{\theta_1,\theta_2\pm\pi}) - (I_{\theta_1\pm\pi,\theta_2} - I_{\theta_1\pm\pi,\theta_2\pm\pi})) \\ &\quad - ((I_{\theta_2,\theta_1} - I_{\theta_2,\theta_1\pm\pi}) - (I_{\theta_2\pm\pi,\theta_1} - I_{\theta_2\pm\pi,\theta_1\pm\pi})) \\ &= \pi(\operatorname{sign}(\theta_2-\theta_1) - \operatorname{sign}(\theta_1-\theta_2)),\end{aligned}$$

i.e.,

$$\hat{\gamma}_1 \cap \hat{\gamma}_2 = \operatorname{sign}(\theta_2-\theta_1) = \gamma_1 \cap \gamma_2.$$

• Case p is the starting point of γ_1 and a middle point of γ_2; we have

$$2\pi\hat{\gamma}_1 \cap \hat{\gamma}_2 = (I_{\theta_1,\theta_2} - I_{\theta_1,\theta_2\pm\pi}) - (I_{\theta_2,\theta_1} - I_{\theta_2\pm\pi,\theta_1}) = \pi\operatorname{sign}(\theta_2-\theta_1),$$

i.e.,

$$\hat{\gamma}_1 \cap \hat{\gamma}_2 = \frac{1}{2}\operatorname{sign}(\theta_2-\theta_1).$$

• Case p is the starting point of γ_1 and of γ_2; we have

$$2\pi\hat{\gamma}_1 \cap \hat{\gamma}_2 = I_{\theta_1,\theta_2} - I_{\theta_2,\theta_1} = \pi\operatorname{sign}(\theta_2-\theta_1) - (\theta_1-\theta_1),$$

i.e.,

$$\hat{\gamma}_1 \cap \hat{\gamma}_2 = \frac{1}{2}\operatorname{sign}(\theta_2-\theta_1) - \frac{1}{2\pi}(\theta_2-\theta_1).$$

Notice that

$$\hat{\gamma}_1 \cap \hat{\gamma}_2 \in \Big[-\frac{1}{2},\frac{1}{2}\Big].$$

If instead of being the starting point p is an ending point, we just revert $\gamma_i \to -\gamma_i$, and if there are several intersection points, we take the algebraic sum. ■

3.2.5. First-kind cycles. First-kind cycles are the images of closed Jordan loops, those verifying $\gamma(a) = \gamma(b)$.

Definition 3.11 (First-kind cycles). Let

$$\mathcal{J}_0 = \{\gamma \in \mathcal{J} \mid \gamma(a) = \gamma(b)\}.$$

We have a map

$$\begin{aligned}\mathcal{J}_0 &\to \mathfrak{M}_1(\mathcal{S}),\\ \gamma &\mapsto \hat{\gamma}.\end{aligned}$$

It can be extended to a linear map on the module of linear combinations of Jordan loops

$$\widehat{\sum_{i=1}^{k} c_i\gamma_i} = \sum_{i=1}^{k} c_i\hat{\gamma}_i.$$

The following theorem is obvious.

Theorem 3.4. *If γ is a Jordan loop, then*

$$\hat{B}(\hat{\gamma}) = \text{holomorphic 1-form on } \Sigma \in \Omega^1(\Sigma).$$

As a corollary of Theorem 3.3, we have the following theorem.

Theorem 3.5 (Intersections). *The intersection of 1st-kind generalized cycles is the same as the usual intersection of the corresponding homology classes of Jordan loops:*

$$\hat{\gamma}_1 \cap \hat{\gamma}_2 = \gamma_1 \cap \gamma_2.$$

3.2.6. Boundary cycles. Assume that Σ has some boundary b that is topologically a circle, oriented so that Σ is to the right of b, and such that $\mathrm{x}(b) = b_0$ is a circle on the base $\underline{\Sigma}$, and we choose a local coordinate on $\underline{\Sigma}$ so that the circle $\mathrm{x}(b)$ is the unit circle $S^1 \subset \mathbb{C}$, and the image of a neighborhood of b is sent to the exterior of the unit disc in $\mathbb{C}$.

Notice that x being a ramified covering map, it may happen that x in a neighborhood of b is not 1:1, but $d_b : 1$. We say that $d_b \geq 1$ is the *winding number* of b.

We have the following definition.

Definition 3.12 (Local coordinate on b).

$$\zeta_b(p) := \mathrm{x}(p)^{1/d_b}.$$

The coordinate ζ_b is well defined in an annulus $1 < |\zeta_b| < r_b$.

Also, we recall that our requirement for meromorphic 1-forms is that they are meromorphic on the interior of Σ, which means that they can be singular on the boundary.

Definition 3.13 (Boundary cycles). We associate a family of generalized cycles to a boundary b as follows.

- Cycle $\mathcal{A}_{b,k}$. For $k \geq 0$, we define

$$\mathcal{A}_{b,k} : \mathfrak{M}^1(\Sigma) \to \mathbb{C},$$

$$\omega \mapsto \langle \mathcal{A}_{b,k}, \omega \rangle = \liminf_{r \to 1+} \int_{\mathrm{x}^{-1}(rS^1)} \zeta_b^k \omega = \liminf_{r \to 1+} \int_{\theta=0}^{2\pi} r^k e^{\mathrm{i}k\theta} \omega.$$

The lim inf is a well-defined projective limit because the integral is independent of r if $1 < r < r_\omega$ with

$$r_\omega \leq \min \left\{ |\zeta_b(p)| \mid p = \text{poles of } \omega \right\} \cup \{r_b\}.$$

- Cycle $\mathcal{B}_{b,k}$. For $k > 0$, we define

$$\mathcal{B}_{b,k} : \mathfrak{M}^1(\Sigma) \to \mathbb{C},$$
$$\omega \mapsto \langle \mathcal{B}_{b,k}, \omega \rangle = \frac{1}{2\pi \mathrm{i} k} \liminf_{r\to 1+} \int_{\mathrm{x}^{-1}(rS^1)} \zeta_b^{-k} \omega.$$

  Again, the lim inf is a well-defined projective limit because the integral is independent of r if $1 < r < r_\omega$.

- Cycle $\mathcal{B}_{b,0}$. For $k = 0$, let o be some generic point of Σ and $\gamma_{o\to\zeta_{b_j}^{-1}(r)}$ a Jordan arc from o to the point of coordinate $\zeta_{b_j} = r$ on the boundary b_j. We define

$$\mathcal{B}_{b,0,o} : \mathfrak{M}^1(\Sigma) \to \mathbb{C},$$
$$\omega \mapsto \langle \mathcal{B}_{b,0,o}, \omega \rangle = \liminf_{r\to 1+} \left(\int_{\hat{\gamma}_{o\to\zeta_{b_j}^{-1}(r)}} \omega + \frac{-1}{2\pi} \int_{\theta=0}^{2\pi} \theta\omega(\zeta_b^{-1}(re^{\mathrm{i}\theta})) - \frac{\ln r}{2\pi \mathrm{i}} \int_{\mathcal{A}_{b_j,0}} \omega \right).$$

  The lim inf is a well-defined projective limit because the right-hand side is independent of r for $1 < r < r_\omega$.

Theorem 3.6. *$\mathcal{A}_{b,k}$ and $\mathcal{B}_{b,k}$ are generalized cycles. We have*

$$\mathcal{A}_{b,k} \cap \mathcal{B}_{b',k'} = \delta_{b,b'}\delta_{k,k'}.$$

Proof. The fact that the lim inf exists because these integrals are independent of r if $1 < r < r_\omega$ implies that the pairing to any meromorphic 1-form is a complex number, so these are linear forms on $\mathfrak{M}^1(\Sigma)$. The fact that integrating $B(\cdot, z_0)$ yields a meromorphic 1-form is again because if $1 < r < |\zeta_b(z_0)|$, the integration path does not contain the pole of $B(\cdot, z_0)$. ■

Remark 3.3 (Poles as boundaries). One can remove a small disc around a pole and make it a boundary, and then, these definitions coincide with the definitions of 2nd-kind cycles associated to poles. This remark is very useful to do surgery of surfaces.

Lemma 3.4 (Fourier series). *Any $\omega \in \mathfrak{M}^1(\Sigma)$ is holomorphic in the tubular neighborhood of b in the annulus $1 < |\zeta_b| < r_\omega$. In the local coordinate ζ_b, we can write*

$$\omega = f(\zeta_b)d\zeta_b = f(re^{\mathrm{i}\theta})(dr + \mathrm{i}rd\theta)e^{\mathrm{i}\theta},$$

where

$$\zeta_b = re^{\mathrm{i}\theta}, \quad r_\omega > r > 1.$$

The function $f(re^{\mathrm{i}\theta})$ *is a periodic* C^∞ *function of* θ *of period* 2π*, for any* r *in a neighborhood of* 1*. It can thus be decomposed as a Fourier series as follows:*

$$\omega(z) = \sum_{k=-\infty}^{\infty} c_{b,k}\zeta_b(z)^{-k-1}d\zeta_b(z), \quad c_{b,k} = \frac{r^k}{2\pi\mathrm{i}}\int_{\theta=0}^{2\pi} e^{k\mathrm{i}\theta}\omega = \frac{1}{2\pi\mathrm{i}}\int_{\theta=0}^{2\pi}\zeta_b^k\omega. \tag{3.8}$$

The Fourier series is absolutely and uniformly convergent for any $r_\omega > r > 1$*, which implies that at large* $k > 0$*, we have*

$$|c_{b,k}| = O(r_\omega^k), \quad |c_{b,-k}| = O(1).$$

Notice that if $k \geq 0$*,*

$$\langle \mathcal{A}_{b,k}, \omega\rangle = \int_{\mathcal{A}_{b,k}} \omega = 2\pi\mathrm{i}c_{b,k} = O(r_\omega^k) \quad \textit{for large } k \geq 0,$$

and if $k > 0$*, we have*

$$\langle \mathcal{B}_{b,k}, \omega\rangle = \int_{\mathcal{B}_{b,k}} \omega = \frac{1}{k}c_{b,-k} = O(1) \quad \textit{for large } k > 0.$$

Proof. Fourier series of periodic C^∞ functions are absolutely convergent. ■

3.3. Inverse map: Forms to cycles

In this section, let us assume that Σ is an open connected surface of some genus $\mathfrak{g}$ with $n \geq 0$ boundaries that are topological circles denoted by $b_i = \partial_i\Sigma$, $i = 1, \ldots, n$. And if $n = 0$, Σ is compact of genus $\mathfrak{g}$ without boundary.

Let us also add the requirement that B is normalized on boundaries; i.e., for each boundary b_j,

$$\hat{B}(\mathcal{A}_{b_j,0}) = 0.$$

Definition 3.14 (Marking). Choose a generic interior point $o \in \Sigma$. Then, we choose n non-intersecting C^1 Jordan arcs $\gamma_{o\to b_j}$ that end at the boundary $b_j = \partial_j\Sigma$ at the point of coordinate $\zeta_{b_j} = 1$. Then, $\pi_1(\Sigma \setminus \bigcup_{j=1}^n \gamma_{o\to b_j})$ has rank $2\mathfrak{g}$, so we can choose a generating family of $2\mathfrak{g}$ non-intersecting Jordan loops starting end ending at o, denoted by $\mathcal{A}_1, \ldots, \mathcal{A}_\mathfrak{g}, \mathcal{B}_1, \ldots, \mathcal{B}_\mathfrak{g}$, that do not intersect each other except at o, and with symplectic intersections

$$\mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j}, \quad \mathcal{A}_i \cap \mathcal{A}_j = 0, \quad \mathcal{B}_i \cap \mathcal{B}_j = 0.$$

Such a choice of $2\mathfrak{g} + n$ Jordan arcs is not unique, so we choose one and call it a “Torelli marking of Jordan loops”.

Definition 3.15 (Fundamental domain). Having chosen a Torelli marking of loops, we define the graph Υ as the union of all arcs

$$\Upsilon = \bigcup_{j=1}^{\mathfrak{g}} \mathcal{A}_j \bigcup_{j=1}^{\mathfrak{g}} \mathcal{B}_j \bigcup_{j=1}^{n} \gamma_{o\to b_j} \bigcup_{j=1}^{n} b_j$$

and the fundamental domain

$$\Sigma_0 = \Sigma \setminus \Upsilon.$$

By our definition, Σ_0 is simply connected.

The boundaries of Σ_0 are

$$\begin{aligned}\partial\Sigma_0 = &\sum_{j=1}^{\mathfrak{g}} \mathcal{A}_j^{\text{right}} - \mathcal{A}_j^{\text{left}} + \sum_{j=1}^{\mathfrak{g}} \mathcal{B}_j^{\text{right}} - \mathcal{B}_j^{\text{left}} \\ &+ \sum_{j=1}^{n} \gamma_{o\to b_j}^{\text{right}} - \gamma_{o\to b_j}^{\text{left}} + \sum_{j=1}^{n} \zeta_{b_j}^{-1}(S_1).\end{aligned}$$

Definition 3.16 (Fundamental bi-cycle). Having chosen a fundamental domain $\Sigma_0 = \Sigma \setminus \Upsilon$, we define

$$\begin{aligned}2\pi\mathrm{i}\hat{\Gamma} = &\sum_{i=1}^{\mathfrak{g}} \hat{\mathcal{A}}_i \otimes \hat{\mathcal{B}}_i - \hat{\mathcal{B}}_i \otimes \hat{\mathcal{A}}_i \\ &+ \sum_{p\in\Sigma}\sum_{k=1}^{\infty} \mathcal{A}_{p,k} \otimes \mathcal{B}_{p,k} - \mathcal{B}_{p,k} \otimes \mathcal{A}_{p,k} \\ &+ \sum_{p\in\Sigma} \mathcal{A}_{p,0} \otimes \hat{\gamma}_{o\to p} - \hat{\gamma}_{o\to p} \otimes \mathcal{A}_{p,0} \\ &+ \sum_{j=1}^{n}\sum_{k=0}^{\infty} \mathcal{A}_{b_j,k} \otimes \mathcal{B}_{b_j,k} - \mathcal{B}_{b_j,k} \otimes \mathcal{A}_{b_j,k},\end{aligned}$$

where, for each $p \in \Sigma$, $\gamma_{o\to p}$ is a once for all chosen Jordan arc from o to p in the fundamental domain. An example is to choose a Jordan arc geodesic for the distance $|\mathrm{y}|$ on Σ (there is typically only finitely many geodesics from o to p).

It may look like an ill-defined uncountably infinite sum of tensor product of generalized cycles. However, as we will see below, it is a well-defined element of $\mathfrak{M}^1(\Sigma)^* \otimes \mathfrak{M}^1(\Sigma)^*$ because when paired with meromorphic forms $\omega_1 \otimes \omega_2$, it reduces to finite or absolutely convergent sums.

Lemma 3.5. *We have*

$$\hat{\Gamma} \in \mathfrak{M}^1(\Sigma)^* \otimes \mathfrak{M}^1(\Sigma)^*$$

Proof. Let $\omega_1 \otimes \omega_2 \in \mathfrak{M}^1(\Sigma) \otimes \mathfrak{M}^1(\Sigma)$. We have

$$\begin{aligned}
\langle \hat{\Gamma}, \omega_1 \otimes \omega_2 \rangle = & \frac{1}{2\pi \mathrm{i}} \sum_{i=1}^{\mathfrak{g}} \int_{\hat{\mathcal{A}}_i} \omega_1 \int_{\hat{\mathcal{B}}_i} \omega_2 - \int_{\hat{\mathcal{B}}_i} \omega_1 \int_{\hat{\mathcal{A}}_i} \omega_2 \\
& + \sum_{p=\text{poles of } \omega_1} \sum_{k=1}^{\deg_p \omega_1 - 1} t_{p,k}(\omega_1) \int_{\mathcal{B}_{p,k}} \omega_2 \\
& - \sum_{p=\text{poles of } \omega_2} \sum_{k=1}^{\deg_p \omega_2 - 1} t_{p,k}(\omega_2) \int_{\mathcal{B}_{p,k}} \omega_1 \\
& + \sum_{p=\text{poles of } \omega_1} \operatorname{Res}_p \omega_1 \int_{\hat{\gamma}_{o\to p}} \omega_2 \\
& - \sum_{p=\text{poles of } \omega_2} \int_{\hat{\gamma}_{o\to p}} \omega_1 \operatorname{Res}_p \omega_2 \\
& + \frac{1}{2\pi \mathrm{i}} \sum_{i=1}^{n} \sum_{k=0}^{\infty} \int_{\mathcal{A}_{b_i,k}} \omega_1 \int_{\mathcal{B}_{b_i,k}} \omega_2 - \int_{\mathcal{B}_{b_i,k}} \omega_1 \int_{\mathcal{A}_{b_i,k}} \omega_2 .
\end{aligned}$$

The term in the last line is an absolutely convergent infinite sum, and all the others are finite sums. This implies that $\hat{\Gamma}$ is a well-defined element of $\mathfrak{M}^1(\Sigma)^* \otimes \mathfrak{M}^1(\Sigma)^*$. ∎

Theorem 3.7 (Riemann bilinear identity). *If $\omega_1, \omega_2 \in \mathfrak{M}^1(\Sigma)$, we have*

$$\langle \hat{\Gamma}, \omega_1 \otimes \omega_2 \rangle = 0.$$

Proof. See Appendix A. ∎

Corollary 3.1. *It satisfies*

$$\langle \hat{\Gamma}, \omega \otimes B(\cdot, z_0) \rangle = \omega(z_0).$$

Proof. Let $z_0 \in \Sigma$ in a chart with some local coordinate $\zeta(z_0)$. Let

$$\omega_2(z) = B(z, z_0)/d\zeta(z_0)$$

it is a meromorphic 1-form $\omega_2 \in \mathfrak{M}^1(\Sigma)$, with a double pole at $z = z_0$ and no other pole, and near the pole, it behaves as follows:

$$\omega_2(z) \sim \frac{1}{(\zeta(z) - \zeta(z_0))^2} d\zeta(z) + \text{holomorphic},$$

and, in particular, it has $t_{z_0,1}(\omega_2) = 1$ and $t_{z_0,0}(\omega_2) = 0$. We then apply Theorem 3.7 to $\omega \otimes \omega_2$, and putting on one side all the terms coming from the poles of ω_2 i.e., from z_0, we have

$$\begin{aligned} d\zeta(z_0) & \sum_{p=\text{poles of } \omega_2} \int_{\mathcal{A}_{p,k}} \omega_2 \int_{\mathcal{B}_{p,k}} \omega \\ & = 2\pi \mathrm{i} d\zeta(z_0) \int_{\mathcal{B}_{z_0,1}} \omega = 2\pi \mathrm{i} \langle \mathrm{ev}_{z_0}, \omega \rangle = 2\pi \mathrm{i} \omega(z_0). \end{aligned}$$

All the other terms applied to ω_2 give the left-hand side of

$$\langle \hat{\Gamma}, \omega \otimes B(\cdot, z_0) \rangle = \omega(z_0). \qquad \blacksquare$$

Definition 3.17 (Operator $\hat{C}$: forms to cycles). Let

$$\begin{aligned} \hat{C} : \mathfrak{M}^1(\Sigma) & \to \mathfrak{M}_1(\mathcal{S}), \\ \omega & \mapsto \hat{\Gamma}(\omega, \cdot). \end{aligned}$$

Remark 3.4. Heuristically, if there would exist a basis $\{\gamma_i\}_{i \in J}$ of $\mathfrak{M}_1(\mathcal{S})$, with intersection matrix $I_{i,j} = \gamma_i \cap \gamma_j$ invertible, we would define

$$\hat{C}(\omega) = \sum_{i,j \in J^2} \frac{1}{2\pi \mathrm{i}} \left(\int_{\gamma_i} \omega \right) (I^{-1})_{i,j} \gamma_j .$$

However, such a basis does not exist. But what the definition of $\hat{C}$ above does, thanks to Riemann bilinear identity, is to provide an adapted basis for each ω, where the sums are actually finite or absolutely convergent.

Theorem 3.8. *$\hat{B}$ is surjective and $\hat{C}$ is a right inverse of $\hat{B}$:*

$$\hat{B} \circ \hat{C} = \mathrm{Id}.$$

Proof. It follows from Corollary 3.1. $\blacksquare$

3.3.1. Lagrangian decomposition.

Theorem 3.9 (Lagrangian decomposition). *We have the Lagrangian decomposition*

$$\mathfrak{M}_1(\mathcal{S}) = \operatorname{Ker} \hat{B} \oplus \operatorname{Im} \hat{C},$$

where both $\operatorname{Ker} \hat{B}$ *and* $\operatorname{Im} \hat{C}$ *are Lagrangian.*

The map $\hat{B} : \mathfrak{M}_1(\mathcal{S}) \to \mathfrak{M}^1(\Sigma)$ is surjective.

We have the exact sequence

$$0 \to \operatorname{Ker} \hat{B} \to \mathfrak{M}_1(\mathcal{S}) \xrightarrow{\hat{B}} \mathfrak{M}^1(\Sigma) \to 0.$$

Proof. It is obvious from the definitions that $\cap$ vanishes on $\operatorname{Ker}\hat{B} \times \operatorname{Ker}\hat{B}$.

Let $\gamma \in \mathfrak{M}_1(\mathcal{S})$ and $\gamma' = \hat{C}(\hat{B}(\gamma))$, we have $\gamma' \in \operatorname{Im}\hat{C}$, and we have

$$\hat{B}(\gamma') = \hat{B} \circ \hat{C}(\hat{B}(\gamma)) = \hat{B}(\gamma).$$

Therefore,

$$\hat{B}(\gamma - \gamma') = 0,$$

which implies $\gamma - \gamma' \in \operatorname{Ker}\hat{B}$. This shows that

$$\mathfrak{M}_1(\mathcal{S}) = \operatorname{Ker}\hat{B} \oplus \operatorname{Im}\hat{C}.$$

It remains to prove that $\hat{C}$ is Lagrangian. We have

$$\begin{aligned} 2\pi\mathrm{i}\hat{C}(\omega_1) \cap \hat{C}(\omega_2) &= \int_{\hat{C}(\omega_1)} \hat{B}(\hat{C}(\omega_2)) - \int_{\hat{C}(\omega_2)} \hat{B}(\hat{C}(\omega_1)) \\ &= \int_{\hat{C}(\omega_1)} \omega_2 - \int_{\hat{C}(\omega_2)} \omega_1 \\ &= \langle \hat{\Gamma}, \omega_1 \otimes \omega_2 \rangle - \langle \hat{\Gamma}, \omega_2 \otimes \omega_1 \rangle \\ &= 2\langle \hat{\Gamma}, \omega_1 \otimes \omega_2 \rangle, \end{aligned}$$

which vanishes due to Theorem 3.7. ∎

Theorem 3.10. *The intersection symplectic form is nondegenerate.*

Proof. Assume that there would exist some $\gamma \in \mathfrak{M}_1(\mathcal{S})$ such that $\forall \tilde{\gamma} \in \mathfrak{M}_1(\mathcal{S})$, we have $\gamma \cap \tilde{\gamma} = 0$. In particular, this implies that

$$\int_\gamma \hat{B}(\tilde{\gamma}) = \int_{\tilde{\gamma}} \hat{B}(\gamma).$$

In particular, with $\tilde{\gamma} = \hat{C}(\omega)$, we have

$$\int_\gamma \omega = \int_\gamma \hat{B} \circ \hat{C}(\omega) = \int_{\hat{C}(\omega)} \hat{B}(\gamma) = \langle \hat{\Gamma}, \omega \otimes \hat{B}(\gamma) \rangle = 0.$$

This implies that, for any ω, we have $\int_\gamma \omega = 0$. In other words, as an element of $\mathfrak{M}^1(\Sigma)^*$, we have $\gamma = 0$.

This implies that $\cap$ is nondegenerate. ∎

Definition 3.18 (Generating cycle of $\operatorname{Ker}\hat{B}$). Let

$$\mathcal{A}(z_0) = \hat{C}(B(\cdot, z_0)).$$

More explicitly,

$$
\begin{aligned}
\mathcal{A}(z_0) = & \sum_{i=1}^{\mathfrak{g}} \Big(\int_{\mathcal{B}_i} B(\cdot, z_0) \Big) \mathcal{A}_i - \Big(\int_{\mathcal{A}_i} B(\cdot, z_0) \Big) \mathcal{B}_i \\
& + \sum_{j=1}^{n} \sum_{k=1}^{\infty} \Big(\int_{\mathcal{B}_{b_j,k}} B(\cdot, z_0) \Big) \mathcal{A}_{b_j,k} - \sum_{j=1}^{n} \sum_{k=1}^{\infty} \Big(\int_{\mathcal{A}_{b_j,k}} B(\cdot, z_0) \Big) \mathcal{B}_{b_j,k} \\
& + \sum_{j=1}^{n} \Big(\int_{\mathcal{B}_{b_j,0}} B(\cdot, z_0) \Big) \mathcal{A}_{a_j,0} - \Big(\int_{\mathcal{A}_{b_j,0}} B(\cdot, z_0) \Big) \mathcal{B}_{a_j,0}.
\end{aligned}
$$

It is a 1-form in z_0, valued in generalized cycles, it belongs to

$$H^0(\Sigma, K_\Sigma \otimes \mathfrak{M}_1(\mathcal{S})).$$

Theorem 3.11. *For every $z_0 \in \Sigma$, $\mathcal{A}(z_0)$ is in* $\operatorname{Ker} \hat{B}$ *(in fact, $\mathcal{A}(z_0) \in T^*_{z_0}\Sigma \otimes \operatorname{Ker} \hat{B}$):*

$$\forall z_0 \in \Sigma,\ z_1 \in \Sigma, \quad \int_{\mathcal{A}(z_0)} B(\cdot, z_1) = 0.$$

Vice versa, $\operatorname{Ker} \hat{B}$ *is generated by $\mathcal{A}(z_0)$, i.e., for every $\gamma \in$* $\operatorname{Ker} \hat{B}$*, we have*

$$\gamma = \int_{z_0 \in \gamma} \mathcal{A}(z_0).$$

Proof. We have

$$\langle \mathcal{A}(z_0), B(\cdot, z_1) \rangle = \langle \hat{\Gamma}, B(\cdot, z_0) \otimes B(\cdot, z_1) \rangle = B(z_0, z_1) - B(z_1, z_0) = 0.$$

Then, we have

$$2\pi \mathrm{i} \mathcal{A}(z_0) \cap \tilde{\gamma} = \int_{\mathcal{A}(z_0)} \hat{B}(\tilde{\gamma}) - \int_{\tilde{\gamma}} \hat{B}(\mathcal{A}(z_0)) = \int_{\mathcal{A}(z_0)} \hat{B}(\tilde{\gamma}).$$

Let $\gamma \in \operatorname{Ker} \hat{B}$. For every $\omega \in \mathfrak{M}^1(\Sigma)$, we have

$$\Big\langle \int_{z_0 \in \gamma} \mathcal{A}(z_0), \omega \Big\rangle = \int_\gamma \int_{\hat{C}(\omega)} B = \int_\gamma \hat{B}(\hat{C}(\omega)) = \int_\gamma \omega.$$

This implies that, as elements of $\mathfrak{M}^1(\Sigma)^*$, we have

$$\gamma = \int_{z_0 \in \gamma} \mathcal{A}(z_0). \qquad \blacksquare$$

4. Topological recursion

The main application framework of the generalized cycles is topological recursion.

4.1. Topological recursion

Topological recursion [7] is a recursive procedure that associates a sequence of forms to a spectral curve:

$$\begin{gathered}\text{Topological Recursion (TR)}\\ \mathcal{S} \mapsto \{\omega_{g,n}(\mathcal{S})\}_{(g,n)\in\mathbb{Z}_+\times\mathbb{Z}_+\setminus\{(0,0)\}},\end{gathered}$$

where $\omega_{0,1} = \mathrm{y}$, $\omega_{0,2} = B$, and for $2g-2+n>0$ and $n>0$,

$$\omega_{g,n} \in H^0(\Sigma^n, K_\Sigma^{\overset{\text{sym}}{\boxtimes} n}(*R)),$$

where R is the set of ramification points of x.

For $n = 0$, $\omega_{g,0}$ is often denoted by F_g and is a scalar $\in \mathbb{C}$:

$$\omega_{g,0}(\mathcal{S}) = F_g(\mathcal{S}).$$

Moreover, we have the following useful lemma.

Lemma 4.1 (Normalized on Ker $\hat{B}$). *We have*

$$\forall \gamma \in \text{Ker}\,\hat{B},\ \forall g \geq 0,\ n \geq 1+\delta_{g,0},\quad \langle \gamma, \omega_{g,n}\rangle_{\pi_1} = 0.$$

Proof. See [7]. Notice that the case $(g,n) = (0,2)$ is the definition of Ker $\hat{B}$. ■

4.2. Deformations of topological recursion

In [7], there is the following theorem.

Theorem 4.1 (Deformations [7]). *Let $\mathcal{S}_t$ for $t \in I = [a,b] \subset \mathbb{R}$ be a smooth family of spectral curves. If one can find some generalized cycle $\Gamma_t \in \mathfrak{M}_1(\mathcal{S}_t)$ such that*

$$\frac{d}{dt}\mathrm{y}_t = \int_{\Gamma_t} B_t,$$
$$\frac{d}{dt}B_t = \int_{\Gamma_t} \omega_{0,3}(\mathcal{S}_t),$$

then one has the special geometry relation for all $(g,n) \neq (0,0)$:

$$\frac{d}{dt}\omega_{g,n}(\mathcal{S}_t) = \int_{\Gamma_t} \omega_{g,n+1}(\mathcal{S}_t).$$

The following lemma follows easily from Lemma 4.1.

Lemma 4.2. *The genearlized cycle Γ_t is not unique, it can be defined modulo* Ker $\hat{B}_t$. *This allows to choose Γ_t in some Lagrangian submanifold $\mathcal{L}_t \subset \mathfrak{M}_1(\mathcal{S}_t)$, transverse to* Ker $\hat{B}_t$.

Let us make the following definition.

Definition 4.1 (Flat cycles). The family Γ_t is said to be "constant" or "flat" if and only if

- it is continuous, i.e., Γ_t is continuous in any finite-dimensional sub-bundle of $\mathfrak{M}^1(\Sigma_t) \to I$,
- we have
$$\frac{d}{dt} \mathrm{x}_{t*}\Gamma_t = 0,$$
where the push-forward $\mathrm{x}_{t*}\Gamma_t \in \mathfrak{M}^1(\underline{\Sigma})^*$ is defined by $\langle \mathrm{x}_{t*}\Gamma_t, \Omega\rangle = \langle \Gamma_t, \mathrm{x}_t^*\Omega\rangle$. In fact, flat families of cycles can be used to define a connection of the bundle of generalized cycles.

Theorem 4.2. *If Γ_t is constant, we can exponentiate the deformation*

$$\omega_{g,n}(\mathcal{S}_{t+h}) = \sum_{m=0}^{\infty} \frac{h^m}{m!} \overbrace{\int_{\Gamma_t} \cdots \int_{\Gamma_t}}^{m} \omega_{g,n+m}(\mathcal{S}_t).$$

The sum over m is absolutely convergent for h small enough.

Proof. This is Taylor series. ∎

4.3. Prepotential

We can understand why $F_0 = \omega_{0,0}$ is ill defined. Consider 2 flows d/dt and $d/d\tilde{t}$, corresponding to 2 generalized cycles Γ and $\tilde{\Gamma}$, assumed constant.

If $\omega_{0,0}$ existed, we would have

$$\frac{d}{dt}\,\omega_{0,0} = \int_{\Gamma} \mathrm{y}, \quad \frac{d}{d\tilde{t}}\,\omega_{0,0} = \int_{\tilde{\Gamma}} \mathrm{y},$$

and taking a second derivative

$$\frac{d}{d\tilde{t}}\frac{d}{dt}\,\omega_{0,0} - \frac{d}{dt}\frac{d}{d\tilde{t}}\,\omega_{0,0} = \int_{\Gamma}\int_{\tilde{\Gamma}} B - \int_{\tilde{\Gamma}}\int_{\Gamma} B = 2\pi \mathrm{i}\Gamma \cap \tilde{\Gamma}.$$

It would vanish iff $\Gamma \cap \tilde{\Gamma} = 0$, i.e., iff Γ and $\tilde{\Gamma}$ are in a same flat Lagrangian $\mathcal{L}$.

In other words, we can define $\omega_{0,0}$ only if we also choose a flat Lagrangian $\mathcal{L}$.

We have from [6] the following definition.

Definition 4.2 (Prepotential). Let $\mathcal{S}$ be a spectral curve and $\mathcal{L}$ a flat Lagrangian. We define

$$F_0(\mathcal{S}, \mathcal{L}) = \frac{1}{2}\big\langle \Pi^{\|\mathcal{L}}_{\mathrm{Ker}\,\hat{B}} \circ \hat{C}(\mathrm{y}), \mathrm{y}\big\rangle,$$

where $\Pi_{\mathcal{L}}^{\|\mathrm{Ker}\,\hat{B}} : \mathfrak{M}_1(\mathcal{S}) \to \mathrm{Ker}\,\hat{B}$ is the projection onto $\mathcal{L}$, parallel to $\mathrm{Ker}\,\hat{B}$.

Theorem 4.3. *For any $\Gamma_t \in \mathcal{L}$ associated to a family $\mathcal{S}_t$, we have*

$$\frac{d}{dt} F_0(\mathcal{S}_t, \mathcal{L}) = \int_{\Gamma_t} \mathrm{y}.$$

Proof. See [6]. ∎

Theorem 4.4. *If we have a multidimensional family $\mathcal{S}_{t_1,\dots,t_k}$, we let*

$$\delta = \sum_{i=1}^{k} \Gamma_i \, dt_i.$$

It is a generalized cycle valued in the cotangent space of the t_is, i.e., a 1-form in the cotangent moduli space, valued in generalized cycles.

We have

$$dF_0(\mathcal{S}, \mathcal{L}) = \langle \delta, \mathrm{y} \rangle = \sum_{i=1}^{k} \langle \Gamma_i, \mathrm{y} \rangle dt_i.$$

Since $\mathcal{L}$ is Lagrangian, we have the Lax equation (zero curvature equation)

$$[d, d] = 0 \Longleftrightarrow \Gamma_i \cap \Gamma_j = 0.$$

Proof. See [6]. ∎

5. Conclusion

The aim of this article was to provide a comprehensive setting of the notion of generalized cycles. Their geometry is fascinating, and they have plenty of applications in integrable systems and topological recursion.

In [7], they were introduced as a way to encode deformations.

They provide a way to unify Miwa–Jimbo equations, Seiberg–Witten and Bertola–Malgrange into a single notion.

It was shown in [5] how to formulate Kontsevich–Soibelman quantum Airy structures [11] in terms of generalized cycles. The formalism is powerful enough to easily provide the generalization of quantum Airy structures to higher-order operators (limited to quadratic operators in Kontsevich–Soibleman).

Generalized cycles are also useful in understanding the relationship between the moduli space of spectral curves and the de Rham moduli space of connections on principal bundles [2]. In [1], we study how, in the $\varepsilon \to 0$ limit, the Goldman cycles project to the present generalized cycles of a spectra curve, which is the Hitchin's map $\det(y - \Phi(x))$.

A. Riemann bilinear identity

Theorem A.1 (Fundamental bi-cycle and Riemann bilinear identity). *It satisfies*

$$\langle \hat{\Gamma}, \omega_1 \otimes \omega_2 \rangle = 0.$$

Proof. The proof uses Riemann bilinear identity, which proceeds as follows.

Let $r > 0$ be small enough such that, for each pole p of ω_1 and/or ω_2, the coordinate ζ_p is well defined in the disc $|\zeta_p| < r$, all these discs are disjoints, and ω_1 and ω_2 are analytic in the pointed disc (i.e., the only pole is at $\zeta_p = 0$ in each disc), and for each boundary b, ω_1 and ω_2 are analytic in the annulus of radius $1 < |\zeta_b| < r$, and all the annuli and discs are disjoint.

Then, in the fundamental domain Σ_0, let us consider Jordan arcs $\gamma_{o\to p}$ for each pole p of ω_1 and/or ω_2, and such that in the discs $|\zeta_p| < r$, the Jordan arc's support is the segment $[0, r]$ oriented from r to 0. We define the truncated arc at raidus r:

$$\gamma^{(r)}_{o\to p} = \gamma_{o\to p} \setminus \zeta_p^{-1}([0, r]).$$

Consider also the small circle of radius r around p, starting at angle $\theta = 0$ and ending at $\theta = 2\pi$:

$$\mathcal{C}^{(r)}_p = \zeta_p^{-1}(\{re^{\mathrm{i}\theta} \mid \theta \in]0, 2\pi[\}).$$

Let

$$\Sigma_0' = \Sigma_0 \Big\backslash \bigcup_{p=\text{poles of } \omega_2} \{\gamma_{o\to p}\}.$$

In Σ_0', consider an antiderivative of ω_2:

$$\phi_2(z) = \int_o^z \omega_2,$$

i.e., $d\phi_2 = \omega_2$. By definition, ϕ_2 is analytic in Σ_0'.

In Σ_0', which is simply connected, the following contour is contractible:

$$\begin{aligned}\gamma = {} & \sum_{i=1}^{\mathfrak{g}} \mathcal{A}_i^{\text{right}} - \mathcal{A}_i^{\text{left}} \\ & + \sum_{i=1}^{\mathfrak{g}} \mathcal{B}_i^{\text{right}} - \mathcal{B}_i^{\text{left}} \\ & + \sum_{p=\text{poles of } \omega_2} \gamma^{(r)\text{right}}_{o\to p} - \gamma^{(r)left}_{o\to p} + \mathcal{C}^{(r)}_p \\ & + \sum_{j=1}^{n} \gamma^{(r)\text{right}}_{o\to b_j} - \gamma^{(r)\text{left}}_{o\to b_j} + \mathcal{C}^{(r)}_{b_j}.\end{aligned}$$

It may, however, enclose the poles of ω_1, and therefore, we have

$$\int_{\partial\Sigma_0'} \omega_1\phi_2 = -2\pi \mathrm{i} \sum_{p=\text{poles of }\omega_1} \mathrm{Res}_p\, \phi_2\omega_1$$

where by "poles of ω_1" we mean the poles of ω_1 inside Σ_0', i.e., those that are not poles of ω_2.

This gives

$$\begin{aligned}
0 &= \sum_{p=\text{ poles of }\omega_1} 2\pi \mathrm{i}\, \mathrm{Res}_p\, \phi_2\omega_1 + \int_{\partial\Sigma_0'} \phi_2\omega_1 \\
&= \sum_{p=\text{ poles of }\omega_1} 2\pi \mathrm{i}\, \mathrm{Res}_p\, \phi_2\omega_1 \\
&\quad + \sum_{p=\text{ poles of }\omega_2} \left(\int_{\hat\gamma^{\text{right}}_{o\to p^{(r)}}} \phi_2(z)\omega_1(z) \right) - \left(\int_{\hat\gamma^{\text{left}}_{o\to p^{(r)}}} \phi_2(z)\omega_1(z) \right) \\
&\quad + \sum_{p=\text{ poles of }\omega_2} \int_{\theta=0, z=\zeta_p^{-1}(re^{\mathrm{i}\theta})}^{2\pi} \phi_2(z)\omega_1(z) \\
&\quad + \sum_{i=1}^{\mathfrak{g}} \left(\int_{\mathcal{A}_i^{\text{right}}} \phi_2(z)\omega_1(z) \right) - \left(\int_{\mathcal{A}_i^{\text{left}}} \phi_2(z)\omega_1(z) \right) \\
&\quad + \sum_{i=1}^{\mathfrak{g}} \left(\int_{\mathcal{B}_i^{\text{right}}} \phi_2(z)\omega_1(z) \right) - \left(\int_{\mathcal{B}_i^{\text{left}}} \phi_2(z)\omega_1(z) \right) \\
&\quad + \sum_{j=1}^{n} \left(\int_{\hat\gamma^{\text{right}}_{o\to p^{(r)}_{b_j}}} \phi_2(z)\omega_1(z) \right) - \left(\int_{\hat\gamma^{\text{left}}_{o\to p^{(r)}_{b_j}}} \phi_2(z)\omega_1(z) \right) \\
&\quad + \sum_{j=1}^{n} \int_{\theta=0, z=\zeta_b^{-1}(re^{\mathrm{i}\theta})}^{2\pi} \phi_2(z)\omega_1(z).
\end{aligned}$$

Let us compute each term.

- Let p be a pole of ω_1, which is not a pole of ω_2; we have

$$\begin{aligned}
&\mathrm{Res}_{z\to p}\, \phi_2(z)\omega_1(z) \\
&= \mathrm{Res}_{z\to p}\, \phi_2(z) \left(\sum_{k=0}^{\deg_p \omega_1 - 1} t_{p,k}(\omega_1)\zeta_p(z)^{-k-1} d\zeta_p(z) + \text{hol. at } p \right) \\
&= \sum_{k=1}^{\deg_p \omega_1 - 1} t_{p,k}(\omega_1)\, \mathrm{Res}_{z\to p}\, \phi_2(z)\zeta_p(z)^{-k-1} d\zeta_p(z) \\
&\quad + t_{p,0}(\omega_1)\, \mathrm{Res}_{z\to p}\, \phi_2(z)\zeta_p(z)^{-1} d\zeta_p(z)
\end{aligned}$$

$$= \sum_{k=1}^{\deg_p \omega_1 - 1} \frac{1}{k} t_{p,k}(\omega_1) \operatorname{Res}_{z\to p} \omega_2(z)\zeta_p(z)^{-k} + t_{p,0}(\omega_1)\phi_2(p)$$

$$= \sum_{k=1}^{\deg_p \omega_1 - 1} t_{p,k}(\omega_1) \int_{\mathcal{B}_{p,k}} \omega_2 + t_{p,0}(\omega_1) \int_{\hat{\gamma}_{o\to p}} \omega_2$$

$$= 2\pi \mathrm{i} \sum_{k=1}^{\deg_p \omega_1 - 1} \int_{\mathcal{A}_{p,k}} \omega_1 \int_{\mathcal{B}_{p,k}} \omega_2 + 2\pi \mathrm{i} \int_{\mathcal{A}_{p,0}} \omega_1 \int_{\hat{\gamma}_{o\to p}} \omega_2.$$

Notice that $\int_{\hat{\gamma}_{o\to p}} \omega_2$ is independent of the choice of Jordan arc $\gamma_{o\to p}$ in Σ_0' because Σ_0' is simply connected and ω_2 is analytic in Σ_0'.
Notice also that $\int_{\mathcal{A}_{p,k}} \omega_2 = 0$, so we can also write

$$\begin{aligned}
&\operatorname{Res}_{z\to p} \phi_2(z)\omega_1(z) \\
&\quad = 2\pi \mathrm{i} \sum_{k=1}^{\deg_p \omega_1 - 1} \left(\int_{\mathcal{A}_{p,k}} \omega_1 \int_{\mathcal{B}_{p,k}} \omega_2 - \int_{\mathcal{B}_{p,k}} \omega_1 \int_{\mathcal{A}_{p,k}} \omega_2 \right) \\
&\qquad + 2\pi \mathrm{i} \left(\int_{\mathcal{A}_{p,0}} \omega_1 \int_{\hat{\gamma}_{o\to p}} \omega_2 - \int_{\hat{\gamma}_{o\to p}} \omega_1 \int_{\mathcal{A}_{p,0}} \omega_2 \right).
\end{aligned}$$

- For the 2nd term, let p be a pole of ω_2. Notice that, for a pole p of ω_2, ϕ_2 is possibly discontinuous across $\gamma_{o\to p}$, with discontinuity

$$\phi_2^{\text{right}}(z) - \phi_2^{\text{left}}(z) = 2\pi \mathrm{i} t_{p,0}(\omega_2) = \int_{\mathcal{A}_{p,0}} \omega_2,$$

which is independent of z. This gives

$$\begin{aligned}
&\left(\int_{\hat{\gamma}^{\text{right}}_{o\to p(r)}} \phi_2(z)\omega_1(z) \right) - \left(\int_{\hat{\gamma}^{\text{left}}_{o\to p(r)}} \phi_2(z)\omega_1(z) \right) \\
&\quad = -2\pi \mathrm{i}\, t_{p,0}(\omega_2) \int_{\hat{\gamma}_{o\to p(r)}} \omega_1 \\
&\quad = -\int_{\mathcal{A}_{p,0}} \omega_2 \int_{\hat{\gamma}_{o\to p(r)}} \omega_1.
\end{aligned}$$

- For p a pole of ω_2, we use the fact that

$$\omega_1(z) \sim \sum_{k=0}^{\deg_p \omega_1 - 1} t_{p,k}(\omega_1)\zeta_p(z)^{-k-1} d\zeta_p(z) + \sum_{k=1}^{\infty} k \int_{\mathcal{B}_{p,k}} \omega_1 \zeta_p(z)^{k-1} d\zeta_p(z)$$

and that

$$\phi_2(z) \sim \int_{\hat\gamma_{o\to p}} \omega_2 + t_{p,0}(\omega_2)(-\pi\mathrm{i} + \ln\zeta_p(z)) - \sum_{k=1}^{\deg_p\omega_2 - 1} \frac{1}{k} t_{p,k}(\omega_2)\zeta_p(z)^{-k}$$
$$+ \sum_{k=1}^{\infty} \zeta_p(z)^k \int_{\mathcal{B}_{p,k}} \omega_2 .$$

This gives

$$\int_{\theta=0, z=\zeta_p^{-1}(re^{\mathrm{i}\theta})}^{2\pi} (\phi_2(z)\omega_1(z))$$
$$= 2\pi\mathrm{i}\Big(\int_{\hat\gamma_{o\to p}} \omega_2 - \pi\mathrm{i} t_{p,0}(\omega_2)\Big) t_{p,0}(\omega_1) - 2\pi\mathrm{i} \sum_{k=1}^{\deg_p\omega_2-1} t_{p,k}(\omega_2)\int_{\mathcal{B}_{p,k}} \omega_1$$
$$+ 2\pi\mathrm{i}\sum_{k=1}^{\deg_p\omega_1-1} t_{p,k}(\omega_1)\int_{\mathcal{B}_{p,k}}\omega_2 + t_{p,0}(\omega_2)\int_{\theta=0,z=\zeta_p^{-1}(re^{\mathrm{i}\theta})}^{2\pi} \ln\zeta_p(z)\omega_1(z)$$
$$= 2\pi^2 t_{p,0}(\omega_2)t_{p,0}(\omega_1)$$
$$+ \int_{\hat\gamma_{o\to p}}\omega_2\int_{\mathcal{A}_{p,0}}\omega_1 + \sum_{k=1}^{\deg_p\omega_1-1}\int_{\mathcal{A}_{p,k}}\omega_1\int_{\mathcal{B}_{p,k}}\omega_2 - \int_{\mathcal{B}_{p,k}}\omega_1\int_{\mathcal{A}_{p,k}}\omega_2$$
$$+ t_{p,0}(\omega_2)\int_{\theta=0,z=\zeta_p^{-1}(re^{\mathrm{i}\theta})}^{2\pi} \ln\zeta_p(z)\omega_1(z).$$

For the last term, we consider ϕ_1 an antiderivative of ω_1 such that $d\phi_1 = \omega_1$, namely,

$$\phi_1(z) = \int_{\hat\gamma_{o\to p}} \omega_1 + t_{p,0}(\omega_1)(-\pi\mathrm{i} + \ln\zeta_p(z)) - \sum_{k=1}^{\deg_p\omega_1 - 1} \frac{1}{k} t_{p,k}(\omega_1)\zeta_p(z)^{-k}$$
$$+ \sum_{k=1}^{\infty} \zeta_p(z)^k \int_{\mathcal{B}_{p,k}} \omega_1 .$$

We have

$$\int_{\theta=0,z=\zeta_p^{-1}(re^{\mathrm{i}\theta})}^{2\pi} \ln\zeta_p(z)\omega_1(z)$$
$$= \Big(\int_{\hat\gamma_{o\to p}}\omega_1 - \pi\mathrm{i}t_{p,0}(\omega_1)\Big)2\pi\mathrm{i} + \frac{1}{2}t_{p,0}(\omega_1)(-4\pi^2 + 4\pi\mathrm{i}\ln r)$$
$$-2\pi\mathrm{i}\sum_{k=1}^{\deg_p\omega_1-1}\frac{r^{-k}}{k}t_{p,k}(\omega_1) + 2\pi\mathrm{i}\sum_{k=1}^{\infty} r^k\int_{\mathcal{B}_{p,k}}\omega_1 = 2\pi\mathrm{i}\phi_1(\zeta_p^{-1}(r)) + \pi\mathrm{i}t_{p,0}(\omega_1).$$

This gives

$$\begin{aligned}
&\int_{\theta=0,z=\zeta_p^{-1}(re^{\mathrm{i}\theta})}^{2\pi} \phi_2(z)\omega_1(z) \\
&\quad + \Big(\int_{\hat\gamma^{\mathrm{right}}_{o\to p(r)}} \phi_2(z)\omega_1(z)\Big) - \Big(\int_{\hat\gamma^{\mathrm{left}}_{o\to p(r)}} \phi_2(z)\omega_1(z)\Big) \\
&= \int_{\hat\gamma_{o\to p}} \omega_2 \int_{\mathcal{A}_{p,0}} \omega_1 + \sum_{k=1}^{\deg_p \omega_1 - 1} \int_{\mathcal{A}_{p,k}} \omega_1 \int_{\mathcal{B}_{p,k}} \omega_2 - \int_{\mathcal{B}_{p,k}} \omega_1 \int_{\mathcal{A}_{p,k}} \omega_2 \\
&\quad + \int_{\mathcal{A}_{p,0}} \omega_2 \Big(- \int_{\hat\gamma_{o\to p(r)}} \omega_1 + \phi_1(\zeta_p^{-1}(r)) \Big) \\
&= \int_{\hat\gamma_{o\to p}} \omega_2 \int_{\mathcal{A}_{p,0}} \omega_1 - \int_{\mathcal{A}_{p,0}} \omega_2 \int_{\hat\gamma_{o\to p}} \omega_1 \\
&\quad + \sum_{k=1}^{\deg_p \omega_1 - 1} \int_{\mathcal{A}_{p,k}} \omega_1 \int_{\mathcal{B}_{p,k}} \omega_2 - \int_{\mathcal{B}_{p,k}} \omega_1 \int_{\mathcal{A}_{p,k}} \omega_2 .
\end{aligned}$$

- $\mathcal{A}_i$ and $\mathcal{B}_i$ cycles. We have

$$\begin{aligned}
&\Big(\int_{\mathcal{A}_i^{\mathrm{right}}} \phi_2(z)\omega_1(z)\Big) - \Big(\int_{\mathcal{A}_i^{\mathrm{left}}} \phi_2(z)\omega_1(z)\Big) \\
&\quad = \int_{\mathcal{A}_i} (\phi_2(z^{\mathrm{right}}) - \phi_2(z^{\mathrm{left}}))\omega_1(z) \\
&\quad = \int_{\mathcal{A}_i} \Big(\int_{\mathcal{B}_i} \omega_2\Big)\omega_1(z) = \int_{\mathcal{A}_i} \omega_1 \int_{\mathcal{B}_i} \omega_2 .
\end{aligned}$$

Similarly, but using that the orientation of $\mathcal{B}_i$ compared to that of $\mathcal{A}_i$ is opposite, we have

$$\Big(\int_{\mathcal{B}_i^{\mathrm{right}}} \phi_2(z)\omega_1(z)\Big) - \Big(\int_{\mathcal{B}_i^{\mathrm{left}}} \phi_2(z)\omega_1(z)\Big) = - \int_{\mathcal{B}_i} \omega_1 \int_{\mathcal{A}_i} \omega_2 .$$

Here, we used the same notation $\mathcal{A}_i$, $\mathcal{B}_i$ for the Jordan loop and its corresponding generalized cycle for lighter formulas.

- Boundaries b_j.
The proof for boundaries is exactly the same as the proof for poles, replacing the Laurent expansion for poles, by the Fourier series, which takes the same form:

$$\omega_1 \sim \sum_{k=0}^{\infty} \Big(\frac{1}{2\pi\mathrm{i}} \int_{\mathcal{A}_{b_j,k}} \omega_1\Big) \zeta_{b_j}^{-k-1} d\zeta_{b_j} + \sum_{k=1}^{\infty} \Big(\int_{\mathcal{B}_{b_j,k}} \omega_1\Big) k \zeta_{b_j}^{k-1} d\zeta_{b_j} .$$

Eventually, all the terms together give

$$\langle \hat{\Gamma}, \omega_1 \otimes \omega_2 \rangle = 0,$$

which completes the proof. ■

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References

[1] J. H. R. Belliard and B. Eynard, Cameral curve topological recursion. 2024, to appear

[2] R. Belliard, B. Eynard, and S. Ribault, The geometry of Casimir W-algebras. *SciPost Phys.* **5** (2018), no. 5, article no. 051

[3] S. Bergman and M. Schiffer, *Kernel functions and elliptic differential equations in mathematical physics*. Academic Press, New York, 1953 Zbl 0053.39003 MR 54140

[4] B. Eynard, Notes about a combinatorial expression of the fundamental second kind differential on an algebraic curve. 2018, arXiv:1805.07247v3

[5] B. Eynard, Topological recursion, airy structures in the space of cycles. 2019, arXiv:1912.03339v2

[6] B. Eynard, The geometry of integrable systems. Tau functions and homology of spectral curves. Perturbative definition. [v1] 2017, [v4] 2019, arXiv:1706.04938v4

[7] B. Eynard and N. Orantin, Invariants of algebraic curves and topological expansion. *Commun. Number Theory Phys.* **1** (2007), no. 2, 347–452 Zbl 1161.14026 MR 2346575

[8] H. M. Farkas and I. Kra, *Riemann surfaces*. Grad. Texts in Math. 71, Springer, New York, 1980 Zbl 0475.30001 MR 0583745

[9] J. D. Fay, *Theta functions on Riemann surfaces*. Lecture Notes in Math. 352, Springer, Berlin, 1973 Zbl 0281.30013 MR 0335789

[10] A. Kokotov and D. Korotkin, Bergmann tau-function on Hurwitz spaces and its applications. [v1] 2003, [v2] 2005, arXiv:math-ph/0310008v1

[11] M. Kontsevich and Y. Soibelman, Airy structures and symplectic geometry of topological recursion. In *Topological recursion and its influence in analysis, geometry, and topology*, pp. 433–489, Proc. Sympos. Pure Math. 100, American Mathematical Society, Providence, RI, 2018 Zbl 1448.53078 MR 3888788

[12] D. Mumford, *Tata lectures on Theta. II*. Mod. Birkhäuser Class., Birkhäuser, Boston, MA, 2007 Zbl 1112.14003 MR 2307768

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