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Hurwitz numbers for reflection groups $G(m, 1, n)$

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Abstract. We build a parallel theory of simple Hurwitz numbers for the reflection groups $G(m, 1, n)$. We study analogs of the cut-and-join operators. An algebraic description as well as a description of Hurwitz numbers in terms of ramified coverings is provided. An explicit formula for them in terms of Schur polynomials is given. In addition, the generating function of $G(m, 1, n)$-Hurwitz numbers is shown to give rise to an m independent-variables τ-function of the KP hierarchy. Finally, we provide an ELSV-type formula for these new Hurwitz numbers. These results extend the results of Fesler (2023).

1. Introduction

Hurwitz numbers were first introduced at the end of the 19th century in the work of A. Hurwitz [12]. Simple Hurwitz numbers $h_{\lambda,m}$ for a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and a non-negative integer m are defined as the number divided by $n!$ of sequences of m transpositions $(\sigma_1, \ldots, \sigma_m)$ such that their product $\sigma_1 \circ \cdots \circ \sigma_m$ belongs to a conjugacy class C_λ in the symmetric group S_n. Hurwitz showed in [12] that equivalently $h_{m,\lambda}$ counts isomorphism classes of ramified coverings of the sphere, with m simple critical values and an extra critical value which has a fixed ramification profile λ. During the last decades, Hurwitz numbers received a lot of attention due to numerous arising applications and connections, as well as the richness of the underlying algebraic theory. I. Goulden and D. Jackson showed that the generating function of Hurwitz numbers satisfies a second-order parabolic PDE called the cut-and-join equation [10]. A. Okounkov showed in [21] that the generating function of Hurwitz numbers is also a τ-function of the KP hierarchy, implying that Hurwitz numbers can be written in terms of Schur polynomials [15]. A striking result was proved in the seminal 2001 paper [7] by T. Ekedahl, S. Lando, M. Shapiro, and A. Vainshtein (and called the ELSV formula since then) built a somewhat unexpected bridge by revealing deep connections of Hurwitz numbers with the geometry of moduli space of complex curves. (Namely, the Hurwitz numbers are the intersection constants of certain characteristic classes.)

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Since then, many variations of Hurwitz numbers have been introduced [2, 9, 18], with the most general version being given by weighted Hurwitz numbers [1]. These generalizations satisfy similar results as the ones for simple Hurwitz numbers. In this article, we are interested in Hurwitz numbers for the family of reflection groups $G(m,1,n)$.

In the last few years, Hurwitz numbers for complex reflection groups have gained more attention, for example, in the work of E. Polak and D. Ross [22], where polynomiality results are studied, the trilogy of papers [4–6] by T. Douvropoulos, J. B. Lewis, and A. H. Morales where a formula for the genus-0 Hurwitz numbers for the complex reflection groups is obtained, and indirectly in the works of P. Johnson in [13] and H. Zhang and J. Zhou [25], where a representation theory approach was used to study wreath product Hurwitz numbers. Our approach, nevertheless, differs from the works quoted above. The present paper can be seen as a generalization of the paper [8] written by the first author, applying similar techniques to the groups $G(m,1,n)$ and showing that the theory of disconnected simple Hurwitz numbers for the complex reflection groups $G(m,1,n)$ is parallel to the classical one.

In Section 2 where we introduce an embedding of $G(m,1,n)$ into S_{mn}, the reflections are sent to the product of m transpositions and power of cycles commuting with the permutation

$$\tau \stackrel{\text{def}}{=} (1\ n+1\ 2n+1 \cdots (m-1)n+1) \cdots (n\ n+n\ 2n+n \cdots (m-1)n+n).$$

We then recall the isomorphism between $G(m,1,n)$ and $\mathbb{Z}/m\mathbb{Z} \wr S_n$. Further, we introduce a convenient description of the conjugacy classes of $G(m,1,n)$, which are indexed by m-tuples of partition

$$\overline{\lambda} = (\lambda_1, \ldots, \lambda_{m-1})$$

(that we call colored partition), and show that this description agrees with the classical description of conjugacy classes (under a well-chosen isomorphism) of $\mathbb{Z}/m\mathbb{Z} \wr S_n$ as in I. Macdonald [19].

In Section 3, we define Hurwitz numbers for the reflection groups $G(m,1,n)$. It is somewhat similar to the classical case where m different types of reflections are used instead of transpositions.

Section 4 is devoted to the description of the cut-and-join operator and its application. We first describe in Theorem 4.3 how multiplying by a reflection affects the conjugacy classes. Then, using the tools introduced in Sections 2 and 3, we show in Proposition 4.6 that, after a well-chosen change of variable, the cut-and-join for the complex reflection group $G(m,1,n)$ is actually a direct sum of m independent classical cut-and-join and $m-1$ sums of Euler fields. This allows us to describe the Hurwitz numbers of the complex reflection group $G(m, 1, n)$ in terms of Schur polynomials (Theorem 4.7).

In Section 5, Theorem 5.3, we show that the generating function of Hurwitz numbers for the reflection group $G(m, 1, n)$ is an m-parameter family of τ-functions of the KP hierarchy, independently in m variables.

Section 6 uses the results from [25] and describes in terms of ramified covering the Hurwitz numbers for reflection group $G(m, 1, n)$, These are compositions of ramified coverings (Proposition 6.4).

Finally, in Section 7, Theorem 7.3 provides an ELSV formula type for the logarithm of the generating function. It is proved using the simple form of the cut-and-join after the change of variables.

To conclude this introduction, it is worth noting that the nice description of Hurwitz numbers for the complex reflection group $G(m, 1, n)$ in terms of Schur polynomials (Theorem 4.7) motivates to look at weighted generalization [11]; this will be done in a subsequent work. In addition, it is worth applying the tools of this article to the complex reflection groups $G(m, p, n)$; this will also be done in some future work.

2. Complex reflection groups $G(m, 1, n)$

2.1. Definitions and embedding into S_{mn}

Let $V = \mathbb{C}^n$ be a complex vector space of dimension n equipped with the standard Hermitian inner product. A *reflection* in V is a unitary automorphism of V of finite order with exactly $n - 1$ eigenvalues equal to 1. A finite subgroup $W \subset GL(V)$ is called a *complex reflection group* if it is generated by reflections.

Let $G(m, 1, n)$ be the complex reflection group described in matrix terms as the $n \times n$ monomial matrices whose non-zero entries are m-th roots of unity. The reflection elements of this group are

- reflections $R_{ij}^{(\alpha)}$ of order 2, which fix the hyperplane $x_j = \exp(2i\pi/m)^\alpha x_i$ for $1 \le i < j \le n$ and $\alpha = 0, 1, \ldots, m - 1$,
- reflections L_i^α of order dividing m, which fix the hyperplane $x_i = 0$ for $i = 1, \ldots, n$ and $\alpha = 1, \ldots, m - 1$.

The reflections $\langle L_1^1, R_{1,2}^{(0)}, \ldots, R_{n,n-1}^{(0)} \rangle$ generate the group $G(m, 1, n)$; see [17] for details.

We define now an embedding of $G(m, 1, n)$ into S_{mn}. Let the permutation of order m be

$$\begin{aligned}\tau \stackrel{\text{def}}{=} {} & (1\ n+1\ 2n+1 \cdots (m-1)n+1)(2\ n+2\ 2n+2 \cdots (m-1)n+2) \\ & \cdots (n\ n+n\ 2n+n\ \cdots (m-1)n+n).\end{aligned}$$

Proposition 2.1. *There exists an embedding $\Psi : G(m,1,n) \hookrightarrow S_{mn}$ such that*

$$\Psi(R_{ij}^{(\alpha)}) = (i\ \tau^{\alpha}(j))(\tau(i)\ \tau^{\alpha+1}(j))\cdots(\tau^{m-1}(i)\ \tau^{\alpha-1}(j))$$

for $\alpha = 0,\dots,m-1$ and $\Psi(L_i^{\alpha}) = (i\ \tau(i)\cdots\tau^{m-1}(i))^{\alpha}$, where the power of α means multiplication of the permutation by itself α times. The image of Ψ is the normalizer of τ:

$$\Psi(G(m,1,n)) = \{\sigma \in S_{mn} \mid \sigma\tau = \tau\sigma\} \overset{\text{def}}{=} \mathrm{Norm}(\tau).$$

Remark 2.2. It is also clear that if $\sigma \in \mathrm{Norm}(\tau)$, then $\tau^i\sigma(\tau^{-1})^i = \sigma$ for $i = 1,\dots,m$.

Remark 2.3. Another way to define this embedding is by considering the faithful action of $G(m,1,n)$ on the nm unit vectors $\mathbf{v}_{j,k} = (0,\dots,0,\exp(2k\pi i/m),0,\dots,0)$, where the non-zero element is in position j and k runs from 0 to $m-1$. This faithful action induces an embedding in S_{mn} given the identification

$$\mathbf{v}_{j,k} \mapsto j + nk$$

with the set $\{1,\dots,mn\}$ on which S_{mn} is acting.

To prove Proposition 2.1, we describe the cycle decomposition of elements in $\mathrm{Norm}(\tau)$.

Introduce a notation more convenient for our purposes:

$$r_{ij}^{(\alpha)} \overset{\text{def}}{=} (i\ \tau^{\alpha}(j))(\tau(i)\ \tau^{\alpha+1}(j))\cdots(\tau^{m-1}(i)\ \tau^{\alpha-1}(j)) \tag{2.1}$$

for all $1 \le i \ne j \le mn$, $i \not\equiv j \mod n$. So, if $1 \le i < j \le n$, then

$$r_{ij}^{(\alpha)} = r_{\tau(i),\tau(j)}^{(\alpha)} = \cdots = r_{\tau^{m-1}(i),\tau^{m-1}(j)}^{(\alpha)} = \Psi(R_{ij}^{(\alpha)}).$$

Also, denote

$$l_i^{\alpha} \overset{\text{def}}{=} (i\ \tau(i)\cdots\tau^{m-1}(i))^{\alpha} \quad \text{for all } 1 \le i \le mn; \tag{2.2}$$

thus $l_i^{\alpha} = l_{\tau(i)}^{\alpha} = \cdots = l_{\tau^{m-1}(i)}^{\alpha} = \Psi(L_i^{\alpha})$ if $i \le n$.

In the following proposition, the power of τ is considered modulo m; denote also the integers

$$k_{\alpha} = \frac{m}{\gcd(m,\alpha)}.$$

Proposition 2.4. *Let $x \in \mathrm{Norm}(\tau)$, and let $x = a_1\cdots a_r$ be its cycle decomposition. Then, for every a_j, $j = 1\cdots r$, one of the following is true:*

(1) *the cycle decomposition contains $m-1$ other cycles $a_{n_1},\dots,a_{n_{m-1}}$ of the same length such that*

$$a_j = \tau a_{n_1}\tau^{-1} = \tau^2 a_{n_2}\tau^{-2} = \cdots = \tau^{m-1}a_{n_{m-1}}\tau^{-(m-1)},$$

or

(2) *for some* $1 \le \alpha \le m-1$, *the cycle* a_j *has the following structure: its length is a multiple of* k_α, *it is* τ^α *invariant, and there are* $\gcd(m, \alpha) - 1$ *other cycles* $a_{j_2}, \ldots, a_{j_{\gcd(m,\alpha)}}$ *having the same length such that* $a_j = \tau a_{j_2} \tau^{-1} = \tau^2 a_{j_3} \tau^{-2} = \cdots = \tau^{\gcd(m,\alpha)-1} a_{\gcd(m,\alpha)} \tau^{-(\gcd(m,\alpha)-1)}$.

The first case will be called a β_0-*cycle*, and the second β-*cycles of type* α and denoted by β_α-cycle for short. The β_0-cycle will play a special role later on in the text; this is why they are extracted from the case (2), where they can be seen as a β_α-cycle with $\alpha = 0$.

Example 2.5. In $\mathbb{Z}/6\mathbb{Z} \wr S_n$,

$$\tau = (1\ n+1 \cdots 5n+1) \cdots (n\ 2n \cdots 6n),$$

and let $a \stackrel{\text{def}}{=} (a_1 \cdots a_k)$ such that $a_i \neq \tau^\alpha(a_j)$ for $i, j = 1, \ldots, k$ and $\alpha = 0, \ldots, 5$; then, we have the following possible β_α-cycles:

- β_0-cycle: $(a)(\tau(a)) \cdots (\tau^5(a))$,
- β_1-cycle: $(a, \tau(a), \tau^2(a), \ldots, \tau^5(a))$,
- β_2-cycle: $(a, \tau^2(a), \tau^4(a))(\tau(a), \tau^3(a), \tau^5(a))$,
- β_3-cycle: $(a, \tau^3(a))(\tau(a), \tau^4(a))(\tau^2(a), \tau^5(a))$,
- β_4-cycle: $(a, \tau^4(a), \tau^2(a))(\tau(a), \tau^5(a), \tau^3(a))$,
- β_5-cycle: $(a, \tau^5(a), \tau^4(a), \ldots, \tau(a))$,

where, for example, in the β_2-cycle, we have that $k_\alpha = k_2 = 3$.

Proof. Let $x = a_1 a_2 \cdots a_r$ be the cycle of decomposition of x, where

$$a_i = (a_{1i} \cdots a_{1m_i}).$$

Given that $\tau^\alpha x \tau^{-\alpha} = x$ for $\alpha = 1, \ldots, m$ by the definition of $\mathrm{Norm}(\tau)$, this implies

$$\begin{aligned} x &= \tau(a_1) \cdots \tau(a_r), \\ x &= \tau^2(a_1) \cdots \tau^2(a_r), \\ &\vdots \\ x &= \tau^{m-1}(a_1) \cdots \tau^{m-1}(a_r). \end{aligned} \tag{2.3}$$

By the uniqueness of the cycle decomposition, every $\tau(a_j), \tau^2(a_j), \ldots, \tau^{m-1}(a_j)$ must be equal to some a_s. If all these cycles are different, they form a β_0 cycle. Let now $1 \le \alpha < m$; then, $\tau^\alpha(a_s) = (\tau^\alpha(a_{1s}) \cdots \tau^\alpha(a_{1m_s}))$, and one has that

$$\tau^\alpha(a_s) \neq \tau^{2\alpha}(a_s) \neq \cdots \neq \tau^{k_\alpha}(a_s).$$

By definition of k_α, we have $\tau^\alpha(\tau^{k_\alpha}(a_s)) = a_s$ as $k_\alpha + \alpha = 0 \mod m$, implying that we have a cycle of the form $a = (a_s, \tau^\alpha(a_s), \tau^{2\alpha}(a_s), \dots, \tau^{k_\alpha}(a_s))$. Furthermore, the condition given in (2.3) and the uniqueness of cycle decomposition enforce having $\gcd(m,\alpha) - 1$ other cycles $a_{j_2}, \dots, a_{j_{\gcd(m,\alpha)}}$ having the same length such that

$$a = \tau a_{j_2}\tau^{-1} = \tau^2 a_{j_3}\tau^{-2} = \cdots = \tau^{\gcd(m,\alpha)-1} a_{\gcd(m,\alpha)}\tau^{-(\gcd(m,\alpha)-1)},$$

and this finishes the proof. ∎

Proof of Proposition 2.1. By Remark 2.3, the map $\Psi : G(m,1,n) \mapsto S_{mn}$ is a group homomorphism and an embedding. We write $\Psi(x) = \tilde{d}_1 \cdots \tilde{d}_N$, where $\tilde{d}_i$ is the image of the reflection d_i under Ψ.

The elements $\Psi(r_{ij}^{(\alpha)})$ and $\Psi(\ell_i^\gamma)$ commute with τ, implying that $\Psi(G(m,1,n)) \subset \mathrm{Norm}(\tau)$.

We are left to show that $\mathrm{Norm}(\tau) \subset \Psi(G(m,1,n))$. To do so, we prove that any $x \in \mathrm{Norm}(\tau)$, which is a product of β_0-cycle and β_α-cycles, can be written in terms of product of $r_{ij}^{(k)}$ and ℓ_i^k.

For a β_0-cycle, with the multiplication from left to right, we have

$$(a_1 \cdots a_j)(\tau(a_1)\cdots\tau(a_j))\cdots(\tau^{m-1}(a_1)\cdots\tau^{m-1}(a_j)) = r^{(0)}_{a_{j-1},a_j}\cdots r^{(0)}_{a_2,a_3} r^{(0)}_{a_1,a_2}.$$

Now, a straightforward computation shows that if $x \in \beta_\alpha$, then $x\ell^\gamma_{a_j} \in \beta_{\alpha+\gamma}$ (see also Section 4.2.2). It follows that, for a β_α-cycle with $a = (a_1, \dots, a_j)$ and $a_i \neq \tau^k(a_\ell)$ for all i, ℓ, k,

$$\begin{aligned}&(a, \tau^\alpha(a), \tau^{2\alpha}(a), \dots, \tau^{(k_\alpha-1)\alpha}(a))(\tau(a), \tau^{\alpha+1}(a), \tau^{2\alpha+1}(a), \dots, \tau^{(k_\alpha-1)\alpha+1}(a))\\ &\cdots(a_j\ \tau(a_j)\ \tau^2(a_j)\ \cdots\ \tau^{m-1}(a_j))^\gamma = r^{(0)}_{a_{j-1},a_j}\cdots r^{(0)}_{a_2,a_3} r^{(0)}_{a_1,a_2}\ell^\gamma_{a_j}\end{aligned}$$

as $r^{(0)}_{a_{j-1},a_j}\cdots r^{(0)}_{a_2,a_3} r^{(0)}_{a_1,a_2} \in \beta_0$. ∎

From now on, and by abuse of notation, we denote by $G(m,1,n)$ both the image $\Psi(G(m,1,n)) \subset S_{mn}$ and $G(m,1,n)$ itself.

2.2. Wreath product, correspondence, and conjugacy classes

There is a natural way to represent the elements of $G(m,1,n)$ in terms of a wreath product [4–6, 24]. Namely, given an element z in $G(m,1,n)$, one can encode it by a pair $[u; g]$ with $u \in S_n$ and $g = (g_1, \dots, g_n) \in (\mathbb{Z}/m\mathbb{Z})^n$ as follows: for $k = 1, \dots, n$, the non-zero entry in column k of z is in row $u(k)$, and the value of the entry is $\exp(\frac{2i\pi g_k}{m})$. The product rule reads

$$[u; g]\cdot[v; b] = [uv; v(g) + b], \quad \text{where } v(g) = (g_{v(1)}, \dots, g_{v(n)}).$$

The description above provides $G(m,1,n)$ with the structure of the wreath product $\mathbb{Z}/m\mathbb{Z} \wr S_n$. The reflections $r_{ij}^{(\alpha)}$ and ℓ_i^α are represented as follows:

$$\begin{aligned} r_{ij}^{(\alpha)} &= [(i,j);0,\dots,0,\alpha,0,\dots,0,-\alpha,0,\dots,0], \\ \ell_i^\alpha &= [(id);0,\dots,0,\alpha,0,\dots,0]. \end{aligned}$$

It is convenient to introduce an action of $\mathbb{Z}/m\mathbb{Z} \wr S_n$ on the set $\{1,\dots,mn\}$ in the following fashion: the element $[\sigma,(g_1,\dots,g_n)]$ acts by permuting the n subsets $\{1+in\}_{i\in\mathbb{Z}/m\mathbb{Z}},\dots,\{n+in\}_{i\in\mathbb{Z}/m\mathbb{Z}}$ in accordance with σ, and inside each subset $\{i, i+n,\dots,i+(m-1)n\}$, the action is described by shifting the elements by g_i positions. This induces a faithful action and, therefore, induces an embedding $\mathbb{Z}/m\mathbb{Z} \wr S_n \hookrightarrow S_{mn}$ (compare with Remark 2.3).

Elements of $\mathbb{Z}/m\mathbb{Z} \wr S_n$ correspond to permutations on the mn elements that preserve the circles (possibly permuting them entirely) with the order of the elements in each individual circle. The reflection $r_{ij}^{(\alpha)}$ would correspond to rotating the i-th circle by α, rotating the j-th circle by $-\alpha$, and permuting them, whereas the reflection l_i^α just rotates the i-th circle by α.

The conjugacy classes of the symmetric group S_n are in one-to-one correspondence with partitions of n, provided by cycle decomposition of the elements. A similar conjugacy class description for $\mathbb{Z}/m\mathbb{Z} \wr S_n$ can be found in [19]. We recall it here for completeness following closely [19] and then translate it into the language developed in the previous section.

Denote by Λ the set of all partitions:

$$\Lambda = \bigcup_{i\geq 0}\{\lambda \mid \lambda \vdash i\}.$$

An *m-colored partition* is a map $\mathbb{Z}/m\mathbb{Z} \to \Lambda$. A pair $p = [u;g]$ with $u \in S_n$ and $g = (g_1,\dots,g_n) \in (\mathbb{Z}/m\mathbb{Z})^n$ defines an m-colored partition as follows: the image of the element $r \in \mathbb{Z}/m\mathbb{Z}$ is a partition formed by lengths of those cycles $(i_1\cdots i_k)$ of $u \in S_n$ such that their *cycle-product* $g_{i_1}+\cdots+g_{i_k}$ equals r. The constructed m-colored partition determines the conjugacy class of $[u,g]$, and two pairs are of the same conjugacy classes if the corresponding colored partitions coincide.

We adopt the following notation for m-colored partitions: for an m-tuple of partitions $\lambda_0,\dots,\lambda_{m-1} \in \Lambda$, write $\overline{\lambda} = \lambda_0|\cdots|\lambda_{m-1}$ for the map

$$\overline{\lambda}\colon \begin{aligned} \mathbb{Z}/m\mathbb{Z} &\to \Lambda \\ r &\mapsto \lambda_r. \end{aligned}$$

We now show that the sets $C_{\overline{\lambda}} = C_{\lambda_0|\cdots|\lambda_{m-1}}$ of elements in $G(m,1,n) \subset S_{mn}$ such that their cycle decomposition contains β_α-cycles with cycle structure λ_α for $\alpha = 0,\dots,m-1$ are the conjugacy classes of $\mathbb{Z}/m\mathbb{Z} \wr S_n$.

We will show this using the previously discussed embedding of $\mathbb{Z}/m\mathbb{Z} \wr S_n$ into S_{mn}. Explicitly, it can be described as follows:

$$[(g_1, \dots, g_n), \left(\begin{smallmatrix} 1 & 2 & 3 & \cdots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \cdots & \sigma(n) \end{smallmatrix}\right)] \mapsto$$
$$\mapsto \left(\begin{smallmatrix} 1 & 2 & 3 & \cdots & n & \cdots & kn+1 & \cdots\cdots & (k+1)n & \cdots \\ \sigma(1)+g_1 n & \sigma(2)+g_2 n & \sigma(3)+g_3 n & \cdots & \sigma(n)+g_n n & \cdots\cdots & (\sigma(1)+kn)+g_1 n & \cdots & (\sigma(n)+kn)+g_n n & \cdots \end{smallmatrix}\right),$$

where the images of the permutation in S_{mn} are considered modulo mn.

Then, let us compute the image of the element $[(g_1, \dots, g_n), (i_1\ i_2\ \cdots\ i_k)] \in \mathbb{Z}/m\mathbb{Z} \wr S_n$ in S_{mn} for an individual cycle $\sigma = (i_1\ i_2\ \cdots\ i_k)$ in S_n. If we try to recover the cycle structure, we will get the following:

$$\begin{aligned} i_1 &\mapsto i_2 + g_{i_1} n \mod mn, \\ i_2 + g_{i_1} n \mod mn &\mapsto (i_3 + g_{i_2} n) + g_{i_1} n \mod mn, \\ &\dots \\ i_k + (g_{i_1} + g_{i_2} + \cdots + g_{i_{k-1}}) n \mod mn &\mapsto i_1 + (g_{i_1} + a_{i_2} + \cdots + g_{i_k}) n \mod mn. \end{aligned}$$

This proceeds until, after repeating this process some l times, $l(g_{i_1} + g_{i_2} + \cdots + g_{i_k})$ becomes divisible by m, and we return back to g_1 completing the cycle. If $l = m$, then the image is this one cycle, and this corresponds to the β_0 cycles in notations from the previous section. However, if $l|m$, then the image will contain m/l cycles received from the constructed one by adding n to all the elements. The resulting permutation type depends on k and $(g_{i_1} + g_{i_2} + \cdots + g_{i_k}) \bmod m$, which is exactly the input from the colored partition. Moreover, $g_1 + \cdots + g_k \bmod m$ is exactly the power α of τ which defines the β_α-cycle.

This finishes the proof of the following proposition.

Proposition 2.6. *The set $C_{\overline{\lambda}}$ is a conjugacy class. Every conjugacy class in $G(m,1,n)$ is $C_{\overline{\lambda}}$ for some partitions $\lambda_0, \dots, \lambda_{m-1}$ such that $|\lambda_0| + \cdots + |\lambda_{m-1}| = n$.*

In particular, the reflections $r_{i,j}^{(k)}$ form the conjugacy class $C_{2,1^{n-2}|\emptyset|\cdots|\emptyset}$ and the reflections ℓ_i^k the conjugacy classes $C_{1^{n-1}|\emptyset|\cdots|1|\emptyset|\cdots}$, where 1 is in position k.

We denote by $\mathcal{R}$ the conjugacy class $C_{2,1^{n-2}|\emptyset|\cdots|\emptyset}$ and by $\mathcal{L}^k$ the conjugacy class $C_{1^{n-1}|\emptyset|\cdots|1|\emptyset|\cdots}$.

3. Hurwitz numbers

Let us consider the group $G(m,1,n)$ for some fixed n and m and a conjugacy class in it given by a colored partition $\overline{\lambda}$ such that $|\lambda_0| + \cdots + |\lambda_{m-1}| = n$.

Definition 3.1. Given that $M = n_0 + n_1 + \cdots + n_{m-1}$, a sequence of reflections $(\sigma_1, \dots, \sigma_M)$ of the group $G(m,1,n)$ is said to have *profile* $(\overline{\lambda}, n_0, n_1, \dots, n_{m-1})$

if $\#\{p \mid \sigma_p \in \mathcal{R}\} = n_0$, $\#\{p \mid \sigma_p \in \mathcal{L}^1\} = n_1, \dots, \#\{p \mid \sigma_p \in \mathcal{L}^{m-1}\} = n_{m-1}$ and $\sigma_1 \cdots \sigma_M \in C_{\overline{\lambda}}$. The *Hurwitz numbers for the group* $G(m,1,n)$ $h_{n_0,n_1,\dots,n_{m-1}\overline{\lambda}}$ count the (normalized) number of sequences of reflections with the given profile $\overline{\lambda}$:

$$h_{n_0,n_1,\dots,n_{m-1}\overline{\lambda}} = \frac{1}{m^n n!}\#\{(\sigma_1,\dots,\sigma_M) \text{ is a sequence of profile } (\overline{\lambda}, n_0, n_1, \dots, n_{m-1})\}.$$

Denote $\mathcal{C}_{\overline{\lambda}} \stackrel{\text{def}}{=} \frac{1}{\#C_{\overline{\lambda}}} \sum_{x \in C_{\overline{\lambda}}} x \in \mathbb{C}[G(m,1,n)]$. By Proposition 2.6, the conjugacy classes are sets $C_{\overline{\lambda}}$ for various $\overline{\lambda}$, so the elements $\mathcal{C}_{\overline{\lambda}}$ form a basis in the center of the group algebra $Z[G(m,1,n)]$. Consider now a ring of polynomials

$$\mathbb{C}[\mathbf{p}] \stackrel{\text{def}}{=} \mathbb{C}[p^{(0)}, p^{(1)}, \dots, p^{(m-1)}],$$

where $p^{(i)} = (\frac{p_1^{(i)}}{m}, \frac{p_2^{(i)}}{m^2}, \dots)$,[1] (by abuse of notation, we drop the factors in front of the variables and write $(p_1^{(i)}, p_2^{(i)}, \dots)$ instead) are m infinite sets of variables. The ring is graded by the total degree where one assumes $\deg p_k^{(i)} = k$ for all $k = 1, 2, \dots$. The linear map Θ defined by

$$\Theta(\mathcal{C}_{\overline{\lambda}}) = \mathbf{p}_{\overline{\lambda}} \stackrel{\text{def}}{=} p_{\lambda_0}^{(0)} p_{\lambda_1}^{(1)}, \dots, p_{\lambda_{m-1}}^{(m-1)}$$

establishes an isomorphism between the vector spaces $Z[G(m,1,n)]$ and the homogeneous component $\mathbb{C}[p^{(0)}, p^{(1)}, \dots, p^{(m-1)}]_n$ of total degree n.

Now, denote by

$$\mathcal{T}_0 := \frac{1}{m} \sum_{\substack{1 \le i < j \le mn \\ 0 \le k \le m-1}} r_{ij}^{(k)} = \#\mathcal{R} \cdot \mathcal{C}_{1^{n-2}2|\emptyset|\cdots\emptyset}$$

and for all $k = 1, \dots, m-1$

$$\mathcal{T}_k := \frac{1}{m} \sum_{1 \le i \le mn} l_i^k = \#\mathcal{L}^k \cdot \mathcal{C}_{1^{n-1}|\emptyset|\cdots|1|\emptyset|\cdots|\emptyset}$$

sums of all elements of the conjugacy classes containing reflections; see equations (2.1) and (2.2) for the definitions of $r_{ij}^{(\alpha)}$ and ℓ_i^k. Now,

$$r_{ij}^{(k)} = r_{\tau(i),\tau(j)}^{(k)} = \cdots = r_{\tau^{m-1}(i),\tau^{m-1}(j)}^{(k)},$$

and $l_i^k = l_{\tau(i)}^k = \cdots = l_{\tau^{m-1}(i)}^k$, so every element is repeated m times in these sums, so is the factor $\frac{1}{m}$. The elements $\mathcal{T}_0$ and $\mathcal{T}_k$ belong to $Z[G(m,1,n)]$, so one can consider

[1] These extra factors $\frac{1}{m^i}$ which seem unnecessary are present to get nicer formula, especially in Corollary 4.2.

linear operators $T_0, T_1, \ldots, T_{m-1} : Z[G(m,1,n)] \to Z[G(m,1,n)]$ of multiplication by $\mathcal{T}_0, \ldots, \mathcal{T}_{m-1}$, respectively. The operators T_i and T_j pairwise commute for all $i, j = 0, \ldots, m-1$.

Let now the family of m operators $\mathcal{C}\mathcal{J}_0, \ldots, \mathcal{C}\mathcal{J}_{m-1}$ be the operators making the following diagrams commutative:

$$\begin{array}{ccc} Z[G(m,1,n)] & \xrightarrow{\times T_i} & Z[G(m,1,n)] \\ \Big\downarrow \simeq & & \Big\downarrow \simeq \\ \mathbb{C}[\mathbf{p}]_n & \xrightarrow[\mathcal{C}\mathcal{J}_i]{} & \mathbb{C}[\mathbf{p}]_n \end{array}, \qquad i = 0, \ldots, m-1.$$

Let the two colored partition $\overline{\lambda}$ and $\overline{\mu}$ such that $|\overline{\lambda}| = |\overline{\mu}| = n$. For an element $\sigma \in C_{\overline{\lambda}}$, define the multiplicity $\langle \overline{\lambda} | \overline{\mu} \rangle_0$ as the number of reflections $r_{i,j}^{(k)}$ (i.e., elements of $\mathcal{R}$) such that $\sigma r_{i,j}^{(k)} \in C_{\overline{\mu}}$. Similarly, define the multiplicity $\langle \overline{\lambda} | \overline{\mu} \rangle_k$ for elements in $\mathcal{L}^k$ and $k = 1, \ldots, m-1$

The following lemma and theorem are proved in the exact same way as in [8]; we refer to this paper for more details.

Lemma 3.2. *Multiplicities do not depend on the choice of* σ.

Theorem 3.3. $T_i \mathcal{C}_{\overline{\lambda}} = \sum_{\overline{\mu}} \langle \overline{\lambda} \mid \overline{\mu} \rangle_i \cdot \mathcal{C}_{\overline{\mu}}$ *for* $i = 0, \ldots, m-1$.

4. Generating function, cut-and-join and explicit formulas

4.1. Generating function

Consider the following generating function for Hurwitz numbers of the complex reflection group $G(m,1,n)$:

$$\mathcal{H}(\beta_0, \beta_1, \ldots, \beta_{m-1}, \mathbf{p}) = \sum_{n_0, \ldots, n_{m-1}} \sum_{\overline{\lambda}} \frac{h_{n_0, \ldots, n_{m-1}, \overline{\lambda}}}{n_0! n_1! \cdots n_{m-1}!} p_{\overline{\lambda}} \beta_0^{n_0} \cdots \beta_{m-1}^{n_{m-1}},$$

where by abuse of notation we use β_i as a variable in the generating function that keeps track of the number of β_i-cycles.

Theorem 4.1. *The generating function* $\mathcal{H}$ *satisfies the family of* m *cut-and-join equations:*

$$\frac{\partial \mathcal{H}}{\partial \beta_0} = \mathcal{C}\mathcal{J}_0(\mathcal{H}), \ldots, \frac{\partial \mathcal{H}}{\partial \beta_{m-1}} = \mathcal{C}\mathcal{J}_{m-1}(\mathcal{H}). \tag{4.1}$$

Proof. Fix a positive integer n and denote by $\mathcal{H}_n$ a degree n homogeneous component of $\mathcal{H}$. The cut-and-join operators preserve the degree, so $\mathcal{H}$ satisfies the cut-and-join equations if and only if $\mathcal{H}_n$ does (for each n).

Let i be any integer between 0 and $m-1$, and let

$$\mathcal{G}_n \stackrel{\text{def}}{=} \sum_{n_0,\dots,n_{m-1}\geq 0} \sum_{\bar{\lambda}:|\bar{\lambda}|=n} \frac{m^n n! h_{n_0,\dots,n_{m-1},\bar{\lambda}}}{n_0!\cdots n_{m-1}!} \mathcal{C}_{\bar{\lambda}} \beta_0^{n_0}\beta_1^{n_1}\cdots\beta_{m-1}^{n_{m-1}} \in \mathbb{C}[G(m,1,n)].$$

An elementary combinatorial reasoning gives

$$\mathcal{G}_n = \sum_{m\geq 0} \frac{\beta_0^{n_0}\beta_1^{n_1}\cdots\beta_{m-1}^{n_{m-1}}}{n_0!\cdots n_{m-1}!} T_0^{n_0}\cdots T_{m-1}^{n_{m-1}}(e_n),$$

where $e_n \in G(m,1,n)$ is the unit element. Clearly,

$$\begin{aligned} T_i(\mathcal{G}_n) &= \sum_{n_0,\dots,n_{m-1}\geq 0} \frac{\beta_0^{n_0}\beta_1^{n_1}\cdots\beta_{m-1}^{n_{m-1}}}{n_0!\cdots n_i!\cdots n_{m-1}!} T_0^{n_0}\cdots (T_i)^{n_i+1}\cdots T_{m-1}^{n_{m-1}}(e_n) \\ &= \sum_{\substack{n_0\cdots\hat{n}_i,\dots,n_{m-1}\geq 0\\ n_i\geq 1}} \frac{\beta_0^{n_0}\cdots\beta_i^{n_i-1}\cdots\beta_{m-1}^{n_{m-1}}}{n_0!\cdots(n_i-1)!\cdots n_{m-1}} T_0^{n_0}\cdots T_i^{n_i}\cdots T_{m-1}^{n_{m-1}}(e_n) \\ &= \frac{\partial\mathcal{G}_n}{\partial\beta_\alpha}. \end{aligned}$$

Applying the isomorphism Ψ, one obtains $\Psi T_i(\mathcal{G}_n) = \Psi(\frac{\partial\mathcal{G}_n}{\partial\beta_\alpha}) = \frac{\partial}{\partial\beta_\alpha}\Psi(\mathcal{G}_n)$. One has $\Psi(\mathcal{G}_n) = \mathcal{H}_n$; hence, $\frac{\partial}{\partial\beta_\alpha}\Psi(\mathcal{G}_n) = \frac{\partial\mathcal{H}_n}{\partial\beta_\alpha}$. By the definition of the cut-and-join operators, $\Psi T_i(\mathcal{G}_n) = \mathcal{C}\mathcal{J}_i(\Psi(\mathcal{G}_n)) = \mathcal{C}\mathcal{J}_i(\mathcal{H}_n)$, and equalities (4.1) follow. ∎

Corollary 4.2. *We have*

$$\mathcal{H}(\beta_0,\dots,\beta_{m-1},\mathbf{p}) = e^{\beta_0\mathcal{C}\mathcal{J}_0+\cdots+\beta_{m-1}\mathcal{C}\mathcal{J}_{m-1}} e^{p_1^{(0)}}. \tag{4.2}$$

Proof. It follows from Definition 3.1 that $h_{0,\dots,0,\bar{\lambda}} = \frac{1}{m^n n!}$ if $\lambda_0 = 1^n$ and $\lambda_i = \emptyset$ for $i = 1,\dots,m-1$, and $h_{0,\dots,0,\bar{\lambda}} = 0$, otherwise. Thus, $\mathcal{H}(0,\dots,0,\mathbf{p}) = e^{p_1^{(0)}}$ and (4.2) follows from Theorem 4.1. ∎

4.2. Explicit formulas

We provide in this section explicit formulas for the cut-and-join and the generating function. In the next subsection, we will relate this cut-and-join operator to the classical case of S_n via a change a variables.

Theorem 4.3. *The m-family of cut-and-join operators for the complex reflection groups $G(m,1,n)$ is given by*

$$\mathcal{C}\mathcal{J}_0 = \frac{1}{2} \sum_{\substack{i,j\geq 1\\ \alpha,\gamma\in\mathbb{Z}/m\mathbb{Z}}} (i+j) p_i^{(\alpha)} p_j^{(\gamma)} \frac{\partial}{\partial p_{i+j}^{(\alpha+\gamma)}} + mij p_{i+j}^{(\alpha+\gamma)} \frac{\partial^2}{\partial p_i^{(\alpha)} p_j^{(\gamma)}} \tag{4.3}$$

and

$$\mathcal{CJ}_k = \sum_{\substack{i \geq 1 \\ \alpha \in \mathbb{Z}/m\mathbb{Z}}} \left(i p_i^{(\alpha+k)} \frac{\partial}{\partial p_i^{(\alpha)}} \right) \quad \textit{for } k = 1, \ldots, m-1. \tag{4.4}$$

Remark 4.4. The operator $\mathcal{CJ}_0$ has $2m^2$ terms, while $\mathcal{CJ}_k$ has m terms for each k.

Example 4.5. For the wreath product $\mathbb{Z}/2\mathbb{Z} \wr S_n$, α and γ can be equal only to 0 or 1. Let $p_i^{(0)} \stackrel{\text{def}}{=} p_i$ and $p_i^{(1)} \stackrel{\text{def}}{=} q_i$. Thus, for $\mathcal{CJ}_0$, one gets

$$\begin{aligned}\mathcal{CJ}_0 =& \frac{1}{2} \sum_{i,j \geq 1} (i+j) \left(p_i p_j \frac{\partial}{\partial p_{i+j}} + p_i q_j \frac{\partial}{\partial q_{i+j}} + q_i p_j \frac{\partial}{\partial q_{i+j}} + q_i q_j \frac{\partial}{\partial p_{i+j}} \right) \\ &+ 2ij \left(p_{i+j} \frac{\partial^2}{\partial p_i p_j} + p_{i+j} \frac{\partial^2}{\partial q_i q_j} + q_{i+j} \frac{\partial^2}{\partial q_i p_j} + q_{i+j} \frac{\partial^2}{\partial p_i q_j} \right),\end{aligned}$$

which after simplification gives

$$\begin{aligned}\mathcal{CJ}_0 = \sum_{i,j=1}^{\infty} \Bigg(&(i+j) p_i q_j \frac{\partial}{\partial q_{i+j}} + 2ij q_{i+j} \frac{\partial^2}{\partial p_i \partial q_j} + ij p_{i+j} \frac{\partial^2}{\partial q_i \partial q_j} \\ &+ \frac{1}{2}(i+j) q_i q_j \frac{\partial}{\partial p_{i+j}} + \frac{1}{2}(i+j) p_i p_j \frac{\partial}{\partial p_{i+j}} + ij p_{i+j} \frac{\partial^2}{\partial p_i \partial p_j} \Bigg),\end{aligned}$$

and we recover the result from [8].

Similarly, for $\mathcal{CJ}_1$, we get

$$\mathcal{CJ}_1 = \sum_{i=1}^{\infty} \left(i p_i \frac{\partial}{\partial q_i} + i q_i \frac{\partial}{\partial p_i} \right),$$

which also agrees with [8]

To prove Theorem 4.3, we will explicitly compute the coefficients $\langle \overline{\lambda} \mid \overline{\mu} \rangle_\nu$ for all possible $\overline{\lambda}$, $\overline{\mu}$ and $\nu = 0, \ldots, m-1$.

Let $\sigma \in S_n$, and $1 \leq a < b \leq n$. The cyclic structure of the product $\sigma(a,b)$ depends on the position of a and b: if a and b are in the same cycle, say σ_1, then σ_1 is split; we call this a cut. If a and b are in different cycles, say σ_1 and σ_2, respectively, then these two cycles are joined.

Let now $\sigma \in G(m,1,n)$, and let ρ be one of the following types of reflections (where elements are modulo $(m-1)n$):

- $\rho = r_{a,b} = (a,b)(\tau(a),\tau(b)) \cdots (\tau^{m-1}(a), \tau^{m-1}(b))$ for $1 \leq a,b \leq (m-1)n$,
- $\rho = \ell_a^k = (a, \tau(a), \tau^2(a), \ldots)^k$ for $1 \leq a \leq (m-1)n$ and for any $k = 1, \ldots, m-1$.

The structure of $\sigma' = \rho\sigma$, where the multiplication is from *left to right*, depends on the structure of σ, i.e., which β_α-cycle σ is, and the positions of a and b.

4.2.1. The operators $\mathcal{CJ}_0$. Let α and γ be such that $0 \leq \alpha, \gamma \leq m-1$, and let the permutation σ with cycle decomposition the product of a β_α-cycle and a β_γ-cycle, where the β_α-cycle is

$$(a_1,\dots,a_{\ell_1},\tau^\alpha(a_1),\dots,\tau^\alpha(a_{\ell_1}),\dots)(\tau(a_1),\dots,\tau(a_{\ell_1}),\tau^{\alpha+1}(a_1),\dots,\tau^{\alpha+1}(a_{\ell_1}),\dots)$$

and the β_γ-cycle is

$$(b_1,\dots,b_{\ell_2},\tau^\gamma(b_1),\dots,\tau^\gamma(b_{\ell_2}),\dots)(\tau(b_1),\dots,\tau(b_{\ell_2}),\tau^{\gamma+1}(b_1),\dots,\tau^{\gamma+1}(b_{\ell_2}),\dots).$$

Consider also the reflection

$$\rho_{a_1,b_1} = (a_1,b_1)(\tau(a_1),\tau(b_1))\cdots(\tau^{m-1}(a_1),\tau^{m-1}(b_1)),$$

where $1 \leq a_1, b_1 \leq (m-1)n$.

A straightforward computation shows that $\rho\sigma$ is a $\beta_{\alpha+\gamma}$-cycle. Indeed,

$$\begin{aligned}
&(\rho\sigma)(a_{\ell_1}) = \sigma(\rho(a_{\ell_1})) = \tau^\alpha(b_1),\\
&\sigma(\rho(\tau^\alpha(b_1))) = \tau^\alpha(b_2),\\
&\vdots\\
&\sigma(\rho(\tau^\alpha(b_{\ell_2}))) = (\tau^{\gamma+\alpha}(a_1)).
\end{aligned}$$

In other words, multiplying by ρ a product of a β_α with a β_γ-cycles gives a $\beta_{\alpha+\gamma}$-cycle, we call this a *join*. Furthermore, ρ being an involution, multiplying the $\beta_{\alpha+\gamma}$-cycle obtained once more by ρ, gives back the product of $\beta_\alpha\beta_\gamma$-cycles; we call this a *cut*. There are m^2 possible joins and m^2 possible cuts counted without multiplicities, so $2m^2$ cases in total. This is true for any α, β and $\alpha+\beta$; these are all the possible cases for the multiplication by a reflection

$$\rho = (a_1,b_1)(\tau(a_1),\tau(b_1))\cdots(\tau^{m-1}(a_1),\tau^{m-1}(b_1)).$$

We now compute the multiplicities $\langle\overline{\lambda} \mid \overline{\mu}\rangle_0$, which will prove the result for $\mathcal{CJ}_0$. Let $\sigma \in C_{\overline{\lambda}}$, where $\overline{\lambda} = (\lambda_0,\dots,\lambda_{m-1})$ and $\lambda_\alpha = 1^{c_{1\alpha}}\cdots n^{c_{n\alpha}}$ that is for every $k = 1,\dots,n$ the element σ contains c_k β_α-cycles of length $k \times \frac{m}{\gcd(m,\alpha)}$. In the same way, define $\sigma\rho \in C_{\overline{\mu}}$. Calculate $\langle\overline{\lambda} \mid \overline{\mu}\rangle_0$ for all $\overline{\mu} = (\mu_0,\dots,\mu_{m-1})$, where $\mu_\gamma = 1^{d_{1\gamma}}\cdots n^{d_{n\gamma}}$.

Case 1: Cut. Let ρ act on elements from a $\beta_{\alpha+\gamma}$-cycle of length $(i+j)\times\frac{m}{\gcd(m,\alpha+\gamma)}$. The total possible numbers of positions for a_1 is the total number of elements in all the $\beta_{\alpha+\gamma}$-cycles of length $(i+j)\times\frac{m}{\gcd(m,\alpha+\gamma)}$, that is,

$$(i+j)\times\gcd(m,\alpha+\gamma)\times\frac{m}{\gcd(m,\alpha+\gamma)}c_{i+j} = m(i+j)c_{i+j}.$$

Since we know the length of the product of β_α with the β_γ of $\rho\sigma$, the position of b_1 as well as the positions of $\tau^\nu(a_1)$ and $\tau^\nu(b_1)$ for $\nu = 1, \dots, m-1$ is unique once the position of a_1 is chosen. Doing like this, one counts every reflection ρ_{a_1,b_1} m times we further divide by two as $\rho_{a_1,b_1} = \rho_{b_1,a_1}$, and the multiplicity is $\frac{1}{2}(i+j)c_{i+j}$

Case 2: Join. Let ρ act on elements from a β_α-cycle of length $i \times \frac{m}{\gcd(m,\alpha)}$ and a β_γ-cycle of length $j \times \frac{m}{\gcd(m,\gamma)}$. The total possible number of positions for a_1 is the total number of elements in all the β_α-cycles of length $i \times \frac{m}{\gcd(m,\alpha)}$, that is, $i \times \gcd(m,\alpha) \times \frac{m}{\gcd(m,\alpha)} c_i = mic_i$. Similarly, for b_1, we get mjc_j total possible number of positions. For the same reason as above, we need to divide by $2m$ to not over count ρ_{a_1,b_1}. We get that the multiplicity is $mijc_ic_j$

The explicit formula of $\mathcal{CJ}_0$ follows from Theorem 3.3 as

$$\mathcal{CJ}_0 p_{\overline{\lambda}} = \sum_{\overline{\mu}} \langle \overline{\lambda} \mid \overline{\mu} \rangle_0 p_{\overline{\mu}}.$$

For the cut, the monomial $p_{\overline{\lambda}}$ contains $p_{i+j}^{(\alpha+\beta)c_{i+j}}$. The exponent of $p_{i+j}^{(\alpha+\beta)c_{i+j}}$ in the monomial $p_{\overline{\mu}}$ is decreased by one, and the exponents of $p_i^{(\alpha)}$ and $p_j^{(\beta)}$ are increased by one. We will have $p_i^{(\alpha)} p_j^{(\gamma)} \frac{\partial}{\partial p_{i+j}^{(\alpha+\gamma)}}$ for the cut, and to have the correct multiplicity, add the coefficient $\frac{i+j}{2}$ before. The join case is similar, and this finishes to the proof for $\mathcal{CJ}_0$.

4.2.2. The operators $\mathcal{CJ}_k$. The reasoning to get the explicit formulas for $\mathcal{CJ}_k$ is similar to the one for $\mathcal{CJ}_0$, but we look instead at the product $\sigma\rho$, where $\rho = (a_1, \tau(a_1), \dots, \tau^{m-1}(a_1))^k$ for $k = 1, \dots, m-1$.

Once more, let σ be the β_α-cycle:

$$\begin{aligned}&(a_1, \dots, a_{\ell_1}, \tau^\alpha(a_1), \dots, \tau^\alpha(a_{\ell_1}), \dots)(\tau(a_1), \dots, \tau(a_{\ell_1}),\\ &\tau^{\alpha+1}(a_1), \dots, \tau^{\alpha+1}(a_{\ell_1}), \dots).\end{aligned}$$

Let also the reflection ρ:

$$\rho = \ell_{a_1}^\nu = (a_1, \tau(a_1), \dots, \tau^{m-1}(a_1))^\nu \quad \text{for some } \nu = 1, \dots, m-1.$$

A straightforward computation shows that $\sigma\rho$ is a $\beta_{\alpha+\nu}$-cycle.

We now compute the multiplicities $\langle \overline{\lambda} \mid \overline{\mu} \rangle_\gamma$ for $\gamma = 1, \dots, m-1$, which will end the proof of Theorem 4.3. As above, let $\sigma \in C_{\overline{\lambda}}$, where $\overline{\lambda} = (\lambda_0, \dots, \lambda_{m-1})$ and $\lambda_\alpha = 1^{c_{1\alpha}} \cdots n^{c_{n\alpha}}$; that is, for every $k = 1, \dots, n$, the element σ contains c_k β_α-cycles of length $k \times \frac{m}{\gcd(m,\alpha)}$. In the same way, define $\sigma\rho \in C_{\overline{\mu}}$. Calculate $\langle \overline{\lambda} \mid \overline{\mu} \rangle_\nu$ for all $\overline{\mu} = (\mu_0, \dots, \mu_{m-1})$, where

$$\mu_\gamma = 1^{d_{1\gamma}} \cdots n^{d_{n\gamma}} \quad \text{and} \quad \nu = 1, \dots, m-1.$$

Let a β_α-cycle of length $i \times \frac{m}{\gcd(m,\alpha)}$. The total possible number of positions for a_1 is the total number of elements in all the β_α-cycles of length $i \times \frac{m}{\gcd(m,\alpha)}$, that is,

$$i \times \gcd(m,\alpha) \times \frac{m}{\gcd(m,\alpha)} c_i = mic_i.$$

We need to divide the multiplicity by m to not over count the reflection $\ell^\nu_{a_1}$ as $\ell^\nu_{a_1} = \ell^\nu_{\tau(a_1)} = \cdots = \ell^\nu_{\tau^{m-1}(a_1)}$, and finally, the multiplicity is ic_i

As done for $\mathcal{CJ}_0$, the monomial $p_{\overline{\lambda}}$ contains $p_i^{(\alpha)c_i}$. The exponent of $p_{i+j}^{(\alpha)c_i}$ in the monomial $p_{\overline{\mu}}$ is decreased by one, and the exponent of $p_i^{(\alpha+k)}$ is increased by one. We finally get $p_i^{(\alpha+k)} \frac{\partial}{\partial p_i^{(\alpha)}}$, and to have the correct multiplicity, add the coefficient i before. This completes the proof of Theorem 4.3.

4.3. Change of variables

We show in this section that, after a suitable change of variable, the operators in (4.3) and in (4.4) reduce to the classical cut-and-join operators denoted by

$$\mathcal{CJ} \stackrel{\text{def}}{=} \frac{1}{2} \sum_{i,j=1}^{\infty} \left(ijp_{i+j} \frac{\partial^2}{\partial p_i \partial p_j} + (i+j) p_i p_j \frac{\partial}{\partial p_{i+j}} \right) \tag{4.5}$$

and Euler fields

$$E \stackrel{\text{def}}{=} \sum_{i=1}^{\infty} i p_i \frac{\partial}{\partial p_i}, \tag{4.6}$$

respectively.

Proposition 4.6. *Let $\xi \stackrel{\text{def}}{=} \exp(\frac{2\pi i}{m})$. Consider the following change of variables:*

$$\begin{aligned}
p_i^{(0)} &= u_i^{(0)} + u_i^{(1)} + \cdots + u_i^{(m-1)}, \\
p_i^{(1)} &= u_i^{(0)} + \xi u_i^{(1)} + \cdots + \xi^{m-1} u_i^{(m-1)}, \\
&\vdots \\
p_i^{(m-1)} &= u_i^{(0)} + \xi^{m-1} u_i^{(1)} + \cdots + \xi u_i^{(m-1)}.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathcal{CJ}_0 &= m(\mathcal{CJ}_{u^{(0)}} + \cdots + \mathcal{CJ}_{u^{(m-1)}}), \\
\mathcal{CJ}_k &= \sum_{\alpha=0}^{m-1} \xi^{k\alpha} E_{u^{(\alpha)}} \stackrel{\text{def}}{=} \mathcal{E}_k \quad k = 1, \ldots, m-1,
\end{aligned}$$

where by $\mathcal{CJ}_{u^{(\alpha)}}$ we denote the operator (4.5) with $u_i^{(\alpha)}$ substituted for p_i, and similarly, $E_{u^{(\alpha)}}$.

Proof. First, for $\mathcal{C}\mathcal{J}_0$, by the chain rule, we have

$$\frac{\partial}{\partial p_i^{(\alpha)}} = \frac{1}{m}\sum_{\nu=0}^{m-1} \xi^{-\nu\alpha}\frac{\partial}{\partial u_i^{(\nu)}}. \tag{4.7}$$

Then, plugging (4.7) into (4.1) gives

$$\begin{aligned}\mathcal{C}\mathcal{J}_0 = \frac{1}{2m} \sum_{\substack{i,j\geq 1\\ \alpha,\gamma\in\mathbb{Z}/m\mathbb{Z}\\ \nu_1,\nu_2,\nu_3\in\mathbb{Z}/m\mathbb{Z}}} & \xi^{\alpha\nu_1+\beta\nu_2-(\alpha+\beta)\nu_3}(i+j)u_i^{(\nu_1)}u_j^{(\nu_2)}\frac{\partial}{\partial u_{i+j}^{(\nu_3)}} \\ & + \xi^{(\alpha+\beta)\nu_1-\alpha\nu_2-\beta\nu_3} miju_{i+j}^{(\nu_1)}\frac{\partial^2}{\partial u_i^{(\nu_2)}u_j^{(\nu_3)}}.\end{aligned} \tag{4.8}$$

We want to express the coefficients in front of the monomials $(i+j)u_i^{(\nu_1)}u_j^{(\nu_2)}\frac{\partial}{\partial u_{i+j}^{(\nu_3)}}$ and $miju_{i+j}^{(\nu_1)}\frac{\partial^2}{\partial u_i^{(\nu_2)}u_j^{(\nu_3)}}$. A straightforward computation gives

$$\begin{aligned}\sum_{\alpha,\gamma\in\mathbb{Z}/m\mathbb{Z}} \xi^{\alpha\nu_1+\beta\nu_2-(\alpha+\beta)\nu_3} &= \sum_{\alpha,\gamma\in\mathbb{Z}/m\mathbb{Z}} \xi^{\alpha(\nu_1-\nu_2)+\beta(\nu_1-\nu_3)} \\ &= \begin{cases} m^2 & \text{if } \nu_1=\nu_2=\nu_3 \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Plugging this last equality into (4.8) and doing a similar reasoning for

$$\sum_{\alpha,\gamma\in\mathbb{Z}/m\mathbb{Z}} \xi^{(\alpha+\beta)\nu_1-\alpha\nu_2-\beta\nu_3}$$

give

$$\mathcal{C}\mathcal{J}_0 = m\sum_{\substack{i,j\geq 1\\ \nu\in\mathbb{Z}/m\mathbb{Z}}} (i+j)u_i^{(\nu)}u_j^{(\nu)}\frac{\partial}{\partial u_{i+j}^{(\nu)}} + miju_{i+j}^{(\nu)}\frac{\partial^2}{\partial u_i^{(\nu)}u_j^{(\nu)}},$$

and this finishes the proof for $\mathcal{C}\mathcal{J}_0$

By the same reasoning, for $\mathcal{C}\mathcal{J}_k$, one gets

$$\mathcal{C}\mathcal{J}_k = \frac{1}{m}\sum_{\substack{i\geq 1\\ \alpha,\nu_1,\nu_2\in\mathbb{Z}/m\mathbb{Z}}} \xi^{(\alpha+k)\nu_1-\alpha\nu_2}u_i^{(\nu_1)}\frac{\partial}{\partial u_i^{(\nu_2)}}; \tag{4.9}$$

this reduces to computing

$$\sum_{\alpha\in\mathbb{Z}/m\mathbb{Z}} \xi^{(\alpha+k)\nu_1-\alpha\nu_2} = \xi^{k\nu_1}\sum_{\alpha\in\mathbb{Z}/m\mathbb{Z}} \xi^{\alpha\nu_1-\alpha\nu_2} = \begin{cases} m & \text{if } \nu_1=\nu_2 \\ 0 & \text{otherwise,}\end{cases}$$

and plugging it into (4.9) finishes the proof. ∎

Theorem 4.7. *For all colored partitions $\overline{\lambda} = \lambda_0|\cdots|\lambda_{m-1}$, the polynomials*

$$s_{\overline{\lambda}}(\mathbf{P}^{(0)},\dots,\mathbf{P}^{(m-1)}) \stackrel{\text{def}}{=} s_{\lambda_0}(\mathbf{P}^{(0)})\cdots s_{\lambda_{m-1}}(\mathbf{P}^{(m-1)}),$$

where s_{λ_i} are Schur polynomials and $\mathbf{P}^{(\alpha)}$ means $(\frac{\sum_\beta \xi^{\alpha\beta} p_1^{(\alpha)}}{m}, \frac{\sum_\beta \xi^{\alpha\beta} p_2^{(\alpha)}}{m},\dots)$, are eigenvectors of $\mathcal{CJ}_0$ and $\mathcal{CJ}_k$ for all $k = 1,\dots,m-1$; the respective eigenvalues are

$$c_0(\overline{\lambda}) \stackrel{\text{def}}{=} m\sum_{i=1}^{\infty}\big(\lambda_{0,i}(\lambda_{0,i}-2i+1)+\lambda_{1,i}(\lambda_{1,i}-2i+1)+\dots+\lambda_{m-1,i}(\lambda_{m-1,i}-2i+1)\big)$$

for $\mathcal{CJ}_0$ and $c_k(\overline{\lambda}) \stackrel{\text{def}}{=} \sum_{i=1}^{\infty}(\lambda_{0,i} + \xi^k\lambda_{1,i} + \dots + \xi^{k(m-1)}\lambda_{m-1,i})$ for $\mathcal{CJ}_k$.

The theorem follows immediately from Proposition 4.6 and the fact that Schur polynomials are eigenvectors of the cut-and-join operator (4.5) (see [15]) and of the Euler field (4.6) ($s_{\overline{\lambda}}$ are weighted homogeneous).

Theorem 4.7 and the Cauchy identity [19] imply that

$$e^{p_1^{(0)}} = e^{u_1^{(0)}}\cdots e^{u_1^{(m-1)}} = \sum_{\overline{\lambda}} s_{\overline{\lambda}}(\mathbf{P}^{(0)},\dots,\mathbf{P}^{(m-1)}) s_{\overline{\lambda}}(1,0,\dots;1,0,\dots).$$

This allows us to prove the following.

Corollary 4.8 (of Theorem 4.7). *We have*

$$\mathcal{H}(\beta_0,\beta_1,\dots,\beta_{m-1},\mathbf{p}) = \sum_{\overline{\lambda}} \exp\big(\beta_0 c_0(\overline{\lambda}) + \sum_{1\le k\le m-1}\beta_k c_k(\overline{\lambda})\big) \times s_{\overline{\lambda}}(1,0,\dots;1,0,\dots) s_{\overline{\lambda}}(\mathbf{P}^{(0)},\dots,\mathbf{P}^{(m-1)}).$$

This is the $G(m,1,n)$-analog of the formula expressing the simple Hurwitz numbers via the Schur polynomials [15].

5. KP hierarchy

The KP hierarchy is one of the best-known and studied integrable systems; see, e.g., [14, 16, 20] for details. It is an infinite system of PDE applied to a formal series $F \in \mathbb{C}[\![t]\!]$, where $t = (t_1, t_2, \dots)$ is a countable collection of variables ("times"). Each equation looks like

$$\frac{\partial^2 F}{\partial t_i \partial t_j} = P_{ij}(F),$$

where P_{ij} is a homogeneous multivariate polynomial with partial derivatives of F used as its arguments. For example, the first two equations of the hierarchy are

$$\begin{aligned} F_{22} &= -\frac{1}{2}F_{11}^2 + F_{31} - \frac{1}{12}F_{1111}, \\ F_{32} &= -F_{11}F_{21} + F_{41} - \frac{1}{6}F_{2111}; \end{aligned}$$

here, $F_{i_1 i_2 \cdots i_n}$ means $\frac{\partial^n F}{\partial t_{i_1} \partial t_{i_2} \cdots \partial t_{i_n}}$. If F is a solution of the hierarchy, its exponential $\tau = e^F$ is called a τ-function.

Example 5.1. Equations of the hierarchy do not contain first-order derivatives, so the function $F(t) = t_1$ is a solution of the KP hierarchy; $\tau = e^{t_1}$ is a τ-function.

Proposition 5.2. *There exists a Lie algebra $\mathcal{G}$ of differential operators on the space $\mathbb{C}[\![p]\!]$ having the following properties.*

(1) *$\mathcal{G}$ is isomorphic to a one-dimensional central extension $\widehat{\mathrm{gl}(\infty)}$ of a Lie algebra $\mathrm{gl}(\infty)$ of infinite matrices $(x_{ij})_{i,j\in\mathbb{Z}}$ having non-zero elements at a finite set of diagonals.*

(2) *The Euler field E and the cut-and-join operator $\mathcal{CJ}$ belong to $\mathcal{G}$.*

(3) *For any τ-function f and any $X \in \mathcal{G}$, the power series $e^X f$ is a τ-function, too.*

See [16] for the definition of $\mathcal{G}$, [23] and [14, Section 6] for the proof.

Theorem 5.3. *The generating function $\mathcal{H}(\beta_0, \beta_1, \ldots, \beta_{m-1}, \mathbf{p})$ is an m-parameter family of τ-functions, independently in the $p^{(i)}$ variables.*

Proof. Denote by $\mathbf{u}_i^\alpha \overset{\text{def}}{=} \sum_{0\le\beta\le m-1} \xi^{\alpha\beta} u_i^\beta$. Equation (4.2) and Proposition 4.6 imply that

$$\begin{aligned} &\mathcal{H}(\beta_0, \ldots, \beta_{m-1}, \mathbf{u}^0, \ldots, \mathbf{u}^{m-1}) \\ &\quad = e^{\beta_0 m(\mathcal{CJ}_{u^{(0)}} + \cdots + \mathcal{CJ}_{u^{(m-1)}})} e^{(\beta_1 \mathcal{E}_1 + \cdots \beta_{m-1}\mathcal{E}_{m-1})} e^{u_1^{(0)}} \cdots e^{u_1^{(m-1)}} \\ &\quad = e^{(\beta_0 \mathcal{CJ}_{u^{(0)}} + \beta_1 E_{u^{(0)}} + \cdots + \beta_{m-1} E_{u^{(0)}})} e^{u^{(0)}} \\ &\qquad \cdots e^{(\beta_0 \mathcal{CJ}_{u^{(m-1)}} + \beta_1 \xi^{m-1} E_{u^{(m-1)}} + \cdots + \beta_{m-1} \xi E_{u^{(m-1)}})} e^{u^{(m-1)}}. \end{aligned}$$

(Operators $\mathcal{CJ}_{u^{(\alpha)}}, \mathcal{CJ}_{u^{(\nu)}}$, $E_{u^{(\alpha')}}$, and $E_{u^{(\nu')}}$ commute because they act on different sets of variables.)

By assertion (2) of Proposition 5.2, $\beta_0 \mathcal{CJ}_{u^{(0)}} + \beta_1 E_{u^{(0)}} + \cdots + \beta_{m-1} E_{u^{(0)}} \in \mathcal{G}_u$ (the Lie algebra $\mathcal{G}$ acting by differential operators on the $u^{(0)}$ variables). By Example 5.1, $e^{u_1^{(0)}}$ is a τ-function (in the $\mathbf{u}^{(0)}$ variables), and so is

$$\mathcal{H}^B(\beta_0, \ldots, \beta_{m-1}, \mathbf{u}^0, \ldots, \mathbf{u}^{m-1})$$

by assertion (3) of the same proposition. The reasoning for the $u^{(\alpha)}$ with

$$\alpha = 1, \dots, m-1$$

variables is the same. ■

6. Hurwitz numbers and ramified covering

6.1. Wreath ramified coverings

In [25], the notion of wreath branched cover for a general wreath product of S_n is introduced; we will follow it closely. For details and proofs, see [25].

In Section 2.2, we showed the correspondence between the conjugacy classes of $G(m,1,n)$ when embedded in S_{mn} and the conjugacy classes of $\mathbb{Z}/m\mathbb{Z} \wr S_n$. In this section, conjugacy classes are subsets of the wreath product $\mathbb{Z}/m\mathbb{Z} \wr S_n$.

A $G(m,1,n)$-ramified cover (p, f) of degree mn over $\mathbb{C}P^1$ is an ordinary ramified cover f of degree n:

$$f : M \to \mathbb{C}P^1$$

with a set of critical values Y together with a smooth curve $\widetilde{M}$ with a left $\mathbb{Z}/m\mathbb{Z}$-action and $\mathbb{Z}/m\mathbb{Z}$-equivariant map ($\mathbb{Z}/m\mathbb{Z}$ acts trivially on M) $p : \widetilde{M} \to M$ such that

$$p' : \widetilde{M} - p^{-1}(f^{-1}(Y)) \to M - f^{-1}(Y),$$

where p' is the restriction of p to $\widetilde{M} - p^{-1}(f^{-1}(Y))$ and is a principal $\mathbb{Z}/m\mathbb{Z}$-bundle map.

Two $G(m,1,n)$-ramified covers

$$F : \widetilde{M} \xrightarrow{p} M \xrightarrow{f} \mathbb{C}P^1$$

and

$$F' : \widetilde{M}' \xrightarrow{p'} M' \xrightarrow{f'} \mathbb{C}P^1$$

are said to be equivalent if there is a pair of biholomorphic maps $\widetilde{\phi} : \widetilde{M} \to \widetilde{M}'$ and $\phi : M \to M'$ such that $\widetilde{\phi}$ is $\mathbb{Z}/m\mathbb{Z}$-equivariant and the following diagram commutes:

$$\begin{array}{ccccc}
\widetilde{M} & \xrightarrow{p} & M & \xrightarrow{f} & \mathbb{C}P^1 \\
\downarrow{\scriptstyle \widetilde{\phi}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle id} \\
\widetilde{M}' & \xrightarrow{p'} & M' & \xrightarrow{f'} & \mathbb{C}P^1.
\end{array}$$

An automorphism of a $G(m,1,n)$-ramified cover (f,p) is a pair of biholomorphic maps $\widetilde{\phi}:\widetilde{M}\to\widetilde{M}$ and $\phi: M\to M$ such that $\widetilde{\phi}$ is $\mathbb{Z}/m\mathbb{Z}$-equivariant and the following diagram commutes,

$$\begin{array}{ccccc} \widetilde{M} & \xrightarrow{p} & M & \xrightarrow{f} & \mathbb{C}P^1 \\ \big\downarrow \widetilde{\phi} & & \big\downarrow \phi & & \big\downarrow id \\ \widetilde{M} & \xrightarrow{p} & M & \xrightarrow{f} & \mathbb{C}P^1. \end{array}$$

An important result of [25] is the description of the monodromy of $G(m,1,n)$-ramified covers. We describe it now. Let (f,p) be a $G(m,1,n)$-ramified cover:

$$\widetilde{M}\xrightarrow{p} M \xrightarrow{f} \mathbb{C}P^1.$$

Taking a critical value $y\in Y$, the preimage $f^{-1}(y)$ is a set of points $x_1,\dots,x_\ell$ with multiplicities $\lambda_1,\dots,\lambda_\ell$, where $\lambda_1+\lambda_2+\cdots+\lambda_\ell=n$, i.e., a partition of n. We can assume without loss of generality that $\lambda_1\le\lambda_2\le\cdots\le\lambda_\ell$. Furthermore, pick a regular point b in $\mathbb{C}P^1$; its preimage by f are n points $q_1,\dots,q_n$. With the map p' being a normal cover, we can identify each fiber $p^{-1}(q_i)$ with $\mathbb{Z}/m\mathbb{Z}$ using local trivialization: choose a small neighborhood U_i of q_i and a diffeomorphism

$$\psi_i : p^{-1}(U_i)\to U_i\times\mathbb{Z}/m\mathbb{Z}, \quad z\to(p(z),\phi_i(z))$$

such that $\phi_i(hz)=h\phi_i(z)$ for any $h\in\mathbb{Z}/m\mathbb{Z}$

Take a loop γ based on b and winding once around y. The monodromy of f along the loop γ is a permutation $(q_{1,1},\dots,q_{1,\lambda_1})\cdots(q_{\ell,1},\dots,q_{\ell,\lambda_\ell})$ of S_n belonging to the conjugacy class indexed by λ. Pick now one of the cycles, say, $(q_{j,1},\dots,q_{j,\lambda_j})$, and denote by γ_i the lifting of γ starting at $q_{j,i}$. Now, starting with an element $z_1\in p^{-1}(q_{j,1})$, the unique lifting $\widetilde{\gamma_1}$ of γ_1 will end at some $z_2\in p^{-1}(q_{j,2})$; continuing so, we get that the lifting $\widetilde{\gamma_{\lambda_j}}$ of γ_{λ_j} will end at some $z_1'\in p^{-1}(q_{j,1})$. Suppose

$$\phi_2(z_2)=g_1\phi_1(z_1),\dots,\phi_{\lambda_j}(z_{\lambda_j})=g_{\lambda_{j-1}}\phi_{\lambda_{j-1}}(z_{\lambda_{j-1}}),\quad \phi_1(z_1')=g_{\lambda_j}\phi_{\lambda_j}(z_{\lambda_j})$$

for $g_1,\dots,g_{\lambda_j}\in\mathbb{Z}/m\mathbb{Z}$; then,

$$g_{\lambda_j}g_{\lambda_{j-1}}\cdots g_1(z_1)=\phi_1(z_1').$$

The product $g_{\lambda_j}g_{\lambda_{j-1}}\cdots g_1(z_1)=\phi_1(z_1')\in\mathbb{Z}/m\mathbb{Z}$ can be viewed as the monodromy of p induced by the loop $\gamma_1\cdots\gamma_{\lambda_j}$ starting and ending at $q_{j,1}$.

Add the following restriction that $\phi_1(z_1)=0\in\mathbb{Z}/m\mathbb{Z}$.

Lemma 6.1 ([25]). *The cycle product $g_{\lambda_j}g_{\lambda_{j-1}}\cdots g_1\in\mathbb{Z}/m\mathbb{Z}$ is determined by $\gamma_1\cdots\gamma_{\lambda_j}$ up to conjugacy.*

Now, given a loop γ winding once around a critical value y, we get an element $\sigma \in S_n$, together with an element $g = (g_1, \dots, g_n) \in (\mathbb{Z}/m\mathbb{Z})^n$. In other words, we get a map

$$\begin{aligned} \mathfrak{M}:\ \pi_1(\mathbb{C}P^1 - Y) &\to \mathbb{Z}/\mathbb{Z}m \wr S_n \\ [\gamma] &\to (\sigma, g), \end{aligned}$$

which is a group homomorphism. This is the monodromy representation associated with the $G(m, 1, n)$-cover (p, f).

Lemma 6.2 ([25]). *The monodromy group of a $G(m, 1, n)$-ramified cover (p, f) of degree mn over $\mathbb{C}P^1$ is a subgroup of the wreath product $\mathbb{Z}/\mathbb{Z}m \wr S_n$, which is determined up to conjugacy class. And it is transitive if M is connected.*

The inverse construction is as follows: given a group homomorphism

$$\mathfrak{M} : \pi_1(\mathbb{C}P^1 - Y) \to \mathbb{Z}/\mathbb{Z}m \wr S_n,$$

we first construct a ramified cover $f : M \to \mathbb{C}P^1$ of degree n, using the composition homomorphism of $\mathfrak{M}$ and the projection map $\mathbb{Z}/\mathbb{Z}m \wr S_n \to S_n$ as monodromy representation. Then, we construct a cover $p : \widetilde{M} \to M$ using the information resulting from the cycle products. This produces a $G(m, 1, n)$-ramified cover (p, f) of degree mn, where $\mathfrak{M}$ is its monodromy representation.

The construction described above gives the following lemma.

Lemma 6.3 ([25]). *The following correspondence is one-to-one between the two sets:*

- *non isomorphic $G(m, 1, n)$-ramified covers (p, f) of $\mathbb{C}P^1$ of degree mn,*
- *$Hom(\pi(\mathbb{C}P^1 - Y), \mathbb{Z}/\mathbb{Z}m \wr S_n)/(\mathbb{Z}/\mathbb{Z}m \wr S_n)$.*

6.2. Hurwitz numbers: The algebro-geometric definition

We now define in an algebro-geometric way Hurwitz numbers for the reflection groups $G(m, 1, n)$.

We say that a $G(m, 1, n)$-ramified cover (p, f) of degree mn over $\mathbb{C}P^1$ has profile $(\overline{\lambda}, n_0, n_1, \dots, n_{m-1})$ if the following hold.

- For each $\alpha = 0, \dots, m-1$, there are n_α other critical values $y_{\alpha,1} \cdots y_{\alpha,n_\alpha}$ in $\mathbb{C}P^1$, and the monodromy over these $y_{\alpha,1} \cdots y_{\alpha,n_\alpha}$ must be a β_α-cycle.
- The monodromy over ∞ must be $\overline{\lambda}$.
- The map $f \circ p$ is unramified outside of $Y = \{\infty\} \cup \{y_{\alpha,i}\}_{\substack{\alpha=0,\dots,m-1 \\ 1\le i\le n_\alpha}}$.

The genus of M can be found by using the Riemann–Hurwitz formula, and we do not require M to be connected. It then follows that the algebro-geometric definition of $h_{n_0,n_1,\dots,n_{m-1}\bar{\lambda}}$ is the number of non-isomorphic $G(m,1,n)$-ramified covers (p,f) of degree mn over $\mathbb{C}P^1$ having profile $(\bar{\lambda}, n_0, n_1, \dots, n_{m-1})$ divided by their weight. In a more compact formula

$$\mathbf{h}_{n_0,n_1,\dots,n_{m-1}\bar{\lambda}} = \sum_{(p,f)} \frac{1}{|\mathrm{Aut}(p,f)|},$$

where the (p,f) have profile $(\bar{\lambda}, n_0, n_1, \dots, n_{m-1})$ and are all distinct.

Proposition 6.4 ([25]). *For the complex reflection group $G(m,1,n)$, the combinatorial definition of Hurwitz numbers $h_{n_0,n_1,\dots,n_{m-1}\bar{\lambda}}$ and the algebro-geometric definitions $\mathbf{h}_{n_0,n_1,\dots,n_{m-1}\bar{\lambda}}$ are equivalent.*

7. ELSV-type formula

Given the description of the family of the cut-and-join equation, it is possible to reduce the problem of computation of the generating function $\mathcal{H}$ to the well-known case of usual Hurwitz numbers. The operators $\mathcal{C}\mathcal{J}_i$ commute for all $i = 0, \dots, m-1$, so we can use the following strategy: first, apply operators $\mathcal{C}\mathcal{J}_k$ to the initial condition and then try to relate the result of the action of $\mathcal{C}\mathcal{J}_0$ on the resulting function to the studied cases.

Proposition 7.1. *We have*

$$e^{\sum_{k=1}^{m-1} \beta_k \mathcal{C}\mathcal{J}_k} e^{p_1^{(0)}} = \prod_{\alpha=0}^{m-1} \exp\big(e^{\sum_{k=1}^{m-1} \beta_k \xi^{k\alpha}} u_1^{(\alpha)}\big).$$

Proof. It is enough to compute $e^{\sum_{k=1}^{m-1} \beta_k \xi^{k\alpha} E_{u^{(\alpha)}}} e^{u_1^{(\alpha)}}$; the result will be obtained by the product of these expressions for all α. Denote the coefficient $\sum_{k=1}^{m-1} \beta_k \xi^{k\alpha}$ by γ_α. Then, we have

$$\begin{aligned} e^{\gamma_\alpha E_{u^{(\alpha)}}} e^{u_1^{(\alpha)}} &= \sum_{k=0}^{\infty} \frac{1}{k!} \gamma_\alpha^k E_{u^{(\alpha)}}^k \sum_{l=0}^{\infty} \frac{1}{l!} (u_1^{(\alpha)})^l = \sum_{k,l=0}^{\infty} \frac{1}{k!l!} \gamma_\alpha^k l^k (u_1^{(\alpha)})^l \\ &= \sum_{l=0}^{\infty} \frac{1}{l!} (u_1^{(\alpha)})^l e^{\gamma_\alpha l} = \exp(e^{\gamma_\alpha} u_1^{(\alpha)}). \end{aligned}$$ ■

Now, consider the action of $\mathcal{C}\mathcal{J}_0$. Notice that the exponent $e^{\mathcal{C}\mathcal{J}_0}$ splits in the product of commuting operators, each depending on $u^{(\alpha)}$ with α fixed only. So, the result can be naturally written down as a product.

Proposition 7.2. *The generating function*

$$e^{\beta_0 m \mathcal{C}\mathcal{J}_{u^{(\alpha)}}} \exp\left(e^{\sum_{k=1}^m \beta_k \xi^{k\alpha}} u_1^{(\alpha)}\right)$$

coincides with the generating function for the disconnected weighted usual simple Hurwitz numbers $e^{G^{(\alpha)}}$, *where*

$$G^{(\alpha)} = \sum_{\substack{g\geq 0, n\geq 1\\ k_1,\ldots,k_n\in\mathbb{N}}} \frac{(m\beta_0)^{2g-2+n+\sum_{i=1}^n k_n} e^{\sum_{i=1}^n k_n \sum_{k=1}^m \beta_k \xi^{k\alpha}}}{(2g-2+n+\sum_{i=1}^n k_n)!} \frac{h_{g;k_1,\ldots,k_n} u_{k_1}^{(\alpha)} \cdots u_{k_n}^{(\alpha)}}{n!}.$$

Proof. Both functions satisfy the same evolution equation and the same initial condition. For the details, see [3, 26]. ∎

Usual Hurwitz numbers are known to satisfy the ELSV-formula [7]: for any $g \geq 0, k_1,\ldots,k_n \in \mathbb{N}$, we have

$$\mathfrak{H}_{g;k_1,\ldots,k_n} = \left(2g-2+n+\sum_{i=1}^n k_i\right)! \prod_{i=1}^n \frac{k_i^{k_i}}{k_i!} \int_{\bar{\mathcal{M}}_{g,n}} \frac{\Lambda_{g,n}}{(1-k_1\psi_1)\cdots(1-k_n\psi_n)},$$

where $\bar{\mathcal{M}}_{g,n}$ is the Deligne–Mumford compactification of the moduli space of genus g curves with n marked points, $\Lambda_{g,n}$ is the total Chern class of the dual to the Hodge vector bundle, and ψ_i are the corresponding ψ-classes. Combining the formulas, we obtain the following theorem.

Theorem 7.3. *The logarithm of the generating function* $\mathcal{H}$ *in variables* $u^{(\alpha)}$ *equals*

$$\log \mathcal{H} = \sum_{\alpha=0}^{m-1} G^{(\alpha)} = \sum_{\alpha=0}^{m-1} \sum_{\substack{g\geq 0, n\geq 1\\ k_1,\ldots,k_n\in\mathbb{N}}} \frac{(m\beta_0)^{2g-2+n+\sum_{i=1}^n k_n} e^{\sum_{i=1}^n k_n \sum_{k=1}^m \beta_k \xi^{k\alpha}}}{n!} \times \prod_{i=1}^n \frac{k_i^{k_i}}{k_i!} \int_{\bar{\mathcal{M}}_{g,n}} \frac{\Lambda_{g,n}}{(1-k_1\psi_1)\cdots(1-k_n\psi_n)} u_{k_1}^{(\alpha)} \cdots u_{k_n}^{(\alpha)}.$$

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