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The null condition and global existence for nonlinear wave equations on slowly rotating Kerr spacetimes

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Abstract. We study a semilinear equation with derivatives satisfying a null condition on slowly rotating Kerr spacetimes. We prove that given sufficiently small initial data, the solution exists globally in time and decays with a quantitative rate to the trivial solution. The proof uses the robust vector field method. It makes use of the decay properties of the linear wave equation on Kerr spacetime, in particular the improved decay rates in the region $\{r \leq t/4\}$.

1. Introduction

In this paper, we consider the global existence for small data for a semilinear equation with null condition on a Kerr spacetime. Kerr spacetimes are stationary axisymmetric asymptotically flat black hole solutions to the vacuum Einstein equations

$$R_{\mu\nu} = 0$$

in $3 + 1$ dimensions. They are parametrized by two parameters (M, a), representing respectively the mass and the angular momentum of a black hole. We study semilinear equations on a Kerr spacetime with $a \ll M$ of the form

$$\Box_{g_K} \Phi = F(D\Phi),$$

where $\Box_{g_K}$ is the Laplace–Beltrami operator for the Kerr metric g_K, and F denotes nonlinear terms that are at least quadratic and satisfy the null condition that we will define in Section 1.2.

The corresponding problem on Minkowski spacetime has been well studied. In 4+1 or higher dimensions, the decay of the linear wave equation is sufficiently fast for one to prove global existence for small data of nonlinear wave equations with any quadratic nonlinearity [16]. However, in 3+1 dimensions, which is also the dimension of physical relevance, the decay rate is only sufficient to prove the almost global existence of solutions [15]. Indeed, a counterexample is known [14] for the equation

$$\Box_m \Phi = (\partial_t \Phi)^2.$$

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Nevertheless, if the quadratic nonlinearity satisfies the null condition defined by Klainerman, it has been proved independently by Christodoulou [4] and Klainerman [17] that any solutions for sufficiently small initial data are global in time. There has been an extensive literature on extensions and variations of the original results, including the cases of the multiple-speed system and the exterior domains ([26], [27], [21], [22]).

The decay rate of the solutions to the linear wave equation on Kerr spacetimes with $a \ll M$ has been proved in [7], [1], [29] and [20]. The known decay outside the set $\{ct^* \leq r \leq Ct^*\}$ is sufficiently strong and the proof (in [7], [1] and [20]) is sufficiently robust that one expects the main obstacle to proving a small data global existence result (if it indeed holds) would come from quantities in the set $\{ct^* \leq r \leq Ct^*\}$. This set, however, approaches the same set in Minkowski spacetime as $t^* \to \infty$ due to the asymptotic flatness of Kerr spacetimes. Therefore, one expects that with a null condition similar to that on Minkowski spacetime, a similar global existence result holds. Indeed, we have (see the precise version in Section 1.2)

Main Theorem 1.1. *Consider $\Box_{g_K}\Phi = F(D\Phi)$ where F satisfies the null condition (see Section 1.2). Then for any initial data that are sufficiently small, the solution exists globally in time.*

Our major motivation for studying the null condition on a Kerr spacetime is the problem of the stability of the Kerr spacetime. It is conjectured that Kerr spacetimes are stable. In the framework of the initial value problem, the stability of Kerr spacetime would mean that for any solution to the vacuum Einstein equations with initial data close to the initial data of a Kerr spacetime, its maximal Cauchy development has an exterior region that approaches a nearby, but possibly different, Kerr spacetime. In the case of the Minkowski spacetime, the null condition has served as a good model problem for the study of the stability of the Minkowski spacetime. We hope that this work will find relevance to the problem of the stability of the Kerr spacetime.

1.1. Some related known results

We turn to some relevant work on linear and nonlinear scalar wave equations on Kerr spacetimes. The decay of solutions to the linear wave equation on Kerr spacetimes has received considerable attention. We mention some results on Kerr spacetimes with $a > 0$ here and refer the readers to [7], [19] for references on the corresponding problem on Schwarzschild spacetimes. There has been a large literature on the mode stability and nonquantitative decay of azimuthal modes (see for example [25], [12], [32], [10], [11] and references in [7]). The first global result for the Cauchy problem was obtained by Dafermos–Rodnianski [6], who proved that for a class of small, axisymmetric, stationary perturbations of Schwarzschild spacetime, which include Kerr spacetimes that rotate sufficiently slowly, solutions to the wave equation are uniformly bounded. Similar results were obtained later using an integrated decay estimate on slowly rotating Kerr spacetimes by Tataru–Tohaneanu [30]. Using the integrated decay estimate, Tohaneanu also proved Strichartz estimates [31].

Decay for general solutions to the wave equation on sufficiently slowly rotating Kerr spacetimes was first proved by Dafermos–Rodnianski [7] with a quantitative rate of $|\Phi| \leq C(t^*)^{-1+Ca}$. A similar result was later obtained by [1] using a physical space construction to obtain an integrated decay estimate. In all of [30], [7] and [1], the integrated decay estimate is proved and plays an important role. All proofs of such estimates rely heavily on the separability of the wave equation, or equivalently, the existence of a Killing tensor on Kerr spacetime. In a recent work [8], Dafermos–Rodnianski prove the nondegenerate energy decay and the pointwise decay assuming the integrated local energy decay estimate and boundedness for the wave equation on an asymptotically flat spacetime. Their work shows a decay rate of $|\Phi| \leq Ct^{-1}$ and improves the rates in [7] and [1]. In a similar framework, but assuming in addition exact stationarity, Tataru [29] proved a local decay rate of $(t^*)^{-3}$ using Fourier-analytic methods. This applies in particular to sufficiently slowly rotating Kerr spacetimes. Dafermos and Rodnianski have recently announced a proof for the decay of solutions to the wave equation on the full range of sub-extremal Kerr spacetimes $a < M$.

For nonlinear equations, global existence for the equation with power nonlinearity $\Box_{g_k}\Phi = \pm|\Phi|^p\Phi$ was initiated in [23] and [24], in which the large data subcritical defocusing case of $p = 2$ is studied. Later, there have been much work on the small data problem in which the sign of the nonlinearity is not important, and the dispersive properties of the linear equation play a crucial role. Global existence was proved for small radial data for $p > 3$ on Reissner–Nordström spacetime [5] and for general small data vanishing on the bifurcate sphere for $p > 2$ [2] on Schwarzschild spacetime. Global existence was also proved for $p = 4$ on Schwarzschild spacetime with general data that has small nondegenerate energy [13]. This was extended to the case of sufficiently slowly rotating Kerr spacetime in [31]. A counterexample is known for the case $0 < p < \sqrt{2}$ [3]. To our knowledge, the present work is the first work on semilinear equations with derivatives on black hole spacetimes.

1.2. The statement of the Main Theorem

Before introducing the null condition and stating the precise version of the Main Theorem, we briefly introduce the necessary concepts and notations on Kerr geometry and the vector field method. See [20] for more details.

The Kerr metric in the Boyer–Lindquist coordinates takes the following form:

$$\begin{aligned} g_K = &-\left(1 - \frac{2M}{r\left(1+\frac{a^2\cos^2\theta}{r^2}\right)}\right)dt^2 + \frac{1+\frac{a^2\cos^2\theta}{r^2}}{1-\frac{2M}{r}+\frac{a^2}{r^2}}dr^2 + r^2\left(1+\frac{a^2\cos^2\theta}{r^2}\right)d\theta^2 \\ &+ r^2\left(1+\frac{a^2}{r^2}+\left(\frac{2M}{r}\right)\frac{a^2\sin^2\theta}{r^2\left(1+\frac{a^2\cos^2\theta}{r^2}\right)}\right)\sin^2\theta d\phi^2 \\ &- 4M\frac{a\sin^2\theta}{r\left(1+\frac{a^2\cos^2\theta}{r^2}\right)}dtd\phi. \end{aligned} \tag{1}$$

Let r_+ be the larger root of $\Delta = r^2 - 2Mr + a^2$. Then $r = r_+$ is the event horizon. In this paper, we will use the coordinate system (t^*, r, θ, ϕ^*) defined by

$$t^* = t + \chi(r)h(r), \quad \text{where} \quad \frac{dh(r)}{dr} = \frac{2Mr}{r^2 - 2Mr + a^2},$$
$$\phi^* = \phi + \chi(r)P(r), \quad \text{where} \quad \frac{dP(r)}{dr} = \frac{a}{r^2 - 2Mr + a^2},$$

where

$$\chi(r) = \begin{cases} 1, & r \le r_Y^- - (r_Y^- - r_+)/2, \\ 0, & r \ge r_Y^- - (r_Y^- - r_+)/4, \end{cases}$$

r_+ is as above and $r_Y^- > r_+$ is a fixed constant very close to r_+, the value of which can be determined from the proof of the energy estimates in [20]. Following the notation in [20], we will use $t^* = \tau$ to denote the t^* slice on which we want to prove estimates, and $t^* = \tau_0$ to denote the t^* slice on which the initial data is posed.

In [20], following [6], various quantities are defined via an explicit identification of the Kerr spacetime with the corresponding Schwarzschild spacetime with the same mass. We recall the identification:

$$r_S^2 - 2Mr_S = r^2 - 2Mr + a^2, \quad t_S + \chi(r_S)2M\log(r_S - 2M) = t^*,$$
$$\theta_S = \theta, \quad \phi_S = \phi^*,$$

where χ is as above.

Define

$$r_S^* = r_S + 2M\log(r_S - 2M) - 3M - 2M\log M,$$
$$\mu = 2M/r_S, \quad u = \tfrac{1}{2}(t_S - r_S^*), \quad v = \tfrac{1}{2}(t_S + r_S^*).$$

We note that the variable u will also be used to quantify decay.

We define in coordinates

$$\underline{L} = \partial_u \quad \text{in the } (u, v, \theta_S, \phi_S) \text{ coordinates},$$
$$L = 2\partial_{t^*} + \chi(r)\frac{a}{Mr_+}\partial_{\phi^*} - \underline{L}.$$

We can now define the "good" and "bad" derivatives. Define

$$\bar{\nabla} \in \left\{\frac{1}{r}\partial_\theta, \frac{1}{r}\partial_\phi\right\}, \quad \bar{D} \in \left\{L, \frac{1}{r}\partial_\theta, \frac{1}{r}\partial_\phi\right\}, \quad D \in \left\{\frac{1}{1-\mu}\underline{L}, L, \frac{1}{r}\partial_\theta, \frac{1}{r}\partial_\phi\right\}.$$

Notice that D spans the whole tangent space and we always have $[D, \partial_{t^*}] = 0$.

We now define the null condition. On Minkowski spacetime, the classical null condition can be defined geometrically by requiring the nonlinearity to have the form

$$A^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi,$$

where A satisfies $A^{\mu\nu}\xi_\mu\xi_\nu = 0$ whenever ξ is null. On Kerr spacetime, we would like to define a notion of the null condition that includes this geometric notion. This is also because many physically relevant semilinear equations satisfy this condition. On the other hand, in order to prove the global existence result, we need to use the vector fields that capture the good derivative. We would therefore like to define the null condition using the vector fields defined in [7], [20], i.e., using D and $\overline{D}$. In particular, we want the nonlinearity to have at least one good, i.e., $\overline{D}$, derivative. This on its own is however inconsistent with the geometric null condition. We therefore allow a term in the quadratic nonlinearity that does not have a good derivative but decays in r.

Definition 1.2. Consider the nonlinearity $F(\Phi, D\Phi, t^*, r, \theta, \phi^*)$. We say that F satisfies the *null condition* if

$$F = \Lambda_0(\Phi, t^*, r, \theta, \phi^*) D\Phi\overline{D}\Phi + \Lambda_1(\Phi, t^*, r, \theta, \phi^*) D\Phi D\Phi + \mathcal{C}(\Phi, D\Phi, t^*, r, \theta, \phi^*),$$

where

$$|D_\Phi^{i_1}\partial_{t^*}^{i_2}\partial_r^{i_3}\partial_\theta^{i_4}\partial_{\phi^*}^{i_5}\Lambda_j| \le C(t^*)^{-i_2} r^{-i_3} \quad \text{for } i_1+i_2+i_3+i_4+i_5 \le 16,\ j = 0, 1,$$
$$|D_\Phi^{i_1}\partial_{t^*}^{i_2}\partial_r^{i_3}\partial_\theta^{i_4}\partial_{\phi^*}^{i_5}\Lambda_1| \le C(t^*)^{-i_2} r^{-1-i_3} \quad \text{for } i_1+i_2+i_3+i_4+i_5 \le 16 \text{ and } r \ge 9t^*/10,$$

and $\mathcal{C}$ denotes a polynomial that is at least cubic in $D\Phi$ (with coefficients in $\Phi, t^*, r, \theta, \phi^*$) satisfying

$$|D_\Phi^{i_1}\partial_{t^*}^{i_2}\partial_r^{i_3}\partial_\theta^{i_4}\partial_{\phi^*}^{i_5}\mathcal{C}| \le C(t^*)^{-i_2} r^{-i_3} \sum_{s=3}^{S} |D\Phi|^s \quad \text{for } i_1+i_2+i_3+i_4+i_5 \le 16.$$

Remark 1.3. The null condition is a special structure for the quadratic nonlinearity. We note that in our case, the restriction is necessary only for $r \ge 9t^*/10$. Moreover, higher order terms should give better estimates and do not need any special structure.

Under this definition of the null condition, global existence holds for small data. Moreover, the solution Φ satisfies pointwise decay estimates. In order to appropriately describe smallness, we introduce the language of compatible currents. Define the energy-momentum tensor

$$T_{\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - \tfrac{1}{2} g_{\mu\nu}\partial^\alpha\Phi\partial_\alpha\Phi.$$

By virtue of the wave equation, T is divergence-free,

$$\nabla^\mu T_{\mu\nu} = 0.$$

For a vector field V, define the compatible currents

$$J_\mu^V(\Phi) = V^\nu T_{\mu\nu}(\Phi), \qquad K^V(\Phi) = \pi_{\mu\nu}^V T^{\mu\nu}(\Phi),$$

where $\pi_{\mu\nu}^V$ is the deformation tensor defined by

$$\pi_{\mu\nu}^V = \tfrac{1}{2}(\nabla_\mu V_\nu + \nabla_\nu V_\mu).$$

In particular, $K^V(\Phi) = \pi^V_{\mu\nu} = 0$ if V is Killing. Since the energy-momentum tensor is divergence-free,

$$\nabla^\mu J^V_\mu(\Phi) = K^V(\Phi).$$

We also define the modified currents

$$J^{V,w}_\mu(\Phi) = J^V_\mu(\Phi) + \frac{1}{8}(w\partial_\mu\Phi^2 - \partial_\mu w\Phi^2),$$
$$K^{V,w}(\Phi) = K^V(\Phi) + \frac{1}{4}w\partial^\nu\Phi\partial_\nu\Phi - \frac{1}{8}\Box_g w\Phi^2.$$

Then

$$\nabla^\mu J^{V,w}_\mu(\Phi) = K^{V,w}(\Phi).$$

In [20], we have used the currents corresponding to N and (Z, w^Z) defined by

$$N = \partial_{t^*} + e(y_1(r)\hat{Y} + y_2(r)\hat{V}), \quad Z = u^2\underline{L} + v^2 L, \quad w^Z = \frac{8tr^*_S(1-2M/r_S)}{r},$$

where

$$y_1(r) = 1 + \frac{1}{(\log(r-r_+))^3}, \quad y_2(r) = \frac{1}{(\log(r-r_+))^3},$$

r_+ is the larger root of $\Delta = r^2 - 2Mr + a^2$, and $\hat{Y}$ and $\hat{V}$ are compactly supported vector fields in a neighborhood of $\{r_+ \leq r \leq r_Y^-\}$ and are null in $\{r_+ \leq r \leq r_Y^-\}$, and e is an appropriately small constant depending only on a (see [20]). Since N is future-directed, we have the pointwise inequality

$$J^N_\mu(\Phi)n^\mu_{\Sigma_{t^*}} \geq 0.$$

In [20] we have shown that there exists a constant C such that

$$\int_{\Sigma_{t^*}} J^{Z,w^Z}_\mu(\Phi)n^\mu_{\Sigma_{t^*}} + C(t^*)^2\int_{\Sigma_{t^*}\cap\{r\leq r_Y^-\}} J^N_\mu(\Phi)n^\mu_{\Sigma_{t^*}} \geq 0.$$

These energy quantities will be used for Φ as well as for derivatives of Φ. We now define the commutators that we will use. ∂_{t^*} is a Killing vector field that is defined as the coordinate vector field with respect to the (t^*, r, θ, ϕ^*) coordinate system. Near the event horizon, we use the commutator $\hat{Y}$ which is compactly supported in $\{r \leq r_Y^+\}$ (where $r_Y^+ > r_Y^-$ is an explicit constant in [20]), null in $\{r_+ \leq r \leq r_Y^-\}$ and transverse to the event horizon (see [20]). $\hat{Y}$ has good positivity property that reflects the celebrated red-shift effect. In the region of large r, we use the commutators $\tilde{\Omega}$. Let Ω_i be a basis of vector fields of rotations in Schwarzschild spacetimes. An explicit realization can be $\Omega = \partial_\phi$, $\sin\phi\partial_\theta \pm \frac{\cos\phi\cos\theta}{\sin\theta}\partial_\phi$. Define $\tilde{\Omega}_i = \chi(r)\Omega_i$ to be cutoff so that it is supported in $\{r > R_\Omega\}$ and equals Ω_i for $r > R_\Omega + 1$ for some large R. We also use the commutator S that would provide an improved decay rate of the solution. It is defined as

$$S = t^*\partial_{t^*} + h(r_S)\partial_r,$$

where

$$h(r_S) = \begin{cases} r_S - 2M, & r_S \sim 2M, \\ r_S^*(1-\mu), & r \geq R, \end{cases}$$

for some large R, and is interpolated so that it is smooth and nonnegative. For the commutators, we also use the notation that

$$\Gamma \in \{\partial_{t^*}, \tilde{\Omega}\}.$$

We are now in a position to state our Main Theorem precisely.

Theorem 1.4. *Consider the equation*

$$\Box_g \Phi = F(\Phi, D\Phi, t^*, r, \theta, \phi^*), \tag{2}$$

where F satisfies the null condition. There exists an ϵ such that if the initial data of Φ satisfies

$$\sum_{i+j+k=16} \int_{\Sigma_{\tau_0}} \left(J_\mu^{Z+CN,w^Z}(\hat{Y}^k \partial_{t^*}^i \tilde{\Omega}^j \Phi) n^\mu_{\Sigma_{\tau_0}} + J_\mu^{Z+CN,w^Z}(\hat{Y}^k S \partial_{t^*}^i \tilde{\Omega}^j \Phi) n^\mu_{\Sigma_{\tau_0}} \right) \leq \epsilon$$

and

$$\sum_{\ell=0}^{13} \left(r|D^\ell \Phi(\tau_0)| + r|D^\ell S\Phi(\tau_0)| \right) \leq \epsilon,$$

then Φ exists globally in time. Moreover, for all $\eta > 0$, we can take a sufficiently small such that the solution Φ obeys the decay estimate

$$|\Phi| \leq C\epsilon r^{-1} u^{-1/2} (t^*)^\eta, \quad |D\Phi| \leq C\epsilon r^{-1} u^{-1} (t^*)^\eta, \quad |\overline{D}\Phi| \leq C\epsilon r^{-1} (t^*)^{-1+\eta} \quad \text{for } r \geq R,$$

$$|\Phi| \leq C_\delta \epsilon (t^*)^{-3/2+\eta} r^\delta, \quad |D\Phi| \leq C_\delta \epsilon (t^*)^{-3/2+\eta} r^{-1/2+\delta} \quad \text{for } r \leq t^*/4.$$

We specialize to a particular case which resembles better the classical null condition [17].

Theorem 1.5. *Consider the equation*

$$\Box_g \Phi = \Gamma(\Phi) A^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi, \tag{3}$$

where A satisfies $A^{\mu\nu}\xi_\mu\xi_\nu = 0$ whenever $\xi \in TK$ is null. Then the statement of Theorem 1.4 holds.

The above formulation is geometric and independent of the choice of coordinates. We note that this condition is obviously satisfied by the wave map equation in the intrinsic formulation.

1.3. The case of Minkowski spacetime

We now outline the proof of the main theorem. In the original proof in [17], many symmetries of Minkowski spacetime are captured and exploited using the vector field method. Kerr spacetime, on the other hand, lacks symmetries and this limits the set of vector fields that are at our disposal. In view of this, we would like to re-examine the proof of the small data global existence result for the nonlinear wave equation with a null condition on Minkowski spacetime, using only the vector fields whose analogues in Kerr spacetimes have been established in previous works. In particular, we would have to avoid using the Lorentz boost.

We first study the decay properties of the solutions to the linear wave equation on Minkowski spacetime. Since the vector field $T = \partial_t$ is Killing and $Z = u^2\partial_u + v^2\partial_v$ is conformally Killing, we see for $w = 8t$ that

$$\int_{\Sigma_t} J^T_\mu(\Phi)n^\mu_{\Sigma_t}, \qquad \int_{\Sigma_t} J^{Z,w^Z}_\mu(\Phi)n^\mu_{\Sigma_t}$$

are conserved in time.

Decay can be proved using the above conserved quantities for $V\Phi$ for appropriate vector fields V. It is proved separately for $r \geq t/2$ and $r \leq t/2$. In the former case, we use the fact that $\Omega_{ij} = x_i\partial_{x_j} + x_j\partial_{x_i}$ is Killing on Minkowski spacetime and hence $\Box_m(\Omega^k\Phi) = 0$. Since Ω has a weight in r, it can be proved that

$$|D\Phi|^2 \leq Cr^{-2}\sum_{k=0}^{2}\int_{\Sigma_t} J^T_\mu(\Omega^k\Phi)n^\mu_{\Sigma_t}.$$

Notice that in this region $r^{-2} \leq Ct^{-2}$. It is known, for example by the representation formula, that this decay rate cannot be improved. In the region $r \leq t/2$, however, the decay rate is better. One can consider the conformal energy

$$\int J^{Z,w^Z}_\mu(\Phi)n^\mu_{\Sigma_t} \geq \int_{\Sigma_t}\left(u^2(\underline{L}\Phi)^2 + v^2(L\Phi)^2 + (u^2+v^2)|\nabla\!\!\!/\Phi|^2 + \left(\frac{u^2+v^2}{r^2}\right)\Phi^2\right),$$

where $u = \frac{1}{2}(t-r)$, $v = \frac{1}{2}(t+r)$. In particular, we have

$$|D\Phi|^2 \leq t^{-2}\int_{\Sigma_t\cap\{r\leq t/2\}} \tau^2(D\Phi)^2 \leq t^{-2}\int J^{Z,w^Z}_\mu(\Phi)n^\mu_{\Sigma_t}.$$

To improve the decay rate in this region, we can consider the equation for $S\Phi = (t\partial_t + r\partial_r)\Phi$ and use the integrated decay estimates as in [19], [20]. This approach allows us to avoid the use of Lorentz boost of [17] and the global elliptic estimates of [18], neither of which has a clear analogue in Kerr spacetimes. On Minkowski spacetime, a local energy decay estimate can be proved using the vector field $\left(1 - \frac{1}{(1+r^2)^{(1+\delta)/2}}\right)\partial_r$ for the linear wave equation [28], which together with the conformal energy yields

$$\begin{aligned}\int_t^{(1.1)t}\int_{\Sigma_{t'}\cap\{r\leq t'/2\}} r^{-1-\delta}J^T_\mu(\Phi)n^\mu_{\Sigma_{t'}}\,dt' &\leq C\int_{\Sigma_\tau\cap\{r\leq t/2\}}(D\Phi)^2\\ &\leq Ct^{-2}\int J^{Z,w^Z}_\mu(\Phi)n^\mu_{\Sigma_t}.\end{aligned}$$

This would imply that there exists a "dyadic" sequence $t_i \sim (1.1)^i t_0$ on which there is better decay

$$\int_{\Sigma_{t_i}\cap\{r\le t/2\}} r^{-1-\delta} J^T_\mu(\Phi) n^\mu_{\Sigma_{t_i}} \le C t_i^{-3} \int J^{Z,w^Z}_\mu(\Phi) n^\mu_{\Sigma_t}.$$

Since S is Killing on Minkowski spacetime, $\Box_m \Phi = 0$ implies $\Box_m(S\Phi) = 0$. Then the above argument would give

$$\int_t^{(1.1)t} \int_{\Sigma_t\cap\{r\le t/2\}} r^{-1-\delta} J^T_\mu(S\Phi) n^\mu_{\Sigma_t} dt \le t^{-2} \int J^{Z,w^Z}_\mu(S\Phi) n^\mu_{\Sigma_t}.$$

Since $S = t\partial_t + r\partial_r$ has a weight in t, we can integrate along the integral curves of S from the "good" t_i slice and get

$$\int_{\Sigma_t\cap\{r\le t/2\}} r^{-1-\delta} J^T_\mu(\Phi) n^\mu_{\Sigma_t} \le C t^{-3} \int J^{Z,w^Z}_\mu(\Phi) n^\mu_{\Sigma_{t_0}}.$$

Together with the use of Ω, we have the pointwise estimate

$$\begin{aligned} |D\Phi|^2 &\le C r^{-1+\delta} \sum_{k=0}^{2} \int_{\Sigma_t\cap\{r\le t/2\}} r^{-1-\delta} J^T_\mu(\Omega^k\Phi) n^\mu_{\Sigma_t} \\ &\le C r^{-1+\delta} t^{-3} \sum_{k=0}^{2} \int J^{Z,w^Z}_\mu(\Omega^k\Phi) n^\mu_{\Sigma_{t_0}}. \end{aligned}$$

We now study how this decay rate can be used for the nonlinear problem. The main idea is to prove the above conservation and decay estimates in a bootstrap setting, showing that the decay to the linear wave equation is sufficiently strong that the nonlinear terms can be treated as error. In this framework, the decay of t^{-1} is borderline and since the decay rate is better when $r \le t/2$, the difficulty arises when dealing with terms in the region $r \ge t/2$. Furthermore, in order to achieve this decay of t^{-1} it is imperative to show that $\int_{\Sigma_t} J^T_\mu(\Phi) n^\mu_{\Sigma_t}$ is uniformly bounded in time.

We now show a heuristic argument. With the inhomogeneous term, the conservation law for the energy now has the error term

$$\int_{\Sigma_t} J^T_\mu(\Phi) n^\mu_{\Sigma_t} \le \int_{\Sigma_{t_0}} J^T_\mu(\Phi) n^\mu_{\Sigma_{t_0}} + \left(\int_{t_0}^t \left(\int_{\Sigma_t} (\Box_m\Phi)^2 \right)^{1/2} dt \right)^2,$$

and that for the conformal energy has the error term

$$\int_{\Sigma_t} J^{Z,w^Z}_\mu(\Phi) n^\mu_{\Sigma_t} \le \int_{\Sigma_{t_0}} J^{Z,w^Z}_\mu(\Phi) n^\mu_{\Sigma_{t_0}} + \left(\int_{t_0}^t \left(\int_{\Sigma_t} (t^2+r^2)(\Box_m\Phi)^2 \right)^{1/2} dt \right)^2$$

Since $\Box_m\Phi$ is quadratic in $D\Phi$, we can use Hölder's inequality on the inside integral to control one term in L^2 and one in L^∞. However, since on the linear level $D\Phi$ is bounded in L^2 and decays as t^{-1} in L^∞, the inhomogeneous term for the estimate for the energy is controlled by

$$\left(\int_{t_0}^{t} t^{-1}\left(\int_{\Sigma_t} J^T_\mu(\Phi)n^\mu_{\Sigma_t}\right)^{1/2} dt\right)^2.$$

This is insufficient to show that the energy is bounded. We therefore need to make use of the null condition. The null condition would allow one to prove

$$\int (D\Phi\overline{D}\Phi)^2 \leq Ct^{-2}\int_{\Sigma_t} J^{Z,w^Z}_\mu(\partial^k\Phi)n^\mu_{\Sigma_t}. \tag{4}$$

In order to prove this estimate, we observe that in the conformal energy, the good derivatives $(\partial_v, \nabla\!\!\!/)$ has better decay rates. In order to use this, we then need to control the conformal energy. Using again the null condition, the inhomogeneous term in the conservation law for the conformal energy can be bounded by

$$\left(\int_{t_0}^{t} t^{-1}\left(\int_{\Sigma_t} J^{Z,w^Z}_\mu(\Phi)n^\mu_{\Sigma_t}\right)^{1/2} dt\right)^2.$$

This would not be sufficient to prove that the conformal energy is bounded, but is sufficient to prove that it grows no faster than t^η for sufficiently small data. This in turn would be sufficient to prove the boundedness of the energy and obtain all the necessary decay rates. In practice, the argument is more complicated as we need to control the higher order energy and conformal energy in order to obtain the decay rates.

1.4. The case of Kerr spacetime

In [7] and [20], all the analogues of the above estimates have been proved in the linear setting in Kerr spacetimes. However, it is apparent from the linear case that several issues arise when we apply a similar strategy to the nonlinear problem on Kerr spacetime.

Among other issues, two difficulties loom large. The first is the lack of symmetries in Kerr spacetimes. While Kerr spacetimes possess the Killing vector field ∂_{t^*}, it is spacelike in a neighborhood of the event horizon and thus does not give nonnegative conserved quantities. The works [6], [7] suggest that we can instead use the vector fields N and Z on Kerr spacetime as substitutes for T and Z on Minkowski spacetime. N is constructed as the Killing vector field ∂_{t^*} added to a small amount of the red-shift vector field near the event horizon. The red-shift vector field, first introduced in [9], takes advantage of the geometry of the event horizon and has been used crucially to obtain decay rates in [9], [6], [7], [19], and [20]. It is one of the few stable features of the Schwarzschild spacetime. The vector field Z approaches the corresponding Z on Minkowski spacetime at the asymptotically flat end and has the weights in r and t^* from which we can prove decay. These vector fields, however, do not correspond to any symmetries of Kerr spacetimes, and therefore, as is already apparent in the linear scenario, the energy estimates would contain error terms that need to be controlled. One consequence is that even in the linear setting, the conformal energy is not bounded. Similar issues arise for the vector field commutators Ω and S, which are crucial to obtaining pointwise decay estimates, whose corresponding error terms at the linear level have been studied in [7], [19], [20]. A further issue that arises in the case of the Kerr spacetime is the lack of good vector field commutators that are useful to obtain control of higher order derivatives. This has been treated in the linear setting in [6] and [7] using ∂_{t^*} and the red-shift vector field as commutators and retrieving all other derivatives via elliptic estimates. In the nonlinear setting, we again use elliptic estimates, noting however that the proof of the elliptic estimates now couples with that of the energy estimates in a bootstrap argument.

Secondly, Kerr spacetimes contain trapped null geodesics. As a consequence, any decay results at the linear level must involve a loss of derivatives. This is manifested in the degeneracy of the integrated decay estimate near $r = 3M$. We note, however, that on the linear level the nondegenerate energy can be proved to be bounded without any loss of derivatives. We therefore prove energy bounds that are consistent with the linear scenario. We would try to prove on the highest level of derivatives only a boundedness result and begin to prove decay results on the level of fewer derivatives. However, as we will see, the nonlinear effect comes into play and it is not possible to prove even the boundedness of the nondegenerate energy at the highest level of derivatives. We can nevertheless show that the energy is bounded by $(t^*)^\eta$. On the level of one less derivative, we can prove that the conformal energy grows no faster than $\tau^{1+\eta}$. Using this fact as we prove the estimates for the nondegenerate energy, we can show that at this level of derivatives, the nondegenerate energy is bounded. This is crucial for obtaining the necessary borderline decay of $(t^*)^{-1}$ in $r \geq t^*/2$, thus allowing us to close the bootstrap argument. Trapping would also cause a loss in derivatives when controlling the error terms arising from the commutation with S. To tackle this problem, we would commute with S only once. With this approach, we would not have an improved decay for $DS\Phi$ in $r \leq t^*/2$. Nevertheless, we can show that the bootstrap can be closed. Here we make use of the fact that as we close the assumptions for $S\Phi$, we are at a level of derivatives of Φ such that the local energy flux decays.

In the next section, we will introduce the energy quantities on Kerr spacetimes that can be thought of as analogues of the energy, conformal energy and integrated local energy. In Section 2, we will state the energy estimates that they satisfy. In Section 4, we will state the elliptic estimates that will be used. Then in Section 5, we prove the necessary L^∞ estimates. With all this preparation, we then prove all the estimates using a bootstrap argument in Section 6. This then easily implies the main theorem in Section 7.

2. The energy quantities

We use three kinds of energy quantities, following the notation in [20]. They represent the nondegenerate energy, the conformal energy and the energy norm for the integrated decay estimate. The nondegenerate energy controls all derivatives:

Proposition 2.1.

$$\int_{\Sigma_\tau} (D\Phi)^2 \leq C \int_{\Sigma_\tau} J^N_\mu(\Phi) n^\mu_{\Sigma_\tau}.$$

The conformal energy gives different weights to different derivatives and this will be crucially used to capture the null condition:

Proposition 2.2.

$$\int_{\Sigma_\tau\cap\{r\geq r_Y^-\}}\left(u^2(\underline{L}\Phi)^2+v^2(L\Phi)^2+(u^2+v^2)|\nabla\!\!\!/\Phi|^2+\left(\frac{u^2+v^2}{r^2}\right)\Phi^2\right)$$
$$\leq C\int_{\Sigma_\tau}J_\mu^{Z+N,w^Z}(\Phi)n^\mu_{\Sigma_\tau}+C^2\tau^2\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}J^N_\mu(\Phi)n^\mu_{\Sigma_\tau}.$$

We use the following notations even though they do not correspond to any vector fields:

Definition 2.3.

$$K^{X_0}(\Phi)=r^{-1-\delta}\mathbb{1}_{\{|r-3M|\geq M/8\}}J^N_\mu(\Phi)n^\mu_{\Sigma_\tau}+r^{-1-\delta}(\partial_r\Phi)^2+r^{-3-\delta}\Phi^2,$$
$$K^{X_1}(\Phi)=r^{-1-\delta}J^N_\mu(\Phi)n^\mu_{\Sigma_\tau}+r^{-3-\delta}\Phi^2.$$

3. The energy estimates

We have proved in [20] the energy estimates for the energy quantities defined in the last section for $\Box_{g_K}\Phi=G$. We have boundedness for the nondegenerate energy:

Proposition 3.1. *Let $G=G_1+G_2$ be any way to decompose the function G. Then*

$$\int_{\Sigma_\tau}J^N_\mu(\Phi)n^\mu_{\Sigma_\tau}+\int_{\mathcal{H}(\tau',\tau)}J^N_\mu(\Phi)n^\mu_{\mathcal{H}^+}+\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}}K^N(\Phi)+\iint_{\mathcal{R}(\tau',\tau)}K^{X_0}(\Phi)$$
$$\leq C\Bigg(\int_{\Sigma_{\tau'}}J^N_\mu(\Phi)n^\mu_{\Sigma_{\tau'}}+\left(\int_{\tau'-1}^{\tau+1}\left(\int_{\Sigma_{t^*}}G_1^2\right)^{1/2}dt^*\right)^2+\iint_{\mathcal{R}(\tau'-1,\tau+1)}G_1^2$$
$$+\sum_{m=0}^{1}\iint_{\mathcal{R}(\tau'-1,\tau+1)}r^{1+\delta}(\partial_{t^*}^m G_2)^2+\sup_{t^*\in[\tau'-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq M/8\}}G_2^2\Bigg).$$

We need an extra derivative for the inhomogeneous term because of trapping. If we know a priori that G is supported away from the trapped region, this loss in derivative is unnecessary.

Proposition 3.2. *Let $G=G_1+G_2$ be any way to decompose the function G. Suppose G_2 is supported away from $\{r:|r-3M|\leq M/8\}$. Then*

$$\int_{\Sigma_\tau}J^N_\mu(\Phi)n^\mu_{\Sigma_\tau}+\int_{\mathcal{H}(\tau',\tau)}J^N_\mu(\Phi)n^\mu_{\mathcal{H}^+}+\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}}K^N(\Phi)+\iint_{\mathcal{R}(\tau',\tau)}K^{X_0}(\Phi)$$
$$\leq C\Bigg(\int_{\Sigma_{\tau'}}J^N_\mu(\Phi)n^\mu_{\Sigma_{\tau'}}+\left(\int_{\tau'-1}^{\tau+1}\left(\int_{\Sigma_{t^*}}G_1^2\right)^{1/2}dt^*\right)^2+\iint_{\mathcal{R}(\tau'-1,\tau+1)}G_1^2$$
$$+\iint_{\mathcal{R}(\tau'-1,\tau+1)}r^{1+\delta}G_2^2\Bigg).$$

The estimates for K^{X_1} were also proved. It is estimated in the same way as K^{X_0} but with an extra derivative.

Proposition 3.3.

$$\iint_{\mathcal{R}(\tau',\tau)} K^{X_1}(\Phi) \le C\Bigg(\sum_{m=0}^{1}\int_{\Sigma_{\tau'}} J^N_\mu(\partial^m_{t^*}\Phi)n^\mu_{\Sigma_{\tau'}} + \sum_{m=0}^{1}\bigg(\int_{\tau'-1}^{\tau+1}\bigg(\int_{\Sigma_{t^*}}(\partial^m_{t^*}G_1)^2\bigg)^{1/2}dt^*\bigg)^2 + \sum_{m=0}^{1}\iint_{\mathcal{R}(\tau'-1,\tau+1)}(\partial^m_{t^*}G_1)^2 + \sum_{m=0}^{2}\iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta}(\partial^m_{t^*}G_2)^2 + \sup_{t^*\in[\tau'-1,\tau+1]}\sum_{m=0}^{1}\int_{\Sigma_{t^*}\cap\{|r-3M|\le M/8\}}(\partial^m_{t^*}G_2)^2\Bigg).$$

As before, if the inhomogeneous term is supported away from the trapped set, we can save a derivative:

Proposition 3.4. *Let $G = G_1 + G_2$ be any way to decompose the function G. Suppose G_2 is supported away from $\{r : |r - 3M| \le M/8\}$. Then*

$$\iint_{\mathcal{R}(\tau',\tau)} K^{X_1}(\Phi) \le C\sum_{m=0}^{1}\Bigg(\int_{\Sigma_{\tau'}} J^N_\mu(\partial^m_{t^*}\Phi)n^\mu_{\Sigma_{\tau'}} + \bigg(\int_{\tau'-1}^{\tau+1}\bigg(\int_{\Sigma_{t^*}}(\partial^m_{t^*}G_1)^2\bigg)^{1/2}dt^*\bigg)^2 + \iint_{\mathcal{R}(\tau'-1,\tau+1)}(\partial^m_{t^*}G_1)^2 + \iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta}(\partial^m_{t^*}G_2)^2\Bigg).$$

The conformal energy satisfies the following estimates:

Proposition 3.5. *For $\delta, \delta' > 0$ sufficiently small and $0 \le \gamma < 1$, there exist $c = c(\delta,\gamma)$ and $C = C(\delta,\gamma)$ such that the following estimate holds for any solution to $\Box_{g_K}\Phi = G$:*

$$\begin{aligned}
c\int_{\Sigma_\tau} J^{Z,w^Z}_\mu(\Phi)n^\mu_{\Sigma_\tau} &+ \tau^2\int_{\Sigma_\tau\cap\{r\le\gamma\tau\}} J^N_\mu(\Phi)n^\mu_{\Sigma_\tau} \\
\le C\int_{\Sigma_{\tau_0}} J^{Z+CN,w^Z}_\mu(\Phi)n^\mu_{\Sigma_{\tau_0}} &+ C\iint_{\mathcal{R}(\tau_0,\tau)} t^* r^{-1+\delta} K^{X_1}(\Phi) \\
&+ C\delta'\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\le t^*/2\}}(t^*)^2 K^{X_0}(\Phi) + C(\delta'+a)\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\le r_Y^-\}}(t^*)^2 K^N(\Phi) \\
&+ C(\delta')^{-1}\bigg(\int_{\tau_0}^{\tau}\bigg(\int_{\Sigma_{t^*}\cap\{r\ge t^*/2\}} r^2G^2\bigg)^{1/2}dt^*\bigg)^2 \\
&+ C(\delta')^{-1}\sum_{m=0}^{1}\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\le 9t^*/10\}}(t^*)^2 r^{1+\delta}(\partial^m_{t^*}G)^2 \\
&+ C(\delta')^{-1}\sup_{t^*\in[\tau_0,\tau]}\int_{\Sigma_{t^*}\cap\{r_Y^-\le r\le 25M/8\}}(t^*)^2G^2.
\end{aligned}$$

Remark 3.6. As in Proposition 3.2, we can save a derivative if we know that the inhomogeneous term is supported away from the trapped region. More precisely, let $G = G_1 + G_2$ be any way to decompose the function G. Suppose G_2 is supported away from $\{r : |r - 3M| \leq M/8\}$. Then we can replace

$$\sum_{m=0}^{1} \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq 9t^*/10\}} (t^*)^2 r^{1+\delta} (\partial_{t^*}^m G)^2 + \sup_{t^*\in[\tau_0,\tau]} \int_{\Sigma_{t^*}\cap\{r_Y^- \leq r\leq 25M/8\}} (t^*)^2 G^2$$

by

$$\sum_{m=0}^{1} \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq 9t^*/10\}} (t^*)^2 r^{1+\delta} (\partial_{t^*}^m G_1)^2 + \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq 9t^*/10\}} (t^*)^2 r^{1+\delta} G_2^2 + \sup_{t^*\in[\tau_0,\tau]} \int_{\Sigma_{t^*}\cap\{r_Y^- \leq r\leq 25M/8\}} (t^*)^2 G_1^2$$

in Proposition 3.5. This follows from a straightforward modification of the proof in [20].

The estimates for K^{X_0} and K^{X_1} can be localized to $r \leq t^*/2$ if we control them by the conformal energy:

Proposition 3.7. (i) (Localized estimate for X_0)

$$\begin{aligned}\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq t^*/2\}} & K^{X_0}(\Phi) \\ & \leq C\left(\tau^{-2}\int_{\Sigma_{\tau'}} J_\mu^{Z+N,w^Z}(\Phi) n^\mu_{\Sigma_{\tau'}} + C\int_{\Sigma_{\tau'}\cap\{r\leq r_Y^-\}} J_\mu^N(\Phi) n^\mu_{\Sigma_{\tau'}}\right) \\ & \quad + C\Bigg(\sum_{m=0}^{1}\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq 9t^*/10\}} r^{1+\delta'}(\partial_{t^*}^m G)^2 \\ & \quad + \sup_{t^*\in[\tau'-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq M/8\}\cap\{r\leq 9t^*/10\}} G^2\Bigg).\end{aligned}$$

(ii) (Localized estimate for X_1)

$$\begin{aligned}\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq t^*/2\}} & K^{X_1}(\Phi) \\ \leq C\Bigg(\tau^{-2}\sum_{m=0}^{1} & \int_{\Sigma_{\tau'}} J_\mu^{Z+N,w^Z}(\partial_{t^*}^m\Phi) n^\mu_{\Sigma_{\tau'}} + C\sum_{m=0}^{1}\int_{\Sigma_{\tau'}\cap\{r\leq r_Y^-\}} J_\mu^N(\partial_{t^*}^m\Phi) n^\mu_{\Sigma_{\tau'}}\Bigg) \\ + C\Bigg(\sum_{m=0}^{2} & \iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq 9t^*/10\}} r^{1+\delta}(\partial_{t^*}^m G)^2 \\ + \sup_{t^*\in[\tau'-1,\tau+1]} & \sum_{m=0}^{1}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq M/8\}\cap\{r\leq 9t^*/10\}} (\partial_{t^*}^m G)^2\Bigg).\end{aligned}$$

Remark 3.8. As before, if $G = G_1 + G_2$ and G_2 is supported outside $\{|r-3M| \leq M/8\}$, we can replace, in Proposition 3.7(i),

$$\sum_{m=0}^{1} \iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq 9t^*/10\}} r^{1+\delta} (\partial_{t^*}^m G_2)^2 + \sup_{t^*\in[\tau'-1,\tau+1]} \int_{\Sigma_{t^*}\cap\{|r-3M|\leq M/8\}\cap\{r\leq 9t^*/10\}} G^2$$

by

$$\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq 9t^*/10\}} r^{1+\delta} G_2^2;$$

and in Proposition 3.7(ii), replace

$$\sum_{m=0}^{2} \iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq 9t^*/10\}} r^{1+\delta} (\partial_{t^*}^m G_2)^2 + \sum_{m=0}^{1} \sup_{t^*\in[\tau'-1,\tau+1]} \int_{\Sigma_{t^*}\cap\{|r-3M|\leq M/8\}\cap\{r\leq 9t^*/10\}} (\partial_{t^*}^m G_2)^2$$

by

$$\sum_{m=0}^{1} \iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq 9t^*/10\}} r^{1+\delta} (\partial_{t^*}^m G_2)^2.$$

4. The elliptic estimates and Hardy inequality

We have also proved in [20] the following elliptic estimates:

Proposition 4.1. *Suppose* $\Box_{g_K} \Phi = G$. *For* $m \geq 1$ *and for any* α, *we have*

(i) (Boundedness of weighted energy)

$$\int_{\Sigma_\tau\cap\{r\geq r_Y^-\}} r^\alpha (D^m\Phi)^2 \leq C_\alpha \Big(\sum_{j=0}^{m-1} \int_{\Sigma_\tau} r^\alpha J_\mu^N (\partial_{t^*}^j \Phi) n_{\Sigma_\tau}^\mu + \sum_{j=0}^{m-2} \int_{\Sigma_\tau} r^\alpha (D^j G)^2 \Big).$$

(ii) (Boundedness of local energy) *For any* $0 < \gamma < \gamma'$,

$$\int_{\Sigma_\tau\cap\{r_Y^-\leq r\leq \gamma t^*\}} r^\alpha (D^m\Phi)^2 \leq C_\alpha \Big(\sum_{j=0}^{m-1} \int_{\Sigma_\tau\cap\{r\leq\gamma' t^*\}} r^\alpha J_\mu^N (\partial_{t^*}^j \Phi) n_{\Sigma_\tau}^\mu + \sum_{j=0}^{m-2} \int_{\Sigma_\tau} r^\alpha (D^j G)^2 \Big).$$

We need a Hardy-type inequality that improves the analogous one in [20]:

Proposition 4.2. *For $R > R'$,*

$$\int_{\Sigma_\tau \cap \{r \geq R\}} r^{\alpha-2}\Phi^2 \leq C \int_{\Sigma_\tau \cap \{r \geq R'\}} r^\alpha J^N_\mu(\Phi) n^\mu_{\Sigma_\tau}.$$

Proof. Let $k(r)$ be defined by solving

$$k'(r, \theta, \phi) = r^{\alpha-2}\,\mathrm{vol}$$

in the region $r \geq R'$, where $\mathrm{vol} = \mathrm{vol}(r, \theta, \phi)$ is the volume density on Σ_τ with r, θ, ϕ coordinates, with boundary condition $k(R', \theta, \phi) = 0$. Now

$$\int_{\Sigma_\tau \cap \{r \geq R\}} r^{\alpha-2}\Phi^2 = \iiint_{R'}^{\infty} k'(r)\Phi^2\, dr\, d\theta\, d\phi \leq -2 \iiint k(r)\Phi\partial_r\Phi\, dr\, d\theta\, d\phi$$
$$\leq 2\Big(\iiint_{R'}^{\infty} \frac{1 + k(r)^2}{1 + k'(r)}(\partial_r\Phi)^2\, dr\, d\theta\, d\phi\Big)^{1/2}\Big(\iiint_{R'}^{\infty} (1 + k'(r))\Phi^2\, dr\, d\theta\, d\phi\Big)^{1/2}$$

Notice that $\mathrm{vol} \sim r^2$, $k(r) \sim r^{\alpha+1}$ and $1 + k'(r) \sim r^\alpha$. Hence $\frac{1+k(r)^2}{1+k'(r)} \sim r^\alpha\,\mathrm{vol}$. The lemma follows. □

With the help of this Hardy inequality, we are able to "localize" the elliptic estimates for $r \geq R$.

Proposition 4.3. *Suppose $\Box_{g_K}\Phi = G$. For $m \geq 1$ and for any α, and any $R > R'$,*

$$\int_{\Sigma_\tau \cap \{r \geq R\}} r^\alpha (D^m\Phi)^2$$
$$\leq C_{\alpha,R,R'}\Big(\sum_{j=0}^{m-1} \int_{\Sigma_\tau \cap \{r \geq R'\}} r^\alpha J^N_\mu(\partial^j_{t^*}\Phi) n^\mu_{\Sigma_\tau} + \sum_{j=0}^{m-2} \int_{\Sigma_\tau} r^\alpha (D^j G)^2\Big).$$

Near the event horizon, elliptic estimates have been proved to control all the derivatives if we have control on the ∂_{t^*} and the $\hat{Y}$ derivatives [6], [7], [20]:

Proposition 4.4. *Suppose $\Box_{g_K}\Phi = G$. For every $m \geq 1$,*

$$\int_{\Sigma_\tau \cap \{r \leq r^-_Y\}} (D^m\Phi)^2$$
$$\leq C\Big(\sum_{j+k \leq m-1} \int_{\Sigma_\tau \cap \{r \leq r^-_Y\}} J^N_\mu(\partial^j_{t^*}\hat{Y}^k\Phi) n^\mu_{\Sigma_\tau} + \sum_{j=0}^{m-2} \int_{\Sigma_\tau \cap \{r \leq r^-_Y\}} (D^j G)^2\Big).$$

This is useful together with the following control for the equation commuted with $\hat{Y}$:

Proposition 4.5. *Suppose* $\Box_{g_K}\Phi = G$. *For every* $k \geq 0$,

$$\begin{aligned}
&\int_{\Sigma_\tau\cap\{r\leq r_Y^+\}} J^N_\mu(\hat{Y}^k\Phi)n^\mu_{\Sigma_\tau} + \int_{\mathcal{H}(\tau',\tau)} J^N_\mu(\hat{Y}^k\Phi)n^\mu_{\Sigma_\tau} + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} J^N_\mu(\hat{Y}^k\Phi)n^\mu_{\Sigma_{t^*}}\\
&\quad\leq C\Bigg(\sum_{j+m\leq k}\int_{\Sigma_{\tau'}\cap\{r\leq r_Y^+\}} J^N_\mu(\partial^j_{t^*}\hat{Y}^m\Phi)n^\mu_{\Sigma_{\tau'}} + \sum_{j=0}^k\int_{\Sigma_\tau\cap\{r\leq r_Y^+\}} J^N_\mu(\partial^j_{t^*}\Phi)n^\mu_{\Sigma_\tau}\\
&\qquad+\sum_{j=0}^k\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq 23M/8\}} J^N_\mu(\partial^j_{t^*}\Phi)n^\mu_{\Sigma_{t^*}} + \sum_{j=0}^k\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq 23M/8\}}(D^jG)^2\Bigg).
\end{aligned}$$

5. Pointwise estimates

We prove pointwise estimates using Sobolev embedding. We will have different estimates in the regions $\{r \geq t^*/4\}$ and $\{r \leq t^*/4\}$.

We first consider $\{r \geq t^*/4\}$. For this region, we will prove five different pointwise estimates. First, we prove a boundedness result for $D\Phi$ (Proposition 5.1) using only standard Sobolev embedding and the elliptic estimates of Proposition 4.1. Then we prove decay estimates of r^{-1} for $D^\ell\Phi$ using the r weight in the vector field commutator $\tilde{\Omega}$ and the nondegenerate energy (Proposition 5.2). It is crucial that this depends only on the nondegenerate energy but not the conformal energy because we will not be able to prove boundedness of the conformal energy (which already is the case in the *linear* situation, see [7], [1], [20]). Notice that Proposition 5.1 does not follow from Proposition 5.2 because the latter requires an extra derivative. This save in derivatives is strictly speaking not necessary for the bootstrap if we have instead assumed an extra derivative of regularity in the initial data. Thirdly, using similar ideas, we will prove the decay of r^{-1} for Φ using $\tilde{\Omega}$ and the conformal energy (Proposition 5.3). Then we prove an extra decay rate of $D\Phi$ using the conformal energy. For any derivatives, we will have an extra decay in the u variable, which degenerates in the wave zone (Proposition 5.5). For the good derivatives, we will have an extra decay in the v variable (Proposition 5.6). This decay rate will be crucial in capturing the good derivative in the null condition.

Proposition 5.1. *For* $r \geq t^*/4$ *we have*

$$|D\Phi|^2 \leq C\Bigg(\sum_{k=0}^2\int_{\Sigma_\tau} J^N_\mu(\partial^k_{t^*}\Phi)n^\mu_{\Sigma_\tau} + \sum_{k=0}^1\int_{\Sigma_\tau}(D^k\Box_{g_K}\Phi)^2\Bigg).$$

Proof. By standard Sobolev embedding in three dimensions and Proposition 4.1,

$$\begin{aligned}
|D\Phi|^2 &\leq C\sum_{k=1}^3\int_{\Sigma_\tau\cap\{r\geq r_Y^-\}}(D^k\Phi)^2\\
&\leq C\Bigg(\sum_{k=0}^2\int_{\Sigma_\tau} J^N_\mu(\partial^k_{t^*}\Phi)n^\mu_{\Sigma_\tau} + \sum_{k=0}^1\int_{\Sigma_\tau}(D^k\Box_{g_K}\Phi)^2\Bigg). \qquad\Box
\end{aligned}$$

We then prove the decay rate of r^{-1} for $D^\ell\Phi$. The idea here is standard: Making use of the commutator $\tilde{\Omega}$, we use the Sobolev embedding on the 2-sphere and then integrate along the r direction.

Proposition 5.2. *For $r \geq t^*/4$ and $\ell \geq 1$, we have*

$$|D^\ell\Phi|^2 \leq Cr^{-2}\Big(\sum_{m=0}^{\ell}\sum_{k=0}^{2}\int_{\Sigma_\tau} J_\mu^N(\partial_{t^*}^m\tilde{\Omega}^k\Phi)n_{\Sigma_\tau}^\mu + \sum_{j=0}^{\ell-1}\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{u'\sim u\}\cap\{r\geq\tau/2\}} (D^j\Box_{g_K}(\tilde{\Omega}^k\Phi))^2\Big).$$

Proof. We have

$$\begin{aligned} r^2|D^\ell\Phi|^2 &\leq C\int_{\mathbb{S}^2}\big((D^\ell\Phi)^2 + (\tilde{\Omega}D^\ell\Phi)^2 + (\tilde{\Omega}^2D^\ell\Phi)^2\big)r^2\,dA \\ &\leq C\sum_{k=0}^{2}\Big(\int_{\mathbb{S}^2(\tilde{r})}(\tilde{\Omega}^kD^\ell\Phi)^2\tilde{r}^2\,dA \\ &\quad + \int_r^{\tilde{r}}\int_{\mathbb{S}^2(r')}\big(|\partial_r\tilde{\Omega}^kD^\ell\Phi\tilde{\Omega}^kD^\ell\Phi|(r')^2 + (\tilde{\Omega}^kD^\ell\Phi)^2r'\big)\,dA\,dr'\Big). \end{aligned}$$

Noticing that $|[D,\tilde{\Omega}]\Phi| \leq C|D\Phi|$, we have

$$\begin{aligned} r^2|D^\ell\Phi|^2 &\leq C\sum_{k=0}^{2}\Big(\int_{\mathbb{S}^2(\tilde{r})}(\tilde{\Omega}^kD^\ell\Phi)^2\tilde{r}^2\,dA \\ &\quad + \int_r^{\tilde{r}}\int_{\mathbb{S}^2(r')}\big(|\partial_rD^\ell\tilde{\Omega}^k\Phi D^\ell\tilde{\Omega}^k\Phi|(r')^2 + (\tilde{\Omega}^kD^\ell\Phi)^2r'\big)\,dA\,dr'\Big) \\ &\leq C\sum_{k=0}^{2}\Big(\int_{\mathbb{S}^2(\tilde{r})}(D^\ell\tilde{\Omega}^k\Phi)^2\tilde{r}^2\,dA \\ &\quad + \int_r^{\tilde{r}}\int_{\mathbb{S}^2(r')}\big(|D^{\ell+1}\tilde{\Omega}^k\Phi D^\ell\tilde{\Omega}^k\Phi|(r')^2 + (D^\ell\tilde{\Omega}^k\Phi)^2r'\big)\,dA\,dr'\Big) \\ &\leq C\sum_{k=0}^{2}\Big(\int_{\mathbb{S}^2(\tilde{r})}(D^\ell\tilde{\Omega}^k\Phi)^2\tilde{r}^2\,dA \\ &\quad + \int_r^{\tilde{r}}\int_{\mathbb{S}^2(r')}\big((D^{\ell+1}\tilde{\Omega}^k\Phi)^2(r')^2 + (D^\ell\tilde{\Omega}^k\Phi)^2(r')^2\big)\,dA\,dr'\Big). \end{aligned}$$

Take $r \leq \tilde{r} \leq r+1$. By Proposition 4.1,

$$\begin{aligned} &\sum_{k=0}^{2}\int_r^{r+1}\int_{\mathbb{S}^2(r')}\big((D^{\ell+1}\tilde{\Omega}^k\Phi)^2(r')^2 + (D^\ell\tilde{\Omega}^k\Phi)^2(r')^2\big)\,dA\,dr' \\ &\quad\leq C\Big(\sum_{m=0}^{\ell}\sum_{k=0}^{2}\int_{\Sigma_\tau} J_\mu^N(\partial_{t^*}^m\Omega^k\Phi)n_{\Sigma_\tau}^\mu + \sum_{j=0}^{\ell-1}\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{u'\sim u\}\cap\{r\geq\tau/2\}}(D^j\Box_{g_K}(\tilde{\Omega}^k\Phi))^2\Big). \end{aligned}$$

By pigeonholing on this we also see that for some $\tilde{r}$,

$$\sum_{k=0}^{2}\int_{\mathbb{S}^2(\tilde{r})}(D^\ell\Omega^k\Phi)^2\tilde{r}^2\,dA$$
$$\leq C\bigg(\sum_{m=0}^{\ell}\sum_{k=0}^{2}\int_{\Sigma_\tau}J^N_\mu(\partial^m_{t^*}\Omega^k\Phi)n^\mu_{\Sigma_\tau}+\sum_{j=0}^{\ell-1}\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{u'\sim u\}\cap\{r\geq\tau/2\}}(D^j\Box_{g_K}(\tilde{\Omega}^k\Phi))^2\bigg).\quad\square$$

We would also like to prove the pointwise decay in r for Φ. However, we need to use the conformal energy as well as the nondegenerate energy. We note that only the decay in r will be used in the bootstrap argument, the decay in u is proved to achieve the decay rate asserted in Theorem 1.

Proposition 5.3. *Consider* $\Box_{g_K}\Phi=G$. *For* $r\geq t^*/4$, *we have*

$$|\Phi|^2\leq Cr^{-2}(1+|u|)^{-1}\bigg(\sum_{k=0}^{2}\int_{\Sigma_\tau}J^{Z+N,w^Z}_\mu(\Omega^k\Phi)n^\mu_{\Sigma_\tau}+C\tau^2\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{r\leq r^-_Y\}}J^N_\mu(\Omega^k\Phi)n^\mu_{\Sigma_\tau}\bigg).$$

Proof. Following the proof of Proposition 5.2, we have

$$r^2|\Phi|^2$$
$$\leq C\sum_{k=0}^{2}\bigg(\int_{\mathbb{S}^2(\tilde{r})}(\tilde{\Omega}^k\Phi)^2\tilde{r}^2\,dA+\bigg|\int_r^{\tilde{r}}\int_{\mathbb{S}^2(r')}\Big(|\tilde{\Omega}^k\Phi D\tilde{\Omega}^k\Phi|(r')^2+(\tilde{\Omega}^k\Phi)^2r'\Big)\,dA\,dr'\bigg|\bigg).$$

We will treat separately the cases $|u|\leq 1$, $u\geq 1$, $u\leq 1$. For $|u|\leq 1$, take $r\leq\tilde{r}\leq r+1$. By Proposition 2.2,

$$\sum_{k=0}^{2}\int_r^{r+1}\int_{\mathbb{S}^2(r')}\Big((D\tilde{\Omega}^k\Phi)^2(r')^2+(\tilde{\Omega}^k\Phi)^2(r')^2\Big)\,dA\,dr'$$
$$\leq C\bigg(\sum_{k=0}^{2}\int_{\Sigma_\tau}J^{Z+N,w^Z}_\mu(\Omega^k\Phi)n^\mu_{\Sigma_\tau}+C\tau^2\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{r\leq r^-_Y\}}J^N_\mu(\Omega^k\Phi)n^\mu_{\Sigma_\tau}\bigg).$$

By pigeonholing on this we also see that for some $\tilde{r}$,

$$\sum_{k=0}^{2}\int_{\mathbb{S}^2(\tilde{r})}(\Omega^k\Phi)^2\tilde{r}^2\,dA$$
$$\leq C\bigg(\sum_{k=0}^{2}\int_{\Sigma_\tau}J^{Z+N,w^Z}_\mu(\Omega^k\Phi)n^\mu_{\Sigma_\tau}+C\tau^2\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{r\leq r^-_Y\}}J^N_\mu(\Omega^k\Phi)n^\mu_{\Sigma_\tau}\bigg).$$

For $u\geq 1$, pick a fixed R and let $\tilde{r}\in[R,R+1]$. Then by a pigeonhole argument, there is some $\tilde{r}$ such that

$$\sum_{k=0}^{2}\int_{\mathbb{S}^2(\tilde{r})}(\tilde{\Omega}^k\Phi)^2\tilde{r}^2\,dA$$
$$\leq Cu^{-2}\bigg(\sum_{k=0}^{2}\int_{\Sigma_\tau}J^{Z+N,w^Z}_\mu(\Omega^k\Phi)n^\mu_{\Sigma_\tau}+C\tau^2\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{r\leq r^-_Y\}}J^N_\mu(\Omega^k\Phi)n^\mu_{\Sigma_\tau}\bigg).$$

By Proposition 2.2,

$$\sum_{k=0}^{2}\int_R^r\int_{\mathbb{S}^2(r')}\big(|\tilde{\Omega}^k\Phi D\tilde{\Omega}^k\Phi|(r')^2+(\tilde{\Omega}^k\Phi)^2r'\big)\,dA\,dr'$$
$$\leq C(rt^*u^{-2}+ru^{-2})\bigg(\sum_{k=0}^{2}\int_{\Sigma_\tau}J_\mu^{Z+N,w^Z}(\Omega^k\Phi)n^\mu_{\Sigma_\tau}$$
$$+C\tau^2\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}J_\mu^N(\Omega^k\Phi)n^\mu_{\Sigma_\tau}\bigg).$$

Using the fact that $t^*\leq Cu$ in this region, we have the desired bound in this region.

Finally, for $u\leq 1$, pick $\tilde{r}\in[-2u,-3u]$. Then by a pigeonhole argument, there is an $\tilde{r}$ such that

$$\sum_{k=0}^{2}\int_{\mathbb{S}^2(\tilde{r})}(\tilde{\Omega}^k\Phi)^2\tilde{r}^2\,dA$$
$$\leq Cu^{-2}\bigg(\sum_{k=0}^{2}\int_{\Sigma_\tau}J_\mu^{Z+N,w^Z}(\Omega^k\Phi)n^\mu_{\Sigma_\tau}+C\tau^2\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}J_\mu^N(\Omega^k\Phi)n^\mu_{\Sigma_\tau}\bigg).$$

By Proposition 2.2,

$$\sum_{k=0}^{2}\int_r^\infty\int_{\mathbb{S}^2(r')}\big(|\tilde{\Omega}^k\Phi D\tilde{\Omega}^k\Phi|(r')^2+(\tilde{\Omega}^k\Phi)^2r'\big)\,dA\,dr'$$
$$\leq C|u|^{-1}\bigg(\sum_{k=0}^{2}\int_{\Sigma_\tau}J_\mu^{Z+N,w^Z}(\Omega^k\Phi)n^\mu_{\Sigma_\tau}+C\tau^2\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}J_\mu^N(\Omega^k\Phi)n^\mu_{\Sigma_\tau}\bigg),$$

which gives the desired bound. □

We would like to use the conformal energy and elliptic estimates to prove decay in the u variable. However, we need to be careful when applying the localized version of the elliptic estimates. In particular, we need to perform a dyadic decomposition in the variable u. We remark that we can prove this for any number of derivatives by iterating the cutoff procedure in the proof of the following proposition. However, as this will not be necessary later, we will be content with the following proposition:

Proposition 5.4. *Suppose* $\Box_{g_K}\Phi=G$. *Let* $r\geq t^*/4$, $\ell=1$ *or* 2 *and* u_0 *be the* u*-coordinate corresponding to the two sphere given by the coordinate functions* (τ,r_0). *Then*

$$\int_{r_0}^{r_0+1}\int_{\mathbb{S}^2(r')}(D^\ell\Phi)^2(r')^2\,dA\,dr'$$
$$\leq C(1+|u_0|)^{-2}\sum_{j=0}^{\ell-1}\bigg(\int_{\Sigma_\tau}J_\mu^{Z+CN}(\partial_{t^*}^j\Phi)n^\mu_{\Sigma_\tau}+C\tau^2\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}J_\mu^N(\partial_{t^*}^j\Phi)n^\mu_{\Sigma_\tau}\bigg)$$
$$+C\sum_{j=0}^{\ell-2}\int_{\Sigma_\tau\cap\{u\sim u_0\}\cap\{r\geq\tau/2\}}(D^jG)^2.$$

Proof. The $\ell = 1$ case is trivial. For $\ell = 2$, we consider separately *Case* 0: $|u_0| \leq C_*$, *Case* 1_k: $2^k \leq u_0 \leq 2^{k+1}$, and *Case* 2_k: $-2^{k+1} \leq u_0 \leq -2^k$, $k \geq \frac{\log C_*}{\log 2}$ for some sufficiently large but fixed C_*. In Case 0, we have $|u| \leq C$ for the range $[r_0, r_0 + 1]$ and hence the proposition is obvious as we have $1 \leq C(1 + |u|)^{-2}$.

For the other cases, we consider a cutoff function $\chi : \mathbb{R} \to \mathbb{R}_{\geq 0}$ which is compactly supported in $[-2, 2]$ and identically 1 in $[-1, 1]$. In Case 1_k (resp. 2_k), we consider $\tilde{\Phi}$ be defined by $\tilde{\Phi}(\tau, r, \theta, \phi) = \chi(2^{-k+3}(r - r_0))\Phi(\tau, r, \theta, \phi)$. Then $\tilde{\Phi}$ is supported in $[r_0 - 2^{k-2}, r_0 + 2^{k-2}]$ and equals Φ in $[r_0 - 2^{k-3}, r_0 + 2^{k-3}]$. On the support of $\tilde{\Phi}$, $|\Box_{g_K} \tilde{\Phi} - G| \leq C \sum_{j=0}^{1} 2^{-(2-j)k} |D^j \Phi|$. Moreover, on the support of $\tilde{\Phi}$,

$$|u - u_0| \leq \frac{1}{2}|r_S^* - (r_0^*)_S| \leq \frac{1}{2}|r - r_0| + \frac{M}{2}\left|\log \frac{r - 2M}{r_0 - 2M}\right| \leq |r - r_0| \leq 2^{k-1}$$

for r_0 sufficiently large (which we can assume for otherwise τ and r must both be bounded, in which case we must be in Case 0 for appropriately chosen C_*). Hence $u \sim 2^k$ (resp. $u \sim -2^k$).

Therefore, by Proposition 4.1(i) applied twice, first to $\tilde{\Phi}$ then to Φ, we have

$$\begin{aligned}
&\int_{r_0}^{r_0+1} \int_{\mathbb{S}^2(r')} (D^2\Phi)^2 (r')^2 \, dA \, dr' \leq \int_{\Sigma_\tau \cap \{r_0 \leq r \leq r_0+1\}} (D^2 \tilde{\Phi})^2 \\
&\quad \leq C \sum_{j=0}^{1} \int_{\Sigma_\tau \cap \{r_0 - 2^{k-3} \leq r \leq r_0 + 2^{k-3}\}} J_\mu^N (\partial_{t^*}^j \tilde{\Phi}) n_{\Sigma_\tau}^\mu \\
&\qquad + C \sum_{j=0}^{1} \int_{\Sigma_\tau \cap \{r_0 - 2^{k-2} \leq r \leq r_0 + 2^{k-2}\}} 2^{-(2-j)2k} (D^j \Phi)^2 + C \int_{\Sigma_\tau \cap \{r_0 - 2^{k-2} \leq r \leq r_0 + 2^{k-2}\}} G^2 \\
&\quad \leq C \sum_{j=0}^{1} \int_{\Sigma_\tau \cap \{r_0 - 2^{k-2} \leq r \leq r_0 + 2^{k-2}\}} (2^{-2k} \Phi^2 + J_\mu^N (\partial_{t^*}^j \Phi) n_{\Sigma_\tau}^\mu) \\
&\qquad + C \int_{\Sigma_\tau \cap \{r_0 - 2^{k-2} \leq r \leq r_0 + 2^{k-2}\}} G^2 \\
&\quad \leq C(1 + |u_0|)^{-2} \sum_{j=0}^{1} \left(\int_{\Sigma_\tau} J_\mu^{Z+CN} (\partial_{t^*}^j \Phi) n_{\Sigma_\tau}^\mu + C\tau^2 \int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} J_\mu^N (\partial_{t^*}^j \Phi) n_{\Sigma_\tau}^\mu \right) \\
&\qquad + C \int_{\Sigma_\tau \cap \{u \sim u_0\}} G^2. \qquad \Box
\end{aligned}$$

Using this we can prove more decay in the u variable:

Proposition 5.5. *Suppose $\Box_{g_K} \Phi = G$. For $r \geq t^*/4$ and $\ell \geq 1$, we have*

$$\begin{aligned}
|D\Phi|^2 &\leq Cr^{-2}(1 + |u|)^{-2} \sum_{j=0}^{1} \sum_{k=0}^{2} \left(\int_{\Sigma_\tau} J_\mu^{Z+CN} (\partial_{t^*}^j \tilde{\Omega}^k \Phi) n_{\Sigma_\tau}^\mu \right. \\
&\quad \left. + C\tau^2 \int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} J_\mu^N (\partial_{t^*}^j \tilde{\Omega}^k \Phi) n_{\Sigma_\tau}^\mu \right) + Cr^{-2} \sum_{k=0}^{2} \int_{\Sigma_\tau \cap \{u' \sim u\} \cap \{r \geq \tau/2\}} (\Box_{g_K} (\tilde{\Omega}^k \Phi))^2.
\end{aligned}$$

Proof. Following the proof of Proposition 5.2, we have

$$r^2|D\Phi|^2 \le C\sum_{k=0}^{2}\Big(\int_{\mathbb{S}^2(\tilde{r})}(D\Omega^k\Phi)^2\tilde{r}^2\,dA + \int_r^{\tilde{r}}\int_{\mathbb{S}^2(r')}\big((D^2\Omega^k\Phi)(D\Omega^k\Phi)(r')^2+(D\Omega^k\Phi)^2r'\big)\,dA\,dr'\Big).$$

Take $r \le \tilde{r} \le r+1$. Then by Proposition 5.4,

$$\begin{aligned}&\sum_{k=0}^{2}\int_r^{r+1}\int_{\mathbb{S}^2(r')}\big((D^2\Omega^k\Phi)(D\Omega^k\Phi)(r')^2+(D\Omega^k\Phi)^2r'\big)\,dA\,dr'\\ &\le C(1+|u|)^{-2}\sum_{j=0}^{1}\sum_{k=0}^{2}\Big(\int_{\Sigma_\tau}J^{Z+CN}_\mu(\partial^j_{t^*}\tilde{\Omega}^k\Phi)n^\mu_{\Sigma_\tau}+C\tau^2\int_{\Sigma_\tau\cap\{r\le r^-_Y\}}J^N_\mu(\partial^j_{t^*}\tilde{\Omega}^k\Phi)n^\mu_{\Sigma_\tau}\Big)\\ &\quad+C\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{u'\sim u\}\cap\{r\ge\tau/2\}}(\Box_{g_K}(\tilde{\Omega}^k\Phi))^2.\end{aligned}$$

By pigeonholing on this we also find that for some $\tilde{r}$,

$$\begin{aligned}\sum_{k=0}^{2}\int_{\mathbb{S}^2(\tilde{r})}(D^\ell\Omega^k\Phi)^2\tilde{r}^2\,dA &\le C(1+|u|)^{-2}\sum_{j=0}^{1}\sum_{k=0}^{2}\Big(\int_{\Sigma_\tau}J^{Z+CN}_\mu(\partial^j_{t^*}\tilde{\Omega}^k\Phi)n^\mu_{\Sigma_\tau}\\ &\qquad+C\tau^2\int_{\Sigma_\tau\cap\{r\le r^-_Y\}}J^N_\mu(\partial^j_{t^*}\tilde{\Omega}^k\Phi)n^\mu_{\Sigma_\tau}\Big)\\ &\quad+C\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{u'\sim u\}\cap\{r\ge\tau/2\}}(\Box_{g_K}(\tilde{\Omega}^k\Phi))^2.\end{aligned}$$

□

We have a better pointwise decay for a "good" derivative:

Proposition 5.6. *For $r \ge t^*/4$, we have*

$$\begin{aligned}|\bar{D}\Phi|^2 \le Cr^{-4}\sum_{k=0}^{2}\sum_{i+j\le 1}\Big(&\int_{\Sigma_\tau}J^N_\mu(S^i\partial^j_{t^*}\Phi)n^\mu_{\Sigma_\tau}+\int_{\Sigma_\tau}J^{Z+CN}_\mu(\partial^j_{t^*}\tilde{\Omega}^k\Phi)n^\mu_{\Sigma_\tau}\\ &+C\tau^2\int_{\Sigma_\tau\cap\{r\le r^-_Y\}}J^N_\mu(\partial^j_{t^*}\tilde{\Omega}^k\Phi)n^\mu_{\Sigma_\tau}+\int_{\Sigma_\tau}(\Box_{g_K}(\tilde{\Omega}^k\Phi))^2\Big)\\ &+Cr^{-2}\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{r\ge\tau/2\}}(\Box_{g_K}(\tilde{\Omega}^k\Phi))^2.\end{aligned}$$

Proof. We have

$$\begin{aligned}
r^2|\bar{D}\Phi|^2 &\le C\int_{\mathbb{S}^2}\big((\bar{D}\Phi)^2+(\tilde{\Omega}\bar{D}\Phi)^2+(\tilde{\Omega}^2\bar{D}\Phi)^2\big)r^2\,dA\\
&\le C\sum_{k=0}^{2}\bigg(\int_{\mathbb{S}^2(\tilde{r})}(\tilde{\Omega}^k\bar{D}\Phi)^2\tilde{r}^2\,dA\\
&\qquad+\int_r^{\tilde{r}}\int_{\mathbb{S}^2(r')}\big(|\partial_r\tilde{\Omega}^k\bar{D}\Phi\tilde{\Omega}^k\bar{D}\Phi|(r')^2+(\tilde{\Omega}^k\bar{D}\Phi)^2r'\big)\,dA\,dr'\bigg).
\end{aligned}$$

Noticing that $|[D,\tilde{\Omega}]\Phi|\le C|D\Phi|$, $|[\bar{D},\tilde{\Omega}]\Phi|\le C(|\bar{D}\Phi|+r^{-1}|D\Phi|)$ and $|[\bar{D},\partial_r]\Phi|\le Cr^{-1}|D\Phi|$, we have

$$\begin{aligned}
r^2|\bar{D}\Phi|^2 &\le C\sum_{k=0}^{2}\bigg(\int_{\mathbb{S}^2(\tilde{r})}(\tilde{\Omega}^k\bar{D}\Phi)^2\tilde{r}^2\,dA\\
&\qquad+\int_r^{\tilde{r}}\int_{\mathbb{S}^2(r')}\big(|\partial_r\bar{D}\tilde{\Omega}^k\Phi\bar{D}\tilde{\Omega}^k\Phi|(r')^2+(\tilde{\Omega}^k\bar{D}\Phi)^2r'\big)\,dA\,dr'\bigg)\\
&\le C\sum_{k=0}^{2}\bigg(\int_{\mathbb{S}^2(\tilde{r})}\big((\bar{D}\tilde{\Omega}^k\Phi)^2+\tilde{r}^{-2}(D\tilde{\Omega}^k\Phi)^2\big)\tilde{r}^2\,dA\\
&\qquad+\int_r^{\tilde{r}}\int_{\mathbb{S}^2(r')}\big((\bar{D}D\tilde{\Omega}^k\Phi)^2+(\bar{D}\tilde{\Omega}^k\Phi)^2+(r')^{-2}(D\tilde{\Omega}^k\Phi)^2\big)(r')^2\,dA\,dr'\bigg).
\end{aligned}\tag{5}$$

The last term already exhibits better decay rate:

$$\int_r^{\tilde{r}}\int_{\mathbb{S}^2(r')}(r')^{-2}(D\tilde{\Omega}^k\Phi)^2(r')^2\,dA\,dr'\le Cr^{-2}\int_{\Sigma_\tau}J^N_\mu(\tilde{\Omega}^k\Phi)n^\mu_{\Sigma_\tau}.$$

We will now show that the energy quantities involving $\bar{D}$ obey better decay rates. This is immediate for the term $\int_r^{\tilde{r}}\int_{\mathbb{S}^2(r')}(\bar{D}\tilde{\Omega}^k\Phi)^2(r')^2\,dA\,dr'$ using the conformal energy:

$$\begin{aligned}
&\int_r^{\tilde{r}}\int_{\mathbb{S}^2(r')}(\bar{D}\tilde{\Omega}^k\Phi)^2(r')^2\,dA\,dr'\\
&\qquad\le Cv^{-2}\bigg(\int_{\Sigma_\tau}J^{Z+CN}_\mu(\tilde{\Omega}^k\Phi)n^\mu_{\Sigma_\tau}+C\tau^2\int_{\Sigma_\tau\cap\{r\le r_Y^-\}}J^N_\mu(\tilde{\Omega}^k\Phi)n^\mu_{\Sigma_\tau}\bigg).
\end{aligned}$$

However, we note that this cannot be shown directly for the term

$$\int_r^{\tilde{r}}\int_{\mathbb{S}^2(r')}(\bar{D}D\tilde{\Omega}^k\Phi)^2(r')^2\,dA\,dr'$$

with the conformal energy because $\Box_{g_K}$ does not commute with derivatives in every direction. In order to remedy this, we use the nondegenerate energy for $S\Phi$. In particular, we use the fact that for $r\ge r_Y^-$, $|\bar{D}\Phi|\le Cv^{-1}(|S\Phi|+u|D\Phi|+vr^{-1}|D\Phi|)$. We have

$$\begin{aligned}
&\int_r^{\tilde r}\int_{\mathbb{S}^2(r')}(\bar D D\tilde\Omega^k\Phi)^2(r')^2\,dA\,dr'\\
&\quad\le C\int_r^{\tilde r}\int_{\mathbb{S}^2(r')}\big((v')^{-2}(SD\tilde\Omega^k\Phi)^2+(u')^2(v')^{-2}(D^2\tilde\Omega^k\Phi)^2\\
&\qquad+(r')^{-2}(D^2\tilde\Omega^k\Phi)^2\big)(r')^2\,dA\,dr'\\
&\quad\le C\int_r^{\tilde r}\int_{\mathbb{S}^2(r')}\big((v')^{-2}(DS\tilde\Omega^k\Phi)^2+(v')^{-2}(D\tilde\Omega^k\Phi)^2\\
&\qquad+(u')^2(v')^{-2}(D^2\tilde\Omega^k\Phi)^2+(r')^{-2}(D^2\tilde\Omega^k\Phi)^2\big)(r')^2\,dA\,dr'.
\end{aligned}$$

Take $r\le\tilde r\le r+1$. We have, for the first two terms,

$$\int_r^{r+1}\int_{\mathbb{S}^2(r')}(v')^{-2}\big((DS\tilde\Omega^k\Phi)^2+(D\tilde\Omega^k\Phi)^2\big)(r')^2\,dA\,dr'\le Cv^{-2}\sum_{j=0}^{1}\int_{\Sigma_\tau}J^N_\mu(S^j\tilde\Omega^k\Phi)n^\mu_{\Sigma_\tau}.$$

The third term can be estimated by Proposition 5.4,

$$\begin{aligned}
&\int_r^{r+1}\int_{\mathbb{S}^2(r')}(u')^2(v')^{-2}(D^2\tilde\Omega^k\Phi)(r')^2\,dA\,dr'\\
&\quad\le Cv^{-2}\sum_{j=0}^{1}\Big(\int_{\Sigma_\tau}J^{Z+CN}_\mu(\partial^j_{t^*}\tilde\Omega^k\Phi)n^\mu_{\Sigma_\tau}+C\tau^2\int_{\Sigma_\tau\cap\{r\le r_Y^-\}}J^N_\mu(\partial^j_{t^*}\tilde\Omega^k\Phi)n^\mu_{\Sigma_\tau}\Big)\\
&\qquad+C\int_{\Sigma_\tau\cap\{u'\sim u\}\cap\{r\ge\tau/2\}}(\Box_{g_K}(\tilde\Omega^k\Phi))^2.
\end{aligned}$$

The fourth term can be estimated elliptically by Proposition 4.1:

$$\int_r^{r+1}\int_{\mathbb{S}^2(r')}(r')^{-2}(D^2\tilde\Omega^k\Phi)^2(r')^2\,dA\,dr'\le Cr^{-2}\Big(\sum_{j=0}^{1}\int_{\Sigma_\tau}J^N_\mu(\partial^j_{t^*}\tilde\Omega^k\Phi)n^\mu_{\Sigma_\tau}+\int_{\Sigma_\tau}(\Box_{g_K}(\tilde\Omega^k\Phi))^2\Big).$$

Collecting all the above estimates and noting that $r\ge\tau/2$, we get

$$\begin{aligned}
&\sum_{k=0}^{2}\int_r^{r+1}\int_{\mathbb{S}^2(r')}\big((\bar DD\tilde\Omega^k\Phi)^2+(\bar D\tilde\Omega^k\Phi)^2+(r')^{-2}(D\tilde\Omega^k\Phi)^2\big)(r')^2\,dA\,dr'\\
&\quad\le Cv^{-2}\sum_{k=0}^{2}\sum_{i+j\le1}\Big(\int_{\Sigma_\tau}J^N_\mu(S^i\partial^j_{t^*}\Phi)n^\mu_{\Sigma_\tau}+\int_{\Sigma_\tau}J^{Z+CN}_\mu(\partial^j_{t^*}\tilde\Omega^k\Phi)n^\mu_{\Sigma_\tau}\\
&\qquad+C\tau^2\int_{\Sigma_\tau\cap\{r\le r_Y^-\}}J^N_\mu(\partial^j_{t^*}\tilde\Omega^k\Phi)n^\mu_{\Sigma_\tau}+\int_{\Sigma_\tau}(\Box_{g_K}(\tilde\Omega^k\Phi))^2\Big)\\
&\qquad+C\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{r\ge\tau/2\}}(\Box_{g_K}(\tilde\Omega^k\Phi))^2. \qquad (6)
\end{aligned}$$

By pigeonholing on this we also deduce that for some $\tilde{r}$,

$$\sum_{k=0}^{2}\int_{\mathbb{S}^2(\tilde{r})}\big((\bar{D}\tilde{\Omega}^k\Phi)^2+\tilde{r}^{-2}(D\tilde{\Omega}^k\Phi)^2\big)\tilde{r}^2\,dA$$
$$\leq Cv^{-2}\sum_{k=0}^{2}\sum_{i+j\leq 1}\bigg(\int_{\Sigma_\tau}J^N_\mu(S^i\partial^j_{t^*}\Phi)n^\mu_{\Sigma_\tau}+\int_{\Sigma_\tau}J^{Z+CN}_\mu(\partial^j_{t^*}\tilde{\Omega}^k\Phi)n^\mu_{\Sigma_\tau}$$
$$+C\tau^2\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}J^N_\mu(\partial^j_{t^*}\tilde{\Omega}^k\Phi)n^\mu_{\Sigma_\tau}+\int_{\Sigma_\tau}(\Box_{g_K}(\tilde{\Omega}^k\Phi))^2\bigg)$$
$$+C\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{r\geq\tau/2\}}(\Box_{g_K}(\tilde{\Omega}^k\Phi))^2. \tag{7}$$

(5)–(7) together imply the proposition. ◻

We now turn to the region $r\leq t^*/4$. We first show a simple Sobolev embedding result.

Proposition 5.7. *Suppose $\Box_{g_K}\Phi=G$. For $\ell\geq 1$ and $r\leq t^*/4$,*

$$|D^\ell\Phi|^2\leq C\bigg(\sum_{j+m\leq\ell+1}\int_{\Sigma_\tau\cap\{r\leq t^*/2\}}J^N_\mu(\partial^m_{t^*}\hat{Y}^j\Phi)n^\mu_{\Sigma_\tau}+\sum_{j=0}^{\ell}\int_{\Sigma_\tau}(D^jG)^2\bigg).$$

We can capture better estimates in r if we use an extra derivative.

Proposition 5.8. *For $\ell\geq 1$ and $r\leq t^*/4$,*

$$|D^\ell\Phi|^2\leq Cr^{-2}\bigg(\sum_{j+m+k\leq\ell+2}\int_{\Sigma_\tau\cap\{r\leq t^*/2\}}J^N_\mu(\partial^m_{t^*}\hat{Y}^j\tilde{\Omega}^k\Phi)n^\mu_{\Sigma_\tau}$$
$$+\sum_{j=0}^{\ell+1-k}\sum_{k=0}^{2}\int_{\Sigma_\tau}(D^j\Box_{g_K}(\tilde{\Omega}^k\Phi))^2\bigg).$$

Proof. We only need to consider the situation when $r\geq R_\Omega+C$. For otherwise, this proposition is implied by Proposition 5.7 since r is finite. We assume from now on that $r\geq R_\Omega+C$. Following the proof of Proposition 5.2, we have

$$r^2|D^\ell\Phi|^2\leq C\sum_{k=0}^{2}\int_{\mathbb{S}^2(\tilde{r})}\bigg((D^\ell\tilde{\Omega}^k\Phi)^2\tilde{r}^2\,dA+\int_{\tilde{r}}^{r}\int_{\mathbb{S}^2(r')}\big((D^{\ell+1}\tilde{\Omega}^k\Phi)^2(r')^2$$
$$+(D^\ell\tilde{\Omega}^k\Phi)^2(r')^2\big)\,dA\,dr'\bigg)$$

Take $r-1\leq\tilde{r}\leq r$. By Propositions 4.1(ii) and 4.4,

$$\sum_{k=0}^{2}\int_{r-1}^{r}\int_{\mathbb{S}^2(r')}\big((D^{\ell+1}\tilde{\Omega}^k\Phi)^2(r')^2+(D^\ell\tilde{\Omega}^k\Phi)^2(r')^2\big)\,dA\,dr'$$
$$\leq C\bigg(\sum_{j+m\leq\ell}\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{r\leq t^*/2\}}J^N_\mu(\partial^m_{t^*}\hat{Y}^j\Omega^k\Phi)n^\mu_{\Sigma_\tau}+\sum_{j=0}^{\ell-1}\sum_{k=0}^{2}\int_{\Sigma_\tau}(D^j\Box_{g_K}(\tilde{\Omega}^k\Phi))^2\bigg).$$

By pigeonholing on this we also infer that for some $\tilde{r}$ with $r-1 \leq \tilde{r} \leq r$,

$$\sum_{k=0}^{2}\int_{\mathbb{S}^2(\tilde{r})}(D\Omega^k\Phi)^2\tilde{r}^2\,dA$$
$$\leq C\Big(\sum_{j+m\leq \ell}\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{r\leq t^*/2\}} J^N_\mu(\partial_{t^*}^m\hat{Y}^j\Omega^k\Phi)n^\mu_{\Sigma_\tau}+\sum_{j=0}^{\ell-1}\sum_{k=0}^{2}\int_{\Sigma_\tau}(D^j\Box_{g_K}(\tilde{\Omega}^k\Phi))^2\Big). \qquad \square$$

We also have pointwise estimates for Φ instead of $D\Phi$ if we use the conformal energy.

Proposition 5.9. *Suppose $\Box_{g_K}\Phi = 0$. For $r \leq t^*/4$,*

$$|\Phi|^2 \leq C\tau^{-2}\Big(\sum_{i+j\leq 2}\int_{\Sigma_\tau} J^{Z+N,w^Z}_\mu(\hat{Y}^i\partial_{t^*}^j\Phi)n^\mu_{\Sigma_\tau} + C\tau^2\sum_{i+j\leq 2}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}} J^N_\mu(\hat{Y}^i\partial_{t^*}^j\Phi)n^\mu_{\Sigma_\tau}\Big).$$

Proof. By Sobolev embedding in three dimensions, for $r \leq t^*/4$,

$$|\Phi|^2 \leq C\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{r\leq t^*/4\}}(D^k\Phi)^2.$$

Then, using the elliptic estimates in Propositions 4.1(ii) and 4.4, we have

$$|\Phi|^2 \leq C\sum_{i+j\leq 2}\int_{\Sigma_\tau\cap\{r\leq t^*/2\}}\big(\Phi^2 + J^N_\mu(\hat{Y}^i\partial_{t^*}^j\Phi)n^\mu_{\Sigma_\tau}\big).$$

Using Proposition 2.2, we conclude the proof. $\square$

We proceed to show that the pointwise estimate is better if we use the vector field commutator S. To this end, we first show that we can control a fixed t^* quantity by an integrated quantity. The proof follows ideas in [19], [20] and applies an integration in the direction of S.

Proposition 5.10. *For any sufficiently regular Φ, not necessarily satisfying any differential equations, and α_0 a constant,*

$$\int_{\Sigma_\tau\cap\{r\leq\tau/4\}} r^{\alpha_0-2}\Phi^2 \leq C\tau^{-1}\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/3\}}\big(r^{\alpha_0-2}\Phi^2 + r^{\alpha_0-2}(S\Phi)^2\big).$$

Proof. To use the estimates for $S\Phi$, we need to integrate along integral curves of S. The following argument imitates that for proving improved decay for the homogeneous equation in [20]. We first find the integral curves by solving the ordinary differential equation

$$\frac{dr_S}{dt^*_S} = \frac{h(r_S)}{t^*_S}$$

where $h(r_S)$ is as in the definition of S. Hence the integral curves are given by

$$\frac{\exp\left(\int_{(r_S)_0}^{r_S}\frac{dr_S'}{h(r_S')}\right)}{t_S^*}=\text{constant},$$

where $(r_S)_0>2M$ can be chosen arbitrarily. Let $\sigma=t^*$, $\rho=\exp\left(\int_{(r_S)_0}^{r_S}\frac{dr_S'}{h(r_S')}\right)/t_S^*$ and consider (σ,ρ,x^A,x^B) as a new system of coordinates. Notice that

$$\partial_\sigma=\frac{h(r_S)}{t_S^*}\partial_{r_S}+\partial_{t_S^*}=\frac{1}{t_S^*}S.$$

Now for each fixed ρ, we have

$$\Phi^2(\tau)\le\Phi^2(\tau')+\left|\int_{\tau'}^{\tau}\frac{1}{\sigma}S(\Phi^2)\,d\sigma\right|.$$

Multiplying by ρ^α and integrating along a finite region of ρ, we get

$$\int_{\rho_1}^{\rho_2}\Phi^2(\tau)\rho^\alpha d\rho\le\int_{\rho_1}^{\rho_2}\Phi^2(\tau')\rho^\alpha d\rho+\int_{\rho_1}^{\rho_2}\int_{\tau'}^{\tau}\left|\frac{2\rho^\alpha}{\sigma}\Phi S\Phi\right|d\sigma d\rho.$$

We choose α so that $\alpha=0$ for $r\le r_Y^-$ and $\alpha=\alpha_0$ for $r\ge R$ and smooth depending on r in between. We would like to change coordinates back to (t_S^*,r_S,x_S^A,x_S^B). Notice that since $h(r_S)$ is everywhere positive, (ρ,τ) would correspond to a point with a larger value of r_S than (ρ,τ'). Therefore,

$$\begin{aligned}\int_{2M}^{(r_S)_2}\Phi^2(\tau)\frac{\exp\left((1+\alpha)\int_{2M}^{r_S}\frac{dr_S'}{h(r_S')}\right)}{\tau h(r_S)}dr_S&\le\int_{2M}^{(r_S)_2}\Phi^2(\tau')\frac{\exp\left((1+\alpha)\int_{(r_S)_0}^{r_S}\frac{dr_S'}{h(r_S')}\right)}{\tau' h(r_S)}dr_S\\&\quad+\int_{\tau'}^{\tau}\int_{2M}^{(r_S)_2}\left|\frac{2}{\sigma}\Phi S\Phi\right|\frac{\exp\left((1+\alpha)\int_{(r_S)_0}^{r_S}\frac{dr_S'}{h(r_S')}\right)}{t^*h(r_S)}dr_S\,dt^*.\end{aligned}$$

We have to compare $\exp\left((1+\alpha(r_S))\int_{(r_S)_0}^{r_S}\frac{dr_S'}{h(r_S')}\right)/h(r_S)$ with the volume form. Very close to the horizon, $h(r_S)=r_S-2M$ and $\alpha(r)=0$. Hence

$$\frac{\exp\left((1+\alpha)\int_{(r_S)_0}^{r_S}\frac{dr_S'}{h(r_S')}\right)}{h(r_S)}=e^{\int_{(r_S)_0}^{r_S}\frac{dr_S'}{h(r_S')}}\left(\frac{1}{r_S-2M}\right)\sim 1.$$

On the other hand, for $r\ge R$, $h(r_S)=(r_S+2M\log(r_S-2M)-3M-2M\log M)(1-\mu)$ and $\alpha(r_S)=\alpha_0$. In particular, for a sufficiently large choice of R, $h(r_S)\sim r_S$. Hence

$$\frac{\exp\left((1+\alpha)\int_{(r_S)_0}^{r_S}\frac{dr_S'}{h(r_S')}\right)}{h(r_S)}\sim\frac{\exp\left((1+\alpha)\int_{(r_S)_0}^{r_S}\frac{dr_S'}{h(r_S')}\right)}{r_S}\sim\frac{r_S^{\alpha_0}}{R}\sim r^{\alpha_0-2}.$$

The corresponding expression on the compact set $[r_Y^-, R]$ is obviously bounded. Hence, since the volume density both on a slice and on a spacetime region is $\sim r^2$, we have

$$\int_{\Sigma_\tau\cap\{r<r_2\}} \frac{\Phi^2(\tau)}{\tau} r^{\alpha_0-2} \leq C\Big(\int_{\Sigma_{\tau'}\cap\{r<r_2\}} \frac{\Phi^2(\tau')}{\tau'} r^{\alpha_0-2} + \iint_{\mathcal{R}(\tau',\tau)\cap\{r<r_2\}} r^{\alpha_0-2}\Big|\frac{2}{(t^*)^2}\Phi S\Phi\Big|\Big).$$

This easily implies the following improved decay for the nondegenerate energy for $\tau' \in [\tau/1.1, \tau]$:

$$\int_{\Sigma_\tau\cap\{r<\tau/4\}} r^{\alpha_0-2}\Phi^2 \leq C\tau^{-1}\Big(\int_{\Sigma_{\tau'}\cap\{r<\tau'/3\}} r^{\alpha_0-2}\Phi^2 + \iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r<t^*/3\}} r^{\alpha_0-2}(S\Phi)^2\Big). \quad (8)$$

By choosing an appropriate $\tilde{\tau}$, we have

$$\int_{\Sigma_{\tilde{\tau}}\cap\{r<\tilde{\tau}/3\}} r^{\alpha_0-2}\Phi^2 \leq C\tau^{-1}\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r<t^*/3\}} r^{\alpha_0-2}\Phi^2.$$

Now, applying (8) with $\tau' = \tilde{\tau}$, we have

$$\begin{aligned}\int_{\Sigma_\tau\cap\{r\leq\tau/4\}} r^{-1-\delta}\Phi^2 &\leq C\tau\Big(\int_{\Sigma_{\tilde{\tau}}\cap\{r<\tilde{\tau}/3\}} \frac{\Phi^2}{\tilde{\tau}} r^{\alpha_0-2} + \iint_{\mathcal{R}(\tilde{\tau},\tau)\cap\{r<t^*/3\}} r^{\alpha_0-2}\Big|\frac{2}{(t^*)^2}\Phi S\Phi\Big|\Big)\\ &\leq C\tau^{-1}\Big(\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/3\}} r^{\alpha_0-2}\Phi^2 + \iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/3\}} r^{\alpha_0-2}(S\Phi)^2\Big),\end{aligned}$$

using Cauchy–Schwarz for the second term. □

By Sobolev embedding, this would give an improved decay estimate in t^* in the region $\{r \leq t^*/4\}$. For the application, we also need an improved decay in r, which we get by commuting with the angular momentum $\tilde{\Omega}$.

Proposition 5.11. *Suppose $\Box_{g_K}\Phi = G$. For $r \leq t^*/4$ and $\ell \geq 1$, we have*

$$\begin{aligned}|D^\ell\Phi|^2 \leq{}& C(t^*)^{-1}r^{-1+\delta}\sum_{i+j\leq\ell-1}\sum_{k=0}^{2}\iint_{\mathcal{R}(t^*/1.1,t^*)\cap\{r\leq t^*/2\}}\big(K^{X_1}(\hat{Y}^i\partial_{t^*}^j\tilde{\Omega}^k\Phi)\\ &+ K^{X_1}(S\hat{Y}^i\partial_{t^*}^j\tilde{\Omega}^k\Phi)\big)\\ &+ C(t^*)^{-1}r^{-1+\delta}\sum_{j=0}^{\ell-1}\sum_{k=0}^{2}\iint_{\mathcal{R}(t^*/1.1,t^*)\cap\{r\leq t^*/2\}} r^{-1-\delta}(D^j\Box_{g_K}(\tilde{\Omega}^k\Phi))^2.\end{aligned}$$

Proof. Using a similar argument as before, except for choosing $\tilde{r} \le r$, we have

$$
\begin{aligned}
r^{1-\delta}|D^\ell\Phi|^2 &\le C\sum_{k=0}^{2}\int_{\mathbb{S}^2}(\tilde{\Omega}^k D^\ell\Phi)^2 r^{1-\delta}\,dA \\
&\le C\Big(\sum_{k=0}^{2}\int_{\mathbb{S}^2(\tilde{r})}(D^\ell\Omega^k\Phi)^2\tilde{r}^{1-\delta}\,dA \\
&\qquad + \int_{\tilde{r}}^{r}\int_{\mathbb{S}^2(r')}\big((D^{\ell+1}\Omega^k\Phi)^2 + (D^\ell\Omega^k\Phi)^2\big)(r')^{1-\delta}\,dA\,dr'\Big).
\end{aligned}
$$

Using Proposition 5.10, we have

$$
\begin{aligned}
\int_{\tilde{r}}^{r}\int_{\mathbb{S}^2(r')}(D^\ell\Omega^k\Phi)^2(r')^{1-\delta}\,dA\,dr' &\le C\int_{\Sigma_\tau\cap\{r\le\tau/4\}} r^{-1-\delta}(D^\ell\Omega^k\Phi)^2 \\
&\le C\tau^{-1}\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\le t^*/3\}}\big(r^{-1-\delta}(D^\ell\Omega^k\Phi)^2 + r^{-1-\delta}(SD^\ell\Omega^k\Phi)^2\big).
\end{aligned}
$$

By first commuting $[D, S]$ and then using Proposition 4.1(ii) and 4.4 on each fixed t^* slice in the integral, we have

$$
\begin{aligned}
&\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\le t^*/3\}}\big(r^{-1-\delta}(D^\ell\Omega^k\Phi)^2 + r^{-1-\delta}(SD^\ell\Omega^k\Phi)^2\big) \\
&\quad\le C\sum_{i+j\le\ell-1}\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\le t^*/2\}} r^{-1-\delta}\big(J^N_\mu(Y^i\partial_{t^*}^{j-i}\tilde{\Omega}^k\Phi)n^\mu_{\Sigma_\tau} \\
&\qquad\qquad + J^N_\mu(SY^i\partial_{t^*}^{j-i}\tilde{\Omega}^k\Phi)n^\mu_{\Sigma_\tau}\big) \\
&\qquad + C\sum_{j=0}^{\ell-1}\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\le t^*/2\}} r^{-1-\delta}(D^j\Box_{g_K}(\tilde{\Omega}^k\Phi))^2 \\
&\quad\le C\sum_{i+j\le\ell-1}\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\le t^*/2\}}\big(K^{X_1}(Y^i\partial_{t^*}^{j-i}\tilde{\Omega}^k\Phi) + K^{X_1}(SY^i\partial_{t^*}^{j-i}\tilde{\Omega}^k\Phi)\big) \\
&\qquad + C\sum_{j=0}^{\ell-1}\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\le t^*/2\}} r^{-1-\delta}(D^j\Box_{g_K}(\tilde{\Omega}^k\Phi))^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&r^{1-\delta}|D^\ell\Phi|^2 \\
&\quad\le C\tau^{-1}\sum_{i+j\le\ell-1}\sum_{k=0}^{2}\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\le t^*/2\}}\big(K^{X_1}(\hat{Y}^i\partial_{t^*}^{j}\tilde{\Omega}^k\Phi) + K^{X_1}(S\hat{Y}^i\partial_{t^*}^{j}\tilde{\Omega}^k\Phi)\big) \\
&\qquad + C\tau^{-1}\sum_{j=0}^{\ell-1}\sum_{k=0}^{2}\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\le t^*/2\}} r^{-1-\delta}(D^j\Box_{g_K}(\tilde{\Omega}^k\Phi))^2. \qquad \square
\end{aligned}
$$

Similar ideas can be used to prove decay of Φ without derivatives, except for a loss in powers of r. This will not be used for the bootstrap argument, but will be used to prove the decay for Φ in the statement of Theorem 1.

Proposition 5.12. *Suppose $\Box_{g_K}\Phi = G$. For $r \leq t^*/4$, we have*

$$|\Phi|^2 \leq C(t^*)^{-1}r^{\delta}\sum_{k=0}^{2}\iint_{\mathcal{R}(t^*/1.1,t^*)\cap\{r\leq t^*/3\}}\big(K^{X_1}(\tilde{\Omega}^k\Phi)+K^{X_1}(S\tilde{\Omega}^k\Phi)\big).$$

Proof. Fix R. Taking $\tilde{r}\in[R,\tau/5]$, we have

$$\begin{aligned} r^{-\delta}|\Phi|^2 &\leq C\sum_{k=0}^{2}\int_{\mathbb{S}^2}(\tilde{\Omega}^k\Phi)^2r^{-\delta}\,dA\\ &\leq C\sum_{k=0}^{2}\bigg(\int_{\mathbb{S}^2(\tilde{r})}(\Omega^k\Phi)^2\tilde{r}^{-\delta}\,dA\\ &\qquad+\bigg|\int_{\tilde{r}}^{r}\int_{\mathbb{S}^2(r')}\big(|\Omega^k\Phi D\Omega^k\Phi|^2(r')^{-\delta}+(\Omega^k\Phi)^2(r')^{-1-\delta}\big)\,dA\,dr'\bigg|\bigg). \end{aligned}$$

There exists $\tilde{r}\in[R,\tau/5]$ such that

$$\int_{\mathbb{S}^2(\tilde{r})}(\Omega^k\Phi)^2\tilde{r}^{-\delta}\,dA\leq\tau^{-1}\int_{r_+}^{\tau/4}\int_{\mathbb{S}^2(r')}(\Omega^k\Phi)^2(r')^{-\delta}\,dA\,dr'.$$

Using Proposition 5.10, we have

$$\begin{aligned} \int_{r_+}^{\tau/4}\int_{\mathbb{S}^2(\tilde{r})}(\Omega^k\Phi)^2(r')^{-\delta}\,dA\,dr' &\leq C\tau\int_{\Sigma_\tau\cap\{r\leq\tau/4\}}r^{-3-\delta}(\Omega^k\Phi)^2\\ &\leq C\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/3\}}\big(r^{-3-\delta}(\Omega^k\Phi)^2+r^{-3-\delta}(S\Omega^k\Phi)^2\big)\\ &\leq C\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/3\}}\big(K^{X_1}(\tilde{\Omega}^k\Phi)+K^{X_1}(S\tilde{\Omega}^k\Phi)\big). \end{aligned}$$

Using Proposition 5.10, we also have

$$\begin{aligned} &\bigg|\int_{\tilde{r}}^{r}\int_{\mathbb{S}^2(r')}\big(|\Omega^k\Phi D\Omega^k\Phi|^2(r')^{-\delta}+(\Omega^k\Phi)^2(r')^{-1-\delta}\big)\,dA\,dr'\bigg|\\ &\quad\leq C\int_{\Sigma_\tau\cap\{r\leq\tau/4\}}\big(r^{-3-\delta}(\Omega^k\Phi)^2+r^{-1-\delta}(D\Omega^k\Phi)^2\big)\\ &\quad\leq C\tau^{-1}\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/3\}}\big(r^{-3-\delta}(\Omega^k\Phi)^2+r^{-3-\delta}(S\Omega^k\Phi)^2+r^{-1-\delta}(D\Omega^k\Phi)^2\\ &\qquad\qquad+r^{-1-\delta}(SD\Omega^k\Phi)^2\big)\\ &\quad\leq C\tau^{-1}\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/3\}}\big(K^{X_1}(\tilde{\Omega}^k\Phi)+K^{X_1}(S\tilde{\Omega}^k\Phi)\big). \end{aligned}$$

Therefore,

$$|\Phi|^2 \leq C\tau^{-1} r^{\delta} \sum_{k=0}^{2} \iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/2\}} \big(K^{X_1}(\tilde{\Omega}^k\Phi) + K^{X_1}(S\tilde{\Omega}^k\Phi)\big). \qquad \square$$

6. Bootstrap

Bootstrap Assumptions (J): We first introduce the bootstrap assumptions corresponding to energy quantities on a fixed t^* slice:

$$\sum_{i+j=16} A_j^{-1} \int_{\Sigma_\tau} J^N_\mu(\partial^i_{t^*}\tilde{\Omega}^j\Phi) n^\mu_{\Sigma_\tau} + \sum_{i+k=16} A_Y^{-1} \int_{\Sigma_\tau} J^N_\mu(\hat{Y}^k\partial^i_{t^*}\Phi) n^\mu_{\Sigma_\tau} \leq \epsilon\tau^{\eta_{16}}, \tag{9}$$

$$\sum_{i+j=15} A_j^{-1} \bigg(\int_{\Sigma_\tau} J^{Z+N,w^Z}_\mu(\partial^i_{t^*}\tilde{\Omega}^j\Phi) n^\mu_{\Sigma_\tau} + C\tau^2 \int_{\Sigma_\tau\cap\{r\leq 9\tau/10\}} J^N_\mu(\partial^i_{t^*}\tilde{\Omega}^j\Phi) n^\mu_{\Sigma_\tau} \bigg) + \sum_{i+k=15} A_Y^{-1}\tau^2 \int_{\Sigma_\tau\cap\{r\leq r_Y^+\}} J^N_\mu(\hat{Y}^k\partial^i_{t^*}\Phi) n^\mu_{\Sigma_\tau} \leq \epsilon\tau^{1+\eta_{15}}, \tag{10}$$

$$\sum_{i+j\leq 14} A_j^{-1} \bigg(\int_{\Sigma_\tau} J^{Z+N,w^Z}_\mu(\partial^i_{t^*}\tilde{\Omega}^j\Phi) n^\mu_{\Sigma_\tau} + C\tau^2 \int_{\Sigma_\tau\cap\{r\leq 9\tau/10\}} J^N_\mu(\partial^i_{t^*}\tilde{\Omega}^j\Phi) n^\mu_{\Sigma_\tau} \bigg) + \sum_{i+k\leq 14} A_Y^{-1}\tau^2 \int_{\Sigma_\tau\cap\{r\leq r_Y^+\}} J^N_\mu(\hat{Y}^k\partial^i_{t^*}\Phi) n^\mu_{\Sigma_\tau} \leq \epsilon\tau^{\eta_{14}}, \tag{11}$$

$$\sum_{i+j\leq 15} A_j^{-1} \int_{\Sigma_\tau} J^N_\mu(\partial^i_{t^*}\tilde{\Omega}^j\Phi) n^\mu_{\Sigma_\tau} \leq \epsilon, \tag{12}$$

$$\sum_{i+j=13} A_{S,j}^{-1} \int_{\Sigma_\tau} J^N_\mu(S\partial^i_{t^*}\tilde{\Omega}^j\Phi) n^\mu_{\Sigma_\tau} + \sum_{i+k=13} A_{S,Y}^{-1} \int_{\Sigma_\tau} J^N_\mu(\hat{Y}^k S\partial^i_{t^*}\Phi) n^\mu_{\Sigma_\tau} \leq \epsilon\tau^{\eta_{S,13}}, \tag{13}$$

$$\sum_{i+j=12} A_{S,j}^{-1} \bigg(\int_{\Sigma_\tau} J^{Z+N,w^Z}_\mu(S\partial^i_{t^*}\tilde{\Omega}^j\Phi) n^\mu_{\Sigma_\tau} + C\tau^2 \int_{\Sigma_\tau\cap\{r\leq 9\tau/10\}} J^N_\mu(S\partial^i_{t^*}\tilde{\Omega}^j\Phi) n^\mu_{\Sigma_\tau} \bigg) + \sum_{i+k=12} A_{S,Y}^{-1}\tau^2 \int_{\Sigma_\tau\cap\{r\leq r_Y^+\}} J^N_\mu(\hat{Y}^k S\partial^i_{t^*}\Phi) n^\mu_{\Sigma_\tau} \leq \epsilon\tau^{1+\eta_{S,12}}, \tag{14}$$

$$\sum_{i+j\leq 11} A_{S,j}^{-1} \bigg(\int_{\Sigma_\tau} J^{Z+N,w^Z}_\mu(S(\partial^i_{t^*}\tilde{\Omega}^j\Phi)) n^\mu_{\Sigma_\tau} + C\tau^2 \int_{\Sigma_\tau\cap\{r\leq 9\tau/10\}} J^N_\mu(S(\partial^i_{t^*}\tilde{\Omega}^j\Phi)) n^\mu_{\Sigma_\tau} \bigg) + \sum_{i+k\leq 12} A_{S,Y}^{-1}\tau^2 \int_{\Sigma_\tau\cap\{r\leq r_Y^+\}} J^N_\mu(\hat{Y}^k S\partial^i_{t^*}\Phi) n^\mu_{\Sigma_\tau} \leq \epsilon\tau^{\eta_{S,11}}, \tag{15}$$

$$\sum_{i+j\leq 12} A_{S,j}^{-1} \int_{\Sigma_\tau} J^N_\mu(S\partial^i_{t^*}\tilde{\Omega}^j\Phi) n^\mu_{\Sigma_\tau} \leq \epsilon. \tag{16}$$

Bootstrap Assumptions (K): We also need bootstrap assumptions for the energy quantities in a spacetime slab:

$$\sum_{i+j=16} A_{X,j}^{-1} \iint_{\mathcal{R}(\tau_0,\tau)} \big(K^{X_0}(\partial_{t^*}^i \tilde{\Omega}^j \Phi) + K^N(\partial_{t^*}^i \tilde{\Omega}^j \Phi)\big) \leq \epsilon \tau^{\eta_{16}}, \tag{17}$$

$$\sum_{i+j=15} A_{X,j}^{-1} \iint_{\mathcal{R}(\tau_0,\tau)} K^{X_1}(\partial_{t^*}^i \tilde{\Omega}^j \Phi) \leq \epsilon \tau^{\eta_{16}}, \tag{18}$$

$$\sum_{i+j\leq 15} A_{X,j}^{-1} \iint_{\mathcal{R}(\tau_0,\tau)} K^{X_0}(\partial_{t^*}^i \tilde{\Omega}^j \Phi) \leq \epsilon, \tag{19}$$

$$\sum_{i+j\leq 14} A_{X,j}^{-1} \iint_{\mathcal{R}(\tau_0,\tau)} K^{X_1}(\partial_{t^*}^i \tilde{\Omega}^j \Phi) \leq \epsilon, \tag{20}$$

$$\sum_{i+j\leq 15} A_{X,j}^{-1} \iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/2\}} \big(K^{X_0}(\partial_{t^*}^i \tilde{\Omega}^j \Phi) + K^N(\partial_{t^*}^i \tilde{\Omega}^j \Phi)\big) \leq \epsilon \tau^{-1+\eta_{15}}, \tag{21}$$

$$\sum_{i+j\leq 14} A_{X,j}^{-1} \iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/2\}} K^{X_1}(\partial_{t^*}^i \tilde{\Omega}^j \Phi) \leq \epsilon \tau^{-1+\eta_{15}}, \tag{22}$$

$$\sum_{i+j\leq 14} A_{X,j}^{-1} \iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/2\}} \big(K^{X_0}(\partial_{t^*}^i \tilde{\Omega}^j \Phi) + K^N(\partial_{t^*}^i \tilde{\Omega}^j \Phi)\big) \leq \epsilon \tau^{-2+\eta_{14}}, \tag{23}$$

$$\sum_{i+j\leq 13} A_{X,j}^{-1} \iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/2\}} K^{X_1}(\partial_{t^*}^i \tilde{\Omega}^j \Phi) \leq \epsilon \tau^{-2+\eta_{14}}, \tag{24}$$

$$\sum_{i+j=13} A_{S,X,j}^{-1} \iint_{\mathcal{R}(\tau_0,\tau)} K^{X_0}(S\partial_{t^*}^i \tilde{\Omega}^j \Phi) \leq \epsilon \tau^{\eta_{S,13}}, \tag{25}$$

$$\sum_{i+j\leq 12} A_{S,X,j}^{-1} \iint_{\mathcal{R}(\tau_0,\tau)} K^{X_0}(S\partial_{t^*}^i \tilde{\Omega}^j \Phi) \leq \epsilon, \tag{26}$$

$$\sum_{i+j+k\leq 12} A_{S,X,j}^{-1} \iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/2\}} K^{X_0}(S\partial_{t^*}^i \tilde{\Omega}^j \Phi) \leq \epsilon \tau^{-1+\eta_{S,12}}, \tag{27}$$

$$\sum_{i+j\leq 11} A_{S,X,j}^{-1} \iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/2\}} K^{X_1}(S\partial_{t^*}^i \tilde{\Omega}^j \Phi) \leq \epsilon \tau^{-1+\eta_{S,12}}, \tag{28}$$

$$\sum_{i+j\leq 11} A_{S,X,j}^{-1} \iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/2\}} K^{X_0}(S\partial_{t^*}^i \tilde{\Omega}^j \Phi) \leq \epsilon \tau^{-2+\eta_{S,11}}, \tag{29}$$

$$\sum_{i+j\leq 10} A_{S,X,j}^{-1} \iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/2\}} K^{X_1}(S\partial_{t^*}^i \tilde{\Omega}^j \Phi) \leq \epsilon \tau^{-2+\eta_{S,11}}. \tag{30}$$

Bootstrap Assumptions (P): We also introduce bootstrap assumptions for the pointwise behavior. For $r \geq t^*/4$,

$$\sum_{j=0}^{13} |\Gamma^j \Phi|^2 \leq BA\epsilon r^{-2}(t^*)^{1+\eta_{14}}, \tag{31}$$

$$\sum_{j=0}^{13} |D\Gamma^j \Phi|^2 \leq BA\epsilon, \tag{32}$$

$$\sum_{\ell=1}^{13-j}\sum_{j=0}^{12}|D^{\ell}\Gamma^{j}\Phi|^{2}\leq BA\epsilon r^{-2}, \tag{33}$$

$$\sum_{j=0}^{8}|D\Gamma^{j}\Phi|^{2}\leq BA\epsilon r^{-2}(t^{*})^{\eta_{14}}(1+|u|)^{-2}, \tag{34}$$

$$\sum_{j=0}^{8}|\bar{D}\Gamma^{j}\Phi|^{2}\leq BA\epsilon r^{-2}(t^{*})^{-2+\eta_{14}}, \tag{35}$$

$$\sum_{j=0}^{6}|S\Gamma^{j}\Phi|^{2}\leq B_{S}A\epsilon r^{-2}(t^{*})^{\eta_{S,11}}, \tag{36}$$

$$\sum_{j=0}^{8}|DS\Gamma^{j}\Phi|^{2}\leq B_{S}A\epsilon r^{-2}, \tag{37}$$

$$\sum_{j=0}^{6}|DS\Gamma^{j}\Phi|^{2}\leq B_{S}A\epsilon r^{-2}(t^{*})^{\eta_{S,11}}(1+|u|)^{-2}. \tag{38}$$

For $r\leq t^{*}/4$,

$$\sum_{j=0}^{13}|\Gamma^{j}\Phi|^{2}\leq BA\epsilon(t^{*})^{-1+\eta_{14}}, \tag{39}$$

$$\sum_{\ell=1}^{14-j}\sum_{j=0}^{13}|D^{\ell}\Gamma^{j}\Phi|^{2}\leq BA\epsilon(t^{*})^{-1+\eta_{14}}, \tag{40}$$

$$\sum_{\ell=1}^{13-j}\sum_{j=0}^{12}|D^{\ell}\Gamma^{j}\Phi|^{2}\leq BA\epsilon(t^{*})^{-2+\eta_{14}}, \tag{41}$$

$$\sum_{\ell=1}^{9-j}\sum_{j=0}^{8}|D^{\ell}\Gamma^{j}\Phi|^{2}\leq BA\epsilon(t^{*})^{-3+\eta_{S,11}}r^{-1+\delta}, \tag{42}$$

$$\sum_{j=0}^{6}|S\Gamma^{j}\Phi|^{2}\leq B_{S}A\epsilon(t^{*})^{-2+\eta_{S,11}}, \tag{43}$$

$$\sum_{\ell=1}^{7-j}\sum_{j=0}^{6}|D^{\ell}S\Gamma^{j}\Phi|^{2}\leq B_{S}A\epsilon r^{-2}(t^{*})^{-2+\eta_{S,11}}. \tag{44}$$

Remark 6.1. Notice that in general, for most of the bootstrap assumptions on Φ, there is a corresponding one on $S\Phi$. The arguments to retrieve these assumptions are quite similar, we only have to estimate the commutator term (in a manner similar to [19], [20]) and track the appropriate constants.

Remark 6.2. Notice that all these assumptions are satisfied initially by the assumption of the theorem.

Remark 6.3. We will bootstrap to improve the constants A_j, $A_{X,j}$, A_S, $A_{S,X,j}$, A_Y, $A_{S,Y}$ and B. The constant B is only used for the bootstrap of the pointwise estimates. The constants satisfy

$$1 \ll B \sim B_S \ll A_0 \ll A_{X,0} \ll A_1 \ll \cdots \ll A_{16} \ll A_{X,16} \ll A_Y \\ \ll A_{S,0} \ll A_{S,X,0} \ll \cdots \ll A_{S,X,13} \ll A_{S,Y},$$

We will use A as a shorthand to denote the maximum of all these constants, i.e., $A_{S,Y}$. We will always assume by taking ϵ small that

$$A\epsilon \ll 1.$$

Moreover, we set the constants so that

$$A_{j-1}/A_j \ll A_{X,j-1}/A_j \ll \delta'\eta^{-1}, \quad \delta \sim \delta' \ll A_j/A_{X,j}.$$

The η's, on the other hand, satisfy

$$\delta \sim \delta' \ll \eta_{16} \ll \eta_{15} \ll \eta_{14} \ll \eta_{S,13} \ll \eta_{S,12} \ll \eta_{S,11}.$$

The η's are chosen so that

$$A_j/A_{X,j} \ll \eta_{14} \ll 1 \quad \text{for all } j.$$

ϵ will be much smaller than any combinations of the other constants.

We will use energy estimates and decay estimates to eventually close the bootstrap. In order to derive the estimates, we consider equations for $\Gamma^k\Phi$. We now introduce the notations that will facilitate the discussion below.

Definition 6.4.

$$G_k = \sum_{|j|=k} |\Box_{g_K}(\Gamma^j\Phi)|, \quad U_k = \sum_{|j|=k} |[\Box_{g_K}, \Gamma^j]\Phi|, \quad N_k = \sum_{|j|=k} |\Gamma^j(\Box_{g_K}\Phi)|.$$

Definition 6.5.

$$G_{\leq k} = \sum_{|j|\leq k} |\Box_{g_K}(\Gamma^j\Phi)|, \quad U_{\leq k} = \sum_{|j|\leq k} |[\Box_{g_K}, \Gamma^j]\Phi|, \quad N_{\leq k} = \sum_{|j|\leq k} |\Gamma^j(\Box_{g_K}\Phi)|.$$

In order to keep track of the constants, we also introduce

Definition 6.6. Define $U_{k,j} = |[\Box_{g_K}, \partial_{t^*}^{k-j}\tilde{\Omega}^j]\Phi|$ and $U_{\leq k,\leq j} = \sum_{j'\leq j, k'\leq k} U_{k',j'}$.

Remark 6.7. We will refer to G as the *inhomogeneous term*, U as the *commutator term* and N as the *nonlinear term*. Clearly we have $G_k \leq U_k + N_k$ and $G_{\leq k} \leq U_{\leq k} + N_{\leq k}$

We now estimate the inhomogeneous terms that will appear in the analysis several times below. It is necessary to study the commutator terms and the nonlinear terms together because the estimates for each depend on the estimates for the other when we use elliptic estimates.

Proposition 6.8. *U_k satisfies the following estimates:*

$$\int_{\Sigma_\tau} r^\alpha (D^\ell U_{k,j})^2 \leq C\Big(\sum_{m=0}^{k+\ell-i}\sum_{i=0}^{j-1}\int_{\Sigma_\tau\cap\{r\geq R_\Omega-1\}} r^{\alpha-4} J^N_\mu(\partial_{t^*}^m\tilde{\Omega}^i\Phi)n^\mu_{\Sigma_\tau} + \sum_{m=0}^{k+\ell-i-1}\sum_{i=0}^{j-1}\int_{\Sigma_\tau} r^{\alpha-4}(D^m N_i)^2\Big) \quad (45)$$

for $\alpha \leq 4$, where it is understood that $\sum_{i=0}^{-1} = 0$.

Proof. The commutator terms are estimated in [19]. Notice that since $\tilde{\Omega}$ is supported away from the trapped set, there is no loss of derivatives in using the integrated decay estimate. We have $U_{k,j}$ supported in $\{r \geq R_\Omega\}$ and

$$|U_{k,j}| \leq C\sum_{i=0}^{j}|\partial_{t^*}^{k-j}[\Box_{g_K}, \tilde{\Omega}^i]\Phi| \leq C\sum_{i=0}^{j-1} r^{-2}\big(|D^2\partial_{t^*}^{k-j}\tilde{\Omega}^i\Phi| + |D\partial_{t^*}^{k-j}\tilde{\Omega}^i\Phi|\big),$$

and therefore

$$|D^\ell U_{k,j}| \leq C\sum_{m=1}^{\ell+2}\sum_{i=0}^{j-1} r^{-2}|D^m\partial_{t^*}^{k-j}\tilde{\Omega}^i\Phi| \leq C\sum_{m=1}^{k+\ell-j+2}\sum_{i=0}^{j-1} r^{-2}|D^m\tilde{\Omega}^i\Phi|,$$

where, as in the statement of the proposition, it is understood that the sum vanishes if $j = 0$. Hence, using the elliptic estimate for $\{r \geq R_\Omega\}$, i.e., Proposition 4.3,

$$\begin{aligned}\int_{\Sigma_\tau} r^\alpha(D^\ell U_{k,j})^2 &\leq C\sum_{m=1}^{k+\ell-j+2}\sum_{i=0}^{j-1}\int_{\Sigma_\tau\cap\{r\geq R_\Omega\}} r^{\alpha-4}(D^m\tilde{\Omega}^i\Phi)^2\\ &\leq C\sum_{m=0}^{k+\ell-i}\sum_{i=0}^{j-1}\int_{\Sigma_\tau\cap\{r\geq R_\Omega-1\}} r^{\alpha-4}J^N_\mu(\partial_{t^*}^m\tilde{\Omega}^i\Phi)n^\mu_{\Sigma_\tau}\\ &\quad + C\sum_{m=0}^{k+\ell-i-1}\sum_{i=0}^{j-1}\int_{\Sigma_\tau\cap\{r\geq R_\Omega-1\}} r^{\alpha-4}\big((D^m U_{i,i})^2 + (D^m N_i)^2\big).\end{aligned} \quad (46)$$

Now we can estimate U_k by induction: Fix any k and we will induct on j. By definition, $U_{k,0} = 0$. Now, assume that for all $k + \ell \leq 16$ and for some $j_0 \geq 1$, we have

$$\sum_{k+\ell\leq M,\ j\leq\min\{j_0-1,k\}}\int_{\Sigma_\tau} r^\alpha(D^\ell U_{k,j})^2 \leq C\Big(\sum_{m=0}^{M-i}\sum_{i=0}^{j_0-2}\int_{\Sigma_\tau\cap\{r\geq R_\Omega-1\}} r^{\alpha-4}J^N_\mu(\partial_{t^*}^m\tilde{\Omega}^i\Phi)n^\mu_{\Sigma_\tau} + \sum_{m=0}^{M-1-i}\sum_{i=0}^{j_0-1}\int_{\Sigma_\tau} r^{\alpha-4}(D^m N_i)^2\Big)$$

for all $\alpha \leq 4$. Then, using (46), we find that for $k + \ell \leq 16$, and $j_0 \leq k$,

$$\int_{\Sigma_\tau} r^\alpha (D^\ell U_{k,j_0})^2 \leq C\Big(\sum_{m=0}^{k+\ell-i}\sum_{i=0}^{j_0-1}\int_{\Sigma_\tau\cap\{r\geq R_\Omega-1\}} r^{\alpha-4} J^N_\mu(\partial_{t^*}^m\tilde{\Omega}^i\Phi)n^\mu_{\Sigma_\tau}$$
$$+\sum_{m=0}^{k+\ell-i-1}\sum_{i=0}^{j_0-1}\int_{\Sigma_\tau} r^{\alpha-4}(D^m N_i)^2\Big).$$

Hence, (45) holds. □

We now estimate the nonlinear term N_k. Since N_k is at least quadratic, we do not need to be precise about the constants A and we will always estimate with the maximum A.

Proposition 6.9. *N_k satisfies the following estimates for fixed $t^* = \tau$:*

$$\sum_{k+\ell=16}\int_{\Sigma_\tau}(D^\ell N_k)^2 \leq CBA^2\epsilon^2\tau^{-2+\eta_{16}},$$
$$\sum_{k+\ell\leq 15}\int_{\Sigma_\tau}(D^\ell N_k)^2 \leq CBA^2\epsilon^2\tau^{-2}.$$

N_k also satisfy the following estimates when integrated over $t^* \in [\tau/1.1, \tau]$:

$$\sum_{k+\ell=15}\iint_{\mathcal{R}(\tau/1.1,\tau)} r^{-1-\delta}(D^\ell N_k)^2 \leq CBA^2\epsilon^2\tau^{-2+\eta_{16}},$$
$$\sum_{k+\ell\leq 14}\iint_{\mathcal{R}(\tau/1.1,\tau)} r^{-1-\delta}(D^\ell N_k)^2 \leq CBA^2\epsilon^2\tau^{-2}.$$

Proof. Here, there is no need to distinguish between the good and bad derivatives. We have

$$|D^\ell N_k| \leq |D^\ell\Gamma^k(\Lambda_i D\Phi D\Phi)| + |\Gamma^k\mathcal{C}|.$$

We claim that the most important terms will be those that are quadratic in $D^{j+1}\Gamma^i\Phi$ or cubic of the form

$$(D^{j_1+1}\Gamma^{i_1}\Phi)(D^{j_2+1}\Gamma^{i_2}\Phi)(\Gamma^{i_3}\Phi)$$

with $i_1 + j_1, i_2 + j_2 \leq 8$. For by the assumptions every term has the form

$$(D^{j_1}\Gamma^{i_1}\Phi)(D^{j_2}\Gamma^{i_2}\Phi)(D^{j_3}\Gamma^{i_3}\Phi)(D^{j_4}\Gamma^{i_4}\Phi)\ldots(D^{j_r}\Gamma^{i_r}\Phi),$$

with $r \geq 2$, at least two j's ≥ 1 and $i + j \leq 9$ for all but at most one factor. If all factors have $i + j \leq 9$ or the factor with $i + j > 9$ has $i \geq 1$, we can reduce to the case $D^{j_1+1}\Gamma^{i_1}\Phi D^{j_2+1}\Gamma^{i_2}\Phi$ by putting all other factors in L^∞ using Bootstrap Assumptions (31), (32), (39) and (40). If the factor with $i + j > 9$ has $i = 0$ we reduce to

$$(D^{j_1+1}\Gamma^{i_1}\Phi)(D^{j_2+1}\Gamma^{i_2}\Phi)(\Gamma^{i_3}\Phi)$$

again by putting all other factors in L^∞ using Bootstrap Assumptions (31), (32), (39) and (40). We have

$$\int_{\Sigma_\tau}(D^\ell N_k)^2 \le C\Big(\sup\sum_{i_1+j_1\le 8}|D^{j_1+1}\Gamma^{i_1}\Phi|^2\Big)\sum_{j_2=0}^{\ell}\sum_{i_2=0}^{k}\int_{\Sigma_\tau}|D^{j_2+1}\Gamma^{i_2}\Phi|^2$$

$$+ C\Big(\sup\sum_{i_1+j_1\le 8}|D^{j_1+1}\Gamma^{i_1}\Phi|^2\Big)\Big(\sup\sum_{i_2+j_2\le 8}r^2|D^{j_2+1}\Gamma^{i_2}\Phi|^2\Big)\sum_{i_3=0}^{k}\int_{\Sigma_\tau}r^{-2}|\Gamma^{i_3}\Phi|^2$$

$$\le CBA\epsilon\tau^{-2}\sum_{i=1}^{\ell+1}\sum_{j=0}^{k}\int_{\Sigma_\tau}(D^i\Gamma^j\Phi)^2$$

using Hardy's inequality in Proposition 4.2

$$\le CBA\epsilon\tau^{-2}\sum_{i+m=0}^{\ell}\sum_{j=0}^{k}\int_{\Sigma_\tau}J^N_\mu(\partial^m_{t^*}\Gamma^j\hat{Y}^i\Phi)n^\mu_{\Sigma_\tau}$$

$$+ CBA\epsilon\tau^{-2}\sum_{i+j\le k+\ell-1}\int_{\Sigma_\tau}\big((D^iU_{\le j})^2+(D^iN_{\le j})^2\big)$$

using the elliptic estimates in Propositions 4.1, 4.4

$$\le CBA\epsilon\tau^{-2}\Big(\sum_{i=0}^{\ell}\sum_{j=0}^{k}\int_{\Sigma_\tau}J^N_\mu(\partial^m_{t^*}\Gamma^j\hat{Y}^i\Phi)n^\mu_{\Sigma_\tau}+\sum_{i+j\le k+\ell-1}\int_{\Sigma_\tau}(D^iN_{\le j})^2\Big),$$

where we have used Proposition 6.8 in the last step. Now, a simple induction would show that

$$\sum_{k+\ell=16}\int_{\Sigma_\tau}(D^\ell N_k)^2 \le CBA^2\epsilon^2\tau^{-2+\eta},\qquad \sum_{k+\ell\le 15}\int_{\Sigma_\tau}(D^\ell N_k)^2\le CBA^2\epsilon^2\tau^{-2}.$$

We now move on to the terms integrated over $t^*\in[\tau/1.1,\tau]$. Arguing as before, and noticing that the elliptic estimate in Proposition 4.1 also allows weights in r, we have

$$\iint_{\mathcal{R}(\tau/1.1,\tau)}r^{-1-\delta}(D^\ell N_k)^2$$

$$\le CBA\epsilon\tau^{-2}\sum_{i+m=0}^{\ell}\sum_{j=0}^{k}\iint_{\mathcal{R}(\tau/1.1,\tau)}r^{-1-\delta}J^N_\mu(\partial^m_{t^*}\Gamma^j\hat{Y}^i\Phi)n^\mu_{\Sigma_\tau}$$

$$+ CBA\epsilon\tau^{-2}\sum_{i+j\le k+\ell-1}\iint_{\mathcal{R}(\tau/1.1,\tau)}r^{-1-\delta}\big((D^iU_{\le j})^2+(D^iN_{\le j})^2\big)$$

$$\le CBA\epsilon\tau^{-2}\sum_{i=0}^{\ell}\sum_{j=0}^{k}\iint_{\mathcal{R}(\tau/1.1,\tau)}r^{-1-\delta}J^N_\mu(\partial^m_{t^*}\Gamma^j\hat{Y}^i\Phi)n^\mu_{\Sigma_\tau}$$

$$+ CBA\epsilon\tau^{-2}\sum_{i+j\le k+\ell-1}\iint_{\mathcal{R}(\tau/1.1,\tau)}r^{-1-\delta}(D^iN_{\le j})^2$$

$$\le CBA\epsilon\tau^{-2}\sum_{i=0}^{\ell}\sum_{j=0}^{k}\iint_{\mathcal{R}(\tau/1.1,\tau)}K^{X_1}(\partial^m_{t^*}\Gamma^j\hat{Y}^i\Phi)$$

$$+ CBA\epsilon\tau^{-2}\sum_{i+j\le k+\ell-1}\iint_{\mathcal{R}(\tau/1.1,\tau)}r^{-1-\delta}(D^iN_{\le j})^2$$

Now, Bootstrap Assumptions (18), (20) and an induction on $k+\ell$ conclude the proof. □

Now, the estimates for N_k will also give improved estimates for U_k via Proposition 6.8:

Proposition 6.10. *The following estimates for U_k on a fixed t^* slice hold for $\alpha \leq 2$:*

$$\sum_{k+\ell=16} \int_{\Sigma_\tau} r^\alpha (D^\ell U_{k,j})^2 \leq C A_{j-1} \epsilon \tau^{\eta_{16}},$$
$$\sum_{k+\ell=15} \int_{\Sigma_\tau} r^\alpha (D^\ell U_{k,j})^2 \leq C A_{j-1} \epsilon \tau^{-1+\eta_{15}},$$
$$\sum_{k+\ell\leq 14} \int_{\Sigma_\tau} r^\alpha (D^\ell U_{k,j})^2 \leq C A_{j-1} \epsilon \tau^{-2+\eta_{14}}.$$

The following estimates for U_k integrated on $[\tau/1.1, \tau]$ also hold for $\alpha \leq 1+\delta$:

$$\sum_{k+\ell=16} \iint_{\mathcal{R}(\tau/1.1,\tau)} r^\alpha (D^\ell U_{k,j})^2 \leq C A_{X,j-1} \epsilon \tau^{\eta_{16}},$$
$$\sum_{k+\ell=15} \iint_{\mathcal{R}(\tau/1.1,\tau)} r^\alpha (D^\ell U_{k,j})^2 \leq C A_{X,j-1} \epsilon \tau^{-1+\eta_{15}},$$
$$\sum_{k+\ell\leq 14} \iint_{\mathcal{R}(\tau/1.1,\tau)} r^\alpha (D^\ell U_{k,j})^2 \leq C A_{X,j-1} \epsilon \tau^{-2+\eta_{14}}.$$

Proof. We first prove the estimates for the terms constant in τ. By Proposition 6.8,

$$\int_{\Sigma_\tau} r^\alpha (D^\ell U_{k,j})^2$$
$$\leq C\Big(\sum_{m=0}^{k+\ell-i} \sum_{i=0}^{j-1} \int_{\Sigma_\tau \cap \{r \geq R_\Omega - 1\}} r^{\alpha-4} J^N_\mu (\partial_{t^*}^m \tilde{\Omega}^i \Phi) n^\mu_{\Sigma_\tau} + \sum_{m=0}^{k+\ell-1-i} \sum_{i=0}^{j-1} \int_{\Sigma_\tau} r^{\alpha-4} (D^m N_i)^2\Big)$$

for $\alpha \leq 4$.

The second term satisfies the required estimate by Proposition 6.9. We estimate the first term. By (9),

$$\sum_{m=0}^{16-i} \sum_{i=0}^{j-1} \int_{\Sigma_\tau \cap \{r \geq R_\Omega - 1\}} r^{\alpha-4} J^N_\mu (\partial_{t^*}^m \tilde{\Omega}^i \Phi) n^\mu_{\Sigma_\tau} \leq C A_{j-1} \epsilon \tau^{\eta_{16}}.$$

By (10) and Proposition 2.2,

$$\sum_{m=0}^{15-i} \sum_{i=0}^{j-1} \int_{\Sigma_\tau \cap \{r \geq R_\Omega - 1\}} r^{\alpha-4} J^N_\mu (\partial_{t^*}^m \tilde{\Omega}^i \Phi) n^\mu_{\Sigma_\tau}$$
$$\leq C\Big(\sum_{m=0}^{15-i} \sum_{i=0}^{j-1} \int_{\Sigma_\tau \cap \{r \leq \tau/2\}} J^N_\mu (\partial_{t^*}^m \tilde{\Omega}^i \Phi) n^\mu_{\Sigma_\tau} + \sum_{m=0}^{15-i} \sum_{i=0}^{j-1} \tau^{-2} \int_{\Sigma_\tau \cap \{r \geq \tau/2\}} J^N_\mu (\partial_{t^*}^m \tilde{\Omega}^i \Phi) n^\mu_{\Sigma_\tau}\Big)$$
$$\leq C A_{j-1} \epsilon \tau^{-1+\eta_{15}}.$$

By (11) and Proposition 2.2,

$$\begin{aligned}&\sum_{m=0}^{14-i}\sum_{i=0}^{j-1}\int_{\Sigma_\tau\cap\{r\geq R_\Omega-1\}} r^{\alpha-4}J^N_\mu(\partial_{t^*}^m\tilde{\Omega}^i\Phi)n^\mu_{\Sigma_\tau}\\ &\leq C\Big(\sum_{m=0}^{14-i}\sum_{i=0}^{j-1}\int_{\Sigma_\tau\cap\{r\leq\tau/2\}} J^N_\mu(\partial_{t^*}^m\tilde{\Omega}^i\Phi)n^\mu_{\Sigma_\tau}+\sum_{m=0}^{14-i}\sum_{i=0}^{j-1}\tau^{-2}\int_{\Sigma_\tau\cap\{r\geq\tau/2\}} J^N_\mu(\partial_{t^*}^m\tilde{\Omega}^i\Phi)n^\mu_{\Sigma_\tau}\Big)\\ &\leq CA_{j-1}\epsilon\tau^{-2+\eta_{14}}.\end{aligned}$$

For the integrated terms, we similarly have, by Proposition 6.8,

$$\begin{aligned}&\iint_{\mathcal{R}(\tau/1.1,\tau)} r^\alpha(D^\ell U_{k,j})^2\\ &\qquad\leq C\Big(\sum_{m=0}^{k+\ell-i}\sum_{i=0}^{j-1}\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\geq R_\Omega-1\}} r^{\alpha-4}J^N_\mu(\partial_{t^*}^m\tilde{\Omega}^i\Phi)n^\mu_{\Sigma_\tau}\\ &\qquad\qquad+\sum_{m=0}^{k+\ell-1-i}\sum_{i=0}^{j-1}\iint_{\mathcal{R}(\tau/1.1,\tau)} r^{\alpha-4}(D^mN_i)^2\Big)\quad\text{for } \alpha\leq 4.\end{aligned}$$

The second term can be estimated by Proposition 6.9. Notice that $r^{-1+\delta}J^N_\mu(\Phi)n^\mu_{\Sigma_\tau}\leq K^{X_0}(\Phi)$. Hence, following the argument above for the fixed τ case, we would have proved the proposition for the case $\alpha\leq 1-\delta$. Nevertheless, with more care, we can improve to $\alpha\leq 1+\delta$. We have

$$\begin{aligned}&\sum_{m=0}^{k+\ell-i}\sum_{i=0}^{j-1}\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\geq R_\Omega\}} r^{\alpha-4}J^N_\mu(\partial_{t^*}^m\tilde{\Omega}^i\Phi)n^\mu_{\Sigma_{t^*}}\\ &\leq C\sum_{m=0}^{k+\ell-i}\sum_{i=0}^{j-1}\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/2\}}\big(K^{X_0}(\partial_{t^*}^m\tilde{\Omega}^i\Phi)+r^{-2+2\delta}K^{X_0}(\partial_{t^*}^m\tilde{\Omega}^i\Phi)\big).\end{aligned}$$

For $k+\ell=16$, this is bounded by $CA_{X,j-1}\epsilon\tau^{\eta_{16}}$ by (17). For $k+\ell=15$, this is bounded by $CA_{X,j-1}\epsilon\tau^{-1+\eta_{15}}$ by (21) and (19). For $k+\ell\leq 14$, this is bounded by $CA_{X,j-1}\epsilon\tau^{-2+\eta_{14}}$ by (23) and (19) since $2\delta\leq\eta_{14}$. □

While the above is sufficient to recover the bootstrap assumptions for the pointwise bounds, we will need improvements to achieve the energy bounds. For the improvements, we study separately the regions $r\leq t^*/4$, $t^*/4\leq r\leq 9t^*/10$ and $r\geq 9t^*/10$. For $r\geq t^*/4$, we will only show the improvement for N_k instead of the derivatives of N_k. Various complications would arise in estimating the derivatives of N_k. For $r\leq t^*/4$, however, we will estimate also the derivatives of N_k as they will be necessary to estimate the error terms arising from commuting with the red-shift vector field.

Proposition 6.11.

$$\sum_{\ell+k=16}\int_{\Sigma_\tau\cap\{r\le\tau/4\}} r^{1-\delta}(D^\ell N_k)^2 \le CBA^2\epsilon^2\tau^{-3+\eta_{S,11}+\eta_{16}},$$
$$\sum_{\ell+k=15}\int_{\Sigma_\tau\cap\{r\le\tau/4\}} r^{1-\delta}(D^\ell N_k)^2 \le CBA^2\epsilon^2\tau^{-4+\eta_{S,11}+\eta_{15}},$$
$$\sum_{\ell+k\le14}\int_{\Sigma_\tau\cap\{r\le\tau/4\}} r^{1-\delta}(D^\ell N_k)^2 \le CBA^2\epsilon^2\tau^{-5+\eta_{S,11}+\eta_{14}}.$$

Proof. As before, we only have to estimate terms quadratic in $D^j\Gamma^i\Phi$ with $j\ge1$ or cubic of the form

$$(D^{j_1+1}\Gamma^{i_1}\Phi)(D^{j_2+1}\Gamma^{i_2}\Phi)(\Gamma^{i_3}\Phi)$$

with $i_1+j_1, i_2+j_2\le 8$. We have

$$\begin{aligned}
&\int_{\Sigma_\tau} r^{1-\delta}(D^\ell N_k)^2\\
&\quad\le C\Big(\sup_{r\le\tau/4}\sum_{i_1+j_1\le8} r^{1-\delta}|D^{j_1+1}\Gamma^{i_1}\Phi|^2\Big)\sum_{j_2=0}^{\ell}\sum_{i_2=0}^{k}\int_{\Sigma_\tau\cap\{r\le\tau/4\}}|D^{j_2+1}\Gamma^{i_2}\Phi|^2\\
&\qquad + C\Big(\sup_{r\le\tau/4}\sum_{i_1+j_1\le8} r^{1-\delta}|D^{j_1+1}\Gamma^{i_1}\Phi|^2\Big)^2\sum_{i_3=0}^{k}\tau^{2\delta}\int_{\Sigma_\tau\cap\{r\le\tau/4\}} r^{-2}|\Gamma^{i_3}\Phi|^2\\
&\quad\le CBA\epsilon\tau^{-3+\eta_{S,11}}\sum_{i=1}^{\ell+1}\sum_{j=0}^{k}\int_{\Sigma_\tau\cap\{r\le\tau/4\}}(D^i\Gamma^j\Phi)^2\\
&\qquad + CBA^2\epsilon^2\tau^{-6+2\eta_{S,11}+2\delta}\sum_{i_3=0}^{k}\int_{\Sigma_\tau}(D\Gamma^{i_3}\Phi)^2
\end{aligned}$$

by Hardy's inequality (notice now that the second term has more decay than we need, so we will drop it from now on)

$$\begin{aligned}
&\quad\le CBA\epsilon\tau^{-3+\eta_{S,11}}\sum_{i+m=0}^{\ell}\sum_{j=0}^{k}\int_{\Sigma_\tau\cap\{r\le\tau/2\}} J^N_\mu(\partial_{t^*}^m\Gamma^j\hat{Y}^i\Phi)n^\mu_{\Sigma_\tau}\\
&\qquad + CBA\epsilon\tau^{-3+\eta_{S,11}}\sum_{i=0}^{\ell-1}\sum_{j=0}^{k}\int_{\Sigma_\tau}\big((D^iU_j)^2+(D^iN_j)^2\big)\\
&\quad\le CBA\epsilon\tau^{-3+\eta_{S,11}}\Big(\sum_{i+j\le k+\ell}\int_{\Sigma_\tau} J^N_\mu(\Gamma^j\hat{Y}^i\Phi)n^\mu_{\Sigma_\tau}+\sum_{i+j\le k+\ell-1}\int_{\Sigma_\tau}(D^iN_j)^2\Big). \qquad (47)
\end{aligned}$$

The proposition would follow from an induction on $k+\ell$ and Bootstrap Assumptions

(9)–(11). The $k+\ell=0$ case also follows from the above computation as we have adopted the notation that $\sum_{i+j\leq -1}=0$. □

We now move to the region $\{t^*/4 \leq r \leq 9t^*/10\}$. In this region, $u \sim t^*$, and therefore we can exploit the decay in the variable u given by the estimates from the conformal energy.

Proposition 6.12.

$$\int_{\Sigma_\tau\cap\{\tau/4\leq r\leq 9\tau/10\}} N_{16}^2 \leq CBA^2\epsilon^2\tau^{-4+\eta_{14}+\eta_{16}},$$

$$\int_{\Sigma_\tau\cap\{\tau/4\leq r\leq 9\tau/10\}} N_{15}^2 \leq CBA^2\epsilon^2\tau^{-5+\eta_{14}+\eta 15},$$

$$\sum_{j=0}^{14}\int_{\Sigma_\tau\cap\{\tau/4\leq r\leq 9\tau/10\}} N_j^2 \leq CBA^2\epsilon^2\tau^{-6+2\eta_{14}}.$$

Proof. Arguing as before, we see that the main terms for the nonlinearity are those that are quadratic in $D\Phi$ or those that are cubic with the form $\Gamma^{i_3}\Phi D\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi$ with $i_1, i_2 \leq 8$. The quadratic terms can be estimated:

$$\begin{aligned}\sum_{i_1=0}^{\lfloor k/2\rfloor}\sum_{i_2=0}^{k}&\int_{\Sigma_\tau\cap\{\tau/4\leq r\leq 9\tau/10\}} |D\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi|^2\\ &\leq C\Big(\sum_{i_1=0}^{8}\sup_{\tau/4\leq r\leq 9\tau/10}|D\Gamma^{i_1}\Phi|^2\Big)\sum_{i_2=0}^{k}\int_{\Sigma_\tau\cap\{\tau/4\leq r\leq 9\tau/10\}}|D\Gamma^{i_2}\Phi|^2\\ &\leq CAB\epsilon\tau^{-4+\eta_{14}}\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{\tau/4\leq r\leq 9\tau/10\}}|D\Gamma^{i}\Phi|^2.\end{aligned}$$

The particular cubic term can be estimated as follows:

$$\begin{aligned}\sum_{i_1,i_2=0}^{\lfloor k/2\rfloor}\sum_{i_3=0}^{k}&\int_{\Sigma_\tau\cap\{\tau/4\leq r\leq 9\tau/10\}} |\Gamma^{i_3}\Phi D\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi|^2\\ &\leq C\Big(\sum_{i_1=0}^{8}\sup_{\tau/4\leq r\leq 9\tau/10}|D\Gamma^{i_1}\Phi|^2\Big)^2\sum_{i_3=0}^{k}\int_{\Sigma_\tau\cap\{\tau/4\leq r\leq 9\tau/10\}}|\Gamma^{i_3}\Phi|^2\\ &\leq CA^2B^2\epsilon^2\tau^{-8+2\eta_{14}}\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{\tau/4\leq r\leq 9\tau/10\}}|\Gamma^{i}\Phi|^2.\end{aligned}$$

In principle, for $k\leq 15$, we can then control the last term using the conformal energy. For $k=16$, however, conformal energy is not available, and we need to use Hardy's inequality:

$$A\epsilon\tau^{-8+2\eta_{14}}\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{\tau/4\leq r\leq 9\tau/10\}}|\Gamma^i\Phi|^2 \leq CA\epsilon\tau^{-6+2\eta_{14}}\sum_{i=0}^{k}\int_{\Sigma_\tau} r^{-2}|\Gamma^i\Phi|^2$$
$$\leq CA\epsilon\tau^{-6+2\eta_{14}}\sum_{i=0}^{k}\int_{\Sigma_\tau} J^N_\mu(\Gamma^i\Phi)n^\mu_{\Sigma_\tau}.$$

The estimates now follow from Bootstrap Assumptions (9)–(11). □

For many applications, we only need a much weaker estimate on N_k. We write down the following proposition which corresponds to the estimates that will be proved for the quantities involving S. This would allow a unified approach in dealing with many estimates with or without S.

Proposition 6.13.

$$\int_{\Sigma_\tau\cap\{r\leq 9\tau/10\}} r^{1-\delta}N_{16}^2 \leq CA^2\epsilon^2\tau^{-3+\eta_{S,11}+\eta_{16}},$$
$$\int_{\Sigma_\tau\cap\{r\leq 9\tau/10\}} r^{1-\delta}N_{\leq 15}^2 \leq CA^2\epsilon^2\tau^{-4+\eta_{S,11}+\eta_{15}}.$$

We now move to the estimates for N_k in the region $\{r \geq 9t^*/10\}$. Here, we need to exploit the null condition:

Proposition 6.14. *For $\alpha = 0$ or 2,*

$$\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} N_{16}^2 \leq CBA^2\epsilon^2\tau^{-2+\eta_{16}},$$
$$\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} r^\alpha N_{15}^2 \leq CBA^2\epsilon^2\tau^{-3+\alpha+\eta_{15}},$$
$$\sum_{j=0}^{14}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} r^\alpha N_j^2 \leq CBA^2\epsilon^2\tau^{-4+\alpha+\eta_{14}}.$$

Proof. Following the argument before, we reduce to quadratic and cubic terms. This time, however, the null condition plays a crucial role. For the quadratic terms, we need to consider

$$\bar{D}\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi,\qquad D\Gamma^{i_1}\Phi\bar{D}\Gamma^{i_2}\Phi,\qquad r^{-1}(D\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi),$$

where $i_1 \geq i_2$. For the cubic terms, we need to consider

$$D\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi D\Gamma^{i_3}\Phi,\qquad \bar{D}\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi\Gamma^{i_3}\Phi.$$

Notice that the first cubic term can be dominated pointwise by quadratic terms of the third type listed above using Bootstrap Assumptions (33) and (42). The second cubic term can also be dominated by the first two types of quadratic terms if $i_3 \leq 13$ by (31) and (39). We can thus assume $i_3 > 13$ and hence $i_1, i_2 \leq 8$. We now estimate the quadratic terms:

$$\begin{aligned}
\sum_{i_2=0}^{\lfloor k/2\rfloor}\sum_{i_1=0}^{k-j}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} & r^\alpha\big(|\bar{D}\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi|^2+|D\Gamma^{i_1}\Phi\bar{D}\Gamma^{i_2}\Phi|^2+r^{-2}|D\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi|^2\big)\\
&\leq C\Big(\sup_{r\geq 9\tau/10}\sum_{i_2=0}^{8}r^2|D\Gamma^{i_2}\Phi|^2\Big)\sum_{i_1=0}^{k}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} r^{\alpha-2}|\bar{D}\Gamma^{i_1}\Phi|^2\\
&\quad+C\Big(\sup_{r\geq 9\tau/10}\sum_{i_2=0}^{8}r^2|\bar{D}\Gamma^{i_2}\Phi|^2\Big)\sum_{i_1=0}^{k}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} r^{\alpha-2}|D\Gamma^{i_1}\Phi|^2\\
&\quad+C\tau^{-2}\Big(\sup_{r\geq 9\tau/10}\sum_{i_2=0}^{8}r^2|D\Gamma^{i_2}\Phi|^2\Big)\sum_{i_1=0}^{k}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} r^{\alpha-2}|D\Gamma^{i_1}\Phi|^2\\
&\leq CAB\epsilon\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} r^{\alpha-2}|\bar{D}\Gamma^{i}\Phi|^2\\
&\quad+CAB\epsilon\tau^{-2+\eta_{14}}\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} r^{\alpha-2}|D\Gamma^{i}\Phi|^2.
\end{aligned}$$

We then estimate the particular cubic term:

$$\begin{aligned}
\sum_{i_3=0}^{k}\sum_{i_1,i_2=0}^{8}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} & r^\alpha(\bar{D}\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi\Gamma^{i_3}\Phi)^2\\
&\leq C\Big(\sup_{r\geq 9\tau/10}\sum_{i_1=0}^{8}r^2|D\Gamma^{i_2}\Phi|^2\Big)\Big(\sup_{r\geq 9\tau/10}\sum_{i_2=0}^{8}r^2|\bar{D}\Gamma^{i_2}\Phi|^2\Big)\\
&\quad\times\sum_{i_3=0}^{k}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} r^{\alpha-4}|\Gamma^{i_1}\Phi|^2\\
&\leq CA^2B^2\epsilon^2\tau^{-2+\eta_{14}}\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} r^{\alpha-4}|\Gamma^{i}\Phi|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} N_{16}^2\\
&\quad\leq CAB\epsilon\tau^{-2}\sum_{i=0}^{16}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}}|\bar{D}\Gamma^i\Phi|^2+CAB\epsilon\tau^{-4+\eta_{14}}\sum_{i=0}^{16}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}}|D\Gamma^i\Phi|^2\\
&\qquad+CA^2B^2\epsilon^2\tau^{-4+\eta_{14}}\sum_{i=0}^{16}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}}r^{-2}|\Gamma^i\Phi|^2\\
&\quad\leq (CAB\epsilon\tau^{-2}+(CAB\epsilon+CA^2B^2\epsilon^2)\tau^{-4+\eta_{14}})\sum_{i=0}^{16}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}}|D\Gamma^i\Phi|^2\\
&\quad\leq CA^2B\epsilon^2\tau^{-2+\eta_{16}},
\end{aligned}$$

and

$$\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} N_k^2$$
$$\leq CAB\epsilon\tau^{-4}\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}}\tau^2|\bar{D}\Gamma^i\Phi|^2+CAB\epsilon\tau^{-4+\eta_{14}}\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}}|D\Gamma^i\Phi|^2$$
$$+CA^2B^2\epsilon^2\tau^{-4+\eta_{14}}\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}}r^{-2}|\Gamma^i\Phi|^2$$
$$\leq CAB\epsilon\tau^{-4}\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}}\tau^2|\bar{D}\Gamma^i\Phi|^2$$
$$+(CAB\epsilon+CA^2B^2\epsilon^2)\tau^{-4+\eta_{14}}\sum_{i=0}^{k}\int_{\Sigma_\tau}|D\Gamma^i\Phi|^2,$$

and

$$\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} r^2N_k^2\leq CAB\epsilon\tau^{-2}\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}}\tau^2|\bar{D}\Gamma^i\Phi|^2$$
$$+CAB\epsilon\tau^{-2+\eta_{14}}\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}}|D\Gamma^i\Phi|^2$$
$$+CA^2B^2\epsilon^2\tau^{-2+\eta_{14}}\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}}r^{-2}|\Gamma^i\Phi|^2$$
$$\leq CAB\epsilon\tau^{-2}\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}}\tau^2|\bar{D}\Gamma^i\Phi|^2$$
$$+(CAB\epsilon+CA^2B^2\epsilon^2)\tau^{-2+\eta_{14}}\sum_{i=0}^{k}\int_{\Sigma_\tau}|D\Gamma^i\Phi|^2.$$

The conclusion follows from Proposition 2.2 and Bootstrap Assumptions (9)–(11). □

From the estimates for U_k and N_k and the L^2-L^∞ estimates in the last section, we get the following pointwise bounds:

Proposition 6.15. *For $r\geq t^*/4$,*

$$\sum_{j=0}^{13}|\Gamma^j\Phi|^2\leq (B/2)A\epsilon r^{-2}(t^*)^{1+\eta_{15}}, \tag{48}$$
$$\sum_{j=0}^{13}|D\Gamma^j\Phi|^2\leq (B/2)A\epsilon, \tag{49}$$

$$\sum_{\ell=1}^{13-j}\sum_{j=0}^{12}|D^\ell\Gamma^j\Phi|^2 \le (B/2)A\epsilon r^{-2}, \tag{50}$$

$$\sum_{j=0}^{8}|D\Gamma^j\Phi|^2 \le (B/2)A\epsilon r^{-2}(t^*)^{\eta_{14}}(1+|u|)^{-2}, \tag{51}$$

$$\sum_{j=0}^{8}|\bar{D}\Gamma^j\Phi|^2 \le (B/2)A\epsilon r^{-2}(t^*)^{-2+\eta_{14}}. \tag{52}$$

For $r \le t^*/4$,

$$\sum_{j=0}^{13}|\Gamma^j\Phi|^2 \le (B/2)A\epsilon(t^*)^{-1+\eta_{15}}, \tag{53}$$

$$\sum_{\ell=1}^{14-j}\sum_{j=0}^{13}|D^\ell\Gamma^j\Phi|^2 \le BA\epsilon(t^*)^{-1+\eta_{15}}, \tag{54}$$

$$\sum_{\ell=1}^{13-j}\sum_{j=0}^{12}|D^\ell\Gamma^j\Phi|^2 \le BA\epsilon(t^*)^{-2+\eta_{14}}, \tag{55}$$

$$\sum_{\ell=1}^{9-j}\sum_{j=0}^{8}|D^\ell\Gamma^j\Phi|^2 \le (B/2)A\epsilon r^{-1+\delta}(t^*)^{-3+\eta_{S,11}}. \tag{56}$$

Proof. (48) is immediate from Proposition 5.3 and Bootstrap Assumptions (10) and (11).

By Proposition 5.1,

$$\sum_{j=0}^{13}|D\Gamma^j\Phi|^2 \le C\left(\sum_{k=0}^{15}\int_{\Sigma_\tau} J^N_\mu(\Gamma^k\Phi)n^\mu_{\Sigma_\tau} + \sum_{k=0}^{1}\int_{\Sigma_\tau}(D^kG_{\le 13})^2\right).$$

Hence we get (49) by Bootstrap Assumption (12) and Propositions 6.9 and 6.10. The constant is improved since $A\epsilon \ll 1$ and $C \ll B$.

By Proposition 5.2,

$$\begin{aligned}\sum_{\ell=1}^{13-j}\sum_{j=0}^{12}&|D^\ell\Gamma^j\Phi|^2\\ &\le Cr^{-2}\left(\sum_{m=0}^{13-j}\sum_{k=0}^{2}\sum_{j=0}^{12}\int_{\Sigma_\tau} J^N_\mu(\partial^m_{t^*}\Omega^k\Gamma^j\Phi)n^\mu_{\Sigma_\tau} + \sum_{m+k\le 10}\int_{\Sigma_\tau}(D^mG_{\le k})^2\right)\\ &\le Cr^{-2}\left(\sum_{j=0}^{11}\int_{\Sigma_\tau} J^N_\mu(\Gamma^j\Phi)n^\mu_{\Sigma_\tau} + \sum_{m+k\le 10}\int_{\Sigma_\tau}(D^mG_{\le k})^2\right).\end{aligned}$$

We hence get (50) by Bootstrap Assumption (12) and Propositions 6.9 and 6.10. The constant is improved since $A\epsilon \ll 1$ and $C \ll B$.

By Proposition 5.5, for $r \geq t^*/4$, we have

$$\sum_{j=0}^{8} |D\Gamma^j \Phi|^2 \leq Cr^{-2}(1+|u|)^{-2} \sum_{m=0}^{1}\sum_{k=0}^{2}\sum_{j=0}^{8}\Big(\int_{\Sigma_\tau} J_\mu^{Z+CN}(\partial_{t^*}^m \tilde{\Omega}^k \Gamma^j \Phi) n_{\Sigma_\tau}^\mu + C\tau^2 \int_{\Sigma_\tau\cap\{r\leq r_Y^-\}} J_\mu^N(\partial_{t^*}^m \tilde{\Omega}^k \Gamma^j \Phi) n_{\Sigma_\tau}^\mu\Big)$$
$$+ Cr^{-2}\sum_{k=0}^{2}\sum_{j=0}^{8}\int_{\Sigma_\tau\cap\{u'\sim u\}\cap\{r\geq\tau/4\}} (\Box_{g_K}(\tilde{\Omega}^k\Gamma^j\Phi))^2$$
$$\leq CA\epsilon\tau^{\eta_{14}} r^{-2}(1+|u|)^{-2} + Cr^{-2}\int\int_{\Sigma_\tau\cap\{u'\sim u\}\cap\{r\geq\tau/4\}} G_{\leq 10}^2$$
by Bootstrap Assumption (11)
$$\leq CA\epsilon\tau^{\eta_{14}} r^{-2}(1+|u|)^{-2} + CA\epsilon(t^*)^{-2+\eta_{14}} r^{-2} + CA^2B\epsilon^2(t^*)^{-2+\eta_{14}} r^{-2}$$
by Propositions 6.13, 6.14 and 6.10
$$\leq (B/2)A\epsilon r^{-2}(t^*)^{\eta_{14}}(1+|u|)^{-2}.$$

Hence we have proved (51).

By Proposition 5.6, for $r \geq t^*/4$, we have

$$\sum_{j=0}^{8} |\bar{D}\Gamma^j\Phi|^2$$
$$\leq Cr^{-4}\sum_{k=0}^{2}\sum_{j=0}^{8}\sum_{i+m\leq 1}\Big(\int_{\Sigma_\tau} J_\mu^N(S^i\partial_{t^*}^m\Gamma^j\Phi)n_{\Sigma_\tau}^\mu + \int_{\Sigma_\tau} J_\mu^{Z+CN}(\partial_{t^*}^m\tilde{\Omega}^k\Gamma^j\Phi)n_{\Sigma_\tau}^\mu + C\tau^2\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}} J_\mu^N(\partial_{t^*}^m\tilde{\Omega}^k\Gamma^j\Phi)n_{\Sigma_\tau}^\mu + \int_{\Sigma_\tau}(\Box_{g_K}(\tilde{\Omega}^k\Gamma^j\Phi))^2\Big)$$
$$+ Cr^{-2}\sum_{k=0}^{2}\sum_{j=0}^{8}\int_{\Sigma_\tau\cap\{r\geq\tau/2\}}(\Box_{g_K}(\tilde{\Omega}^k\Gamma^j\Phi))^2$$
$$\leq CA\epsilon r^{-4}(t^*)^{\eta_{14}} + CA\epsilon(t^*)^{-2+\eta_{14}}r^{-2} + CA^2B\epsilon^2(t^*)^{-2+\eta_{14}}r^{-2}$$
$$\leq (B/2)A\epsilon r^{-2}(t^*)^{-2+\eta_{14}}.$$

Hence we have proved (52) and completed the proof for $r \geq t^*/4$.

We now move to the pointwise estimates in the region $r \leq t^*/4$. (53) follows directly from Proposition 5.9 and Bootstrap Assumptions (10) and (11). By Proposition 5.7,

$$\sum_{\ell=1}^{14-j}\sum_{j=0}^{13}|D^\ell\Gamma^j\Phi|^2$$
$$\leq C\Big(\sum_{i+j\leq 15}\int_{\Sigma_\tau\cap\{r\leq t^*/2\}} J_\mu^N(\hat{Y}^i\Gamma^j\Phi)n_{\Sigma_\tau}^\mu + \sum_{\ell=1}^{14-j}\sum_{j=0}^{13}\int_{\Sigma_\tau}(D^\ell G_{\leq j})^2\Big),$$

where we have used the fact that $[\hat{Y}, \Gamma] = 0$. Hence (54) follows from Bootstrap Assumptions (10) and (11) and Propositions 6.9 and 6.10. Next, (55) follows similarly to (54): by Proposition 5.7,

$$\sum_{\ell=1}^{13-j}\sum_{j=0}^{12}|D^\ell\Gamma^j\Phi|^2$$
$$\leq C\Big(\sum_{i+j\leq 14}\int_{\Sigma_\tau\cap\{r\leq t^*/2\}} J^N_\mu(\hat{Y}^i\Gamma^j\Phi)n^\mu_{\Sigma_\tau}+\sum_{\ell=1}^{13-j}\sum_{j=0}^{13}\int_{\Sigma_\tau}(D^\ell G_{\leq j})^2\Big).$$

Hence (55) follows from Bootstrap Assumption (11) and Propositions 6.9 and 6.10.

Finally, by Proposition 5.10, for $r\leq t^*/4$, we have

$$\sum_{\ell=1}^{9-j}\sum_{j=0}^{8}|D^\ell\Gamma^j\Phi|^2$$
$$\leq C(t^*)^{-1}r^{-1+\delta}\sum_{i+j\leq 10}\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/2\}}\big(K^{X_1}(Y^i\Gamma^j\Phi)+K^{X_1}(SY^i\Gamma^j\Phi)$$
$$+r^{-1-\delta}(D^iG_j)^2\big)$$
$$\leq CA\epsilon\tau^{-3+\eta_{S,11}}r^{-1+\delta}+CA^2\epsilon^2\tau^{-3+\eta_{14}}r^{-1+\delta}\leq (B/2)A\epsilon\tau^{-3+\eta_{S,11}}r^{-1+\delta},$$

where in the third line we have used Bootstrap Assumptions (24) and (30) and Propositions 6.13, 6.14 and 6.10. The only caveat is that when using (30), the vector fields $\hat{Y}$ and S are in different order. However, since $[S,\hat{Y}]\sim D$, we can estimate the commutator term by (24). □

Now that we have proved the L^∞ bounds, we will replace the constant B in Bootstrap Assumptions (31)–(35), (39), (42) by C in the sequel. Notice that we have originally assumed $B\ll A_0$ and therefore $C\ll A_0$ still holds. We now proceed to recover Bootstrap Assumptions (K) that do not involve the commutators Y or S. We first retrieve (21)–(24). Notice also that we will retrieve (17) and (19) later together with (9) and (12).

Proposition 6.16.

$$\sum_{i+j\leq 15}A^{-1}_{X,j}\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/2\}}\big(K^{X_0}(\partial^i_{t^*}\tilde{\Omega}^j\Phi)+K^N(\partial^i_{t^*}\tilde{\Omega}^j\Phi)\big)\leq\frac{\epsilon}{2}\tau^{-1+\eta_{15}},\quad (57)$$

$$\sum_{i+j\leq 14}A^{-1}_{X,j}\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/2\}}K^{X_1}(\partial^i_{t^*}\tilde{\Omega}^j\Phi)\leq\frac{\epsilon}{2}\tau^{-1+\eta_{15}},\quad (58)$$

$$\sum_{i+j\leq 14}A^{-1}_{X,j}\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/2\}}\big(K^{X_0}(\partial^i_{t^*}\tilde{\Omega}^j\Phi)+K^N(\partial^i_{t^*}\tilde{\Omega}^j\Phi)\big)\leq\frac{\epsilon}{2}\tau^{-2+\eta_{14}},\quad (59)$$

$$\sum_{i+j\leq 13}A^{-1}_{X,j}\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/2\}}K^{X_1}(\partial^i_{t^*}\tilde{\Omega}^j\Phi)\leq\frac{\epsilon}{2}\tau^{-2+\eta_{14}}.\quad (60)$$

Proof. We first prove the estimates involving X_0, i.e., (57) and (59). By Proposition 3.7(i) and the Remark following it and the fact that $|\partial^m_{t^*}N_k|\leq|N_{\leq k+m}|$, we have

$$
\begin{aligned}
&\sum_{i+j\leq 15} A_{X,j}^{-1}\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/2\}}\big(K^{X_0}(\partial_{t^*}^i\tilde{\Omega}^j\Phi)+K^N(\partial_{t^*}^i\tilde{\Omega}^j\Phi)\big)\\
&\leq C\sum_{i+j\leq 15} A_{X,j}^{-1}\Big(\tau^{-2}\int_{\Sigma_{\tau/1.1}} J_\mu^{Z+N,w^Z}(\partial_{t^*}^i\tilde{\Omega}^j\Phi)n^\mu_{\Sigma_{\tau/1.1}}\\
&\qquad\qquad +C\int_{\Sigma_{\tau/1.1}\cap\{r\leq r_Y^-\}} J_\mu^N(\partial_{t^*}^i\tilde{\Omega}^j\Phi)n^\mu_{\Sigma_{\tau/1.1}}\Big)\\
&\quad +C\sum_{i+j\leq 15} A_{X,j}^{-1}\Big(\iint_{\mathcal{R}(\tau/1.1-1,\tau+1)\cap\{r\leq 9t^*/10\}} r^{1+\delta}N_{\leq 16}^2\\
&\qquad\qquad +\sup_{t^*\in[\tau/1.1-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq M/8\}} N_{\leq 16}^2\\
&\qquad\qquad +\iint_{\mathcal{R}(\tau/1.1-1,\tau+1)\cap\{r\leq 9t^*/10\}} r^{1+\delta}(U_{\leq 15,\leq j})^2\Big)\\
&\leq C\sum_{i+j\leq 15}(A_{X,j}^{-1}A_j\epsilon\tau^{-1+\eta_{15}}+A_{X,j}^{-1}A^2\epsilon^2\tau^{-2+\eta_{S,11}+\eta_{16}+2\delta}+CA_{X,j}^{-1}A_{X,j-1}\epsilon\tau^{-1+\eta_{15}})\\
&\leq(\epsilon/2)\tau^{-1+\eta_{15}},
\end{aligned}
$$

by Propositions 6.13, 6.14 and 6.10. Notice that our integrated estimates for U in Proposition 6.10 are only for $[\tau/1.1,\tau]$. Nevertheless, for the region $[\tau/1.1-1,\tau/1.1]\cap[\tau,\tau+1]$, we can integrate over the fixed τ estimate in the same proposition. By Proposition 3.7(i) and the Remark following it and the fact that $|\partial_{t^*}^m N_k|\leq|N_{\leq k+m}|$, we have

$$
\begin{aligned}
&\sum_{i+j\leq 14} A_{X,j}^{-1}\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/2\}}\big(K^{X_0}(\partial_{t^*}^i\tilde{\Omega}^j\Phi)+K^N(\partial_{t^*}^i\tilde{\Omega}^j\Phi)\big)\\
&\leq C\sum_{i+j\leq 14} A_{X,j}^{-1}\Big(\tau^{-2}\int_{\Sigma_{\tau/1.1}} J_\mu^{Z+N,w^Z}(\partial_{t^*}^i\tilde{\Omega}^j\Phi)n^\mu_{\Sigma_{\tau/1.1}}\\
&\qquad\qquad +C\int_{\Sigma_{\tau/1.1}\cap\{r\leq r_Y^-\}} J_\mu^N(\partial_{t^*}^i\tilde{\Omega}^j\Phi)n^\mu_{\Sigma_{\tau/1.1}}\Big)\\
&\quad +C\sum_{i+j\leq 14} A_{X,j}^{-1}\Big(\iint_{\mathcal{R}(\tau/1.1-1,\tau+1)\cap\{r\leq 9t^*/10\}} r^{1+\delta}N_{\leq 15}^2\\
&\qquad\qquad +\sup_{t^*\in[\tau/1.1-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq M/8\}} N_{\leq 15}^2\\
&\qquad\qquad +\iint_{\mathcal{R}(\tau/1.1-1,\tau+1)\cap\{r\leq 9t^*/10\}} r^{1+\delta}(U_{\leq 14,\leq j})^2\Big)\\
&\leq C\sum_{i+j\leq 14}(A_{X,j}^{-1}A_j\epsilon\tau^{-2+\eta_{14}}+A_{X,j}^{-1}A^2\epsilon^2\tau^{-3+\eta_{S,11}+\eta_{15}+2\delta}+A_{X,j}^{-1}A_{X,j-1}\epsilon\tau^{-2+\eta_{14}})\\
&\leq(\epsilon/2)\tau^{-2+\eta_{14}}.
\end{aligned}
$$

The proof of (58) and (60) proceeds in an identical manner. Notice that using Proposition 3.7(ii), the right hand side when we estimate (58) (respectively (60)) is identical to that when we estimate (57) (respectively (59)). □

Now we move on to retrieving Bootstrap Assumptions (J) with better constants:

Proposition 6.17.

$$\sum_{i+j=16} A_j^{-1} \int_{\Sigma_\tau} J^N_\mu(\partial_{t^*}^i \tilde{\Omega}^j \Phi) n^\mu_{\Sigma_\tau} \leq (\epsilon/4)\tau^{\eta_{16}}, \tag{61}$$

$$\sum_{i+j\leq 15} A_j^{-1} \int_{\Sigma_\tau} J^N_\mu(\partial_{t^*}^i \tilde{\Omega}^j \Phi) n^\mu_{\Sigma_\tau} \leq (\epsilon/2), \tag{62}$$

$$\sum_{i+j=16} A_{X,j}^{-1} \left(\iint_{\mathcal{R}(\tau_0,\tau)} K^{X_0}(\partial_{t^*}^i \tilde{\Omega}^j \Phi) + \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}} K^N(\partial_{t^*}^i \tilde{\Omega}^j \Phi) \right) \leq (\epsilon/2)\tau^{\eta_{16}}, \tag{63}$$

$$\sum_{i+j=15} A_{X,j}^{-1} \iint_{\mathcal{R}(\tau_0,\tau)} K^{X_1}(\partial_{t^*}^i \tilde{\Omega}^j \Phi) \leq (\epsilon/2)\tau^{\eta_{16}}, \tag{64}$$

$$\sum_{i+j\leq 15} A_{X,j}^{-1} \iint_{\mathcal{R}(\tau_0,\tau)} K^{X_0}(\partial_{t^*}^i \tilde{\Omega}^j \Phi) \leq \epsilon/2, \tag{65}$$

$$\sum_{i+j\leq 14} A_{X,j}^{-1} \iint_{\mathcal{R}(\tau_0,\tau)} K^{X_1}(\partial_{t^*}^i \tilde{\Omega}^j \Phi) \leq \epsilon. \tag{66}$$

Proof. We will prove the slightly stronger statements with $A_{X,j}$ replaced by A_j. Using Propositions 3.1 and 3.2, we have

$$\begin{aligned}
&\sum_{i+j=16} A_j^{-1} \Bigg(\int_{\Sigma_\tau} J^N_\mu(\partial_{t^*}^i \tilde{\Omega}^j \Phi) n^\mu_{\Sigma_\tau} + \iint_{\mathcal{R}(\tau_0,\tau)} K^{X_0}(\partial_{t^*}^i \tilde{\Omega}^j \Phi) \\
&\qquad\qquad + \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}} K^N(\partial_{t^*}^i \tilde{\Omega}^j \Phi) \Bigg) \\
&\leq C \sum_{i+j=16} A_j^{-1} \Bigg(\int_{\Sigma_{\tau_0}} J^N_\mu(\partial_{t^*}^i \tilde{\Omega}^j \Phi) n^\mu_{\Sigma_{\tau'}} + \left(\int_{\tau_0-1}^{\tau+1} \left(\int_{\Sigma_{t^*}} N_{16}^2 \right)^{1/2} dt^* \right)^2 \\
&\qquad + \iint_{\mathcal{R}(\tau_0-1,\tau+1)} N_{16}^2 + \iint_{\mathcal{R}(\tau_0-1,\tau+1)} r^{1+\delta} U_{16,j}^2 \\
&\qquad + \sup_{t^*\in[\tau_0-1,\tau+1]} \int_{\Sigma_{t^*}\cap\{|r-3M|\leq M/8\}} U_{16,j}^2 \Bigg) \\
&\leq C \sum_{i+j=16} A_j^{-1} (\epsilon + A^2\epsilon^2\eta_{16}^{-1}\tau^{\eta_{16}} + A_{X,j-1}\epsilon\tau^{\eta_{16}}) \leq (\epsilon/4)\tau^{\eta_{16}}.
\end{aligned}$$

We now turn to the estimates for $\sum_{j=0}^{15} |\Gamma^j \Phi|$. We have

$$
\begin{aligned}
&\sum_{i+j\leq 15} A_j^{-1}\Big(\int_{\Sigma_\tau} J^N_\mu(\partial_{t^*}^i\tilde{\Omega}^j\Phi)n^\mu_{\Sigma_\tau}+\iint_{\mathcal{R}(\tau_0,\tau)} K^{X_0}(\partial_{t^*}^i\tilde{\Omega}^j\Phi)\Big)\\
&\quad\leq C\sum_{i+j\leq 15} A_j^{-1}\Big(\int_{\Sigma_{\tau_0}} J^N_\mu(\partial_{t^*}^i\tilde{\Omega}^j\Phi)n^\mu_{\Sigma_{\tau_0}}+\Big(\int_{\tau_0-1}^{\tau+1}\Big(\int_{\Sigma_{t^*}} N_{\leq 15}^2\Big)^{1/2}dt^*\Big)^2\\
&\qquad+\iint_{\mathcal{R}(\tau_0-1,\tau+1)} N_{\leq 15}^2+\iint_{\mathcal{R}(\tau_0-1,\tau+1)} r^{1+\delta}U^2_{\leq 15,\leq j}\\
&\qquad+\sup_{t^*\in[\tau_0-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq M/8\}} U^2_{\leq 15,\leq j}\Big)\\
&\quad\leq C\sum_{i+j\leq 15} A_j^{-1}(\epsilon+A^2\epsilon^2+CA_{X,j-1}\epsilon)\leq \epsilon/2.
\end{aligned}
$$

It now remains to show (64) and (66). By Proposition 3.4 they can be estimated by exactly the same terms as (63) and (65) respectively. The proposition hence follows. □

We now move on to control the conformal energy and close the part of Bootstrap Assumption (10) without $\hat{Y}$.

Proposition 6.18.

$$
\sum_{i+j=15} A_j^{-1}\Big(\int_{\Sigma_\tau} J^{Z+N,w^Z}_\mu(\partial_{t^*}^i\tilde{\Omega}^j\Phi)n^\mu_{\Sigma_\tau}+C\tau^2\int_{\Sigma_\tau\cap\{r\leq 9\tau/10\}} J^N_\mu(\partial_{t^*}^i\tilde{\Omega}^j\Phi)n^\mu_{\Sigma_\tau}\Big)\leq(\epsilon/4)\tau^{1+\eta_{15}}. \tag{67}
$$

Proof. By Proposition 3.5,

$$
\begin{aligned}
&\sum_{i+j=15} A_j^{-1}\Big(\int_{\Sigma_\tau} J^{Z,w^Z}_\mu(\partial_{t^*}^i\tilde{\Omega}^j\Phi)n^\mu_{\Sigma_\tau}+C\tau^2\int_{\Sigma_\tau\cap\{r\leq 9\tau/10\}} J^N_\mu(\partial_{t^*}^i\tilde{\Omega}^j\Phi)n^\mu_{\Sigma_\tau}\Big)\\
&\leq C\sum_{i+j=15} A_j^{-1}\Big(\int_{\Sigma_{\tau_0}} J^{Z+CN,w^Z}_\mu(\partial_{t^*}^i\tilde{\Omega}^j\Phi)n^\mu_{\Sigma_{\tau_0}}\\
&\qquad+\delta'\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq t^*/2\}}(t^*)^2K^{X_0}(\partial_{t^*}^i\tilde{\Omega}^j\Phi)\\
&\qquad+(\delta'+a)\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}}(t^*)^2K^N(\partial_{t^*}^i\tilde{\Omega}^j\Phi)\\
&\qquad+(\delta')^{-1}\Big(\iint_{\mathcal{R}(\tau_0,\tau)} t^*r^{-1+\delta}K^{X_1}(\partial_{t^*}^i\tilde{\Omega}^j\Phi)\\
&\quad+\sum_{m=0}^{1}\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq 9t^*/10\}}(t^*)^2r^{1+\delta}(\partial_{t^*}^m N_{15})^2+\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq 9t^*/10\}}(t^*)^2r^{1+\delta}U^2_{15,j}\\
&\quad+\Big(\int_{\tau_0}^{\tau}\Big(\int_{\Sigma_{t^*}\cap\{r\geq t^*/2\}} r^2G^2_{15,j}\Big)^{1/2}dt^*\Big)^2+\sup_{t^*\in[\tau_0,\tau]}\int_{\Sigma_{t^*}\cap\{r_Y^-\leq r\leq 25M/8\}}(t^*)^2N^2_{15}\Big)\Big).
\end{aligned}
$$

We will estimate the terms one by one. First, the term with the initial data, i.e., the very first term, is clearly bounded by $C(\sum_j A_j^{-1})\epsilon$. Second, we consider the two $(t^*)^2K$ terms. To this end, we define as before $\tau_0 \le \tau_1 \le \cdots \le \tau_n = \tau$ with $\tau_{i+1} \le (1.1)\tau_i$, and $n \sim \log(\tau - \tau_0)$ is the minimum such that this can be done. Thus, these two terms can be bounded, using Bootstrap Assumption (21)

$$\begin{aligned}
&\delta' \sum_{i+j=15} A_j^{-1} \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\le t^*/2\}} (t^*)^2 K^{X_0}(\partial_{t^*}^i \tilde{\Omega}^j \Phi) \\
&\qquad\qquad + (\delta' + a) \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\le r_Y^-\}} (t^*)^2 K^N(\partial_{t^*}^i \tilde{\Omega}^j \Phi) \\
&\quad \le C \sum_{i+j=15} A_j^{-1} \sum_{k=0}^{n-1} \Bigg(\delta' \tau_k^2 \iint_{\mathcal{R}(\tau_k,\tau_{k+1})\cap\{r\le t^*/2\}} K^{X_0}(\partial_{t^*}^i \tilde{\Omega}^j \Phi) \\
&\qquad\qquad + (\delta' + a)\tau_k^2 \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\le r_Y^-\}} K^N(\partial_{t^*}^i \tilde{\Omega}^j \Phi) \Bigg) \\
&\quad \le C\left(\sum_j \frac{A_{X,j}}{A_j}\right)\epsilon(2\delta' + a)\tau^{1+\eta_{15}}.
\end{aligned}$$

This is acceptable since $a, \delta' \ll A_j/A_{X,j}$. Third, the term with $t^* r^{-1+\delta} K$ can be bounded using Bootstrap Assumption (18):

$$\begin{aligned}
&(\delta')^{-1} \sum_{i+j=15} A_j^{-1} \iint_{\mathcal{R}(\tau_0,\tau)} t^* r^{-1+\delta} K^{X_1}(\partial_{t^*}^i \tilde{\Omega}^j \Phi) \\
&\le C(\delta')^{-1} \sum_{i+j=15} A_j^{-1} \sum_{k=0}^{n-1} \tau_k \iint_{\mathcal{R}(\tau_k,\tau_{k+1})} K^{X_1}(\partial_{t^*}^i \tilde{\Omega}^j \Phi) \le C(\delta')^{-1}\left(\sum_j \frac{A_{X,j}}{A_j}\right)\tau^{1+\eta_{16}}.
\end{aligned}$$

This is acceptable since $\eta_{16} \ll \eta_{15}$ and therefore the constant can be improved for τ large. Fourth, the integrals involving N_{15} can be bounded using Propositions 6.13 and 6.14:

$$\begin{aligned}
&C(\delta')^{-1} A_0^{-1} \sum_{m=0}^{1} \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\le 9t^*/10\}} (t^*)^2 r^{1+\delta} (\partial_{t^*}^m N_{15})^2 \\
&\qquad \le C A^2 A_0^{-1} \epsilon^2 (\delta')^{-1} \int_{\tau_0}^{\tau} (t^*)^{-1+\eta_{S,11}+\eta_{15}+2\delta} dt^* \le C A^2 A_0^{-1} \epsilon^2 (\delta')^{-1} \tau^{\eta_{S,11}+\eta_{15}+2\delta}, \\
&C(\delta')^{-1} A_0^{-1} \left(\int_{\tau_0}^{\tau} \left(\int_{\Sigma_{t^*}\cap\{r\ge t^*/2\}} r^2 N_{15}^2 \right)^{1/2} dt^* \right)^2 \le C A^2 A_0^{-1} \epsilon^2 (\delta')^{-1} \tau^{1+\eta_{15}}, \\
&C(\delta')^{-1} A_0^{-1} \sup_{t^*\in[\tau_0,\tau]} \int_{\Sigma_{t^*}\cap\{r_Y^-\le r\le 25M/8\}} (t^*)^2 N_{15}^2 \le C A^2 A_0^{-1} \epsilon^2 (\delta')^{-1}.
\end{aligned}$$

These are all acceptable since ϵ would beat all the constants. Fifth, for the commutator terms $U_{15,j}^2$, we estimate by Proposition 6.10:

$$(\delta')^{-1}A_j^{-1}\iint_{\mathcal{R}(\tau_0,\tau)}(t^*)^2r^{1+\delta}(U_{15,j})^2 \leq C\sum_{i=0}^{n-1}(\delta')^{-1}\iint_{\mathcal{R}(\tau_i,\tau_{i+1})}\tau_i^2r^{1+\delta}(U_{15,j})^2$$
$$\leq C(\delta')^{-1}\frac{A_{X,j-1}}{A_j}\tau^{1+\eta_{15}},$$

and

$$(\delta')^{-1}A_j^{-1}\left(\int_{\tau_0}^{\tau}\left(\int_{\Sigma_{t^*}\cap\{r\geq t^*/2\}}r^2U_{15,j}^2\right)^{1/2}dt^*\right)^2 \leq C(\delta')^{-1}\frac{A_{j-1}}{A_j}\tau.$$

Since $A_{j-1}/A_j \ll A_{X,j-1}/A_j \ll \delta'$, all terms are acceptable. □

With 14 or less derivatives, the conformal energy behaves better. We now close the part of Bootstrap Assumption (11) without $\hat{Y}$.

Proposition 6.19.

$$\sum_{i+j\leq 14}A_j^{-1}\left(\int_{\Sigma_\tau}J_\mu^{Z+N,w^Z}(\partial_{t^*}^i\tilde{\Omega}^j\Phi)n_{\Sigma_\tau}^\mu + C\tau^2\int_{\Sigma_\tau\cap\{r\leq 9\tau/10\}}J_\mu^N(\partial_{t^*}^i\tilde{\Omega}^j\Phi)n_{\Sigma_\tau}^\mu\right)$$
$$\leq (\epsilon/4)\tau^{\eta_{14}}. \qquad (68)$$

Proof. By Proposition 3.5, and noticing that U is supported away from $\{|r-3M|\leq M/8\}$, we have

$$\sum_{i+j\leq 14}A_j^{-1}\left(\int_{\Sigma_\tau}J_\mu^{Z+N,w^Z}(\partial_{t^*}^i\tilde{\Omega}^j\Phi)n_{\Sigma_\tau}^\mu + C\tau^2\int_{\Sigma_\tau\cap\{r\leq 9\tau/10\}}J_\mu^N(\partial_{t^*}^i\tilde{\Omega}^j\Phi)n_{\Sigma_\tau}^\mu\right)$$
$$\leq C\sum_{i+j\leq 14}A_j^{-1}\Bigg(\int_{\Sigma_{\tau_0}}J_\mu^{Z+CN,w^Z}(\partial_{t^*}^i\tilde{\Omega}^j\Phi)n_{\Sigma_{\tau_0}}^\mu$$
$$+\delta'\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq t^*/2\}}(t^*)^2K^{X_0}(\partial_{t^*}^i\tilde{\Omega}^j\Phi)$$
$$+(\delta'+a)\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}}(t^*)^2K^N(\partial_{t^*}^i\tilde{\Omega}^j\Phi)$$
$$+(\delta')^{-1}\Bigg(\iint_{\mathcal{R}(\tau_0,\tau)}t^*r^{-1+\delta}K^{X_1}(\partial_{t^*}^i\tilde{\Omega}^j\Phi)$$
$$+\sum_{m=0}^{1}\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq 9t^*/10\}}(t^*)^2r^{1+\delta}(\partial_{t^*}^mN_{\leq 15})^2$$
$$+\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq 9t^*/10\}}(t^*)^2r^{1+\delta}U_{\leq 15,\leq j}^2+\left(\int_{\tau_0}^{\tau}\left(\int_{\Sigma_{t^*}\cap\{r\geq t^*/2\}}r^2G_{\leq 15,\leq j}^2\right)^{1/2}dt^*\right)^2$$
$$+\sup_{t^*\in[\tau_0,\tau]}\int_{\Sigma_{t^*}\cap\{r_Y^-\leq r\leq 25M/8\}}(t^*)^2N_{\leq 15}^2\Bigg)\Bigg).$$

As before, we estimate each term one by one. First, the term with initial data is clearly bounded by $C\sum_j A_j\epsilon$. Second, the $(t^*)^2K$ terms can be bounded, using (23) and dividing the interval into $\tau_0 < \tau_1 < \cdots < \tau_n = \tau$ as before, by

$$C\frac{A_{X,j}}{A_j}\epsilon(2\delta'+a)\sum_{i=0}^{n-1}\tau_i^{\eta_{14}} \leq C\frac{A_{X,j}}{A_j}\epsilon\eta_{14}^{-1}(2\delta+a)\tau^{\eta_{14}}.$$

This is acceptable since $a, \delta' \ll A_j/A_{X,j}$. Third, the $t^*r^{-1+\delta}K$ term can be bounded, using Bootstrap Assumptions (20) and (22), by

$$\begin{aligned}
&(\delta')^{-1}\sum_{i+j\leq 14}A_j^{-1}\iint_{\mathcal{R}(\tau_0,\tau)}t^*r^{-1+\delta}K^{X_1}(\partial_{t^*}^i\tilde{\Omega}^j\Phi)\\
&\quad\leq(\delta')^{-1}\sum_{i+j\leq 14}A_j^{-1}\Bigg(\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq t^*/2\}}t^*K^{X_1}(\partial_{t^*}^i\tilde{\Omega}^j\Phi)\\
&\qquad\qquad+\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\geq t^*/2\}}t^*r^{-1+\delta}K^{X_1}(\partial_{t^*}^i\tilde{\Omega}^j\Phi)\Bigg)\\
&\quad\leq C(\delta')^{-1}\sum_{i+j\leq 14}A_j^{-1}\sum_{k=0}^{n-1}\Bigg(\tau_k\iint_{\mathcal{R}(\tau_k,\tau_{k+1})\cap\{r\leq t^*/2\}}K^{X_1}(\partial_{t^*}^i\tilde{\Omega}^j\Phi)\\
&\qquad\qquad+\tau_k^\delta\iint_{\mathcal{R}(\tau_k,\tau_{k+1})\cap\{r\geq t^*/2\}}K^{X_1}(\partial_{t^*}^i\tilde{\Omega}^j\Phi)\Bigg)\\
&\quad\leq C(\delta')^{-1}\sum_j\frac{A_{X,j}}{A_j}\eta_{15}^{-1}\tau^{\eta_{15}},
\end{aligned}$$

which is acceptable for τ large since $\eta_{15} \ll \eta_{14}$.

Fourth, the integrals involving $N_{\leq 14}$ can be bounded using Propositions 6.13 and 6.14 by noticing that $|\partial_{t^*}N_{\leq 14}| \leq CN_{\leq 15}$:

$$\begin{aligned}
&C(\delta')^{-1}A_0^{-1}\sum_{m=0}^{1}\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq 9t^*/10\}}(t^*)^2r^{1+\delta}(\partial_{t^*}^mN_{\leq 14})^2\\
&\qquad\leq CA^2A_0^{-1}\epsilon^2(\delta')^{-1}\int_{\tau_0}^{\tau}(t^*)^{-2+\eta_{S,11}+\eta_{15}+2\delta}dt^* \leq CA^2A_0^{-1}\epsilon^2(\delta')^{-1},\\
&C(\delta')^{-1}A_0^{-1}\left(\int_{\tau_0}^{\tau}\left(\int_{\Sigma_{t^*}\cap\{r\geq t^*/2\}}r^2N_{15}^2\right)^{1/2}dt^*\right)^2 \leq CA^2A_0^{-1}\epsilon^2(\delta')^{-1}\eta_{15}^{-1}\tau^{\eta_{15}},\\
&C(\delta')^{-1}A_0^{-1}\sup_{t^*\in[\tau_0,\tau]}\int_{\Sigma_{t^*}\cap\{r_Y^-\leq r\leq 25M/8\}}(t^*)^2N_{15}^2 \leq CA^2A_0^{-1}\epsilon^2(\delta')^{-1},
\end{aligned}$$

which is acceptable since $\epsilon \ll A\delta\eta_{15}^{-1}$. Fifth, for the commutator terms $U^2_{\leq 14,\leq j}$, we estimate by Proposition 6.10:

$$(\delta')^{-1}A_j^{-1}\iint_{\mathcal{R}(\tau_0,\tau)}(t^*)^2r^{1+\delta}(U_{\leq 14,\leq j})^2 \leq C\sum_{i=0}^{n-1}(\delta')^{-1}A_{X,j-1}\tau_i^2\tau_i^{-2+\eta_{14}}$$

$$\leq C(\delta')^{-1}\eta_{14}^{-1}\frac{A_{X,j-1}}{A_j}\tau^{\eta_{14}},$$

$$(\delta')^{-1}A_j^{-1}\left(\int_{\tau_0}^{\tau}\left(\int_{\Sigma_{t^*}\cap\{r\geq t^*/2\}}r^2U_{\leq 14,j}^2\right)^{1/2}dt^*\right)^2 \leq C(\delta')^{-1}\frac{A_{j-1}}{A_j}\eta_{14}^{-1}\tau.$$

Since $A_{j-1}/A_j \ll A_{X,j-1}/A_j \ll \delta'\eta_{14}^{-1}$, all terms are acceptable. □

We now consider terms involving commutation with $\hat{Y}$ and recover Bootstrap Assumptions (9)–(11).

Proposition 6.20.

$$\sum_{i+k=16}\int_{\Sigma_\tau}J_\mu^N(\hat{Y}^k\partial_{t^*}^i\Phi)n^\mu_{\Sigma_\tau} \leq \frac{A_Y}{4}\epsilon\tau^{\eta_{16}},$$

$$\sum_{i+k=15}\tau^2\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}J_\mu^N(\hat{Y}^k\partial_{t^*}^i\Phi)n^\mu_{\Sigma_\tau} \leq \frac{A_Y}{4}\epsilon\tau^{1+\eta_{15}},$$

$$\sum_{i+k\leq 14}\tau^2\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}J_\mu^N(\hat{Y}^k\partial_{t^*}^i\Phi)n^\mu_{\Sigma_\tau} \leq \frac{A_Y}{4}\epsilon\tau^{\eta_{14}}.$$

Proof. The idea is to use Proposition 13 and the fact that it gives control of an integrated-in-time quantity. From this we can extract a good slice to improve the constant. By Proposition 4.5,

$$\sum_{i+k=16}\left(\int_{\Sigma_\tau\cap\{r\leq r_Y^+\}}J_\mu^N(\hat{Y}^k\partial_{t^*}^i\Phi)n^\mu_{\Sigma_\tau}+\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}}J_\mu^N(\hat{Y}^k\partial_{t^*}^i\Phi)n^\mu_{\Sigma_{t^*}}\right)$$

$$\leq C\Bigg(\sum_{j+m\leq 16}\int_{\Sigma_{\tau'}\cap\{r\leq r_Y^+\}}J_\mu^N(\partial_{t^*}^j\hat{Y}^m\Phi)n^\mu_{\Sigma_{\tau'}}+\sum_{j=0}^{16}\int_{\Sigma_\tau\cap\{r\leq r_Y^+\}}J_\mu^N(\partial_{t^*}^j\Phi)n^\mu_{\Sigma_\tau}$$

$$+\sum_{j=0}^{16}\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq 23M/8\}}J_\mu^N(\partial_{t^*}^j\Phi)n^\mu_{\Sigma_{t^*}}+\sum_{i+j\leq 16}\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq 23M/8\}}(D^iG_{\leq j,0})^2\Bigg)$$

$$\leq CA_Y(\tau')^{\eta_{16}}+CA_0\tau^{\eta_{16}}+CA^2\epsilon^2(\tau')^{-1+\eta_{16}},$$

by (9), (17) and Proposition 6.9. Take $\tau'=\tau-A_0$. Then

$$\sum_{i+k=16}\iint_{\mathcal{R}(\tau-A_0,\tau)\cap\{r\leq r_Y^-\}}J_\mu^N(\hat{Y}^k\partial_{t^*}^i\Phi)n^\mu_{\Sigma_{t^*}} \leq CA_Y\tau^{\eta_{16}}+CA_0\tau^{\eta_{16}}+CA^2\epsilon^2\tau^{-1+\eta_{16}}.$$

Hence there is some $\tilde{\tau}\in[\tau-A_0,\tau]$ such that

$$\sum_{i+k=16}\int_{\Sigma_{\tilde{\tau}}\cap\{r\leq r_Y^-\}}J_\mu^N(\hat{Y}^k\partial_{t^*}^i\Phi)n^\mu_{\Sigma_{\tilde{\tau}}} \leq CA_YA_0^{-1}\tau^{\eta_{16}}+C\tau^{\eta_{16}}+CA^2\epsilon^2\tau^{-1+\eta_{16}}.$$

We also have, by (9) and the elliptic estimates in Proposition 4.1,

$$\begin{aligned}
&\sum_{i+k=16}\int_{\Sigma_{\tilde\tau}\cap\{r_Y^-\le r\le r_Y^+\}} J^N_\mu(\hat Y^k\partial_{t^*}^i\Phi)n^\mu_{\Sigma_{\tilde\tau}}\\
&\qquad\le C\sum_{i=0}^{16}\int_{\Sigma_{\tilde\tau}\cap\{r_Y^-\le r\le t^*/2\}} J^N_\mu(\partial_{t^*}^i\Phi)n^\mu_{\Sigma_{\tilde\tau}}+\sum_{i+j\le 15}\int_{\Sigma_{\tilde\tau}}(D^iG_{\le j,0})^2\\
&\qquad\le CA_0\tau^{\eta_{16}}+CA^2\epsilon^2\tau^{-1+\eta_{16}}.
\end{aligned}$$

Now reapplying Proposition 4.5, from $\tilde\tau$ to τ, we get

$$\begin{aligned}
&\sum_{i+k=16}\int_{\Sigma_\tau\cap\{r\le r_Y^+\}} J^N_\mu(\hat Y^k\partial_{t^*}^i\Phi)n^\mu_{\Sigma_\tau}\\
&\qquad\le C\sum_{j+m\le 16}\int_{\Sigma_{\tilde\tau}\cap\{r\le r_Y^+\}} J^N_\mu(\partial_{t^*}^j\hat Y^m\Phi)n^\mu_{\Sigma_{\tilde\tau}}+CA_0\tau^{\eta_{16}}+CA^2\epsilon^2\\
&\qquad\le CA_YA_0^{-1}\tau^{\eta_{16}}+CA_0\tau^{\eta_{16}}+CA^2\epsilon^2\tau^{-1+\eta_{16}}.
\end{aligned}$$

Since $C\ll A_0\ll A_Y$, we get the first statement in the proposition. The derivations for the other bounds are identical, with the constants and exponents replaced appropriately. □

From this we can also derive some integrated estimates for $Y^k\Gamma^j\Phi$. This will be useful in controlling the commutator $[\Box_{g_K}, S]$.

Proposition 6.21.

$$\begin{aligned}
\sum_{i+k=16}\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\le r_Y^-\}} J^N_\mu(\hat Y^k\partial_{t^*}^i\Phi)n^\mu_{\Sigma_\tau}&\le A_Y\epsilon\tau^{\eta_{16}},\\
\sum_{i+k=15}\tau^2\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\le r_Y^-\}} J^N_\mu(\hat Y^k\partial_{t^*}^i\Phi)n^\mu_{\Sigma_\tau}&\le A_Y\epsilon\tau^{1+\eta_{15}},\\
\sum_{i+k\le 14}\tau^2\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\le r_Y^-\}} J^N_\mu(\hat Y^k\partial_{t^*}^i\Phi)n^\mu_{\Sigma_\tau}&\le A_Y\epsilon\tau^{\eta_{14}}.
\end{aligned}$$

Proof. This is a direct consequence of Propositions 4.5, 6.9, 6.20, as well as Bootstrap Assumptions (9)–(11). □

We will finally proceed to the quantities associated to the vector field S. Recall from [20] that for large values of r,

$$\begin{aligned}
&\left|[\Box_{g_K},S]\Phi-\left(2+\frac{r^*\mu}{r}\right)\Box_g\Phi-\frac{2}{r}\left(\frac{r^*}{r}-1-\frac{2r^*\mu}{r}\right)\partial_{r^*}\Phi-2\left(\left(\frac{r^*}{r}-1\right)-\frac{3r^*\mu}{2r}\right)\not\Delta\Phi\right|\\
&\qquad\le Car^{-2}\sum_{k=1}^{2}|D^k\Phi|.
\end{aligned}$$

and that for finite values of r, we have

$$|[\Box_{g_K}, S]\Phi| \leq C \sum_{k=1}^{2} |D^k \Phi|.$$

Moreover, all the coefficients in the commutator term obey the same estimates (with a different constant) upon differentiation. Therefore,

$$\Box_{g_K}(S\Gamma^k\Phi) = V_k + S(U_k) + S(N_k),$$

where

$$(D^\ell V_k)^2 \leq Cr^{-4}(\log r)^2\Big(\sum_{j=1}^{\ell+1}(D^j\Gamma^{k+1}\Phi)^2 + \sum_{j=1}^{\ell+2}(D^j\Gamma^k\Phi)^2\Big).$$

We will now estimate these three terms separately. We first estimate the V_k terms:

Proposition 6.22. *For $\alpha \leq 2$,*

$$\sum_{\ell+k\leq 13}\int_{\Sigma_\tau} r^\alpha (D^\ell V_{\leq k})^2 \leq CA_Y\epsilon\tau^{-2+\eta_{14}+\delta}.$$

For $\alpha \leq 1+\delta$,

$$\sum_{\ell+k=13}\iint_{\mathcal{R}(\tau/1.1,\tau)} r^\alpha (D^\ell V_{\leq k})^2 \leq CA_Y\epsilon\tau^{-1+\eta_{15}+\delta},$$
$$\sum_{\ell+k\leq 12}\iint_{\mathcal{R}(\tau/1.1,\tau)} r^\alpha (D^\ell V_{\leq k})^2 \leq CA_Y\epsilon\tau^{-2+\eta_{14}+\delta}.$$

Proof. By the elliptic estimates in Propositions 4.1 and 4.4, we have, for $\alpha \leq 2$,

$$\begin{aligned}&\sum_{\ell+k\leq 13}\int_{\Sigma_\tau} r^\alpha (D^\ell V_{\leq k})^2\\ &\leq C\sum_{i+j\leq 12}\int_{\Sigma_\tau} r^{\alpha-4+\delta}J^N_\mu(\hat{Y}^i\Gamma^j\Phi)n^\mu_{\Sigma_\tau} + C\sum_{i+j\leq 11}\int_{\Sigma_\tau} r^{\alpha-4+\delta}(D^iG_{\leq j})^2\\ &\leq C(A_Y\epsilon + A^2\epsilon^2)\tau^{-2+\eta_{14}+\delta},\end{aligned}$$

where we have used Propositions 6.9 and 6.10, Bootstrap Assumptions (11) (for $r \leq 9t^*/10$) and (12) (for $r \geq 9t^*/10$).

By the elliptic estimates in Propositions 4.1 and 4.4, we have

$$\begin{aligned}\iint_{\mathcal{R}(\tau/1.1,\tau)} r^\alpha (D^\ell V_{\leq k})^2 &\leq C\sum_{i+j=0}^{\ell+k+1}\iint_{\mathcal{R}(\tau/1.1,\tau)} r^{\alpha-4+\delta}J^N_\mu(\hat{Y}^i\Gamma^j\Phi)n^\mu_{\Sigma_\tau}\\ &\quad + C\sum_{i+j=0}^{\ell+k}\iint_{\mathcal{R}(\tau/1.1,\tau)} r^{\alpha-4+\delta}(D^iG_{\leq j})^2.\end{aligned}$$

We first consider the case $\ell + k = 13$. For the first term, we divide into $r \leq t^*/2$ (which we estimate by (22) and Proposition 6.21) and $r \geq t^*/2$ (which we estimate using the extra decay in r by (19)). The second term contains the U_k and the N_k part. The U_k

part can be estimated by Proposition 6.10. The N_k part can be estimated by Proposition 6.9. The $\ell + k \leq 12$ case is completely analogous, replacing Bootstrap Assumption (22) by (24). □

We then proceed to the estimates for $S(U_k)$. Notice that when we prove the estimates for the derivatives for $S(U_k)$, the derivatives for $S(N_k)$ will be involved. As in the proof of the estimates for U_k, we will first prove estimates for the derivatives of $S(U_k)$ depending on $S(N_k)$, and close the estimates after we control $S(N_k)$.

Proposition 6.23. *The following estimates for $S(U_k)$ on a fixed t^* slice hold for $\alpha \leq 2$:*

$$\sum_{k+\ell=13} \int_{\Sigma_\tau} r^\alpha \big(D^\ell(S(U_{k,j}))\big)^2 \leq C A_{S,j-1} \epsilon \tau^{\eta_{S,13}} + \sum_{m=1}^{13-j} \int_{\Sigma_\tau} (D^m S(N_{\leq j-1}))^2,$$

$$\sum_{k+\ell=12} \int_{\Sigma_\tau} r^\alpha \big(D^\ell(S(U_{k,j}))\big)^2 \leq C A_{S,j-1} \epsilon \tau^{-1+\eta_{S,12}} + \sum_{m=1}^{12-j} \int_{\Sigma_\tau} (D^m S(N_{\leq j-1}))^2,$$

$$\sum_{k+\ell\leq 11} \int_{\Sigma_\tau} r^\alpha \big(D^\ell(S(U_{k,j}))\big)^2 \leq C A_{S,j-1} \epsilon \tau^{-2+\eta_{S,11}} + \sum_{m=1}^{11-j} \int_{\Sigma_\tau} (D^m S(N_{\leq j-1}))^2.$$

The following estimates for $S(U_k)$ integrated on $[\tau/1.1, \tau]$ also hold for $\alpha \leq 1+\delta$:

$$\sum_{k+\ell=13} \iint_{\mathcal{R}(\tau/1.1,\tau)} r^\alpha \big(D^\ell(S(U_{k,j}))\big)^2 \leq C A_{S,X,j-1} \epsilon \tau^{\eta_{S,13}} + \sum_{m=1}^{13-j} \iint_{\mathcal{R}(\tau/1.1,\tau)} r^{-2} (D^m S(N_{\leq j-1}))^2,$$

$$\sum_{k+\ell=12} \iint_{\mathcal{R}(\tau/1.1,\tau)} r^\alpha \big(D^\ell(S(U_{k,j}))\big)^2 \leq C A_{S,X,j-1} \epsilon \tau^{-1+\eta_{S,12}} + \sum_{m=1}^{12-j} \iint_{\mathcal{R}(\tau/1.1,\tau)} r^{-2} (D^m S(N_{\leq j-1}))^2,$$

$$\sum_{k+\ell\leq 11} \iint_{\mathcal{R}(\tau/1.1,\tau)} r^\alpha \big(D^\ell(S(U_{k,j}))\big)^2 \leq C A_{S,X,j-1} \epsilon \tau^{-2+\eta_{S,11}} + \sum_{m=1}^{11-j} \iint_{\mathcal{R}(\tau/1.1,\tau)} r^{-2} (D^m S(N_{\leq j-1}))^2.$$

Proof. Notice that $D^\ell(S(U_{k,j}))$ is supported in $\{r \geq R_\Omega\}$ and satisfies

$$\begin{aligned}|D^\ell(S(U_{k,j}))| &\leq C \sum_{m=1}^{\ell+2} \sum_{i=0}^{j-1} r^{-2} \big(|D^m S \partial_{t^*}^{k-j} \tilde{\Omega}^i \Phi| + |D^m \partial_{t^*}^{k-j} \tilde{\Omega}^i \Phi|\big) \\ &\leq C \sum_{m=1}^{\ell+k-j+2} \sum_{i=0}^{j-1} r^{-2} \big(|D^m S \tilde{\Omega}^i \Phi| + |D^m \tilde{\Omega}^i \Phi|\big).\end{aligned}$$

We can ignore the last term because it appears already in $D^\ell U_{k,j}$ and can be estimated by Proposition 6.10. We have

$$\begin{aligned}
\int_{\Sigma_\tau} r^\alpha (D^\ell(S(U_{k,j})))^2 &\leq C \sum_{m=1}^{\ell+k-j+2} \sum_{i=0}^{j-1} \int_{\Sigma_\tau\cap\{r\geq R_\Omega\}} r^{\alpha-4} (D^m S\tilde{\Omega}^i \Phi)^2 \\
&\leq C \sum_{m=1}^{\ell+k-j+1} \sum_{i=0}^{j-1} \int_{\Sigma_\tau\cap\{r\geq R_\Omega-1\}} r^{\alpha-4} J^N_\mu(\partial_{t^*}^m S\tilde{\Omega}^i\Phi) n^\mu_{\Sigma_\tau} \\
&\quad + C \sum_{m=1}^{\ell+k-j} \sum_{i=0}^{j-1} \int_{\Sigma_\tau} r^{-2} (D^m \Box_{g_K}(S\tilde{\Omega}^i\Phi))^2 \\
&\leq C \sum_{m=1}^{\ell+k-j+1} \sum_{i=0}^{j-1} \int_{\Sigma_\tau\cap\{r\geq R_\Omega-1\}} r^{\alpha-4} J^N_\mu(\partial_{t^*}^m S\tilde{\Omega}^i\Phi) n^\mu_{\Sigma_\tau} \\
&\quad + C \sum_{m=1}^{\ell+k-j} \int_{\Sigma_\tau} r^{-2} \big((D^m S(U_{\leq j-1,\leq j-1}))^2 + (D^m S(N_{\leq j-1}))^2 + (D^m V_{\leq j-1})^2\big).
\end{aligned}$$

We now apply the bootstrap assumptions. By Bootstrap Assumption (13) and Proposition 6.22,

$$\begin{aligned}
&\sum_{k+\ell=13} \int_{\Sigma_\tau} r^\alpha \big(D^\ell(S(U_{k,j}))\big)^2 \\
&\quad \leq C A_{S,j-1}\epsilon\tau^{\eta_{S,13}} + \sum_{m=1}^{13-j} \int_{\Sigma_\tau} \big((D^m S(U_{\leq j-1,\leq j-1}))^2 + (D^m S(N_{\leq j-1}))^2\big).
\end{aligned}$$

By Bootstrap Assumption (12) (for $r \geq t^*/2$), (14) (for $r \leq t^*/2$) and Proposition 6.22,

$$\begin{aligned}
&\sum_{k+\ell=12} \int_{\Sigma_\tau} r^\alpha \big(D^\ell(S(U_{k,j}))\big)^2 \\
&\quad \leq C A_{S,j-1}\epsilon\tau^{-1+\eta_{S,12}} + \sum_{m=1}^{12-j} \int_{\Sigma_\tau} \big((D^m S(U_{\leq j-1,\leq j-1}))^2 + (D^m S(N_{\leq j-1}))^2\big).
\end{aligned}$$

By Bootstrap Assumption (12) (for $r \geq t^*/2$), (15) (for $r \leq t^*/2$) and Proposition 6.22,

$$\begin{aligned}
&\sum_{k+\ell\leq 11} \int_{\Sigma_\tau} r^\alpha \big(D^\ell(S(U_{k,j}))\big)^2 \\
&\quad \leq C A_{S,j-1}\epsilon\tau^{-2+\eta_{S,11}} + \sum_{m=1}^{11-j} \int_{\Sigma_\tau} \big((D^m S(U_{\leq j-1,\leq j-1}))^2 + (D^m S(N_{\leq j-1}))^2\big).
\end{aligned}$$

Noticing that $U_{k,0} = 0$, we can deduce the first three statements in the proposition using induction on j (see proof of Proposition 6.8). For the integrated-in-time estimate, we note that $r^{-1-\delta} J^N_\mu(\Gamma^i \Phi) \le C K^{X_0}(\Gamma^i \Phi)$ and use Bootstrap Assumptions (25)–(27) and (29) (see proof of Propositions 6.8 and 6.10). ◻

We then move on to the $S(N_k)$ terms; first we will prove an estimate for the derivatives of $S(N_k)$. The decay rate here is not optimal, but would be sufficient to close the bootstrap argument. Our approach here is to prove the decay rate that is driven only by the pointwise decay of $D^\ell \Phi$ but not by that of $D^\ell S\Phi$. The latter can, in principle, be done by similar methods, but we will skip it since it will not be necessary. In subsequent propositions, we will then prove refined decay rate for $S(N_k)$ (without derivatives) as well as for $D^\ell S(N_k)$ restricted to the region $r \le t^*/4$.

Proposition 6.24. *$S(N_k)$ satisfies the following estimates for any fixed $t^* = \tau$:*

$$\sum_{k+\ell=13} \int_{\Sigma_\tau} (D^\ell S(N_k))^2 \le C B_S A^2 \epsilon^2 \tau^{-2+\eta_{S,11}},$$
$$\sum_{k+\ell\le 12} \int_{\Sigma_\tau} (D^\ell S(N_k))^2 \le C A^2 \epsilon^2 \tau^{-2+\eta_{14}+\delta}.$$

$S(N_k)$ also satisfies the following integrated estimates over $t^* \in [\tau/1.1, \tau]$:

$$\sum_{k+\ell=12} \iint_{\mathcal{R}(\tau/1.1,\tau)} r^{-1-\delta} (D^\ell S(N_k))^2 \le C B_S A^2 \epsilon^2 \tau^{-2+\eta_{S,11}},$$
$$\sum_{k+\ell\le 11} \iint_{\mathcal{R}(\tau/1.1,\tau)} r^{-1-\delta} (D^\ell S(N_k))^2 \le C A^2 \epsilon^2 \tau^{-2+\eta_{14}+\delta}.$$

Proof. We would like to do a reduction similar to how we estimated N_k. Clearly, only the quadratic and cubic terms matter and we only need to consider terms that contain S, for the other terms are already controlled by the estimates of N_k. We will call terms that are already in N_k "good". The only cubic terms that are relevant are those which contain $S\Gamma^i\Phi$ since in the terms with $D^{j+1}S\Gamma^i\Phi$, we can put all but one other factor in L^∞ using Bootstrap Assumptions (31), (33), (39) and (40). Notice also that the conditions for $D_\Phi \Lambda_0$, $D_\Phi \Lambda_1$ and $D_\Phi \mathcal{C}$ in the definition of the null condition guarantee that the bounds do not deteriorate if S acts on the coefficients. The relevant terms are

$$(D^{j_1} S\Gamma^{i_1}\Phi)(D^{j_2}\Gamma^{i_2}\Phi) \quad \text{and} \quad (D^{j_1}\Gamma^{i_1}\Phi)(D^{j_2}\Gamma^{i_2}\Phi)(S\Gamma^{i_3}\Phi).$$

We first treat the case $k+\ell \le 12$. In this case we will always put factors without S in L^∞. We have

$$\begin{aligned}
&\int_{\Sigma_\tau}(D^\ell S(N_k))^2\\
&\leq C\sum_{i_1+i_2+j_1+j_2\leq k+\ell+2,\, j_1,j_2\geq 1}\int_{\Sigma_\tau}(D^{j_1}S\Gamma^{i_1}\Phi)^2(D^{j_2}\Gamma^{i_2}\Phi)^2\\
&\quad+C\sum_{i_1+i_2+i_3+j_1+j_2\leq 14,\, j_1,j_2\geq 1}\int_{\Sigma_\tau}(D^{j_1}\Gamma^{i_1}\Phi)^2(D^{j_2}\Gamma^{i_2}\Phi)^2(S\Gamma^{i_3}\Phi)^2+\text{good terms}\\
&\leq C\Big(\sum_{i+j\leq k+\ell+1,\, j\geq 1}\sup(D^j\Gamma^i\Phi)^2\Big)\sum_{i+j\leq k+\ell+1,\, j\geq 1}\int_{\Sigma_\tau}(D^jS\Gamma^i\Phi)^2\\
&\quad+C\Big(\sum_{i+j\leq k+\ell+1,\, j\geq 1}\sup(D^j\Gamma^i\Phi)^2\Big)\Big(\sum_{i+j\leq k+\ell+1,\, j\geq 1}\sup r^2(D^j\Gamma^i\Phi)^2\Big)\\
&\quad\times\sum_{i\leq k+\ell}\int_{\Sigma_\tau}r^{-2}(S\Gamma^i\Phi)^2+\text{good terms}\\
&\leq CA\epsilon\tau^{-2+\eta_{14}}\sum_{i+j\leq k+\ell+1,\, j\geq 1}\int_{\Sigma_\tau}(D^jS\Gamma^i\Phi)^2\\
&\quad+CA^2\epsilon^2\tau^{-2+\eta_{14}}\sum_{i\leq k+\ell}\int_{\Sigma_\tau}J^N_\mu(S\Gamma^i\Phi)n^\mu_{\Sigma_\tau}+\text{good terms}\\
&\quad\text{using Bootstrap Assumptions (32), (33), (37), (41) and (44) and Proposition 4.2}\\
&\leq CA\epsilon\tau^{-2+\eta_{14}}\sum_{i\leq k+\ell}\int_{\Sigma_\tau}\big(J^N_\mu(S\Gamma^i\Phi)n^\mu_{\Sigma_\tau}+J^N_\mu(\Gamma^i\Phi)n^\mu_{\Sigma_\tau}\big)\\
&\quad+CA\epsilon\tau^{-2+\eta_{14}}\sum_{i+j\leq k+\ell-1}\int_{\Sigma_\tau}\big((D^iU_{\leq j})^2+(D^iS(U_{\leq j}))^2+(D^iN_{\leq j})^2\\
&\qquad\qquad+(D^iS(N_{\leq j}))^2+(D^iV_{\leq j})^2\big).
\end{aligned}$$

We now apply the estimates for the inhomogeneous terms, i.e., Propositions 6.9, 6.10, 6.22, 6.23. Since $k+\ell\leq 12$, we have

$$\begin{aligned}
&\int_{\Sigma_\tau}(D^\ell S(N_k))^2\\
&\leq CA\epsilon\tau^{-2+\eta_{14}}\sum_{i\leq 12}\int_{\Sigma_\tau}\big(J^N_\mu(S\Gamma^i\Phi)n^\mu_{\Sigma_\tau}+J^N_\mu(\Gamma^i\Phi)n^\mu_{\Sigma_\tau}\big)+CA^2\epsilon^2\tau^{-2+\eta_{14}+\delta}\\
&\quad+CA\epsilon\tau^{-2+\eta_{14}}\sum_{i+j\leq k+\ell-1}\int_{\Sigma_\tau}(D^iS(N_{\leq j}))^2.
\end{aligned}$$

The desired estimates then follow from an induction, together with Bootstrap Assumptions (12) and (16), since according to this notation $\sum_{i+j=0}^{-1}=0$.

We then treat the case $k+\ell=13$. In this case it is possible to have 14 derivatives falling on the factor with Φ, which hence cannot be controlled in L^∞. However, in this scenario, we must have

$$\sum_{i+j=14}(DS\Phi)(D^j\Gamma^i\Phi)$$

and therefore $DS\Phi$ can be controlled in L^∞ by Bootstrap Assumptions (37) and (44). In short, we have

$$\begin{aligned}
&\sum_{k+\ell=13}\int_{\Sigma_\tau}(D^\ell S(N_k))^2\\
&\quad\le C\sum_{i_1+i_2+j_1+j_2\le 15,\ j_1,j_2\ge 1}\int_{\Sigma_\tau}(D^{j_1}S\Gamma^{i_1}\Phi)^2(D^{j_2}\Gamma^{i_2}\Phi)^2\\
&\qquad+C\sum_{i_1+i_2+i_3+j_1+j_2\le 15,\ j_1,j_2\ge 1}\int_{\Sigma_\tau}(D^{j_1}\Gamma^{i_1}\Phi)^2(D^{j_2}\Gamma^{i_2}\Phi)^2(S\Gamma^{i_3}\Phi)^2+\text{good terms}\\
&\quad\le C\Big(\sum_{i+j\le 14,\ j\ge 1}\sup(D^j\Gamma^i\Phi)^2\Big)\sum_{i+j\le 14,\ j\ge 1}\int_{\Sigma_\tau}(D^jS\Gamma^i\Phi)^2\\
&\qquad+(\sup(DS\Phi)^2)\sum_{i+j=14,\ j\ge 1}\int_{\Sigma_\tau}(D^j\Gamma^i\Phi)^2\\
&\qquad+C\Big(\sum_{i+j\le 14,\ j\ge 1}\sup(D^j\Gamma^i\Phi)^2\Big)\Big(\sum_{i+j\le 14,\ j\ge 1}\sup r^2(D^j\Gamma^i\Phi)^2\Big)\\
&\qquad\times\sum_{i\le 13}\int_{\Sigma_\tau}r^{-2}(S\Gamma^i\Phi)^2\\
&\qquad+\text{good terms}\\
&\quad\le CA\epsilon\tau^{-2+\eta_{14}}\sum_{i+j\le 14,\ j\ge 1}\int_{\Sigma_\tau}(D^jS\Gamma^i\Phi)^2\\
&\qquad+CB_SA\epsilon\tau^{-2+\eta_{S,11}}\sum_{i+j=14,\ j\ge 1}\int_{\Sigma_\tau}(D^j\Gamma^i\Phi)^2\\
&\qquad+CA^2\epsilon^2\tau^{-2+\eta_{14}}\sum_{i\le 13}\int_{\Sigma_\tau}J^N_\mu(S\Gamma^i\Phi)n^\mu_{\Sigma_\tau}+\text{good terms}
\end{aligned}$$

using Bootstrap Assumption (32), (33), (37), (41) and (44) and Proposition 4.2

$$\begin{aligned}
&\quad\le CA\epsilon\tau^{-2+\eta_{14}}\sum_{i\le 13}\int_{\Sigma_\tau}J^N_\mu(S\Gamma^i\Phi)n^\mu_{\Sigma_\tau}+CB_SA\epsilon\tau^{-2+\eta_{S,11}}\sum_{i\le 13}\int_{\Sigma_\tau}J^N_\mu(\Gamma^i\Phi)n^\mu_{\Sigma_\tau}\\
&\qquad+CA\epsilon\tau^{-2+\eta_{14}}\sum_{\ell+k\le 12}\int_{\Sigma_\tau}\big((D^\ell U_{\le k})^2+(D^iS(U_{\le k}))^2+(D^\ell N_{\le k})^2\\
&\qquad\qquad+(D^\ell S(N_{\le k}))^2+(D^\ell V_{\le k})^2\big).
\end{aligned}$$

We now apply the estimates for the inhomogeneous terms, i.e., Propositions 6.9, 6.10, 6.22, 6.23:

$$\begin{aligned}
&\sum_{k+\ell=13}\int_{\Sigma_\tau}(D^\ell S(N_k))^2\\
&\quad\leq CA\epsilon\tau^{-2+\eta_{14}}\sum_{i\leq 13}\int_{\Sigma_\tau}J^N_\mu(S\Gamma^i\Phi)n^\mu_{\Sigma_\tau}+CB_SA\epsilon\tau^{-2+\eta_{S,11}}\sum_{i\leq 13}\int_{\Sigma_\tau}J^N_\mu(\Gamma^i\Phi)n^\mu_{\Sigma_\tau}\\
&\qquad+CA\epsilon\tau^{-2+\eta_{14}}\sum_{\ell+k\leq 12}\int_{\Sigma_\tau}(D^\ell S(N_{\leq k}))^2,
\end{aligned}$$

which is acceptable. The estimates for the terms integrated in t^* are proved analogously, noting that the elliptic estimate in Proposition 4.1 would allow for weight in r, and using the second parts of Propositions 6.9, 6.10, 6.22 and 6.23. □

This would allow us to close the estimates for $S(U_k)$ from Proposition 6.23.

Proposition 6.25. *The following estimates for $S(U_k)$ on a fixed t^* slice hold for $\alpha\leq 2$:*

$$\sum_{k+\ell=13}\int_{\Sigma_\tau}r^\alpha\big(D^\ell(S(U_{k,j}))\big)^2\leq CA_{S,j-1}\epsilon\tau^{\eta_{S,13}},$$

$$\sum_{k+\ell=12}\int_{\Sigma_\tau}r^\alpha\big(D^\ell(S(U_{k,j}))\big)^2\leq CA_{S,j-1}\epsilon\tau^{-1+\eta_{S,12}},$$

$$\sum_{k+\ell\leq 11}\int_{\Sigma_\tau}r^\alpha\big(D^\ell(S(U_{k,j}))\big)^2\leq CA_{S,j-1}\epsilon\tau^{-2+\eta_{S,11}}.$$

The following estimates for $S(U_k)$ integrated on $[\tau/1.1,\tau]$ also hold for $\alpha\leq 1+\delta$:

$$\sum_{k+\ell=13}\iint_{\mathcal{R}(\tau/1.1,\tau)}r^\alpha\big(D^\ell(S(U_{k,j}))\big)^2\leq CA_{S,X,j-1}\epsilon\tau^{\eta_{S,13}},$$

$$\sum_{k+\ell=12}\iint_{\mathcal{R}(\tau/1.1,\tau)}r^\alpha\big(D^\ell(S(U_{k,j}))\big)^2\leq CA_{S,X,j-1}\epsilon\tau^{-1+\eta_{S,12}},$$

$$\sum_{k+\ell\leq 11}\iint_{\mathcal{R}(\tau/1.1,\tau)}r^\alpha\big(D^\ell(S(U_{k,j}))\big)^2\leq CA_{S,X,j-1}\epsilon\tau^{-2+\eta_{S,11}}.$$

Proof. This follows directly from Proposition 6.23 and 6.24. □

In the region $\{r\leq t^*/4\}$, we have refined decay rates for $D^\ell S(N_k)$:

Proposition 6.26.

$$\sum_{k+\ell=13}\int_{\Sigma_\tau\cap\{r\le t^*/4\}} r^{1-\delta}(D^\ell S(N_k))^2 \le CA^2\epsilon^2\tau^{-3+\eta_{S,11}},$$
$$\sum_{k+\ell\le 12}\int_{\Sigma_\tau\cap\{r\le t^*/4\}} r^{1-\delta}(D^\ell S(N_k))^2 \le CA^2\epsilon^2\tau^{-4+\eta_{S,12}+\eta_{S,11}}.$$

Proof. Take $k+\ell\le 13$. Notice that $|[D,S]\Phi|\le C|D\Phi|$.

We would like to do a reduction similar to how we estimated N_k. Clearly, only the quadratic and cubic terms matter and we only need to consider terms that contain S, for the other terms are already controlled by the estimates of N_k. Notice also as before that the conditions in the null condition guarantee that the bounds do not deteriorate if S acts on the coefficients. The relevant terms are

$$\begin{aligned}
&(D^{j_1}S\Gamma^{i_1}\Phi)(D^{j_2}\Gamma^{i_2}\Phi), && j_1, j_2\ge 1,\\
&(D^{j_1}S\Gamma^{i_1}\Phi)(D^{j_2}\Gamma^{i_2}\Phi)(\Gamma^{i_3}\Phi), && j_1, j_2\ge 1,\ i_3>8,\\
&(D^{j_1}\Gamma^{i_1}\Phi)(D^{j_2}\Gamma^{i_2}\Phi)(S\Gamma^{i_3}\Phi), && j_1, j_2\ge 1,\ i_3>8.
\end{aligned}$$

We first tackle the quadratic terms:

$$\begin{aligned}
&\sum_{i_1+\ell_1\le 7,\ \ell_1\ge 1}\ \sum_{i_2+j_2\le k+\ell+1,\ j_2\ge 1}\int_{\Sigma_\tau\cap\{r\le\tau/4\}} r^{1-\delta}\big(|D^{j_1}S\Gamma^{i_1}\Phi D^{j_2}\Gamma^{i_2}\Phi|^2\\
&\qquad\qquad\qquad + |D^{j_1}\Gamma^{i_1}\Phi D^{j_2}S\Gamma^{i_2}\Phi|^2\big)\\
&\le C\Big(\sup_{r\le\tau/4}\sum_{i+j\le 7,\ j\ge 1} r^{1-\delta}|D^jS\Gamma^i\Phi|^2\Big)\sum_{i+j\le k+\ell+1,\ j\ge 1}\int_{\Sigma_\tau\cap\{r\le\tau/4\}}|D^j\Gamma^i\Phi|^2\\
&\quad + C\Big(\sup_{r\le\tau/4}\sum_{i+j\le 7,\ j\ge 1} r^{1-\delta}|D^j\Gamma^i\Phi|^2\Big)\sum_{i+j\le k+\ell+1,\ j\ge 1}\int_{\Sigma_\tau\cap\{r\le\tau/4\}}|D^jS\Gamma^i\Phi|^2\\
&\le CA\epsilon\tau^{-2+\eta_{S,11}}\sum_{i+j\le k+\ell}\int_{\Sigma_\tau\cap\{r\le 9\tau/10\}} J^N_\mu(\hat{Y}^j\Gamma^i\Phi)\\
&\quad + CA\epsilon\tau^{-3+\eta_{S,11}}\sum_{i+j\le k+\ell}\int_{\Sigma_\tau\cap\{r\le 9\tau/10\}} J^N_\mu(\hat{Y}^jS\Gamma^i\Phi)\\
&\quad + CA\epsilon\tau^{-2+\eta_{S,11}}\sum_{i+j\le k+\ell-1}\int_{\Sigma_\tau\cap\{r\le 9\tau/10\}}\big((D^iU_j)^2+(D^iN_j)^2\big)\\
&\quad + CA\epsilon\tau^{-3+\eta_{S,11}}\sum_{i+j\le k+\ell-1}\int_{\Sigma_\tau\cap\{r\le 9\tau/10\}}\big((D^iS(U_j))^2+(D^iS(N_j))^2+(D^iV_j)^2\big)
\end{aligned}$$

by Bootstrap Assumptions (33) and (35) and the elliptic estimates of Propositions 4.1 and 4.4. Since $k+\ell-1\le 12$, the inhomogeneous terms can be bounded using Propositions

6.9, 6.10, 6.22, 6.24 and 6.25 to be

$$\leq CA^2\epsilon^2\tau^{-4+\eta_{S,12}+\eta_{S,11}}.$$

We then move on to the cubic terms:

$$\begin{aligned}
&\sum_{i_1+j_1\leq 7,\, j_1\geq 1}\ \sum_{i_2+j_2\leq 7,\, j_2\geq 1}\ \sum_{i_3=0}^{k}\int_{\Sigma_\tau\cap\{r\leq 9\tau/10\}} r^{1-\delta}\big((D^{j_1}S\Gamma^{i_1}\Phi D^{j_2}\Gamma^{i_2}\Phi\Gamma^{i_3}\Phi)^2 \\
&\qquad\qquad + (D^{j_1}\Gamma^{i_1}\Phi D^{j_2}\Gamma^{i_2}\Phi S\Gamma^{i_3}\Phi)^2\big) \\
&\quad\leq C\Big(\sup_{r\leq\tau/4}\sum_{i+j\leq 7,\, j\geq 1} r^2(D^jS\Gamma^i\Phi)^2\Big)\Big(\sup_{r\leq\tau/4}\sum_{i+j\leq 7,\, j\geq 1} r^{1-\delta}(D^j\Gamma^i\Phi)^2\Big) \\
&\qquad\times\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\leq\tau/4\}} r^{-2}(\Gamma^{i_3}\Phi)^2 \\
&\qquad + C\Big(\sup_{r\leq\tau/4}\sum_{i+j\leq 7,\, j\geq 1} r^{1-\delta}(D^j\Gamma^i\Phi)^2\Big)^2\tau^{1+\delta}\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\leq\tau/4\}} r^{-2}(S\Gamma^i\Phi)^2 \\
&\quad\leq CA^2\epsilon^2\tau^{-5+2\eta_{S,11}}\sum_{i=0}^{k}\int_{\Sigma_\tau}(D\Gamma^i\Phi)^2 + CA^2\epsilon^2\tau^{-5+2\eta_{S,11}+\delta}\sum_{i=0}^{k}\int_{\Sigma_\tau}(DS\Gamma^i\Phi)^2,
\end{aligned}$$

by Bootstrap Assumptions (33) and (35), which now clearly decays better than we need by using Bootstrap Assumptions (12), (13) and (16). Therefore,

$$\begin{aligned}
\int_{\Sigma_\tau\cap\{r\leq\tau/4\}} r^{1-\delta}(D^\ell S(N_k))^2 &\leq CA\epsilon\tau^{-2+\eta_{S,11}}\sum_{i+j\leq k+\ell}\int_{\Sigma_\tau\cap\{r\leq 9\tau/10\}} J^N_\mu(\hat{Y}^j\Gamma^i\Phi) \\
&\quad + CA\epsilon\tau^{-3+\eta_{S,11}}\sum_{i+j\leq k+\ell}\int_{\Sigma_\tau\cap\{r\leq 9\tau/10\}} J^N_\mu(\hat{Y}^jS\Gamma^i\Phi) \\
&\quad + CA^2\epsilon^2\tau^{-4+\eta_{S,12}+\eta_{S,11}}.
\end{aligned}$$

The proposition follows from Bootstrap Assumptions (11) and (13)–(15). □

A similar decay rate can be proved in the region $\{r\leq 9t^*/10\}$, if we do not require the estimate for the derivatives:

Proposition 6.27.

$$\begin{aligned}
\int_{\Sigma_\tau\cap\{r\leq 9t^*/10\}} r^{1-\delta}(S(N_{13}))^2 &\leq CA^2\epsilon^2\tau^{-3+\eta_{S,11}}, \\
\sum_{k=0}^{12}\int_{\Sigma_\tau\cap\{r\leq 9t^*/10\}} r^{1-\delta}(S(N_k))^2 &\leq CA^2\epsilon^2\tau^{-4+\eta_{S,12}+\eta_{S,11}}.
\end{aligned}$$

Proof. Take $k \leq 13$. The proof follows very closely that of the previous proposition, by noting that we have similar pointwise decay estimates in the region (without higher derivatives) by Bootstrap Assumptions (34) and (38). As in the previous proposition, the relevant terms are

$$(DS\Gamma^{i_1}\Phi)(D\Gamma^{i_2}\Phi),$$
$$(DS\Gamma^{i_1}\Phi)(D\Gamma^{i_2}\Phi)(\Gamma^{i_3}\Phi), \quad i_3 > 8,$$
$$(D\Gamma^{i_1}\Phi)(D\Gamma^{i_2}\Phi)(S\Gamma^{i_3}\Phi), \quad i_3 > 8.$$

We first tackle the quadratic terms:

$$\begin{aligned}
&\sum_{i_1=0}^{6}\sum_{i_2=0}^{k}\int_{\Sigma_\tau\cap\{r\leq 9\tau/10\}} r^{1-\delta}\big(|DS\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi|^2+|D\Gamma^{i_1}\Phi DS\Gamma^{i_2}\Phi|^2\big)\\
&\quad\leq C\Big(\sup_{r\leq 9\tau/10}\sum_{i=0}^{6} r^{1-\delta}|DS\Gamma^i\Phi|^2\Big)\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\leq 9\tau/10\}}|D\Gamma^i\Phi|^2\\
&\qquad+C\Big(\sup_{r\leq 9\tau/10}\sum_{i=0}^{6} r^{1-\delta}|D\Gamma^i\Phi|^2\Big)\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\leq 9\tau/10\}}|DS\Gamma^i\Phi|^2\\
&\quad\leq CA^2\epsilon^2\tau^{-4+\eta_{14}+\eta_{S,11}}+CA\epsilon\tau^{-3+\eta_{S,11}}\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\leq 9\tau/10\}}|DS\Gamma^i\Phi|^2.
\end{aligned}$$

We then move on to the cubic terms:

$$\begin{aligned}
&\sum_{i_1,i_2=0}^{6}\sum_{i_3=0}^{k}\int_{\Sigma_\tau\cap\{r\leq 9\tau/10\}} r^{1-\delta}\big((DS\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi\Gamma^{i_3}\Phi)^2+(D\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi S\Gamma^{i_3}\Phi)^2\big)\\
&\quad\leq C\Big(\sup_{r\leq 9\tau/10}\sum_{i=0}^{6} r^{2}(DS\Gamma^i\Phi)^2\Big)\Big(\sup_{r\leq 9\tau/10}\sum_{i=0}^{6} r^{1-\delta}(D\Gamma^i\Phi)^2\Big)\\
&\qquad\times\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\leq 9\tau/10\}} r^{-2}(\Gamma^i\Phi)^2\\
&\qquad+C\Big(\sup_{r\leq 9\tau/10}\sum_{i=0}^{6} r^{1-\delta}(D\Gamma^i\Phi)^2\Big)^2\tau^{1+\delta}\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\leq 9\tau/10\}} r^{-2}(S\Gamma^i\Phi)^2\\
&\quad\leq CA^2\epsilon^2\tau^{-5+2\eta_{S,11}}\sum_{i=0}^{k}\int_{\Sigma_\tau}(D\Gamma^i\Phi)^2+CA^2\epsilon^2\tau^{-5+2\eta_{S,11}+\delta}\sum_{i=0}^{k}\int_{\Sigma_\tau}(DS\Gamma^i\Phi)^2\\
&\quad\leq CA^3\epsilon^3\tau^{-5+2\eta_{S,11}}+CA^2\epsilon^2\tau^{-5+2\eta_{S,11}+\delta}\sum_{i=0}^{k}\int_{\Sigma_\tau}(DS\Gamma^i\Phi)^2.
\end{aligned}$$

Therefore,

$$\int_{\Sigma_\tau\cap\{r\leq\tau/4\}} r^{1-\delta}(S(N_k))^2 \leq CA^2\epsilon^2\tau^{-4+\eta_{14}+\eta_{S,11}}$$
$$+CA\epsilon\tau^{-3+\eta_{S,11}}\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\leq 9\tau/10\}}(DS\Gamma^i\Phi)^2+CA^2\epsilon^2\tau^{-5+2\eta_{S,11}+\delta}\sum_{i=0}^{k}\int_{\Sigma_\tau}(DS\Gamma^i\Phi)^2.$$

The proposition follows from Bootstrap Assumptions (13)–(15). □

We then move on to the region $\{r \geq 9t^*/10\}$.

Proposition 6.28. *For* $\alpha = 0$ *or* 2*,*

$$\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}}(S(N_{13}))^2 \leq CA^2\epsilon^2\tau^{-2+\eta_{S,13}},$$
$$\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} r^\alpha(S(N_{12}))^2 \leq CA^2\epsilon^2\tau^{-3+\alpha+\eta_{S,12}},$$
$$\sum_{k=0}^{11}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} r^{1-\delta}(S(N_k))^2 \leq CA^2\epsilon^2\tau^{-4+\alpha+\eta_{S,11}}.$$

Proof. Take $k \leq 13$. Following the reduction before and noticing that $[D, S] \sim D$ and $[\bar{D}, S] \sim \bar{D}$, we have to consider the quadratic terms

$$\bar{D}S\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi,\quad \bar{D}\Gamma^{i_1}\Phi DS\Gamma^{i_2}\Phi,\quad DS\Gamma^{i_1}\Phi\bar{D}\Gamma^{i_2}\Phi,$$
$$D\Gamma^{i_1}\Phi\bar{D}S\Gamma^{i_2}\Phi,\quad r^{-1}(DS\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi),\quad r^{-1}(D\Gamma^{i_1}\Phi DS\Gamma^{i_2}\Phi),$$

for $i_1 \geq i_2$ and the cubic terms

$$\bar{D}\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi S\Gamma^{i_3}\Phi,\quad D\Gamma^{i_1}\Phi\bar{D}\Gamma^{i_2}\Phi S\Gamma^{i_3}\Phi,\quad r^{-1}(D\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi S\Gamma^{i_3}\Phi).$$

For these cubic terms, we can assume $i_1, i_2 \leq 6$, for otherwise $i_3 \leq 6$ and we can control the last factor in the sup norm and reduce to the quadratic terms above. The cubic terms

$$\bar{D}S\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi\Gamma^{i_3}\Phi,\quad \bar{D}\Gamma^{i_1}\Phi DS\Gamma^{i_2}\Phi\Gamma^{i_3}\Phi,\quad DS\Gamma^{i_1}\Phi\bar{D}\Gamma^{i_2}\Phi\Gamma^{i_3}\Phi),$$
$$D\Gamma^{i_1}\Phi\bar{D}S\Gamma^{i_2}\Phi\Gamma^{i_3}\Phi,\quad r^{-1}(DS\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi\Gamma^{i_3}\Phi),\quad r^{-1}(D\Gamma^{i_1}\Phi DS\Gamma^{i_2}\Phi\Gamma^{i_3}\Phi)$$

are irrelevant here because $i_3 \leq 13$ and we can thus control the last factor in the sup norm to reduce to the quadratic terms above. As before, we also have terms that do not have S (from $S\Lambda$ or from the commutators $[D, S]$, $[\bar{D}, S]$), but they already appear in N_k and we will use the estimates proved for N_k in Proposition 6.14.

We first estimate the quadratic terms. The crucial technical point here is that we do not have an improved pointwise decay estimate for $\bar{D}S\Gamma^i\Phi$ because we have used S in the proof of Proposition 5.6 and we are only commuting with S once. Nevertheless, since $k \leq 13$, we can instead put $D\Gamma^i\Phi$ in L^∞. We have

$$
\begin{aligned}
&\sum_{i_2=0}^{\lfloor k/2\rfloor}\sum_{i_1=0}^{k}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} r^\alpha(\text{quadratic terms})^2\\
&\quad\leq C\Big(\sup_{r\geq 9\tau/10}\sum_{i_2=0}^{6} r^2|D\Gamma^{i_2}\Phi|^2\Big)\sum_{i_1=0}^{k}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} r^{\alpha-2}|\bar{D}S\Gamma^{i_1}\Phi|^2\\
&\qquad+C\Big(\sup_{r\geq 9\tau/10}\sum_{i_2=0}^{6} r^2|DS\Gamma^{i_2}\Phi|^2\Big)\sum_{i_1=0}^{k}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} r^{\alpha-2}|\bar{D}\Gamma^{i_1}\Phi|^2\\
&\qquad+C\Big(\sup_{r\geq 9\tau/10}\sum_{i_2=0}^{6} r^2|\bar{D}\Gamma^{i_2}\Phi|^2\Big)\sum_{i_1=0}^{k}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} r^{\alpha-2}|DS\Gamma^{i_1}\Phi|^2\\
&\qquad+C\Big(\sup_{r\geq 9\tau/10}\sum_{i_1=0}^{k} r^2|D\Gamma^{i_1}\Phi|^2\Big)\sum_{i_2=0}^{6}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} r^{\alpha-2}|\bar{D}S\Gamma^{i_2}\Phi|^2\\
&\qquad+C\tau^{-2}\Big(\sup_{r\geq 9\tau/10}\sum_{i_2=0}^{6} r^2|D\Gamma^{i_2}\Phi|^2\Big)\sum_{i_1=0}^{k}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} r^{\alpha-2}|DS\Gamma^{i_1}\Phi|^2\\
&\qquad+C\tau^{-2}\Big(\sup_{r\geq 9\tau/10}\sum_{i_2=0}^{6} r^2|DS\Gamma^{i_2}\Phi|^2\Big)\sum_{i_1=0}^{k}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} r^{\alpha-2}|D\Gamma^{i_1}\Phi|^2\\
&\quad\leq CA\epsilon\sum_{i_1=0}^{k}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} r^{\alpha-2}(|\bar{D}S\Gamma^{i_1}\Phi|^2+|\bar{D}\Gamma^{i_1}\Phi|^2)\\
&\qquad+CA\epsilon\tau^{-2+\eta_{14}}\sum_{i_1=0}^{k}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} r^{\alpha-2}|DS\Gamma^{i_1}\Phi|^2\\
&\qquad+CA\epsilon\tau^{-4+\alpha+\eta_{S,11}}\sup_{r\geq 9\tau/10}\sum_{i_1=0}^{k} r^2|D\Gamma^{i_1}\Phi|^2+CA\epsilon\tau^{-4+\alpha}.
\end{aligned}
$$

We then estimate the cubic terms:

$$
\begin{aligned}
&\sum_{i_1,i_2=0}^{\lfloor k/2\rfloor}\sum_{i_3=0}^{k}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} r^\alpha(\text{cubic terms})^2\\
&\qquad\leq C\Big(\sup_{r\geq 9\tau/10}\sum_{i_1=0}^{6}(r^2\bar{D}\Gamma^{i_1}\Phi)^2\Big)\Big(\sup_{r\geq 9\tau/10}\sum_{i_2=0}^{6} r^2(D\Gamma^{i_2}\Phi)^2\Big)\\
&\qquad\quad\times\sum_{i_3=0}^{k}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} r^{\alpha-4}(S\Gamma^{i_3}\Phi)^2\\
&\qquad\quad+C\tau^{-2}\Big(\sup_{r\geq 9\tau/10}\sum_{i_1=0}^{6} r^2(D\Gamma^{i_1}\Phi)^2\Big)^2\sum_{i_3=0}^{k}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} r^{\alpha-4}(S\Gamma^{i_3}\Phi)^2\\
&\qquad\leq CA^2\epsilon^2\tau^{-2+\eta_{14}}\sum_{i_3=0}^{k}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}} r^{\alpha-2}(DS\Gamma^{i_3}\Phi)^2,
\end{aligned}
$$

which is better than the estimates obtained for the quadratic terms. We hence focus on the quadratic terms and spell out explicitly what the estimates amount to for different values of k and α:

$$\sum_{i_2=0}^{6}\sum_{i_1=0}^{13}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}}(\text{quadratic terms})^2\leq CA^2\epsilon^2\tau^{-2+\eta_{S,13}}+CA^2\epsilon^2\tau^{-4+\eta_{S,11}+\eta_{16}},$$
$$\sum_{i_2=0}^{6}\sum_{i_1=0}^{12}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}}(\text{quadratic terms})^2\leq CA^2\epsilon^2\tau^{-3+\eta_{S,12}}+CA^2\epsilon^2\tau^{-4+\eta_{S,11}},$$
$$\sum_{i_2=0}^{6}\sum_{i_1=0}^{11}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}}(\text{quadratic terms})^2\leq CA^2\epsilon^2\tau^{-4+\eta_{S,11}}+CA^2\epsilon^2\tau^{-4+\eta_{S,11}},$$
$$\sum_{i_2=0}^{6}\sum_{i_1=0}^{12}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}}r^2(\text{quadratic terms})^2\leq CA^2\epsilon^2(\tau^{-1+\eta_{S,12}}+\tau^{-2+\eta_{14}}+\tau^{-2+\eta_{S,11}}),$$
$$\sum_{i_2=0}^{6}\sum_{i_1=0}^{11}\int_{\Sigma_\tau\cap\{r\geq 9\tau/10\}}r^2(\text{quadratic terms})^2\leq CA^2\epsilon^2(\tau^{-2+\eta_{S,11}}+\tau^{-2+\eta_{14}}).\qquad\square$$

With the estimates for the inhomogeneous terms for the equations involving S, we can now retrieve the bootstrap assumptions involving S. We will follow the order that we proved the estimates without S, namely, first proving the pointwise estimates, then the integrated estimates, then the energy estimates and finally the energy estimates involving also $\hat{Y}$. Noticing that $U_{k,j}$ (respectively N_k) and $S(U_{k,j})$ (respectively $S(N_k)$) satisfy similar estimates (see Propositions 6.13, 6.14, 6.10, 6.27, 6.28 and 6.25), we focus on showing that the estimates for V_k are enough to close the bootstrap assumptions. We now prove the pointwise estimates and retrieve Bootstrap Assumptions (37), (38), (43) and (44).

Proposition 6.29. *For $r\geq t^*/4$,*

$$\sum_{j=0}^{8}|DS\Gamma^j\Phi|^2\leq (B_S/2)A\epsilon r^{-2},\tag{69}$$
$$\sum_{j=0}^{6}|DS\Gamma^j\Phi|^2\leq (B_S/2)A\epsilon r^{-2}(t^*)^{\eta_{S,11}}(1+|u|)^{-2}.\tag{70}$$

For $r\leq t^*/4$,

$$\sum_{j=0}^{6}|S\Gamma^j\Phi|^2\leq (B_S/2)A\epsilon(t^*)^{-2+\eta_{15}},\tag{71}$$
$$\sum_{\ell=1}^{7-j}\sum_{j=0}^{6}|D^\ell S\Gamma^j\Phi|^2\leq (B_S/2)A\epsilon r^{-2}(t^*)^{-2+\eta_{15}}.\tag{72}$$

Proof. The proof of the estimates for $r \geq t^*/4$ (i.e. (69) and (70)) is completely analogous to Proposition 6.15, with the use of Propositions 6.10, 6.13, 6.14 replaced by Propositions 6.22, 6.25, 6.27, 6.28 appropriately. Notice especially that the estimates in Proposition 6.22 for V are better than those in Proposition 6.25 for SU and are thus acceptable.

(71) follows directly from Proposition 5.9 and Bootstrap Assumptions (11) and (15). Here, we need to use also (11) because we would need to permute S with ∂_{t^*} and would get terms that do not contain S.

(72) follows directly from Proposition 5.8, Bootstrap Assumptions (11) and (15), as well as Propositions 6.10, 6.13, 6.22, 6.25 and 6.26 to control the inhomogeneous terms. As before, (11) and Propositions 6.10, 6.13 are used to control the terms arising from $[S, \partial_{t^*}]$. Notice here that the decay rate for $\sum_{\ell=1}^{7-j}\sum_{j=0}^{6}|D^\ell S\Gamma^j\Phi|^2$ is not as good as that for $\sum_{\ell=1}^{9-j}\sum_{j=0}^{8}|D^\ell\Gamma^j\Phi|^2$ because in proving the decay rate for $\sum_{\ell=1}^{9-j}\sum_{j=0}^{8}|D^\ell\Gamma^j\Phi|^2$, we have used the quantities associated to $S\Phi$, while we do not have estimates for $S^2\Phi$ at our disposal. □

As before, once we have proved the L^∞ bounds, we will replace the constant B_S by C.

Proposition 6.30.

$$\sum_{i+j+k\leq 12} A_{S,X,j}^{-1}\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/2\}} K^{X_0}(S\partial_{t^*}^i\tilde{\Omega}^j\Phi) \leq (\epsilon/2)\tau^{-1+\eta_{S,12}}, \tag{73}$$

$$\sum_{i+j\leq 11} A_{S,X,j}^{-1}\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/2\}} K^{X_1}(S\partial_{t^*}^i\tilde{\Omega}^j\Phi) \leq (\epsilon/2)\tau^{-1+\eta_{S,12}}, \tag{74}$$

$$\sum_{i+j\leq 11} A_{S,X,j}^{-1}\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/2\}} K^{X_0}(S\partial_{t^*}^i\tilde{\Omega}^j\Phi) \leq (\epsilon/2)\tau^{-2+\eta_{S,11}}, \tag{75}$$

$$\sum_{i+j\leq 10} A_{S,X,j}^{-1}\iint_{\mathcal{R}(\tau/1.1,\tau)\cap\{r\leq t^*/2\}} K^{X_1}(S\partial_{t^*}^i\tilde{\Omega}^j\Phi) \leq (\epsilon/2)\tau^{-2+\eta_{S,11}}. \tag{76}$$

Proof. This follows exactly as Proposition 6.16 except for replacing the use of Propositions 6.10, 6.13 and 6.14 with Propositions 6.22, 6.25, 6.27 and 6.28. □

Proposition 6.31.

$$\sum_{i+j=13} A_{S,j}^{-1}\int_{\Sigma_\tau} J_\mu^N(S\partial_{t^*}^i\tilde{\Omega}^j\Phi)n_{\Sigma_\tau}^\mu \leq (\epsilon/4)\tau^{\eta_{S,13}}, \tag{77}$$

$$\sum_{i+j\leq 12} A_{S,j}^{-1}\int_{\Sigma_\tau} J_\mu^N(S\partial_{t^*}^i\tilde{\Omega}^j\Phi)n_{\Sigma_\tau}^\mu \leq \epsilon/2, \tag{78}$$

$$\sum_{i+j=13} A_{S,X,j}^{-1}\iint_{\mathcal{R}(\tau_0,\tau)} K^{X_0}(S\partial_{t^*}^i\tilde{\Omega}^j\Phi) \leq (\epsilon/2)\tau^{\eta_{S,13}}, \tag{79}$$

$$\sum_{i+j\leq 12} A_{S,X,j}^{-1}\iint_{\mathcal{R}(\tau_0,\tau)} K^{X_0}(S\partial_{t^*}^i\tilde{\Omega}^j\Phi) \leq \epsilon/2. \tag{80}$$

Proof. This follows exactly as Proposition 6.17 except for replacing the use of Propositions 6.10, 6.13 and 6.14 with Propositions 6.22, 6.25, 6.27 and 6.28. □

Proposition 6.32.

$$\sum_{i+j=12} A_{S,j}^{-1}\left(\int_{\Sigma_\tau} J_\mu^{Z+N,w^Z}(S\partial_{t^*}^i\tilde{\Omega}^j\Phi)n_{\Sigma_\tau}^\mu + C\tau^2\int_{\Sigma_\tau\cap\{r\le 9\tau/10\}} J_\mu^N(S\partial_{t^*}^i\tilde{\Omega}^j\Phi)n_{\Sigma_\tau}^\mu\right) \le (\epsilon/4)\tau^{1+\eta_{S,12}}. \quad (81)$$

Proof. This follows exactly as Proposition 6.18 except for replacing the use of Propositions 6.10, 6.13 and 6.14 with Propositions 6.22, 6.25, 6.27 and 6.28. □

Proposition 6.33.

$$\sum_{i+j\le 11} A_{S,j}^{-1}\left(\int_{\Sigma_\tau} J_\mu^{Z+N,w^Z}(S(\partial_{t^*}^i\tilde{\Omega}^j\Phi))n_{\Sigma_\tau}^\mu + C\tau^2\int_{\Sigma_\tau\cap\{r\le 9\tau/10\}} J_\mu^N(S(\partial_{t^*}^i\tilde{\Omega}^j\Phi))n_{\Sigma_\tau}^\mu\right) \le (\epsilon/4)\tau^{\eta_{S,11}} \quad (82)$$

Proof. This follows exactly as Proposition 6.19 except for replacing the use of Propositions 6.10, 6.13 and 6.14 with Propositions 6.22, 6.25, 6.27 and 6.28. □

To close the bootstrap argument we need finally to consider energy quantities with both S and $\hat{Y}$.

Proposition 6.34.

$$\sum_{i+k=13} A_{S,Y}^{-1}\int_{\Sigma_\tau} J_\mu^N(\hat{Y}^k S\partial_{t^*}^i\Phi)n_{\Sigma_\tau}^\mu \le (\epsilon/4)\tau^{\eta_{S,13}},$$

$$\sum_{i+k=12} A_{S,Y}^{-1}\tau^2\int_{\Sigma_\tau\cap\{r\le r_Y^-\}} J_\mu^N(\hat{Y}^k S\partial_{t^*}^i\Phi)n_{\Sigma_\tau}^\mu \le (\epsilon/4)\tau^{1+\eta_{S,12}},$$

$$\sum_{i+k\le 11} A_{S,Y}^{-1}\tau^2\int_{\Sigma_\tau\cap\{r\le r_Y^-\}} J_\mu^N(\hat{Y}^k S\partial_{t^*}^i\Phi)n_{\Sigma_\tau}^\mu \le (\epsilon/4)\tau^{\eta_{S,11}}.$$

Proof. This follows exactly as Proposition 6.20 except for replacing the use of Propositions 6.13 with Propositions 6.22 and 6.27. □

7. Proof of Theorem 1

Now all the bootstrap assumptions are closed and all the estimates hold. The solution hence exists globally by a standard local existence argument that we omit here. The decay estimates of the derivatives of Φ claimed in the theorem are restatements of (51), (52), (33). The decay estimates follow from the use of Proposition 5.3 and (6.19) for $r \ge R$ and Proposition 5.12 and (60) for $r \le t^*/4$.

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