Ann. Inst. H. Poincaré D
Comb. Phys. Interact. 13 (2026), 775–843
DOI 10.4171/AIHPD/218
© 2025 Association Publications de l'Institut Henri Poincaré
Published by EMS Press
This work is licensed under a CC BY 4.0 license

Enumeration of maps with tight boundaries and the Zhukovsky transformation

Jérémie Bouttier, Emmanuel Guitter, and Grégory Miermont

Abstract. We consider maps with tight boundaries, i.e., maps whose boundaries have minimal length in their homotopy class, and discuss the properties of their generating functions $T^{(g)}_{\ell_1,\dots,\ell_n}$ for fixed genus g and prescribed boundary lengths $\ell_1,\dots,\ell_n$, with a control on the degrees of inner faces. We find that these series appear as coefficients in the expansion of $\omega^{(g)}_n(z_1,\dots,z_n)$, a fundamental quantity in the Eynard–Orantin theory of topological recursion, thereby providing a combinatorial interpretation of the Zhukovsky transformation used in this context. This interpretation results from the so-called trumpet decomposition of maps with arbitrary boundaries. In the planar bipartite case, we obtain a fully explicit formula for $T^{(0)}_{2\ell_1,\dots,2\ell_n}$ from the Collet–Fusy formula. We also find recursion relations satisfied by $T^{(g)}_{\ell_1,\dots,\ell_n}$, which consist in adding an extra tight boundary, keeping the genus g fixed. Building on a result of Norbury and Scott, we show that $T^{(g)}_{\ell_1,\dots,\ell_n}$ is equal to a parity-dependent quasi-polynomial in $\ell_1^2,\dots,\ell_n^2$ times a simple power of the basic generating function R. In passing, we provide a bijective derivation in the case $(g,n)=(0,3)$, generalizing a recent construction of ours to the non-bipartite case.

1. Introduction

1.1. Context and motivations

We pursue our investigation, started in [5, 6], of the enumerative properties of maps with tight boundaries. In this episode, we add a new twist to the story, by making an unexpected connection with the theory of topological recursion [12, 13]. In particular, we find a combinatorial interpretation of the Zhukovsky transformation, which was so far used as an analytical tool in this theory.

Let us first recall some context. The enumeration of maps (graphs embedded in surfaces) is a venerable topic in combinatorics, initiated by Tutte in his famous series of "Census" papers, see [22] and references therein. Accounts of further developments

Mathematics Subject Classification 2020: 05A15 (primary); 05A19 (secondary).
Keywords: map enumeration, maps with tight boundaries, Zhukovsky transformation, trumpet decomposition, topological recursion, quasi-polynomiality.

may be found, for instance, in [15,20]. A fertile connection with random matrices was made in [7], and it is essentially its ramifications, motivated by the study of 2D quantum gravity—see, e.g., [2, 11, 24] for overviews—that led to topological recursion.

In the paper [5], we found a surprisingly simple formula for the generating function of planar bipartite maps with three *tight* boundaries (as we will explain in more detail below, a boundary is said tight if it has minimal length in its homotopy class). The tightness property seems to play a role, as without it the corresponding generating function becomes slightly more involved [10]. One may, therefore, wonder whether a similar phenomenon occurs for maps of other topologies (higher genus, more boundaries).

This question was already explored in the limit case of *tight maps*. Colloquially speaking, a tight map is a map in which every face is seen as a boundary, and is forced to be tight. The counting of tight maps was actually investigated first by Norbury [17], as tight maps are nothing but fatgraphs describing lattice points in the moduli space of curves. A remarkable *quasi-polynomiality phenomenon* occurs in this problem: if we denote by $N_{g,n}(b_1, \dots, b_n)$ the number of tight maps of genus g with n boundaries of lengths $b_1, \dots, b_n$, then Norbury showed that it is a quasi-polynomial of degree $3g - 3 + n$ in $b_1^2, \dots, b_n^2$ depending on the parities, and we gave in [6] a bijective construction of these quasi-polynomials in the planar case $g = 0$.

In a further paper [18], Norbury and Scott observed that the quasi-polynomiality phenomenon can be related to a specific feature in topological recursion, namely, that the “x” meromorphic function attached to the “spectral curve” takes the particular form

$$x(z) := \alpha + \gamma(z + z^{-1}). \tag{1}$$

This form is nothing but the so-called Zhukovsky transformation [25]. The case of tight maps corresponds to taking the second “y” function equal to the Zhukovsky variable z, but Norbury and Scott showed that the quasi-polynomiality subsists when y is modified.

In this paper, we relate the observation of Norbury and Scott to the enumeration of maps with tight boundaries. At the combinatorial level, the Zhukovsky transformation $z \mapsto x(z)$ corresponds to a substitution, namely, we find that it translates the natural idea of transforming a map with tight boundaries into a map with arbitrary boundaries by gluing a “trumpet” onto each boundary (see Figure 2 below for an illustration). By a trumpet we mean a planar map with two boundaries, one of them being tight while the other is arbitrary. We use the term by analogy with [19], where a similar notion is introduced in the context of hyperbolic geometry. The coefficients α and γ appearing in (1) are related to the “basic” generating functions R and S defined in the next section by $\alpha = S$ and $\gamma = R^{1/2}$.

By this approach, we deduce the interesting property that the generating function $T^{(g)}_{\ell_1,\dots,\ell_n}$ of maps of genus g and n tight boundaries of lengths $\ell_1,\dots,\ell_n$ has the general form

$$T^{(g)}_{\ell_1,\dots,\ell_n} = \tau^{(g)}_{\ell_1,\dots,\ell_n}\gamma^{\ell_1+\cdots+\ell_n}$$

with $\tau^{(g)}_{\ell_1,\dots,\ell_n}$ a quasi-polynomial of degree $3g-3+n$ in $\ell_1^2,\dots,\ell_n^2$ depending on the parities, and $\gamma = R^{1/2}$ as before. In particular, in the planar bipartite case, we will derive a fully explicit expression for $T^{(0)}_{2\ell_1,\dots,2\ell_n}$ from the Collet–Fusy formula [10].

Let us conclude this section by mentioning the recent paper [9], which introduces the notion of tight boundaries in the context of hyperbolic geometry and makes the connection with the so-called JT gravity [19]. A very similar polynomiality phenomenon occurs therein, which makes us believe that we are looking at the same reality from different angles.

1.2. Basic definitions

Let us now introduce precisely the main definitions and conventions used in this paper.

Combinatorial notions: Maps with boundaries and their automorphisms. For g a nonnegative integer, a *map* of genus g is a cellular embedding of a finite connected multigraph into the closed orientable surface of genus g, considered up to orientation-preserving homeomorphism. By cellular embedding we mean that the graph (which consists of *vertices* and *edges*) is drawn on the surface without edge crossings, and that the connected components of the complement of the graph, called *faces*, are homeomorphic to open disks. A map of genus zero is said *planar*.

A *map with boundaries* is a map in which we mark some faces and vertices and call them *boundaries*. We will use the terms *boundary-face* and *boundary-vertex* when we want to specify the type of a boundary. A face which is not a boundary is called an *inner* face. The *length* of a boundary is by definition equal to its degree (number of incident edge sides) for a boundary-face, and to 0 for a boundary-vertex. A boundary-face is said *rooted* if one of its incident corners is marked. A boundary-vertex cannot be rooted.

An *automorphism* of a map with boundaries is a graph isomorphism that also preserves the map structure, in the following sense.

- The images of the edges pointing away from a given vertex, listed cyclically in clockwise order, are the edges pointing away from the image vertex, also in cyclic clockwise order. In particular, the graph isomorphism also induces a bijection of the set of faces onto itself, respecting the incidence relations with vertices and edges.

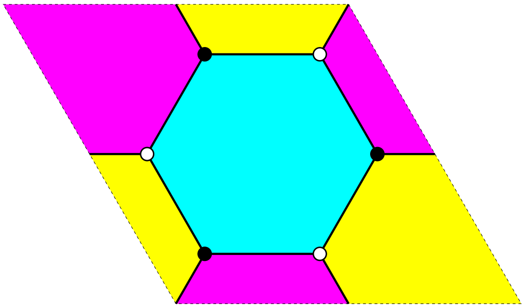

Figure 1. A toric map, with three distinguished boundary-faces of degree 6, admitting an automorphism group of order 3. There are only 2 rooted maps resulting from marking one of the 6 corners incident to the cyan face, and we must account for this fact by weighing this map by the inverse automorphism group order factor $1/3$.

- The marked elements, i.e., the (possibly rooted) boundary-faces and boundary-vertices, are preserved.

The automorphisms of a given map with boundaries form a group, which is often reduced to the identity element. For instance, this holds for a rooted map of any genus, or for a planar map possessing at least three distinguished boundaries. Contrary to the former situation, the latter is not true in higher genera: Figure 1 provides a counterexample in genus 1.

Topological notions: Boundaries and homotopy. We think of boundaries as representing holes in the surface. Precisely, we create a puncture inside each boundary-face, while we draw a small circle around each boundary-vertex: the interior of the circle is removed but the edges incident to the vertex remain connected by the circle. See [5, Figure 2] for an illustration.

Given a map, possibly with boundaries, a *path* is a sequence of consecutive edges. It defines a curve on the underlying surface provided that, if the path visits a boundary-vertex, we choose a direction for "circumventing" the corresponding puncture in the surface. A path will always be assumed to contain this data in the following. A path is said *closed* if it starts and ends at the same vertex. A path is said *simple* if it does not visit the same vertex twice (except at its endpoints for a simple closed path). The *contour* of a face is the closed path formed by its incident edges. Two closed paths on the map are said (freely) *homotopic* to one another if their corresponding curves can be continuously deformed into one another on the punctured surface. A closed path is said *homotopic* to a boundary if, on the underlying surface of the map, the curve associated with the path can be contracted onto the puncture associated with the boundary. A boundary-face is said *tight* (resp., *strictly tight*) if its degree is not larger (resp., is strictly smaller) than the number of edges of any other closed path homotopic to it. A boundary-vertex is considered as (strictly) tight by convention.

Generating functions. Let $t, t_1, t_2, t_3, \ldots$ be a collection of formal variables. We attach a weight t to each vertex which is not a boundary, and a weight t_i to each inner face of degree i, $i = 1, 2, 3, \ldots$. The global weight of a map is the product of the weights of all its (non-boundary-) vertices and faces, multiplied by the inverse of the order of the automorphism group of the map. This last factor yields better combinatorial properties for the corresponding generating functions, see the caption of Figure 1 for a quick explanation.

We now introduce two important quantities denoted by R and S, which are formal power series in $t, t_1, t_2, t_3, \ldots$. The first one R is the generating function of planar maps with two distinguished boundaries of length 1. The second one S is that of planar maps with two boundaries, one of length 1 and the other of length 0. It is known[1] that, as formal power series, R and S are determined by the following equations:

$$\begin{aligned} R &= t + \sum_{i\geq 1} t_i\, [z^{-1}]\left(z + S + \frac{R}{z}\right)^{i-1}, \\ S &= \sum_{i\geq 1} t_i\, [z^{0}]\left(z + S + \frac{R}{z}\right)^{i-1}. \end{aligned} \tag{2}$$

A simplification occurs in the *essentially bipartite* case $t_1 = t_3 = t_5 = \cdots = 0$, which corresponds to imposing that every inner face has even degree. Indeed, we then have $S = 0$ and R satisfies the single equation

$$R = t + \sum_{j\geq 1} t_{2j} \binom{2j-1}{j} R^j. \tag{3}$$

In this case, R can also be understood as the generating function of bipartite planar maps with two unrooted boundaries, one of length 2 and the other of length 0.

1.3. Outline

The remainder of this paper is organized as follows. Section 2 is devoted to the trumpet decomposition which relates maps with arbitrary boundaries to maps with tight boundaries. We first describe in Section 2.1 the bijection for maps with boundary-faces only, before analyzing its enumerative consequences in Section 2.2. The case

[1] See, for instance, [3] for a proof in the dual language. Beware that t is a weight per edge in this reference, one has to set $t_i = t\, g_i$ to match the notations.

of maps having also boundary-vertices is discussed in Section 2.3. In Section 2.4, we derive from the Collet–Fusy formula an explicit expression for the generating function of planar bipartite maps with tight boundaries. Section 3 is devoted to recursion relations satisfied by the generating functions of maps with tight boundaries. These recursion relations consist in adding an extra tight boundary, preserving the genus. We first discuss in Section 3.1 the addition of a boundary-vertex, and then that of a boundary-face in Section 3.2. In Section 3.3, we check that the explicit expression for planar bipartite maps found in Section 2.4 satisfies these recursion relations. Section 4 is devoted to the quasi-polynomiality phenomenon: we first consider maps with boundary-faces only in Section 4.1, revisiting the result of Norbury and Scott. We then treat the case of maps having also boundary-vertices in Section 4.2: the proof is by induction on the number of boundaries, using the recursion relations of Section 3 for the induction step and results from topological recursion for the initialization. For the sake of clarity, we begin with the case of maps with even face degrees. The extension to the general case follows the same strategy but requires extra technical steps which are offloaded to the appendices. Concluding remarks are gathered in Section 5. Additional material is contained in the appendices: Appendix A generalizes the formula of [5] for the generating function of planar maps with three tight boundaries (i.e., "tight pair of pants") to the non-bipartite case. Appendices B and C contain the proof of the induction step and the initialization step, respectively, for the quasi-polynomiality phenomenon. Appendix D gives a combinatorial expression for the derivatives of inverse functions of several variables, used in Appendix C.

2. The trumpet decomposition

In this section, we discuss a bijective decomposition of a map with boundaries into a map with tight boundaries and a collection of annular maps with one strictly tight boundary (a *trumpet*). This generalizes [5, Proposition 6.5], which dealt with planar maps with three boundaries, to arbitrary genera and number of boundaries.

2.1. The decomposition theorem

Let us introduce some notation. A *trumpet* is a map M of genus 0 with two boundary-faces f, F, such that the boundary-face F is rooted, and such that the boundary-face f, called the *mouthpiece*, is unrooted and strictly tight. Recall from Section 1.2 that this means that the contour of f is the *unique* closed path of minimal length among all closed paths of M that are (freely) homotopic to f. By considering a leftmost geodesic from the root corner of the face F aimed towards the face f, we

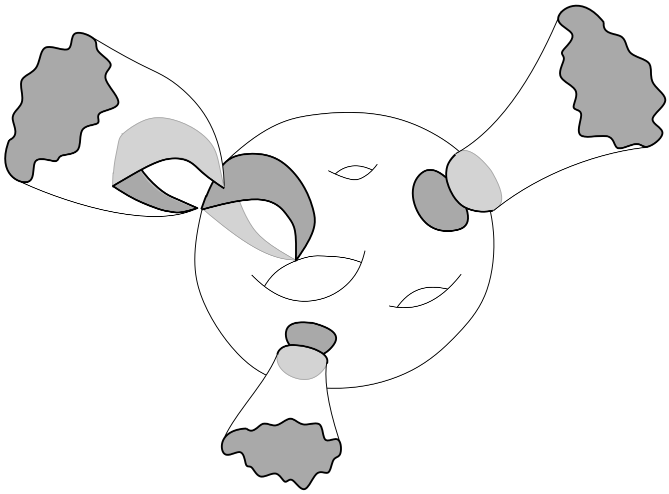

Figure 2. An illustration of the decomposition of a map of genus 3 with three external faces into a map with tight boundaries and with the same topology and a brassband of three trumpets. We emphasize in this picture that while the mouthpieces of the trumpets always have simple boundaries by Lemma 2.1, the (tight) external faces of the central map need not have simple boundaries.

may canonically distinguish one corner incident to f. We state the following simple but crucial fact.

Lemma 2.1. *The contour of the boundary of the mouthpiece of a trumpet is simple.*

This lemma is a consequence of the fact that the trumpet M is drawn on a topological cylinder, and is characteristic of this topology: as can be seen, for example, in Figure 2, tight faces need not have simple boundaries in maps of arbitrary topology (precisely, of negative Euler characteristic $\chi = 2 - 2g - n$, where g is the genus and n is the number of boundaries).

Proof. We may assume that the trumpet M is embedded in the plane, with the puncture of the mouthpiece f set at point 0, and the puncture of the rooted face F sent to infinity. We let $\mathbb{D}$ be the unit open disk in $\mathbb{R}^2$ and $\mathbb{S}^1 = \partial\mathbb{D}$ be the unit circle.

The contour of f is a closed path, which we orient arbitrarily and denote by c. Plainly, it admits a simple closed subpath c' not reduced to a single vertex, and we aim at showing that $c' = c$. By the Jordan–Schoenflies theorem, c' cuts the plane into two domains D (bounded) and $\mathbb{R}^2 \setminus \bar{D}$ (unbounded), and moreover, c' viewed as a continuous injective mapping $\mathbb{S}^1 \to \mathrm{Im}(c')$ can be extended into a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ sending $\mathbb{D}$ to D. If 0 does not belong to D, it follows c' is homotopic to the trivial path in the punctured plane $\mathbb{R}^2 \setminus \{0\}$, so that c is homotopic to a strictly shorter path, contradicting the tightness assumption of f. Therefore, it must be that $0 \in D$, so that c' is homotopic to $\mathbb{S}^1$ and therefore generates the fundamental group $\pi_1(\mathbb{R}^2 \setminus \{0\}) \simeq \mathbb{Z}$. Since c, being the contour of the face f that contains 0, is also a generator, this implies that c' is either homotopic to c or to its reversed path. But the tightness of f shows that the length of c' is at least that of c, and the only possibility is that c' is equal to c. ∎

Next, let M_0 be a map of genus g with n tight rooted boundary-faces $f_1^0, \dots, f_n^0$, and let M_i, $1 \le i \le n$ be a sequence of trumpets (a *brassband*), with respective boundary-faces f_i, F_i, where f_i denotes the mouthpiece. We assume that the length of f_i and the length of f_i^0 are equal for $1 \le i \le n$.

For every $i \in \{1, \dots, n\}$, we orient the contour of the faces f_i^0, f_i in such a way that f_i^0 lies to the right, and f_i lies to the left of their oriented contours. Then, since the boundary of f_i is a simple closed path, and since f_i, f_i^0 have the same length, one may identify these boundaries, in such a way that the root of f_i^0 and the distinguished corner of f_i (obtained from the root of F_i as explained above) are matched, by gluing edges sequentially as they appear in contour order, for our choice of orientation. This results in a map $M = \Phi(M_0, M_1, \dots, M_n)$ with genus g and with the n rooted boundary faces $F_1, \dots, F_n$ inherited from $M_1, \dots, M_n$.

Theorem 2.2 (Trumpet decomposition of maps). *Let g, n be integers such that $g \ge 0$, $n \ge 1$, and $(g,n) \ne (0,1)$ or equivalently $\chi = 2 - 2g - n \le 0$, and let $L_1, \ell_1, \dots, L_n, \ell_n$ be fixed positive integers. Then, the mapping Φ is a bijection between*

- *sequences $(M_0, M_1, \dots, M_n)$, where M_0 is a map of genus g with n rooted tight boundaries of lengths ℓ_i, $1 \le i \le n$ and for $1 \le i \le n$, M_i is a trumpet with mouthpiece length ℓ_i and whose other external face has length L_i and*
- *maps M of genus g with n rooted boundaries $F_1, \dots, F_n$ of lengths $L_1, \dots, L_n$, and such that for $1 \le i \le n$, the minimal length of a closed path homotopic to F_i is equal to ℓ_i.*

This result is illustrated in Figure 2. As already mentioned above, this is a generalization of [5, Proposition 6.5]. The main difficulty comes from constructing the reverse mapping $\Phi^{-1}(M)$, where M is a map of genus g with n boundary-faces $F_1, \dots, F_n$. This consists in considering, for $i \in \{1, 2, \dots, n\}$, the set $\mathcal{C}^{(i)}_{\min}(M)$ of closed paths in M that are homotopic to F_i, and that have minimal possible length, and to define an order $\prec^{(i)}$ that makes $\mathcal{C}^{(i)}_{\min}(M)$ a lattice. In words, we have $c \prec^{(i)} c'$ if c is closer to F_i than c', although this description requires some interpretation, which is done in [5] by working in the universal cover of M. Since $\mathcal{C}^{(i)}_{\min}(M)$ is clearly a finite set, this implies that there is a smallest element c^i in $\mathcal{C}^{(i)}_{\min}(M)$, that we call the outermost minimal closed path homotopic to F_i. Cutting along these outermost closed paths, for every $i \in \{1, \dots, n\}$, produces n trumpets $M_1, \dots, M_n$ and a remaining map M_0 with n tight boundaries $f_1^0, \dots, f_n^0$, whose lengths match with those of the corresponding mouthpieces.

We will not repeat the detailed argument here and refer the interested reader to [5, Section 6.1].

2.2. Enumerative consequences

Theorem 2.2 admits the following immediate corollary in terms of generating functions (recall that we attach a weight t to each vertex which is not a boundary, and a weight t_i to each inner face of degree i, $i = 1, 2, 3, \dots$).

Corollary 2.3. *For $g \geq 0$ and $L_1, \dots, L_n$ positive integers ($n \geq 1$), let us denote by $F^{(g)}_{L_1,\dots,L_n}$ the generating function of maps of genus g with n rooted boundary-faces of lengths $L_1, \dots, L_n$, and by $\widehat{T}^{(g)}_{L_1,\dots,L_n}$ the generating function of those maps whose boundaries are tight*[2]. *Then, for $(g, n) \neq (0, 1)$ or equivalently $\chi = 2 - 2g - n \leq 0$, we have*

$$F^{(g)}_{L_1,\dots,L_n} = \sum_{\ell_1,\dots,\ell_n \geq 1} A_{L_1,\ell_1} \cdots A_{L_n,\ell_n} \widehat{T}^{(g)}_{\ell_1,\dots,\ell_n}, \tag{4}$$

where $A_{L,\ell}$ is the generating function of trumpets with rooted boundary of length L and mouthpiece of length ℓ, where the vertices incident to the mouthpiece do not receive a weight t.

Note that, by their very definition, $F^{(g)}_{L_1,\dots,L_n}$ and $\widehat{T}^{(g)}_{L_1,\dots,L_n}$ are symmetric functions of $L_1, \dots, L_n$. The usefulness of the statement above comes from the fact that $A_{L,\ell}$ admits an explicit expression.

Proposition 2.4 ([4, Section 9.3]). *For any two positive integers L, ℓ, we have*

$$A_{L,\ell} = [z^\ell]\left(z + S + \frac{R}{z}\right)^L. \tag{5}$$

Note that the notation from [4] differs from that of the present paper: precisely, Proposition 2.4 is implied by formula (9.18) in this reference, in the case $d = 0$, $d' = \ell$ and $n = L$. The right-hand side of (5) is then nothing but the three-step path generating function $P_\ell(L; R^{(0)}, S^{(0)})$, beware that the variable z used here does not have the same meaning as in [4].

[2] By convention, we use a hat symbol to indicate that the boundaries are *rooted*. Later on, we will consider the hatless generating function $T^{(g)}_{L_1,\dots,L_n}$, where the boundaries are *unrooted*, see (11).

Let us now introduce the "grand" generating function

$$W_n^{(g)}(x_1, \dots, x_n) := \sum_{L_1,\dots,L_n \geq 1} \frac{F^{(g)}_{L_1,\dots,L_n}}{x_1^{L_1+1} \cdots x_n^{L_n+1}}, \tag{6}$$

which is a formal power series in $x_1^{-1}, \dots, x_n^{-1}$ ($n \geq 1$). Then, following [12, Definition 3.3.1, p. 87], we define for $(g, n) \neq (0, 1), (0, 2)$[3], or equivalently $\chi = 2 - 2g - n < 0$, the quantity

$$\omega_n^{(g)}(z_1, \dots, z_n) := W_n^{(g)}\big(x(z_1), \dots, x(z_n)\big) x'(z_1) \cdots x'(z_n),$$

where

$$x(z) := R^{1/2}(z + z^{-1}) + S$$

is precisely the Zhukovsky transformation discussed in Section 1.1.

Theorem 2.5 (Combinatorial interpretation of $\omega_n^{(g)}$). *The series $\omega_n^{(g)}(z_1, \dots, z_n)$ is a well-defined power series in $z_1^{-1}, \dots, z_n^{-1}$ of the form*

$$\omega_n^{(g)}(z_1, \dots, z_n) = \sum_{\ell_1,\dots,\ell_n \geq 1} \frac{\hat{\tau}^{(g)}_{\ell_1,\dots,\ell_n}}{z_1^{\ell_1+1} \cdots z_n^{\ell_n+1}}. \tag{7}$$

Its coefficients are related to the generating functions of maps with rooted tight boundaries by

$$\hat{T}^{(g)}_{\ell_1,\dots,\ell_n} = \hat{\tau}^{(g)}_{\ell_1,\dots,\ell_n} R^{(\ell_1+\cdots+\ell_n)/2}. \tag{8}$$

Proof. Observe that

$$x(z)^{-1} = \frac{z^{-1}}{R^{1/2} + Sz^{-1} + R^{1/2}z^{-2}} = \frac{z^{-1}}{R^{1/2}} - \frac{Sz^{-2}}{R} + \cdots$$

can be seen as a formal power series in z^{-1} without constant coefficient, so that for any L, $x'(z)/x(z)^{L+1}$ is a series containing only powers of z^{-1} larger than or equal to $L + 1$. This shows that

$$\omega_n^{(g)}(z_1, \dots, z_n) = \sum_{L_1,\dots,L_n \geq 1} F^{(g)}_{L_1,\dots,L_n} \frac{x'(z_1)}{x(z_1)^{L_1+1}} \cdots \frac{x'(z_n)}{x(z_n)^{L_n+1}} \tag{9}$$

is a well-defined power series in $z_1^{-1}, \dots, z_n^{-1}$ of the form (7): the only terms of (9) contributing to the coefficient $\hat{\tau}^{(g)}_{\ell_1,\dots,\ell_n}$ of $z_1^{-\ell_1-1} \cdots z_n^{-\ell_n-1}$ are those with $1 \leq L_i \leq \ell_i$, for all i.

[3] Some conventional terms have to be added in these cases.

Now, we claim that, for any formal Laurent series $F(x)$ in x^{-1}, we have the identity $[x^{-1}]F(x) = [z^{-1}]F(x(z))x'(z)$. Indeed, by linearity it suffices to check this identity for $F(x) = x^k$, $k \in \mathbb{Z}$: both sides are equal to 1 for $k = -1$, and vanish for $k \neq -1$ (since $x(z)^k x'(z) = \frac{d}{dz}\frac{x(z)^{k+1}}{k+1}$ and a derivative contains no monomial in z^{-1}). Applying the identity for each variable of the multivariate formal Laurent series $x_1^{L_1} \cdots x_n^{L_n} W_n^{(g)}(x_1, \ldots, x_n)$, we obtain

$$\begin{aligned} F^{(g)}_{L_1,\ldots,L_n} &= [x_1^{-1} \cdots x_n^{-1}] x_1^{L_1} \cdots x_n^{L_n} W_n^{(g)}(x_1, \ldots, x_n) \\ &= [z_1^{-1} \cdots z_n^{-1}] x(z_1)^{L_1} \cdots x(z_n)^{L_n} \omega_n^{(g)}(z_1, \ldots, z_n) \\ &= \sum_{\ell_1,\ldots,\ell_n \geq 1} \left([z_1^{\ell_1}]x(z_1)^{L_1}\right) \cdots \left([z_n^{\ell_n}]x(z_n)^{L_n}\right) \hat{\tau}^{(g)}_{\ell_1,\ldots,\ell_n} \\ &= \sum_{\ell_1,\ldots,\ell_n \geq 1} A_{L_1,\ell_1} \cdots A_{L_n,\ell_n} R^{(\ell_1+\cdots+\ell_n)/2} \hat{\tau}^{(g)}_{\ell_1,\ldots,\ell_n}, \end{aligned}$$

where $A_{L,\ell}$ is as in (5). Comparing with (4), and noting that the matrix $(A_{L,\ell})_{L,\ell \geq 1}$ is unitriangular, hence invertible, we get equality (8), as desired. ∎

Remark 2.6. As we have $x(z) = x(z^{-1})$ and $x'(z) = -x'(z^{-1})/z^2$, we may alternatively view $\omega_n^{(g)}(z_1, \ldots, z_n)$ as a formal power series in $z_1, \ldots, z_n$ with expansion

$$\omega_n^{(g)}(z_1, \ldots, z_n) = (-1)^n \sum_{\ell_1,\ldots,\ell_n \geq 1} \hat{T}^{(g)}_{\ell_1,\ldots,\ell_n} R^{-(\ell_1+\cdots+\ell_n)/2} z_1^{\ell_1 - 1} \cdots z_n^{\ell_n - 1}. \quad (10)$$

As we will see below, $\omega_n^{(g)}(z_1, \ldots, z_n)$ is actually a rational function of $z_1, \ldots, z_n$, and (7) and (10) are two different expansions of it (at infinity and at zero, respectively).

2.3. Allowing boundary-vertices

Theorem 2.2 deals with the case where all boundaries of the maps at hand are faces. There is a more general statement that also deals with both boundary-faces and boundary-vertices. Let M_0 be a map of genus g with $n = m + s$ distinguished tight boundaries, the first m of which are rooted faces $f_1^0, \ldots, f_m^0$, the remaining s being vertices, for some $m, s \geq 0$. Let M_i, $1 \leq i \leq m$ be a brassband with respective boundary-faces f_i, F_i, where f_i denotes the mouthpiece. We assume that the lengths of f_i and of f_i^0 are equal for $1 \leq i \leq m$. We let $M = \Phi(M_0, M_1, \ldots, M_m)$ be the map obtained as above by identifying the boundaries of f_i and f_i^0 for $1 \leq i \leq m$. Then, the map M has m rooted boundary-faces and s boundary-vertices, inherited from M_0.

Theorem 2.7. *Let g, m, s be nonnegative integers such that $n = m + s \geq 1$ and $\chi = 2 - 2g - n \leq 0$. Let $L_1, \ell_1, \ldots, L_m, \ell_m$ be fixed positive integers. Then, the mapping Φ is a bijection between*

- *sequences $(M_0, M_1, \ldots, M_m)$, where M_0 is a map of genus g with n distinguished tight boundaries, the first m of which are rooted boundary-faces of lengths ℓ_i,*

$1 \le i \le m$, the remaining s being boundary-vertices, and for $1 \le i \le m$, M_i is a trumpet with mouthpiece length ℓ_i and whose other external face has length L_i and

- *maps M of genus g with n distinguished boundaries, the first m of which are rooted boundary-faces $F_1, \ldots, F_m$ of lengths $L_1, \ldots, L_m$, the remaining s being boundary-vertices, and such that for $1 \le i \le m$, the minimal length of a closed path homotopic to F_i is equal to ℓ_i.*

The justification for this statement is exactly the same as for Theorem 2.2. The inverse bijection consists in cutting the map M along the outermost minimal closed paths that are homotopic to the boundary-faces, considering *both* boundary-faces and boundary-vertices as punctures. In particular, the boundary-vertices are never included in the trumpets $M_1, \ldots, M_m$ that are cut away.

We now consider the enumerative consequences of Theorem 2.7. In order to put boundary-faces and boundary-vertices on a same footing, we denote by $T^{(g)}_{\ell_1,\ldots,\ell_n}$ the generating function of maps of genus g with n *unrooted* tight boundaries of lengths $\ell_1, \ldots, \ell_n$. As the generating function $\widehat{T}^{(g)}_{\ell_1,\ldots,\ell_n}$ considered above, it is symmetric in $\ell_1, \ldots, \ell_n$, and these two quantities are related by

$$\widehat{T}^{(g)}_{\ell_1,\ldots,\ell_n} = \ell_1 \cdots \ell_n T^{(g)}_{\ell_1,\ldots,\ell_n} \tag{11}$$

as there are $\ell_1 \cdots \ell_n$ ways to root the boundaries, up to the symmetry factor corresponding to the inverse of the automorphism group order[4]. In particular, there are zero ways if one ℓ_i vanishes: as said in Section 1.2, a boundary-vertex cannot be rooted. Thus, $T^{(g)}_{\ell_1,\ldots,\ell_n}$ contains a priori more information than $\widehat{T}^{(g)}_{\ell_1,\ldots,\ell_n}$. In particular, for given integers $m, s \ge 0$ not both equal to 0, and positive integers $\ell_1, \ldots, \ell_m$, the generating function of maps with m rooted tight boundary-faces of lengths $\ell_1, \ldots, \ell_m$ and with s boundary-vertices is given by $(\ell_1 \cdots \ell_m) T^{(g)}_{\ell_1,\ldots,\ell_m,0,\ldots,0}$, with s zero indices. With this remark at hand, we can now state the generating function counterpart to Theorem 2.7 as follows.

Corollary 2.8. *Let g, m, s be nonnegative integers such that $n = m + s \ge 1$ and $\chi = 2 - 2g - n \le 0$. Let $L_1, \ldots, L_m$ be fixed positive integers. Then, we have*

$$\frac{\partial^s F^{(g)}_{L_1,\ldots,L_m}}{\partial t^s} = \sum_{\ell_1,\ldots,\ell_m \ge 1} A_{L_1,\ell_1} \cdots A_{L_m,\ell_m} (\ell_1 \cdots \ell_m) T^{(g)}_{\ell_1,\ldots,\ell_m,0,\ldots,0}, \tag{12}$$

[4]Note that the objects enumerated by the generating series $T^{(g)}_{\ell_1,\ldots,\ell_n}$ are not rooted, and therefore can have nontrivial automorphism groups. This is why we have to weigh them by the inverse of the automorphism group order, as discussed in the introduction.

where there are s indices equal to 0 in the last term, and where we reuse the notations from Corollary 2.3. For $m = 0$, the relation reads

$$\frac{\partial^s F^{(g)}}{\partial t^s} = T^{(g)}_{0,\ldots,0},$$

with s indices equal to 0, and where $F^{(g)}$ is the generating function of maps of genus g without boundaries.

For g, m, s as above, we introduce the following generalization of the grand generating function (6):

$$\begin{aligned} W^{(g)}_{m,s}(x_1,\ldots,x_m) &:= \sum_{L_1,\ldots,L_m} \frac{\partial^s F^{(g)}_{L_1,\ldots,L_m}}{\partial t^s} \cdot \frac{1}{x_1^{L_1+1}\cdots x_m^{L_m+1}} \\ &= \frac{\partial^s}{\partial t^s} W^{(g)}_m(x_1,\ldots,x_m), \end{aligned}$$

and, assuming further that $\chi < 0$, the quantity

$$\omega^{(g)}_{m,s}(z_1,\ldots,z_m) := W^{(g)}_{m,s}(x(z_1),\ldots,x(z_m))x'(z_1)\cdots x'(z_m).$$

For $m = 0$, we have $\omega^{(g)}_{0,s} = W^{(g)}_{0,s} = \partial^s F^{(g)}/\partial t^s$. Corollary 2.8 admits the following consequence, analogous to Theorem 2.5, whose proof is adapted from the latter in a straightforward way.

Theorem 2.9. *For integers $\ell_1,\ldots,\ell_n$, define*

$$\tau^{(g)}_{\ell_1,\ldots,\ell_n} := T^{(g)}_{\ell_1,\ldots,\ell_n} R^{-(\ell_1+\cdots+\ell_n)/2} \tag{13}$$

with $T^{(g)}_{\ell_1,\ldots,\ell_n}$ the generating function of maps with unrooted tight boundaries defined above. Then, we have the expansion

$$\omega^{(g)}_{m,s}(z_1,\ldots,z_m) = \sum_{\ell_1,\ldots,\ell_m \geq 1} \frac{\ell_1\cdots\ell_m \tau^{(g)}_{\ell_1,\ldots,\ell_m,0,\ldots,0}}{z_1^{\ell_1+1}\cdots z_m^{\ell_m+1}},$$

where there are s trailing zeros in the indices of $\tau^{(g)}$. For $m = 0$, we have $\omega^{(g)}_{0,s} = \tau^{(g)}_{0,\ldots,0} = T^{(g)}_{0,\ldots,0}$.

Theorems 2.5 and 2.9 provide combinatorial interpretations for the coefficients of $\omega^{(g)}_n$ and $\omega^{(g)}_{m,s}$ in terms of generating series of maps with tight boundaries. However, they do not provide any information about the structural properties of $\tau^{(g)}_{\ell_1,\ldots,\ell_n}$. This is the goal of the upcoming Section 4, where we will see that, for $2 - 2g - n < 0$, these are quasi-polynomials in the variables $\ell_1,\ldots,\ell_n$, and that $\tau^{(g)}_{\ell_1,\ldots,\ell_m,0,\ldots,0}$ (with s zero indices) indeed corresponds to the evaluation of this quasi-polynomial function of $n = m + s$ variables when the last s ones are set to 0.

Remark 2.10. The case $(g,n)=(0,2)$ is special, since we have for $\ell_1>0$

$$\begin{aligned}\tau^{(0)}_{\ell_1,\ell_2} &= \frac{\delta_{\ell_1,\ell_2}}{\ell_1},\\ T^{(0)}_{\ell_1,\ell_2} &= \frac{R^{\ell_1}\delta_{\ell_1,\ell_2}}{\ell_1},\\ \widehat{T}^{(0)}_{\ell_1,\ell_2} &= \ell_1 R^{\ell_1}\delta_{\ell_1,\ell_2},\end{aligned} \tag{14}$$

which are *not* quasi-polynomials in ℓ_1, ℓ_2. Indeed, it is clear that these quantities vanish for $\ell_1\neq\ell_2$, and we get

$$\ell_1 T^{(0)}_{\ell_1,\ell_1} = R^{\ell_1}$$

by taking $d=0$, $n=d'=\ell_1$ in [4, equation (9.19)].

2.4. A formula for the series of planar bipartite maps with tight boundaries

In this section, we give an explicit expression for the generating function $T^{(0)}_{\ell_1,\dots,\ell_n}$ of planar maps with n unrooted tight boundaries of lengths $\ell_1,\dots,\ell_n$, in the *bipartite case* where the ℓ_i are even and where the weights $t_1,t_3,t_5,\dots$ for inner faces of odd degrees are set to zero. To emphasize this latter restriction we will use the notation $|_{\mathrm{bip}}$ in the following equations. Recall that the basic generating function R is here determined by (3).

Our input is the Collet–Fusy formula [10, Theorem 1.1] for the generating function of planar bipartite maps with n rooted, not necessarily tight, boundaries of prescribed lengths. In our present notations, it reads

$$F^{(0)}_{2L_1,\dots,2L_n}\Big|_{\mathrm{bip}} = \frac{L_1\cdots L_n}{L_1+\cdots+L_n}\prod_{i=1}^{n}\binom{2L_i}{L_i}\frac{\partial^{n-2}}{\partial t^{n-2}}R^{L_1+\cdots+L_n} \tag{15}$$

for any $n\geq 1$ and positive integers $L_1,\dots,L_n$. Note that the formula makes sense for $n=1$, upon understanding that $\frac{\partial^{-1}}{\partial t^{-1}}$ means integrating over t. For $n=2$, it is a simple exercise to check that the formula is consistent with Corollary 2.3 and (14). We concentrate on the case $n\geq 3$ from now on.

We will deduce from the Collet–Fusy formula an expression for $T^{(0)}_{2\ell_1,\dots,2\ell_n}|_{\mathrm{bip}}$, using Corollary 2.8. Interestingly, our expression involves a certain family of polynomials that we encountered previously in [6]. Namely, for any integers $k\geq 0$ and $n\geq 1$, let us consider the multivariate polynomial in the variables $\ell_1,\dots,\ell_n$

$$p_k(\ell_1,\ell_2,\dots,\ell_n) := \sum_{\substack{k_1,k_2,\dots,k_n\geq 0\\ k_1+k_2+\cdots+k_n=k}} p_{k_1}(\ell_1)q_{k_2}(\ell_2)\cdots q_{k_n}(\ell_n), \tag{16}$$

where, on the right-hand side, all factors except the first one are q_{k_i}'s, and where the univariate polynomials p_k and q_k are defined by

$$
\begin{aligned}
p_k(\ell) &:= \frac{1}{(k!)^2}\prod_{i=1}^{k}(\ell^2-i^2) = \binom{\ell-1}{k}\binom{\ell+k}{k},\\
q_k(\ell) &:= \frac{1}{(k!)^2}\prod_{i=0}^{k-1}(\ell^2-i^2) = \binom{\ell}{k}\binom{\ell+k-1}{k}
\end{aligned}
\tag{17}
$$

with the convention $p_0(\ell) = q_0(\ell) = 1$, and with $\binom{x}{k} = x(x-1)\cdots(x-k+1)/k!$ viewed as a polynomial in x. Clearly, $p_k(\ell_1,\ell_2,\ldots,\ell_n)$ is a polynomial in the variables $\ell_1^2,\ldots,\ell_n^2$. Moreover, as discussed in [6, Proposition 2.1], it is a symmetric polynomial and it satisfies the consistency relation

$$
p_k(\ell_1,\ell_2,\ldots,\ell_n,0) = p_k(\ell_1,\ell_2,\ldots,\ell_n). \tag{18}
$$

For $k = 1,2,3$, we have

$$
\begin{aligned}
p_1(\ell_1,\ldots,\ell_n) &= \left(\sum_{i=1}^{n}\ell_i^2\right)-1,\\
p_2(\ell_1,\ldots,\ell_n) &= \frac{1}{4}\left(\sum_{i=1}^{n}\ell_i^4\right)+\sum_{i<j}\ell_i^2\ell_j^2-\frac{5}{4}\left(\sum_{i=1}^{n}\ell_i^2\right)+1,\\
p_3(\ell_1,\ldots,\ell_n) &= \frac{1}{36}\left(\sum_{i=1}^{n}\ell_i^6\right)+\frac{1}{4}\Big(\sum_{i\neq j}\ell_i^4\ell_j^2\Big)+\sum_{i<j<h}\ell_i^2\ell_j^2\ell_h^2\\
&\quad -\frac{7}{18}\left(\sum_{i=1}^{n}\ell_i^4\right)-\frac{3}{2}\Big(\sum_{i<j}\ell_i^2\ell_j^2\Big)+\frac{49}{36}\left(\sum_{i=1}^{n}\ell_i^2\right)-1.
\end{aligned}
\tag{19}
$$

We will also need the so-called partial exponential Bell polynomials, defined for any nonnegative integers $n \geq k \geq 1$ by

$$
\begin{aligned}
&B_{n,k}(r_1,r_2,\ldots,r_{n-k+1})\\
&\quad := \sum_{\substack{j_1,j_2,\ldots,j_{n-k+1}\geq 0\\ j_1+j_2+\cdots+j_{n-k+1}=k\\ j_1+2j_2+\cdots+(n-k+1)j_{n-k+1}=n}} \frac{n!}{j_1!j_2!\cdots j_{n-k+1}!}\left(\frac{r_1}{1!}\right)^{j_1}\left(\frac{r_2}{2!}\right)^{j_2}\\
&\qquad\qquad \cdots\left(\frac{r_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}.
\end{aligned}
$$

Combinatorially, $B_{n,k}(r_1,r_2,\ldots)$ is the generating series of partitions of an n-element set into k blocks, where we attach a weight r_i to each block of size i. Table 1 lists the first few Bell polynomials. They appear in the following classical formula.

$n \backslash k$	1	2	3	4
1	r_1			
2	r_2	r_1^2		
3	r_3	$3r_1r_2$	r_1^3	
4	r_4	$4r_1r_3 + 3r_2^2$	$6r_1^2r_2$	r_1^4

Table 1. The first few partial exponential Bell polynomials $B_{n,k}(r_1, r_2, \ldots)$.

Proposition 2.11 (Faà di Bruno's formula). *Let f, g be sufficiently smooth functions of one variable, and let n be a positive integer. Then, we have*

$$\frac{d^n}{dt^n}(f \circ g)(t) = \sum_{k=1}^{n} f^{(k)}(g(t)) B_{n,k}\big(g'(t), g''(t), \ldots, g^{(n-k+1)}(t)\big).$$

See, for instance, the discussion in [14, Example III.24]. We may now state the main result of this section.

Theorem 2.12. *For $n \geq 3$ and $\ell_1, \ldots, \ell_n$ nonnegative integers, the generating function of bipartite planar maps with n unrooted tight boundaries of lengths $2\ell_1, \ldots, 2\ell_n$ is equal to*

$$T^{(0)}_{2\ell_1,\ldots,2\ell_n}\big|_{\mathrm{bip}} = R^{\ell_1+\cdots+\ell_n} \sum_{k=0}^{n-3} k!\, p_k(\ell_1, \ldots, \ell_n) b_{n-2,k+1} + \frac{(-1)^n (n-3)!}{t^{n-2}} \delta_{\ell_1+\cdots+\ell_n,0} \tag{20}$$

with R given by (3), and where we introduce the shorthand notation

$$b_{n,k} := R^{-k} B_{n,k}(R', R'', \ldots) = B_{n,k}\left(\frac{R'}{R}, \frac{R''}{R}, \ldots\right) \tag{21}$$

with $R', R'', \ldots$ the successive derivatives of R with respect to the vertex weight t.

For $n = 3$, we recover $T^{(0)}_{2\ell_1,2\ell_2,2\ell_3}|_{\mathrm{bip}} = R^{\ell_1+\ell_2+\ell_3-1} R' - t^{-1}\delta_{\ell_1+\ell_2+\ell_3,0}$ consistently with [5, Theorem 1.1], while for $n = 4$, we find

$$T^{(0)}_{2\ell_1,2\ell_2,2\ell_3,2\ell_4}\big|_{\mathrm{bip}} = R^{\ell_1+\ell_2+\ell_3+\ell_4}\left(\left(\ell_1^2 + \ell_2^2 + \ell_3^2 + \ell_4^2 - 1\right)\frac{R'^2}{R^2} + \frac{R''}{R}\right) + \frac{\delta_{\ell_1+\cdots+\ell_4,0}}{t^2}. \tag{22}$$

Longer expressions for $n = 5, 6$ may be written straightforwardly from (19) and Table 1.

Remark 2.13. When setting the weights $t_2, t_4, \dots$ for inner faces to zero, we have $R = t$, $R' = 1$ and $R^{(j)} = 0$ for $j \geq 2$. Thus, in this case, we have $b_{n,k} = t^{-k}\delta_{n,k}$, hence, $T^{(0)}_{2\ell_1,\dots,2\ell_n}$ reduces to $(n-3)!\,p_{n-3}(\ell_1,\dots,\ell_n)t^{\ell_1+\dots+\ell_n-n+2}$ if at least one ℓ_i is non-zero, and vanishes otherwise since $p_{n-3}(0,\dots,0) = (-1)^{n-3}$ so the rightmost term in (20) cancels the result. This is consistent with [6, Theorem 2.3] since the tight maps considered in this reference are nothing but maps with tight boundaries and without inner faces. Note that the exponent of t is consistent with Euler's relation.

Proof of Theorem 2.12. Let us first consider the case where all ℓ_i are zero: then $T^{(0)}_{0,\dots,0}$ is simply the generating function of bipartite planar maps with $n \geq 3$ marked vertices. For $n = 3$, it is equal to $\frac{\partial}{\partial t}\ln(R/t)$, see, e.g., [5, Appendix A], and for larger n, we simply have to take more derivatives with respect to t, namely,

$$T^{(0)}_{0,\dots,0}\big|_{\text{bip}} = \frac{\partial^{n-2}}{\partial t^{n-2}}\ln(R/t) = \frac{\partial^{n-2}}{\partial t^{n-2}}\ln R + \frac{(-1)^n(n-3)!}{t^{n-2}}.$$

Now, by applying Faà di Bruno's formula, we get

$$\begin{aligned} T^{(0)}_{0,\dots,0}\big|_{\text{bip}} &= \sum_{k=1}^{n-2}(-1)^{k-1}(k-1)!R^{-k}B_{n-2,k}(R', R'', \dots) + \frac{(-1)^n(n-3)!}{t^{n-2}} \\ &= \sum_{k=0}^{n-3}(-1)^k k!\, b_{n-2,k+1} + \frac{(-1)^n(n-3)!}{t^{n-2}}. \end{aligned}$$

This gives the wanted formula for $\ell_1 = \dots = \ell_n = 0$ since, by (17) and (18), we have $p_k(0,\dots,0) = p_k(0) = (-1)^k$.

Let us now assume that at least one ℓ_i is non-zero, and let $m \geq 1$ be the number of non-zero ℓ_i, and $s = n - m$ the number of zero ℓ_i. Without loss of generality, we can assume that $\ell_1,\dots,\ell_m > 0$ and $\ell_i = 0$ for $i > m$. We note that, in the bipartite case, the trumpet generating function of Proposition 2.4 vanishes whenever its indices are of different parities (since $S = 0$), and reads for even indices

$$A_{2L,2\ell} = \binom{2L}{L-\ell}R^{L-\ell}. \tag{23}$$

Thus, Corollary 2.8 reduces to

$$\frac{\partial^s}{\partial t^s}F^{(0)}_{2L_1,\dots,2L_m}\big|_{\text{bip}} = \sum_{\ell_1,\dots,\ell_m\geq 1}(2\ell_1)A_{2L_1,2\ell_1}\cdots(2\ell_m)A_{2L_m,2\ell_m}T^{(0)}_{2\ell_1,\dots,2\ell_m,0,\dots,0}\big|_{\text{bip}} \tag{24}$$

(where we append s zeros after $2\ell_m$). On the other hand, by the Collet–Fusy formula and by Faà di Bruno's formula, we have

$$\begin{aligned}\frac{\partial^s}{\partial t^s}&F^{(0)}_{2L_1,\dots,2L_m}\Big|_{\mathrm{bip}}\\ &= \frac{L_1\cdots L_m}{L_1+\cdots+L_m}\binom{2L_1}{L_1}\cdots\binom{2L_m}{L_m}\frac{\partial^{n-2}}{\partial t^{n-2}}R^{L_1+\cdots+L_m}\\ &= \sum_{k=1}^{n-2} L_1\cdots L_m(L_1+\cdots+L_m-1)_{k-1}\binom{2L_1}{L_1}\cdots\binom{2L_m}{L_m}R^{L_1+\cdots+L_m}b_{n-2,k},\end{aligned} \tag{25}$$

where $(L)_k := L(L-1)\cdots(L-k+1)$ denotes the falling factorial. We will rewrite this expression in the same form as the right-hand side of (24), so as to identify $T^{(0)}_{2\ell_1,\dots,2\ell_m,0,\dots,0}|_{\mathrm{bip}}$. To this end, we proceed as in [6, Section 3.1] and use the Chu–Vandermonde identity to write

$$\begin{aligned}(L_1&+L_2+\cdots+L_m-1)_{k-1}\\ &= (k-1)!\sum_{\substack{k_1,k_2,\dots,k_m\geq 0\\ k_1+k_2+\cdots+k_m=k-1}}\binom{L_1-1}{k_1}\binom{L_2}{k_2}\cdots\binom{L_m}{k_m}.\end{aligned}$$

Plugging into (25), we obtain

$$\begin{aligned}\frac{\partial^s}{\partial t^s}&F^{(0)}_{2L_1,\dots,2L_m}\Big|_{\mathrm{bip}}\\ &= \sum_{k=1}^{n-2}(k-1)!R^{L_1+\cdots+L_m}b_{n-2,k}\\ &\times\sum_{\substack{k_1,\dots,k_m\geq 0\\ k_1+\cdots+k_m=k-1}} L_1\binom{L_1-1}{k_1}\binom{2L_1}{L_1}L_2\binom{L_2}{k_2}\binom{2L_2}{L_2}\cdots L_m\binom{L_m}{k_m}\binom{2L_m}{L_m}.\end{aligned} \tag{26}$$

Now, by [6, Lemma 3.1] and by (23)[5], we have for $L>0$

$$\begin{aligned}L\binom{L-1}{k}\binom{2L}{L} &= \sum_{\ell=1}^{L}(2\ell)\binom{2L}{L-\ell}p_k(\ell) = \sum_{\ell=1}^{L}(2\ell)A_{2L,2\ell}R^{\ell-L}p_k(\ell),\\ L\binom{L}{k}\binom{2L}{L} &= \sum_{\ell=1}^{L}(2\ell)\binom{2L}{L-\ell}q_k(\ell) = \sum_{\ell=1}^{L}(2\ell)A_{2L,2\ell}R^{\ell-L}q_k(\ell).\end{aligned} \tag{27}$$

[5]Beware that the notation $A_{\ell,m}$ used in [6] differs from that of the present paper.

Therefore, by the definition (16) of p_k, we may rewrite (26) as

$$\frac{\partial^s}{\partial t^s} F^{(0)}_{2L_1,\dots,2L_m}\Big|_{\mathrm{bip}} = \sum_{\ell_1,\dots,\ell_m \geq 1} (2\ell_1) A_{2L_1,2\ell_1} \cdots (2\ell_m) A_{2L_m,2\ell_m} \\ \times R^{\ell_1+\cdots+\ell_m} \sum_{k=1}^{n-2} (k-1)!\, p_{k-1}(\ell_1,\dots,\ell_m) b_{n-2,k}, \tag{28}$$

and by (18), we may replace $p_{k-1}(\ell_1,\dots,\ell_m)$ by $p_{k-1}(\ell_1,\dots,\ell_m,0,\dots,0)$ in order to have n variables. Comparing this expression with (24), using the invertibility of the unitriangular matrix $(A_{2L,2\ell})_{L,\ell\geq 1}$ and doing a change of variable $k-1 \to k$ finally give the wanted formula (20), since the term $\frac{(-1)^n(n-3)!}{t^{n-2}}$ is absent for $m \geq 1$. ∎

A variant of the Collet–Fusy formula (15) holds in the *quasi-bipartite case* where exactly two of the boundaries have odd lengths, i.e., when exactly two of the L_i, say, L_1 and L_2, are half-integers. In this case, the factors $\binom{2L_1}{L_1}\binom{2L_2}{L_2}$ appearing on the right-hand side should be replaced by $4\binom{2L_1-1}{L_1-1/2}\binom{2L_2-1}{L_2-1/2}$. Correspondingly, we have the following analogue of Theorem 2.12.

Proposition 2.14. *For $n \geq 3$, ℓ_1, ℓ_2 positive half-integers and $\ell_3,\dots,\ell_n$ nonnegative integers, the generating function of quasi-bipartite planar maps with n unrooted tight boundaries of lengths $2\ell_1,\dots,2\ell_n$ is equal to*

$$T^{(0)}_{2\ell_1,\dots,2\ell_n}\Big|_{\mathrm{bip}} = R^{\ell_1+\cdots+\ell_n} \sum_{k=0}^{n-3} k!\, \tilde{p}_k(\ell_1,\ell_2;\ell_3,\dots,\ell_n) b_{n-2,k+1}, \tag{29}$$

where, following [6, Section 2.1.3]*, we define*

$$\tilde{p}_k(\ell_1,\ell_2;\ell_3,\dots,\ell_n) := \sum_{\substack{k_1,k_2,\dots,k_n\geq 0\\ k_1+k_2+\cdots+k_n=k}} \tilde{p}_{k_1}(\ell_1)\tilde{p}_{k_2}(\ell_2) q_{k_3}(\ell_3)\cdots q_{k_n}(\ell_n)$$

with q_k as in (17) *and with*

$$\tilde{p}_k(\ell) := \frac{1}{(k!)^2} \prod_{i=1}^{k} \left(\ell^2 - \left(i-\frac{1}{2}\right)^2\right) = \binom{\ell-\frac{1}{2}}{k}\binom{\ell+k-\frac{1}{2}}{k}.$$

Proof. Starting with the quasi-bipartite variant of the Collet–Fusy formula, we proceed as in the proof of Theorem 2.12. Using Faà di Bruno's formula, we obtain the expression (25) with the binomial factors $\binom{2L_1}{L_1}\binom{2L_2}{L_2}$ modified as discussed above. Now, we expand the falling factorial as follows:

$$(L_1+\cdots+L_m-1)_{k-1} \\ = (k-1)! \sum_{\substack{k_1,\dots,k_m\geq 0\\ k_1+\cdots+k_m=k-1}} \binom{L_1-\frac{1}{2}}{k_1}\binom{L_2-\frac{1}{2}}{k_2}\binom{L_3}{k_3}\cdots\binom{L_m}{k_m},$$

which leads to a variant of (26) in which the factors $\binom{L_1-1}{k_1}\binom{2L_1}{L_1}\binom{L_2}{k_2}\binom{2L_2}{L_2}$ are replaced by $4\binom{L_1-1/2}{k_1}\binom{2L_1-1}{L_1-1/2}\binom{L_2-1/2}{k_2}\binom{2L_2-1}{L_2-1/2}$. By [6, (3.9)], we have for L a positive half-integer

$$2L\binom{L-\frac{1}{2}}{k}\binom{2L-1}{L-\frac{1}{2}} = \sum_{\ell=\frac{1}{2}}^{L}(2\ell)\binom{2L}{L-\ell}\tilde{p}_k(\ell) = \sum_{\ell=\frac{1}{2}}^{L}(2\ell)A_{2L,2\ell}R^{\ell-L}\tilde{p}_k(\ell),$$

where the sum runs over the positive half-integers ℓ between $\frac{1}{2}$ and L. Using again the second line of (27) to expand the remaining binomial factors, we arrive at a variant of (28) with p_{k-1} replaced by $\tilde{p}_{k-1}$, giving the wanted result. ■

Let us consider the case of four boundaries. We have $\tilde{p}_1(\ell_1,\ell_2;\ell_3,\ell_4) = \ell_1^2 + \cdots + \ell_4^2 - \frac{1}{2}$ and $\tilde{p}_0(\ell_1,\ell_2;\ell_3,\ell_4) = 1$; hence, from Proposition 2.14, we see that, in the quasi-bipartite case, the term -1 appearing in the expression (22) should be replaced by $-\frac{1}{2}$. In contrast, we found after a lengthy computation (which we omit here) that (22) holds as is when all ℓ_i are half-integers, i.e., when all four boundaries have odd lengths. This is consistent with the discussion of the lattice count polynomial $N_{0,4}$ at the end of [17, Section 1], since $T^{(0)}_{2\ell_1,2\ell_2,2\ell_3,2\ell_4}$ reduces to $N_{0,4}(2\ell_1,2\ell_2,2\ell_3,2\ell_4)$ when we set the weights for inner faces to zero.

3. Recursion relations for series of maps with tight boundaries

In this section, we give a number of recursion relations for the generating function $T^{(g)}_{\ell_1,\dots,\ell_n}$, as obtained by adding an extra tight boundary to a map with pre-existing boundaries. We will need the following proposition, proved bijectively in Appendix A.

Proposition 3.1. *Let ℓ_1, ℓ_2, and ℓ_3 be nonnegative integers, and let $T^{(0)}_{\ell_1,\ell_2|\ell_3}$ denote the generating function of planar maps with three labeled distinct tight unrooted boundaries of lengths ℓ_1, ℓ_2, ℓ_3, the third being strictly tight, where we attach a weight t per vertex different from a boundary-vertex and not incident to the third boundary (and for all $k \geq 1$, a weight t_k per inner face of degree k). Then, for $\ell_1+\ell_2 > \ell_3$, we have*

$$T^{(0)}_{\ell_1,\ell_2|\ell_3} = \begin{cases} R^{(\ell_1+\ell_2-\ell_3)/2-1}\frac{\partial R}{\partial t} & \text{if } \ell_1+\ell_2-\ell_3 \text{ is even,} \\ R^{(\ell_1+\ell_2-\ell_3-1)/2}\frac{\partial S}{\partial t} & \text{if } \ell_1+\ell_2-\ell_3 \text{ is odd.}\end{cases}$$

3.1. Adding an extra boundary-vertex

Marking a vertex in a trumpet. Take L, ℓ positive integers. Recall that $A_{L,\ell}$ is the generating function for trumpets with rooted boundary of length L and mouthpiece of

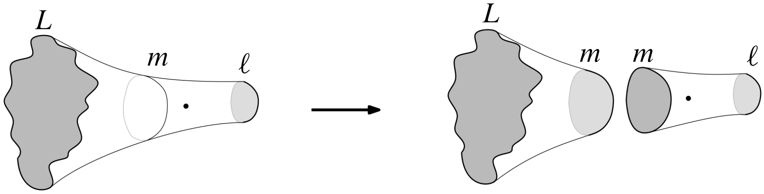

Figure 3. An illustration of the decomposition of a trumpet with mouthpiece of length ℓ endowed with an extra marked vertex, viewed as a boundary vertex. It results in two pieces: a trumpet with mouthpiece of length $m > \ell$ and a map with three boundaries counted by $m\, T^{(0)}_{m,0|\ell}$.

length ℓ. The quantity $\frac{\partial A_{L,\ell}}{\partial t}$ is, therefore, the generating function for such trumpets with a marked vertex which is not incident to the mouthpiece (recall that vertices of the mouthpiece do not receive the weight t). It has the following expression.

Proposition 3.2. *We have*

$$\begin{aligned}\frac{\partial A_{L,\ell}}{\partial t} &= \sum_{m>\ell} A_{L,m} m T^{(0)}_{m,0|\ell} \\ &= \sum_{\substack{m=\ell+1 \\ m=\ell+1 \bmod 2}}^{L} A_{L,m} m\, R^{\frac{m-\ell-1}{2}} \frac{\partial S}{\partial t} + \sum_{\substack{m=\ell+2 \\ m=\ell+2 \bmod 2}}^{L} A_{L,m} m\, R^{\frac{m-\ell}{2}-1} \frac{\partial R}{\partial t}.\end{aligned} \tag{30}$$

Combinatorial proof. Considering the marked vertex as a boundary-vertex, we obtain a planar map with three boundaries which me may decompose as follows: consider the outermost minimal closed path homotopic to the boundary of length L. Cutting along this path produces a trumpet whose mouthpiece has length $m > \ell$, see Figure 3. The remaining map with three tight boundaries is precisely a map enumerated by $T^{(0)}_{m,0|\ell}$, endowed canonically with one of its m rootings (inherited from the rooting of the boundary face of length L). ■

It is interesting to note that the expression in the second line in equation (30) may be obtained in a purely computational way as follows.

Computational proof. From the explicit expression (5) of $A_{L,\ell}$, we have

$$\begin{aligned}\frac{\partial A_{L,\ell}}{\partial t} &= \frac{\partial}{\partial t}\left([z^\ell]\left(z + S + \frac{R}{z}\right)^L\right) \\ &= [z^\ell] L \left(z + S + \frac{R}{z}\right)^{L-1} \left(\frac{\partial S}{\partial t} + \frac{1}{z}\frac{\partial R}{\partial t}\right) \\ &= L\, A_{L-1,\ell} \frac{\partial S}{\partial t} + L\, A_{L-1,\ell+1} \frac{\partial R}{\partial t}.\end{aligned}$$

To obtain the second line of (30), it is enough to prove the equality

$$L\, A_{L-1,\ell} = \sum_{p\geq 0}(\ell+2p+1)R^p A_{L,\ell+2p+1},$$

which itself follows from iterating the identity

$$L\, A_{L-1,\ell} = (\ell+1)A_{L,\ell+1} + R\, L\, A_{L-1,\ell+2} \tag{31}$$

p times until p reaches a value such that $\ell+2p > L-1$. As for this last identity (31), upon moving its last term to the left-hand side, it is simply obtained by extracting the z^ℓ coefficient of the straightforward identity

$$\left(1-\frac{R}{z^2}\right)\frac{\partial}{\partial S}\left(z+S+\frac{R}{z}\right)^L = \frac{\partial}{\partial z}\left(z+S+\frac{R}{z}\right)^L. \qquad \blacksquare$$

Marking a vertex in a map with tight boundaries. We start with equation (12) with $m=n$ and $s=0$, which we differentiate with respect to t. We get

$$\begin{aligned}\frac{\partial F^{(g)}_{L_1,\ldots,L_n}}{\partial t} &= \sum_{\ell_1,\ldots,\ell_n\geq 1} A_{L_1,\ell_1}\cdots A_{L_n,\ell_n}(\ell_1\cdots\ell_n)\frac{\partial T^{(g)}_{\ell_1,\ldots,\ell_n}}{\partial t}\\ &\quad + \sum_{i=1}^{n}\sum_{\ell_1,\ldots,\ell_n\geq 1} A_{L_1,\ell_1}\cdots\frac{\partial A_{L_i,\ell_i}}{\partial t}\cdots A_{L_n,\ell_n}(\ell_1\cdots\ell_n)T^{(g)}_{\ell_1,\ldots,\ell_n}.\end{aligned}$$

From (30), the general term in the final sum over i may be written as

$$\begin{aligned}&\sum_{\ell_1,\ldots,\ell_n\geq 1} A_{L_1,\ell_1}\cdots\Big(\sum_{m_i>\ell_i} A_{L_i,m_i} m_i\, T^{(0)}_{m_i,0|\ell_i}\Big)\\ &\qquad\qquad\qquad\qquad \cdots A_{L_n,\ell_n}(\ell_1\cdots\ell_i\cdots\ell_n)T^{(g)}_{\ell_1,\ldots,\ell_i,\ldots,\ell_n}\\ &= \sum_{\ell_1,\ldots,\ell_n\geq 1} A_{L_1,\ell_1}\cdots A_{L_i,\ell_i}\cdots A_{L_n,\ell_n}(\ell_1\cdots\ell_i\cdots\ell_n)\\ &\qquad\qquad\qquad\qquad \times\Big(\sum_{m_i<\ell_i} m_i\, T^{(0)}_{\ell_i,0|m_i}\, T^{(g)}_{\ell_1,\ldots,m_i,\ldots,\ell_n}\Big),\end{aligned}$$

where, to go from the left-hand side to the right-hand side, we exchanged the dummy summation variables ℓ_i and m_i, as well as the order of summation. Comparing with the expression for $\frac{\partial F^{(g)}_{L_1,\ldots,L_n}}{\partial t}$ coming from (12) with $m=n$ and $s=1$, namely,

$$\frac{\partial F^{(g)}_{L_1,\ldots,L_n}}{\partial t} = \sum_{\ell_1,\ldots,\ell_n\geq 1} A_{L_1,\ell_1}\cdots A_{L_n,\ell_n}(\ell_1\cdots\ell_n)T^{(g)}_{\ell_1,\ldots,\ell_n,0},$$

and using the invertibility of the unitriangular matrix $(A_{L,\ell})_{L,\ell\geq 1}$, we obtain the following identity

$$T^{(g)}_{\ell_1,\ldots,\ell_n,0} = \frac{\partial T^{(g)}_{\ell_1,\ldots,\ell_n}}{\partial t} + \sum_{i=1}^{n} \sum_{m_i<\ell_i} m_i\, T^{(0)}_{\ell_i,0|m_i}\, T^{(g)}_{\ell_1,\ldots,m_i,\ldots,\ell_n}.$$

This identity has the following nice combinatorial interpretation: consider a map counted by $T^{(g)}_{\ell_1,\ldots,\ell_n,0}$, and let v be the boundary-vertex corresponding to the last 0. We claim that there exists at most one $i = 1,\ldots,n$ such that there exists a closed path of length $m_i < \ell_i$ separating v and the ith boundary from the others. Indeed, if two such i_1 and i_2 existed, we could construct a closed path contradicting the tightness of the i_1th or the i_2th boundary (see Figure 4). If no such i exists, v can be changed into a regular vertex without affecting the tightness of the other boundaries, and we get a map counted by $\frac{\partial}{\partial t} T^{(g)}_{\ell_1,\ldots,\ell_n}$. Otherwise, if exactly one such i exists, we can consider the "extremal" shortest closed path separating v and the ith boundary from the others. By cutting along this path, we get on the one hand a map counted by $T^{(0)}_{\ell_i,0|m_i}$ (it is a pair of pants with one tight boundary of length ℓ_i, one boundary-vertex and one strictly tight boundary of length m_i) and on the other hand a map counted by $T^{(g)}_{\ell_1,\ldots,m_i,\ldots,\ell_n}$. There are m_i ways to glue these maps back together.

Note that the above reasoning also works when ℓ_i is allowed to take the value 0, in which case the sum over m_i vanishes. Putting things together, we arrive at the following proposition.

Proposition 3.3. *Let g, n be nonnegative integers. We have, for $\chi = 2-2g-n \leq 0$ and $\ell_1,\ldots,\ell_n$ nonnegative integers,*

$$\begin{aligned} T^{(g)}_{\ell_1,\ldots,\ell_n,0} &= \frac{\partial T^{(g)}_{\ell_1,\ldots,\ell_n}}{\partial t} + \sum_{i=1}^{n} \sum_{m_i<\ell_i} m_i\, T^{(0)}_{\ell_i,0|m_i}\, T^{(g)}_{\ell_1,\ldots,m_i,\ldots,\ell_n} \\ &= \frac{\partial T^{(g)}_{\ell_1,\ldots,\ell_n}}{\partial t} + \sum_{i=1}^{n} \Bigg(\sum_{\substack{m_i=1 \\ m_i=\ell_i-1 \bmod 2}}^{\ell_i-1} m_i\, R^{\frac{\ell_i-m_i-1}{2}} \frac{\partial S}{\partial t} T^{(g)}_{\ell_1,\ldots,m_i,\ldots,\ell_n} \\ &\qquad + \sum_{\substack{m_i=1 \\ m_i=\ell_i-2 \bmod 2}}^{\ell_i-2} m_i\, R^{\frac{\ell_i-m_i}{2}-1} \frac{\partial R}{\partial t} T^{(g)}_{\ell_1,\ldots,m_i,\ldots,\ell_n} \Bigg). \end{aligned} \tag{32}$$

Note that, for $n = 0$, the relation reads

$$T^{(g)}_{0} = \partial T^{(g)}_{\varnothing}/\partial t = \partial F^{(g)}/\partial t,$$

where $T^{(g)}_{\varnothing} = F^{(g)}$ is the generating function of maps of genus g without boundaries.

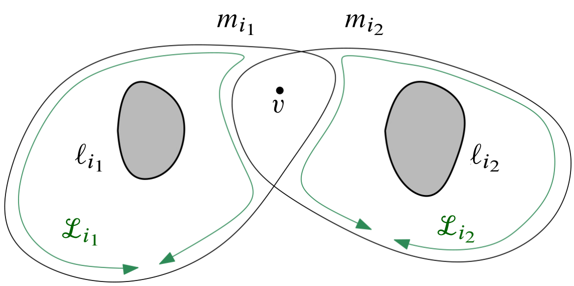

Figure 4. A sketch of why $m_{i_1} < \ell_{i_1}$ and $m_{i_2} < \ell_{i_2}$ are mutually exclusive: indeed, it would imply $\mathcal{L}_{i_1} < \ell_{i_1}$ or $\mathcal{L}_{i_2} < \ell_{i_2}$, in contradiction with the fact that the boundary-faces of length ℓ_1 and ℓ_2 are both tight.

3.2. Adding an extra boundary-face

Marking an extra boundary-face in a trumpet. Take L, ℓ, M positive integers. The quantity $M\frac{\partial A_{L,\ell}}{\partial t_M}$ is the generating function for trumpets with rooted boundary of length L, mouthpiece of length ℓ and with an extra marked rooted boundary-face of length M.

Proposition 3.4. *We have*

$$M\frac{\partial A_{L,\ell}}{\partial t_M} = \sum_{m>\ell}\sum_{\ell_0\geq 1} A_{L,m}A_{M,\ell_0} m\,\ell_0\, T^{(0)}_{m,\ell_0|\ell}. \tag{33}$$

The proof is in all points similar to that of Proposition 3.2 and amounts to performing a trumpet decomposition of the map with three boundaries of length L, M, and ℓ.

Marking extra boundary-faces via boundary insertion operators. We now wish to find a recurrence relation relating $T^{(g)}_{m_1,\ldots,m_n,m_{n+1}}$ to $T^{(g)}_{m_1,\ldots,m_n}$ for m_{n+1} a positive integer.

For m a positive integer, let us define the *tight boundary insertion operator* D_m by

$$D_m := \sum_{M=1}^{m} (A^{-1})_{m,M}\frac{M}{m}\frac{\partial}{\partial t_M}, \tag{34}$$

where A^{-1} is the inverse of the unitriangular matrix $A = (A_{M,m})_{M,m\geq 1}$. It satisfies

$$M\frac{\partial}{\partial t_M} = \sum_{m=1}^{M} m\,A_{M,m}D_m. \tag{35}$$

We start with the identity

$$L_{n+1}\frac{\partial F^{(g)}_{L_1,\ldots,L_n}}{\partial t_{L_{n+1}}} = F^{(g)}_{L_1,\ldots,L_n,L_{n+1}}$$
$$= \sum_{\ell_1,\ldots,\ell_{n+1}\geq 1} A_{L_1,\ell_1}\cdots A_{L_{n+1},\ell_{n+1}}(\ell_1\cdots\ell_{n+1})T^{(g)}_{\ell_1,\ldots,\ell_{n+1}}. \quad (36)$$

Alternatively, we may write

$$L_{n+1}\frac{\partial F^{(g)}_{L_1,\ldots,L_n}}{\partial t_{L_{n+1}}}$$
$$= \sum_{\ell_1,\ldots,\ell_n\geq 1} A_{L_1,\ell_1}\cdots A_{L_n,\ell_n}(\ell_1\cdots\ell_n)L_{n+1}\frac{\partial T^{(g)}_{\ell_1,\ldots,\ell_n}}{\partial t_{L_{n+1}}}$$
$$+\sum_{i=1}^{n}\sum_{\ell_1,\ldots,\ell_n\geq 1} A_{L_1,\ell_1}\cdots\left(L_{n+1}\frac{\partial A_{L_i,\ell_i}}{\partial t_{L_{n+1}}}\right)\cdots A_{L_n,\ell_n}(\ell_1\cdots\ell_n)T^{(g)}_{\ell_1,\ldots,\ell_n}. \quad (37)$$

From (33), the general term in the final sum over i may be written as

$$\sum_{\ell_1,\ldots,\ell_n\geq 1} A_{L_1,\ell_1}\cdots\Big(\sum_{m_i>\ell_i}\sum_{\ell_{n+1}\geq 1} A_{L_i,m_i}A_{L_{n+1},\ell_{n+1}}m_i\,\ell_{n+1}T^{(0)}_{m_i,\ell_{n+1}|\ell_i}\Big)\cdots A_{L_n,\ell_n}$$
$$\times(\ell_1\cdots\ell_n)T^{(g)}_{\ell_1,\ldots,\ell_n}$$
$$= \sum_{\ell_1,\ldots,\ell_{n+1}\geq 1} A_{L_1,\ell_1}\cdots A_{L_{n+1},\ell_{n+1}}(\ell_1\cdots\ell_{n+1})$$
$$\times\Big(\sum_{m_i<\ell_i} m_i\,T^{(0)}_{\ell_i,\ell_{n+1}|m_i}\,T^{(g)}_{\ell_1,\ldots,m_i,\ldots,\ell_n}\Big),$$

where, to go from the left-hand side to the right-hand side, we exchanged the dummy summation variables ℓ_i and m_i, as well as the order of summation. As for the first term on the right-hand side of (37), using (35), it may be rewritten as

$$\sum_{\ell_1,\ldots,\ell_{n+1}\geq 1} A_{L_1,\ell_1}\cdots A_{L_{n+1},\ell_{n+1}}(\ell_1\cdots\ell_{n+1})D_{\ell_{n+1}}T^{(g)}_{\ell_1,\ldots,\ell_n}.$$

Gathering all the terms in (37), comparing with (36), and noting that we may incorporate Proposition 3.3 upon defining

$$D_0 := \frac{\partial}{\partial t},$$

we arrive at the following identification.

Proposition 3.5. *Let g, n be nonnegative integers. We have, for $\chi = 2-2g-n \leq 0$ and $\ell_1, \dots, \ell_{n+1}$ nonnegative integers,*

$$T^{(g)}_{\ell_1,\dots,\ell_n,\ell_{n+1}} = D_{\ell_{n+1}} T^{(g)}_{\ell_1,\dots,\ell_n} + \sum_{i=1}^{n} \sum_{m_i<\ell_i} m_i \, T^{(0)}_{\ell_i,\ell_{n+1}|m_i} T^{(g)}_{\ell_1,\dots,m_i,\dots,\ell_n}. \tag{38}$$

If ℓ_{n+1} is even, this reads

$$\begin{aligned} T^{(g)}_{\ell_1,\dots,\ell_n,\ell_{n+1}} &= D_{\ell_{n+1}} T^{(g)}_{\ell_1,\dots,\ell_n} \\ &\quad + \sum_{i=1}^{n} \Bigg(\sum_{\substack{m_i=1 \\ m_i=\ell_i-1 \bmod 2}}^{\ell_i-1} m_i \, R^{\frac{\ell_i+\ell_{n+1}-m_i-1}{2}} \frac{\partial S}{\partial t} T^{(g)}_{\ell_1,\dots,m_i,\dots,\ell_n} \\ &\qquad + \sum_{\substack{m_i=1 \\ m_i=\ell_i-2 \bmod 2}}^{\ell_i-2} m_i \, R^{\frac{\ell_i+\ell_{n+1}-m_i}{2}-1} \frac{\partial R}{\partial t} T^{(g)}_{\ell_1,\dots,m_i,\dots,\ell_n} \Bigg), \end{aligned}$$

while if ℓ_{n+1} is odd, it reads instead

$$\begin{aligned} T^{(g)}_{\ell_1,\dots,\ell_n,\ell_{n+1}} &= D_{\ell_{n+1}} T^{(g)}_{\ell_1,\dots,\ell_n} \\ &\quad + \sum_{i=1}^{n} \Bigg(\sum_{\substack{m_i=1 \\ m_i=\ell_i-1 \bmod 2}}^{\ell_i-1} m_i \, R^{\frac{\ell_i+\ell_{n+1}-m_i}{2}-1} \frac{\partial R}{\partial t} T^{(g)}_{\ell_1,\dots,m_i,\dots,\ell_n} \\ &\qquad + \sum_{\substack{m_i=1 \\ m_i=\ell_i-2 \bmod 2}}^{\ell_i-2} m_i \, R^{\frac{\ell_i+\ell_{n+1}-m_i-1}{2}} \frac{\partial S}{\partial t} T^{(g)}_{\ell_1,\dots,m_i,\dots,\ell_n} \Bigg). \end{aligned}$$

For $n=0$, these formulas read $T^{(g)}_{\ell_1} = D_{\ell_1} T^{(g)}_{\varnothing} = D_{\ell_1} F^{(g)}$.

Again, the identity (38) has a clear combinatorial interpretation. Consider a map counted by $T^{(g)}_{\ell_1,\dots,\ell_{n+1}}$. Then, there exists at most one $i \in \{1,\dots,n\}$ such that there exists a closed path of length $m_i < \ell_i$ separating the ith and the $(n+1)$th boundaries from the others, for a reason similar to that illustrated in Figure 4. If exactly one such i exists, we can consider the extremal shortest separating closed path and cut along it. We get, on the one hand, a map counted by $T^{(0)}_{\ell_i,\ell_{n+1}|m_i}$, i.e., a pair of pants with two tight boundaries of lengths ℓ_i and ℓ_{n+1}, and one strictly tight boundary of length m_i, and, on the other hand, a map counted by $T^{(g)}_{\ell_1,\dots,m_i,\dots,\ell_n}$. There are m_i ways to glue back these maps together. If no such i exists, the $(n+1)$th boundary can be changed into an inner face without affecting the tightness of the other boundaries. We deduce that such maps are counted by $D_{\ell_{n+1}} T^{(g)}_{\ell_1,\dots,\ell_n}$, which explains, in retrospect, the name of the operator D_m.

Remark 3.6. For $g = 0$ and $n = 2$, notice that Proposition 3.5 relates the generating function of tight cylinders $T^{(0)}_{\ell_1,\ell_2}$, given by (14) for $\ell_1, \ell_2 > 0$, and $T^{(0)}_{0,0} = \ln(R/t)$, with that of tight pairs of pants given by Theorem A.1.

3.3. Consistency with the expression for the planar bipartite case

We take $n \geq 3$ and check that the explicit expression (20) for $T^{(0)}_{2\ell_1,\dots,2\ell_n}|_{\mathrm{bip}}$ in the planar bipartite case is consistent with the above relations (32) and (38) describing the addition of an extra boundary-vertex and an extra boundary-face, respectively.

Adding an extra boundary-vertex. The relation (32) of Proposition 3.3 reduces in the planar bipartite case to

$$
\begin{aligned}
T^{(0)}_{2\ell_1,\dots,2\ell_n,0}\big|_{\mathrm{bip}} &= \frac{\partial}{\partial t} T^{(0)}_{2\ell_1,\dots,2\ell_n}\big|_{\mathrm{bip}} \\
&\quad + \sum_{i=1}^{n} \sum_{0<m_i<\ell_i} (2m_i) R^{\ell_i - m_i - 1} R' T^{(0)}_{2\ell_1,\dots,2m_i,\dots,2\ell_n}\big|_{\mathrm{bip}}
\end{aligned} \tag{39}
$$

with R given by (3). Let us see how this relation is indeed satisfied by the expression (20) for $T^{(0)}_{2\ell_1,\dots,2\ell_n}|_{\mathrm{bip}}$.

By (20), we have indeed

$$
\begin{aligned}
T^{(0)}_{2\ell_1,\dots,2\ell_n,0}\big|_{\mathrm{bip}} &= R^{\ell_1+\dots+\ell_n} \sum_{k=0}^{n-2} k!\, p_k(\ell_1,\dots,\ell_n) b_{n-1,k+1} \\
&\quad + \frac{(-1)^{n+1}(n-2)!}{t^{n-1}} \delta_{\ell_1+\dots+\ell_n,0},
\end{aligned}
$$

while

$$
\begin{aligned}
\frac{\partial}{\partial t} T^{(0)}_{2\ell_1,\dots,2\ell_n}\big|_{\mathrm{bip}} &= R^{\ell_1+\dots+\ell_n-1} \sum_{k=0}^{n-3} k!\, p_k(\ell_1,\dots,\ell_n) \\
&\quad \times \Big((\ell_1+\dots+\ell_n-k-1) R' b_{n-2,k+1} + \sum_{j\geq 1} R^{(j+1)} b^{[j]}_{n-2,k+1} \Big) \\
&\quad + \frac{(-1)^{n+1}(n-2)!}{t^{n-1}} \delta_{\ell_1+\dots+\ell_n,0},
\end{aligned}
$$

where $R^{(i)}$ denotes the ith derivative of R with respect to t while $b^{[j]}_{n,k}$ is the derivative of Bell polynomial $B_{n,k}$ with respect to its jth variable, evaluated at $(R'/R, R''/R, \dots)$. By the recursion relation

$$
b_{n-1,k+1} = \frac{R'}{R} b_{n-2,k} + \sum_{j\geq 1} \frac{R^{(j+1)}}{R} b^{[j]}_{n-2,k+1},
$$

which is easily derived from the combinatorial interpretation of the Bell polynomials in terms of partitions (with the convention that $b_{n,k} = 0$ if $k > n$), we deduce that

$$\begin{aligned} &T^{(0)}_{2\ell_1,\ldots,2\ell_n,0}\big|_{\mathrm{bip}} - \frac{\partial}{\partial t} T^{(0)}_{2\ell_1,\ldots,2\ell_n}\big|_{\mathrm{bip}} \\ &\quad = R^{\ell_1+\cdots+\ell_n-1} R' \\ &\qquad \times \sum_{k=0}^{n-2} k!\, p_k(\ell_1,\ldots,\ell_n)\big(b_{n-2,k} - (\ell_1+\cdots+\ell_n-k-1) b_{n-2,k+1}\big). \end{aligned} \tag{40}$$

On the other hand, we have

$$\begin{aligned} &\sum_{i=1}^{n} \sum_{0<m_i<\ell_i} (2m_i) R^{\ell_i-m_i-1} R' T^{(0)}_{2\ell_1,\ldots,2m_i,\ldots,2\ell_n}\big|_{\mathrm{bip}} \\ &\quad = R^{\ell_1+\cdots+\ell_n-1} R' \\ &\qquad \times \sum_{k=0}^{n-2} \sum_{i=1}^{n} \sum_{0<m_i<\ell_i} (2m_i) k!\, p_k(\ell_1,\ldots,m_i,\ldots,\ell_n) b_{n-2,k+1}. \end{aligned} \tag{41}$$

Comparing (40) and (41), we see that relation (39) is then a direct consequence of the following recursion relation for $p_k(\ell_1,\ldots,\ell_n)$.

Proposition 3.7 (String equation). *For nonnegative integers $\ell_1,\ldots,\ell_n$, we have*

$$\begin{aligned} (k+1) p_{k+1}(\ell_1,\ldots,\ell_n) &= (\ell_1+\cdots+\ell_n-k-1) p_k(\ell_1,\ldots,\ell_n) \\ &\quad + \sum_{i=1}^{n} \sum_{0<m_i<\ell_i} (2m_i) p_k(\ell_1,\ldots,m_i,\ldots,\ell_n). \end{aligned} \tag{42}$$

This latter recursion is proved in [6, Proposition 2.2].

Adding an extra boundary-face of even length. We now wish to check the consistency of the expression (20) for $T^{(0)}_{2\ell_1,\ldots,2\ell_n}|_{\mathrm{bip}}$ with the recursion relation (38) of Proposition 3.5 which, in the planar bipartite case, reduces to

$$\begin{aligned} &T^{(0)}_{2\ell_1,\ldots,2\ell_n,2\ell_{n+1}}\big|_{\mathrm{bip}} \\ &\quad = D_{2\ell_{n+1}} T^{(0)}_{2\ell_1,\ldots,2\ell_n}\big|_{\mathrm{bip}} \\ &\qquad + \sum_{i=1}^{n} \sum_{0<m_i<\ell_i} (2m_i) R^{\ell_i-m_i+\ell_{n+1}-1} R' T^{(0)}_{2\ell_1,\ldots,2m_i,\ldots,2\ell_n}\big|_{\mathrm{bip}}. \end{aligned} \tag{43}$$

As a prerequisite to prove (43) from (20), we note that D_{2m} satisfies the Leibniz product rule, namely, $D_{2m}(XY) = (D_{2m}X)Y + X(D_{2m}Y)$, and moreover that the action of D_{2m} on derivatives of R is given by

$$D_{2m}R^{(j)} = T^{(0)}_{2m,2,\underbrace{0,\dots,0}_{j+1\text{ times}}}\big|_{\text{bip}} = R^{m+1}\sum_{k=0}^{j} k!q_k(m)b_{j+1,k+1}. \tag{44}$$

Here, we may justify the first equality by noting that, in the bipartite case, $R^{(j)}$ can be seen as the generating function of maps with one unrooted boundary-face of length 2 and $j+1$ boundary-vertices. Since D_{2m} creates an extra tight boundary-face of length $2m$, and since the other boundaries (which have lengths 2 and 0 only) are automatically tight, we indeed obtain the maps counted by $T^{(0)}_{2m,2,\underbrace{0,\dots,0}_{j+1\text{ times}}}\big|_{\text{bip}}$.

As for the second equality, it is a particular instance of (20), simplified via the identity $p_k(m,1,0,\dots,0) = p_k(m,1) = p_k(m) + p_{k-1}(m) = q_k(m)$.

By (20), we now have

$$\begin{aligned}
&T^{(0)}_{2\ell_1,\dots,2\ell_n,2\ell_{n+1}}\big|_{\text{bip}}\\
&\quad = R^{\ell_1+\cdots+\ell_n+\ell_{n+1}}\sum_{k=0}^{n-2} k!p_k(\ell_1,\dots,\ell_n,\ell_{n+1})b_{n-1,k+1}\\
&\quad = R^{\ell_1+\cdots+\ell_n+\ell_{n+1}}\sum_{k'\geq 0}\sum_{k''\geq 0}(k'+k'')!p_{k''}(\ell_1,\dots,\ell_n)q_{k'}(\ell_{n+1})b_{n-1,k'+k''+1}
\end{aligned} \tag{45}$$

(throughout this computation, we will leave the summation ranges as implicit as possible, since they are naturally enforced by the vanishing of the summands). On the other hand, using (44), we have

$$\begin{aligned}
D_{2\ell_{n+1}}T^{(0)}_{2\ell_1,\dots,2\ell_n}\big|_{\text{bip}} &= R^{\ell_1+\cdots+\ell_n-1}\sum_{k\geq 0} k!p_k(\ell_1,\dots,\ell_n)\\
&\qquad\times\Big((\ell_1+\cdots+\ell_n-k-1)(D_{2\ell_{n+1}}R)b_{n-2,k+1}\\
&\qquad\qquad+\sum_{j\geq 1}(D_{2\ell_{n+1}}R^{(j)})b^{[j]}_{n-2,k+1}\Big)\\
&= R^{\ell_1+\cdots+\ell_n+\ell_{n+1}}\sum_{k\geq 0} k!p_k(\ell_1,\dots,\ell_n)\\
&\qquad\times\Big((\ell_1+\cdots+\ell_n-k-1)\frac{R'}{R}b_{n-2,k+1}\\
&\qquad\qquad+\sum_{j\geq 1}\sum_{k'\geq 0} k'!q_{k'}(\ell_{n+1})b_{j+1,k'+1}b^{[j]}_{n-2,k+1}\Big).
\end{aligned} \tag{46}$$

To compare the last two displays (45) and (46), we may write

$$\binom{k'+k''}{k'} b_{n-1,k'+k''+1} = \sum_{j\geq 0} \binom{n-2}{j} b_{j+1,k'+1} b_{n-2-j,k''}$$
$$= \frac{R'}{R}\delta_{k',0} b_{n-2,k''} + \sum_{j\geq 1} b_{j+1,k'+1} b^{[j]}_{n-2,k''+1}$$

upon interpreting the lhs as a sum over partitions of $\{1,\dots,n-1\}$ into $k'+k''+1$ blocks, $k'+1$ of which are marked, the element $n-1$ being always in a marked block: the first equality is obtained by summing over the number j of elements other than $n-1$ that are in a marked block, and the second equality is obtained by noting that $\binom{n-2}{j} b_{n-2-j,k''} = b^{[j]}_{n-2,k''+1}$ for $j\geq 1$. We deduce that the sum appearing at the end of (46), with k replaced by k'', can be rewritten as

$$\sum_{j\geq 1}\sum_{k'\geq 0} k'! q_{k'}(\ell_{n+1}) b_{j+1,k'+1} b^{[j]}_{n-2,k''+1}$$
$$= \sum_{k'\geq 0} \binom{k'+k''}{k'} k'! q_{k'}(\ell_{n+1}) b_{n-1,k'+k''+1} - \frac{R'}{R} b_{n-2,k''},$$

and hence,

$$T^{(0)}_{2\ell_1,\dots,2\ell_n,2\ell_{n+1}}\big|_{\text{bip}} - D_{2\ell_{n+1}} T^{(0)}_{2\ell_1,\dots,2\ell_n}\big|_{\text{bip}}$$
$$= R^{\ell_1+\cdots+\ell_n+\ell_{n+1}-1} R'$$
$$\times \sum_{k\geq 0} k! p_k(\ell_1,\dots,\ell_n)\big(b_{n-2,k} - (\ell_1+\cdots+\ell_n-k-1) b_{n-2,k+1}\big).$$

By comparing with (40), we see that

$$T^{(0)}_{2\ell_1,\dots,2\ell_n,2\ell_{n+1}}\big|_{\text{bip}} - D_{2\ell_{n+1}} T^{(0)}_{2\ell_1,\dots,2\ell_n}\big|_{\text{bip}}$$
$$= R^{\ell_{n+1}} \left(T^{(0)}_{2\ell_1,\dots,2\ell_n,0}\big|_{\text{bip}} - \frac{\partial}{\partial t} T^{(0)}_{2\ell_1,\dots,2\ell_n}\big|_{\text{bip}} \right),$$

and we immediately deduce that (43) is again a direct consequence of the string equation (42).

Remark 3.8. We may similarly check the explicit expression (29) for $T^{(0)}_{2\ell_1,\dots,2\ell_n}|_{\text{bip}}$ in the planar quasi-bipartite case is also consistent with the relations (32) and (38) for the addition of an extra boundary. This property is now a direct consequence of the string equation relating $\tilde{p}_{k+1}(\ell_1,\ell_2;\ell_3,\dots,\ell_n)$ to $\tilde{p}_k(\ell_1,\ell_2;\ell_3,\dots,\ell_n)$, see [6, Proposition 2.10].

4. The quasi-polynomiality phenomenon

We have seen in Section 2.4 that the generating function of planar bipartite maps with (at least three) tight boundaries is an even polynomial in the boundary lengths times a power of the basic generating function R. As mentioned in the introduction, this is a particular case of a more general quasi-polynomiality phenomenon.

Precisely, given a positive integer n, let us call *parity-dependent quasi-polynomial in n variables* a function f of n integer variables such that there exists a (necessarily unique) family of polynomials $(f_I)_I$ indexed by subsets I of $\{1,\dots,n\}$ such that $f(m_1,\dots,m_n) = f_I(m_1,\dots,m_n)$ whenever the m_i with $i \in I$ are odd integers, and the m_i with $i \notin I$ are even integers. By *coefficients* of f we mean the collection of coefficients of the polynomials f_I, which may belong to an arbitrary ring; by *total degree* of f we mean the maximum of the total degrees of the f_I. It is convenient to allow the value $n = 0$, in which case it is understood that f is just a constant element of the ring at hand. A parity-dependent quasi-polynomial is said *symmetric* if it is invariant under any permutation of its variables.

In order to state the main result of this section, we also introduce the quantity

$$\chi(\mathcal{M}_{g,n}) := \begin{cases} (-1)^{n-1}(n-3)! & \text{if } g = 0, \\ (-1)^{n-1}\frac{(2g+n-3)!}{(2g-2)!}\zeta(1-2g) & \text{if } g > 0, \end{cases}$$

which, for $2-2g-n < 0$, is the Euler characteristic of the moduli space $\mathcal{M}_{g,n}$ [16].

Theorem 4.1. *Let $g \geq 0$, $n \geq 1$ be integers such that $2-2g-n < 0$. Then, there exists a symmetric parity-dependent quasi-polynomial $\mathfrak{T}_n^{(g)}$ in n variables, of total degree $3g-3+n$, such that, for any nonnegative integers $\ell_1,\dots,\ell_n$, we have*

$$T^{(g)}_{\ell_1,\dots,\ell_n} = R^{\frac{\ell_1+\cdots+\ell_n}{2}}\mathfrak{T}_n^{(g)}(\ell_1^2,\dots,\ell_n^2) - \chi(\mathcal{M}_{g,n})t^{2-2g-n}\delta_{\ell_1+\cdots+\ell_n,0}. \tag{47}$$

The coefficients of $\mathfrak{T}_n^{(g)}$ are rational functions with rational coefficients of the quantities $\frac{R^{(k)}}{R}$ and $\frac{S^{(k)}}{\sqrt{R}}$, $k = 1,\dots,3g-2+n$, where $R^{(k)}$, $S^{(k)}$ denote the kth derivatives of the series R, S with respect to the vertex weight t.

Recall from Remark 2.10 that there is no quasi-polynomiality phenomenon for $(g,n) = (0,2)$. The quasi-polynomiality phenomenon was first observed in the language of topological recursion by Norbury and Scott [18], who showed that the coefficients $\hat{\tau}^{(g)}_{\ell_1,\dots,\ell_n}$ appearing in the series expansion (7) of $\omega_n^{(g)}(z_1,\dots,z_n)$ are quasi-polynomials in $\ell_1,\dots,\ell_n$ which are odd in each variable. We review their result in Section 4.1: combining it with the trumpet decomposition yields (47) in the case where n and all ℓ_i are non-zero.

However, the quasi-polynomiality of $\hat{\tau}^{(g)}_{\ell_1,\dots,\ell_n}$ does not imply a priori of $\tau^{(g)}_{\ell_1,\dots,\ell_n}$, as there may be pathologies when *some* variables are set to 0. To circumvent this problem, and show that $\tau^{(g)}_{\ell_1,\dots,\ell_n}$ indeed coincides with an even quasi-polynomial except when *all* variables are set to 0, we will make use in Section 4.2 of the tight boundary insertion relations established in Section 3. The proof is by induction on the number of boundaries n, and we still use results from topological recursion to initialize the induction.

Remark 4.2. In fact, Theorem 4.1 also makes sense for $n = 0$, and states that for any $g \geq 2$, the generating function $T^{(g)}_{\varnothing} = F^{(g)}$ of maps of genus g without boundaries is equal to

$$F^{(g)} = \mathfrak{F}^{(g)} - \chi(\mathcal{M}_{g,0})t^{2-2g},$$

with $\mathfrak{F}^{(g)}$ a rational function with rational coefficients of $\frac{R^{(k)}}{R}$, $\frac{S^{(k)}}{\sqrt{R}}$, and $k = 1, \dots, 3g-2$. As we will see below, this result is obtained by combining the property known from topological recursion that $F^{(g)}$ can be expressed in terms of the so-called moments, with an expression of the moments in terms of the derivatives of R and S. In the case $g = 1$, we no longer have rational functions but rather logarithms, see Theorem C.1.

Remark 4.3. When setting the weights $t_1, t_2, \dots$ for inner faces to zero, i.e., when we consider *tight maps* for which every face is a tight boundary, then $\mathfrak{T}^{(g)}_n(\ell_1^2, \dots, \ell_n^2)$ reduces to t^{2-2g-n} times Norbury's lattice count (quasi-)polynomial $N_{g,n}(\ell_1, \dots, \ell_n)$ [17]. It is shown in this reference that $N_{g,n}(0, \dots, 0) = \chi(\mathcal{M}_{g,n})$. This is consistent, in view of (47), with the fact that $T^{(g)}_{0,\dots,0}$ vanishes when we set the weights for inner faces to zero (there are no tight maps having only boundary-vertices and no faces).

4.1. Maps without boundary-vertices

In this section, we establish the quasi-polynomiality property (47) in the case where n and all ℓ_i are non-zero. We will do so by combining Theorem 2.5 with known results coming from the theory of topological recursion. One reason behind the fact that this theory is formulated in terms of $\omega^{(g)}_n$ instead of $W^{(g)}_n$ is that the former are rational functions of their n variables with a simple pole structure; see, for instance, [12, Section 3.3.1]. It was recognized by Norbury and Scott that this simple pole structure amounts to the following proposition.

Proposition 4.4. *For $g \geq 0$, $n \geq 1$, and $2 - 2g - n < 0$, the coefficients $\hat{\tau}^{(g)}_{\ell_1,\dots,\ell_n}$ appearing in the expansion (7) of $\omega^{(g)}_n$ are of the form*

$$\hat{\tau}^{(g)}_{\ell_1,\dots,\ell_n} = \ell_1 \cdots \ell_n \mathfrak{T}^{(g)}_n(\ell_1^2, \dots, \ell_n^2) \tag{48}$$

with $\mathfrak{T}^{(g)}_n$ a parity-dependent quasi-polynomial in n variables of total degree $3g - 3 + n$.

The above proposition is a reformulation of [18, Theorem 1][6], in which the quantity $\mathfrak{T}_n^{(g)}(\ell_1^2,\dots,\ell_n^2)$ is denoted by $N_n^g(\ell_1,\dots,\ell_n)$. Note that we do not characterize the structure of the coefficients of $\mathfrak{T}_n^{(g)}$ at this stage. The relation (48) implies that (47) holds when n and all ℓ_i are non-zero, by (8) and (11).

For completeness, let us sketch a proof of Proposition 4.4 based on the results available in [12]. We start with a technical lemma.

Lemma 4.5. *Let $P(n)$ be a polynomial of degree at most $2d+1$, which is equivalent to the series $F(z) := \sum_{n\geq 0} \frac{P(n)}{z^{n+1}}$ being a rational function of the form $N(z)/(z-1)^{2d+2}$, with N a polynomial of degree at most $2d+1$. Then, the following properties are equivalent:*

(i) *P is odd,*

(ii) *F is such that $F(z) = z^{-2}F(z^{-1})$,*

(iii) *N satisfies $N(z) = z^{2d}N(z^{-1})$, i.e., it is a self-reciprocal (palindromic) polynomial.*

Proof. By [21, Proposition 4.2.3][7], the rational function $F(z)$ admits around 0 the expansion $F(z) = -\sum_{n\geq 1} P(-n)z^{n-1}$, hence, we have

$$z^{-2}F(z^{-1}) = -\sum_{n\geq 1} \frac{P(-n)}{z^{n+1}}.$$

This implies the equivalence between (i), (ii): indeed, if P is odd, then $z^{-2}F(z^{-1}) = \sum_{n\geq 1} z^{-n-1}P(n) = F(z)$, since $P(0)=0$, and conversely, if $F(z) = z^{-2}F(z^{-1})$ then by comparing the expansions, we obtain $P(0)=0$ and $-P(-n) = P(n)$ for every $n\geq 1$, implying that P is odd. The equivalence between (ii) and (iii) is straightforward. ∎

---

[6] Precisely, this theorem considers the expansion of $\omega_n^{(g)}(z_1,\dots,z_n)$ around $z_i = 0$. Here, our $\hat{\tau}^{(g)}_{\ell_1,\dots,\ell_n}$ are rather defined in terms of the expansion around $z_i = \infty$. But the two expansions can be related through antisymmetry properties of $\omega_n^{(g)}$, see (51).

[7] Short rederivation: we consider the vector space $\mathcal{L}$ of formal Laurent series of the form $\sum_{n\in\mathbb{Z}} a_n z^n$. We cannot multiply two elements of this space in general, but we can multiply an element of $\mathcal{L}$ by a polynomial in z, and this operation is linear. Let us consider in particular the series $\sum_{n\in\mathbb{Z}} \frac{P(n)}{z^{n+1}} \in \mathcal{L}$, and the polynomial $(z-1)^{2d+2}$. Their product vanishes since P is a polynomial of degree at most $2d+1$; hence, it is annihilated by the $(2d+2)$th power of the forward difference operator. We deduce that $(z-1)^{2d+2}\sum_{n\geq 0}\frac{P(n)}{z^{n+1}} = -(z-1)^{2d+2}\sum_{n\leq -1}\frac{P(n)}{z^{n+1}}$. The left-hand side may be interpreted as a product in the ring $\mathbb{C}((z^{-1}))$ and is thus equal to $N(z)\in\mathbb{C}[z]$. Therefore, the right-hand side is also equal to $N(z)$, and we obtain an identity valid in $\mathbb{C}[[z]]$. Multiplying by the inverse of $(z-1)^{2d+2}$ in $\mathbb{C}[[z]]$ (which differs from that in $\mathbb{C}((z^{-1}))$!) and changing n in $-n$, we obtain the wanted series identity.

Proof of Proposition 4.4. According to [12, Lemma 3.3.3, p. 89, Theorem 3.5.1, p. 121, and the last equation on p. 140][8], $\omega_n^{(g)}$ has a partial fraction decomposition of the form

$$\omega_n^{(g)}(z_1,\dots,z_n) = \sum_{\varepsilon\in\{+1,-1\}^n} \sum_{\substack{d\in\mathbb{N}^n\\ d_1+\cdots+d_n\le 6g-6+2n}} \frac{c_{\varepsilon,d}}{(z_1-\varepsilon_1)^{d_1+2}\cdots(z_n-\varepsilon_n)^{d_n+2}}, \tag{49}$$

where the $c_{\varepsilon,d}$ are some coefficients depending on the spectral curve. By the negative binomial formula, we have

$$\frac{1}{(z-\varepsilon)^{d+2}} = \sum_{\ell\ge 1}\binom{\ell}{d+1}\frac{\varepsilon^{\ell-d-1}}{z^{\ell+1}}.$$

Viewing $\binom{\ell}{d+1}$ as a polynomial in ℓ, this implies that, in the expansion (7) of $\omega_n^{(g)}$, the $\hat{\tau}$ coefficients are of the form

$$\hat{\tau}^{(g)}_{\ell_1,\dots,\ell_n} = \sum_{\varepsilon\in\{+1,-1\}^n} Q^{(g,\varepsilon)}(\ell_1,\dots,\ell_n)\varepsilon_1^{\ell_1}\cdots\varepsilon_n^{\ell_n}, \tag{50}$$

where $Q^{(g,\varepsilon)}$ is a polynomial of degree at most $6g-6+3n$. It remains to check that these polynomials are indeed odd in each variable, so that we may write, for some polynomials $P^{(g,\varepsilon)}$ of degree at most $3g-3+n$,

$$Q^{(g,\varepsilon)}(\ell_1,\dots,\ell_n) = \ell_1\cdots\ell_n\, P^{(g,\varepsilon)}(\ell_1^2,\dots,\ell_n^2).$$

Thanks to the symmetry in $\ell_1,\dots,\ell_n$, it suffices to check that $Q^{(g,\varepsilon)}$ is odd in ℓ_1. For this, we use the antisymmetry property of [12, Lemma 3.3.2, p. 88], which entails that

$$\omega_n^{(g)}(z_1,z_2,\dots,z_n) = z_1^{-2}\omega_n^{(g)}(z_1^{-1},z_2,\dots,z_n). \tag{51}$$

In the expansion (49), each term of the sum over ε must be separately invariant under the antisymmetry mapping $f(z_1)\mapsto z_1^{-2}f(z_1^{-1})$, since it corresponds to the contribution to $\omega_n^{(g)}$ having poles at $z_1=\varepsilon_1,\dots,z_n=\varepsilon_n$. Such term corresponds to the term $Q^{(g,\varepsilon)}(\ell_1,\dots,\ell_n)\varepsilon_1^{\ell_1}\cdots\varepsilon_2^{\ell_2}$ in (50) and we claim that $Q^{(g,\varepsilon)}$ is odd in ℓ_1. By the transformation $z_i\mapsto\varepsilon_i z_i$, it suffices to consider the contribution of $\varepsilon=(1,\dots,1)$ to $\omega_n^{(g)}$, namely,

$$F(z_1) := \sum_{\substack{d\in\mathbb{N}^n\\ d_1+\cdots+d_n\le 6g-6+2n}} \frac{c_{(1,\dots,1),d}}{(z_1-1)^{d_1+2}\cdots(z_n-1)^{d_n+2}},$$

[8] There seems to be a discrepancy between Theorem 3.5.1 and the equation on page 140: $\sum d_i$ is at most $6g-6+2n$ in the former, and $6g-4+2n$ in the latter. We assume that the former is correct since it holds for pairs of pants ($g=0, n=3$) and lids ($g=1, n=1$).

satisfying

$$F(z_1) = z_1^{-2} F(z_1^{-1}).$$

By Lemma 4.5, it expands as $\sum_{\ell_1 \geq 1} \frac{P(\ell_1)}{z_1^{\ell_1+1}}$ with P an odd polynomial in ℓ_1, which implicitly depends on the variables $z_2, \ldots, z_n$. Further expanding in these extra variables, the coefficient of $\frac{1}{z_2^{\ell_2+1} \cdots z_n^{\ell_n+1}}$, which is nothing but $Q^{(g,(1,\ldots,1))}(\ell_1, \ldots, \ell_n)$, must be odd in ℓ_1 as wanted, hence the oddness property is established. As $m_i = \ell_i^2$ has the same parity as ℓ_i, we conclude that (48) holds with

$$\mathfrak{T}_n^{(g)}(m_1, \ldots, m_n) = \sum_{\varepsilon \in \{+1,-1\}^n} P^{(g,\varepsilon)}(m_1, \ldots, m_2) \varepsilon_1^{m_1} \cdots \varepsilon_n^{m_n},$$

which is indeed a parity-dependent quasi-polynomial in n variables of total degree $3g - 3 + n$. ■

4.2. Maps with boundary-vertices

The purpose of this section is to complete the proof of Theorem 4.1, by including the case where some of the boundary lengths $\ell_1, \ldots, \ell_n$ vanish, and by showing the structure properties of the coefficients of the quasi-polynomial $\mathfrak{T}_n^{(g)}$. We proceed by induction on n, by using the recursion relations of Section 3 for the induction step. As a warmup, let us first consider the easier case of maps with even face degrees. Recall the relation (13) between the T's and the τ's, and the notation $|_{\mathrm{bip}}$ from Section 2.4 which indicates that we set the weights $t_1, t_3, \ldots$ for inner faces of odd degrees to zero.

Proposition 4.6. *Let g, n be nonnegative integers such that $2 - 2g - n < 0$. There exist $Q_n^{(g)}(x_1, \ldots, x_{3g-2+n}; u_1, \ldots, u_n) \in \mathbb{Q}(x_1, \ldots, x_{3g-2+n})[u_1, \ldots, u_n]$ and a constant $c(g,n) \in \mathbb{Q}$ such that, for any integers $\ell_1, \ldots, \ell_n$, we have*

$$\tau^{(g)}_{2\ell_1,\ldots,2\ell_n}\big|_{\mathrm{bip}} = Q_n^{(g)}\left(\frac{R'}{R}, \ldots, \frac{R^{(3g-2+n)}}{R}; \ell_1^2, \ldots, \ell_n^2\right) - c(g,n) t^{2-2g-n} \delta_{\ell_1+\cdots+\ell_n,0}. \tag{52}$$

Proof. As mentioned above, we proceed by induction on n for a fixed value of g. The induction step relies on the recursion relations of Propositions 3.3 and 3.5 which, for $t_1 = t_3 = \cdots = 0$ (so that $S = 0$), reduce to

$$\begin{aligned} T^{(g)}_{2\ell_1,\ldots,2\ell_n,2\ell_{n+1}}\big|_{\mathrm{bip}} &= D_{2\ell_{n+1}} T^{(g)}_{2\ell_1,\ldots,2\ell_n}\big|_{\mathrm{bip}} \\ &\quad + \sum_{i=1}^{n} \sum_{0<m_i<\ell_i} (2m_i) R^{\ell_i - m_i + \ell_{n+1} - 1} R' T^{(g)}_{2\ell_1,\ldots,2m_i,\ldots,2\ell_n}\big|_{\mathrm{bip}}. \end{aligned} \tag{53}$$

Here, D_m is the tight boundary insertion operator defined at (34), reading for $S = 0$

$$D_{2\ell} = \begin{cases} \sum_{i=1}^{\ell} (-1)^{\ell+i} \binom{\ell+i-1}{\ell-i} R^{\ell-i} \frac{\partial}{\partial t_{2i}} & \text{if } \ell > 0, \\ \frac{\partial}{\partial t} & \text{if } \ell = 0. \end{cases}$$

In the second line of (53), it is understood that, if $\ell_i = 0$, then the corresponding sum over $0 < m_i < \ell_i$ vanishes. In terms of the τ's, this relation is rewritten as

$$\begin{aligned} &\tau^{(g)}_{2\ell_1,\dots,2\ell_n,2\ell_{n+1}}\Big|_{\mathrm{bip}} \\ &\quad = \frac{D_{2\ell_{n+1}} \tau^{(g)}_{2\ell_1,\dots,2\ell_n}\Big|_{\mathrm{bip}}}{R^{\ell_{n+1}}} \\ &\qquad + \frac{R'}{R} \sum_{i=1}^{n} \Big(\ell_i \tau^{(g)}_{2\ell_1,\dots,2\ell_n}\Big|_{\mathrm{bip}} + \sum_{0<m_i<\ell_i} (2m_i) \tau^{(g)}_{2\ell_1,\dots,2m_i,\dots,2\ell_n}\Big|_{\mathrm{bip}} \Big). \end{aligned} \tag{54}$$

Here, we have used the Leibniz product rule for the differential operator $D_{2\ell_{n+1}}$, and the relation $D_{2\ell} R = R^{\ell} R'$ which is the case $j = 0$ of (44), recalling the definition (21) of $b_{n,k}$.

Let us now sketch the rough idea of the induction, ignoring the pathological term of (52) for now.

- From the recursion relation (54), it is relatively straightforward that if $\tau^{(g)}_{2\ell_1,\dots,2\ell_n}$ is polynomial in $\ell_1^2, \dots, \ell_n^2$, then $\tau^{(g)}_{2\ell_1,\dots,2\ell_n,2\ell_{n+1}}$ is also polynomial in $\ell_1^2, \dots, \ell_n^2$: it is clear for the first term since the linear operator $D_{2\ell_{n+1}}$ does not act on these variables, while the second term involves a summation which turns out to preserve the wanted polynomiality property.
- Regarding the new variable ℓ_{n+1}, we use the assumption that the coefficients of $\tau^{(g)}_{2\ell_1,\dots,2\ell_n}$ are rational in the derivatives of R: acting on this quantity with the differential operator $D_{2\ell_{n+1}}$ and using the chain rule, it follows from (44) that $R^{-\ell_{n+1}} D_{2\ell_{n+1}} \tau^{(g)}_{2\ell_1,\dots,2\ell_n}$ is polynomial in ℓ_{n+1}^2, also with rational coefficients in the derivatives of R.

Let us now make this argument precise. Assuming that the expression (52) holds at rank n, we plug it into the right-hand side of (54), and analyze the two terms separately. The second term is easier to deal with, since it does not depend on ℓ_{n+1} nor receives a contribution from the pathological term of (52). It equals

$$\frac{R'}{R} \sum_{i=1}^{n} \Big(\ell_i Q_n^{(g)}(\dots, \ell_i^2, \dots) + \sum_{0<m_i<\ell_i} (2m_i) Q_n^{(g)}(\dots, m_i^2, \dots) \Big),$$

where, for readability, we only display the u_i variable of $Q_n^{(g)}$. Now, it is an exercise (solved in Lemma B.1 below) that for every univariate polynomial $\mathfrak{q}$, the quantity

$\ell\mathfrak{q}(\ell^2) + \sum_{0<m<\ell}(2m)\mathfrak{q}(m^2)$ is again a polynomial in ℓ^2. Therefore, the second term of (54) is of the wanted form. To deal with the first term, we apply the chain rule to obtain

$$\frac{D_{2\ell_{n+1}}\tau^{(g)}_{2\ell_1,\ldots,2\ell_n}\big|_{\mathrm{bip}}}{R^{\ell_{n+1}}} = \sum_{j=1}^{3g-2+n} \frac{D_{2\ell_{n+1}}(\frac{R^{(j)}}{R})}{R^{\ell_{n+1}}} \frac{\partial Q_n^{(g)}(\ldots)}{\partial x_j} - c(g,n)\frac{D_{2\ell_{n+1}}t^{2-2g+n}}{R^{\ell_{n+1}}}\delta_{\ell_1+\cdots+\ell_n,0}.$$

Here, the first term is clearly polynomial in $\ell_1^2,\ldots,\ell_n^2$. Moreover, it is also a polynomial in ℓ_{n+1}^2 by (44), which also entails that the coefficients are rational in $R^{(k)}/R$ for $1 \le k \le 3g-2+(n+1)$. Finally, the second term in the last display is zero unless $\ell_1,\ldots,\ell_{n+1}$ all vanish, in which case it is equal to

$$-c(g,n)\frac{\partial}{\partial t}t^{2-2g-n} = -c(g,n+1)t^{2-2g-(n+1)},$$

where $c(g,n+1) = (2-2g+n)c(g,n)$. This completes the proof of the induction step: if the statement of Proposition 4.6 is true for (g,n), then it is true for $(g,n+1)$.

It remains to initialize the induction. For $g=0$, we initialize the induction at $n=3$ by using the explicit formula of [5, Theorem 1.1], which is the bipartite version of Theorem A.1 in Appendix A. For $g \ge 1$, we rely on the consequences of the topological recursion that are described in [8]. Assume first that $g \ge 2$, in which case we initialize the induction at $n=0$: in view of Remark 4.2, we only have to check that the generating function $F^{(g)}|_{\mathrm{bip}} = \tau^{(g)}_{\varnothing}|_{\mathrm{bip}}$ of essentially bipartite maps of genus g without boundary is a rational function of $R^{(k)}/R$, $1 \le k \le 3g-2$. By [8, (31) and (32)], there exists a polynomial $\widetilde{P}_g$ with rational coefficients in $3g-3$ variables such that for $t=1$ we have

$$F^{(g)}\big|_{\mathrm{bip}}[t=1] = \widetilde{P}_g(0,\ldots,0) - \left(\frac{1}{\bar{M}_0}\right)^{2g-2}\widetilde{P}_g\left(\frac{\bar{M}_1}{\bar{M}_0},\ldots,\frac{\bar{M}_{3g-3}}{\bar{M}_0}\right),$$

where $\bar{M}_p$ are the so-called *moments*, defined at [8, (30)]. Using Euler's relation, we can restore the dependence on t by the formula

$$F^{(g)}\big|_{\mathrm{bip}} = t^{2-2g}F^{(g)}\big|_{\mathrm{bip}}[t=1](t_2, tt_4, t^2t_6,\ldots).$$

Consequently, letting $\bar{M}_p^{(0)} = t\bar{M}_p(t_2,tt_4,t^2t_6,\ldots)$, in accordance with [8, (43)], we have

$$F^{(g)}\big|_{\mathrm{bip}} = t^{2-2g}\widetilde{P}_g(0,\ldots,0) - \left(\frac{1}{\bar{M}_0^{(0)}}\right)^{2g-2}\widetilde{P}_g\left(\frac{\bar{M}_1^{(0)}}{\bar{M}_0^{(0)}},\ldots,\frac{\bar{M}_{3g-3}^{(0)}}{\bar{M}_0^{(0)}}\right). \quad (55)$$

Finally, [8, Lemma 9] shows that $\bar{M}_0^{(0)} = R/R'$, while we have

$$\frac{\bar{M}_p^{(0)}}{\bar{M}_0^{(0)}} = \frac{\mathsf{T}_p(\frac{R'}{R}, \dots, \frac{R^{(p+1)}}{R})}{(R'/R)^{2p}}$$

for some homogeneous polynomial T_p of degree $2p$. This entails that $F^{(g)}|_{\text{bip}}$ is of the wanted form (52): the second term in (55) is of the form $Q_0^{(g)}(\frac{R'}{R}, \dots, \frac{R^{(3g-2)}}{R})$ with $Q_0^{(g)} \in \mathbb{Q}(x_1, \dots, x_{3g-2})$ (there are no variables u_i for $n = 0$), and the first term is of the form $t^{2-2g} c(g, 0)$ with $c(g, 0) = -\widetilde{P}_g(0, \dots, 0)$.

In the case $g = 1$, we rely on [8, (31)], stating, in our notation, that $F^{(1)} = \ln(tR'/R)/12$. This entails, again by the boundary insertion formula and by (44), that

$$\tau_{2\ell}^{(1)}\Big|_{\text{bip}} = \frac{D_{2\ell} F^{(1)}}{R^\ell} = \frac{D_{2\ell}(\frac{R'}{R})}{12\frac{R'}{R}R^\ell} + \frac{\delta_{\ell,0}}{12t} = \frac{1}{12}\left((\ell^2 - 1)\frac{R'}{R} + \frac{R''}{R'}\right) + \frac{\delta_{\ell,0}}{12t}, \quad (56)$$

which is of the wanted form (52), with $c(1, 1) = -1/12$. This concludes the proof of Proposition 4.6. ∎

Let us record in passing the following interesting consequence of (56), which calls for a bijective interpretation.

Proposition 4.7. *The generating function of essentially bipartite maps of genus* 1 *with one tight boundary of length* 2ℓ *is given by*

$$T_{2\ell}^{(1)}\Big|_{\text{bip}} = \frac{R^\ell}{12}\left((\ell^2 - 1)\frac{R'}{R} + \frac{R''}{R'}\right) + \frac{\delta_{\ell,0}}{12t}. \quad (57)$$

We now state the non bipartite analogue of Proposition 4.6.

Proposition 4.8. *Let g, n be nonnegative integers such that $2 - 2g - n < 0$. There exist a parity-dependent quasi-polynomial $\mathfrak{T}_n^{(g)}$ in n variables, whose coefficients are rational functions with rational coefficients of the quantities $\frac{R^{(k)}}{R}$ and $\frac{S^{(k)}}{\sqrt{R}}$ for $k = 1, \dots, 3g - 2 + n$, and a constant $c(g, n) \in \mathbb{Q}$ such that, for any integers $\ell_1, \dots, \ell_n$, we have*

$$\tau_{\ell_1,\dots,\ell_n}^{(g)} = \mathfrak{T}_n^{(g)}(\ell_1^2, \dots, \ell_n^2) - c(g, n) t^{2-2g-n} \delta_{\ell_1+\dots+\ell_n,0}.$$

The proof, by induction, is in the same spirit as that of Proposition 4.6, but involves extra difficulties. For this reason, we offload it to the appendices: Appendix B is devoted to the induction step and Appendix C to the initialization.

End of the proof of Theorem 4.1. In view of Proposition 4.8 and of the relation (13) between the T's and the τ's, the only tasks left to show are that $c(g, n) = \chi(\mathcal{M}_{g,n})$ and that $\mathfrak{T}_n^{(g)}$ has total degree $3g - 3 + n$ for every $n \geq 1$. As discussed in Remark 4.3, the first assertion is a consequence of [17, Theorem 2], which states that the lattice count polynomial $N_{g,n}$ evaluates to $\chi(\mathcal{M}_{g,n})$ when all its variables are set to zero. The second assertion follows from Proposition 4.4. We believe that it should also be possible to keep track of the degree of $\mathfrak{T}_n^{(g)}$ in our inductive proof, at the price of a refinement of Proposition 4.8, by looking at the joint homogeneity properties of $\mathfrak{T}_n^{(g)}$ in the variables ℓ_i^2 as well as the derivatives of R and S, similarly to what is stated in Theorem C.2 for $n = 0$. ∎

5. Conclusion and discussion

In this paper, we have seen that, within the theory of topological recursion applied to the enumeration of maps, the expansion of the fundamental quantity $\omega_n^{(g)}(z_1, \dots, z_n)$ has a simple relation with the generating functions $T^{(g)}_{\ell_1,\dots,\ell_n}$ of maps of genus g with n tight boundaries of lengths $\ell_1, \dots, \ell_n$. This coincidence is combinatorially explained by the trumpet decomposition. We have found recursion relations, also of combinatorial nature, expressing $T^{(g)}_{\ell_1,\dots,\ell_{n+1}}$ in terms of $T^{(g)}_{\ell_1,\dots,\ell_n}$. Finally, we have seen that, when at least one ℓ_i is non-zero, $T^{(g)}_{\ell_1,\dots,\ell_n}$ is equal to a parity-dependent quasi-polynomial in the variables $\ell_1^2, \dots, \ell_n^2$ times a simple power of R.

At this stage, a tantalizing question is whether one can derive *bijectively* an expression for $T^{(g)}_{\ell_1,\dots,\ell_n}$. So far, only the situation $(g, n) = (0, 3)$ has been treated in [5] for the essentially bipartite case, and in Appendix A for the general case. In particular, the remarkably simple formulas (22) for $(g, n) = (0, 4)$ and (57) for $(g, n) = (1, 1)$, in the case of maps with faces of even degree only, are still waiting for a bijective proof. One may also attempt to find a combinatorial interpretation of (20), given that its constituents $p_k(\ell_1, \dots, \ell_n)$ and $b_{n-2,k+1}$ have themselves interpretations in terms of tight maps [6] and set partitions, respectively.

A. Bijective enumeration of tight pairs of pants

The purpose of this appendix is to establish the following theorem.

Theorem A.1. *Let a, b, and c be nonnegative integers or half-integers. Then, the generating function $T^{(0)}_{2a,2b,2c}$ of planar maps with three labeled distinct tight boundaries of lengths $2a$, $2b$, $2c$, counted with a weight t per vertex different from a boundary-vertex and, for all $k \geq 1$, a weight t_k per inner face of degree k, is given by*

$$T^{(0)}_{2a,2b,2c} = \begin{cases} R^{-1}\frac{\partial R}{\partial t} - t^{-1} & \text{if } a = b = c = 0, \\ R^{a+b+c-1}\frac{\partial R}{\partial t} & \text{if } a+b+c \in \mathbb{Z}_{>0}, \\ R^{a+b+c-\frac{1}{2}}\frac{\partial S}{\partial t} & \text{if } a+b+c \in \mathbb{Z}_{>0} - \frac{1}{2}, \end{cases} \tag{58}$$

where R and S are the formal power series in $t, t_1, t_2, t_3, \dots$ defined in Section 1.2.

This theorem is an extension of [5, Theorem 1.1] to non bipartite maps. Note that, in this reference (which the reader is invited to consult for any definition not found in the current paper), we used the notation $T_{a,b,c}$ instead of the present notation $T^{(0)}_{2a,2b,2c}$. The (essentially) bipartite case is recovered upon taking $t_1 = t_3 = t_5 = \cdots = 0$ which, as noted above, implies that R satisfies (3) and that S vanishes. In that case, $T^{(0)}_{2a,2b,2c}$ is non-zero if and only if $a + b + c$ is an integer, as wanted.

To prove Theorem A.1, we will use the well-known fact that any map M can be transformed into a bipartite map by doing a *subdivision* of every edge, that is adding a new vertex at the midpoint of every edge, which is thus split into two. We denote by $\phi(M)$ the resulting *subdivided map*. The vertices of $\phi(M)$ consist of *regular vertices* (those originally in M) and *midpoint vertices* (those added by the subdivision, which have degree 2). The faces of $\phi(M)$ are those of M, only their degree is doubled.

We then proceed by "conjugating" the bijective decomposition of [5] by ϕ. Indeed, let M be a planar map with three tight boundaries of lengths $2a, 2b, 2c$: clearly, $\phi(M)$ is a bipartite planar map with three tight boundaries. We may then apply one of two possible decompositions described in [*ibid.*, Sections 5.6 and 5.7], namely,

- the decomposition of type I if $a \leq b + c$ and $b \leq c + a$ and $c \leq a + b$,
- the decomposition of type II if $a \geq b + c$ or $b \geq c + a$ or $c \geq a + b$.

In both cases, we obtain a quintuple consisting of five pieces: two so-called bigeodesic triangles, and three bigeodesic diangles. The vertices of these pieces are those of $\phi(M)$, and we see that a midpoint vertex of $\phi(M)$ remains of degree 2 in each piece to which it belongs: indeed, the decomposition is made by cutting along bigeodesics, and distinct bigeodesics may only merge at vertices of degree at least 3, which are necessarily regular. As a consequence of this remark, each piece admits a (unique) preimage by ϕ, and so, our decomposition amounts to cutting M itself into five pieces, denoted by $(D_1, D_2, D_3, T_1, T_2)$ in Figure 5.

While this approach seems very natural, the main difficulty is to characterize precisely which sorts of pieces we may obtain in the decomposition. Recall from [*ibid.*, Sections 2.3 and 2.4] that bigeodesic diangles and triangles are planar maps with one boundary-face having several (four and six, respectively) distinguished incident corners, half of them being the so-called *attachment points*. Let us here call *tips* the other distinguished corners: by the above remark, we see that the tips of the pieces obtained in the decomposition of $\phi(M)$ are always incident to regular vertices. However, the attachment points may be incident to either regular or midpoint vertices. We are thus led to introduce the following definitions.

In a map M, a *midpoint corner* is a corner of the subdivided map $\phi(M)$ which is incident to a midpoint vertex. A *regular corner* is a corner in the usual sense, it corresponds to a corner of $\phi(M)$ incident to a regular vertex. We define the distance $d_M(c, c')$ between two (regular or midpoint) corners c, c' of M as *half* the graph distance between their incident vertices in $\phi(M)$. The distance between two regular corners, or between two midpoint corners, is thus an integer, while the distance between a regular corner and a midpoint corner is a half-integer. The notion of geodesic boundary interval of [*ibid.*, Section 2.1] extends naturally to midpoint corners.

A *generalized bigeodesic diangle* D is defined as in [*ibid.*, Section 2.3] except that we no longer require D to be bipartite, and that we allow that zero, one or two among the attachment points c_{12}, c_{21} are midpoint corners. We still require that the tips c_1, c_2 are regular corners of D. See the left of Figure 6 for an illustration. The *exceedance* of D is defined as

$$e(D) := d_D(c_{12}, c_1) - d_D(c_{21}, c_1) = d_D(c_{21}, c_2) - d_D(c_{12}, c_2).$$

This quantity is half the exceedance of $\phi(D)$, which is clearly a bigeodesic diangle in the original sense. When $e(D)$ is an integer, the two attachment points are of the same kind (regular or midpoint), and they are of different kinds when $e(D)$ is a half-integer.

A *generalized bigeodesic triangle* T is defined as in [*ibid.*, Section 2.4] except that we no longer require T to be bipartite, and that we allow the attachment points c_{12}, c_{23}, c_{31} to be midpoint corners. We still require that the tips c_1, c_2, c_3 are regular corners of T. Recall that the definition of a bigeodesic triangle implies that $d_T(c_{12}, c_1) = d_T(c_{31}, c_1)$ (and circular permutations thereof), and hence all the attachment points must be of the same kind. The triangle is said *odd* if c_{12}, c_{23}, c_{31} are midpoint corners, and *even* if they are regular corners. See the right of Figure 6 for an illustration. Clearly, $\phi(T)$ is a bigeodesic triangle in the usual sense.

With these new definitions at hand, by applying [*ibid.*, Theorem 3.1] to $\phi(M)$, we arrive at the following proposition.

Proposition A.2. *Let M be a planar map with three tight boundaries of lengths $2a$, $2b$, $2c$.*

If $a \le b + c$, $b \le c + a$, and $c \le a + b$, then by the type I decomposition of [5, Section 5.6], *M is in one-to-one correspondence with a quintuple $(D_1, D_2, D_3, T_1, T_2)$, where D_1, D_2, D_3 are generalized bigeodesic triangles of respective exceedances $e(D_1) = a + c - b$, $e(D_2) = a + b - c$, $e(D_3) = b + c - a$, and T_1, T_2 are generalized bigeodesic triangles, with the constraints that the attachment points which are identified together in Figure 5 must be of the same (regular or midpoint) kind, and that not all elements of this quintuple are reduced to the vertex-map.*

If $b \ge a + c$, say, then the type II decomposition of [ibid., Section 5.7] produces a quintuple $(D_1, D_2, D_3, T_1, T_2)$ with the same properties, except that the exceedances are now $e(D_1) = b - a - c$, $e(D_2) = 2a$, $e(D_3) = 2c$.

For both type I and type II, we have the relation

$$e(D_1) + e(D_2) + e(D_3) = a + b + c. \tag{59}$$

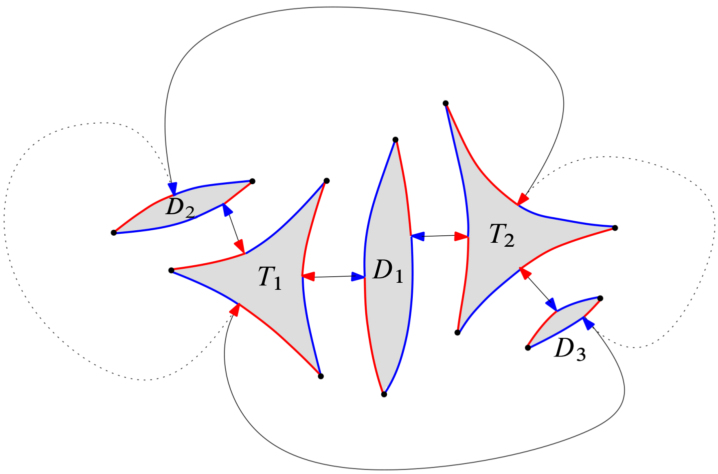

Figure 5. A summary representation of the bijection of [5] between maps with three tight boundaries and quintuples $(D_1, D_2, D_3, T_1, T_2)$ made of three bigeodesic diangles D_1, D_2, D_3, and two bigeodesic triangles T_1, T_2. The solid lines indicate the identification between attachment points for type I. For type II, the two outer identification lines have to be replaced by the dotted lines.

The above proposition encompasses a number of different cases. When $a+b+c$ is an integer, all the exceedances are integers, and therefore, all twelve attachment points are of the same kind. If they are all regular (resp., midpoint) corners, then T_1 and T_2 are both even (resp., odd) triangles. When $a+b+c$ is a half-integer, $e(D_1)$ is a half-integer and hence T_1 and T_2 have different "parities". The other exceedances $e(D_2)$ and $e(D_3)$ are half-integers for type I, and integers for type II.

Let us now discuss the enumerative consequences of Proposition A.2. To this end, it is necessary to introduce the generating functions of the various objects that may arise in the situations described above. Generally speaking, we assign a weight t_k per inner face of degree k (before subdivision), and a weight t per vertex (of the map, i.e., per regular vertex of the subdivided map). Following the conventions of [*ibid.*], vertices that lie on strictly geodesic boundaries (displayed in red in the figures), and that are not incident to attachment points, do not receive the weight t. We denote by Y (resp., $\widetilde{Y}$) the generating function of generalized bigeodesic triangles that are even (resp., odd). We then treat the case of diangles in the following proposition.

Proposition A.3. *Let X denote the generating function of generalized bigeodesic diangles with exceedance* 0*, whose attachment points are both regular corners. Then, the following holds.*

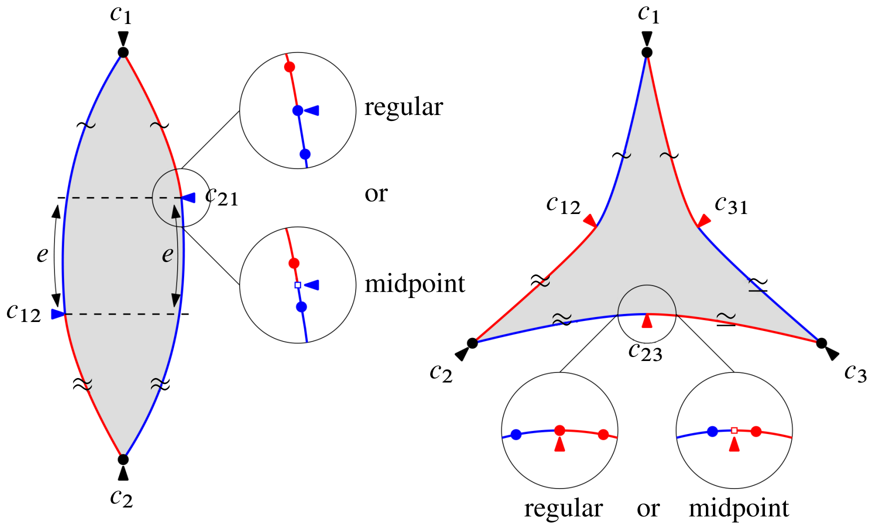

Figure 6. Schematic representation of a generalized bigeodesic diangle (left) and triangle (right). The tips c_1, c_2, c_3 are regular corners, incident to vertices (shown as solid disks). The attachment points $c_{12}, \ldots$ are either regular corners or midpoint corners, incident to midpoint vertices of the subdivided map (shown as small squares). We use the same red/blue coloring convention as [5, Figures 2.2 and 2.5].

The generating function of generalized bigeodesic diangles with exceedance $e \in \mathbb{Z}_{\geq 0}$, whose attachment points are both regular corners, is equal to $R^e X$.

The generating function of generalized bigeodesic diangles with exceedance $e \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$, where the first attachment point is a regular corner and the second a midpoint corner, is equal to $R^{e+\frac{1}{2}} X/t$.

The generating function of generalized bigeodesic diangles with exceedance $e \in \mathbb{Z}_{\geq 0}$, whose attachment points are both midpoint corners, is equal to $R^{e+1} X/t^2$.

Proof. The first statement is obtained by a straightforward adaptation of the proof of [*ibid.*, Proposition 2.2], noting that the arguments do not require that the maps be bipartite, and that R can still be interpreted as the generating function for elementary slices.

The second statement follows from the first one, by observing that there is a bijection between the generalized bigeodesic diangles whose exceedance is $e \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$ and whose second attachment point is a midpoint corner, and diangles whose exceedance is $e + \frac{1}{2}$ and whose both attachment points are regular corners. Indeed, looking at Figure 6, we simply have to move the second attachment point c_{21} to the nearest (red) regular vertex in the direction of c_1, and we need to divide by t to unweigh this regular vertex.

Finally, the third statement follows from a similar observation: we now move the two attachment points c_{12} and c_{21} to their nearest regular vertices in the direction of c_2 and c_1, respectively, resulting in a diangle of exceedance $e+1$, and the weight $1/t^2$ is needed to unweigh these regular vertices. ■

Proof of Theorem A.1. We apply Proposition A.2, discussing separately the different cases that appear there. Let us first consider the case $a+b+c \in \mathbb{Z}_{>0}$: we may write

$$\begin{aligned} T^{(0)}_{2a,2b,2c} &= \frac{1}{t^6}(R^{e(D_1)}X)(R^{e(D_2)}X)(R^{e(D_3)}X)Y^2 \\ &\quad + \frac{R^{e(D_1)+1}X}{t^2}\,\frac{R^{e(D_2)+1}X}{t^2}\,\frac{R^{e(D_3)+1}X}{t^2}\widetilde{Y}^2 \\ &= R^{a+b+c}\frac{X^3(Y^2+R^3\widetilde{Y}^2)}{t^6}. \end{aligned} \tag{60}$$

Here, the term in Y^2 (resp., $\widetilde{Y}^2$) corresponds to the situation where both T_1 and T_2 are even (resp., odd) triangles, and the division by t^6 in the first term arises from the identification between the vertices incident to the attachment points. Note that the second equality holds both for type I and type II, thanks to (59).

The formula (60) remains valid in the case $a=b=c=0$, except that we have to remove a spurious term t^{-1} corresponding to the contribution of the quintuple whose all elements are reduced to the vertex map.

Let us now consider the case $a+b+c \in \mathbb{Z}_{\geq 0}+\frac{1}{2}$: then the discussion differs slightly between type I and type II. For type I, we may write

$$\begin{aligned} T^{(0)}_{2a,2b,2c} &= \frac{1}{t^3}\,\frac{R^{e(D_1)+\frac{1}{2}}X}{t}\,\frac{R^{e(D_2)+\frac{1}{2}}X}{t}\,\frac{R^{e(D_3)+\frac{1}{2}}X}{t}(2Y\widetilde{Y}) \\ &= R^{a+b+c-\frac{1}{2}}\frac{2R^2X^3Y\widetilde{Y}}{t^6}. \end{aligned}$$

Here, the factor $2Y\widetilde{Y}$ enumerates the pairs (T_1, T_2) of different parities. Note that in this situation, all diangles have exactly one regular attachment point, and the factor $1/t^3$ arises from the identification between the vertices at these attachment points. For type II, assuming without loss of generality that $b \geq a+c$ as in Proposition A.2, we may write

$$\begin{aligned} T^{(0)}_{2a,2b,2c} &= \frac{1}{t^3}\,\frac{R^{e(D_1)+\frac{1}{2}}X}{t}\Bigg((R^{e(D_2)}X)\frac{R^{e(D_3)+1}X}{t^2}Y\widetilde{Y} \\ &\qquad\qquad + \frac{R^{e(D_2)+1}X}{t^2}(R^{e(D_3)}X)\widetilde{Y}Y\Bigg) \\ &= R^{a+b+c-\frac{1}{2}}\frac{2R^2X^3Y\widetilde{Y}}{t^6}. \end{aligned}$$

Here, the first (resp., second) term corresponds to the case where T_1 is even (resp., odd), so that D_2 has two (resp., zero) regular attachment points, D_3 has zero (resp., two) regular attachment points. In all cases, D_1 has exactly one regular attachment point.

Gathering all these cases together, we arrive at the unified formula

$$T^{(0)}_{2a,2b,2c} + t^{-1}\delta_{a+b+c=0} = \begin{cases} R^{a+b+c-1}\frac{RX^3(Y^2+R^3\widetilde{Y}^2)}{t^6} & \text{if } a+b+c \in \mathbb{Z}_{\geq 0}, \\ R^{a+b+c-\frac{1}{2}}\frac{2R^2X^3Y\widetilde{Y}}{t^6} & \text{if } a+b+c \in \mathbb{Z}_{\geq 0}+\frac{1}{2}. \end{cases}$$

We conclude by noting that

$$\frac{\partial R}{\partial t} = T^{(0)}_{1,1,0} = \frac{RX^3(Y^2+R^3\widetilde{Y}^2)}{t^6} \quad \text{and} \quad \frac{\partial S}{\partial t} = T^{(0)}_{1,0,0} = \frac{2R^2X^3Y\widetilde{Y}}{t^6},$$

since R (resp., S) counts planar maps with two distinguished boundaries both of length 1 (resp., one of length 1 and the other of length 0): differentiating with respect to t has the effect of adding a third boundary of length 0, and boundaries of lengths 0 or 1 are tautologically tight. ■

We now state a variant of Theorem A.1 allowing to count tight pairs of pants with some boundaries strictly tight, under certain assumptions.

Proposition A.4. *Let a, b, and c be nonnegative integers or half-integers, and let $T^{(0)}_{2a,2b|2c}$ (resp., $T^{(0)}_{2a|2b,2c}$) denote the generating function of planar maps with three labeled distinct tight boundaries of lengths $2a$, $2b$, $2c$, the third (resp., the second and the third) being strictly tight, where we attach a weight t per vertex different from a boundary-vertex and not incident to the third boundary (resp., to the second nor to the third boundary) and, for all $k \geq 1$, a weight t_k per inner face of degree k.*

Then, for $a+b>c$, we have

$$T^{(0)}_{2a,2b|2c} = \begin{cases} R^{a+b-c-1}\frac{\partial R}{\partial t} & \text{if } a+b+c \in \mathbb{Z}_{>0}, \\ R^{a+b-c-\frac{1}{2}}\frac{\partial S}{\partial t} & \text{if } a+b+c \in \mathbb{Z}_{>0}-\frac{1}{2}, \end{cases}$$

while, for $a>b+c$, we have

$$T^{(0)}_{2a|2b,2c} = \begin{cases} R^{a-b-c-1}\frac{\partial R}{\partial t} & \text{if } a+b+c \in \mathbb{Z}_{>0}, \\ R^{a-b-c-\frac{1}{2}}\frac{\partial S}{\partial t} & \text{if } a+b+c \in \mathbb{Z}_{>0}-\frac{1}{2}. \end{cases}$$

Observe that, in these two expressions, the assumptions $a+b>c$ and $a>b+c$ are, respectively, crucial to avoid having terms involving negative powers of t. It would be interesting to obtain formulas for $T^{(0)}_{2a,2b|2c}$ and $T^{(0)}_{2a|2b,2c}$ without these assumptions, but this is beyond the scope of this paper. Note that, whenever $a>b+c$, the boundaries of lengths b, c cannot touch each other.

Proof of Proposition A.4. In view of Theorem A.1, it suffices to establish that

$$T^{(0)}_{2a,2b,2c} = \begin{cases} R^{2c} T^{(0)}_{2a,2b|2c} & \text{if } a+b>c, \\ R^{2b+2c} T^{(0)}_{2a|2b,2c} & \text{if } a>b+c. \end{cases}$$

Let us first assume $a+b>c$ and establish the formula for $T^{(0)}_{2a,2b|2c}$. If $c=0$ then there is nothing to prove, as a boundary-vertex is always considered strictly tight. Assuming $c>0$, let M be a planar map with three labeled distinct tight boundaries of lengths $2a$, $2b$, $2c$. The third boundary is a face, which we denote by F_3. We then cut M along the *innermost* minimal closed path C homotopic to F_3. This path can be precisely defined as the *maximal* element of the lattice $\mathcal{C}^{(3)}_{\min}$, discussed in [5, Section 6.1]. We will justify in Lemma A.5 below that C is simple, hence by cutting we obtain two pieces: a planar map with three tight boundaries of lengths $2a$, $2b$, $2c$, the third being strictly tight, and a planar map with two tight boundaries both of length $2c$. Moreover, the boundary resulting from this cutting operation can be canonically rooted: indeed, since planar maps with three boundaries have a trivial automorphism group, we may canonically select a distinguished corner incident to C. This decomposition is clearly bijective. Turning to generating functions, we decide to assign the weight t for vertices on C to the second map. It is known [4, Section 9.3] that planar maps with two tight boundaries both of length $2c$, one of which is rooted, have generating function R^{2c}. We conclude that

$$T^{(0)}_{2a,2b,2c} = R^{2c} T^{(0)}_{2a,2b|2c}$$

as wanted.

For $a>b+c$, we may perform a similar decomposition with the boundary of length $2b$, giving the relation

$$T^{(0)}_{2a,2b|2c} = R^{2b} T^{(0)}_{2a|2b,2c}. \qquad \blacksquare$$

Let us now justify that the closed path C used in the above proof is simple.

Lemma A.5. *Let M be a planar map with three labeled distinct tight boundaries of lengths ℓ_1, ℓ_2, ℓ_3. If $\ell_1+\ell_2>\ell_3>0$ then every closed path on M of length ℓ_3 which is homotopic to the third boundary is simple.*

Proof. Let C be a closed path on M of length $\ell>0$. We may decompose C into a (finite) sequence of *simple* closed paths $\widehat{C}=(C_1,C_2,\ldots,C_n)$ with *positive lengths*, for instance, by doing a "loop-erasure" (in this construction, a closed path reduced to a single edge followed once in both ways is regarded as simple).

To be explicit, choose an arbitrary vertex v_0 on C and an orientation, and let us denote by $v_0, e_1, v_1, e_2, v_2, \ldots, e_\ell, v_\ell = v_0$ the successive vertices and edges of M

visited by C. We then apply the following iterative procedure. Initialize γ as the path reduced to v_0, and $\widehat{C}$ as an empty sequence. Then, for every step i from 1 to ℓ, append the edge e_i and v_i to γ:

- if γ is simple, proceed to the next step,
- if γ is not simple, then it necessarily consists of a simple path γ' from v_0 to v_i (possibly of length zero if $v_i = v_0$) and of a simple closed path C' with positive length: set $\gamma = \gamma'$, append C' to the sequence $\widehat{C}$, and proceed to the next step.

At the last step, the path γ is reduced to $v_0 = v_\ell$, and $\widehat{C}$ is the sequence we are looking for. There is an interesting combinatorial structure of *heap* underlying this decomposition, see, e.g., [23, Proposition 6.3], but for our purposes, we need only remark that the sum of the lengths of the C_i's is ℓ.

Now, let us assume that C is homotopic to the third boundary. We claim that at least one of the following assertions is true:

(i) there exists i such that C_i is homotopic to the third boundary,

(ii) there exists i, j such that C_i and C_j are homotopic to the first and second boundaries, respectively.

Indeed, each C_i is either contractible or homotopic to one of the boundaries, as we justify in Lemma A.7 below. It is not possible that all C_i are contractible or homotopic to the first boundary, as otherwise C would be homotopic to a multiple of the latter. Similarly for the second boundary. Hence, either (i) or (ii) must hold.

Let us now assume $\ell = \ell_3$. As the boundaries are assumed tight, in case (i) we find that $\widehat{C}$ consists of a single element ($n = 1$) so that $C = C_1$ is simple. In case (ii) C would have length at least $\ell_1 + \ell_2$, but this is excluded by the assumption $\ell_1 + \ell_2 > \ell_3$. ∎

Remark A.6. In the case where ℓ_1, say, vanishes, Lemma A.5 and the above proof still hold, upon using the convention discussed in Section 1.2 and [5, Figure 2] of replacing the boundary-vertex by a small circle made of edges of length zero. Observe that the boundary-vertex cannot be incident to the boundary-face of length ℓ_3, as otherwise following the contour of that face but circumventing the boundary-vertex in the other direction would yield a path of length ℓ_3 that is homotopic to the boundary-face of length ℓ_2, contradicting the tightness assumption.

The following fact, used in the proof of Lemma A.5, seems well known in some communities, but we rederive it for convenience.

Lemma A.7. *Let M be a planar map with three boundaries. Then, every simple closed path on M is either contractible or homotopic to one of the boundaries.*

Proof. We may draw M in the plane, with the puncture of one of the boundaries sent to infinity. Let C be a simple closed path on M. By the Jordan–Schoenflies theorem, C delimits a bounded domain D homeomorphic to a disk, which may contain zero, one or two punctures. If D contains zero puncture, then C is contractible. If D contains one puncture, then C is homotopic to the corresponding boundary. Finally, if D contains two punctures, then C is homotopic to the boundary corresponding to the puncture at infinity. ∎

B. Quasi-polynomiality in the general case: Induction step

This appendix and the next one are devoted to the proof of Proposition 4.8 by induction on n. Here, we check that if the statement of the proposition is true for n boundaries, then it is also true for $n+1$ boundaries. We proceed in three steps.

B.1. A discrete integration lemma for polynomials

We first record an elementary discrete integration lemma. It is certainly well known, but the derivation is a nice variation on the Bernoulli–Faulhaber formulas so we give a full proof.

Lemma B.1. *Fix an integer $k \geq 0$. Then, the sums*

$$\sum_{\substack{0<m<\ell \\ m\in 2\mathbb{Z}}} m^{2k+1} + \frac{\ell^{2k+1}}{2}, \qquad \sum_{\substack{0<m<\ell \\ m\in 2\mathbb{Z}+1}} m^{2k+1}$$

are even polynomials in the even integer variable ℓ, and the sums

$$\sum_{\substack{0<m<\ell \\ m\in 2\mathbb{Z}}} m^{2k+1}, \qquad \sum_{\substack{0<m<\ell \\ m\in 2\mathbb{Z}+1}} m^{2k+1} + \frac{\ell^{2k+1}}{2}$$

are even polynomials in the odd integer variable ℓ.

Proof. First, suppose that $\ell = 2n$ is even. By the Bernoulli–Faulhaber formula, we have

$$\begin{aligned}\sum_{\substack{0<m<\ell \\ m\in 2\mathbb{Z}}} m^{2k+1} + \frac{\ell^{2k+1}}{2} &= 2^{2k+1}\left(\sum_{q=1}^{n} q^{2k+1} - n^{2k+1} + \frac{n^{2k+1}}{2}\right) \\ &= 2^{2k+1}\left(\frac{1}{2k+2}\sum_{i=0}^{2k+1}\binom{2k+2}{i} B_i n^{2k+2-i} - \frac{n^{2k+1}}{2}\right).\end{aligned}$$

Since $B_1 = 1/2$ and all other Bernoulli numbers with odd indices vanish, we obtain that this is a polynomial in $n^2 = (\ell/2)^2$. Similarly, a simple consequence of the Bernoulli–Faulhaber formula, subtracting the terms with even indices, is that

$$\sum_{\substack{0<m<\ell \\ m\in 2\mathbb{Z}+1}} m^{2k+1} = \frac{2^{2k+2}}{k+1}\sum_{i=0}^{2k+1}\binom{2k+2}{i} B_i n^{2k+2-i}(2^{1-i}-1)$$

and we see that the term involving B_1 vanishes. Using again the vanishing properties of Bernoulli numbers of odd indices, this is a polynomial in $n^2 = (\ell/2)^2$.

We now study the case where $\ell = 2n-1$ is odd. Note that

$$\sum_{\substack{0<m<\ell \\ m\in 2\mathbb{Z}}} m^{2k+1} = 2^{2k+1}\sum_{0<q<n} q^{2k+1},$$

which, by another formula of Faulhaber, is a polynomial in the variable $n(n-1) = (\ell^2-1)/4$, hence an even polynomial in ℓ. Finally, simply note that, since ℓ is odd,

$$\begin{aligned}\sum_{\substack{0<m<\ell \\ m\in 2\mathbb{Z}+1}} m^{2k+1} + \frac{\ell^{2k+1}}{2} &= \sum_{\substack{0<m\le\ell \\ m\in 2\mathbb{Z}+1}} m^{2k+1} - \frac{\ell^{2k+1}}{2} \\ &= \frac{1}{2}\Bigg(\sum_{\substack{0<m<\ell \\ m\in 2\mathbb{Z}+1}} m^{2k+1} + \sum_{\substack{0<m\le\ell \\ m\in 2\mathbb{Z}+1}} m^{2k+1}\Bigg),\end{aligned}$$

since the last term is the average of the first two equal terms. By a final use of the Bernoulli–Faulhaber formula, this quantity if of the form $(P(n) + P(n+1))$, where P is a polynomial and $2n-1 = \ell - 2$, so it can be put in the form $P((\ell-1)/2) + P((\ell+1)/2)$, which is an even polynomial in ℓ by Newton's formula. ∎

Corollary B.2. *Let $P \in \mathbb{Q}[x]$ be a polynomial. Then,*

$$\sum_{\substack{0<m<\ell \\ m\in 2\mathbb{Z}}} mP(m^2) + \frac{\ell}{2}P(\ell^2), \qquad \sum_{\substack{0<m<\ell \\ m\in 2\mathbb{Z}+1}} mP(m^2),$$

are even polynomials of the even integer variable ℓ, and

$$\sum_{\substack{0<m<\ell \\ m\in 2\mathbb{Z}}} mP(m^2), \qquad \sum_{\substack{0<m<\ell \\ m\in 2\mathbb{Z}+1}} mP(m^2) + \frac{\ell}{2}P(\ell^2),$$

are even polynomials of the odd integer variable ℓ.

B.2. Quasi-polynomiality for tight boundary insertions

Next, we prove the following result on the boundary insertion operator D_ℓ.

Proposition B.3. *For every $i \geq 0$, the quantities*

$$R^{-\frac{\ell}{2}} \frac{D_\ell R^{(i)}}{R}, \quad R^{-\frac{\ell}{2}} \frac{D_\ell S^{(i)}}{\sqrt{R}}$$

are parity-dependent quasi-polynomials in the variable ℓ^2 whose coefficients are polynomial functions of $R^{(j)}/R$, $S^{(j)}/\sqrt{R}$, $1 \leq j \leq i+1$.

Proof. We prove this statement by induction on i, using the fact that for every $\ell \geq 0$,

$$D_\ell R^{(i)} = T^{(0)}_{\ell,1,1,0_i}, \quad D_\ell S^{(i)} = T^{(0)}_{\ell,1,0_{i+1}},$$

where we use the simplifying notation $0_i = 0, \dots, 0$ for a string of i consecutive zeros. We initialize the induction at $i = 0$ by noting that, by our formula (58) for tight pairs of pants,

$$D_\ell R = R^{\frac{\ell}{2}+1} \times \begin{cases} \frac{R'}{R} & \text{if } \ell \text{ is even,} \\ \frac{S'}{\sqrt{R}} & \text{if } \ell \text{ is odd,} \end{cases} \qquad D_\ell S = R^{\frac{\ell+1}{2}} \times \begin{cases} \frac{S'}{\sqrt{R}} & \text{if } \ell \text{ is even,} \\ \frac{R'}{R} & \text{if } \ell \text{ is odd.} \end{cases} \tag{61}$$

Suppose that the statement holds up to some given value of i, namely, we may find polynomials $P_{j,2}$, $P_{j,3}$, $Q_{j,1}$, $Q_{j,2}$ in $\mathbb{Q}[x_1, \dots, x_{j+1}, y_1, \dots, y_{j+1}, u]$ (the second index will refer to the number of boundary-faces of odd degree in the expressions below) such that, for $0 \leq j \leq i$, and letting ε be 0 if ℓ is even and 1 if ℓ is odd, we have

$$D_\ell R^{(j)} = R^{\frac{\ell}{2}+1} P_{j,2+\varepsilon}\left(\frac{R^{(\leq j+1)}}{R}, \frac{S^{(\leq j+1)}}{\sqrt{R}}, \ell^2\right),$$
$$D_\ell S^{(j)} = R^{\frac{\ell+1}{2}} Q_{j,1+\varepsilon}\left(\frac{R^{(\leq j+1)}}{R}, \frac{S^{(\leq j+1)}}{\sqrt{R}}, \ell^2\right).$$

Here, we use the shorthand notation $R^{(\leq j+1)}$, $S^{(\leq j+1)}$ for the variables $(R^{(k)}, 1 \leq k \leq j+1)$ and $(S^{(k)}, 1 \leq k \leq j+1)$.

Now, to prove the statement at rank $i+1$, we write

$$D_\ell R^{(i+1)} = T^{(0)}_{\ell,1,1,0_{i+1}} = \frac{\partial}{\partial t} T^{(0)}_{\ell,1,1,0_i} + \sum_{0<m<\ell} m T^{(0)}_{\ell,0|m} T^{(0)}_{m,1,1,0_i}, \tag{62}$$

$$D_\ell S^{(i+1)} = T^{(0)}_{\ell,1,0_{i+2}} = \frac{\partial}{\partial t} T^{(0)}_{\ell,1,0_{i+1}} + \sum_{0<m<\ell} m T^{(0)}_{\ell,0|m} T^{(0)}_{m,1,0_{i+1}}, \tag{63}$$

where the second equality on each line is a consequence of Proposition 3.3. Let us first analyze the right-hand side of (62). Its first term is equal to

$$\frac{\partial}{\partial t}T^{(0)}_{\ell,1,1,0_i} = \left(\frac{\ell}{2}+1\right)R^{\frac{\ell}{2}+1}\frac{R'}{R}P_{i,2+\varepsilon}\left(\frac{R^{(\leq i+1)}}{R},\frac{S^{(\leq i+1)}}{\sqrt{R}},\ell^2\right) \tag{64}$$
$$+ R^{\frac{\ell}{2}+1}\sum_{k=1}^{i+1}\left(\frac{R^{(k)}}{R}\right)'\frac{\partial P_{i,2+\varepsilon}}{\partial x_k}\left(\frac{R^{(\leq i+1)}}{R},\frac{S^{(\leq i+1)}}{\sqrt{R}},\ell^2\right)$$
$$+ R^{\frac{\ell}{2}+1}\sum_{k=1}^{i+1}\left(\frac{S^{(k)}}{\sqrt{R}}\right)'\frac{\partial P_{i,2+\varepsilon}}{\partial y_k}\left(\frac{R^{(\leq i+1)}}{R},\frac{S^{(\leq i+1)}}{\sqrt{R}},\ell^2\right).$$

Note that $(R^{(k)}/R)' = (R^{(k+1)}/R) - (R^{(k)}/R)(R'/R)$ and $(S^{(k)}/\sqrt{R})' = (S^{(k+1)}/\sqrt{R}) - (1/2)(S^{(k)}/\sqrt{R})(R'/R)$ and therefore the last two sums are quasi-polynomial in ℓ^2 with coefficients in $\mathbb{Q}[R^{(\leq i+2)}/R, S^{(\leq i+2)}/\sqrt{R}]$. In order to study the last sum in (62), it is easier to split cases.

Assume that ℓ is even ($\varepsilon = 0$). The last sum in (62) then equals

$$\sum_{0<m<\ell} mT^{(0)}_{\ell,0|m}T^{(0)}_{m,1,1,0_i}$$
$$= \sum_{\substack{0<m<\ell\\ m\in 2\mathbb{Z}}} mR^{\frac{\ell-m}{2}}\frac{R'}{R}R^{\frac{m}{2}+1}P_{i,2}\left(\frac{R^{(\leq i+1)}}{R},\frac{S^{(\leq i+1)}}{\sqrt{R}},m^2\right) \tag{65}$$
$$+ \sum_{\substack{0<m<\ell\\ m\in 2\mathbb{Z}+1}} mR^{\frac{\ell-m}{2}}\frac{S'}{\sqrt{R}}R^{\frac{m}{2}+1}P_{i,3}\left(\frac{R^{(\leq i+1)}}{R},\frac{S^{(\leq i+1)}}{\sqrt{R}},m^2\right). \tag{66}$$

By Corollary B.2, the term (66), and the sum of the term (65) and (64) (for $\varepsilon = 0$) is of the wanted form of $R^{\frac{\ell}{2}+1}$ times an even polynomial in the variable ℓ^2, with coefficients in $\mathbb{Q}[R^{(\leq i+1)}/R, S^{(\leq i+1)}/\sqrt{R}]$. Putting things together, we see that (62) indeed yields an even polynomial expression in the even variable ℓ.

Assume that ℓ is odd ($\varepsilon = 1$). This time, we have

$$\sum_{0<m<\ell} mT^{(0)}_{\ell,0|m}T^{(0)}_{m,1,1,0_i}$$
$$= \sum_{\substack{0<m<\ell\\ m\in 2\mathbb{Z}}} mR^{\frac{\ell-m}{2}}\frac{S'}{\sqrt{R}}R^{\frac{m}{2}+1}P_{i,2}\left(\frac{R^{(\leq i+1)}}{R},\frac{S^{(\leq i+1)}}{\sqrt{R}},m^2\right) \tag{67}$$
$$+ \sum_{\substack{0<m<\ell\\ m\in 2\mathbb{Z}+1}} mR^{\frac{\ell-m}{2}}\frac{R'}{R}R^{\frac{m}{2}+1}P_{i,3}\left(\frac{R^{(\leq i+1)}}{R},\frac{S^{(\leq i+1)}}{\sqrt{R}},m^2\right). \tag{68}$$

We then apply Corollary B.2 to the term (67) on the one hand, and to the sum of the term (68) and (64) (for $\varepsilon = 1$) on the other hand. These are of the wanted form of $R^{\frac{\ell}{2}+1}$ times an even polynomial in ℓ^2, with coefficients in $\mathbb{Q}[R^{(\leq i+1)}/R, S^{(\leq i+1)}/\sqrt{R}]$. Putting things together, we see that (62) indeed yields an even polynomial expression in the odd variable ℓ. This completes the induction step for $D_\ell R^{(i+1)}$.

We now analyze the right-hand side of (63) in a similar way. Its first term is equal to

$$\begin{aligned}\frac{\partial}{\partial t} T^{(0)}_{\ell,1,0_{i+1}} &= \frac{\ell+1}{2} \cdot R^{\frac{\ell+1}{2}} \frac{R'}{R} Q_{i,1+\varepsilon}\left(\frac{R^{(\leq i+1)}}{R}, \frac{S^{(\leq i+1)}}{\sqrt{R}}, \ell^2\right) \\ &\quad + R^{\frac{\ell+1}{2}} \sum_{k=1}^{i+1} \left(\frac{R^{(k)}}{R}\right)' \frac{\partial Q_{i,1+\varepsilon}}{\partial x_k}\left(\frac{R^{(\leq i+1)}}{R}, \frac{S^{(\leq i+1)}}{\sqrt{R}}, \ell^2\right) \\ &\quad + R^{\frac{\ell+1}{2}} \sum_{k=1}^{i+1} \left(\frac{S^{(k)}}{\sqrt{R}}\right)' \frac{\partial Q_{i,1+\varepsilon}}{\partial y_k}\left(\frac{R^{(\leq i+1)}}{R}, \frac{S^{(\leq i+1)}}{\sqrt{R}}, \ell^2\right).\end{aligned} \tag{69}$$

Again, the last two sums are quasi-polynomial in the variable ℓ^2 with coefficients in $\mathbb{Q}[R^{(\leq i+2)}/R, S^{(\leq i+2)}/\sqrt{R}]$. To analyze the last sum in (63) we again split cases. Assume that ℓ is even ($\varepsilon = 0$). Then, we have

$$\sum_{0<m<\ell} m T^{(0)}_{\ell,0|m} T^{(0)}_{m,1,0_{i+1}}$$

$$= \sum_{\substack{0<m<\ell \\ m\in 2\mathbb{Z}}} m R^{\frac{\ell-m}{2}} \frac{R'}{R} R^{\frac{m+1}{2}} Q_{i,1}\left(\frac{R^{(\leq i+1)}}{R}, \frac{S^{(\leq i+1)}}{\sqrt{R}}, m^2\right) \tag{70}$$

$$+ \sum_{\substack{0<m<\ell \\ m\in 2\mathbb{Z}+1}} m R^{\frac{\ell-m}{2}} \frac{S'}{\sqrt{R}} R^{\frac{m+1}{2}} Q_{i,2}\left(\frac{R^{(\leq i+1)}}{R}, \frac{S^{(\leq i+1)}}{\sqrt{R}}, m^2\right). \tag{71}$$

We apply Corollary B.2 to the term (71), and to the sum of the term (70) and (69) (for $\varepsilon = 0$), showing that they are of the wanted form of $R^{\frac{\ell+1}{2}}$ times an even polynomial in the variable ℓ^2, with coefficients in $\mathbb{Q}[R^{(\leq i+1)}/R, S^{(\leq i+1)}/\sqrt{R}]$. Finally, assume that ℓ is odd ($\varepsilon = 1$). This time, we have

$$\sum_{0<m<\ell} m T^{(0)}_{\ell,0|m} T^{(0)}_{m,1,0_{i+1}}$$

$$= \sum_{\substack{0<m<\ell \\ m\in 2\mathbb{Z}}} m R^{\frac{\ell-m}{2}} \frac{S'}{\sqrt{R}} R^{\frac{m+1}{2}} Q_{i,1}\left(\frac{R^{(\leq i+1)}}{R}, \frac{S^{(\leq i+1)}}{\sqrt{R}}, m^2\right) \tag{72}$$

$$+ \sum_{\substack{0<m<\ell \\ m\in 2\mathbb{Z}+1}} m R^{\frac{\ell-m}{2}} \frac{R'}{R} R^{\frac{m+1}{2}} Q_{i,2}\left(\frac{R^{(\leq i+1)}}{R}, \frac{S^{(\leq i+1)}}{\sqrt{R}}, m^2\right). \tag{73}$$

We then apply Corollary B.2 to the term (72) on the one hand, and to the sum of the term (73) and (69) (for $\varepsilon = 1$) on the other hand. These are of the wanted form of $R^{\frac{\ell}{2}+1}$ times an even polynomial in the variable ℓ^2, with coefficients in $\mathbb{Q}[R^{(\leq i+1)}/R, S^{(\leq i+1)}/\sqrt{R}]$. Putting things together, we see that (63) indeed yields an even polynomial expression in the odd variable ℓ. This completes the induction step for $D_\ell S^{(i+1)}$, and the proof of Proposition B.3. ■

From the fact that D_m is a differential operator, note that

$$D_\ell\left(\frac{R^{(i)}}{R}\right) = \frac{D_\ell R^{(i)}}{R} - \frac{R^{(i)}}{R}\frac{D_\ell R}{R}, \qquad D_\ell\left(\frac{S^{(i)}}{\sqrt{R}}\right) = \frac{D_\ell S^{(i)}}{\sqrt{R}} - \frac{1}{2}\frac{S^{(i)}}{\sqrt{R}}\frac{D_\ell R}{R} \tag{74}$$

are of the form of $R^{\ell/2}$ times a parity-dependent quasi-polynomial in ℓ^2, and with coefficients in $\mathbb{Q}[R^{(\leq i+1)}/R, S^{(\leq i+1)}/\sqrt{R}]$.

B.3. End of the proof of the induction step

Let us now assume that the statement of Proposition 4.8 is true for (g, n), we will show that it is then true for $(g, n+1)$. Let us introduce the shorthand notation

$$d_{g,n} := 3g - 2 + n.$$

By the induction hypothesis, for every subset I of $\{1, \dots, n\}$, there exists

$$P_{n,I}^{(g)} \in \mathbb{Q}(x_1, \dots, x_{d_{g,n}}, y_1, \dots, y_{d_{g,n}})[u_1, \dots, u_n]$$

such that, whenever the ℓ_i with $i \in I$ are odd integers, and the ℓ_i with $i \notin I$ are even integers, we have

$$\tau_{\ell_1,\dots,\ell_n}^{(g)} = P_{n,I}^{(g)}\left(\frac{R^{(\leq d_{g,n})}}{R}, \frac{S^{(\leq d_{g,n})}}{\sqrt{R}}, \ell_1^2, \dots, \ell_n^2\right) - c(g,n)t^{2-2g-n}\delta_{\ell_1+\dots+\ell_n,0}. \tag{75}$$

The $P_{n,I}^{(g)}$ correspond to the family of polynomials associated with the parity-dependent quasi-polynomial $\mathfrak{T}_n^{(g)}$, but the extra variables $x_1, \dots, x_{d_{g,n}}, y_1, \dots, y_{d_{g,n}}$ are useful to keep track of the dependency in the derivatives of R and S. In what follows, to lighten notation, we omit the mention of these extra variables, except when we differentiate with respect to them.

Recall that Propositions 3.3 and 3.5 assert that

$$T_{\ell_1,\dots,\ell_{n+1}}^{(g)} = D_{\ell_{n+1}} T_{\ell_1,\dots,\ell_n}^{(g)} + \sum_{i=1}^{n} \sum_{0<m_i<\ell_i} m_i T_{\ell_i,\ell_{n+1}|m_i}^{(0)} T_{\ell_1,\dots,m_i,\dots,\ell_n}^{(g)} \tag{76}$$

with $D_0 = \frac{\partial}{\partial t}$ and D_m as in (34) for $m > 0$. We plug (75) into the above relation. Since $D_{\ell_{n+1}}$ is a differential operator that annihilates the variable t except when $\ell_{n+1} = 0$, the first term on the right-hand side of (76) is equal to

$$\begin{aligned} D_{\ell_{n+1}} &\left(R^{\frac{\ell_1+\cdots+\ell_n}{2}} P^{(g)}_{n,I}(\ell_1^2,\ldots,\ell_n^2) - c(g,n) t^{2-2g-n} \delta_{\ell_1+\cdots+\ell_n,0} \right) \\ &= \frac{\ell_1+\cdots+\ell_n}{2} \cdot R^{\frac{\ell_1+\cdots+\ell_n}{2}} \frac{D_{\ell_{n+1}} R}{R} P^{(g)}_{n,I}(\ell_1^2,\ldots,\ell_n^2) \\ &\quad + R^{\frac{\ell_1+\cdots+\ell_n}{2}} \sum_{r=1}^{d_{g,n}} \left(D_{\ell_{n+1}} \left(\frac{R^{(r)}}{R} \right) \frac{\partial P^{(g)}_{n,I}}{\partial x_r}(\ell_1^2,\ldots,\ell_n^2) \right. \\ &\qquad\qquad \left. + D_{\ell_{n+1}} \left(\frac{S^{(r)}}{\sqrt{R}} \right) \frac{\partial P^{(g)}_{n,I}}{\partial y_r}(\ell_1^2,\ldots,\ell_n^2) \right) \\ &\quad - c(g,n+1) t^{2-2g-(n+1)} \delta_{\ell_1+\cdots+\ell_n+\ell_{n+1},0}, \end{aligned} \tag{77}$$

where we set $c(g,n+1) = (2-2g-n)c(g,n)$. By Proposition B.3 and its consequence discussed around (74), the third line of (77) is of the wanted form $R^{\frac{\ell_1+\cdots+\ell_{n+1}}{2}}$ times a quasi-polynomial in $\ell_1^2,\ldots,\ell_{n+1}^2$. We then split the last term of (76) into pieces.

Assume first that ℓ_{n+1} is even, in which case, the quantity $D_{\ell_{n+1}} R/R$ appearing in the second line of (77) equals $R^{\ell_{n+1}/2} R'/R$. The last term of (76) is then

$$\sum_{\substack{i=1\\ \ell_i \in 2\mathbb{Z}}}^{n} \sum_{\substack{0<m_i<\ell_i\\ m_i\in 2\mathbb{Z}}} m_i R^{\frac{\ell_i+\ell_{n+1}-m_i}{2}} \frac{R'}{R} R^{\frac{\ell_1+\cdots+m_i+\cdots+\ell_n}{2}} \times P^{(g)}_{n,I}(\ell_1^2,\ldots,m_i^2,\ldots,\ell_n^2) \tag{78}$$

$$+ \sum_{\substack{i=1\\ \ell_i \in 2\mathbb{Z}}}^{n} \sum_{\substack{0<m_i<\ell_i\\ m_i\in 2\mathbb{Z}+1}} m_i R^{\frac{\ell_i+\ell_{n+1}-m_i}{2}} \frac{S'}{\sqrt{R}} R^{\frac{\ell_1+\cdots+m_i+\cdots+\ell_n}{2}} \times P^{(g)}_{n,I\cup\{i\}}(\ell_1^2,\ldots,m_i^2,\ldots,\ell_n^2) \tag{79}$$

$$+ \sum_{\substack{i=1\\ \ell_i \in 2\mathbb{Z}+1}}^{n} \sum_{\substack{0<m_i<\ell_i\\ m_i\in 2\mathbb{Z}}} m_i R^{\frac{\ell_i+\ell_{n+1}-m_i}{2}} \frac{S'}{\sqrt{R}} R^{\frac{\ell_1+\cdots+m_i+\cdots+\ell_n}{2}} \times P^{(g)}_{n,I\setminus\{i\}}(\ell_1^2,\ldots,m_i^2,\ldots,\ell_n^2) \tag{80}$$

$$+ \sum_{\substack{i=1\\ \ell_i \in 2\mathbb{Z}+1}}^{n} \sum_{\substack{0<m_i<\ell_i\\ m_i\in 2\mathbb{Z}+1}} m_i R^{\frac{\ell_i+\ell_{n+1}-m_i}{2}} \frac{R'}{R} R^{\frac{\ell_1+\cdots+m_i+\cdots+\ell_n}{2}} \times P^{(g)}_{n,I}(\ell_1^2,\ldots,m_i^2,\ldots,\ell_n^2). \tag{81}$$

By the discrete integration of Corollary B.2, we see that (79) and (80) are polynomial expressions in their, respectively, even and odd variables ℓ_i^2, while we should combine (78) and (81) with the corresponding first terms of (77) (which is made possible by the fact that $D_{\ell_{n+1}}$ has a factor R'/R) to get similar polynomial expressions.

If, on the other hand, ℓ_{n+1} is odd, then $D_{\ell_{n+1}} R/R$ equals $R^{\ell_{n+1}/2} S'/\sqrt{R}$. The last term of (76) is then

$$\begin{aligned}
&\sum_{\substack{i=1\\ \ell_i \in 2\mathbb{Z}}}^{n} \sum_{\substack{0<m_i<\ell_i\\ m_i \in 2\mathbb{Z}}} m_i R^{\frac{\ell_i+\ell_{n+1}-m_i}{2}} \frac{S'}{\sqrt{R}} R^{\frac{\ell_1+\cdots+m_i+\cdots+\ell_n}{2}} \\
&\qquad\qquad \times P_{n,I}^{(g)}(\ell_1^2,\ldots,m_i^2,\ldots,\ell_n^2) \qquad (82)\\
&+ \sum_{\substack{i=1\\ \ell_i \in 2\mathbb{Z}}}^{n} \sum_{\substack{0<m_i<\ell_i\\ m_i \in 2\mathbb{Z}+1}} m_i R^{\frac{\ell_i+\ell_{n+1}-m_i}{2}} \frac{R'}{R} R^{\frac{\ell_1+\cdots+m_i+\cdots+\ell_n}{2}} \\
&\qquad\qquad \times P_{n,I\cup\{i\}}^{(g)}(\ell_1^2,\ldots,m_i^2,\ldots,\ell_n^2) \qquad (83)\\
&+ \sum_{\substack{i=1\\ \ell_i \in 2\mathbb{Z}+1}}^{n} \sum_{\substack{0<m_i<\ell_i\\ m_i \in 2\mathbb{Z}}} m_i R^{\frac{\ell_i+\ell_{n+1}-m_i}{2}} \frac{R'}{R} R^{\frac{\ell_1+\cdots+m_i+\cdots+\ell_n}{2}} \\
&\qquad\qquad \times P_{n,I\setminus\{i\}}^{(g)}(\ell_1^2,\ldots,m_i^2,\ldots,\ell_n^2) \qquad (84)\\
&+ \sum_{\substack{i=1\\ \ell_i \in 2\mathbb{Z}+1}}^{n} \sum_{\substack{0<m_i<\ell_i\\ m_i \in 2\mathbb{Z}+1}} m_i R^{\frac{\ell_i+\ell_{n+1}-m_i}{2}} \frac{S'}{\sqrt{R}} R^{\frac{\ell_1+\cdots+m_i+\cdots+\ell_n}{2}} \\
&\qquad\qquad \times P_{n,I}^{(g)}(\ell_1^2,\ldots,m_i^2,\ldots,\ell_n^2). \qquad (85)
\end{aligned}$$

By the discrete integration Corollary B.2, we see that (83) and (84) are polynomial expressions in their, respectively, even and odd variables ℓ_i^2, while we should combine (82) and (85) with the corresponding first terms of (77) to get similar polynomial expressions.

From this discussion, we conclude that the statement of Proposition 4.8 holds for $(g, n+1)$, which concludes the proof of the induction step.

C. Quasi-polynomiality in the general case: Initialization

We now initialize the induction for proving Proposition 4.8: for $g = 0$, $g = 1$, and $g \geq 2$ we should, respectively, show that the statement holds for $n = 3$, $n = 1$, and $n = 0$.

The fact that the statement holds in the planar case $(g, n) = (0, 3)$ is a direct consequence of Theorem A.1 and of the relation (13) between the T's and the τ's. To treat the case of higher genera, we will need some results from the theory of topological recursion presented in [12], and more precisely, we will need the expression of the generating function of maps in genus $g \geq 1$ based on the so-called *method of moments* [1]. Our approach can be seen as an extension of the analysis of Budd [8, Section 3.4] to the non bipartite case (but only considering the non-irreducible case). Recall that $F^{(g)} = F^{(g)}_{\varnothing} = T^{(g)}_{\varnothing} = \tau^{(g)}_{\varnothing}$ denote the generating series of maps of genus g without boundaries.

C.1. Genus one

Let us first deal with the case $g = 1$. Our main input will be the following.

Theorem C.1. *The generating function for maps of genus one without boundary is given by*

$$F^{(1)} = \frac{1}{24} \ln\left(\left(\frac{R'}{R}\right)^2 - \left(\frac{S'}{\sqrt{R}}\right)^2\right) + \frac{\ln(t)}{12}.$$

From this, we can easily deduce Proposition 4.8 in the case $(g,n) = (1,1)$. Indeed, note that, by Propositions 3.3 and 3.5, we have $T^{(1)}_\ell = D_\ell F^{(1)}$, which equals

$$T^{(1)}_\ell = \frac{\frac{R'}{R} D_\ell(\frac{R'}{R}) - \frac{S'}{\sqrt{R}} D_\ell(\frac{S'}{\sqrt{R}})}{12((\frac{R'}{R})^2 - (\frac{S'}{\sqrt{R}})^2)} + \frac{\delta_{\ell,0}}{12t}$$

and by Proposition B.3—see also (74)—this is of the wanted form of $R^{\ell/2}$ times a quasipolynomial in ℓ with quasiperiod 2, whose coefficients are rational functions of $R^{(i)}/R, S^{(i)}/\sqrt{R}, i \in \{1,2\}$, minus a pathological term for $\ell = 0$.

Proof of Theorem C.1. Let us recall Eynard's notation[9] from [12]. We set (p. 58 and Theorem 3.1.2, p. 62) $x(z) = S + \sqrt{R}(z + 1/z)$ ($S = \alpha$, $R = \gamma^2$), and (pp. 59–60, Definition 3.1.1, p. 63 and Theorem 3.1.2, p. 62)

$$\begin{aligned} y(z) &= W^{(0)}_1(x(z)) - \frac{V'(x(z))}{2} \\ &= -\frac{1}{2} \sum_{k \geq 1} u_k (z^k - z^{-k}), \end{aligned}$$

[9] Eynard puts a bound on the degrees, and excludes faces of degrees 1 and 2, but this restriction is in fact inessential, as it results from Eynard's choice of considering the map generating functions as formal power series in the vertex weight t whose coefficients are polynomials in the face weights. Working instead with a weight per edge lifts this restriction since the set of maps with a given number of edges is finite, and we believe that the reasonings of [12] extend to this setting.

where

$$\begin{aligned}u_k &= S\delta_{k,0} + \sqrt{R}\delta_{k,1} \\ &\quad - \sum_{i\geq k+1} t_i \sum_{j=k}^{\lfloor (i+k-1)/2\rfloor} \binom{i-1}{j,\, j-k,\, i-1+k-2j} R^{j-k/2} S^{i-1+k-2j}. \end{aligned} \tag{86}$$

Note that the latter is related to the generating function V_{k-1} for $(k-1)$-slices introduced in [4]: for $k \geq 2$, we have

$$V_{k-1} = -R^{k/2} u_k.$$

On the other hand, we have $u_0 = 0$ and $u_1 = t/\sqrt{R}$, as seen by applying the multinomial formulas in (2). We define the 0th order *moments* by

$$\begin{aligned}M_{+,0} &= \frac{-y'(1)}{\sqrt{R}}, \\ M_{-,0} &= \frac{-y'(-1)}{\sqrt{R}}.\end{aligned}$$

Now, [12, Theorem 3.4.6] expresses the generating function of toric maps in terms of $M_{\pm,0}$ as

$$F^{(1)} = -\frac{1}{24}\ln\left(\frac{\gamma^2 y'(1) y'(-1)}{t^2}\right) = -\frac{1}{24}\ln\left(\frac{R^2 M_{+,0} M_{-,0}}{t^2}\right). \tag{87}$$

On the other hand, [12, Theorem 3.3.4] gives

$$\begin{aligned}\omega_3^{(0)}(z_1,z_2,z_3) &= \frac{1}{2RM_{+,0}} \frac{1}{(z_1-1)^2(z_2-1)^2(z_3-1)^2} \\ &\quad - \frac{1}{2RM_{-,0}} \frac{1}{(z_1+1)^2(z_2+1)^2(z_3+1)^2},\end{aligned}$$

which we now know to count pairs of pants. Namely, this is

$$\sum_{\ell_1,\ell_2,\ell_3\geq 1} \left(\frac{1}{2RM_{+,0}} + \frac{(-1)^{\ell_1+\ell_2+\ell_3}}{2RM_{-,0}}\right) \frac{\ell_1\ell_2\ell_3}{z_1^{\ell_1+1} z_2^{\ell_2+1} z_3^{\ell_3+1}}.$$

Comparing with our formula for tight pairs of pants (58), we obtain

$$\tau^{(0)}_{\ell_1,\ell_2,\ell_3} = \begin{cases} \dfrac{1}{2RM_{+,0}} + \dfrac{1}{2RM_{-,0}} = \dfrac{R'}{R} & \text{if } \ell_1+\ell_2+\ell_3 \in 2\mathbb{Z}, \\[2ex] \dfrac{1}{2RM_{+,0}} - \dfrac{1}{2RM_{-,0}} = \dfrac{S'}{\sqrt{R}} & \text{if } \ell_1+\ell_2+\ell_3 \in 2\mathbb{Z}+1, \end{cases}$$

where $\ell_1, \ell_2, \ell_3 > 0$. This shows that

$$\frac{1}{RM_{+,0}} = -\frac{1}{\gamma y'(1)} = \frac{R'}{R} + \frac{S'}{\sqrt{R}}, \quad \frac{1}{RM_{-,0}} = -\frac{1}{\gamma y'(-1)} = \frac{R'}{R} - \frac{S'}{\sqrt{R}}. \tag{88}$$

Plugging this into (87) yields Theorem C.1. ∎

C.2. Higher moments, higher genera

The generating functions for maps in higher genera are given by higher-order moments, which are given [12, p. 64], for $h \geq 1$, by

$$M_{\pm,h} = \frac{-1}{R^{(h+1)/2} M_{\pm,0}} \sum_{k \geq h+1} (\pm 1)^{k+h+1} u_k \binom{k+h}{2h+1}.$$

We define the renormalized version

$$\bar{M}_{\pm,h} = R^{(h+1)/2} M_{\pm,h} M_{\pm,0} = -\sum_{k \geq h+1} (\pm 1)^{k+h+1} u_k \binom{k+h}{2h+1}. \tag{89}$$

One should note that [12, Corollary 3.5.1] introduces the quantities $R^{h/2} M_{\pm,h}$ that are called "dimensionless", we will see that these are indeed natural quantities to consider, due to the following result[10].

Theorem C.2 ([12, Corollary 3.5.1]). *There exists $\mathcal{P}_g \in \mathbb{Q}[x_+, x_-, (y_{+,h}, y_{-,h}, 1 \leq h \leq 3g-3)]$ that is homogeneous of degree $2g-2$ in the first two variables, and such that $\mathcal{P}_g(x_+, x_-, (\mu^h, \mu^h, 1 \leq h \leq 3g-3))$ is of degree $3g-3$ in μ, and $c(g,0) \in \mathbb{Q}$, such that*

$$F^{(g)} = \mathcal{P}_g\left(\frac{1}{RM_{+,0}}, \frac{1}{RM_{-,0}}, (R^{h/2} M_{+,h}, R^{h/2} M_{-,h}, 1 \leq h \leq 3g-3)\right) - c(g,0) t^{2-2g}.$$

We would like to show that, for $h \geq 1$, the "dimensionless" moments $R^{h/2} M_{\pm,h}$ are rational functions of $R^{(k)}/R, S^{(k)}/\sqrt{R}$ for $1 \leq k \leq h+1$. Our goal will be to get rid of the variables t_i appearing implicitly in the u_k in the previous expression, and

[10] There seems to be an inaccuracy in the statement of [12, Corollary 3.5.1]. Indeed, the latter is stated without the boundary term involving t^{2-2g} only, which is, however, present in other similar statements such as [*ibid.*, Theorems 3.4.3 and 3.4.9]. In these statements, the coefficient $\frac{B_{2g}}{2g(2-2g)}$ is indeed equal to $\chi(\mathcal{M}_{g,0})$ as wanted. It seems that this boundary term got omitted at some stage.

express this purely in terms of R, S and their derivatives. Our approach is inspired by that of [8, Section 3.4].

First, for $h \geq 1$, we rewrite, using (89) and (86),

$$\begin{aligned}\bar{M}_{\pm,h} &= \sum_{k\geq h+1} (\pm 1)^{k+h+1} \sum_{i\geq 1} t_i \binom{k+h}{2h+1} \\ &\qquad\qquad \times \sum_j \binom{i-1}{j, j-k, i-1+k-2j} R^{j-k/2} S^{i-1+k-2j} \\ &= (\pm 1)^h \sum_{i\geq h+2} t_i \sum_l (\pm 1)^{i-l} R^{\frac{i-l-1}{2}} S^l \\ &\qquad\qquad \times \sum_k \binom{k+h}{2h+1} \binom{i-1}{\frac{i-1+k-l}{2}, \frac{i-1-k-l}{2}, l},\end{aligned} \tag{90}$$

where the second expression comes by changing the variable j into $l = i-1+k-2j$ and permuting the sums. To alleviate notations, we do not specify some summation ranges as they are naturally enforced by the vanishing of the binomial and multinomial coefficients and by the requirement that their arguments are nonnegative integers. In particular, in the second expression, $k+1$ has the same parity as $i-l$, which allows to reorganize the term $(\pm 1)^{k+h+1}$ as displayed.

C.2.1. A binomial identity. Following [8, Lemma 7], it is convenient to simplify the sum over k appearing in (90). Let us first rewrite

$$\sum_k \binom{k+h}{2h+1} \binom{i-1}{\frac{i-1+k-l}{2}, \frac{i-1-k-l}{2}, l} = \binom{i-1}{l} \sum_k \binom{k+h}{2h+1} \binom{i-l-1}{\frac{i-l-1+k}{2}},$$

where we recall that the sum is over nonnegative integers k with the same parity as $i-l-1$. Writing $j = i-l$, we treat the sum over k on the right-hand side by the following lemma.

Lemma C.3. *For any nonnegative integer h, there exist two polynomials $Q_h^{[0]}$ and $Q_h^{[1]}$ of degree $h+1$ such that, denoting by $\varepsilon = \varepsilon(j) \in \{0,1\}$ the parity of j, we have*

$$\sum_k \binom{k+h}{2h+1} \binom{j-1}{\frac{j-1+k}{2}} = \binom{j-1}{\frac{j-\varepsilon}{2}} Q_h^{[\varepsilon]}\left(\frac{j-\varepsilon}{2}\right), \tag{91}$$

where we sum over nonnegative k such that $k+\varepsilon$ is odd.

Thanks to this lemma, the sum over k in (90) is expressed as

$$\sum_k \binom{k+h}{2h+1} \binom{i-1}{\frac{i-1+k-l}{2}, \frac{i-1-k-l}{2}, l} = \binom{i-1}{\frac{i-l-\varepsilon}{2}, \frac{i-l+\varepsilon-2}{2}, l} Q_h^{[\varepsilon]}\left(\frac{i-l-\varepsilon}{2}\right),$$

which leads to the expression

$$\bar{M}_{\pm,h} = (\pm 1)^h \sum_{\varepsilon\in\{0,1\}} (\pm 1)^\varepsilon \sum_{i\geq 2} t_i \sum_{l=i+\varepsilon \bmod 2} \times R^{\frac{i-l-1}{2}} S^l \binom{i-1}{\frac{i-l-\varepsilon}{2}, \frac{i-l+\varepsilon-2}{2}, l} Q_h^{[\varepsilon]}\left(\frac{i-l-\varepsilon}{2}\right) \tag{92}$$

that we will further study in the next subsection. In passing, we note that

$$Q_h^{[0]}(1) = 0 \quad \text{for every } h \geq 1, \tag{93}$$

as seen by taking $j = 2$ in (91): on the left-hand side, the sum is over all positive odd k, and each term vanishes.

Proof of Lemma C.3. Assume first that j is even, that is, $\varepsilon = 0$, so that the summation is over odd $k = 2k' + 1$. Then, writing $2j' = j$, equation (91) is rewritten as

$$\sum_{k'\geq 0} \binom{2k'+h+1}{2h+1}\binom{2j'-1}{j'+k'} = \binom{2j'-1}{j'} Q_h^{[0]}(j'), \tag{94}$$

where

$$Q_h^{[0]} = Q_h(0,\cdot)$$

is the polynomial of degree $h+1$ of [8, Lemma 7] specialized at $b = 0$. Alternatively, we may recover the expression of $Q_h^{[0]}$ by the following independent argument. First, note that, in the notation of [6], we have

$$\binom{x+h}{2h+1} = \frac{h!^2}{(2h+1)!} x p_h(x), \tag{95}$$

so that, using the notation

$$\mathsf{A}_{\ell,k} = \frac{2k}{2\ell}\binom{2\ell}{\ell+k}, \quad \ell, k \in \frac{1}{2}\mathbb{Z} \tag{96}$$

from [6, (3.2)], we can rewrite the left-hand side of (94) in the form

$$\begin{aligned} &\frac{h!^2}{(2h+1)!}\sum_{k'} 2(k'+1/2) p_h(2(k'+1/2))\binom{2(j'-1/2)}{j'-1/2+k'+1/2} \\ &= (2j'-1)\frac{h!^2}{(2h+1)!}\sum_{k'} \mathsf{A}_{j'-1/2,k'+1/2}\, p_h(2(k'+1/2)). \end{aligned}$$

Taking $m = k' + 1/2$, the polynomial $p_h(2m)$ is of degree h in m^2, so it can be expressed in the basis of the polynomials $\tilde{p}_r(m), 0 \leq r \leq h$ defined in [6, (2.18)], in the form $p_h(2m) = \sum_{r=0}^{h} \tilde{\alpha}_{h,r} \tilde{p}_r(m)$, which, based on [6, (3.9)], yields the expression

$$
\begin{aligned}
&(2j'-1)\frac{h!^2}{(2h+1)!}\sum_{k'}\sum_{r=0}^{h}\tilde{\alpha}_{h,r}\mathsf{A}_{j'-1/2,k'+1/2}\tilde{p}_r(k'+1/2)\\
&\quad=(2j'-1)\frac{h!^2}{(2h+1)!}\sum_{r=0}^{h}\tilde{\alpha}_{h,r}\sum_{k'}\mathsf{A}_{j'-1/2,k'+1/2}\tilde{p}_r(k'+1/2)\\
&\quad=(2j'-1)\frac{h!^2}{(2h+1)!}\sum_{r=0}^{h}\tilde{\alpha}_{h,r}\binom{j'-1}{r}\binom{2j'-2}{j'-1}\\
&\quad=\binom{2j'-1}{j'}\frac{h!^2}{(2h+1)!}j'\sum_{r=0}^{h}\tilde{\alpha}_{h,r}\binom{j'-1}{r}
\end{aligned}
$$

which, since $j' = j/2$, is indeed of the wanted form $\binom{j-1}{j/2} Q_h^{[0]}(j/2)$ for some polynomial $Q_h^{[0]}$ of degree $h+1$.

For j odd, i.e., $\varepsilon = 1$, we argue in a similar (and slightly simpler) way. The summation is now over even $k = 2k'$, and we write $2j' = j-1$, so that the left-hand side of (91) reads

$$
\sum_{k'}\binom{2k'+h}{2h+1}\binom{2j'}{j'+k'} = 2j'\frac{h!^2}{(2h+1)!}\sum_{k'}\mathsf{A}_{j',k'}p_h(2k'),
$$

where we use (95) and (96) to pass to the right-hand side. This time, we take $m = k'$ and we express the polynomial $p_h(2m)$ in the basis of the polynomials $p_r(m), 0 \le r \le h$, in the form $p_h(2m) = \sum_{r=0}^{h} \alpha_{h,r} p_r(m)$, which yields, now using [6, (3.7)], the expression

$$
\begin{aligned}
&2j'\frac{h!^2}{(2h+1)!}\sum_{k'}\sum_{r=0}^{h}\alpha_{h,r}\mathsf{A}_{j',k'}p_r(k')\\
&\quad=2j'\frac{h!^2}{(2h+1)!}\sum_{r=0}^{h}\alpha_{h,r}\sum_{k'}\mathsf{A}_{j',k'}p_r(k')\\
&\quad=2j'\frac{h!^2}{(2h+1)!}\sum_{r=0}^{h}\alpha_{h,r}\binom{j'-1}{r}\binom{2j'-1}{j'}\\
&\quad=\binom{2j'}{j'}\frac{h!^2}{(2h+1)!}j'\sum_{r=0}^{h}\alpha_{h,r}\binom{j'-1}{r},
\end{aligned}
$$

which is again of the wanted form $\binom{j-1}{(j-1)/2} Q_h^{[1]}((j-1)/2)$ for some polynomial $Q_h^{[1]}$ of degree $h+1$. ∎

C.2.2. Relating the moments with the recursions determining R, S. The expression (92) for the moments can be compared to the equations (2) defining R and S, which we may write in the more compact and symmetric form

$$\mathbf{Z}(\mathbf{U}(\mathbf{t})) = \mathbf{t},$$

where $\mathbf{t} = (t_0, t_1)$, $t_0 = t$, $\mathbf{U} = (R, S)$ and $\mathbf{Z} = (Z_0, Z_1)$ is the pair of bivariate series defined by

$$Z_0(r,s) = r - \sum_{i\geq 2} t_i \sum_{l=i \bmod 2} \binom{i-1}{\frac{i-l}{2}, \frac{i-l}{2}-1, l} r^{\frac{i-l}{2}} s^l \tag{97}$$

and

$$Z_1(r,s) = s - \sum_{i\geq 2} t_i \sum_{l=i+1 \bmod 2} \binom{i-1}{\frac{i-l-1}{2}, \frac{i-l-1}{2}, l} r^{\frac{i-l-1}{2}} s^l.$$

Combining this with (92), we obtain that, for every $h \geq 1$,

$$(\pm 1)^h \bar{M}_{\pm,h} = - \sum_{\varepsilon\in\{0,1\}} (\pm 1)^\varepsilon R^{\frac{\varepsilon-1}{2}} Q_h^{[\varepsilon]}(r\partial_r) Z_\varepsilon(r,s)|_{r=R,s=S}, \tag{98}$$

where we note that the term r in (97) is annihilated by the differential operator $Q_h^{[0]}(r\partial_r)$ thanks to (93). By Leibniz's formula, this means that the renormalized moments are linear combinations of the expressions

$$R^{k+\frac{\varepsilon-1}{2}} \frac{\partial^k Z_\varepsilon}{\partial r^k}(R,S), \quad 1 \leq k \leq h+1, \ \varepsilon \in \{0,1\}. \tag{99}$$

We now discuss the structure of these expressions.

C.2.3. Structure of derivatives of Z. To this end, we need to better understand the partial derivatives of R, S with respect to t_0, t_1. We recall that $R^{(k)}$ and $S^{(k)}$ correspond to the derivatives with respect to $t = t_0$.

Proposition C.4. *There exists a family of polynomials $P^{(a,b)}(x_k, y_k, 1 \leq k \leq a+b)$ such that, for any integers $a, b \geq 0$, we have*

$$\frac{\partial^{a+b} R}{\partial t_0^a \partial t_1^b} = R^{\frac{b}{2}+1} P^{(a,b)}\left(\frac{R^{(k)}}{R}, \frac{S^{(k)}}{\sqrt{R}}, 1 \leq k \leq a+b\right) \tag{100}$$

and, if moreover $(a,b) \neq (0,0)$,

$$\frac{\partial^{a+b} S}{\partial t_0^a \partial t_1^b} = R^{\frac{b+1}{2}} P^{(a+1,b-1)}\left(\frac{R^{(k)}}{R}, \frac{S^{(k)}}{\sqrt{R}}, 1 \leq k \leq a+b\right). \tag{101}$$

It is defined by induction as follows: we have $P^{(a,0)} = x_a$ *for every* $a \geq 0$*, with the convention that* $x_0 = 1$*,* $P^{(a+1,-1)} = y_a$ *for every* $a \geq 1$*, and, for every* $a, b \geq 0$*,*

$$P^{(a,b+1)} = \left(\left(\frac{b}{2}+1\right)y_1 + \sum_{k=1}^{a+b}\left(\sum_{i=0}^{k-1}\binom{k}{i}x_i y_{k+1-i}\partial_{x_k} + \left(x_{k+1} - \frac{y_1 y_k}{2}\right)\partial_{y_k}\right)\right)P^{(a,b)}. \tag{102}$$

Moreover, the polynomial $P^{(a,b)}(\bar{x}_k^k, \bar{y}_k^k, 1 \leq k \leq a+b)$ *in the variables* $\bar{x}_k, \bar{y}_k$ *is homogeneous of degree* $a+b$*.*

Proof. We first make the trivial observation that for $b = 0$, (100) holds with $P^{(a,0)} = x_a$ for $a \geq 0$, and (101) holds with $P^{(a+1,-1)} = y_a$ for $a \geq 1$. Moreover, by the combinatorial definition of R and S given in Section 1.2, the derivatives in (100) and (101) are of the form $T^{(0)}_{1,\ldots,1,0,\ldots,0}$ in both cases, with a terms equal to 0 and $b+2$ terms equal to 1 in the first case, and with $a+1$ terms equal to 0 and $b+1$ terms equal to 1 in the second case. This shows that, for $a \geq 0$ and $b \geq 1$, (101) is nothing but (100) applied to $a+1$ and $b-1$ instead of a, b. Furthermore, we deduce from this discussion and Theorem A.1 the identities

$$\frac{\partial R}{\partial t_1} = RS', \quad \frac{\partial S}{\partial t_1} = R', \tag{103}$$

which are consistent with the particular values $P^{(0,1)} = y_1$ and $P^{(1,0)} = x_1$.

In view of these preliminary remarks, it now suffices to establish (100) for general b, which we will do by induction. Assume that it holds at rank $b \geq 0$, and let us establish it at rank $b+1$. To this end, we write

$$\frac{\partial^{a+b+1}R}{\partial t_0^a \partial t_1^{b+1}} = \frac{\partial}{\partial t_1}\left(R^{\frac{b}{2}+1}P^{(a,b)}\left(\frac{R^{(k)}}{R}, \frac{S^{(k)}}{\sqrt{R}}, 1 \leq k \leq a+b\right)\right),$$

and we expand the right-hand side using the Leibniz and chain rules. The first term in the expansion, obtained by differentiating $R^{b/2+1}$, is equal to

$$\left(\frac{b}{2}+1\right)R^{\frac{b}{2}}\frac{\partial R}{\partial t_1}P^{(a,b)} = \left(\frac{b}{2}+1\right)R^{\frac{b+1}{2}+1}\frac{S'}{\sqrt{R}}P^{(a,b)}. \tag{104}$$

The term obtained by differentiating with respect to the variable $x_k = R^{(k)}/R$ in $P^{(a,b)}$ is equal to

$$R^{\frac{b}{2}+1}\partial_{x_k}P^{(a,b)}\frac{\partial}{\partial t_1}\left(\frac{R^{(k)}}{R}\right),$$

where we may write

$$\frac{\partial}{\partial t_1}\left(\frac{R^{(k)}}{R}\right) = \frac{\frac{\partial R^{(k)}}{\partial t_1}}{R} - \frac{\frac{\partial R}{\partial t_1}R^{(k)}}{R^2} = \frac{\frac{\partial^k (RS')}{\partial t_0^k}}{R} - \frac{S'R^{(k)}}{R}$$
$$= \sqrt{R}\sum_{i=0}^{k-1}\binom{k}{i}\frac{R^{(i)}}{R}\frac{S^{(k+1-i)}}{\sqrt{R}}$$

by (103) and Leibniz's formula. Finally, the term obtained by differentiating with respect to the variable $y_k = S^{(k)}/\sqrt{R}$ in $P^{(a,b)}$ is equal to

$$R^{\frac{b}{2}+1}\partial_{y_k} P^{(a,b)} \frac{\partial}{\partial t_1}\left(\frac{S^{(k)}}{\sqrt{R}}\right),$$

where we may write

$$\frac{\partial}{\partial t_1}\left(\frac{S^{(k)}}{\sqrt{R}}\right) = \frac{\frac{\partial S^{(k)}}{\partial t_1}}{\sqrt{R}} - \frac{\frac{\partial R}{\partial t_1}S^{(k)}}{2R^{3/2}} = \sqrt{R}\left(\frac{R^{(k+1)}}{R} - \frac{1}{2}\frac{S'}{\sqrt{R}}\frac{S^{(k)}}{\sqrt{R}}\right). \tag{105}$$

Putting (104)–(105) together, we obtain that (100) is valid for $(a, b+1)$, where $P^{(a,b+1)}$ is given by (102), as wanted. The statement about the degree and homogeneity of $P^{(a,b)}(\bar{x}_k^k, \bar{y}_k^k, 1 \le k \le a+b)$ is readily derived by induction, using (102). ■

At this point, we use the formula (107) for derivatives of inverses discussed in Appendix D, applied to $\mathbf{f} = \mathbf{U}$ and $\mathbf{g} = \mathbf{Z}$ (for $n = 2$ in the notation therein):

$$\frac{\partial^k Z_\varepsilon}{\partial r^k}(\mathbf{U}(\mathbf{t})) = \sum_{\tau\in\mathcal{T}_\varepsilon^{\mathbf{0}}(k)}\prod_{v\in\tau}(d_{\mathbf{t}}\mathbf{U})^{-1}_{i_v,j_v}\prod_{v\in\tau:k_v(\tau)>0}\frac{1}{k_v(\tau)!}\frac{\partial^{k_v(\tau)}U_{j_v}}{\partial t_{i_{v1}}\cdots\partial t_{i_{vk_v(\tau)}}}(\mathbf{t}), \tag{106}$$

where $\mathcal{T}_\varepsilon^{\mathbf{0}}(k)$ is the family of planted Pólya trees τ with k leaves, and where the bottom and top half-edges of the edge e_v below the vertex $v \in \tau$ are labeled by elements $i_v, j_v \in \{0, 1\}$, in such a way that $i_\varnothing = \varepsilon$, and $i_v = 0$ for every leaf $v \in \tau$.

Let us look more closely at the contribution of a given tree $\tau \in \mathcal{T}_\varepsilon^{\mathbf{0}}(k)$ to the preceding sum. Note that, by (61), we have

$$d_{\mathbf{t}}\mathbf{U} = \begin{pmatrix} \frac{\partial R}{\partial t_0} & \frac{\partial R}{\partial t_1} \\ \frac{\partial S}{\partial t_0} & \frac{\partial S}{\partial t_1}\end{pmatrix} = \begin{pmatrix} R' & RS' \\ S' & R'\end{pmatrix}.$$

We use the comatrix formula to compute

$$(d_{\mathbf{t}}\mathbf{U}^{-1})_{i,j} = R^{-1-\frac{i}{2}+\frac{j}{2}}\frac{\mathcal{D}(i,j)}{(\frac{R'}{R})^2 - (\frac{S'}{\sqrt{R}})^2},$$

where $\mathcal{D}(i,j)$ is R'/R if $i+j$ is even, and $-S'/\sqrt{R}$ otherwise. Hence, the contribution of edges of τ—the first product in (106)—is

$$\prod_{v\in\tau} R^{-1-\frac{i_v}{2}+\frac{j_v}{2}} \frac{\mathcal{D}(i_v,j_v)}{(\frac{R'}{R})^2-(\frac{S'}{\sqrt{R}})^2},$$

while the contribution of internal vertices – the second product – is, by Proposition C.4,

$$\prod_{v\in\tau:k_v(\tau)>0} \frac{R^{\frac{\sum_{l=1}^{k_v(\tau)} i_{vl}}{2}+1-\frac{j_v}{2}}}{k_v(\tau)!} P^{(\sum_{l=1}^{k_v(\tau)}(1-i_{vl})+j_v, \sum_{l=1}^{k_v(\tau)} i_{vl}-j_v)} \times \left(\frac{R^{(l)}}{R}, \frac{S^{(l)}}{\sqrt{R}}, 1\le l\le k_v(\tau)\right).$$

The exponents in R have many cancellations and result in a global exponent of

$$-k-i_\varnothing/2+\sum_{v\in\tau:k_v(\tau)=0} j_v/2=-k-\varepsilon/2,$$

since $\tau\in\mathcal{T}_\varepsilon^{\mathbf{0}}(k)$. The rest of the contribution of τ is a homogeneous polynomial of degree $2\#\tau-1$ (in the sense of Proposition C.4, that is, where the variables of index l are considered to have degree l) times $((R'/R)^2-(S'/\sqrt{R})^2)^{-\#\tau}$. Note that $\#\tau$ is maximal and equal to $2k-1$ when τ is a binary tree. Hence, we reduce the sum on the right-hand side of (106) to a common denominator by multiplying and dividing the contribution of τ by $((R'/R)^2-(S'/\sqrt{R})^2)^{2k-1-\#\tau}$, which is thus of the form

$$R^{-k-\varepsilon/2} \frac{\Pi^{\tau}_{\pm,k}(\frac{R^{(l)}}{R}, \frac{S^{(l)}}{\sqrt{R}}, 1\le l\le k)}{\left((\frac{R'}{R})^2-(\frac{S'}{\sqrt{R}})^2\right)^{2k-1}},$$

where $\Pi^{\tau}_{\pm,k}$ is a homogeneous polynomial of degree $4k-3$. After multiplying by R^k, summing over all possible $\tau\in\mathcal{T}_\varepsilon^{\mathbf{0}}(k)$, and applying (98) and the remark leading to (99), we finally obtain that

$$\bar{M}_{\pm,h}=R^{-1/2} \frac{\Pi_{\pm,h}(\frac{R^{(k)}}{R}, \frac{S^{(k)}}{\sqrt{R}}, 1\le k\le h+1)}{\left((\frac{R'}{R})^2-(\frac{S'}{\sqrt{R}})^2\right)^{2h+1}},$$

where $\Pi_{\pm,h}$ are homogeneous polynomials of degree $4h+1$. Coming back to the "dimensionless" quantities $R^{h/2}M_{\pm,h}=\bar{M}_{\pm,h}/(\sqrt{R}M_{\pm,0})$, those are, recalling (88),

$$R^{h/2}M_{\pm,h}=\left(\frac{R'}{R}\pm\frac{S'}{\sqrt{R}}\right) \frac{\Pi_{\pm,h}(\frac{R^{(k)}}{R}, \frac{S^{(k)}}{\sqrt{R}}, 1\le k\le h+1)}{\left((\frac{R'}{R})^2-(\frac{S'}{\sqrt{R}})^2\right)^{2h+1}}.$$

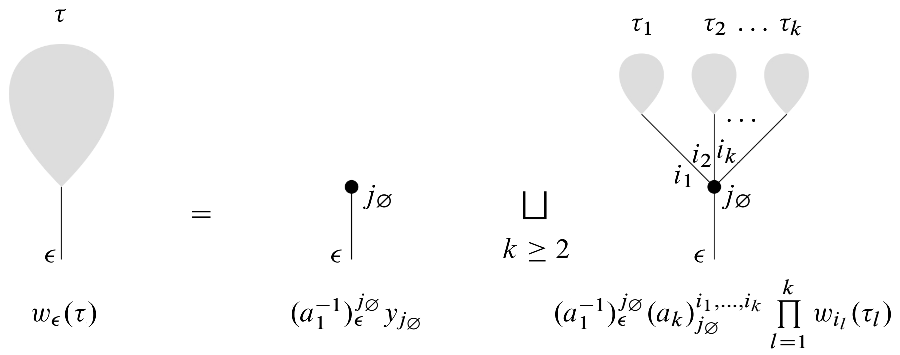

Figure 7. An illustration of the tree structure of derivatives of inverses.

Combining this with Theorem C.2, this concludes the proof that Proposition 4.8 holds for $g \geq 2$ and $n = 0$.

D. Higher-order differentials of inverse functions of several variables

Here, we provide a general discussion revisiting Lagrange inversion that is used in Section C.2.3. The results are probably known in some other form, the paper by Warren Johnson cited in [8] pointing as far as Sylvester.

Fix $n \geq 1$ and let $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$ be formal variables. We let $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))$ and $\mathbf{g}(\mathbf{y}) = (g_1(\mathbf{y}), \ldots, g_n(\mathbf{y}))$ be two families of n elements in $K[[\mathbf{x}]]$ and $K[[\mathbf{y}]]$, respectively, where K is some given field. We assume that $\mathbf{f}$ and $\mathbf{g}$ are compositional inverses, that is, $\mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{x}$ and $\mathbf{f}(\mathbf{g}(\mathbf{y})) = \mathbf{y}$. By classical consideration, given $\mathbf{f}$, the compositional inverse $\mathbf{g}$ exists if and only if $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ and the differential of $\mathbf{f}$ at $\mathbf{0}$ is invertible, which we assume.

Let us write the expansion

$$\mathbf{f}(\mathbf{x}) = a_1\mathbf{x} - a_2(\mathbf{x}, \mathbf{x}) - a_3(\mathbf{x}, \mathbf{x}, \mathbf{x}) - \cdots,$$

where $a_1, -a_2, -a_3, \ldots$ are, up to a multiplicative coefficient, the successive differentials of $\mathbf{f}$ at $\mathbf{0}$, which are, respectively, k-linear symmetric. Using tensor notations, we explicitly have

$$(a_1)_i^j = \frac{\partial f_i}{\partial x_j}(\mathbf{0}), \quad (a_k)_i^{j_1,\ldots,j_k} = -\frac{1}{k!}\frac{\partial^k f_i}{\partial x_{j_1} \cdots \partial x_{j_k}}(\mathbf{0}), \quad k \geq 2.$$

This yields, after substituting $\mathbf{g}(\mathbf{y})$ to $\mathbf{x}$,

$$\mathbf{g}(\mathbf{y}) = a_1^{-1}\mathbf{y} + a_1^{-1}a_2(\mathbf{g}(\mathbf{y}), \mathbf{g}(\mathbf{y})) + a_1^{-1}a_3(\mathbf{g}(\mathbf{y}), \mathbf{g}(\mathbf{y}), \mathbf{g}(\mathbf{y})) + \cdots.$$

This former expression has a natural tree interpretation, illustrated in Figure 7. We use the Ulam–Harris notation, where a tree τ is a subset of the set of integer words, rooted at the empty word $\varnothing$. Every vertex $v \in \tau$ has an arity (number of children) denoted by $k_v(\tau)$, and these children are the words $v1, v2, \ldots, vk_v(\tau)$. The trees considered are planted Pólya trees (with an extra edge pointing to the root vertex $\varnothing$), in which the arity $k_v(\tau)$ of every vertex $v \in \tau$ is an element of $\{0, 2, 3, 4, \ldots\}$. We let $\mathcal{T}$ be the family of such trees in which the bottom and top half-edges of the parent edge of v is decorated with two elements $i_v, j_v \in \{1, \ldots, n\}$. For a given $\varepsilon \in \{1, \ldots, n\}$, we also let $\mathcal{T}_\varepsilon$ be the set of those trees for which $i_\varnothing = \varepsilon$. Then,

$$g_\varepsilon(\mathbf{y}) = \sum_{\tau \in \mathcal{T}_\varepsilon} \prod_{v \in \tau} (a_1^{-1})_{i_v}^{j_v} (a_{k_v(\tau)})_{j_v}^{i_{v1}, i_{v2}, \ldots, i_{vk_v(\tau)}},$$

with the convention that $(a_0)_j^\varnothing = y_j$, that is, the contributions of the variable $\mathbf{y}$ come from the leaves of the trees. The partial derivatives of order k of g_ε come from the finite family $\mathcal{T}_\varepsilon(k)$ of elements of $\mathcal{T}_\varepsilon$ with k leaves. More precisely, for a given composition $\mathbf{c} = (c_1, \ldots, c_n)$ of k with n parts, one has

$$\frac{\partial^k g_\varepsilon}{\partial x_1^{c_1} \cdots \partial x_n^{c_n}}(\mathbf{0}) = \sum_{\tau \in \mathcal{T}_\varepsilon^{\mathbf{c}}} \prod_{v \in \tau} (a_1^{-1})_{i_v}^{j_v} \prod_{v \in \tau : k_v(\tau) > 0} (a_{k_v(\tau)})_{j_v}^{i_{v1}, i_{v2}, \ldots, i_{vk_v(\tau)}},$$

where $\mathcal{T}_\varepsilon^{\mathbf{c}}$ is the set of trees $\tau \in \mathcal{T}_\varepsilon(k)$ with c_i leaves v labeled by $j_v = i$, for $1 \le i \le n$.

By applying this reasoning to the series $\mathbf{f}(\mathbf{x} + \mathbf{x}') - \mathbf{f}(\mathbf{x})$ and to $\mathbf{g}(\mathbf{y} + \mathbf{y}') - \mathbf{g}(\mathbf{y})$ instead, we obtain

$$\frac{\partial^k g_\varepsilon}{\partial x_1^{c_1} \cdots \partial x_n^{c_n}}(\mathbf{f}(\mathbf{x})) = \sum_{\tau \in \mathcal{T}_\varepsilon^{\mathbf{c}}(k)} \prod_{v \in \tau} (d_{\mathbf{x}} \mathbf{f}^{-1})_{i_v}^{j_v} \prod_{v \in \tau : k_v(\tau) > 0} \frac{-1}{k_v(\tau)!} \frac{\partial^{k_v(\tau)} f_{j_v}}{\partial x_{i_{v1}} \cdots \partial x_{i_{vk_v(\tau)}}}(\mathbf{x}). \tag{107}$$

Acknowledgments. Many thanks are due to the anonymous referee for suggesting valuable improvements to this paper. JB acknowledges the hospitality of the Laboratoire de Physique of ENS de Lyon during the initial stages of this work. This work was completed while GM was holding a visiting professor position at the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

Funding. This work was supported by the Agence Nationale de la Recherche via the Grant nos. ANR-18-CE40-0033 “Dimers”, ANR-19-CE48-0011 “Combiné”, and ANR-23-CE48-0018 “CartesEtPlus”.

References

[1] J. Ambjørn, L. Chekhov, C. F. Kristjansen, and Y. Makeenko, Matrix model calculations beyond the spherical limit. *Nuclear Phys. B* **404** (1993), no. 1-2, 127–172 Zbl 1043.81636 MR 1232337

[2] J. Ambjørn, B. Durhuus, and T. Jonsson, *Quantum geometry. A statistical field theory approach*. Cambridge Monogr. Math. Phys., Cambridge University Press, Cambridge, 1997 Zbl 0993.82500 MR 1465433

[3] J. Bouttier, P. Di Francesco, and E. Guitter, Census of planar maps: From the one-matrix model solution to a combinatorial proof. *Nuclear Phys. B* **645** (2002), no. 3, 477–499 Zbl 0999.05052 MR 1938319

[4] J. Bouttier and E. Guitter, On irreducible maps and slices. *Combin. Probab. Comput.* **23** (2014), no. 6, 914–972 Zbl 1303.05086 MR 3265834

[5] J. Bouttier, E. Guitter, and G. Miermont, Bijective enumeration of planar bipartite maps with three tight boundaries, or how to slice pairs of pants. *Ann. H. Lebesgue* **5** (2022), 1035–1110 Zbl 1516.05043 MR 4526246

[6] J. Bouttier, E. Guitter, and G. Miermont, On quasi-polynomials counting planar tight maps. *Comb. Theory* **4** (2024), no. 1, article no. 12 Zbl 1547.05021 MR 4770591

[7] E. Brézin, C. Itzykson, G. Parisi, and J. B. Zuber, Planar diagrams. *Comm. Math. Phys.* **59** (1978), no. 1, 35–51 Zbl 0997.81548 MR 0471676

[8] T. Budd, Irreducible metric maps and Weil–Petersson volumes. *Comm. Math. Phys.* **394** (2022), no. 2, 887–917 Zbl 1495.05061 MR 4469409

[9] T. Budd and B. Zonneveld, Topological recursion of the Weil–Petersson volumes of hyperbolic surfaces with tight boundaries. *J. Math. Phys.* **65** (2024), no. 9, article no. 092302 Zbl 1548.57054 MR 4796626

[10] G. Collet and É. Fusy, A simple formula for the series of bipartite and quasi-bipartite maps with boundaries. In *24th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2012)*, pp. 607–618, Discrete Math. Theor. Comput. Sci. Proc., Association for Discrete Mathematics and Theoretical Computer Science, Nancy, 2012 Zbl 1412.05014 MR 2958033

[11] P. Di Francesco, P. Ginsparg, and J. Zinn-Justin, 2D gravity and random matrices. *Phys. Rep.* **254** (1995), no. 1-2, 1–133 MR 1320471

[12] B. Eynard, *Counting surfaces*. Prog. Math. Phys. 70, Birkhäuser/Springer, Cham, 2016 Zbl 1338.81005 MR 3468847

[13] B. Eynard and N. Orantin, Invariants of algebraic curves and topological expansion. *Commun. Number Theory Phys.* **1** (2007), no. 2, 347–452 Zbl 1161.14026 MR 2346575

[14] P. Flajolet and R. Sedgewick, *Analytic combinatorics*. Cambridge University Press, Cambridge, 2009 Zbl 1165.05001 MR 2483235

[15] I. P. Goulden and D. M. Jackson, *Combinatorial enumeration*. Dover, Mineola, NY, 2004 Zbl 1099.05005 MR 2079788

[16] J. Harer and D. Zagier, The Euler characteristic of the moduli space of curves. *Invent. Math.* **85** (1986), no. 3, 457–485 Zbl 0616.14017 MR 0848681

[17] P. Norbury, Counting lattice points in the moduli space of curves. *Math. Res. Lett.* **17** (2010), no. 3, 467–481 Zbl 1225.32023 MR 2653682

[18] P. Norbury and N. Scott, Polynomials representing Eynard–Orantin invariants. *Q. J. Math.* **64** (2013), no. 2, 515–546 Zbl 1269.30047 MR 3063520

[19] P. Saad, S. H. Shenker, and D. Stanford, JT gravity as a matrix integral. 2019, arXiv:1903.11115v1

[20] G. Schaeffer, Planar maps. In *Handbook of enumerative combinatorics*, pp. 335–395, Discrete Math. Appl. (Boca Raton), CRC Press, Boca Raton, FL, 2015 Zbl 1326.05036 MR 3409346

[21] R. P. Stanley, *Enumerative combinatorics. Volume 1*. 2nd edn., Cambridge Stud. Adv. Math. 49, Cambridge University Press, Cambridge, 2012 Zbl 1247.05003 MR 2868112

[22] W. T. Tutte, A census of planar maps. *Canadian J. Math.* **15** (1963), 249–271 Zbl 0115.17305 MR 0146823

[23] G. X. Viennot, Heaps of pieces. I. Basic definitions and combinatorial lemmas. In *Combinatoire énumérative (Montreal, Que., 1985/Quebec, Que., 1985)*, pp. 321–350, Lecture Notes in Math. 1234, Springer, Berlin, 1986 Zbl 0618.05008 MR 0927773

[24] E. Witten, Two-dimensional gravity and intersection theory on moduli space. In *Surveys in differential geometry (Cambridge, MA, 1990)*, pp. 243–310, Lehigh University, Bethlehem, PA, 1991 Zbl 0757.53049 MR 1144529

[25] N. Zhukovskii, Über die Konturen der Tragflächen der Drachenflieger. *Zeitschrift für Flugtechnik und Motorluftschiffahrt* **1** (1910), 281–284

Communicated by Gaëtan Borot

Received 5 July 2024; revised 24 September 2025.

Jérémie Bouttier
Sorbonne Université and Université Paris Cité, CNRS, IMJ-PRG, 75005 Paris; Université Paris-Saclay, CNRS, CEA, Institut de Physique Théorique, 91191 Gif-sur-Yvette, France; jeremie.bouttier@imj-prg.fr

Emmanuel Guitter
Université Paris-Saclay, CNRS, CEA, Institut de Physique Théorique, 91191 Gif-sur-Yvette, France; emmanuel.guitter@ipht.fr

Grégory Miermont
École Normale Supérieure de Lyon, UMPA, CNRS UMR 5669, 46 allée d'Italie, 69364 Lyon Cedex 07; Institut Universitaire de France, 103 boulevard Saint-Michel, 75005 Paris, France; gregory.miermont@ens-lyon.fr