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Linearization of quasistatic fracture evolution in brittle materials

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Abstract. We prove a linearization result for quasistatic fracture evolution in nonlinear elasticity. As the stiffness of the material tends to infinity, we show that rescaled displacement fields and their associated crack sets converge to a solution of quasistatic crack growth in linear elasticity without any a priori assumptions on the geometry of the crack set. This result corresponds to the evolutionary counterpart of the static linearization result [Friedrich, Math. Eng. 2 (2020), 75–100], where a Griffith model for nonsimple brittle materials has been considered featuring an elastic energy which also depends suitably on the second gradient of the deformations. The proof relies on a careful study of unilateral global minimality, as determined by the nonlinear evolutionary problem, and its linearization together with a variant of the jump transfer lemma in GSBD [Friedrich and Solombrino, Ann. Inst. H. Poincaré C Anal. Non Linéaire 35 (2018), 27–64].

1. Introduction

A crucial question for nonlinear models in materials science consists in establishing the range of validity of suitably linearized approximations. Indeed, large strain models are often challenging to treat, both analytically and numerically, due to their nonconvex behavior, whereas linearized models are significantly easier and, in many situations, still allow one to accurately predict observed phenomena. The last decades have witnessed remarkable progress in providing rigorous results relating models with different strain regimes in terms of Γ-convergence [21]. After the seminal work on elastostatics [25], this question was explored in various directions, among others for incompressible material models [46, 48], atomistic settings [10, 57], residually stressed materials [55], problems without Dirichlet boundary conditions [49], homogenization [45,51], brittle fracture [3,33, 35], plasticity [52], or multiwell energies allowing for phase transformations [2,4,27,56]. Beyond the static framework, we refer to evolutionary results on elastodynamics [1], viscoelasticity [9, 37, 59], or plasticity [50].

The purpose of this paper is to prove a linearization result for the quasistatic evolution of brittle fracture. We consider a variational model for fracture introduced by Francfort

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and Marigo [32] based on Griffith's idea that crack formation and propagation is the result of the competition between an elastic bulk energy and a surface energy proportional to the size of the crack. Our goal is to show that, in dimension two, crack growth in nonlinear elasticity can be rigorously approximated by crack growth in linear elasticity.

Moving within the frame of variational models for fracture, we start by introducing the underlying energies. For a reference domain $\Omega \subseteq \mathbb{R}^2$, we consider nonlinear energies of the form

$$\mathcal{E}_\varepsilon[y, K] := \frac{1}{\varepsilon^2} \int_\Omega (W(\nabla y) + \varepsilon^{2(1-\beta)} |\nabla^2 y|^2) \, \mathrm{d}x + \kappa \mathcal{H}^1(K), \tag{1.1}$$

where $y \in W^{2,2}_{\mathrm{loc}}(\Omega \setminus K; \mathbb{R}^2)$ is the deformation, W is a frame-indifferent stored energy density, $\beta \in (0, 1)$, $K \subseteq \overline{\Omega}$ is the crack set (on which y is allowed to be discontinuous), κ denotes the fracture toughness, and $\mathcal{H}^1$ is the Hausdorff measure penalizing the length of the crack. The prefactor $1/\varepsilon^2$ represents a high stiffness of the material depending on a small parameter $\varepsilon > 0$. For technical reasons discussed below, we consider an energy in the frame of *nonsimple materials*, referring to the fact that the elastic energy depends additionally on the second gradient of the deformation. However, as the contribution of the Hessian vanishes in the limit $\varepsilon \to 0$, the effective energy is described by a linear Griffith fracture energy without second gradients. Precisely, using Γ-convergence, in [35] the first author showed that the above energies converge as $\varepsilon \to 0$ to their linearized counterpart

$$\mathcal{E}[u, K] := \int_\Omega \frac{1}{2} \mathbb{C} e(u) : e(u) \, \mathrm{d}x + \kappa \mathcal{H}^1(K), \tag{1.2}$$

where $u \in W^{1,2}_{\mathrm{loc}}(\Omega \setminus K; \mathbb{R}^2)$ is the displacement approximated by $u \approx (y - \mathrm{id})/\varepsilon$ (id is the identity mapping) and $\mathbb{C} = \mathrm{D}^2 W(\mathrm{Id})$ is a fourth-order elasticity tensor. Whereas the models (1.1)–(1.2) correspond to the so-called *strong formulation*, the result in [35] is actually formulated in the weak setting of the functions spaces GSBV (for (1.1)) and GSBD (for (1.2)).

In this work, our goal is to enhance the understanding of the relation between the models (1.1)–(1.2) by showing that solutions of quasistatic crack growth associated with the nonlinear energies converge to ones related to the linear energy; see our main result in Theorem 2.2. This provides a rigorous justification for the fact that crack propagation in stiff materials (described in terms of $1/\varepsilon^2$) is well approximated by using a linear theory. We also point out that the energetic rescaling in the energy (1.1) may be found by looking at the nonlinear fracture energy ($\varepsilon = 1$) on progressively larger domains Ω/ε, referred to as Bažant's law [53].

Linearization for variational fracture evolutions has already received some attention in the literature. Negri and Zanini [54] showed that the nonlinear quasistatic fracture evolution converges to the linear counterpart when the path of the crack is restricted to a line segment. Subsequent work by Negri and Toader [53] showed that this convergence still holds under a weaker assumption on the cracks (effectively, the crack is made of finitely many nonintersecting regular arcs). Our conclusions significantly improve these results: We show convergence in the passage from nonlinear-to-linearized models with *no assumptions* on the geometry of the crack.

The study of quasistatic crack evolutions was initiated in [26] for a two-dimensional model with restrictive assumptions on the crack topology. A breakthrough existence result in SBV is due to Francfort and Larsen [31] in the simplified setting of antiplanar displacements (meaning scalar displacements with control on the full gradient). Subsequently, this result was generalized to nonlinear elasticity [23], including the setting of noninterpenetration [24]. In all these works, the quasistatic fracture evolution is described in terms of (i) an irreversible crack path, (ii) unilateral global minimality of the deformation/displacement, and (iii) an energy-balance equation. Essential to the verification of (ii) is the so-called *jump transfer lemma*, allowing the authors to carry over unilateral minimality for approximate free discontinuity problems to their limits, where "unilateral" refers to the irreversible nature of the process.

Surprisingly, despite its paramount relevance for realistic engineering applications, the theory of linear fracture models is considerably less developed. This is due to technical difficulties inherent in the presence of symmetrized gradients. Indeed, due to the unknown and potentially rough discontinuity sets and the lack of Korn's inequality, there is no control on the skew symmetric part $(\nabla u)^\top - \nabla u$ of the strain, leading to an analytically more intricate formulation in the larger space of special functions of bounded deformation (SBD) [6]. Only recently, departing from Dal Maso's seminal paper [22] on the generalized space GSBD, an enormous amount of effort has been invested to develop the theory. Technical advances regarding functional inequalities [11, 12, 34], compactness theorems [5, 15], and lower semicontinuity [28, 40] have allowed a variety of authors to prove existence of Griffith energy minimizers in the strong formulation [13, 14, 19] with higher regularity [8, 38, 47], or existence of minimizers in anisotropic and heterogeneous settings [16, 39].

Most important for our work is the existence proof of a solution to the quasistatic evolution for cracks in dimension two [41]. In the linear case, it was essential to develop another jump transfer lemma suited for settings where only control on the symmetric gradient is available. Using a variant of this jump transfer lemma is at the core of our approach dealing with the passage from nonlinear to linearized theory.

We proceed by describing the ingredients of the proof in more detail. To see that solutions of the nonlinear problem associated with the energy $\mathcal{E}_\varepsilon$ converge to a solution of linear quasistatic crack growth with energy $\mathcal{E}$, we prove (i) irreversibility, (ii) minimality, and (iii) the energy balance for the limit displacement and crack. First, to identify a limiting displacement we need to ensure that deformation gradients are sufficiently close to the identity. Given that the domain can fracture into multiple pieces, we are forced to identify the limit up to modification, which essentially transforms the deformation back to the identity by a piecewise rigid motion related to a Caccioppoli partition of the reference domain. This technique has already been employed in a variety of works, in both a nonlinear [33, 35] and linearized setting [17, 41, 58]. The proof of (i) then follows by design.

The proof of (ii) is significantly more involved. Beyond the jump transfer lemma, carrying minimality from nonlinear to linear problems requires a delicate look at the linearization procedure. Formally, given $(y_\varepsilon, \Gamma_\varepsilon)$ minimizing $\mathcal{E}_\varepsilon$, we can define a limit displacement $u = \lim_{\varepsilon\to 0}(y_\varepsilon - \mathrm{id})/\varepsilon$ and crack $\Gamma = \lim_{\varepsilon\to 0}\Gamma_\varepsilon$. To show that minimality is inherited by the limit, we want to show that $\mathcal{E}[u,\Gamma] \leq \mathcal{E}[u+\phi, \Gamma\cup\Gamma_\phi]$ for all test pairs (ϕ, Γ_ϕ). For this, we use a Taylor expansion to linearize the nonlinear minimality condition $\mathcal{E}_\varepsilon[y_\varepsilon, \Gamma_\varepsilon] \leq \mathcal{E}_\varepsilon[y_\varepsilon + \phi_\varepsilon, \Gamma_\varepsilon \cup \Gamma_{\phi_\varepsilon}]$ for test pairs $(\phi_\varepsilon, \Gamma_{\phi_\varepsilon})$ approximating a fixed (ϕ, Γ_ϕ). The challenge is that, within this expansion, we must carefully control the error on the right-hand side to ensure it disappears in the limit $\varepsilon \to 0$. The key ingredients are a quantitative description of the almost-minimality of the rescaled displacement $(y_\varepsilon - \mathrm{id})/\varepsilon$ (see Lemma 5.1), the jump transfer lemma of [41], and a density result for $GSBD^2$ functions with prescribed boundary values; see Theorem 3.2.

The proof of the energy balance (iii) follows the now classical idea which relies on global minimality and a Riemann sum argument. Yet in our setting the expression representing the work of the external loads is more intricate if there are regions determined by the above-mentioned Caccioppoli partition where the deformation is not close to the identity but to a different rigid motion. In this case, we will show that this piece is (almost) relaxed to an equilibrium configuration and thus no longer contributes to the change of the total energy. Rigorously, this is done by using our analysis from (ii) to derive an approximate variational inequality that can pass to the limit in the time-integrated energy-balance equation; see Corollary 5.3 and Lemma 6.1.

As a technical point on our result, we introduce a time-step parameter $\tau > 0$ (later encoded as Δ_ε in equation (2.1)) and a "stiffness" parameter $\varepsilon > 0$. For ε fixed, we use τ to construct a time-discretized approximate solution to the nonlinear quasistatic crack growth. Then we pass to the limit jointly in $\tau \to 0$ and $\varepsilon \to 0$ to derive a solution of linear quasistatic crack growth. This means that at no point do we actually use a true solution in the nonlinear setting. (Note that existence of such is not guaranteed by [23] due to the presence of the second-gradient term in (1.1).) Importantly, however, in our analysis, there is no constraint on the relation between τ and ε. Thus, one can expect that we can pass to the limit as $\tau \to 0$ first and then $\varepsilon \to 0$. Although we believe that this would in general be possible, the rigorous execution of this procedure is beyond the scope of this paper.

Our result comes with three reasonable caveats: First, to isolate a rotation about which we may linearize, we require the nonlinear energy to be nonsimple and penalize the Hessian of the deformation. Yet we emphasize that this penalization is small and vanishes in the limit $\varepsilon \to 0$ (see Corollary 5.5). A possible alternative would be to modify the geometric rigidity estimate by Friesecke et al. [42], but this is intricate in this setting as the domain regularity is determined by the a priori irregular crack set. Existing applications using [42] for fracture require fixed crack geometries [53], a curvature penalization of the crack [36], or an involved modification procedure of deformations and crack sets on multiple scales [33]. In particular, the latter modification is incompatible with the irreversibility condition (i).

Second, our result is restricted to the plane. This is essentially because the piecewise Korn inequality [34] has only been proven in the plane. We use this inequality in two places: the jump transfer lemma and a density result with Dirichlet conditions. With more work, we believe that it is possible to extend the density result to higher dimensions as the role of the piecewise Korn inequality therein is to approximate GSBD^2 by L^2-functions. On the other hand, extending the jump transfer lemma to higher dimensions will require fundamentally new ideas.

Third, our model does not account for noninterpenetration and, in particular, for simplicity we require W to be locally Lipschitz, preventing the blow-up of $W(M)$ as $\det M \to 0$. The necessary techniques to implement noninterpenetration in the framework of crack growth in finite elasticity are already available [24] (based on [30]) and, in the static case, the linearization has also recently been studied [3]. The extension to the evolutionary setting will require nontrivial adaptations and is a subject of future research.

The paper is organized as follows. In Section 2, we introduce the model and formulate the main result. Section 3 is subject to some preliminaries, namely compactness and density results in GSBD and the jump transfer lemma from [41]. In Section 4 we derive a priori estimates for the nonlinear evolutions and obtain compactness for the rescaled displacement fields. Then Sections 5 and 6 are devoted to the stability of minimality and an approximate energy balance as $\varepsilon \to 0$. Section 7 contains the proof of the main result. Finally, in Appendixes A and B we collect proofs that have been omitted in the paper.

2. Setting and main results

In this section we first provide some basic notation. Then we introduce the nonlinear quasistatic time-discretized evolution for crack growth using a time-discrete scheme. Next we define a notion of solution for quasistatic crack growth in linear elasticity. Finally, we state our main convergence result relating the nonlinear to the linear setting.

2.1. Basic notation

We introduce some basic notation:

- We let $\Omega \subseteq \Omega' \subseteq \mathbb{R}^2$ be bounded Lipschitz domains such that $\Omega' \setminus \overline{\Omega}$ is also a Lipschitz set.
- $B_r(x) \subseteq \mathbb{R}^d$, $d \geq 2$, denotes the ball centered at $x \in \mathbb{R}^d$ with radius $r > 0$. Similarly, for a set A, we define $B_\delta(A) := \{x \in \mathbb{R}^d : |x - y| < \delta \text{ for some } y \in A\}$.
- We write id for the identity map on $\mathbb{R}^2$ and Id for its derivative, viewed as an element of $\mathbb{R}^{2\times 2}$.
- We denote by $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$ and $\mathbb{R}^{2\times 2}_{\mathrm{skew}}$ symmetric and skew-symmetric matrices, respectively. We set $SO(2) = \{F \in \mathbb{R}^{2\times 2} : F^\top F = \mathrm{Id},\ \det F = 1\}$.
- We call an affine map $Ax + b$ with $A \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$ and $b \in \mathbb{R}^2$ an *infinitesimal rigid motion*.

- χ_A denotes the characteristic function of a set $A \subseteq \mathbb{R}^2$.
- We denote the two-dimensional Lebesgue measure and the one-dimensional Hausdorff measure by $\mathcal{L}^2$ and $\mathcal{H}^1$, respectively.
- We write ∂^* for the essential boundary of a set taken always with respect to the set Ω'; see [7, Def. 3.60].
- We say that a partition $(P_j)_{j\in\mathbb{N}}$ of $\Omega' \subseteq \mathbb{R}^2$ is a *Caccioppoli partition* of Ω' if

$$\sum_j \mathcal{H}^1(\partial^* P_j) < \infty.$$

- We assume the reader to be familiar with the function spaces BV ([7, Def. 3.1]), SBV ([7, Sect. 4.1]), GSBV ([7, Def. 4.26]), GSBD ([22, Def. 4.2]), and GSBD^2 ([44]). We further define ([35, Sect. 2.1])

$$\mathrm{GSBV}^2_2(\Omega;\mathbb{R}^2) := \{y \in \mathrm{GSBV}^2(\Omega;\mathbb{R}^2) : \nabla y \in \mathrm{GSBV}^2(\Omega;\mathbb{R}^{2\times 2})\}.$$

- We use the notation $\langle\cdot,\cdot\rangle$ for the Euclidean inner product on $\mathbb{R}^2$, $\mathbb{R}^{2\times2}$, and $\mathbb{R}^{2\times2\times2\times2}$. When an elastic tensor $\mathbb{C} \in \mathbb{R}^{2\times2\times2\times2}$ is involved, we follow the convention to use the notation ":" for the inner product between two matrices in $\mathbb{R}^{2\times2}$.
- If we do not specify the domain over which a norm is taken, we always implicitly mean Ω', i.e., $\|\cdot\|_{L^2} := \|\cdot\|_{L^2(\Omega')}$. Likewise, we often do not specify the range of a function within the norm.
- For $a, b \in \mathbb{R}^2$, we define $a \odot b := (a \otimes b + b \otimes a)/2$ as the symmetric outer product of a and b.
- For a vector-valued function f, we write f^i for the ith component of f.
- We use the symbol $\lesssim$ to say that the inequality $\leq$ holds up to a constant which may depend on the potential, the boundary data, and the domain. We follow the convention that generic constants C may change from line to line.

2.2. Nonlinear quasistatic fracture evolution

We begin by constructing a time-discretized evolution for a nonlinear model based on the energy (1.1), see e.g., [23] or [31]. Let $(\varepsilon_i)_i$ be an arbitrary sequence with $\varepsilon_i \to 0$ as $i \to \infty$. For convenience, with an abuse of notation, we will simply use ε in place of ε_i and write $\varepsilon \to 0$. Taking our time interval to be $[0, 1]$ without loss of generality, we partition the time interval using the collection of points

$$I_\varepsilon := (t_i^\varepsilon)_{i=0}^{n(\varepsilon)} = \{0 = t_0^\varepsilon < t_1^\varepsilon < \cdots < t_{n(\varepsilon)}^\varepsilon = 1\}.$$

We impose that the partitions created by I_ε are nested, i.e., $I_\varepsilon \subseteq I_{\varepsilon'}$ when $\varepsilon' < \varepsilon$, and that they refine, in the sense that

$$\Delta_\varepsilon := \max_{1\leq i\leq n(\varepsilon)} t_i^\varepsilon - t_{i-1}^\varepsilon \tag{2.1}$$

vanishes as ε tends to zero. Note that the union of the partitions $I_\infty := \bigcup_\varepsilon I_\varepsilon$ is a countable dense subset of $[0, 1]$. We also introduce the shorthand notation

$$I_\varepsilon^t := \{\tau \in I_\varepsilon : \tau \le t\} \quad \text{and} \quad I_\infty^t := \{\tau \in I_\infty : \tau \le t\}.$$

We let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain. As is typically done for fracture problems, we impose displacement boundary conditions by considering a larger set Ω' containing Ω: We fix a boundary datum

$$h \in \mathrm{W}^{1,1}((0,1);\mathrm{W}^{2,2}(\Omega';\mathbb{R}^2)) \cap \mathrm{L}^\infty((0,1);\mathrm{W}^{2,\infty}(\Omega';\mathbb{R}^2)), \tag{2.2}$$

and define

$$\mathcal{S}_n^\varepsilon := \{y \in \mathrm{GSBV}_2^2(\Omega';\mathbb{R}^2) : y = \mathrm{id} + \varepsilon h_n^\varepsilon \text{ on } \Omega' \setminus \overline{\Omega}\},$$

where $h_n^\varepsilon := h(t_n^\varepsilon)$. We assume that also Ω' and $\Omega' \setminus \overline{\Omega}$ are Lipschitz sets.

Following [35, Sect. 2.1], we define the energy

$$\begin{cases} \dfrac{1}{\varepsilon^2}\displaystyle\int_{\Omega'} W(\nabla y)\,\mathrm{d}x + \dfrac{1}{\varepsilon^{2\beta}}\displaystyle\int_{\Omega'} |\nabla^2 y|^2\,\mathrm{d}x + \kappa\mathcal{H}^1(J_y) & \text{if } J_{\nabla y} \subseteq J_y, \\ \infty & \text{otherwise,} \end{cases} \tag{2.3}$$

for $y \in \mathrm{GSBV}_2^2(\Omega';\mathbb{R}^2)$ and some $\beta \in (2/3, 1)$, $\kappa > 0$. This corresponds to the weak formulation of the energy introduced in equation (1.1). Here, $W : \mathbb{R}^{2\times 2} \to [0, \infty)$ satisfies

(W1) W is locally Lipschitz continuous with

$$|W(A) - W(B)| \le C(1 + \max\{|A|, |B|\})|A - B|$$

for some $C > 0$, and there is $r > 0$ such that W is C^3 in the neighborhood $B_{2r}(\mathrm{SO}(2))$ of $\mathrm{SO}(2)$.

(W2) Frame indifference: $W(RF) = W(F)$ for all $F \in \mathbb{R}^{2\times 2}$, $R \in \mathrm{SO}(2)$.

(W3) $W(F) \ge c\,\mathrm{dist}^2(F, \mathrm{SO}(2))$ for some $c > 0$ for all $F \in \mathbb{R}^{2\times 2}$, $W(F) = 0$ if and only if $F \in \mathrm{SO}(2)$.

Note that the local Lipschitz continuity in (W1) is a consequence of the common assumption $|\mathrm{D}W(A)| \lesssim (1 + |A|)$. We emphasize that (W1) excludes $W(A) \to +\infty$ as $\det A \to 0$.

Considering a continuous function y on $\mathbb{R}^2$ that is C^2 in $\{x_2 > 0\}$ and $\{x_2 < 0\}$ and satisfies $J_y = \emptyset$ and $J_{\nabla y} = \{x_2 = 0\} \cap \Omega$, one can approximate this by $y_\eta := y + \eta \mathrm{id}\chi_{\{x_2>0\}}$ for small $\eta > 0$ to see that, as $\eta \to 0$, the condition $J_{\nabla y_\eta} \subseteq J_{y_\eta}$ is not stable under convergence in measure. Thus, for existence of minimizers of the energy (2.3), we must pass to its relaxation

$$\mathrm{E}_\varepsilon(y) := \frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla y)\,\mathrm{d}x + \frac{1}{\varepsilon^{2\beta}}\int_{\Omega'} |\nabla^2 y|^2\,\mathrm{d}x + \kappa\mathcal{H}^1(J_y \cup J_{\nabla y}); \tag{2.4}$$

see [35, Prop. 2.1]. Starting from initial conditions $y_0^\varepsilon \in \mathcal{S}_0^\varepsilon$ with bounded energy, i.e.,

$$\sup_{\varepsilon > 0} \mathrm{E}_\varepsilon(y_0^\varepsilon) < \infty, \tag{2.5}$$

assuming that $y_k^\varepsilon \in \mathcal{S}_k^\varepsilon$ for $k = 0, \ldots, n-1$ have already been found, we iteratively define y_n^ε as a minimizer of the energy

$$\frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla y)\,\mathrm{d}x + \frac{1}{\varepsilon^{2\beta}}\int_{\Omega'}|\nabla^2 y|^2\,\mathrm{d}x + \kappa\mathcal{H}^1\Big((J_y \cup J_{\nabla y}) \setminus \bigcup_{k=0}^{n-1}(J_{y_k^\varepsilon} \cup J_{\nabla y_k^\varepsilon})\Big) \quad (2.6)$$

over $y \in \mathcal{S}_n^\varepsilon$. We emphasize that the integrals in the above minimization could be over Ω or Ω' due to the boundary conditions, but the jump sets are always taken relative to Ω'. The existence of a minimizer for the problem (2.6) follows from an adaptation of [35, Thm. 2.2], which gives existence of minimizers on an open set. Precisely, the only difference is that we remove the energy of the crack from the previous time steps. One can account for this difference by approximating $\bigcup_{k=0}^{n-1}(J_{y_k^\varepsilon} \cup J_{\nabla y_k^\varepsilon})$ from inside with compact sets K and applying the compactness result [35, Thm. 3.2] on $\Omega' \setminus K$ and then diagonalizing as $K \nearrow \bigcup_{k=0}^{n-1}(J_{y_k^\varepsilon} \cup J_{\nabla y_k^\varepsilon})$. The technique used here is similar to the proof of [23, Thm. 2.8].

We define the *discrete quasistatic fracture evolution* $y_\varepsilon\colon [0,1] \times \Omega' \to \mathbb{R}^2$ with parameter $\varepsilon > 0$ as the piecewise constant interpolation of the minimizers, namely

$$y_\varepsilon(t) = y_n^\varepsilon \text{ for } t \in [t_n^\varepsilon, t_{n+1}^\varepsilon), \quad \text{for each } 0 \le n \le n(\varepsilon) - 1, \;\; y_\varepsilon(1) = y_{n(\varepsilon)}^\varepsilon. \quad (2.7)$$

Note that if h_ε denotes the piecewise constant interpolation of h with respect to I_ε, which means

$$h_\varepsilon(t) = h(t_n^\varepsilon) \quad \text{for } t \in [t_n^\varepsilon, t_{n+1}^\varepsilon),$$

then y_ε satisfies the boundary condition $y_\varepsilon(t) = h_\varepsilon(t)$. Lastly, we introduce the shorthand notation

$$\Gamma_\varepsilon(t) := \bigcup_{\tau \in I_\varepsilon^t} J_{y_\varepsilon(\tau)} \cup J_{\nabla y_\varepsilon(\tau)} \quad (2.8)$$

for the total crack at time t. Note then that, as a consequence of the minimization problem (2.6), for all $t \in [0,1]$, $y_\varepsilon(t)$ is a minimizer of the functional

$$\frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla z)\,\mathrm{d}x + \frac{1}{\varepsilon^{2\beta}}\int_{\Omega'}|\nabla^2 z|^2\,\mathrm{d}x + \kappa\mathcal{H}^1((J_z \cup J_{\nabla z}) \setminus \Gamma_\varepsilon(t)) \quad (2.9)$$

among all $z \in \mathrm{GSBV}_2^2(\Omega';\mathbb{R}^2)$ with $z = h_\varepsilon(t)$ on $\Omega' \setminus \overline{\Omega}$.

2.3. Linear quasistatic fracture evolution

We introduce the notation

$$\mathbb{C} := \mathrm{D}^2 W(\mathrm{Id}) \quad \text{and} \quad Q(A) := \mathbb{C}A : A, \quad (2.10)$$

and note that $\mathbb{C}$ is a symmetric positive-definite fourth-order tensor by (W2) and (W3). We define the linearized energy as

$$\mathrm{E}(u) := \int_{\Omega'} \frac{1}{2} Q(e(u))\,\mathrm{d}x + \kappa\mathcal{H}^1(J_u). \quad (2.11)$$

As shown in [35], E is the Γ-limit of E_ε as $\varepsilon \to 0$ for a suitable notion of convergence of the deformations. Based on this energy, we now introduce the notion of a quasistatic fracture evolution in the setting of linear elasticity; see also [41, Thm. 3.1]. As before, we take the time interval to be [0, 1].

Definition 2.1. Let h be as in (2.2) and let $u_0 \in GSBD^2(\Omega')$ with $u_0 = h(0)$ on $\Omega' \setminus \overline{\Omega}$. We say that displacements $t \mapsto u(t) \in GSBD^2(\Omega')$ and cracks $t \mapsto \Gamma(t) \subseteq \Omega'$ are a (linear) quasistatic fracture evolution with the boundary condition h and initial condition u_0 if the following conditions are satisfied:

(i) *(Initial condition).* The displacement $u(0) = u_0$ minimizes the energy (2.11) among all $v \in GSBD^2(\Omega')$ with $v = h(0)$ on $\Omega' \setminus \overline{\Omega}$, and $\Gamma(0) = J_{u(0)}$.

(ii) *(Displacement and boundary conditions).* The displacement satisfies

$$e(u) \in L^\infty((0,1); L^2(\Omega'; \mathbb{R}^{2\times 2}_{\mathrm{sym}})),$$

and at each time it holds that $J_{u(t)} \subseteq \Gamma(t)$ up to an $\mathcal{H}^1$-nullset and $u(t) = h(t)$ in $\Omega' \setminus \overline{\Omega}$.

(iii) *(Irreversibility).* It holds that $\Gamma(t) \subseteq \Gamma(t')$ if $t \leq t'$.

(iv) *(Minimality).* The displacement is a global minimizer of the elastic energy given the existing crack, in the sense that

$$\int_{\Omega'} \frac{1}{2}\mathbb{C}e(u(t)) : e(u(t))\,\mathrm{d}x \leq \int_{\Omega'} \frac{1}{2}\mathbb{C}e(v) : e(v)\,\mathrm{d}x + \kappa\mathcal{H}^1(J_v \setminus \Gamma(t))$$

for all $v \in GSBD^2(\Omega')$ with $v = h(t)$ in $\Omega' \setminus \overline{\Omega}$.

(v) *(Energy balance).* The change in energy is equal to the work done on the system as captured by

$$\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t \int_{\Omega'} \mathbb{C}e(u(s)) : e(\partial_t h(s))\,\mathrm{d}x\,\mathrm{d}s,$$

where by

$$\mathcal{E}(t) := \int_{\Omega'} \frac{1}{2} Q\big(e(u(t))\big)\,\mathrm{d}x + \kappa\mathcal{H}^1(\Gamma(t)) \tag{2.12}$$

we denote the total energy at time $t \in [0, 1]$.

We remark that existence of linear quasistatic fracture evolutions for the Griffith energy has been shown in [41] by means of time-discrete approximations (analogous to (2.6)).

2.4. Main result

To describe the convergence of the deformations y_ε to a displacement u, we need to introduce a proper type of convergence. Typically, in linearization results, one considers the

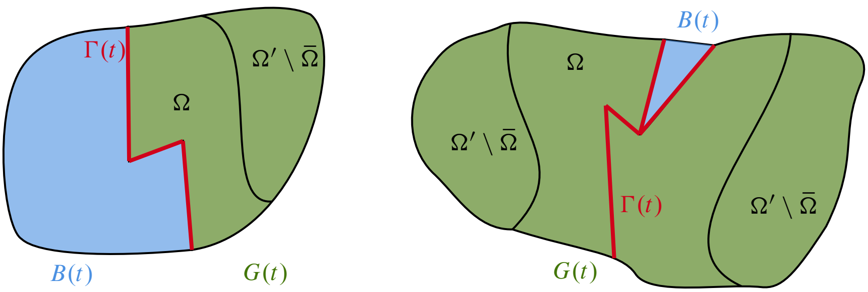

Figure 1. Two examples of the "bad set" $B(t)$ seen in blue.

asymptotic behavior of the rescaled displacements $\bar{u}_\varepsilon := (y_\varepsilon - \mathrm{id})/\varepsilon$. However, as the material may fracture, it is not possible to linearize around the single rigid motion given by id as pieces may be broken off. On these pieces, compactness cannot be expected and we essentially have no information except for the fact that the elastic energy should vanish asymptotically due to the minimality of y_ε in the sense of (2.6).

Let us now describe the convergence towards a limit $u(t) \in \mathrm{GSBD}^2(\Omega')$ for $t \in [0,1]$. For a crack $\Gamma(t) \subseteq \bar{\Omega} \cap \Omega'$ with $\mathcal{H}^1(\Gamma(t)) < \infty$, by $B(t) \subseteq \Omega$ we denote the largest set of finite perimeter (with respect to set inclusion) which satisfies $\partial^* B(t) \subseteq \Gamma(t)$ up to an $\mathcal{H}^1$-negligible set (see Lemma A.1 for existence and uniqueness, and recall that ∂^* is with respect to Ω'). This set represents the "broken-off pieces", and by $G(t) := \Omega' \setminus B(t)$ instead we denote the "good set", which in particular has $\Omega' \setminus \bar{\Omega} \subseteq G(t)$. Intuitively, if a point $x \in \Omega$ is connected to $\Omega' \setminus \bar{\Omega}$ in $\Omega' \setminus \Gamma(t)$, then $x \in G(t)$. See Lemma A.1 for details and Figure 1 for an illustrated example.

Now, we say that y_ε converges to u, and write $y_\varepsilon \rightsquigarrow u$, if for every $t \in [0,1]$,

$$\begin{aligned} \bar{u}_\varepsilon(t) := \frac{y_\varepsilon(t) - \mathrm{id}}{\varepsilon} &\to u(t) \text{ in measure on } G(t), \\ e(\bar{u}_\varepsilon(t)) &\to e(u(t)) \text{ in measure on } G(t), \end{aligned} \tag{2.13}$$

and

$$\frac{1}{\varepsilon^2} \int_{B(t)} \mathrm{dist}^2(\nabla y_\varepsilon(t), SO(2))\, dx \to 0. \tag{2.14}$$

We are now ready to state our main result.

Theorem 2.2 (Linearization of quasistatic fracture evolution). *Let $\Omega \subseteq \Omega' \subseteq \mathbb{R}^2$ be bounded Lipschitz domains such that $\Omega' \setminus \bar{\Omega}$ is a Lipschitz set. Let $u_0 \in \mathrm{GSBD}^2(\Omega')$ with $u_0 = h(0)$ on $\Omega' \setminus \bar{\Omega}$ such that u_0 minimizes (2.11) among all $v \in \mathrm{GSBD}^2(\Omega')$ with $v = h(0)$ on $\Omega' \setminus \bar{\Omega}$. Let y_ε be approximate solutions to nonlinear quasistatic fracture as defined in (2.7) with well-prepared initial data $(y_0^\varepsilon)_\varepsilon$ in the sense that $(y_0^\varepsilon - \mathrm{id})/\varepsilon \to u_0$ in measure on Ω' and* $\limsup_{\varepsilon\to 0} \mathrm{E}_\varepsilon(y_0^\varepsilon) \leq \mathrm{E}(u_0)$.

Then there exists a linear quasistatic fracture evolution as in Definition 2.1, denoted by $(u(t), \Gamma(t))_{t\in[0,1]}$, *such that*

$$\Gamma(t) = \bigcup_{\tau\in I^t_\infty} J_{u(\tau)} \quad \text{for all } t \in [0,1]$$

and, up to a subsequence in ε *(not relabeled),* $y_\varepsilon \rightsquigarrow u$.

Moreover, denoting the total energy along the evolution y_ε *by*

$$\mathcal{E}_\varepsilon(t) := \frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla y_\varepsilon(t))\,\mathrm{d}x + \frac{1}{\varepsilon^{2\beta}}\int_{\Omega'} |\nabla^2 y_\varepsilon(t)|^2\,\mathrm{d}x + \kappa\mathcal{H}^1(\Gamma_\varepsilon(t)), \tag{2.15}$$

we find

$$\mathcal{E}_\varepsilon(t) \to \mathcal{E}(t) \quad \text{for all } t \in [0,1]. \tag{2.16}$$

Remark 2.3. We make the following remarks regarding the main result.

(i) The existence of well-prepared initial conditions is a consequence of the Γ-convergence of E_ε to E; see [35, Thm. 2.7].

(ii) More precisely, convergence (2.16) can be refined to separate convergence of elastic and crack energies; see (7.7) for details.

(iii) If we accept that we modify the sequence $(y_\varepsilon)_\varepsilon$ to account for the broken-off pieces, then we can actually show a much stronger convergence than $y_\varepsilon \rightsquigarrow u$. This modification procedure is however rather involved and is done in equations (4.11), (4.12), (4.16), and (4.18). The resulting improved convergence is then seen in item (C1) and Remark 7.2.

The proof of this result will be our focus for the rest of the paper and it is split into manageable pieces. In Section 3 we introduce a variety of tools that will be essential to our analysis. In Section 4 we address compactness of the sequence $(y_\varepsilon)_\varepsilon$ and its rescaled displacements $(u_\varepsilon)_\varepsilon$ to find a candidate displacement u that might be a linear quasistatic fracture evolution in the sense of Definition 2.1. In Section 5 we prove minimality for the limit displacement u at times $t \in I_\infty$. Likewise, in Section 6 we show that this displacement satisfies two approximate energy-balance relations. We complete the proof of Theorem 2.2 in Section 7. This is done by extending minimality for the limit displacement to all times and showing that the approximate energy-balance relations give rise to the desired energy-balance equation.

3. Preliminaries: Compactness, density, and jump transfer

In this section we record some auxiliary results needed to prove our main theorem.

First, we state a compactness result, which is [41, Thm. 6.1], but with extra information extrapolated from the proof.

Theorem 3.1. *Let $\Omega \subseteq \Omega' \subseteq \mathbb{R}^2$ be bounded Lipschitz domains and $(h_n)_n, h \subseteq \mathrm{W}^{1,2}(\Omega'; \mathbb{R}^2)$ such that $h_n \to h$ in $\mathrm{W}^{1,2}(\Omega'; \mathbb{R}^2)$. Let $(u_n)_{n \in \mathbb{N}} \subseteq \mathrm{GSBD}^2(\Omega')$ be a sequence with*

$$\|e(u_n)\|_{\mathrm{L}^2(\Omega')} + \mathcal{H}^1(J_{u_n}) \leq C < \infty$$

and $u_n = h_n$ on $\Omega' \setminus \overline{\Omega}$. Then there is a subsequence (not relabeled), some $u \in \mathrm{GSBD}^2(\Omega')$ with $u = h$ on $\Omega' \setminus \overline{\Omega}$, a sequence of sets $\mathcal{S}_n := (S^n_j)_{j \geq 0}$ contained in Ω such that $\mathcal{S}_n \cup (\Omega' \setminus \bigcup_{j \geq 0} S^n_j)$ is a Caccioppoli partition of Ω', and a sequence of infinitesimal rigid motions $(a^n_j)_{j \geq 1}$ such that

(i) $u_n - \chi_{S^n_0}(u_n - h_n) - \sum_{j=1}^{\infty} a^n_j \chi_{S^n_j} \to u$ *pointwise almost everywhere in Ω',*

(ii) $e(u_n) \rightharpoonup e(u)$ *weakly in $\mathrm{L}^2(\Omega'; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$,*

(iii) $\mathcal{H}^1(J_u \cap U) \leq \liminf_{n\to\infty} \mathcal{H}^1(J_{u_n} \cap U)$ *for all open Lipschitz subsets $U \subseteq \Omega'$,*

(iv) $\mathcal{H}^1(\bigcup_{j=0}^{\infty} \partial^* S^n_j \setminus J_{u_n}) \to 0$, *and*

(v) $\mathcal{L}^2(S^n_0) \to 0$.

Proof. This is an immediate consequence of the compactness result [41, Thm. 6.1] combined with the proper bookkeeping. More precisely, the form of the functions in (i) can be found in [41, eq. (65)], where the component S^n_0 there is denoted by E^l_k; see [41, eq. (66)]. The rest of the properties then follow immediately from the proof. ∎

We denote by $\mathcal{W}(\Omega'; \mathbb{R}^2)$ the collection of functions $u \in \mathrm{GSBD}^2(\Omega')$ such that J_u is closed and included in a finite union of closed connected pieces of C^1-curves, and $u \in C^\infty(\Omega' \setminus J_u; \mathbb{R}^2) \cap \mathrm{W}^{2,\infty}(\Omega' \setminus J_u; \mathbb{R}^2)$. Note that functions $u \in \mathcal{W}(\Omega'; \mathbb{R}^2)$ automatically lie in $\mathrm{GSBV}^2_2(\Omega'; \mathbb{R}^2)$ and satisfy $\mathcal{H}^1(J_{\nabla u} \setminus J_u) = 0$. We present the following density result whose proof will be given in Appendix B.

Theorem 3.2. *Given $h \in \mathrm{W}^{2,\infty}(\Omega'; \mathbb{R}^2)$ and $v \in \mathrm{GSBD}^2(\Omega')$ with $v = h$ on $\Omega' \setminus \overline{\Omega}$, there exists $(w_\delta)_\delta \subseteq \mathcal{W}(\Omega'; \mathbb{R}^2)$ such that $w_\delta = h$ on $\Omega' \setminus \overline{\Omega}$ and, as $\delta \to 0$,*

(D1) $w_\delta \to v$ *in measure on Ω',*

(D2) $e(w_\delta) \to e(v)$ *in $\mathrm{L}^2(\Omega'; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$, and*

(D3) $\mathcal{H}^1(J_{w_\delta} \triangle J_v) \to 0$.

By keeping track of the bound on the second derivatives, we can also deduce the following refined *jump transfer lemma*.

Theorem 3.3 (Refined jump transfer in GSBD^2). *Let $\Omega \subseteq \Omega' \subseteq \mathbb{R}^2$ be bounded Lipschitz domains such that $\Omega' \setminus \overline{\Omega}$ has Lipschitz boundary. Let $\ell \in \mathbb{N}$ and let $(h^l_n)_n \subseteq \mathrm{W}^{2,\infty}(\Omega'; \mathbb{R}^2)$ be bounded sequences for $l = 1, \dots, \ell$. Let $(u^l_n)_n$ be sequences in $\mathrm{GSBD}^2(\Omega')$ and $u^l \in \mathrm{GSBD}^2(\Omega')$ such that*

(1) $\|e(u^l_n)\|_{\mathrm{L}^2} + \mathcal{H}^1(J_{u^l_n}) \leq M$ *for all $n \in \mathbb{N}$,*

(2) $u^l_n \to u^l$ *in measure in Ω', $u^l_n = h^l_n$ on $\Omega' \setminus \overline{\Omega}$,*

for all $l = 1, \ldots, \ell$. Then there exists a subsequence in n (not relabeled) with the following property: For each $\phi \in \mathcal{W}(\Omega'; \mathbb{R}^2)$, there is a sequence $(\phi_n)_n \subseteq \mathrm{GSBV}^2_2(\Omega'; \mathbb{R}^2)$ with $\phi_n = \phi$ on $\Omega' \setminus \overline{\Omega}$ such that, as $n \to \infty$,

(i) $\phi_n \to \phi$ *in measure in* Ω',

(ii) $e(\phi_n) \to e(\phi)$ *in* $\mathrm{L}^2(\Omega'; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$,

(iii) $\mathcal{H}^1(((J_{\phi_n} \cup J_{\nabla\phi_n}) \setminus \bigcup_{l=1}^{\ell} J_{u_n^l}) \setminus (J_\phi \setminus \bigcup_{l=1}^{\ell} J_{u^l})) \to 0$,

(iv) $\|\nabla\phi_n\|_{\mathrm{L}^\infty} \le C\|\nabla\phi\|_{\mathrm{L}^\infty}$ *and* $\|\nabla^2\phi_n\|_{\mathrm{L}^\infty} \le C\|\nabla^2\phi\|_{\mathrm{L}^\infty}$.

For the proof, we can closely follow [41, Thm. 5.1], where the statement has been proven without item (iv) and without the jump of the gradient in (iii). To obtain these additions, we adapt the extension lemma in [41, Lem. 5.2] in the following way.

Lemma 3.4. *For $r_1, r_2 > 0$, let R^+, R^-, and R be given by a common rotation and translation of*

$$(0, r_1) \times (0, r_2), \quad (0, r_1) \times (-r_2/2, 0), \quad \text{and} \quad (0, r_1) \times (-r_2/2, r_2), \tag{3.1}$$

respectively. Let $\phi \in \mathrm{GSBV}^2_2(R^+; \mathbb{R}^2)$ with $\mathcal{H}^1(J_{\nabla\phi} \setminus J_\phi) = 0$. Then there is an extension of ϕ onto R^- given by $\hat{\phi} \in \mathrm{GSBV}^2_2(R; \mathbb{R}^2)$ satisfying the properties

(i) $\mathcal{H}^1(J_{\hat{\phi}}) \le C\mathcal{H}^1(J_\phi)$,

(ii) $\|\nabla\hat{\phi}\|_{\mathrm{L}^\infty(R)} \le C\|\nabla\phi\|_{\mathrm{L}^\infty(R^+)}$,

(iii) $\|\nabla^2\hat{\phi}\|_{\mathrm{L}^\infty(R)} \le C\|\nabla^2\phi\|_{\mathrm{L}^\infty(R^+)}$, *and*

(iv) $\mathcal{H}^1(J_{\nabla\hat{\phi}} \setminus J_{\hat{\phi}}) = 0$

for some universal constant $C \ge 1$ independent of R and ϕ.

Proof. We may assume R^+, R^-, and R are as in equation (3.1), respectively. For $x \in R^-$ we define $\hat{\phi}$ by

$$\hat{\phi}(x) = 3\phi(x_1, -x_2) - 2\phi(x_1, -2x_2).$$

To see that no jump is created by $\hat{\phi}$ and $\nabla\hat{\phi}$ along $\{x_2 = 0\}$, we show that the traces of $\hat{\phi}$ and $\nabla\hat{\phi}$ from inside R^- agree with the traces of ϕ and $\nabla\phi$ from inside R^+. It is immediate that $\phi(x_1, 0) = \hat{\phi}(x_1, 0)$ for $x_1 \in (0, r_1)$. Additionally, the absolutely continuous derivatives are given by

$$\begin{aligned}
\partial_{x_1}\hat{\phi}(x) &= 3\partial_{x_1}\phi(x_1, -x_2) - 2\partial_{x_1}\phi(x_1, -2x_2),\\
\partial_{x_2}\hat{\phi}(x) &= -3\partial_{x_2}\phi(x_1, -x_2) + 4\partial_{x_2}\phi(x_1, -2x_2),
\end{aligned}$$

so we have continuity in the variable x_2 at $\{x_2 = 0\}$, proving $\nabla\phi(x_1, 0) = \nabla\hat{\phi}(x_1, 0)$ for $x_1 \in (0, r_1)$. Given that $\hat{\phi}$ is defined by reflection and creates no additional jump on $\{x_2 = 0\}$ the conclusion of the lemma follows. ■

Proof of Theorem 3.3. Inspecting the proof of the jump transfer lemma in GSBD^2 ([41, Thm. 5.1]), we see that ϕ_n is defined using a reflection argument [41, Lem. 5.2] on a finite, but increasing number of squares. Using instead the reflection from Lemma 3.4, we see that the additional properties are satisfied by the construction. ■

4. Compactness of the nonlinear evolution

We first obtain a priori estimates on the sequence $(y_\varepsilon)_\varepsilon$, essentially coming from an approximate energy-balance relation. With these bounds, we apply techniques developed in [35] to identify a limiting displacement at countable times $t \in I_\infty$.

4.1. A priori estimates

As in [23,31], we establish some a priori estimates on the total energy $\mathcal{E}_\varepsilon$ defined in (2.15).

Lemma 4.1 (A priori estimate). *Letting y_ε and Γ_ε be defined as in (2.7) and (2.8), respectively, the following estimates are satisfied:*

$$\sup_{\varepsilon>0, t\in[0,1]} \mathcal{E}_\varepsilon(t) < \infty, \tag{4.1}$$

and, for all $t_n^\varepsilon \in I_\varepsilon$,

$$\mathcal{E}_\varepsilon(t_n^\varepsilon) - \sigma_\varepsilon \leq \mathrm{E}_\varepsilon(y_0^\varepsilon) + \frac{1}{\varepsilon}\int_0^{t_n^\varepsilon}\int_{\{\nabla y_\varepsilon \in B_r(\mathrm{SO}(2))\}} \langle \mathrm{D}W(\nabla y_\varepsilon), \nabla \partial_t h\rangle \,\mathrm{d}x\,\mathrm{d}t \tag{4.2}$$

for $\sigma_\varepsilon \to 0$ as $\varepsilon \to 0$ (independently of the chosen time t_n^ε), where r is as in (W1).

Proof. Let $t \in I_\infty$ and ε be sufficiently small such that $t \in I_\varepsilon$. The function $\mathrm{id} + \varepsilon h_n^\varepsilon$ is an admissible competitor for the minimization problem (2.6) without any jumps and thus

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla y_n^\varepsilon)\,\mathrm{d}x + \frac{1}{\varepsilon^{2\beta}}\int_{\Omega'}|\nabla^2 y_n^\varepsilon|^2\,\mathrm{d}x + \kappa\mathcal{H}^1\Big((J_{y_n^\varepsilon}\cup J_{\nabla y_n^\varepsilon})\setminus\bigcup_{k=0}^{n-1}(J_{y_k^\varepsilon}\cup J_{\nabla y_k^\varepsilon})\Big)\\ \leq \frac{1}{\varepsilon^2}\int_{\Omega'} W(\mathrm{Id}+\varepsilon\nabla h_n^\varepsilon)\,\mathrm{d}x + \frac{1}{\varepsilon^{2\beta}}\int_{\Omega'}|\varepsilon\nabla^2 h_n^\varepsilon|^2\,\mathrm{d}x.\end{aligned} \tag{4.3}$$

By assumption (2.2) and (W3), we have $h \in \mathrm{L}^\infty((0,1);\mathrm{W}^{2,\infty}(\Omega';\mathbb{R}^2))$ and $W(\mathrm{Id}) = 0$, $\mathrm{D}W(\mathrm{Id}) = 0$. Therefore, by Taylor's formula, using that W is C^3 near the identity, we get the inequality

$$W(\mathrm{Id}+\varepsilon\nabla h_n^\varepsilon) \lesssim \varepsilon^2|\nabla h_n^\varepsilon|^2.$$

Thus, we may estimate the right-hand side of (4.3), up to a constant, by

$$\int_{\Omega'}|\nabla h_n^\varepsilon|^2\,\mathrm{d}x + \varepsilon^{2-2\beta}\int_{\Omega'}|\nabla^2 h_n^\varepsilon|^2\,\mathrm{d}x,$$

which stays uniformly bounded. Consequently, we have established that, uniformly for $t \in [0,1]$,

$$\frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla y_\varepsilon(t))\,\mathrm{d}x + \frac{1}{\varepsilon^{2\beta}}\int_{\Omega'}|\nabla^2 y_\varepsilon(t)|^2\,\mathrm{d}x \leq C < \infty. \tag{4.4}$$

To obtain a uniform bound on the measure of the cumulative crack, we compare the deformation at time t_n^ε to the deformation at time t_{n-1}^ε. Precisely, noting that $y_{n-1}^\varepsilon + \varepsilon(h_n^\varepsilon - h_{n-1}^\varepsilon)$ is an admissible competitor for the minimization problem (2.6), we get

$$
\begin{aligned}
&\frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla y_n^\varepsilon)\,\mathrm{d}x + \frac{1}{\varepsilon^{2\beta}}\int_{\Omega'}|\nabla^2 y_n^\varepsilon|^2\,\mathrm{d}x + \kappa\mathcal{H}^1\Big((J_{y_n^\varepsilon}\cup J_{\nabla y_n^\varepsilon})\setminus\bigcup_{k=0}^{n-1}(J_{y_k^\varepsilon}\cup J_{\nabla y_k^\varepsilon})\Big)\\
&\quad\le \frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla y_{n-1}^\varepsilon + \varepsilon\nabla(h_n^\varepsilon - h_{n-1}^\varepsilon))\,\mathrm{d}x\\
&\qquad + \frac{1}{\varepsilon^{2\beta}}\int_{\Omega'}|\nabla^2 y_{n-1}^\varepsilon + \varepsilon\nabla^2(h_n^\varepsilon - h_{n-1}^\varepsilon)|^2\,\mathrm{d}x.
\end{aligned}\tag{4.5}
$$

Take $r > 0$ as in (W1) so that W is C^3 in $B_{2r}(\mathrm{SO}(2))$ and define the preimage

$$
\mathcal{N}_{n-1}^\varepsilon := \{\nabla y_{n-1}^\varepsilon \in B_r(\mathrm{SO}(2))\}.
$$

We want to apply Taylor's formula to the first summand on the right-hand side of (4.5) and use the local Lipschitz continuity of W to argue that the integral asymptotically vanishes as $\varepsilon \to 0$ outside of $\mathcal{N}_{n-1}^\varepsilon$, after summing all times.

First we take care of the integral over $\Omega' \setminus \mathcal{N}_{n-1}^\varepsilon$. By the local Lipschitz estimate for W in (W1), the uniform control on the boundary data (2.2), the energy estimate (4.4), (W3), and Hölder's inequality we bound

$$
\begin{aligned}
&\frac{1}{\varepsilon^2}\int_{\Omega'\setminus\mathcal{N}_{n-1}^\varepsilon} W(\nabla y_{n-1}^\varepsilon + \varepsilon\nabla(h_n^\varepsilon - h_{n-1}^\varepsilon))\,\mathrm{d}x\\
&\quad\le \frac{1}{\varepsilon^2}\int_{\Omega'\setminus\mathcal{N}_{n-1}^\varepsilon} W(\nabla y_{n-1}^\varepsilon)\\
&\qquad + C(1 + |\nabla y_{n-1}^\varepsilon| + \varepsilon\|\nabla(h_n^\varepsilon - h_{n-1}^\varepsilon)\|_{\mathrm{L}^\infty})\varepsilon|\nabla(h_n^\varepsilon - h_{n-1}^\varepsilon)|\,\mathrm{d}x\\
&\quad\le \frac{1}{\varepsilon^2}\int_{\Omega'\setminus\mathcal{N}_{n-1}^\varepsilon} W(\nabla y_{n-1}^\varepsilon)\,\mathrm{d}x + C\int_{t_{n-1}^\varepsilon}^{t_n^\varepsilon}\int_{\Omega'\setminus\mathcal{N}_{n-1}^\varepsilon}\frac{1}{\varepsilon}(1 + |\nabla y_{n-1}^\varepsilon|)|\nabla\partial_t h|\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le \frac{1}{\varepsilon^2}\int_{\Omega'\setminus\mathcal{N}_{n-1}^\varepsilon} W(\nabla y_{n-1}^\varepsilon)\,\mathrm{d}x\\
&\qquad + C\int_{t_{n-1}^\varepsilon}^{t_n^\varepsilon}\Big(\frac{1}{\varepsilon^2}\int_{\Omega'\setminus\mathcal{N}_{n-1}^\varepsilon}(1 + |\nabla y_{n-1}^\varepsilon|)^2\,\mathrm{d}x\Big)^{1/2}\|\nabla\partial_t h\|_{\mathrm{L}^2(\Omega'\setminus\mathcal{N}_{n-1}^\varepsilon)}\,\mathrm{d}t\\
&\quad\le \frac{1}{\varepsilon^2}\int_{\Omega'\setminus\mathcal{N}_{n-1}^\varepsilon} W(\nabla y_{n-1}^\varepsilon)\,\mathrm{d}x + C\int_{t_{n-1}^\varepsilon}^{t_n^\varepsilon}\Big(\frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla y_{n-1}^\varepsilon)\,\mathrm{d}x\Big)^{1/2}\|\nabla\partial_t h\|_{\mathrm{L}^2(\Omega'\setminus\mathcal{N}_{n-1}^\varepsilon)}\,\mathrm{d}t\\
&\quad\le \frac{1}{\varepsilon^2}\int_{\Omega'\setminus\mathcal{N}_{n-1}^\varepsilon} W(\nabla y_{n-1}^\varepsilon)\,\mathrm{d}x + C\int_{t_{n-1}^\varepsilon}^{t_n^\varepsilon}\|\nabla\partial_t h\|_{\mathrm{L}^2(\Omega'\setminus\mathcal{N}_{n-1}^\varepsilon)}\,\mathrm{d}t.
\end{aligned}\tag{4.6}
$$

In the neighborhood $\mathcal{N}_{n-1}^{\varepsilon}$, we apply Taylor's formula, the differentiability in time of h, and Minkowski's integral inequality to obtain

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\int_{\mathcal{N}_{n-1}^{\varepsilon}} W(\nabla y_{n-1}^{\varepsilon} + \varepsilon\nabla(h_n^{\varepsilon} - h_{n-1}^{\varepsilon}))\,\mathrm{d}x \\
&\quad\leq \frac{1}{\varepsilon^2}\int_{\mathcal{N}_{n-1}^{\varepsilon}} W(\nabla y_{n-1}^{\varepsilon})\,\mathrm{d}x + \frac{1}{\varepsilon}\int_{t_{n-1}^{\varepsilon}}^{t_n^{\varepsilon}}\int_{\mathcal{N}_{n-1}^{\varepsilon}} \langle \mathrm{D}W(\nabla y_{n-1}^{\varepsilon}), \nabla\partial_t h\rangle\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + C\int_{\mathcal{N}_{n-1}^{\varepsilon}} |\nabla(h_n^{\varepsilon} - h_{n-1}^{\varepsilon})|^2\,\mathrm{d}x \\
&\quad\leq \frac{1}{\varepsilon^2}\int_{\mathcal{N}_{n-1}^{\varepsilon}} W(\nabla y_{n-1}^{\varepsilon})\,\mathrm{d}x + \frac{1}{\varepsilon}\int_{t_{n-1}^{\varepsilon}}^{t_n^{\varepsilon}}\int_{\mathcal{N}_{n-1}^{\varepsilon}} \langle \mathrm{D}W(\nabla y_{n-1}^{\varepsilon}), \nabla\partial_t h\rangle\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + C\left(\int_{t_{n-1}^{\varepsilon}}^{t_n^{\varepsilon}} \|\nabla\partial_t h\|_{\mathrm{L}^2}\,\mathrm{d}t\right)^2.
\end{aligned} \tag{4.7}$$

By expanding the square in the integral involving the second derivatives, we therefore deduce by using the estimates (4.6) and (4.7), and again the differentiability in time of h, that the right-hand side of inequality (4.5) is controlled by

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla y_{n-1}^{\varepsilon})\,\mathrm{d}x + \frac{1}{\varepsilon^{2\beta}}\int_{\Omega'}|\nabla^2 y_{n-1}^{\varepsilon}|^2\,\mathrm{d}x + \frac{1}{\varepsilon}\int_{t_{n-1}^{\varepsilon}}^{t_n^{\varepsilon}}\int_{\mathcal{N}_{n-1}^{\varepsilon}} \langle \mathrm{D}W(\nabla y_{\varepsilon}), \nabla\partial_t h\rangle\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + C\int_{t_{n-1}^{\varepsilon}}^{t_n^{\varepsilon}} \|\nabla\partial_t h\|_{\mathrm{L}^2(\Omega'\setminus\mathcal{N}_{n-1}^{\varepsilon})}\,\mathrm{d}t + \frac{2}{\varepsilon^{2\beta-1}}\int_{t_{n-1}^{\varepsilon}}^{t_n^{\varepsilon}}\int_{\Omega'}\langle\nabla^2 y_{\varepsilon}, \nabla^2\partial_t h\rangle\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + C\left(\int_{t_{n-1}^{\varepsilon}}^{t_n^{\varepsilon}} \|\nabla\partial_t h\|_{\mathrm{L}^2}\,\mathrm{d}t\right)^2 + \frac{1}{\varepsilon^{2\beta-2}}\left(\int_{t_{n-1}^{\varepsilon}}^{t_n^{\varepsilon}} \|\nabla^2\partial_t h\|_{\mathrm{L}^2}\,\mathrm{d}t\right)^2,
\end{aligned} \tag{4.8}$$

where we also employed the notation in (2.7). Noting that the first two terms in (4.8) are the same as the first two terms on the left-hand side of (4.5) with $n-1$ in place of n, we will iterate the above reasoning to derive the a priori estimate. Starting from the energy

$$\frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla y_n^{\varepsilon})\,\mathrm{d}x + \frac{1}{\varepsilon^{2\beta}}\int_{\Omega'}|\nabla^2 y_n^{\varepsilon}|^2\,\mathrm{d}x + \kappa\mathcal{H}^1\left(\bigcup_{k=0}^{n}(J_{y_k^{\varepsilon}} \cup J_{\nabla y_k^{\varepsilon}})\right),$$

we iteratively apply inequality (4.5), together with the fact that at every time step the right-hand side of inequality (4.5) is controlled by the term (4.8), to deduce

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla y_n^{\varepsilon})\,\mathrm{d}x + \frac{1}{\varepsilon^{2\beta}}\int_{\Omega'}|\nabla^2 y_n^{\varepsilon}|^2\,\mathrm{d}x + \kappa\mathcal{H}^1\left(\bigcup_{k=0}^{n}(J_{y_k^{\varepsilon}} \cup J_{\nabla y_k^{\varepsilon}})\right) \\
&\quad\leq \mathrm{E}_{\varepsilon}(y_0^{\varepsilon}) + \frac{1}{\varepsilon}\int_0^{t_n^{\varepsilon}}\int_{\{\nabla y_{\varepsilon}\in B_r(\mathrm{SO}(2))\}} \langle \mathrm{D}W(\nabla y_{\varepsilon}), \nabla\partial_t h\rangle\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

$$+ C \int_0^1 \Big(\|\nabla \partial_t h\|_{L^2(\{\nabla y_\varepsilon \notin B_r(SO(2))\})} + \omega(\Delta_\varepsilon) \Big(\|\nabla \partial_t h\|_{L^2} + \frac{1}{\varepsilon^{2\beta-2}} \|\nabla^2 \partial_t h\|_{L^2} \Big) \Big)\, dt$$
$$+ \int_0^1 \int_{\Omega'} \frac{2}{\varepsilon^{2\beta-1}} |\langle \nabla^2 y_\varepsilon, \nabla^2 \partial_t h \rangle| \, dx \, dt, \tag{4.9}$$

where E_ε is defined in (2.4) and $\omega(\Delta_\varepsilon)$ vanishes as ε tends to zero since integrable functions are absolutely continuous. By the energy estimate (4.4) and (W3), the measure of $\Omega' \setminus \mathcal{N}^\varepsilon_{n-1}$ vanishes uniformly in n since

$$\mathcal{L}^2(\Omega' \setminus \mathcal{N}^\varepsilon_{n-1}) \le \frac{1}{r^2} \int_{\Omega' \setminus \mathcal{N}^\varepsilon_{n-1}} \mathrm{dist}^2(\nabla y^\varepsilon_{n-1}, SO(2))\, dx \lesssim \int_{\Omega'} W(\nabla y^\varepsilon_{n-1})\, dx \lesssim \varepsilon^2. \tag{4.10}$$

Applying Hölder's inequality to the last summand, using the uniform bound on the data (2.2) and energy (4.4), $\beta < 1$, and that the measure of $\{\nabla y_\varepsilon \notin B_r(SO(2))\}$ vanishes uniformly in time by inequality (4.10), the last two lines of the estimate (4.9) vanish as ε tends to zero, thus yielding a sequence $\sigma_\varepsilon \to 0$ independent of t^ε_n such that (4.2) holds.

To show the uniform bound (4.1), we are left with arguing that the last term of inequality (4.2) stays uniformly bounded. Denote by π the (possibly not unique) projection onto SO(2). Since DW is Lipschitz in a neighborhood of SO(2) and $DW(\pi \nabla y_\varepsilon) = 0$, we deduce that, if $\mathrm{dist}(\nabla y_\varepsilon, SO(2)) < r$, then

$$\begin{aligned} |\langle DW(\nabla y_\varepsilon), \nabla \partial_t h \rangle| &= |\langle DW(\nabla y_\varepsilon) - DW(\pi \nabla y_\varepsilon), \nabla \partial_t h \rangle| \lesssim |\nabla y_\varepsilon - \pi \nabla y_\varepsilon| |\nabla \partial_t h| \\ &= \mathrm{dist}(\nabla y_\varepsilon, SO(2)) |\nabla \partial_t h| \lesssim \sqrt{W(\nabla y_\varepsilon)} |\nabla \partial_t h|, \end{aligned}$$

where the last inequality is due to the lower growth bound (W3). Applying the previous inequality and Hölder's inequality to the last term on the right-hand side of inequality (4.2), we obtain

$$\begin{aligned} &\frac{1}{\varepsilon} \int_0^{t^\varepsilon_n} \int_{\{\nabla y_\varepsilon \in B_r(SO(2))\}} \langle DW(\nabla y_\varepsilon), \nabla \partial_t h \rangle \, dx \, dt \\ &\quad \lesssim \int_0^1 \Big(\int_{\Omega'} \frac{1}{\varepsilon^2} W(\nabla y_\varepsilon)\, dx \Big)^{1/2} \|\nabla \partial_t h\|_{L^2}\, dt. \end{aligned}$$

By the control on the elastic energy (4.4) we get that this term stays uniformly bounded. Inserting this information back into inequality (4.2) and using that the initial energies are bounded, see (2.5), we thus have shown the desired a priori estimate (4.1). ∎

Remark 4.2. We note that the reasoning used to derive inequality (4.2) actually shows that, for $t^\varepsilon_n, t^\varepsilon_m \in I_\varepsilon$ with $t^\varepsilon_m < t^\varepsilon_n$, we have

$$\mathcal{E}_\varepsilon(t^\varepsilon_n) - \sigma_\varepsilon \le \mathcal{E}_\varepsilon(t^\varepsilon_m) + \frac{1}{\varepsilon} \int_{t^\varepsilon_m}^{t^\varepsilon_n} \int_{\{\nabla y_\varepsilon \in B_r(SO(2))\}} \langle DW(\nabla y_\varepsilon), \nabla \partial_t h \rangle \, dx \, dt$$

for $\sigma_\varepsilon \to 0$ as $\varepsilon \to 0$ (independently of t^ε_m and t^ε_n).

4.2. From deformations to rescaled displacements

In this subsection we define rescaled displacements that are sufficiently close to a rotation everywhere in Ω'. Following [35], this requires the introduction of a partition of Ω', where on each piece y_ε is sufficiently close to a rigid motion. Beyond accounting for rigidity, we will need to carefully zero-out the displacement in a vanishingly small region in Ω. This will later ensure that minimality is stable when passing from deformations to displacements as $\varepsilon \to 0$; see Section 5.

Choosing $\gamma \in (2/3, \beta)$, for any time $t \in [0,1]$, we apply the reasoning in [35, Thm. 2.3] to the sequence $(y_\varepsilon(t))_\varepsilon$ to obtain the modified sequence

$$y_\varepsilon^{\mathrm{rot}}(t) := \sum_{j=1}^{\infty} R_j^\varepsilon(t) y_\varepsilon(t) \chi_{P_j^\varepsilon(t)} \in \mathrm{GSBV}_2^2(\Omega'; \mathbb{R}^2), \tag{4.11}$$

for a Caccioppoli partition $\mathcal{P}_\varepsilon(t) := (P_j^\varepsilon(t))_j$ of Ω' and $(R_j^\varepsilon(t))_j \subseteq \mathrm{SO}(2)$, such that $R_j^\varepsilon(t) = \mathrm{Id}$ on all elements $P_j^\varepsilon(t)$ intersecting the boundary in the sense of $\mathcal{L}^2(P_j^\varepsilon(t) \cap (\Omega' \setminus \overline{\Omega})) > 0$. We point out that R_j^ε and the Caccioppoli partition $\mathcal{P}_\varepsilon$ are piecewise constant in time on the intervals determined by I_ε (as in (2.7)). (We frequently omit (t) if no confusion arises.) As many of the properties of this modified sequence are determined solely by the uniform energy bound in (4.1), it holds uniformly for all $t \in [0,1]$ that

- (R1) $y_\varepsilon^{\mathrm{rot}} = \mathrm{id} + \varepsilon h_\varepsilon$ on $\Omega' \setminus \overline{\Omega}$,
- (R2) $\mathcal{H}^1((J_{y_\varepsilon^{\mathrm{rot}}} \cup J_{\nabla y_\varepsilon^{\mathrm{rot}}}) \setminus (J_{y_\varepsilon} \cup J_{\nabla y_\varepsilon})) \le \mathcal{H}^1((\Omega' \cap \bigcup_{j=1}^\infty \partial^* P_j^\varepsilon) \setminus J_{\nabla y_\varepsilon}) \lesssim \varepsilon^{\beta-\gamma}$,
- (R3) $\|e(y_\varepsilon^{\mathrm{rot}}) - \mathrm{Id}\|_{L^2} \lesssim \varepsilon$,
- (R4) $\|\nabla y_\varepsilon^{\mathrm{rot}} - \mathrm{Id}\|_{L^2} \lesssim \varepsilon^\gamma$, and
- (R5) $|\nabla y_\varepsilon^{\mathrm{rot}} - \mathrm{Id}| \le \max\{C\varepsilon^\gamma, 2\,\mathrm{dist}(\nabla y_\varepsilon, \mathrm{SO}(2))\}$ pointwise in Ω' for fixed $C > 0$.

The last item is a consequence of [35, eq. (4.11)]. As a result of these properties, the rescaling

$$u_\varepsilon^{\mathrm{aux}}(t) := \frac{1}{\varepsilon}(y_\varepsilon^{\mathrm{rot}}(t) - \mathrm{id}) \in \mathrm{GSBV}_2^2(\Omega'; \mathbb{R}^2) \tag{4.12}$$

has boundary value h_ε on $\Omega' \setminus \overline{\Omega}$. However, this will only be an auxiliary object since for the sequence we are actually interested in we will have to multiply the function by a suitable cutoff; see the comments below and Lemma 4.3. As a consequence of items (R2)–(R3) and the a priori estimate (4.1), the function $u_\varepsilon^{\mathrm{aux}}$ satisfies

$$\sup_{\varepsilon>0} \sup_{t\in[0,1]} (\|e(u_\varepsilon^{\mathrm{aux}}(t))\|_{L^2} + \mathcal{H}^1(J_{u_\varepsilon^{\mathrm{aux}}(t)})) < \infty \tag{4.13}$$

and

$$\mathcal{H}^1((J_{u_\varepsilon^{\mathrm{aux}}} \cup J_{\nabla u_\varepsilon^{\mathrm{aux}}}) \setminus (J_{y_\varepsilon} \cup J_{\nabla y_\varepsilon})) \lesssim \varepsilon^{\beta-\gamma}. \tag{4.14}$$

For the linearization results later, we will need to apply a Taylor expansion of W at the identity with respect to the direction $\varepsilon \nabla u_\varepsilon^{\mathrm{aux}}$. In order to make sure that the error does not explode, we introduce a cutoff function that cuts out the vanishingly small region where $\nabla u_\varepsilon^{\mathrm{aux}}$ is too big. Whereas this procedure is now standard in the derivation of linearized models, see for example [10, 25, 35, 42, 56], in our setting it is important that the removed region is not only small in volume but *also controlled in surface*, which requires refined arguments. The cutoff function is precisely introduced in the lemma below and is based on ideas used by the first author in [35, Thm. 2.7]. We use the notation $u(x,t)$ for $u(t)$ evaluated at x.

Lemma 4.3. *There exists a collection of piecewise constant functions $\eta_\varepsilon\colon[0,1]\to(0,\infty)$, indexed by $\varepsilon>0$, associated characteristic functions $\chi_{\omega_\varepsilon(t)}$ with*

$$\omega_\varepsilon(t) := \big\{x \in \Omega' : |\nabla u_\varepsilon^{\mathrm{aux}}(x,t)|_\infty < \eta_\varepsilon(t)\big\},$$

where $|A|_\infty := \max_{ij}|A_{ij}|$ is the infinity norm of a given matrix, with the following properties holding uniformly for $t\in[0,1]$:

(ω1) $\chi_{\omega_\varepsilon(t)} \to 1$ *in measure (or $\mathrm{L}^1(\Omega')$),*

(ω2) $\mathcal{H}^1(\partial^*\omega_\varepsilon(t)\setminus J_{\nabla u_\varepsilon^{\mathrm{aux}}(t)}) \to 0$,

(ω3) $\varepsilon\eta_\varepsilon^3(t)\to 0$, *and*

(ω4) $\varepsilon^{1-\gamma}\eta_\varepsilon(t)\to\infty$.

Proof. Let the parameters $2/3<\gamma<\beta<1$ be given and define the sequences

$$\theta_\varepsilon^- := \varepsilon^{(9\gamma-10)/12} \quad \text{and} \quad \theta_\varepsilon^+ := \varepsilon^{(\gamma-2)/4}.$$

We note that $-1/3<(\gamma-2)/4<(9\gamma-10)/12<\gamma-1$. In particular, $\theta_\varepsilon^- < \theta_\varepsilon^+$ for $\varepsilon<1$. If $\eta_\varepsilon\in[\theta_\varepsilon^-,\theta_\varepsilon^+]$ for all $\varepsilon>0$ and $t\in[0,1]$, then properties (ω3) and (ω4) will hold, along with

$$\varepsilon^{\beta-1}/(\theta_\varepsilon^+ - \theta_\varepsilon^-) \to 0. \tag{4.15}$$

Indeed, for (ω3), we estimate

$$\varepsilon\eta_\varepsilon^3 \le \varepsilon^{1+(3\gamma-6)/4} = \varepsilon^{(3\gamma-2)/4},$$

which vanishes because $\gamma>2/3$. To see convergence (ω4), we compute

$$\varepsilon^{1-\gamma}\eta_\varepsilon \ge \varepsilon^{1-\gamma+(9\gamma-10)/12} = \varepsilon^{(2-3\gamma)/12},$$

which diverges since $\gamma>2/3$. Lastly, for convergence (4.15) we estimate

$$\frac{\varepsilon^{\beta-1}}{\varepsilon^{(\gamma-2)/4}-\varepsilon^{(9\gamma-10)/12}} = \frac{\varepsilon^{\beta-1-(\gamma-2)/4}}{1-\varepsilon^{(9\gamma-10)/12-(\gamma-2)/4}} = \frac{\varepsilon^{(-2+4\beta-\gamma)/4}}{1-\varepsilon^{(6\gamma-4)/12}},$$

which vanishes because $4\beta-\gamma>3\gamma>2$ and $6\gamma>4$.

From (ω4) together with inequality (R4), for any η_ε in this range, we directly deduce via Markov's inequality that $\mathcal{L}^2(\Omega'\setminus\omega_\varepsilon)\to 0$ uniformly in time, which is (ω1).

To prove convergence ($\omega 2$), it suffices to estimate the perimeter for fixed $\varepsilon > 0$ and $t \in I_\varepsilon$ by an appropriate sequence $\rho_\varepsilon \to 0$. Using the coarea formula for GSBV-functions from [7, Thm. 4.34] (suppressing dependence on t), and Hölder's inequality, we deduce that for all $i, j \in \{1, 2\}$ we have

$$
\begin{aligned}
&\frac{1}{\theta_\varepsilon^+ - \theta_\varepsilon^-} \int_{\theta_\varepsilon^-}^{\theta_\varepsilon^+} \mathcal{H}^1(\partial^* \{\partial_i (u_\varepsilon^{\mathrm{aux}})^j > s\} \setminus J_{\nabla u_\varepsilon^{\mathrm{aux}}}) \, \mathrm{d}s \\
&\quad \le \frac{1}{\theta_\varepsilon^+ - \theta_\varepsilon^-} \int_{-\infty}^{\infty} \mathcal{H}^1(\partial^* \{\partial_i (u_\varepsilon^{\mathrm{aux}})^j > s\} \setminus J_{\nabla u_\varepsilon^{\mathrm{aux}}}) \, \mathrm{d}s \\
&\quad \le \frac{1}{\theta_\varepsilon^+ - \theta_\varepsilon^-} \|\nabla^2 u_\varepsilon^{\mathrm{aux}}\|_{\mathrm{L}^1(\Omega')} = \frac{1}{\theta_\varepsilon^+ - \theta_\varepsilon^-} \frac{1}{\varepsilon} \|\nabla^2 y_\varepsilon\|_{\mathrm{L}^2(\Omega')} \\
&\quad \lesssim \frac{\varepsilon^{\beta - 1}}{\theta_\varepsilon^+ - \theta_\varepsilon^-},
\end{aligned}
$$

where the last inequality uses that the energy of y_ε stays uniformly bounded due to Lemma 4.1. Applying the same reasoning in $[-\theta_\varepsilon^+, -\theta_\varepsilon^-]$, this implies that we can choose $\eta_\varepsilon \in [\theta_\varepsilon^-, \theta_\varepsilon^+]$ such that

$$
\begin{aligned}
&\mathcal{H}^1(\partial^* \omega_\varepsilon \setminus J_{\nabla u_\varepsilon^{\mathrm{aux}}}) \\
&\quad \le \sum_{ij} \mathcal{H}^1(\partial^* \{\partial_i (u_\varepsilon^{\mathrm{aux}})^j \ge \eta_\varepsilon\} \setminus J_{\nabla u_\varepsilon^{\mathrm{aux}}}) + \mathcal{H}^1(\partial^* \{\partial_i (u_\varepsilon^{\mathrm{aux}})^j \le -\eta_\varepsilon\} \setminus J_{\nabla u_\varepsilon^{\mathrm{aux}}}) \\
&\quad \le C \frac{\varepsilon^{\beta - 1}}{\theta_\varepsilon^+ - \theta_\varepsilon^-} =: \rho_\varepsilon,
\end{aligned}
$$

where we used that $\partial^*(\Omega' \setminus \omega_\varepsilon) = \partial^* \omega_\varepsilon$, $\mathcal{H}^1(\partial^*(A \cup B)) \le \mathcal{H}^1(\partial^* A) + \mathcal{H}^1(\partial^* B)$, and

$$
\Omega' \setminus \omega_\varepsilon = \bigcup_{ij} \{\partial_i (u_\varepsilon^{\mathrm{aux}})^j \ge \eta_\varepsilon\} \cup \{\partial_i (u_\varepsilon^{\mathrm{aux}})^j \le -\eta_\varepsilon\}.
$$

Note that ρ_ε vanishes uniformly in time by (4.15), which finishes the proof. ∎

Finally, recalling the definition of $u_\varepsilon^{\mathrm{aux}}$ in (4.12), we define the modified displacements by

$$
u_\varepsilon(t) := \chi_{\omega_\varepsilon(t)} u_\varepsilon^{\mathrm{aux}}(t), \tag{4.16}
$$

where ω_ε is as in Lemma 4.3. Note that, since $\eta_\varepsilon \to \infty$ and $h \in \mathrm{L}^\infty((0,1); \mathrm{W}^{2,\infty}(\Omega'; \mathbb{R}^2))$, we have that $\Omega' \setminus \overline{\Omega} \subseteq \omega_\varepsilon$ for ε sufficiently small independently of time. Thus, u_ε still satisfies the boundary conditions $u_\varepsilon = h_\varepsilon$ on $\Omega' \setminus \overline{\Omega}$. Moreover, as a consequence of convergence ($\omega 2$) and estimate (4.14), we obtain

$$
\mathcal{H}^1((J_{u_\varepsilon} \cup J_{\nabla u_\varepsilon}) \setminus (J_{y_\varepsilon} \cup J_{\nabla y_\varepsilon})) \to 0 \quad \text{uniformly in time as } \varepsilon \to 0. \tag{4.17}
$$

4.3. Compactness of the displacements

We use a diagonal argument to identify the target displacement $u(t)$ for $t \in I_\infty$ as a suitable limit of u_ε defined in (4.16); see also (4.11) and (4.12). Motivated by Theorem 3.1 (i), the function

$$v_\varepsilon(t) := u_\varepsilon(t) - \chi_{S_0^\varepsilon(t)}(u_\varepsilon(t) - h_\varepsilon(t)) - \sum_j a_j^\varepsilon(t)\chi_{S_j^\varepsilon(t)}, \tag{4.18}$$

for a suitable collection $\mathcal{S}_\varepsilon(t) := (S_j^\varepsilon(t))_{j\geq 0}$ of disjoint sets of finite perimeter and infinitesimal rigid motions $(a_j^\varepsilon(t))_{j\geq 1}$, is of special importance to us since by subtraction of a piecewise infinitesimal rigid motion pointwise a.e. convergence can be guaranteed. We also note that $v_\varepsilon(t) = h_\varepsilon(t)$ on $\Omega' \setminus \overline{\Omega}$ by construction. Using that the previous modifications barely increase the jump set, see inequality (4.17), and the compactness property Theorem 3.1 (iv), we see that, for each $t \in [0, 1]$,

$$\mathcal{H}^1(J_{v_\varepsilon(t)} \setminus (J_{y_\varepsilon(t)} \cup J_{\nabla y_\varepsilon(t)})) \quad \to 0 \quad \text{as } \varepsilon \to 0. \tag{4.19}$$

Remark 4.4 (Limit displacement I). We define an auxiliary limit displacement.

Limit for $t \in I_\infty$. We apply the compactness Theorem 3.1 to the sequence $(u_\varepsilon(t))_\varepsilon$ which yields for a subsequence (not relabeled, t-dependent) a sequence $(v_\varepsilon(t))_\varepsilon$ as in equation (4.18) and a function $u(t) \in \mathrm{GSBD}^2(\Omega')$ with $u(t) = h(t)$ on $\Omega' \setminus \overline{\Omega}$ such that

- (C1) $v_\varepsilon(t) \to u(t)$ pointwise almost everywhere in Ω',
- (C2) $e(u_\varepsilon(t)) \rightharpoonup e(u(t))$ weakly in $\mathrm{L}^2(\Omega'; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$,
- (C3) $\mathcal{H}^1(J_{u(t)} \cap U) \leq \liminf_{\varepsilon\to 0} \mathcal{H}^1(J_{u_\varepsilon(t)} \cap U)$ for all open Lipschitz subsets $U \subseteq \Omega'$.

Note that by additionally using the compactness properties (iv) and (v) from Theorem 3.1, we obtain that $v_\varepsilon(t)$ in place of $u_\varepsilon(t)$ also enjoys the properties (C2) and (C3).

Limit for $t \in [0, 1] \setminus I_\infty$. We point out that we can apply the same reasoning to find a limit displacement $\hat{u}(t)$ at any time $t \in [0, 1]$ such that $u_\varepsilon(t)$ converges to $\hat{u}(t)$ in the sense above. We denote this displacement by $\hat{u}$ to emphasize that a single subsequence in ε cannot be used for all t, given that $[0, 1]$ is uncountable, and there is no reason that $\hat{u}$ should be measurable in time. However, we define $\hat{u}(t) := u(t)$ for $t \in I_\infty$.

Finally, the function u is defined as follows.

Definition 4.5 (Limit displacement II and the evolving crack). We can finally define the limiting displacement u satisfying $u(t) = h(t)$ on $\Omega' \setminus \overline{\Omega}$ for all $t \in [0, 1]$.

Limit for $t \in I_\infty$. Let $u(t)$ be as in Remark 4.4 for $t \in I_\infty$. By a diagonalization argument, we can find a single subsequence in ε that satisfies (C1)–(C3) for all times $t \in I_\infty$.

Limit for $t \in [0, 1] \setminus I_\infty$. Take $(t_p)_{p\in\mathbb{N}} \subseteq I_\infty$ such that $t_p \uparrow t$. The L^2-norm of the symmetric gradients and the size of the jump sets of the sequence $(u(t_p))_p$ are uniformly bounded by the (not shown yet) lower semicontinuity (5.1) and the a priori estimates in Lemma 4.1. Thus, by the compactness Theorem 3.1, we can find a subsequence (not relabeled) and a function $u(t) \in \mathrm{GSBD}^2(\Omega')$ such that $u(t_p)$ converges to $u(t)$ in the sense of Theorem 3.1. Regularity of the boundary condition h in (2.2) ensures that $u(t) = h(t)$ on $\Omega' \setminus \overline{\Omega}$.

Crack set. Moreover, we introduce the irreversible crack at time $t \in [0, 1]$ as

$$\Gamma(t) := \bigcup_{\tau \in I^t_\infty} J_{u(\tau)}. \tag{4.20}$$

We introduce two notions for the limit displacement as each is amenable to different tools allowing us to recover different properties in the limit. We emphasize that the displacements u and $\hat{u}$ constructed above may not be unique or even measurable (in space-time) and, a priori, $u \neq \hat{u}$. However, in addition to minimality, Lemma 7.1 shows that the linear strain $e(u)$ is in fact measurable and is uniquely identified by the approximate subsequence u_ε. For this, we prove that the other way of obtaining the limit, namely the function $\hat{u}$ from Remark 4.4, gives rise to the same symmetric gradient. In fact, since $e(u) = e(\hat{u})$, we will be able to show that the limit $u(t) = \hat{u}(t)$ is uniquely determined on the good set $G(t)$ still attached to the boundary $\Omega' \setminus \overline{\Omega}$; see convergence (2.13) above.

5. Stability of minimality as $\varepsilon \to 0$

In this section we prove that u (and $\hat{u}$) found in Remark 4.4 and Definition 4.5 are minimizers of the Griffith energy, excluding any previously created crack.

5.1. Lower semicontinuity of the energies

Recall E defined in equation (2.11). In this subsection, our first goal is to show that for all $t \in I_\infty$ it holds that

$$\mathrm{E}(u(t)) \leq \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{\Omega'} W(\nabla y_\varepsilon(t)) \,\mathrm{d}x + \kappa \mathcal{H}^1(J_{y_\varepsilon(t)} \cup J_{\nabla y_\varepsilon(t)}), \tag{5.1}$$

where $u(t)$ is as in Definition 4.5.

For the proof of the lower semicontinuity (5.1), we use the same strategy as in the proof of the Γ-liminf inequality [35, Thm. 2.7] (differences arising as we do not "zero-out" regions that travel off to infinity). First, we note that on ω_ε (see Lemma 4.3), we have that $\varepsilon|\nabla u^{\mathrm{aux}}_\varepsilon| \leq \varepsilon\eta_\varepsilon \to 0$; see convergence ($\omega$3). As $\mathrm{D}W(\mathrm{Id}) = 0$ and as the tangent space of SO(2) at the identity is the space of skew-symmetric matrices, we have that $Q(A) = Q((A + A^\top)/2)$ (recall (2.10)). Remember the definition of u_ε in (4.16); see also (4.11)–(4.12). By applying a Taylor expansion in a ball of radius $\varepsilon\eta_\varepsilon$, we thus have on ω_ε via the rotational invariance of W that

$$W(\nabla y_\varepsilon) = W(\nabla y^{\mathrm{rot}}_\varepsilon) = W(\mathrm{Id} + \varepsilon \nabla u_\varepsilon) \geq \varepsilon^2 \frac{1}{2} Q(e(u_\varepsilon)) - C\varepsilon^3 \eta^3_\varepsilon.$$

By definition (4.16), the weak convergence of $e(u_\varepsilon)$ to $e(u)$ (see Definition 4.5 for $t \in I_\infty$), and the convexity of Q, it follows that (suppressing the dependence on $t \in I_\infty$)

$$\begin{aligned}\int_{\Omega'} \frac{1}{2} Q(e(u))\,\mathrm{d}x &\le \liminf_{\varepsilon\to 0} \int_{\Omega'} \frac{1}{2} Q(e(u_\varepsilon))\,\mathrm{d}x = \liminf_{\varepsilon\to 0} \int_{\omega_\varepsilon} \frac{1}{2} Q(e(u_\varepsilon^{\mathrm{aux}}))\,\mathrm{d}x \\ &\le \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \int_{\Omega'} W(\nabla y_\varepsilon)\,\mathrm{d}x + C\varepsilon\eta_\varepsilon^3. \end{aligned} \tag{5.2}$$

By convergence (ω4) the last summand vanishes. Combining the fact that the modifications barely increase the jump set by inequality (4.17) with the lower semicontinuity estimate in (C3) we get

$$\mathcal{H}^1(J_{u(t)} \cap U) \le \liminf_{\varepsilon\to 0} \mathcal{H}^1((J_{y_\varepsilon(t)} \cup J_{\nabla y_\varepsilon(t)}) \cap U) \tag{5.3}$$

for all open Lipschitz sets $U \subseteq \Omega'$. Now the estimates (5.2) and (5.3) yield the desired lower semicontinuity (5.1).

As in [31], we also argue that the total crack energy is lower semicontinuous, but we use a different argument here. Precisely, recalling the definition of $\Gamma(t)$ in (4.20), for all $t \in I_\infty$, it holds that

$$\mathcal{H}^1(\Gamma(t)) \le \liminf_{\varepsilon\to 0} \mathcal{H}^1(\Gamma_\varepsilon(t)) \le C. \tag{5.4}$$

The last inequality has already been established by the a priori estimate (4.1). Thus, we are left with showing the first inequality. First, we note that it suffices to show that, given any number of points $t_1, \dots, t_n \in I_\infty^t$, we have

$$\mathcal{H}^1\left(\bigcup_{k=1}^n J_{u(t_k)}\right) \le \liminf_{\varepsilon\to 0} \mathcal{H}^1(\Gamma_\varepsilon(t))$$

since we can then argue by continuity from below sending $n \to \infty$. Let such points be given. Then, given any $\epsilon > 0$, by a standard measure theory argument (Lemma A.4) we find open, pairwise disjoint Lipschitz sets $U_1, \dots, U_n$ such that

$$\begin{aligned}\mathcal{H}^1\left(\bigcup_{k=1}^n J_{u(t_k)}\right) - \epsilon &\le \sum_{k=1}^n \mathcal{H}^1(J_{u(t_k)} \cap U_k) \\ &\le \sum_{k=1}^n \liminf_{\varepsilon\to 0} \mathcal{H}^1((J_{y_\varepsilon(t_k)} \cup J_{\nabla y_\varepsilon(t_k)}) \cap U_k) \\ &\le \liminf_{\varepsilon\to 0} \mathcal{H}^1\left(\bigcup_{k=1}^n J_{y_\varepsilon(t_k)} \cup J_{\nabla y_\varepsilon(t_k)}\right) \\ &\le \liminf_{\varepsilon\to 0} \mathcal{H}^1(\Gamma_\varepsilon(t)). \end{aligned} \tag{5.5}$$

Here, the second inequality follows from inequality (5.3).

5.2. Almost minimality

The goal of this subsection is to prove that the functions u_ε are "almost" minimizers of the linearized functional, in the sense of the following lemma. An essential piece of this result is the quantification of the meaning of "almost" for a given test function, which is strong enough to pass to the limit in both the minimality property and an associated variational inequality, see Lemmas 5.4 and 6.1 below.

Lemma 5.1. *For any $t \in I_\infty$ and any $\phi \in \mathrm{GSBV}^2_2(\Omega';\mathbb{R}^2)$ with $\phi = 0$ on $\Omega' \setminus \overline{\Omega}$ and $|\nabla\phi| \in \mathrm{L}^3(\Omega')$, there exists a sequence $\rho_\varepsilon = \rho_\varepsilon(\phi) \to 0$ such that we have*

$$\int_{\Omega'} \frac{1}{2} Q\big(e(u_\varepsilon(t))\big)\,\mathrm{d}x - \rho_\varepsilon \leq \int_{\Omega'} \frac{1}{2} Q\big(e(u_\varepsilon(t)+\phi)\big)\,\mathrm{d}x + \kappa\mathcal{H}^1((J_\phi \cup J_{\nabla\phi}) \setminus \Gamma_\varepsilon(t)).$$

Moreover, there exist $C > 0$ and $\delta_\varepsilon \to 0$ independently of t and ϕ such that

$$\rho_\varepsilon \leq C\delta_\varepsilon\Big(1 + \int_{\Omega'} |\nabla\phi|^2\,\mathrm{d}x + \int_{\Omega'} |\nabla^2\phi|^2\,\mathrm{d}x\Big) + C\int_{\Omega'} \varepsilon|\nabla\phi|^3\,\mathrm{d}x. \tag{5.6}$$

Proof. The idea of the proof is to use the minimality property (2.6) for y_ε and to apply the linearization procedure, but to keep track of the terms we throw away. For notational simplicity, we suppress the dependence on t for y_ε and u_ε.

Take ε sufficiently small such that $t \in I_\varepsilon$ and let $z \in \mathrm{GSBV}^2_2(\Omega';\mathbb{R}^2)$ with $z = \mathrm{id} + \varepsilon h_\varepsilon$ on $\Omega' \setminus \overline{\Omega}$. Since y_ε is a minimizer of the functional (2.9), we have

$$\begin{aligned} &\frac{1}{\varepsilon^2}\int W(\nabla y_\varepsilon)\,\mathrm{d}x + \frac{1}{\varepsilon^{2\beta}}\int |\nabla^2 y_\varepsilon|^2\,\mathrm{d}x \\ &\quad\leq \frac{1}{\varepsilon^2}\int W(\nabla z)\,\mathrm{d}x + \frac{1}{\varepsilon^{2\beta}}\int |\nabla^2 z|^2\,\mathrm{d}x + \kappa\mathcal{H}^1((J_z \cup J_{\nabla z}) \setminus \Gamma_\varepsilon(t)). \end{aligned} \tag{5.7}$$

We first estimate the left-hand side of this inequality. As in inequality (5.2), we obtain via a Taylor expansion that

$$\frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla y_\varepsilon)\,\mathrm{d}x \geq \frac{1}{\varepsilon^2}\int_{\Omega'} \chi_{\omega_\varepsilon} W(\mathrm{Id} + \varepsilon u_\varepsilon)\,\mathrm{d}x \geq \int_{\Omega'} \frac{1}{2} Q(e(u_\varepsilon))\,\mathrm{d}x - C\varepsilon\eta_\varepsilon^3.$$

Therefore, by choosing $\delta_\varepsilon \geq \varepsilon\eta_\varepsilon^3$ in inequality (5.6) we conclude that

$$\begin{aligned} &\frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla y_\varepsilon)\,\mathrm{d}x + \frac{1}{\varepsilon^{2\beta}}\int_{\Omega'} |\nabla^2 y_\varepsilon|^2\,\mathrm{d}x + \rho_\varepsilon \\ &\quad\geq \int_{\Omega'} \frac{1}{2} Q(e(u_\varepsilon))\,\mathrm{d}x + \frac{1}{\varepsilon^{2\beta}}\int_{\Omega'} |\nabla^2 y_\varepsilon|^2\,\mathrm{d}x. \end{aligned} \tag{5.8}$$

Now let us take care of the right-hand side of inequality (5.7). For ϕ as in the statement, define the sequence $z_\varepsilon := \mathrm{id} + \varepsilon(u_\varepsilon + \phi)$. The function z_ε is an admissible competitor in (5.7). Below we will show that there exists a sequence $\rho_\varepsilon \to 0$ satisfying inequality (5.6) such that

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla z_\varepsilon)\,\mathrm{d}x + \frac{1}{\varepsilon^{2\beta}}\int_{\Omega'}|\nabla^2 z_\varepsilon|^2\,\mathrm{d}x + \kappa\mathcal{H}^1((J_{z_\varepsilon}\cup J_{\nabla z_\varepsilon})\setminus\Gamma_\varepsilon(t)) \qquad (5.9)\\
&\quad\le \frac{1}{2}\int_{\Omega'} Q(e(u_\varepsilon+\phi))\,\mathrm{d}x + \frac{1}{\varepsilon^{2\beta}}\int_{\Omega'}|\nabla^2 y_\varepsilon|^2\,\mathrm{d}x + \kappa\mathcal{H}^1((J_\phi\cup J_{\nabla\phi})\setminus\Gamma_\varepsilon(t)) + \rho_\varepsilon.
\end{aligned}$$

Once this is shown, combining this with inequality (5.8) and the minimization property (5.7), we obtain the desired result (for $2\rho_\varepsilon$ in place of ρ_ε).

Let us now come to the proof of inequality (5.9). Similarly to the a priori estimate in Lemma 4.1, we want to perform a Taylor expansion on $W(\nabla z_\varepsilon)$ in a neighborhood of the identity and use the growth conditions of W away from the identity. Let $r\in(0,1)$ be such that W is C^3 on $B_{2r}(\mathrm{Id})$. Moreover, recall from the choice of the cutoff (see (ω3)) that $\varepsilon|\nabla u_\varepsilon|\to 0$ uniformly. In particular, this means that for ε sufficiently small, $\varepsilon|\nabla\phi|<r$ already implies $\varepsilon|\nabla(u_\varepsilon+\phi)|<2r$. We start by estimating the integral on $\{\varepsilon|\nabla\phi|\le r\}$ using a Taylor expansion. This yields

$$\begin{aligned}
\frac{1}{\varepsilon^2}\int_{\{\varepsilon|\nabla\phi|\le r\}} W(\nabla z_\varepsilon)\,\mathrm{d}x &\le \int_{\Omega'}\frac{1}{2}Q(e(u_\varepsilon+\phi))\,\mathrm{d}x + C\varepsilon\int_{\Omega'}|\nabla(u_\varepsilon+\phi)|^3\,\mathrm{d}x\\
&\le \int_{\Omega'}\frac{1}{2}Q(e(u_\varepsilon+\phi))\,\mathrm{d}x + C\varepsilon\eta_\varepsilon^3 + C\varepsilon\int_{\Omega'}|\nabla\phi|^3\,\mathrm{d}x.
\end{aligned}$$

The last two terms can be absorbed into ρ_ε while satisfying (5.6), due to the choice of the cutoff parameter η_ε in (ω3), if we choose $\delta_\varepsilon\ge\varepsilon\eta_\varepsilon^3$.

Next, we estimate away from the identity. By the local Lipschitz continuity of W, see (W1), Young's inequality, and the fact that $|\nabla u_\varepsilon|\lesssim\eta_\varepsilon$, which implies $\varepsilon|\nabla(u_\varepsilon+\phi)|\gtrsim 1$ on $\{\varepsilon|\nabla\phi|\ge r\}$ for ε sufficiently small, we have

$$\begin{aligned}
\frac{1}{\varepsilon^2}\int_{\{\varepsilon|\nabla\phi|\ge r\}} W(\nabla z_\varepsilon)\,\mathrm{d}x &= \frac{1}{\varepsilon^2}\int_{\{\varepsilon|\nabla\phi|\ge r\}} W(\mathrm{Id}+\varepsilon\nabla(u_\varepsilon+\phi))\,\mathrm{d}x\\
&\lesssim \frac{1}{\varepsilon^2}\int_{\{\varepsilon|\nabla\phi|\ge r\}}(1+\varepsilon|\nabla(u_\varepsilon+\phi)|)\varepsilon|\nabla(u_\varepsilon+\phi)|\,\mathrm{d}x\\
&\lesssim \int_{\{\varepsilon|\nabla\phi|\ge r\}}|\nabla u_\varepsilon|^2\,\mathrm{d}x + \int_{\{\varepsilon|\nabla\phi|\ge r\}}|\nabla\phi|^2\,\mathrm{d}x\\
&\lesssim \frac{\eta_\varepsilon^2\varepsilon^2}{r^2}\int_{\Omega'}|\nabla\phi|^2\,\mathrm{d}x + \frac{\varepsilon}{r}\int_{\Omega'}|\nabla\phi|^3\,\mathrm{d}x. \qquad (5.10)
\end{aligned}$$

By convergence (ω3), the term $\eta_\varepsilon^2\varepsilon^2$ vanishes as ε tends to zero. Thus, by choosing $\delta_\varepsilon\ge\eta_\eta^2\varepsilon^2$, the term (5.10) can be absorbed into ρ_ε; see inequality (5.6).

Now let us take care of the second derivative term $\nabla^2 z_\varepsilon$. We introduce a parameter $\varepsilon^{1-2\beta}\ll\theta_\varepsilon\ll\varepsilon^{-1}$. Then, by expanding the square and by applying Young's inequality to the resulting product, we have

$$\begin{aligned}\frac{1}{\varepsilon^{2\beta}}\int_{\Omega'}|\nabla^2 z_\varepsilon|^2\,\mathrm{d}x &= \frac{\varepsilon^2}{\varepsilon^{2\beta}}\int_{\Omega'}|\nabla^2(u_\varepsilon+\phi)|^2\,\mathrm{d}x\\ &\le \frac{1}{\varepsilon^{2\beta}}\int_{\Omega'}|\nabla^2 y_\varepsilon|^2\,\mathrm{d}x + \frac{\theta_\varepsilon\varepsilon}{\varepsilon^{2\beta}}\int_{\Omega'}|\nabla^2 y_\varepsilon|^2\,\mathrm{d}x\\ &\quad+\Big(\frac{\varepsilon^2}{\varepsilon^{2\beta}}+\frac{\varepsilon}{\theta_\varepsilon\varepsilon^{2\beta}}\Big)\int_{\Omega'}|\nabla^2\phi|^2\,\mathrm{d}x,\end{aligned}$$

where we also used that $\varepsilon|\nabla^2 u_\varepsilon|\le|\nabla^2 y_\varepsilon|$ by the modifications (4.11), (4.12), and (4.16). By choice of θ_ε and using $\beta<1$, the last summand vanishes in the limit. Moreover, since $\varepsilon^{-2\beta}\int_{\Omega'}|\nabla^2 y_\varepsilon|^2\,\mathrm{d}x$ stays uniformly bounded by Lemma 4.1 and $\theta_\varepsilon\varepsilon\to 0$, the second summand vanishes in the limit. Both can be absorbed into ρ_ε as in inequality (5.6) if we choose

$$\delta_\varepsilon \ge \theta_\varepsilon\varepsilon\Big(\sup_{t\in[0,1]}\frac{1}{\varepsilon^{2\beta}}\int_{\Omega'}|\nabla^2 y_\varepsilon(t)|^2\,\mathrm{d}x\Big)+\Big(\frac{\varepsilon^2}{\varepsilon^{2\beta}}+\frac{\varepsilon}{\theta_\varepsilon\varepsilon^{2\beta}}\Big).$$

Lastly, we take care of the jump set by estimating

$$\begin{aligned}&\mathcal{H}^1((J_{z_\varepsilon}\cup J_{\nabla z_\varepsilon})\setminus\Gamma_\varepsilon(t))\\ &\qquad\le\mathcal{H}^1((J_\phi\cup J_{\nabla\phi})\setminus\Gamma_\varepsilon(t))+\mathcal{H}^1((J_{u_\varepsilon}\cup J_{\nabla u_\varepsilon})\setminus\Gamma_\varepsilon(t)).\end{aligned}\tag{5.11}$$

The last term can again be absorbed into ρ_ε since modifications are barely increasing the jump set; see inequality (4.17) and definition (2.8). We collect inequalities (5.10)–(5.11) to deduce inequality (5.9). This concludes the proof. ∎

Remark 5.2. Let $t\in I_\infty$. In the case that the competitor is of the form $\psi\in\mathrm{GSBV}^2_2(\Omega';\mathbb{R}^2)$ with $\psi=h_\varepsilon(t)$ on $\Omega'\setminus\overline{\Omega}$ and $|\nabla\psi|\in L^3(\Omega')$, we immediately deduce from the proof that we instead get the inequality

$$\int_{\Omega'}\frac{1}{2}Q\big(e(u_\varepsilon(t))\big)\,\mathrm{d}x-\rho_\varepsilon\le\int_{\Omega'}\frac{1}{2}Q(e(\psi))\,\mathrm{d}x+\kappa\mathcal{H}^1((J_\psi\cup J_{\nabla\psi})\setminus\Gamma_\varepsilon(t)),$$

where the sequence ρ_ε can be bounded for some $C>0$ and $\delta_\varepsilon\to 0$ independently of t and ψ by

$$\rho_\varepsilon\le C\delta_\varepsilon+C\varepsilon^{2(1-\beta)}\int_{\Omega'}|\nabla^2\psi|^2\,\mathrm{d}x+C\int_{\Omega'}\varepsilon|\nabla\psi|^3\,\mathrm{d}x.\tag{5.12}$$

Since we have an approximate minimization property, we also have an approximate variational inequality, which we will need later for the energy balance.

Corollary 5.3. *There exists a constant $C>0$ such that for every $t\in[0,1]$ and every $\phi\in\mathrm{GSBV}^2_2(\Omega';\mathbb{R}^2)$ with $\phi=0$ on $\Omega'\setminus\overline{\Omega}$, $\mathcal{H}^1(J_{\nabla\phi}\setminus J_\phi)=0$, and $|\nabla\phi|\in L^3(\Omega')$, we have*

$$\Big|\int_{\Omega'}\mathbb{C}e(u_\varepsilon(t)):e(\phi)\,\mathrm{d}x\Big|^2\le C\max\Big\{1,\int_{\Omega'}\frac{1}{2}Q(e(\phi))\,\mathrm{d}x\Big\}(\kappa\mathcal{H}^1(J_\phi\setminus\Gamma_\varepsilon(t))+\rho_\varepsilon(\phi)),$$

where $(\rho_\varepsilon(\phi))_\varepsilon$ denotes the sequence depending on ϕ given in (5.6) satisfying $\rho_\varepsilon(\phi)\to 0$.

Proof. For the sake of brevity, we write $H_\varepsilon := \kappa\mathcal{H}^1(J_\phi \setminus \Gamma_\varepsilon(t))$. If

$$\int_{\Omega'} \frac{1}{2} Q(e(\phi))\,\mathrm{d}x \le H_\varepsilon + \rho_\varepsilon(\phi), \tag{5.13}$$

the desired estimate follows from Hölder's inequality and the uniform energy bound (4.13), where the constant depends on $\mathbb{C}$ (see (2.10)).

Otherwise, for every parameter $1 > \theta > 0$, the function $\theta\phi$ is an admissible competitor for Lemma 5.1. Thus,

$$\begin{aligned} 0 &\le \frac{1}{\theta}\Big(\int_{\Omega'} \frac{1}{2} Q\big(e(u_\varepsilon(t) + \theta\phi)\big)\,\mathrm{d}x - \int_{\Omega'} \frac{1}{2} Q\big(e(u_\varepsilon(t))\big)\,\mathrm{d}x + H_\varepsilon + \rho_\varepsilon(\theta\phi)\Big) \\ &= \int_{\Omega'} \mathbb{C}e(u_\varepsilon(t)) : e(\phi)\,\mathrm{d}x + \theta \int_{\Omega'} \frac{1}{2} Q(e(\phi))\,\mathrm{d}x + \frac{1}{\theta}(H_\varepsilon + \rho_\varepsilon(\theta\phi)). \end{aligned}$$

We choose

$$\theta_0 = (H_\varepsilon + \rho_\varepsilon(\phi))^{1/2}\Big(\int_{\Omega'} \frac{1}{2} Q(e(\phi))\,\mathrm{d}x\Big)^{-1/2}$$

and note that $\theta_0 < 1$ since we ruled out inequality (5.13). Plugging this into the above inequality yields

$$\begin{aligned} &- \int_{\Omega'} \mathbb{C}e(u_\varepsilon(t)) : e(\phi)\,\mathrm{d}x \\ &\le (H_\varepsilon + \rho_\varepsilon(\phi))^{1/2}\Big(\int_{\Omega'} \frac{1}{2} Q(e(\phi))\,\mathrm{d}x\Big)^{1/2} + \frac{H_\varepsilon + \rho_\varepsilon(\theta_0\phi)}{(H_\varepsilon + \rho_\varepsilon(\phi))^{1/2}}\Big(\int_{\Omega'} \frac{1}{2} Q(e(\phi))\Big)^{1/2}. \end{aligned}$$

By observing inequality (5.6), we note that, since $\theta_0 < 1$, we have $\rho_\varepsilon(\theta_0\phi) \le \rho_\varepsilon(\phi)$, and thus we obtain

$$- \int_{\Omega'} \mathbb{C}e(u_\varepsilon(t)) : e(\phi)\,\mathrm{d}x \le 2(H_\varepsilon + \rho_\varepsilon(\phi))^{1/2}\Big(\int_{\Omega'} \frac{1}{2} Q(e(\phi))\,\mathrm{d}x\Big)^{1/2}.$$

Repeating the argument with $-\phi$ then yields the desired inequality. ∎

5.3. Transfer of minimality

We now want to pass to the limit $\varepsilon \to 0$ in Lemma 5.1 to argue that the auxiliary limit $\hat{u}(t)$ given by Remark 4.4 is a minimizer of the Griffith energy accounting for the existing crack, in analogy to [31, Lem. 3.3]. The proof is similar, but since we are not in the antiplanar case, we require the jump transfer lemma for GSBD given in Theorem 3.3 (a refinement of [41, Thm. 5.1]).

Lemma 5.4. *For each $t \in [0, 1]$, the displacement $\hat{u}(t)$ from Remark 4.4 minimizes*

$$\int_{\Omega'} \frac{1}{2} Q(e(v))\,\mathrm{d}x + \kappa\mathcal{H}^1(J_v \setminus (\Gamma(t) \cup J_{\hat{u}(t)})) \tag{5.14}$$

among all $v \in \mathrm{GSBD}^2(\Omega')$ with $v = h(t)$ on $\Omega' \setminus \overline{\Omega}$.

We point out that, for $t \in I_\infty$, $J_{\hat{u}(t)}$ is redundant in (5.14) as $\hat{u}(t) = u(t)$. Naturally, this lemma will be critical for verifying the minimality property (iv) of Definition 2.1, but further, the analysis at times $t \notin I_\infty$ will help to correctly identify the displacement gradient $e(u) \in \mathrm{L}^\infty((0,1); \mathrm{L}^2(\Omega; \mathbb{R}^{2\times 2}_{\mathrm{sym}}))$.

Proof of Lemma 5.4. Let $t \in [0,1]$. By the density result in Theorem 3.2 (applied for $v - \hat{u}(t)$), it suffices to prove the minimality of $\hat{u}(t)$ with respect to competitors $\hat{u}(t) + \phi$ for $\phi \in \mathcal{W}(\Omega'; \mathbb{R}^2)$ with $\phi = 0$ on $\Omega' \setminus \overline{\Omega}$.

We begin by approximating the crack $\Gamma(t)$ with finitely many jump sets. Precisely, letting $\eta > 0$ and recalling that the total crack has finite length, see (5.4), we can find $t_1, \dots, t_p \in I^t_\infty$ such that

$$\mathcal{H}^1\Big(\bigcup_{k=1}^{p} J_{u(t_k)} \cup J_{\hat{u}(t)}\Big) \geq \mathcal{H}^1(\Gamma(t) \cup J_{\hat{u}(t)}) - \eta. \tag{5.15}$$

Choose $t_\varepsilon \in I_\varepsilon$ as the largest element of I_ε which is smaller than t. As u_ε and y_ε are piecewise constant in time, we find $u_\varepsilon(t) = u_\varepsilon(t_\varepsilon)$ and $y_\varepsilon(t) = y_\varepsilon(t_\varepsilon)$. Then, if ε is sufficiently small so that $(t_k)_{k=1}^p \subseteq I_\varepsilon$, by Lemma 5.1 (applied at time $t_\varepsilon \in I_\infty$) we have for all $\phi \in \mathrm{GSBV}^2_2(\Omega'; \mathbb{R}^2)$ with $\phi = 0$ on $\Omega' \setminus \overline{\Omega}$ and $|\nabla\phi| \in \mathrm{L}^3(\Omega')$ that

$$\begin{aligned}
&\int_{\Omega'} \frac{1}{2} Q\big(e(u_\varepsilon(t))\big)\,\mathrm{d}x - \rho_\varepsilon \\
&\quad\leq \int_{\Omega'} \frac{1}{2} Q\big(e(u_\varepsilon(t) + \phi)\big)\,\mathrm{d}x \\
&\qquad + \kappa\mathcal{H}^1\Bigg((J_\phi \cup J_{\nabla\phi}) \setminus \Bigg(\bigcup_{k=1}^{p} (J_{y_\varepsilon(t_k)} \cup J_{\nabla y_\varepsilon(t_k)}) \cup (J_{y_\varepsilon(t)} \cup J_{\nabla y_\varepsilon(t)})\Bigg)\Bigg),
\end{aligned} \tag{5.16}$$

where $\rho_\varepsilon = \rho_\varepsilon(\phi)$ satisfies (5.6).

Now fix $\phi \in \mathcal{W}(\Omega'; \mathbb{R}^2)$ with $\phi = 0$ on $\Omega' \setminus \overline{\Omega}$. Since for every $1 \leq k \leq p$ the sequence $v_\varepsilon(t_k)$ (defined in equation (4.18)) converges in measure to $u(t_k)$, has uniformly bounded jump-set measure, and has uniformly L^2-bounded symmetric gradients, and the same holds true for $v_\varepsilon(t)$ with respect to $\hat{u}(t)$, we can apply the GSBD jump transfer lemma (see Theorem 3.3) to ϕ to find a sequence $(\phi_\varepsilon)_\varepsilon \subseteq \mathrm{GSBV}^2_2(\Omega'; \mathbb{R}^2)$ such that $\phi_\varepsilon = 0$ on $\Omega' \setminus \overline{\Omega}$ and

(i) $\phi_\varepsilon \to \phi$ in measure,

(ii) $e(\phi_\varepsilon) \to e(\phi)$ strongly in $\mathrm{L}^2(\Omega'; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$,

(iii) $\limsup_{\varepsilon\to 0} \mathcal{H}^1(((J_{\phi_\varepsilon} \cup J_{\nabla\phi_\varepsilon}) \setminus (\bigcup_{k=1}^p J_{v_\varepsilon(t_k)} \cup J_{v_\varepsilon(t)})))$
$\leq \mathcal{H}^1((J_\phi \setminus (\bigcup_{k=1}^p J_{u(t_k)} \cup J_{\hat{u}(t)}))),$

(iv) $\|\nabla\phi_\varepsilon\|_{\mathrm{L}^\infty} \leq C\|\nabla\phi\|_{\mathrm{L}^\infty}$ and $\|\nabla^2\phi_\varepsilon\|_{\mathrm{L}^\infty} \leq C\|\nabla^2\phi\|_{\mathrm{L}^\infty}$.

Since ϕ_ε is an admissible competitor within inequality (5.16), we find

$$\begin{aligned}
&\int_{\Omega'} \frac{1}{2} Q\big(e(u_\varepsilon(t))\big)\,\mathrm{d}x - \rho_\varepsilon \\
&\quad\leq \int_{\Omega'} \frac{1}{2} Q\big(e(u_\varepsilon(t)+\phi_\varepsilon)\big)\,\mathrm{d}x \\
&\qquad + \kappa \mathcal{H}^1\bigg((J_{\phi_\varepsilon} \cup J_{\nabla\phi_\varepsilon}) \setminus \Big(\bigcup_{k=1}^{p} (J_{y_\varepsilon(t_k)} \cup J_{\nabla y_\varepsilon(t_k)}) \cup (J_{y_\varepsilon(t)} \cup J_{\nabla y_\varepsilon(t)})\Big)\bigg), \quad (5.17)
\end{aligned}$$

where the error $\rho_\varepsilon = \rho_\varepsilon(\phi_\varepsilon)$ is estimated by inequality (5.6) with a vanishing sequence $\delta_\varepsilon \to 0$ and a constant $C > 0$ as

$$\rho_\varepsilon \leq C\delta_\varepsilon\Big(1 + \int_{\Omega'} |\nabla\phi_\varepsilon|^2\,\mathrm{d}x + \int_{\Omega'} |\nabla^2\phi_\varepsilon|^2\,\mathrm{d}x\Big) + C\int_{\Omega'} \varepsilon|\nabla\phi_\varepsilon|^3\,\mathrm{d}x.$$

By the uniform bound on $\|\nabla\phi_\varepsilon\|_{L^\infty}$ and $\|\nabla^2\phi_\varepsilon\|_{L^\infty}$ given in item (iv), we see that ρ_ε vanishes as ε tends to zero. By subtracting $\int_{\Omega'} \frac{1}{2} Q(e(u_\varepsilon(t)))\,\mathrm{d}x$ on both sides of inequality (5.17), we get

$$\begin{aligned}
-\rho_\varepsilon \leq &\int_{\Omega'} \mathbb{C}e(u_\varepsilon(t)) : e(\phi_\varepsilon)\,\mathrm{d}x + \int_{\Omega'} \frac{1}{2} Q(e(\phi_\varepsilon))\,\mathrm{d}x \qquad (5.18)\\
&+ \kappa\mathcal{H}^1\bigg((J_{\phi_\varepsilon} \cup J_{\nabla\phi_\varepsilon}) \setminus \Big(\bigcup_{k=1}^{p} J_{v_\varepsilon(t_k)} \cup J_{v_\varepsilon(t)}\Big)\bigg) \\
&+ \sum_{k=1}^{p} \kappa\mathcal{H}^1(J_{v_\varepsilon(t_k)} \setminus (J_{y_\varepsilon(t_k)} \cup J_{\nabla y_\varepsilon(t_k)})) + \kappa\mathcal{H}^1(J_{v_\varepsilon(t)} \setminus (J_{y_\varepsilon(t)} \cup J_{\nabla y_\varepsilon(t)})).
\end{aligned}$$

The last line converges to zero as $\varepsilon \to 0$ using that the modifications barely increase the jump set as determined by equation (4.19). Now passing $\varepsilon \to 0$ in inequality (5.18), we can pair the weak convergence of $e(u_\varepsilon)$ with the strong convergence of $e(\phi_\varepsilon)$ (see item (ii)) and use the jump set estimate (iii) to obtain

$$0 \leq \int_{\Omega'} \mathbb{C}e(\hat{u}(t)) : e(\phi)\,\mathrm{d}x + \int_{\Omega'} \frac{1}{2} Q(e(\phi))\,\mathrm{d}x + \kappa\mathcal{H}^1\bigg(J_\phi \setminus \Big(\bigcup_{k=1}^{p} J_{u(t_k)} \cup J_{\hat{u}(t)}\Big)\bigg).$$

We add $\int_{\Omega'} \frac{1}{2} Q(e(\hat{u}(t)))\,\mathrm{d}x$ on both sides of the inequality and note that $J_\phi \setminus J_{\hat{u}(t)} = J_{\hat{u}(t)+\phi} \setminus J_{\hat{u}(t)}$, which yields the desired inequality (5.14) for $v = \hat{u}(t) + \phi$ up to an arbitrary $\eta > 0$ coming from the approximation (5.15). Sending $\eta \to 0$ concludes the minimality property. ∎

5.4. Elastic energy convergence

As a consequence of Lemma 5.4, we have established that $\hat{u}(t)$ is a minimizer with respect to its own jump set in the linearized functional, i.e., one may replace $\Gamma(t) \cup J_{\hat{u}(t)}$ by $J_{\hat{u}(t)}$

in the minimization problem (5.14). In a similar fashion, the functions y_ε are minimizers with respect to their own jump set of the nonlinear functional; cf. (2.9). This minimality property now induces convergence of the elastic energies. Whereas for Griffith energies this simply follows from [35] along with the fundamental theorem of Γ-convergence, in the present evolutionary setting with irreversibility condition, this is a consequence of the jump transfer lemma (see Theorem 3.3).

Corollary 5.5 (Elastic energy convergence). *For all $t \in [0,1]$, we have the convergence of elastic energies*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla y_\varepsilon(t))\,\mathrm{d}x + \frac{1}{\varepsilon^{2\beta}}\int_{\Omega'} |\nabla^2 y_\varepsilon(t)|^2\,\mathrm{d}x = \int_{\Omega'} \frac{1}{2} Q\big(e(\hat{u}(t))\big)\,\mathrm{d}x, \tag{5.19}$$

again along a t-dependent subsequence for $t \notin I_\infty$. In particular, we have

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{2\beta}}\int_{\Omega'} |\nabla^2 y_\varepsilon(t)|^2\,\mathrm{d}x = 0. \tag{5.20}$$

Proof. First, from lower semicontinuity, see inequality (5.2), we obtain the estimate

$$\begin{aligned}\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla y_\varepsilon(t))\,\mathrm{d}x + \frac{1}{\varepsilon^{2\beta}}\int_{\Omega'} |\nabla^2 y_\varepsilon(t)|^2\,\mathrm{d}x &\ge \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla y_\varepsilon(t))\,\mathrm{d}x \\ &\ge \int_{\Omega'} \frac{1}{2} Q\big(e(\hat{u}(t))\big)\,\mathrm{d}x. \end{aligned}\tag{5.21}$$

(Note that the proof was only formulated for $t \in I_\infty$, but still applies to $t \notin I_\infty$.) Since $y_\varepsilon(t)$ is a minimizer of the functional (2.9), for the reverse limit superior inequality it suffices to find a recovery sequence $z_\varepsilon \in \mathrm{GSBV}^2_2(\Omega';\mathbb{R}^2)$ with $z_\varepsilon = \mathrm{id} + \varepsilon h_\varepsilon(t)$ on $\Omega' \setminus \overline{\Omega}$ such that

$$\begin{aligned}&\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla z_\varepsilon)\,\mathrm{d}x + \frac{1}{\varepsilon^{2\beta}}\int_{\Omega'} |\nabla^2 z_\varepsilon|^2\,\mathrm{d}x + \kappa\mathcal{H}^1((J_{z_\varepsilon} \cup J_{\nabla z_\varepsilon}) \setminus \Gamma_\varepsilon(t)) \\ &\quad \le \int_{\Omega'} \frac{1}{2} Q(e(\hat{u}(t)))\,\mathrm{d}x. \end{aligned}\tag{5.22}$$

To find this sequence, we approximate $\hat{u}(t)$ and apply the jump transfer lemma (Theorem 3.3) to the approximations. Precisely, by Theorem 3.2 we find a sequence $(w_n)_n \subseteq \mathcal{W}(\Omega';\mathbb{R}^2)$ with $w_n = h(t)$ on $\Omega' \setminus \overline{\Omega}$ such that, as $n \to \infty$, we have

(1) $w_n \to \hat{u}(t)$ in measure,

(2) $e(w_n) \to e(\hat{u}(t))$ in $\mathrm{L}^2(\Omega';\mathbb{R}^{2\times 2}_{\mathrm{sym}})$, and

(3) $\mathcal{H}^1(J_{w_n} \triangle J_{\hat{u}(t)}) \to 0$.

Observe that the sequence $v_\varepsilon(t)$ defined in equation (4.18) converges in measure to $\hat{u}(t)$, has uniformly bounded jump-set measure, and has uniformly L^2-bounded symmetric gradients. For all $n \in \mathbb{N}$, we apply the jump transfer lemma (Theorem 3.3) to the function $\phi = w_n$ which yields a sequence $(w^n_\varepsilon)_\varepsilon \subseteq \mathrm{GSBV}^2_2(\Omega';\mathbb{R}^2)$ with $w^n_\varepsilon = h(t)$ on $\Omega' \setminus \overline{\Omega}$ such that

(1′) $e(w_\varepsilon^n) \to e(w_n)$ in $\mathrm{L}^2(\Omega';\mathbb{R}^{2\times 2}_{\mathrm{sym}})$,

(2′) $\limsup_{\varepsilon\to 0} \mathcal{H}^1((J_{w_\varepsilon^n} \cup J_{\nabla w_\varepsilon^n}) \setminus J_{v_\varepsilon(t)}) \le \mathcal{H}^1(J_{w_n} \setminus J_{\hat u(t)})$,

(3′) $\|\nabla w_\varepsilon^n\|_{\mathrm{L}^\infty} \lesssim \|\nabla w_n\|_{\mathrm{L}^\infty}$ and $\|\nabla^2 w_\varepsilon^n\|_{\mathrm{L}^\infty} \lesssim \|\nabla^2 w_n\|_{\mathrm{L}^\infty}$.

We define $z_\varepsilon^n := \mathrm{id} + \varepsilon(w_\varepsilon^n + h_\varepsilon(t) - h(t))$. Then $z_\varepsilon^n = \mathrm{id} + \varepsilon h_\varepsilon(t)$ on $\Omega' \setminus \overline{\Omega}$. Moreover, using the boundedness of w_ε^n and h given by (3′) and (2.2), respectively, the comparison of the jump sets (2′) along with (4.19), $\beta < 1$, and $h_\varepsilon(t) \to h(t)$ in $\mathrm{W}^{1,2}(\Omega';\mathbb{R}^2)$, we obtain through a Taylor expansion,

$$\begin{aligned}\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla z_\varepsilon^n)\,\mathrm{d}x + \frac{1}{\varepsilon^{2\beta}}\int_{\Omega'} |\nabla^2 z_\varepsilon^n|^2\,\mathrm{d}x + \kappa\mathcal{H}^1((J_{z_\varepsilon^n} \cup J_{\nabla z_\varepsilon^n}) \setminus \Gamma_\varepsilon(t))\\ \le \int_{\Omega'} \frac{1}{2} Q(e(w_n))\,\mathrm{d}x + \kappa\mathcal{H}^1(J_{w_n} \setminus J_{\hat u(t)}).\qquad (5.23)\end{aligned}$$

By the approximation of $\hat u(t)$ through w_n (see items (2) and (3)), the limit superior in n of the right-hand side of inequality (5.23) can be estimated by

$$\limsup_{n\to\infty} \int_{\Omega'} \frac{1}{2} Q(e(w_n))\,\mathrm{d}x + \kappa\mathcal{H}^1(J_{w_n} \setminus J_{\hat u(t)}) \le \int_{\Omega'} \frac{1}{2} Q\big(e(\hat u(t))\big)\,\mathrm{d}x. \qquad (5.24)$$

By a diagonal argument and by combining inequalities (5.23) and (5.24) we get the existence of a sequence $n(\varepsilon) \to \infty$ such that $z_\varepsilon^{n(\varepsilon)}$ satisfies (5.22). This finishes the proof of the energy convergence (5.19).

To show the convergence (5.20), we simply note that by the elastic energy convergence (5.19) and inequality (5.21) the contribution of the second-derivative integral must vanish. ∎

Corollary 5.6. *For each $t \in [0, 1]$, we have that $e(u_\varepsilon(t))$ and $e(u_\varepsilon^{\mathrm{aux}}(t))$ converge to $e(\hat u(t))$ in $\mathrm{L}^2(\Omega';\mathbb{R}^{2\times 2}_{\mathrm{sym}})$ (up to a t-dependent subsequence in ε for $t \notin I_\infty$).*

Proof. Using the convergence of elastic energies in Corollary 5.5, one can argue precisely as in the proof of [3, Thm. 3.11] to conclude that $e(u_\varepsilon^{\mathrm{aux}}(t)) \to e(\hat u(t))$ in $\mathrm{L}^2(\Omega';\mathbb{R}^{2\times 2}_{\mathrm{sym}})$. As $\chi_{\Omega'\setminus\omega_\varepsilon} \to 0$ by convergence (ω1), it also follows that $e(u_\varepsilon(t)) \to e(\hat u(t))$ in $\mathrm{L}^2(\Omega;\mathbb{R}^{2\times 2}_{\mathrm{sym}})$. ∎

6. Approximate energy balance

We begin the preliminary steps of the proof of the energy-balance equation of (v) in Definition 2.1. To this end, recall the definition of the total energy in (2.12). To derive the energy-balance equation, we make rigorous the idea that once a piece of material is disconnected from the Dirichlet boundary, it will relax to an undeformed equilibrium configuration. Precisely, from the approximate variational inequality in Corollary 5.3 we will deduce that the partition elements P_j^ε which do not intersect the Dirichlet boundary do not contribute to the energy-balance equation in the limit, in the sense of the following lemma.

Lemma 6.1. *For $t \in [0,1]$, let $(P_j^\varepsilon(t))_j$ be the Caccioppoli partition from equation (4.11) and let $(\widehat{R}_j^\varepsilon(t))_j \subseteq \mathrm{SO}(2)$ be any associated collection of matrices that is piecewise constant in time with respect to the same partition of $[0,1]$. For all $t \in [0,1]$, we have*

$$\lim_{\varepsilon\to 0}\left|\int_0^t \sum_{P_j^\varepsilon(s)\in\mathcal{P}_\varepsilon^{\mathrm{int}}(s)} \int_{P_j^\varepsilon(s)} \mathbb{C}e(u_\varepsilon(s)) : e(\widehat{R}_j^\varepsilon(s)\partial_t h(s))\,\mathrm{d}x\,\mathrm{d}s\right| = 0, \tag{6.1}$$

where $\mathcal{P}_\varepsilon^{\mathrm{int}}(s) := \{P_j^\varepsilon(s) : \mathcal{L}^2(P_j^\varepsilon(s)\cap(\Omega'\setminus\overline{\Omega})) = 0\}$ are the interior partition pieces for all $s \in [0,1]$.

As will be seen in the proof, the only constraint on the test function

$$\sum_{P_j^\varepsilon(s)\in\mathcal{P}_\varepsilon^{\mathrm{int}}(s)} \chi_{P_j^\varepsilon(s)}\widehat{R}_j^\varepsilon(s)\partial_t h(s)$$

above is that it is piecewise regular with respect to the partition and 0 on the boundary. Formally, this result shows that for the limit Caccioppoli partition $\mathcal{P}$ with $P_j \in \mathcal{P}^{\mathrm{int}}$, defined as above, and a regular test function ϕ, one has $\int_{P_j} \mathbb{C}e(u) : e(\phi)\,\mathrm{d}x = 0$, which implies that u is an infinitesimal rigid motion on P_j, meaning the material has fully relaxed to an equilibrium configuration on P_j.

Proof of Lemma 6.1. We know by (2.2) that $h \in \mathrm{W}^{1,1}((0,1);\mathrm{W}^{2,2}(\Omega';\mathbb{R}^2))$. Since $e(u_\varepsilon)$ is uniformly L^2-bounded (see estimate (4.13) and (4.16)), via an approximation it thus suffices to prove the claim under the assumption $h \in \mathrm{C}^\infty([0,1];\mathrm{W}^{2,2}(\Omega';\mathbb{R}^2))$ (note that this is just an assumption on the regularity of h explicitly appearing in (6.1), not the boundary data).

Let $s \in [0,1]$. We apply the approximate variational inequality in Corollary 5.3 to the function

$$\phi_\varepsilon(s) = \sum_{P_j^\varepsilon(s)\in\mathcal{P}_\varepsilon^{\mathrm{int}}(s)} \chi_{P_j^\varepsilon(s)}\widehat{R}_j^\varepsilon(s)\partial_t h(s)$$

to get that the term on the left-hand side of equality (6.1) can be estimated by the limit of

$$C\int_0^t \left(\max\left\{1, \int_{\Omega'}\frac{1}{2}Q(e(\phi_\varepsilon))\,\mathrm{d}x\right\}\left(\kappa\mathcal{H}^1(J_{\phi_\varepsilon}\setminus\Gamma_\varepsilon(t)) + \rho_\varepsilon(\phi_\varepsilon)\right)\right)^{1/2}\mathrm{d}s.$$

Since $h \in \mathrm{C}^\infty([0,1];\mathrm{W}^{2,2}(\Omega';\mathbb{R}^2))$, the term $\int_{\Omega'} Q(e(\phi_\varepsilon))\,\mathrm{d}x$ stays uniformly bounded in time. We have $J_{\phi_\varepsilon(s)} \subseteq \bigcup_j \partial^* P_j^\varepsilon(s)$ and thus by construction of the Caccioppoli partition for the rotations, see (R2), we have

$$\mathcal{H}^1(J_{\phi_\varepsilon(s)}\setminus\Gamma_\varepsilon(s)) \lesssim \varepsilon^{\beta-\gamma}$$

uniformly in time. Thus, by recalling the estimate for ρ_ε in (5.6), it suffices to show that

$$\int_0^t\left(\delta_\varepsilon\left(1+\int_{\Omega'}|\nabla\partial_t h|^2\,\mathrm{d}x + \int_{\Omega'}|\nabla^2\partial_t h|^2\,\mathrm{d}x\right) + \int_{\Omega'}\varepsilon|\nabla\partial_t h|^3\,\mathrm{d}x\right)^{1/2}\mathrm{d}s \to 0 \tag{6.2}$$

as $\varepsilon \to 0$. By the Sobolev embedding, we know that $\nabla\partial_t h \in \mathrm{L}^\infty((0,1);\mathrm{L}^3(\Omega';\mathbb{R}^{2\times 2}))$, where we again used $h \in \mathrm{C}^\infty([0,1];\mathrm{W}^{2,2}(\Omega';\mathbb{R}^2))$. Thus, equation (6.2) holds true, which finishes the proof. ∎

As the next step for the desired energy balance, we derive inequalities describing the limiting energy evolution. This lemma, in formulation and proof, closely follows [31, Lem. 3.7]. Recall the definitions in (2.12) and (2.15).

Lemma 6.2. *Assume that the initial conditions are well prepared in the sense that*

$$\limsup_{\varepsilon\to 0} \mathrm{E}_\varepsilon(y_0^\varepsilon) \le \mathcal{E}(0). \tag{6.3}$$

Then, for any $t \in I_\infty$, we respectively have the lower and upper estimates

$$\mathcal{E}(t) \le \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t) \le \mathcal{E}(0) + \liminf_{\varepsilon\to 0} \int_0^t \int_{\Omega'} \mathbb{C}e(u_\varepsilon(s)) : e(\partial_t h(s))\,\mathrm{d}x\,\mathrm{d}s \tag{6.4}$$

and

$$\mathcal{E}(t) \ge \mathcal{E}(0) + \limsup_{\varepsilon\to 0} \sum_{k=0}^{N(\varepsilon)-1} \int_{t_k^\varepsilon}^{t_{k+1}^\varepsilon} \int_{\Omega'} \mathbb{C}e(u(t_{k+1}^\varepsilon)) : e(\partial_t h(s))\,\mathrm{d}x\,\mathrm{d}s, \tag{6.5}$$

where $N(\varepsilon)$ is chosen such that $t = t_{N(\varepsilon)}^\varepsilon$. Furthermore, $\mathcal{E}$ has no negative jumps.

Existence of well-prepared initial conditions follows from Remark 2.3.

Proof. While the lim sup-inequality is proven exactly as in [31, Lem. 3.7] (exploiting Lemma 5.4), we have to modify the proof for the liminf-inequality. In [31, Lem. 3.7], it is an immediate consequence of the computation performed for the a priori estimate. Instead, due to the rotational invariance of W, we have more work to do: By the frame indifference (W2), the derivative $\mathrm{D}W$ satisfies

$$\mathrm{D}W(A) = R^\top \mathrm{D}W(RA) \tag{6.6}$$

for any rotation $R \in SO(2)$. Thus, even though $W(\nabla y_\varepsilon) = W(\nabla y_\varepsilon^{\mathrm{rot}})$ holds, we have

$$\mathrm{D}W(\nabla y_\varepsilon) \ne \mathrm{D}W(\nabla y_\varepsilon^{\mathrm{rot}})$$

in general. For this reason, we need to throw the rotations onto $\nabla\partial_t h$ and argue that the pieces where the rotation is not the identity already have negligible contribution due to Lemma 6.1.

Step 1 *(Lower estimate).* Let us first prove the limit inferior estimate (6.4). As mentioned before, we proceed similarly to the a priori estimate. Let $t \in I_\infty$ and ε be sufficiently small such that $t \in I_\varepsilon$. From the a priori estimate (4.2), we see that there exists some vanishing sequence σ_ε such that

$$\frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla y_\varepsilon(t))\,\mathrm{d}x + \frac{1}{\varepsilon^{2\beta}}\int_{\Omega'}|\nabla^2 y_\varepsilon(t)|^2\,\mathrm{d}x + \kappa\mathcal{H}^1(\Gamma_\varepsilon(t)) - \sigma_\varepsilon \le \mathrm{E}_\varepsilon(y_0^\varepsilon) + \frac{1}{\varepsilon}\int_0^t\int_{\{\nabla y_\varepsilon\in B_r(\mathrm{SO}(2))\}}\langle \mathrm{D}W(\nabla y_\varepsilon), \nabla\partial_t h\rangle\,\mathrm{d}x\,\mathrm{d}s, \tag{6.7}$$

where r is as in item (W1). To handle the integral over time in (6.7), we again apply Taylor's formula. For this, we recall that $\mathrm{D}W(\mathrm{Id}) = 0$ and we note that

$$\mathcal{N}_\varepsilon := \big\{\nabla y_\varepsilon \in B_r(\mathrm{SO}(2))\big\} \subseteq \big\{\varepsilon|\nabla u_\varepsilon^{\mathrm{aux}}| \le 2r\big\} \tag{6.8}$$

for ε sufficiently small by construction of the rotations (see (R5)), and the definition (4.12) of $u_\varepsilon^{\mathrm{aux}}$ (we suppress the dependence on t for $\mathcal{N}_\varepsilon$). Thus, by using the rotational property (6.6) and equations (4.11)–(4.12), on $\mathcal{N}_\varepsilon$ we get that

$$\begin{aligned}\langle \mathrm{D}W(\nabla y_\varepsilon), \nabla\partial_t h\rangle &= \Big\langle \mathrm{D}W(\nabla y_\varepsilon^{\mathrm{rot}}), \sum_j \chi_{P_j^\varepsilon} R_j^\varepsilon \nabla\partial_t h\Big\rangle \\ &= \Big\langle \mathrm{D}W(\mathrm{Id} + \varepsilon\nabla u_\varepsilon^{\mathrm{aux}}), \sum_j \chi_{P_j^\varepsilon} R_j^\varepsilon \nabla\partial_t h\Big\rangle \\ &\le \varepsilon\Big(\mathbb{C}e(u_\varepsilon^{\mathrm{aux}}) : \sum_j \chi_{P_j^\varepsilon} e(R_j^\varepsilon\partial_t h)\Big) + C\varepsilon^2|\nabla u_\varepsilon^{\mathrm{aux}}|^2|\nabla\partial_t h|. \end{aligned}\tag{6.9}$$

Since $d = 2$, the space $\mathrm{W}^{2,2}(\Omega')$ embeds continuously into $\mathrm{W}^{1,p}(\Omega')$ for every $p \in [1,\infty)$, so by the regularity of the boundary data (2.2), we have $\nabla\partial_t h \in \mathrm{L}^1((0,1);\mathrm{L}^p(\Omega';\mathbb{R}^{2\times 2}))$ for every $p \in [1,\infty)$. Moreover, we recall from inequality (R4) that $\|\nabla u_\varepsilon^{\mathrm{aux}}\|_{\mathrm{L}^2} \lesssim \varepsilon^{\gamma-1}$. To control the last term of (6.9), we recall (6.8) and then apply Lemma A.3 to find some $\alpha > 0$ and $q \in [1,\infty)$ such that

$$\varepsilon\int_0^t\int_{\mathcal{N}_\varepsilon}|\nabla u_\varepsilon^{\mathrm{aux}}|^2|\nabla\partial_t h|\,\mathrm{d}x\,\mathrm{d}s \lesssim \varepsilon^\alpha\int_0^t\|\nabla\partial_t h\|_{\mathrm{L}^q(\Omega')}\,\mathrm{d}s, \tag{6.10}$$

which vanishes as ε goes to zero. Looking to the term involving the elastic tensor $\mathbb{C}$ within (6.9), for any $p \in (2,\infty)$ with $p' = p/(p-1)$, we estimate

$$\Big|\int_0^t\int_{\Omega'}(1-\chi_{\mathcal{N}_\varepsilon})\mathbb{C}e(u_\varepsilon^{\mathrm{aux}}) : \sum_j \chi_{P_j^\varepsilon} e(R_j^\varepsilon\partial_t h)\,\mathrm{d}x\,\mathrm{d}s\Big| \lesssim \int_0^t \|e(u_\varepsilon^{\mathrm{aux}})\|_{\mathrm{L}^{p'}(\Omega'\setminus\mathcal{N}_\varepsilon)}\|\nabla\partial_t h\|_{\mathrm{L}^p(\Omega')}\,\mathrm{d}s.$$

As $\mathcal{L}^2(\Omega'\setminus\mathcal{N}_\varepsilon) \lesssim \varepsilon^2$ uniformly in time (see (4.10)) and $e(u_\varepsilon^{\mathrm{aux}})$ is uniformly in time $\mathrm{L}^{p'}$-equiintegrable due to the L^2-bound in (4.13), the above term vanishes as $\varepsilon \to 0$.

Combining this with (6.8)–(6.10) we get

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^t \int_{\mathcal{N}_\varepsilon} \langle \mathrm{D}W(\nabla y_\varepsilon), \nabla \partial_t h\rangle \,\mathrm{d}x\,\mathrm{d}s \\
&\quad \le \liminf_{\varepsilon\to 0} \int_0^t \int_{\Omega'} \mathbb{C}e(u_\varepsilon^{\mathrm{aux}}) : \sum_j \chi_{P_j^\varepsilon} e(R_j^\varepsilon \partial_t h)\,\mathrm{d}x\,\mathrm{d}s. \qquad (6.11)
\end{aligned}$$

Likewise, as $\mathcal{L}^2(\Omega'\setminus\omega_\varepsilon)$ vanishes uniformly in time, we may replace $u_\varepsilon^{\mathrm{aux}}$ by u_ε in the integral on the right-hand side.

Due to the well-preparedness of the initial conditions in the sense of inequality (6.3) and the lower semicontinuity of the total energies (see inequalities (5.2) and (5.4)), we take the limit inferior as $\varepsilon\to 0$ of (6.7) and use (6.11) (with u_ε in place of $u_\varepsilon^{\mathrm{aux}}$) to recover

$$\begin{aligned}
\mathcal{E}(t) &\le \liminf_{\varepsilon\to 0}\Big[\frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla y_\varepsilon(t))\,\mathrm{d}x + \frac{1}{\varepsilon^{2\beta}}\int_{\Omega'} |\nabla^2 y_\varepsilon(t)|^2\,\mathrm{d}x + \kappa\mathcal{H}^1(\Gamma_\varepsilon(t))\Big] \\
&\le \mathcal{E}(0) + \liminf_{\varepsilon\to 0}\int_0^t\int_{\Omega'} \mathbb{C}e(u_\varepsilon) : \sum_j \chi_{P_j^\varepsilon} e(R_j^\varepsilon\partial_t h)\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

In view of (2.15), to obtain the correct estimate (6.4), we must get rid of the rotations in the above estimate. First, with the notation in Lemma 6.1, for partition elements that intersect the boundary, i.e., $P_j^\varepsilon \notin \mathcal{P}_\varepsilon^{\mathrm{int}}$, we have $R_j^\varepsilon = \mathrm{Id}$; see equation (4.11) and the subsequent explanation. For the remaining elements $P_j^\varepsilon \in \mathcal{P}_\varepsilon^{\mathrm{int}}$ that do not intersect the boundary, we apply Lemma 6.1 twice: once to remove the broken-off pieces with $\widehat{R}_j^\varepsilon = R_j^\varepsilon$ and subsequently with $\widehat{R}_j^\varepsilon = \mathrm{Id}$ to put them back in the inequality. This yields the claim.

Step 2 *(Upper estimate).* As mentioned, we can follow the proof of [31, Lem. 3.7], up to replacing the full gradient with the symmetric gradient. We have written it down in Appendix B for the convenience of the reader. ∎

Remark 6.3. One can also show that for each $t', t\in[0,1]$ with $t'<t$ it holds that

$$\mathcal{E}_\varepsilon(t) - \sigma_\varepsilon \le \mathcal{E}_\varepsilon(t') + \int_{t'}^t \int_{\Omega'} \mathbb{C}e(u_\varepsilon(s)) : e(\partial_t h(s))\,\mathrm{d}x\,\mathrm{d}s \qquad (6.12)$$

for $\sigma_\varepsilon\to 0$ as $\varepsilon\to 0$ (independently of t' and t). Indeed, for $t', t\in I_\varepsilon$ this follows by applying the linearization argument used above to the inequality found in Remark 4.2. For general $t, t'\in[0,1]$, we can choose $t_\varepsilon, t'_\varepsilon\in I_\varepsilon$ with $|t-t_\varepsilon|, |t'-t'_\varepsilon|\le\Delta_\varepsilon$ (see (2.1)) and $\mathcal{E}_\varepsilon(t)=\mathcal{E}_\varepsilon(t_\varepsilon)$, $\mathcal{E}_\varepsilon(t')=\mathcal{E}_\varepsilon(t'_\varepsilon)$. Then we use inequality (6.12) for t_ε and t'_ε and observe that, due to the uniform bounds on the symmetric gradients (2.2) and (4.13), the integral $\int_{t'}^t$ can be replaced by $\int_{t'_\varepsilon}^{t_\varepsilon}$ up to an error that can be absorbed into σ_ε.

7. Proof of Theorem 2.2

We complete the proof of the linearization result. Recall (i)–(v) of Definition 2.1, which must be shown for u and Γ from Definition 4.5. First, (i) (initial condition) follows from the choice of u_0 and the well-preparedness of $(y_0^\varepsilon)_\varepsilon$ as stated in Theorem 2.2. (For sequences of initial data provided by [35, Thm. 2.7], see Remark 2.3 (i), no modifications in (4.11) and (4.16) are needed, i.e., $u_\varepsilon(0) = \bar{u}_\varepsilon(0) = (y_\varepsilon^0 - \mathrm{id})/\varepsilon$, which converges to u_0 in measure on Ω'.) By definition of $\Gamma(t)$, the crack will satisfy (iii) (irreversibility). In Lemma 7.1, we prove that it satisfies (ii) (displacement and boundary conditions) and (iv) (minimality), along with the convergence $e(u_\varepsilon) \to e(u)$. Then, in Lemma 7.3, we prove (v) (energy balance). Finally, we will show energy convergence and $y_\varepsilon \rightsquigarrow u$.

Lemma 7.1. *Let u and Γ be as in Definition 4.5. We have $e(u) \in \mathrm{L}^\infty((0,1);\mathrm{L}^2(\Omega';\mathbb{R}^{2\times 2}_{\mathrm{sym}}))$ and $u(t) = h(t)$ in $\Omega' \setminus \bar{\Omega}$ as well as $\mathcal{H}^1(\Gamma(t)) < \infty$ for all $t \in [0,1]$. Furthermore, for any $t \in [0,1]$, it holds that*

$$\int_{\Omega'} \frac{1}{2} Q\big(e(u(t))\big)\,\mathrm{d}x \le \int_{\Omega'} \frac{1}{2} Q(e(v))\,\mathrm{d}x + \kappa \mathcal{H}^1(J_v \setminus \Gamma(t)) \tag{7.1}$$

among all $v \in \mathrm{GSBD}^2(\Omega')$ such that $v = h(t)$ on $\Omega' \setminus \bar{\Omega}$. Moreover,

$$J_{u(t)} \subseteq \Gamma(t) \ \text{ up to a set of } \mathcal{H}^1\text{-measure zero.} \tag{7.2}$$

Finally, without taking subsequences in ε, we have that for almost every $t \in [0,1]$, both $e(u_\varepsilon(t))$ and $e(u_\varepsilon^{\mathrm{aux}}(t))$ converge to $e(u(t))$ in $\mathrm{L}^2(\Omega';\mathbb{R}^{2\times 2}_{\mathrm{sym}})$.

Proof. The proof is the same (up to obvious modifications) as the proof of [31, Lem. 3.8], except for the strong convergence. We therefore prove only the strong convergence and the measurability here. For convenience of the reader, the rest of the proof can be found in Appendix B.

We want to show the strong convergence of the symmetric gradients by comparing the functions $u(t)$ and $\hat{u}(t)$ from Remark 4.4 and Definition 4.5. The minimizing property of $\hat{u}(t)$ from Lemma 5.4 yields in particular that it minimizes

$$\int_{\Omega'} \frac{1}{2} Q(e(v))\,\mathrm{d}x$$

among all $v \in \mathrm{GSBD}^2(\Omega')$ with $v = h(t)$ on $\Omega' \setminus \bar{\Omega}$ and $J_v \subseteq \Gamma(t) \cup J_{\hat{u}(t)}$. By $u(t) = h(t)$ in $\Omega' \setminus \bar{\Omega}$ and (7.2) (as shown in Appendix B), the function $u(t)$ is an admissible competitor, from which we deduce that

$$\int_{\Omega'} \frac{1}{2} Q\big(e(\hat{u}(t))\big)\,\mathrm{d}x \le \int_{\Omega'} \frac{1}{2} Q\big(e(u(t))\big)\,\mathrm{d}x. \tag{7.3}$$

The main point now is to prove the reverse inequality for a.e. $t \in [0,1]$. In fact, once this is shown, we must have $e(u(t)) = e(\hat{u}(t))$ almost everywhere due to the strict convexity of Q on $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$ which implies that a minimizer over the convex subspace

$$\big\{e(v) : v \in \mathrm{GSBD}^2(\Omega'),\ v = h(t) \text{ in } \Omega' \setminus \bar{\Omega}, \text{ and } J_v \subseteq \Gamma(t) \cup J_{\hat{u}(t)}\big\} \subseteq \mathrm{L}^2(\Omega';\mathbb{R}^{2\times 2}_{\mathrm{sym}})$$

must be unique. Therefore, by Corollary 5.6, $e(u_\varepsilon(t)) \to e(\hat{u}(t)) = e(u(t))$ in $\mathrm{L}^2(\Omega'; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ up to a t-dependent subsequence. The analog convergence of $e(u_\varepsilon^{\mathrm{aux}}(t))$ follows from the same corollary. However, because the limit $e(u(t))$ is independent of the subsequence ε, Urysohn's property (if all subsequences converge to the same point, then the whole sequence converges to this point) shows that we do not need to take a subsequence in ε to conclude $e(u_\varepsilon(t)) \to e(u(t))$ in $\mathrm{L}^2(\Omega'; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$. As we now have pointwise convergence in time, an application of Mazur's lemma says the pointwise and weak limits are the same. Thus, using the uniform boundedness (4.13), $e(u_\varepsilon) \rightharpoonup e(u)$ in $\mathrm{L}^p((0,1); \mathrm{L}^2(\Omega; \mathbb{R}^{2\times 2}_{\mathrm{sym}}))$ for any $1 < p < \infty$, so in particular, $e(u)$ is space-time measurable, and we finally have $e(u) \in \mathrm{L}^\infty((0,1); \mathrm{L}^2(\Omega; \mathbb{R}^{2\times 2}_{\mathrm{sym}}))$.

We are left to prove the reverse of inequality (7.3). To this end, we consider

$$l_\varepsilon(t) := \mathcal{H}^1(\Gamma_\varepsilon(t)).$$

The functions l_ε are nondecreasing and uniformly bounded (see (4.1)), thus by Helly's theorem we find a nondecreasing function λ on $[0,1]$ such that for a subsequence of ε (not relabeled), we have $l_\varepsilon \to \lambda$ pointwise. Let H be the at most countable set of discontinuity points of λ. Let $t \notin H$. If $t \in I_\infty$, (7.3) follows directly as $u(t) = \hat{u}(t)$ by Remark 4.4. We thus assume $t \notin I_\infty$ and let $(t_p)_p \subseteq I_\infty$ be as from Definition 4.5. Using the approximate minimality of $u_\varepsilon(t_p)$ given by Remark 5.2 with the admissible competitor $u_\varepsilon(t) + h_\varepsilon(t_p) - h_\varepsilon(t)$ along with convergence (4.17), we get

$$\begin{aligned}
&\int_{\Omega'} \frac{1}{2} Q\big(e(u_\varepsilon(t_p))\big)\,\mathrm{d}x - \rho_\varepsilon \\
&\quad \le \int_{\Omega'} \frac{1}{2} Q\big(e(u_\varepsilon(t) + h_\varepsilon(t_p) - h_\varepsilon(t))\big)\,\mathrm{d}x + \kappa \mathcal{H}^1((J_{u_\varepsilon(t)} \cup J_{\nabla u_\varepsilon(t)}) \setminus \Gamma_\varepsilon(t_p)) \\
&\quad \le \int_{\Omega'} \frac{1}{2} Q\big(e(u_\varepsilon(t) + h_\varepsilon(t_p) - h_\varepsilon(t))\big)\,\mathrm{d}x + \kappa (l_\varepsilon(t) - l_\varepsilon(t_p)) + \sigma_\varepsilon,
\end{aligned} \tag{7.4}$$

where $\sigma_\varepsilon \to 0$ as $\varepsilon \to 0$ and where by inequality (5.12) we can estimate

$$\begin{aligned}
\rho_\varepsilon \lesssim \delta_\varepsilon + \varepsilon^{2(1-\beta)} &\int_{\Omega'} |\nabla^2 (u_\varepsilon(t) + h_\varepsilon(t_p) - h_\varepsilon(t))|^2\,\mathrm{d}x \\
&+ \int_{\Omega'} \varepsilon |\nabla (u_\varepsilon(t) + h_\varepsilon(t_p) - h_\varepsilon(t))|^3\,\mathrm{d}x.
\end{aligned}$$

The first summand vanishes since $\delta_\varepsilon \to 0$. The second summand vanishes by (5.20) in Corollary 5.5 and because $h \in \mathrm{L}^\infty((0,1); \mathrm{W}^{2,\infty}(\Omega'; \mathbb{R}^2))$. The third summand vanishes by the choice of the cutoff, see ($\omega 3$), the definition of u_ε in (4.16), and again by the regularity of h.

Sending ε to zero in inequality (7.4) yields by lower semicontinuity on the left-hand side and the strong convergence given by Corollary 5.6 that

$$\int_{\Omega'} \frac{1}{2} Q\big(e(u(t_p))\big)\,\mathrm{d}x \le \int_{\Omega'} \frac{1}{2} Q\big(e(\hat{u}(t) + h(t_p) - h(t))\big)\,\mathrm{d}x + \kappa(\lambda(t) - \lambda(t_p)).$$

Sending $p \to \infty$ yields the desired inequality, again by lower semicontinuity and the choice of t as a continuity point of λ. This completes the proof. ∎

Remark 7.2. The result of Lemma 7.1 may be strengthened in the sense that $e(u_\varepsilon(t))$ and $e(u_\varepsilon^{\mathrm{aux}}(t))$ converge to $e(u(t)) = e(\hat{u}(t))$ in $\mathrm{L}^2(\Omega'; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ for *all* $t \in [0,1]$. Indeed, the proof above shows that this convergence and the identification of $e(u)$ with $e(\hat{u})$ holds on $[0,1] \setminus H$, where H is the set of discontinuity points for λ. Moreover, by Corollary 5.6 this convergence also always holds on I_∞ as $u = \hat{u}$ on I_∞; see Remark 4.4. Therefore, to obtain the claimed property, it suffices to show that $H \subseteq I_\infty$. In the proof of Theorem 2.2 below, we will see that $\lambda(t) = l(t) := \mathcal{H}^1(\Gamma(t))$ for a.e. $t \in [0,1]$. Assuming this for the moment, since λ is monotone increasing and l is continuous at all times $t \notin I_\infty$ (see equation (4.20)), this implies that $\lambda(t) = l(t)$ for all $t \notin I_\infty$, and then $H \subseteq I_\infty$.

The next lemma establishes the desired energy balance.

Lemma 7.3. *The function $\mathcal{E}$ is absolutely continuous on* $[0,1]$ *and satisfies the energy balance*

$$\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t \int_{\Omega'} \mathbb{C}e(u(s)) : e(\partial_t h)\,\mathrm{d}x\,\mathrm{d}s.$$

Proof. Relying on Lemma 6.2 and inequality (7.1) in Lemma 7.1, the proof is exactly the same as in [31, Lem. 3.9]. ∎

We close with the proof of Theorem 2.2.

Proof of Theorem 2.2. By combining the choice of u_0, Definition 4.5, Lemmas 7.1 and 7.3, we find that $(u(t), \Gamma(t))_{t\in[0,1]}$ satisfies (i)–(v) in Definition 2.1. It remains to prove convergence of the total energies and the rescaled deformations.

Step 1 *(Convergence of the total energies).* By Lemma 7.1, $e(u_\varepsilon) \overset{*}{\rightharpoonup} e(u)$ in the space $\mathrm{L}^\infty((0,1); \mathrm{L}^2(\Omega'; \mathbb{R}^{2\times 2}_{\mathrm{sym}}))$ up to a subsequence, so that using Lemma 7.3, we deduce from (6.4) that $\mathcal{E}(t) = \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t)$ for all $t \in I_\infty$. Since this holds for any further subsequence in ε, we recover

$$\mathcal{E}(t) = \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t) \qquad \text{for all } t \in I_\infty. \tag{7.5}$$

To obtain convergence at any time $t \in [0,1]$, take $t' \in I_\infty$ with $t' < t$ and use Remark 6.3 with equation (7.5) to find

$$\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t) \le \mathcal{E}(t') + \int_{t'}^t \int_{\Omega'} \mathbb{C}e(u(s)) : e(\partial_t h(s))\,\mathrm{d}x\,\mathrm{d}s.$$

Taking $t' \uparrow t$ and using the continuity of $\mathcal{E}$ (from Lemma 7.3), we have $\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t) \le \mathcal{E}(t)$. Analogously, we find $\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t) \ge \mathcal{E}(t)$ by taking $t' > t$, thereby concluding convergence of the energies at every time. Using the convergence of the energy along with Corollary 5.5 and the fact that $e(\hat{u}) = e(u)$ a.e. on $[0,1]$ (as seen in the proof of Lemma 7.1), for a.e. $t \in [0,1]$ we get that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\int_{\Omega'} W(\nabla y_\varepsilon(t))\,\mathrm{d}x + \frac{1}{\varepsilon^{2\beta}}\int_{\Omega'} |\nabla^2 y_\varepsilon(t)|^2\,\mathrm{d}x = \int_{\Omega'} \frac{1}{2} Q\big(e(u(t))\big)\,\mathrm{d}x \tag{7.6}$$

and

$$\lim_{\varepsilon\to 0} \mathcal{H}^1(\Gamma_\varepsilon(t)) = \mathcal{H}^1(\Gamma(t)). \tag{7.7}$$

In view of (7.6) and (7.7), we can now apply Remark 7.2 (which still needed $\lambda(t) = l(t)$ for a.e. $t \in [0,1]$ for its conclusion) and we infer that $e(u(t)) = e(\hat{u}(t))$ in $\mathrm{L}^2(\Omega';\mathbb{R}^{2\times 2}_{\mathrm{sym}})$ for *all* $t \in [0,1]$. Again using Corollary 5.5, this then implies that (7.7) holds for all $t \in [0,1]$.

Step 2 *(Convergence of the rescaled deformations).* To show $y_\varepsilon \rightsquigarrow u$, i.e., to confirm convergences (2.13) and (2.14), we will construct sequences of deformations, which are almost minimizers for the minimization problem (2.6), and use these to identify the limit of $\bar{u}_\varepsilon(t) := (y_\varepsilon(t) - \mathrm{id})/\varepsilon$ on $G(t)$.

Recalling the definition of $\mathcal{P}^{\mathrm{int}}_\varepsilon$ in Lemma 6.1, we define $D^1_\varepsilon(t) = \bigcup_{P^\varepsilon_j(t)\notin\mathcal{P}^{\mathrm{int}}_\varepsilon(t)} P^\varepsilon_j(t)$ and

$$y^*_\varepsilon(t) = \chi_{D^1_\varepsilon(t)} y_\varepsilon(t) + \chi_{\Omega'\setminus D^1_\varepsilon(t)}\mathrm{id}, \quad u^*_\varepsilon(t) = \frac{1}{\varepsilon}(y^*_\varepsilon(t) - \mathrm{id}).$$

Recalling the definition of $\bar{u}_\varepsilon$ and the definition in (4.12), this shows $u^{\mathrm{aux}}_\varepsilon(t)\chi_{D^1_\varepsilon(t)} = u^*_\varepsilon(t)$, and

$$u^*_\varepsilon(t) = \bar{u}_\varepsilon(t) \quad \text{on } D^1_\varepsilon(t). \tag{7.8}$$

We apply Theorem 3.1 to the sequence $(u^*_\varepsilon(t))_\varepsilon$ to find collections of sets $\mathcal{S}_\varepsilon = (S^\varepsilon_j)_{j\geq 0}$ contained in Ω and a limit function $u^*(t) \in GSBD^2(\Omega')$ for the modified sequence; see Theorem 3.1 (i). Denoting $D^2_\varepsilon(t) := \Omega' \setminus \bigcup_{j\geq 0} S^\varepsilon_j$, we introduce $D_\varepsilon(t) := D^1_\varepsilon(t) \cap D^2_\varepsilon(t)$ and, for each $\eta \in \mathbb{R}^2$, define $u^*_{\varepsilon,\eta}(t) := \chi_{D_\varepsilon(t)} u^*_\varepsilon(t) + \chi_{\Omega'\setminus D_\varepsilon(t)}\eta$. By Lemma 4.1, inequality (R2), Theorem 3.1 (iv), and compactness of sets of finite perimeters, we find $D(t) \subseteq \Omega'$ such that $\chi_{D_\varepsilon(t)} \to \chi_{D(t)}$ in $\mathrm{L}^1(\Omega')$ for a t-dependent subsequence of ε. Since $D_\varepsilon(t) \supseteq \Omega' \setminus \overline{\Omega}$ for all $\varepsilon > 0$, we have $D(t) \supseteq \Omega' \setminus \overline{\Omega}$. Moreover, by the already applied Theorem 3.1, we have, for a subsequence depending on t, that

$$u^*_{\varepsilon,\eta}(t) \to u^*_\eta(t) := \chi_{D(t)} u^*(t) + \chi_{\Omega'\setminus D(t)}\eta \quad \text{in measure on } \Omega', \tag{7.9}$$

where $u^*(t)$ is the limit of the sequence $(u^*_\varepsilon(t))_\varepsilon$ found above. Note that $u^*_\eta(t) = h(t)$ on $\Omega' \setminus \overline{\Omega}$ and $\mathcal{H}^1(J_{u^*_{\varepsilon,\eta}(t)} \setminus \Gamma_\varepsilon(t)) \to 0$. Thus, with the same proof as for Lemma 5.4, $u^*_\eta(t)$ is a minimizer of the functional

$$\int_{\Omega'} \frac{1}{2} Q(e(v))\,\mathrm{d}x + \kappa\mathcal{H}^1\big(J_v \setminus (\Gamma(t) \cup J_{u^*_\eta(t)})\big)$$

among all $v \in GSBD^2(\Omega')$ with $v = h(t)$ on $\Omega' \setminus \overline{\Omega}$.

To identify the strain $e(u^*_\eta)$, we first repeat the lower semicontinuity argument in inequalities (5.4)–(5.5), by adding the jump set of $u^*_\eta(t)$ to the union on the left-hand side. This gives

$$\mathcal{H}^1(\Gamma(t) \cup J_{u^*_\eta(t)}) \le \liminf_{\varepsilon\to 0} \mathcal{H}^1(\Gamma_\varepsilon(t)).$$

More precisely, if $t \notin I_\infty$, we perform this estimate with $\Gamma(\tau)$ in place of $\Gamma(t)$ on the left-hand side for any $\tau \in I_\infty$, $\tau \le t$, and then we pass to the limit $\tau \uparrow t$ using the continuity of $t \mapsto \mathcal{H}^1(\Gamma(t))$ on $[0,1] \setminus I_\infty$. This along with (7.7) (for every $t \in [0,1]$) shows $J_{u^*_\eta(t)} \subseteq \Gamma(t)$ up to an $\mathcal{H}^1$-negligible set. Then arguing as for inequality (7.3) in Lemma 7.1, since $u^*_\eta(t)$ and $u(t)$ solve the same minimization problem with the crack $\Gamma(t)$ fixed, we have that

$$e(u^*_\eta(t)) = e(u(t)) \quad \text{for every } t \in [0,1]. \tag{7.10}$$

In view of the definition of $u^*_\eta(t)$, see (7.9), for a particular choice of $\eta \in \mathbb{R}^2$ (actually, for a.e. choice) we get $\partial^* D(t) \subseteq J_{u^*_\eta(t)} \subseteq \Gamma(t)$ up to an $\mathcal{H}^1$-negligible set. Then, by the definition of $G(t)$ preceding (2.13) and the fact that $D(t) \supseteq \Omega' \setminus \overline{\Omega}$, this shows that $G(t) \subseteq D(t)$ up to an $\mathcal{L}^2$-negligible set. Therefore, $\mathcal{L}^2(G(t) \setminus D_\varepsilon(t)) \to 0$ and thus, as $\bar{u}_\varepsilon(t) = u^*_\varepsilon(t) = u^*_{\varepsilon,\eta}(t)$ on $D_\varepsilon(t)$ by equation (7.8), by (7.9) we deduce

$$\bar{u}_\varepsilon(t) \to u^*(t) \quad \text{in measure on } G(t). \tag{7.11}$$

In view of equation (7.10), [18, Thm. A.1] shows that $u(t) - u^*_0(t)$ is a piecewise rigid function on a Caccioppoli partition $\mathcal{P}$ of Ω' with boundaries contained in $J_{u(t)} \cup J_{u^*_0(t)} \subseteq \Gamma(t)$. Let $\mathcal{P}_D$ be the elements of $\mathcal{P}$ whose intersection with $\Omega' \setminus \overline{\Omega}$ have positive measure. Due to the boundary conditions, $u(t) - u^*_0(t) = 0$ on every $P \in \mathcal{P}_D$. Further, we must have that $G(t) \subset \bigcup_{P\in\mathcal{P}_D} P$, otherwise $[G(t) \cap (\bigcup_{P\in\mathcal{P}_D} P)]^c$ contradicts the maximality of the broken-off piece $B(t)$. Altogether, this shows $u(t) - u^*_0(t) = 0$ on $G(t)$. Using convergence (7.11) and $u^*(t) = u^*_0(t)$ on $G(t)$, this argument uniquely identifies the limit of $\bar{u}_\varepsilon(t)$ on $G(t)$ for any subsequence as $u(t)$, thereby implying the first part of (2.13).

As $e(u^{\mathrm{aux}}_\varepsilon(t))$ converges to $e(u(t))$ in $\mathrm{L}^2(\Omega'; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ for all $t \in [0,1]$ by Lemma 7.1 and Remark 7.2, we particularly find $e(u^{\mathrm{aux}}_\varepsilon(t)) \to e(u(t))$ in measure on $G(t)$. As $u^{\mathrm{aux}}_\varepsilon(t) = \bar{u}_\varepsilon(t)$ on $D_\varepsilon(t)$ by equation (7.8) and $\mathcal{L}^2(G(t) \setminus D_\varepsilon(t)) \to 0$, the second part of (2.13) follows (noting that this reasoning holds for every subsequence).

Finally, we come to the proof of convergence (2.14). For any $t \in [0,1]$, recalling $\partial^* G(t) \subseteq \Gamma(t)$ up to an $\mathcal{H}^1$-negligible set and using $\chi_{G(t)} u(t)$ as a competitor in inequality (7.1), we find that

$$e(u(t)) = 0 \quad \text{on } B(t).$$

Then, repeating the lower semicontinuity argument (5.2) on $G(t)$, we find

$$\begin{aligned}\int_{\Omega'} \frac{1}{2} Q\big(e(u(t))\big)\,\mathrm{d}x &= \int_{G(t)} \frac{1}{2} Q\big(e(u(t))\big)\,\mathrm{d}x \le \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \int_{G(t)} W(\nabla y_\varepsilon(t))\,\mathrm{d}x \\ &= \int_{\Omega'} \frac{1}{2} Q\big(e(u(t))\big)\,\mathrm{d}x - \limsup_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \int_{B(t)} W(\nabla y_\varepsilon(t))\,\mathrm{d}x,\end{aligned}$$

where the last identity follows from Corollary 5.5 and the fact that $e(\hat{u}(t)) = e(u(t))$ for every $t \in [0, 1]$ (see Remark 7.2). This along with (W3) shows convergence (2.14), which finishes the proof. ■

A. Bad sets, integral bounds, and measure theory

Lemma A.1 (Bad set). *Let $\Omega \subseteq \Omega' \subseteq \mathbb{R}^2$ be bounded Lipschitz domains such that $\Omega' \setminus \overline{\Omega}$ is also a Lipschitz set, and suppose $\Gamma \subseteq \Omega' \cap \overline{\Omega}$ is a Borel set with $\mathcal{H}^1(\Gamma) < \infty$. Then there exists a unique maximal measurable set $B \subseteq \Omega$ (up to $\mathcal{L}^2$-equivalence class and with respect to set inclusion) such that $\partial^* B \subseteq \Gamma$ (up to an $\mathcal{H}^1$-nullset). If Γ is additionally closed, we have*

$$B = \{x \in \Omega : x \text{ is not path-connected to } \Omega' \setminus \overline{\Omega} \text{ in } \Omega' \setminus \Gamma\}. \tag{A.1}$$

Remark A.2. It is convenient for the proof if we denote by ∂^* the *essential boundary* of a set (see [7, Def. 3.60]), which is defined without assuming the set to be of finite perimeter. Note however that if $\mathcal{H}^1(\partial^* A) < \infty$, then by [29, Thm. 4.5.11], A is already a set of finite perimeter and the essential and reduced boundary agree up to a set of $\mathcal{H}^1$-measure zero.

Proof of Lemma A.1. We first prove the existence of a maximal set by using the direct method. Let

$$M := \{B \subseteq \Omega \text{ measurable} : \mathcal{H}^1(\partial^* B \setminus \Gamma) = 0\}.$$

Note that the empty set is an element of M and by boundedness of Ω, the Lebesgue measure of sets in M is uniformly bounded. Moreover, the essential boundary of every $B \in M$ is bounded with respect to $\mathcal{H}^1$. Thus, for a maximizing sequence $(B_n)_n \subseteq M$ with respect to $\mathcal{L}^2$, we find a subsequence (not relabeled) and some measurable set B such that $B_n \to B$ in measure. By this convergence it follows that $\mathcal{L}^2(B) = \max_{B' \in M} \mathcal{L}^2(B')$. Using lower semicontinuity, we have for every compact set $K \subseteq \Gamma$ that

$$\mathcal{H}^1(\partial^* B \setminus K) \le \liminf_{n\to\infty} \mathcal{H}^1(\partial^* B_n \setminus K) \le \mathcal{H}^1(\Gamma \setminus K).$$

Sending $K \to \Gamma$ yields that $\mathcal{H}^1(\partial^* B \setminus \Gamma) = 0$, and thus $B \in M$. Moreover, B is maximal: If $B' \in M$, then $B \cup B' \in M$ since $\partial^*(B \cup B') \subseteq \partial^* B \cup \partial^* B'$. Thus, by $\mathcal{L}^2$-maximality of B, we must already have $\mathcal{L}^2(B' \setminus B) = 0$. This also proves the uniqueness.

Now let us assume that Γ is closed. We want to show equation (A.1). Denoting by $\widehat{B}$ the right-hand side of (A.1), we aim at checking $B = \widehat{B}$ with B above. First, we show $\widehat{B} \in M$. Assume by contradiction that there exists $x \in \partial^* \widehat{B} \setminus \Gamma$. Then, by the closedness of Γ we find some $\delta > 0$ such that $B_\delta(x) \subseteq \Omega' \setminus \Gamma$. Moreover, by the definition of the essential boundary, we find some $y \in B_\delta(x) \setminus \widehat{B}$. By definition of $\widehat{B}$ and because $B_\delta(x) \subseteq \Omega' \setminus \Gamma$, by connecting elements $z \in B_{\delta(x)}$ with y by a straight segment, this yields that $B_\delta(x)$ is also path-connected to $\Omega' \setminus \overline{\Omega}$ in $\Omega' \setminus \Gamma$. Thus $B_\delta(x) \subseteq \Omega' \setminus \widehat{B}$, a contradiction to $x \in \partial^* \widehat{B}$. We thus conclude that $\partial^* \widehat{B} \subseteq \Gamma$, i.e., $\widehat{B} \in M$.

To conclude $B = \widehat{B}$, it suffices to show that $\widehat{B}$ is maximal. Let $B' \in M$ and suppose by contradiction that $x \in B' \setminus \widehat{B}$ is a point of Lebesgue density 1 of $B' \setminus \widehat{B}$. Then we find a continuous path γ between x and $\Omega' \setminus \overline{\Omega}$ in $\Omega' \setminus \Gamma$. Since γ is compact and Γ closed, we find some $\delta > 0$ such that $B_\delta(\gamma) \subseteq \Omega' \setminus \Gamma$. Since we have chosen x as a point of density 1 of $B' \setminus \widehat{B}$, we have $\mathcal{L}^2(B_\delta(\gamma) \cap B') > 0$. Moreover, because $\Omega' \setminus \overline{\Omega}$ is open, γ ends in $\Omega' \setminus \overline{\Omega}$, and $B' \subseteq \Omega$, we have $\mathcal{L}^2(B_\delta(\gamma) \setminus B') > 0$. As a consequence, we must have $\mathcal{H}^1(\partial^* B' \cap B_\delta(\gamma)) > 0$. This is a contradiction to $B' \in M$ because $\Gamma \cap B_\delta(\gamma) = \emptyset$. ■

Lemma A.3 (L^p-bounds). *Let $\Omega \subseteq \mathbb{R}^2$ with $\mathcal{L}^2(\Omega) < \infty$, $f_\varepsilon \in \mathrm{L}^2(\Omega; [0, \infty))$ be a sequence such that $\|f_\varepsilon\|_{\mathrm{L}^2(\Omega)} \lesssim \varepsilon^{\gamma-1}$ for some $\gamma \in (2/3, 1)$, and let $g \in \mathrm{L}^p(\Omega; [0, \infty))$ for every $p \in [1, \infty)$. Then we find an exponent $q \in (1, \infty)$ and some $\alpha > 0$ such that*

$$\varepsilon \int_{\{\varepsilon f_\varepsilon \lesssim 1\}} f_\varepsilon^2 g \,\mathrm{d}x \lesssim \varepsilon^\alpha \|g\|_{\mathrm{L}^q(\Omega)}.$$

Proof. Let $0 < \zeta < 2$ and let $q \in (1, \infty)$ be such that $1/q + (2 - \zeta)/2 = 1$. By Hölder's inequality we have

$$\begin{aligned} \varepsilon \int_{\{\varepsilon f_\varepsilon \lesssim 1\}} f_\varepsilon^2 g \,\mathrm{d}x &\lesssim \varepsilon^{1-\zeta} \int_\Omega f_\varepsilon^{2-\zeta} g \,\mathrm{d}x \\ &\le \varepsilon^{1-\zeta} \|f_\varepsilon\|_{\mathrm{L}^2(\Omega)}^{2-\zeta} \|g\|_{\mathrm{L}^q(\Omega)} \\ &\lesssim \varepsilon^{1-\zeta+(\gamma-1)(2-\zeta)} \|g\|_{\mathrm{L}^q(\Omega)}. \end{aligned}$$

Since

$$\alpha := 1 - \zeta + (\gamma - 1)(2 - \zeta) = -1 + 2\gamma - \gamma\zeta$$

and $-1 + 2\gamma > 0$, we can choose $\zeta > 0$ sufficiently small such that $\alpha > 0$, which finishes the proof. ■

Lemma A.4 (Measure theory). *Let $A_1, A_2 \subseteq \mathbb{R}^2$ be Borel sets with $\mathcal{H}^1(A_1), \mathcal{H}^1(A_2) < \infty$. Then, for all $\epsilon > 0$, we find open Lipschitz sets U_1, U_2 such that $U_1 \cap U_2 = \emptyset$ and*

$$\mathcal{H}^1(A_1 \cup A_2) - \epsilon \le \mathcal{H}^1(A_1 \cap U_1) + \mathcal{H}^1(A_2 \cap U_2).$$

Proof. The measure $\mu := \mathcal{H}^1 \llcorner_{A_1 \cup A_2}$ is a Radon measure. Thus, by inner regularity, we find compact sets $K_1 \subseteq A_1$, $K_2 \subseteq A_2 \setminus A_1$ such that $\mu(A_1 \setminus K_1), \mu((A_2 \setminus A_1) \setminus K_2) \le \epsilon/2$. Since $K_1 \cap K_2 = \emptyset$ and both are compact, we have $\mathrm{dist}(K_1, K_2) =: \delta > 0$. We therefore find open, disjoint Lipschitz sets U_1 and U_2 such that $K_1 \subseteq U_1$ and $K_2 \subseteq U_2$. Then

$$\begin{aligned} \mathcal{H}^1(A_1 \cup A_2) &= \mathcal{H}^1(A_1) + \mathcal{H}^1(A_2 \setminus A_1) \\ &\le \mathcal{H}^1(A_1 \cap K_1) + \mathcal{H}^1((A_2 \setminus A_1) \cap K_2) + \epsilon \\ &\le \mathcal{H}^1(A_1 \cap U_1) + \mathcal{H}^1((A_2 \setminus A_1) \cap U_2) + \epsilon, \end{aligned}$$

which finishes the proof. ■

B. Proofs

In this section we collect the remaining proofs that have been omitted in the paper.

Proof of Theorem 3.2. We first observe that by [34, Thm. 2.1, eq. (2.4)] we find a sequence $\tilde{v}_k \in \mathrm{GSBD}^2(\Omega) \cap \mathrm{L}^2(\Omega;\mathbb{R}^2)$ such that $E_k := \{x \in \Omega : \tilde{v}_k(x) \neq v(x)\}$ satisfies $\mathcal{L}^2(E_k) + \mathcal{H}^1(\partial^* E_k) \to 0$ as $k \to \infty$. Therefore, the sequence

$$v_k := \begin{cases} \tilde{v}_k \chi_{\Omega \setminus E_k} & \text{on } \Omega, \\ h & \text{on } \Omega' \setminus \overline{\Omega}, \end{cases}$$

lies in $\mathrm{GSBD}^2(\Omega') \cap \mathrm{L}^2(\Omega';\mathbb{R}^2)$ and satisfies

$$v_k \to v \text{ in measure on } \Omega', \quad e(v_k) \to e(v) \text{ in } \mathrm{L}^2(\Omega';\mathbb{R}^{2\times 2}_{\mathrm{sym}}), \quad \mathcal{H}^1(J_{v_k} \triangle J_v) \to 0.$$

By means of a diagonal argument this shows that it suffices to prove the statement for functions $v \in \mathrm{GSBD}^2(\Omega') \cap \mathrm{L}^2(\Omega';\mathbb{R}^2)$. (Here, we use that the convergence in measure is metrizable.) By another diagonal argument it suffices to find a sequence $(w_\delta)_\delta$ as in the statement satisfying items (D1), (D2) and item (D3) is replaced by

$$\limsup_{\delta \to 0} \mathcal{H}^1(J_{w_\delta} \triangle J_v) \le \eta \tag{B.1}$$

for an arbitrary $\eta > 0$.

Let us from now on assume that $v \in \mathrm{GSBD}^2(\Omega') \cap \mathrm{L}^2(\Omega';\mathbb{R}^2)$ with $v = h$ on $\Omega' \setminus \overline{\Omega}$ and fix $\eta > 0$. Let $\partial_D \Omega := \partial\Omega \cap \Omega'$, and let $\partial^*_D \Omega \subseteq \partial_D \Omega$ be the set of differentiablity points of $\partial_D \Omega$, where the corresponding outer normal is denoted by $\nu(x)$ for $x \in \partial^*_D \Omega$. By Rademacher's theorem we have $\mathcal{H}^1(\partial_D \Omega \setminus \partial^*_D \Omega) = 0$. We introduce a fine cover of $\partial^*_D \Omega \setminus J_v$ up to a set of negligible $\mathcal{H}^1$-measure: Consider all closed squares $Q_r(x) \subset\subset \Omega'$ centered at $x \in \partial^*_D \Omega$ with radius r and two sides parallel to $\nu(x)$ such that

$$\begin{aligned} &\text{(i)} \quad \mathcal{H}^1(Q_r(x) \cap J_v) \le \eta r, \\ &\text{(ii)} \quad \mathcal{H}^1(Q_r(x) \cap \partial_D \Omega) \ge 2r. \end{aligned} \tag{B.2}$$

This indeed provides a fine cover as J_v has $\mathcal{H}^1$-density zero $\mathcal{H}^1$-a.e. in $\partial_D \Omega^* \setminus J_v$. We now apply the Besicovitch covering theorem with respect to $\mathcal{H}^1 \llcorner_{\partial^*_D \Omega \setminus J_v}$ which induces a finite disjoint collection $(Q_{r_i}(x_i))_{i=1}^N$ with $(x_i)_{i=1}^N \subseteq \partial^*_D \Omega \setminus J_v$, abbreviated by $(Q_i)_{i=1}^N$ in the sequel, such that (B.2) holds for each Q_i and

$$\mathcal{H}^1\left((\partial^*_D \Omega \setminus J_v) \setminus \bigcup_{i=1}^N Q_i \right) \le \eta. \tag{B.3}$$

Following the proof of [43, Prop. 2.5], we show that for each Q_i there exists a sequence $(z_\delta^i)_\delta \subseteq \mathcal{W}(Q_i;\mathbb{R}^2)$ such that

$$\begin{aligned} &\text{(i)} \quad \|z_\delta^i - v\|_{L^2(Q_i)} \to 0 \text{ as } \delta \to 0, \\ &\text{(ii)} \quad \|e(z_\delta^i) \to e(v)\|_{L^2(Q_i)} \to 0 \text{ as } \delta \to 0, \\ &\text{(iii)} \quad \mathcal{H}^1(J_{z_\delta^i}) \lesssim \eta r_i, \\ &\text{(iv)} \quad z_\delta^i = h \text{ on } Q_i \setminus \overline{\Omega}. \end{aligned} \tag{B.4}$$

To this end, we fix Q_i and drop the index i for convenience, i.e., we write Q as well as x and r. We assume without restriction that $\nu(x) = e_2$ and $Q = (-1,1)^2$. We choose a Lipschitz function $f\colon(-1,1) \to \mathbb{R}$ such that

$$\Omega \cap Q = \big\{y \in Q : y_2 < f(y_1)\big\}, \quad \partial_D\Omega \cap Q = \big\{y \in Q : y_2 = f(y_1)\big\}.$$

Let $v_\delta(y) := v(y + \delta e_2)$ and observe that

$$v_\delta \to v \text{ in } L^2(Q;\mathbb{R}^2) \quad \text{and} \quad e(v_\delta) \to e(v) \text{ in } L^2(Q;\mathbb{R}^{2\times 2}_{\mathrm{sym}}) \quad \text{as } \delta \to 0. \tag{B.5}$$

By [44, Thm. 3.1] (see also [20, Thm. 1.1] for control on higher Sobolev norms) we choose $(w_\delta)_\delta \in \mathcal{W}(Q;\mathbb{R}^2)$ such that

$$\|v_\delta - w_\delta\|_{L^2(Q)} + \|e(v_\delta) - e(w_\delta)\|_{L^2(Q)} \le \delta^2, \quad \mathcal{H}^1(J_{w_\delta} \cap Q) \lesssim r\eta, \tag{B.6}$$

where we used that $\mathcal{H}^1(J_v \cap Q) \le r\eta$, see inequality (B.2) (i), and the definition of v_δ which implies that $\mathcal{H}^1(J_{v_\delta} \cap Q) \le \mathcal{H}^1(J_v \cap Q)$ due to the regularity of the boundary data. Let ψ_δ be a cut-off function with $\psi_\delta = 1$ on $\{y_2 \ge f(y_1) - \delta/3\}$, $\psi_\delta = 0$ on $\{y_2 \le f(y_1) - \delta/2\}$, and $\|\nabla\psi_\delta\|_\infty \lesssim 1/\delta$. Define $z_\delta := \psi_\delta h + (1-\psi_\delta)w_\delta$. By construction and $h \in W^{2,\infty}(\Omega';\mathbb{R}^2)$ we have $z_\delta \in \mathcal{W}(Q;\mathbb{R}^2)$, $\mathcal{H}^1(J_{z_\delta} \cap Q) \lesssim r\eta$, and that $z_\delta = h$ on $Q \setminus \overline{\Omega}$. Thus, points (iii) and (iv) of (B.4) hold. We estimate

$$\begin{aligned} \|z_\delta - v\|_{L^2(Q)} \le{}& \|(\psi_\delta h + (1-\psi_\delta)v) - v\|_{L^2(Q)} \\ &+ \|v_\delta - w_\delta\|_{L^2(Q)} + \|v_\delta - v\|_{L^2(Q)}. \end{aligned} \tag{B.7}$$

This shows $z_\delta \to v$ in $L^2(Q;\mathbb{R}^2)$ as $\delta \to 0$ by (B.5)–(B.6), the fact that $\mathcal{L}^2(\operatorname{supp}\psi_\delta \cap \Omega) \to 0$, and $v = h$ on $\Omega' \setminus \overline{\Omega}$. Thus, it remains to check (B.4) (ii). We note that $e(z_\delta) = \psi_\delta e(h) + (1-\psi_\delta)e(w_\delta) + (h - w_\delta) \odot \nabla\psi_\delta$. By an argument similar to (B.7) employing (B.5)–(B.6), we get $\|\psi_\delta e(h) + (1-\psi_\delta)e(w_\delta) - e(v)\|_{L^2(Q)} \to 0$. On the other hand, setting $\Psi_\delta := \{0 < \psi_\delta < 1\}$, by the first inequality in (B.6) and the Lipschitz continuity of h, we get

$$\begin{aligned} \|(h - w_\delta) \odot \nabla\psi_\delta\|_{L^2(Q)} &\lesssim \delta^{-1}\|w_\delta - h\|_{L^2(\Psi_\delta)} \\ &\lesssim \delta^{-1}\|v_\delta - h\|_{L^2(\Psi_\delta)} + \delta \\ &= \delta^{-1}\|v(\cdot + \delta e_2) - h\|_{L^2(\Psi_\delta)} + \delta \\ &\lesssim \|\nabla h\|_\infty \mathcal{L}^2(\Psi_\delta)^{1/2} + \delta, \end{aligned}$$

where we used that $v(\cdot + \delta e_2) = h(\cdot + \delta e_2)$ on Ψ_δ. Since this vanishes as $\delta \to 0$, (B.4) (ii) follows.

After having found the functions z^i_δ satisfying (B.4), we are now ready to construct the sequence $(w_\delta)_\delta$. To this end, we choose slightly smaller squares $Q'_i \subset\subset Q_i$ for $i = 1, \dots, N$ such that

$$\mathcal{H}^1\Big(\partial_D\Omega \cap \bigcup_{i=1}^N (Q_i \setminus Q'_i)\Big) \le \eta, \tag{B.8}$$

and we choose cut-off functions $(\varphi_i)_{i=1}^N \subseteq C^\infty(\Omega)$ such that $\varphi_i = 1$ on $Q'_i \cap \Omega$ and $\varphi_i = 0$ on $\Omega \setminus Q_i$. Finally, we let $\varphi_0 := 1 - \sum_{i=1}^N \varphi_i$. By [44, Thm. 3.1] (see also [20, Thm. 1.1] for control on higher Sobolev norms) we find a sequence $(z^0_\delta)_\delta \subseteq \mathcal{W}(\Omega;\mathbb{R}^2)$ with

$$\|z^0_\delta - v\|_{L^2(\Omega)} + \|e(z^0_\delta) - e(v)\|_{L^2(\Omega)} + \mathcal{H}^1((J_{z^0_\delta} \triangle J_v) \cap \Omega) \to 0 \quad \text{as } \delta \to 0. \tag{B.9}$$

We define $w_\delta \in \mathcal{W}(\Omega';\mathbb{R}^2)$ by

$$w_\delta = \begin{cases} \displaystyle\sum_{i=0}^N \varphi_i z^i_\delta & \text{on } \Omega, \\ h & \text{on } \Omega' \setminus \overline{\Omega}. \end{cases}$$

By construction and by (B.4) (iv) we find $J_{w_\delta} \cap Q'_i = J_{z^i_\delta} \cap Q'_i$ for all $i = 1, \dots, N$. Furthermore, by adding a vanishingly small constant to z^0_δ, we may ensure $J_{w_\delta} \supseteq \partial_D\Omega \setminus \bigcup_i Q_i$. Therefore, we get

$$\begin{aligned} &\mathcal{H}^1(J_{w_\delta} \triangle J_v) \le \mathcal{H}^1((J_{z^0_\delta} \triangle J_v) \cap \Omega) \\ &\qquad + \sum_{i=1}^N (\mathcal{H}^1(J_{z^i_\delta} \cap Q_i) + \mathcal{H}^1(J_v \cap Q_i)) + \mathcal{H}^1\Big((\partial_D\Omega \setminus J_v) \setminus \bigcup_i Q'_i\Big). \end{aligned}$$

Thus, combining (B.2) (i), (B.3), (B.4) (iii), (B.8), and (B.9) we find

$$\limsup_{\delta \to 0} \mathcal{H}^1(J_{w_\delta} \triangle J_v) \lesssim \eta + \eta \sum_{i=1}^N r_i \lesssim \eta,$$

where in the last step we also used (B.2) (ii) and the fact that $\mathcal{H}^1(\partial_D\Omega) < \infty$. This shows (B.1). Combining (B.4) (i) and (B.9), we find (D1). Finally, since

$$e(w_\delta) = \sum_{i=0}^N \varphi_i e(z^i_\delta) + \nabla\varphi_i \odot z^i_\delta$$

on Ω and $e(v)$ can be written as $e(v) = \sum_{i=0}^N \varphi_i e(v) + \nabla\varphi_i \odot v$, by using (B.4) (i), (ii) and (B.9) we conclude (D2). ∎

Proof of the upper bound in Lemma 6.2. We want to show the upper semicontinuity (6.5). As we will prove, this is a consequence of u being a minimizer with respect to its own jump set; see Lemma 5.4. Let $0 \leq s \leq t$ with $s, t \in I_\infty$ and, for ε sufficiently small, suppose that $n = N(\varepsilon)$ is such that $t = t^\varepsilon_{N(\varepsilon)}$. Then the function $u(t) + h(s) - h(t)$ is an admissible competitor of the minimization problem (5.14) at time s. Thus,

$$\begin{aligned}\int_{\Omega'} \frac{1}{2} Q\big(e(u(s))\big)\,\mathrm{d}x & \\ \leq \int_{\Omega'} \frac{1}{2} Q\big(e(u(t))\big) &+ \mathbb{C}e(u(t)) : e(h(s) - h(t)) + \frac{1}{2} Q\big(e(h(s) - h(t))\big)\,\mathrm{d}x \\ &+ \kappa \mathcal{H}^1(J_{u(t)} \setminus \Gamma(s)),\end{aligned}$$

which is equivalent to

$$\begin{aligned}\mathcal{E}(s) \leq \mathcal{E}(t) + \int_{\Omega'} \mathbb{C}e(u(t)) : e(h(s) - h(t)) + \frac{1}{2} Q\big(e(h(s) - h(t))\big)\,\mathrm{d}x \\ + \kappa\big(\mathcal{H}^1(\Gamma(s)) + \mathcal{H}^1(J_{u(t)} \setminus \Gamma(s)) - \mathcal{H}^1(\Gamma(t))\big).\end{aligned}$$

Note that the term in the brackets is nonpositive since $\Gamma(s) \subseteq \Gamma(t)$ and $J_{u(t)} \subseteq \Gamma(t)$. Thus, since h is continuous in time with respect to $\|\cdot\|_{\mathrm{W}^{2,2}}$, we find that $\mathcal{E}$ has no negative jumps as claimed in the lemma. Moreover, letting $s = t^\varepsilon_{N(\varepsilon)-1}$ and iterating the above estimate (and re-using the variable s), we get that for all $\varepsilon > 0$ sufficiently small,

$$\mathcal{E}(t) \geq \mathcal{E}(0) + \sum_{k=0}^{N(\varepsilon)-1} \int_{t^\varepsilon_k}^{t^\varepsilon_{k+1}} \int_{\Omega'} \mathbb{C}e(u(t^\varepsilon_{k+1})) : e(\partial_t h(s))\,\mathrm{d}x\,\mathrm{d}s - \omega(\Delta_\varepsilon) \int_0^t \|\nabla \partial_t h\|_{\mathrm{L}^2}\,\mathrm{d}s$$

for a function ω which vanishes as Δ_ε tends to zero as in (4.9). Taking the limit superior gives the desired inequality (6.5). ■

Proof of minimality and jump-set inclusion of Lemma 7.1. The bound on $\mathcal{H}^1(\Gamma(t))$ is an immediate consequence of the lower semicontinuity (5.4). The fact that $u(t) = h(t)$ on $\Omega' \setminus \overline{\Omega}$ for all $t \in [0, 1]$ follows from Definition 4.5.

Minimality. For $t \in I_\infty$ the minimality follows from Lemma 5.4. Let us now prove the minimality (7.1) for $t \in [0, 1] \setminus I_\infty$. We know by Lemma 5.4 that for all $t_p \in I_\infty$ (used to define $u(t)$ in Definition 4.5) and all $v \in \mathrm{GSBD}^2(\Omega')$ with $v = h(t_p)$ on $\Omega' \setminus \overline{\Omega}$ it holds that

$$\int_{\Omega'} \frac{1}{2} Q\big(e(u(t_p))\big)\,\mathrm{d}x \leq \int_{\Omega'} \frac{1}{2} Q(e(v))\,\mathrm{d}x + \kappa \mathcal{H}^1(J_v \setminus \Gamma(t_p)).$$

Take $w \in \mathrm{GSBD}^2(\Omega')$ with $w = h(t)$ on $\Omega' \setminus \overline{\Omega}$ and test the above inequality with $w + h(t_p) - h(t)$ to obtain

$$\int_{\Omega'} \frac{1}{2} Q\big(e(u(t_p))\big)\,\mathrm{d}x \leq \int_{\Omega'} \frac{1}{2} Q\big(e(w + h(t_p) - h(t))\big)\,\mathrm{d}x + \kappa \mathcal{H}^1(J_w \setminus \Gamma(t_p)).$$

By weak convergence, in the limit $p \to \infty$, the left-hand side can be bounded from below by the integral $\int_{\Omega'} \frac{1}{2} Q(e(u(t)))\,\mathrm{d}x$. The integral on the right-hand side converges to the integral $\int_{\Omega'} \frac{1}{2} Q(e(w))\,\mathrm{d}x$, and since $t \notin I_\infty$, we have by continuity from above of $\mathcal{H}^1$ that the measure of the jump set converges to $\mathcal{H}^1(J_w \setminus \Gamma(t))$. This proves the desired minimality.

Jump-set inclusion. Next we argue that the inclusion (7.2) holds. If this were not true, then we could find a point $x \in J_{u(t)} \setminus \Gamma(t)$ that has one-dimensional density 1 with respect to the set $J_{u(t)}$ and density 0 with respect to $\Gamma(t)$ (see [7, Thm. 2.83 and eq. (2.41)]). Thus, for some small fixed $r > 0$, we would have

$$\mathcal{H}^1(B_r(x) \cap J_{u(t)}) \geq r/2 \quad \text{and} \quad \mathcal{H}^1(B_r(x) \cap \Gamma(t)) < r/2.$$

Since $J_{u(t_p)} \subseteq \Gamma(t)$, we know that

$$\mathcal{H}^1(B_r(x) \cap J_{u(t_p)}) < r/2.$$

Applying the lower semicontinuity of jump sets as $p \to \infty$, see Theorem 3.1 (iii) for $U = B_r(x)$, leads to a contradiction. ∎

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