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Non-invariance of Gaussian measures under the 2D Euler flow

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Abstract. In this article we consider the two-dimensional incompressible Euler equations on $\mathbb{T}^2$ and seek to characterize the Gaussian measures of jointly independent Fourier coefficients which are invariant or not. We characterize all such measures with sufficiently rapidly decaying coefficients as exactly those supported only on shear flows or cellular flows. For $H^\sigma(\mathbb{T}^2)$ ($\sigma > 3$) regularities, we give a checkable sufficient condition for non-invariance and show that this condition holds on an open and dense set in suitable topologies (and so is generic in a Baire category sense) and check the condition using interval arithmetic for a basic example. We conjecture that the same characterization at high regularities holds in all H^σ at least down to $\sigma > 0$. We also pose a few other related conjectures, which we believe to be approachable.

1. Introduction

1.1. The 2D Euler equations and vorticity formulation

In this article we consider the two-dimensional incompressible Euler equations posed on $\mathbb{T}^2 = (\mathbb{R}/(2\pi\mathbb{Z}))^2$ in vorticity formulation, written here as

$$\begin{cases} \partial_t \Omega + u \cdot \nabla \Omega = 0 & \text{for } (t,x) \in (0,\infty) \times \mathbb{T}^2, \\ \Omega(0,x) = \Omega_0(x) & \text{for } x \in \mathbb{T}^2, \end{cases} \tag{E}$$

where $\Omega\colon \mathbb{T}^2 \to \mathbb{R}$ is the scalar vorticity, and $u\colon \mathbb{T}^2 \to \mathbb{R}^2$ is the velocity, which is related to Ω via the Biot–Savart law,

$$\begin{cases} u = \nabla^\perp(-\Delta)^{-1}\Omega, \\ \Omega = \nabla \wedge u = \partial_1 u_2 - \partial_2 u_1, \end{cases}$$

where $\nabla^\perp = (-\partial_2, \partial_1)$.

For $\sigma > 1$, the vorticity equations (E) are globally well posed in $\mathcal{C}^0([0,\infty), H^\sigma(\mathbb{T}^2))$ (originally due to Kato [47], following the older classical work in $C^{0,\alpha}$ Hölder spaces by Hölder and Wolibner [41,77]). The central question regarding the 2D incompressible Euler

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equations is to understand the long-time dynamics in various settings. One of the main questions is to determine whether one has the formation of small scales in the vorticity. It is widely expected that "generic" solutions to the 2D Euler equations create smaller scales, leading to unbounded growth in the H^s and $C^{0,\alpha}$ norms. Even more is expected: that generic orbits are not pre-compact in L^2 due to the loss of enstrophy (L^2 mass of vorticity) to higher frequencies. The latter is known as Šverák's conjecture (see, e.g., [27, Conjecture 3]). See the recent review article [27] and Section 1.4 below for more discussion.

One natural question one can ask about the long-time dynamics of a PDE is how to find measures which are left invariant under its flow. In this article we are specifically interested in the possible invariance of Gaussian measures under the flow Φ_t of (E). It is known that the Gibbs measure, formally written "$\mathcal{Z}^{-1}\exp(-\frac{1}{2}\|\Omega\|_{L^2_x}^2)\mathrm{d}\Omega$" of (E) – namely the spatial white noise measure, supported on vorticities which are $H^{-1-\varepsilon}_{\mathrm{loc}}$ for all $\varepsilon > 0$ – is invariant under the flow in a suitable sense; see [1, 34]. In the non-periodic setting [22], Gaussian invariant measures have been constructed for the two-dimensional Euler equations at very low regularity (typically $\Omega \in H^{\beta}(\mathbb{R}^2)$, $\beta < -2$).

In this article we seek to characterize the invariant Gaussian measures with independent Fourier coefficients at higher regularities. One expects, and we conjecture, that the only such invariant measures are supported on the trivial stationary states: shear flows and cellular flows. This article seeks to make progress on this conjecture. We resolve this conjecture entirely in the case of measures with very rapidly decaying coefficients (slightly faster than exponential). In H^{σ} for $\sigma > 3$, we provide a numerically checkable sufficient condition for non-invariance, and show that the generic (in the Baire category sense) choice of such measures is non-invariant. We also use a computer-assisted proof via interval arithmetic to explicitly check the condition for an example measure. See Conjecture 1.10 for related open directions and Section 1.4 for more discussion about the motivations and existing literature.

1.2. Random initial data and invariant measures

Let $(\Xi, \mathcal{A}, \mathbb{P})$ be a probability space.

In this article we solve (E) with Gaussian random initial data Ω_0 for (E) of the form

$$\Omega_0^{\omega}(x) = \sum_{n\in\mathbb{Z}^2} a_n g_n^{\omega} e^{in\cdot x}, \tag{1.1}$$

where the $(g_n)_{n\in\mathbb{Z}^2}$ satisfy some independence assumptions and are identically distributed standard complex Gaussian random variables, and $(a_n)_{n\in\mathbb{Z}^2}$ is a complex-valued sequence. Note that replacing $a_n \mapsto a_n e^{i\theta_n}$ for any sequence of phases $\theta_n \in \mathbb{R}$ defines the same random initial data in law. For this reason we restrict to real-valued a_n without loss of generality.

Let us be more precise and let $\mathbb{Z}^2_+ := \{(n_1, n_2) \in \mathbb{Z}^2,\ n_1 > 0 \text{ or } n_1 = 0,\ n_2 \geqslant 0\}$. We will work under the following assumption for the $(g_n)_{n\in\mathbb{Z}^2}$.

Assumption 1.1. For all $n \in \mathbb{Z}^2_+$, $g_n := r_n + i s_n$ where $\{r_n, s_n\}_{n\in\mathbb{Z}^2_+}$ are independent and identically distributed real standard Gaussians $\mathcal{N}(0,1)$, and $g_{-n} = \bar{g}_n$ for all $n \in \mathbb{Z}^2_+$.

Let us use the norm defined by

$$\|(a_n)\|_{h^\sigma} := \left(\sum_{n\in\mathbb{Z}^2} \langle n\rangle^{2\sigma} |a_n|^2 \right)^{1/2},$$

and make the following assumptions on the $(a_n)_{n\in\mathbb{Z}^2}$, which are tailored to ensure that almost-surely $\Omega_0^\omega \in H^\sigma$.

Assumption 1.2. We assume that $(a_n)_{n\in\mathbb{Z}^2} \in h^\sigma$, where

$$h^\sigma = \big\{(a_n)_{n\in\mathbb{Z}^2} : a_n \in \mathbb{R},\ a_0 = 0,\ a_{-n} = a_n,\ \|(a_n)\|_{h^\sigma} < \infty\big\}.$$

We recall that h^σ is a Hilbert space equipped with the inner product associated with the complete norm $\|\cdot\|_{h^\sigma}$.

Given the isotropic nature of (E) we also consider the *radially symmetric* sequences belonging to the space

$$h^\sigma_{\mathrm{rad}} = \big\{(a_n) \in h^\sigma : a_m = a_n, \text{ for all } m, n \text{ such that } |m| = |n|\big\}.$$

Remark 1.3. Assumptions 1.1 and 1.2 ensure that (1.1) is real valued. The condition $a_0 = 0$ ensures that $\int_{\mathbb{T}^2} \Omega_0(x)\,\mathrm{d}x = 0$, which should be the case since $\Omega_0 = \nabla \wedge u(0,\cdot)$.

Let us denote by $\mu_{(a_n)}$ the law that the random variable (1.1) induces on $L^2(\mathbb{T}^2, \mathbb{R})$ (assuming $\sigma \geqslant 0$, and that $(g_n)_{n\in\mathbb{Z}^2}$ satisfies Assumption 1.1). The set of such corresponding measures is therefore defined by

$$\mathcal{M}^\sigma := \big\{\mu_{(a_n)},\ (a_n)_{n\in\mathbb{Z}^2} \in h^\sigma\big\}, \tag{1.2}$$

as well as its radial counterpart $\mathcal{M}^\sigma_{\mathrm{rad}} := \{\mu_{(a_n)},\ (a_n)_{n\in\mathbb{Z}^2} \in h^\sigma_{\mathrm{rad}}\}$.

We remark that for any $\mu \in \mathcal{M}^\sigma$ there holds

$$\mathbb{E}_\mu[\|\Omega\|^2_{H^\sigma}] := \int_{L^2} \|\Omega\|^2_{H^\sigma}\,\mathrm{d}\mu(\Omega) = \mathbb{E}[\|\Omega_0^\omega\|^2_{H^\sigma}] = \sum_{n\in\mathbb{Z}^2} \langle n\rangle^{2\sigma}|a_n|^2 < \infty, \tag{1.3}$$

which implies that $\operatorname{supp}\mu \subset H^\sigma(\mathbb{T}^2)$.

Let us recall that a measure μ on H^σ is said to be invariant under a flow Φ_t if for any $t \in \mathbb{R}$ there holds $(\Phi_t)_*\mu = \mu$, i.e., for all measurable $A \subset H^\sigma$,

$$\mu(\Omega_0 : \Omega(t) \in A) = \mu(\Omega_0 : \Omega_0 \in A).$$

We will use the notation $\mathfrak{m}(\nabla)$ as the Fourier multiplier given by $\mathfrak{m}(n)$ for each $n \in \mathbb{Z}^2$.

At an informal level, one might expect a Gaussian measure ν written as[1]

$$\nu = \mathcal{Z}^{-1} \exp(-\|\mathfrak{m}(\nabla)\Omega\|_{L^2}^2)\, d\Omega \tag{1.4}$$

to be invariant under the flow of (E) if and only if the following conditions hold:

(i) The support of $\mathfrak{m}$ is invariant under (E);

(ii) For smooth solutions with the same Fourier support as $\mathfrak{m}$, $\|\mathfrak{m}(\nabla)\Omega\|_{L^2}^2$ is conserved by the flow of (E).

However, it is not clear how to justify this in general, and moreover, it is not known how to characterize all such conservation laws (one expects energy and enstrophy to be the only conservation laws that can be expressed in this manner; however, this remains open).

1.3. Main results

We first state our main technical lemma, which is a set of criteria that detects measures in $\mathcal{M}^\sigma$ which are non-invariant under the flow of (E). For any sequence (a_n) and Fourier multiplier $\mathfrak{m}\colon \mathbb{Z}^2 \to \mathbb{C}$, we introduce $\gamma^{\mathfrak{m}}$:

$$\gamma^{\mathfrak{m}}_{(a_n)} = \frac{1}{(2\pi)^4} \sum_{n,q\in\mathbb{Z}^2} |a_n|^2 |a_q|^2 \frac{(q\cdot n^\perp)^2}{2} \Big(\frac{1}{|n|^2} - \frac{1}{|q|^2}\Big) \beta^{\mathfrak{m}}_{n,q}, \tag{1.5}$$

where $\beta^{\mathfrak{m}}_{n,q}$ is

$$\begin{aligned} \beta^{\mathfrak{m}}_{n,q} = {} & |\mathfrak{m}(n)|^2 \Big(\frac{1}{|q|^2} - \frac{1}{|q+n|^2}\Big) + |\mathfrak{m}(q)|^2 \Big(\frac{1}{|q+n|^2} - \frac{1}{|n|^2}\Big) \\ & + |\mathfrak{m}(n+q)|^2 \Big(\frac{1}{|n|^2} - \frac{1}{|q|^2}\Big). \end{aligned} \tag{1.6}$$

We omit the (a_n) when it is clear from the context. We denote the special cases γ_s for the case $\mathfrak{m}(n) = \langle n\rangle^s$ and γ^{n_0} for the case $\mathfrak{m}(n) = \mathbf{1}_{n=n_0}$ (and similarly for β). Note that $\mathfrak{m}$ does not play the same role as the $\mathfrak{m}$ appearing in (1.4). We refer to Section 2.1 for an account of the standard notation that we use in this article.

We prove the following result concerning the measures $\mu_{(a_n)}$ under assumptions on the $\gamma^{\mathfrak{m}}_{(a_n)}$.

Theorem A. *Let $\sigma > 3$ and $\mu = \mu_{(a_n)} \in \mathcal{M}^\sigma$, where $\mathcal{M}^\sigma$ is defined by (1.2). Let A satisfy $|\mathfrak{m}(n)| \lesssim \langle n\rangle^s$ for some $0 \le s < \sigma - 3$, and let $\gamma^{\mathfrak{m}} = \gamma^{\mathfrak{m}}_{(a_n)}$ be defined by (1.5) with (1.6). Then, if $\gamma^{\mathfrak{m}} \neq 0$, μ is not invariant under the flow of (E).*

The proof basically proceeds by showing that initial conditions sampled from any invariant measure $\mu_{(a_n)}$ with $\gamma^{\mathfrak{m}}_{(a_n)} \neq 0$ will satisfy

$$\mathbb{E}[\|\mathfrak{m}(\nabla)\Omega^\omega(t)\|_{L^2}^2] - \mathbb{E}[\|\mathfrak{m}(\nabla)\Omega_0^\omega\|_{L^2}^2] \approx \gamma^{\mathfrak{m}}_{(a_n)} t^2 + \mathcal{O}(t^3) \quad \text{as } t \to 0,$$

[1] One can rigorously construct this measure as the image measure of a cylindrical Wiener process.

leading to a contradiction. This is done by justifying a Taylor expansion in t for the evolution of the vorticity along the flow of (E). It is important to note that the proof uses the assumption of invariance of the measure and the a priori estimates it implies in order to justify this approximation. Factors arise in the proof that would otherwise not be integrable, such as terms of the form $\mathbb{E}[\exp(c\|\Omega_0^\omega\|_{H^s}^3)]$; a similar idea is used in [24].

Remark 1.4. Our proof of Theorem A takes advantage of the diagonal covariance structure of the Gaussian measures that we consider in this article. We believe however that it should be possible to derive similar conditions for Gaussian measures with non-diagonal covariance operators.

We give several applications of this lemma. The first is a complete characterization of all such Gaussian invariant measures with at least slightly ultra-analytic regularity.

Theorem B. *Let $\mu = \mu_{(a_n)}$, where the sequence $(a_n)_{n\in\mathbb{Z}^2}$ satisfies Assumption 1.2, and the decay assumption $|a_n|^{\frac{1}{n}} \xrightarrow[|n|\to\infty]{} 0$.*

Then μ is invariant under the flow of (E) if and only if $\mathrm{supp}(a_n)$ is included in a line of $\mathbb{Z}^2$ passing through the origin, or included in a circle centered at the origin; in both cases, μ is supported on steady states of (E).

Remark 1.5. Note the above theorem includes the case of (a_n) compactly supported in $\mathbb{Z}^2$.

Remark 1.6. The decay assumption essentially amounts to asking for decay faster than exponential. Such a condition is met for instance by the sequence $a_n := \exp(-|n|\log|n|)$.

Remark 1.7. It was proven in [31] that the only stationary solutions of (E) with compactly supported Fourier transforms are the ones whose Fourier transform support is contained in a line in $\mathbb{Z}^2$ or in a circle of $\mathbb{Z}^2$ centered at the origin. Theorem B can be considered an extension of this result to invariant measures.

Although we expect the result to hold, the proof of Theorem B does not apply to lower regularities, e.g., Sobolev space regularity. However, we can prove that the non-invariance is at least generic in the Baire category sense.

Our next task is to study the condition $\gamma_{s,(a_n)} = 0$ (therefore in the case $\mathfrak{m}(n) = \langle n\rangle^s$) and prove that for a large class of $(a_n)_{n\in\mathbb{Z}^2}$ belonging to h^σ, there holds $\gamma_{s,(a_n)} \neq 0$ for some $s \in (0,\sigma]$. To this end, let us introduce the set

$$K^\sigma := \big\{(a_n)_{n\in\mathbb{Z}^2} \in h^\sigma, \text{ such that there exists } s \in (0,\sigma-3) : \gamma_{s,(a_n)} \neq 0\big\}.$$

Denote by $\mathcal{K}^\sigma \subset \mathcal{M}^\sigma$ the set of corresponding measures. Similarly, we also define K^σ_{rad} and $\mathcal{K}^\sigma_{\mathrm{rad}}$. Theorem A guarantees that all measures in $\mathcal{K}^\sigma$ are not invariant under the two-dimensional Euler flow. Furthermore, we have the following.

Theorem C (Genericity in the Baire sense). *Let $\sigma > 3$.*

(i) *The set K^σ (resp. K^σ_{rad}) is an open, dense set in h^σ (resp. h^σ_{rad}).*

(ii) *Similarly, $\mathcal{K}^\sigma$ (resp. $\mathcal{K}^\sigma_{\mathrm{rad}}$) is an open dense set of $\mathcal{M}^\sigma$ (resp. $\mathcal{M}^\sigma_{\mathrm{rad}}$) equipped with the topology induced by the Wasserstein W_1 distance associated with the H^σ metric.*

We recall the Kantorovich–Rubinstein duality formulation of the Wasserstein distance: if $\varphi: H^\sigma \to \mathbb{R}$, we define

$$\begin{aligned} \|\varphi\|_{\mathrm{Lip}} &:= \sup_{f \in H^\sigma} |\varphi(f)| + \sup_{f,g \in H^\sigma} \frac{|\varphi(f) - \varphi(g)|}{\|f - g\|_{H^\sigma}}, \\ W_1(\mu, \nu) &= \sup_{\substack{\varphi: H^\sigma \to \mathbb{R} \\ \|\varphi\|_{\mathrm{Lip}} \leqslant 1}} \left| \int_{H^\sigma} \varphi(u) \, \mathrm{d}\nu(u) - \int_{H^\sigma} \varphi(u) \, \mathrm{d}\mu(u) \right|. \end{aligned} \tag{1.7}$$

Remark 1.8. It follows that the set of all measures in $\mathcal{M}^\sigma$ which are invariant under the two-dimensional Euler flow are included in a closed, nowhere dense set of $\mathcal{M}^\sigma$ (in the Wasserstein-1 topology), which is hence a meager set in the sense of Baire category.

We point out that this notion of genericity is strong enough to ensure that many mutually singular measures are included.[2] This will follow from the Feldman–Hájek theorem.

Corollary 1.9 (Mutually singular non-invariant measures). *Let $\sigma > 3$. Every open set $U \subset h^\sigma$ contains an uncountable set $I \subset U$ such that the corresponding measures $(\mu_\xi)_{\xi \in I} \subset \mathcal{M}^\sigma$ are not invariant under the two-dimensional Euler equation and are pairwise mutually singular.*

One obvious disadvantage of genericity results is that they provide no specific examples. However, as the condition in Theorem A can be checked using validated numerics [74], one can provide computer-assisted proofs for given sequences. In Section 6.5 we carry this out on the following explicit example.

Theorem D. *Let $(a_n)_{n \in \mathbb{Z}^2}$ be defined by $a_0 = 0$ and $a_n = \frac{1}{\langle n \rangle^5 \log(3 + \langle n \rangle^2)}$ for $n \neq 0$. Then the measure $\mu = \mu_{(a_n)} \in \mathcal{M}^4_{\mathrm{rad}}$ satisfies $\gamma_{s,(a_n)} > 0$ for some $s \in (0, 1)$ and is therefore not invariant.*

We end with a few conjectures. The first conjecture is that the characterization in Theorem B holds at all regularities. Notice that Conjecture 1.10 specifically relates to Gaussian measures for which the Fourier modes are mutually independent. It is possible that one might be able to construct invariant Gaussian measures which are of a different nature (e.g., supported on quasi-periodic motions). These should also be "non-generic" in some sense, but it is less clear how to characterize this in generality at the moment.

Let us recall that we say that a measure μ is quasi-invariant under a flow Φ_t if for all $t \in \mathbb{R}$ the push-forward measure $\mu_t := (\Phi_t)_* \mu$ is equivalent to μ, i.e., $\mu_t \ll \mu \ll \mu_t$.

[2] In particular, such a family of measures cannot consist of measures that have densities with respect to a common reference measure.

Note that investigating quasi-invariance of Gaussian measures, and not only invariance, is very important, due to the fact that the existence of quasi-invariant measures (in some quantitative way) often leads to new information on the long-time dynamics; see, e.g., the discussion in Section 1.4.2 and Remark 1.14.

Conjecture 1.10. *Let $\sigma > 0$ (or possibly even $\sigma > -1$ if a suitable local well-posedness theory can be established). A measure $\mu = \mu_{(a_n)} \in \mathcal{M}^\sigma$ is quasi-invariant under the flow of (E) if and only if the sequence $(a_n)_{n\in\mathbb{Z}^2}$ is supported in a line of $\mathbb{Z}^2$ passing through the origin or on a circle centered at the origin.*

Remark 1.11. Proving non-invariance (rather than non-quasi-invariance) might be a much easier task.

Remark 1.12. Note that for $\sigma \in (0, 1)$, the random initial data (1.1) is almost surely $W^{\sigma,p}$ for all $p < \infty$, and in particular, the initial conditions are almost surely $C^{0,\alpha}$ for some $\alpha > 0$ (see, e.g., [45]). Hence, these initial data lie in a classical local well-posedness space. For $\sigma = 0$ the initial conditions are almost-surely in $L^2 \cap \mathrm{BMO} \setminus L^\infty$ [45, Chapter 5]; however, we are unsure whether it lies in a local well-posedness class (though we believe it likely does; see [76] for well-posedness results in borderline regularity spaces). For $\sigma < 0$, initial conditions are almost-surely distribution valued, and we are not aware of any theory that would even provide local existence of solutions.

Remark 1.13. In the recent work [23], the author constructs a large set of smooth initial data (random Gaussian initial data of the form (1.1)), which give rise to global solutions to the Euler equations posed on the torus of size L. It is proven that in the large box limit $L \gg 1$, the probability distributions of the associated solutions to the Euler equations remain approximately Gaussian, but nothing is said about quasi-invariance of these Gaussian measures. The setting considered in [23] differs from ours, and is closely connected to a large-box effect. Note that, though it does not seem likely, it could be the case that the evolution of a Gaussian measure under the Euler flow remains Gaussian, but that this evolution is not *quasi-invariant*.

Remark 1.14. Conjecture 1.10 would be in contrast with the non-linear dispersive equation setting, for which many quasi-invariance results are known for similar Gaussian measures; see, e.g., [36,63,65,75]. To the best of the authors' knowledge, non-quasi-invariance of (sufficiently regular) Gaussian measures has only been proven for few examples, the first being a non-dispersive ODE, namely $i\partial_t u = |u|^2 u$ posed on the torus, which is explicitly solvable,[3] see [63, Theorem 1.6]; and the second being the proof that the Gaussian measure formally given by $\mu_s = \exp(-\frac{1}{2}\|u\|_{H^s}^2)\,\mathrm{d}u$ is not quasi-invariant under the flow Ψ_t of the cubic Szegő equation $i\partial_t u = \Pi_{>0}(|u|^2 u)$ on $\mathbb{T}$ for $s \in (\frac{1}{2}, 1) \setminus \{\frac{3}{4}\}$, see [21]. We also refer to [20,39] for results related to the stochastic Navier–Stokes equations.

[3] Note that the method used in [63] seems rather difficult to implement for the Euler equations.

In general, we believe that it would be interesting to further investigate the possible non-quasi-invariance of Gaussian measures.

As our article will make clear, the proof of non-invariance does not require $\gamma_{s,(a_n)} > 0$, but only that $\gamma_{s,(a_n)} \neq 0$. The proof suggests (but does not quite prove) that the H^s norm of the randomly initialized data grows on average for short time if $\gamma_{s,(a_n)} > 0$, but if $\gamma_{s,(a_n)} < 0$ it is actually decaying. We are not aware of any example of sequence $(a_n)_{n\in\mathbb{Z}^2}$ such that $\gamma_{s,(a_n)} < 0$. Specifically, we conjecture that such examples do not exist in $\mathcal{M}^\sigma$ and that data randomly initialized in this manner will grow in H^s on average. We are unsure under what conditions one could conjecture that short-time norm growth will happen almost surely.

Conjecture 1.15. *Let $\sigma > 0$. If $(a_n)_{n\in\mathbb{Z}^2} \in h^\sigma$ is not exclusively supported in a line passing through the origin or a circle centered at the origin, then for initial data sampled using (1.1) there exists $C(s) > 0$ such that there holds*

$$\mathbb{E}\|\Omega^\omega(t)\|_{H^s} \geqslant \mathbb{E}\|\Omega_0^\omega\|_{H^s} + Ct^2,$$

for all $t \in [0, t_0)$ and all $s \in (0, \sigma]$, where $t_0 = t_0((a_n), s)$.

Of course, we also believe the following variant of Šverák's conjecture:

$$\lim_{t\to\infty} \mathbb{E}\|\Omega^\omega(t)\|_{H^s} = +\infty,$$

and even the original version: that the random orbits $\{\Omega^\omega(t)\}_{t\in\mathbb{R}}$ are almost-surely not pre-compact in L^2. However, these stronger conjectures seem very far out of reach at the current time, whereas Conjectures 1.10 and 1.15 seem likely to be manageable.

1.4. Related literature and motivation for the results and conjectures

1.4.1. Creation of small scales in 2D Euler. As mentioned in the introduction, it is classical that (E) is well posed in $\mathcal{C}^0([0,\infty), H^\sigma(\mathbb{T}^2))$ whenever $\sigma > 1$. Moreover, in [81] it was proven that for all $t \geqslant 0$,

$$\|\Omega(t)\|_{H^\sigma} \leqslant C(\Omega_0)e^{e^{Ct}} \tag{1.8}$$

(a similar double-exponential bound for $C^{0,\alpha}$ was proved earlier by Wolibner [77]). Optimality of the bound (1.8) in generality remains an open question. On $\mathbb{T}^2$, the examples of gradient growth are generally based on the Bahouri–Chemin cross [2], which uses a hyperbolic point in the flow to drive rapid growth of the gradient at a single point. Exponential growth can be obtained [82], and double exponential growth can be maintained on finite time [25]. If one replaces $\mathbb{T}^2$ with the unit disk of $\mathbb{R}^2$, Kiselev and Šverák showed that the double exponential growth (1.8) is sharp [51] (see [79] for the extension to any symmetrical smooth domain). In some sense, these examples are also based on the Bahouri–Chemin cross, but also inspired by the computations of finite-time singularity of the 3D Euler equations in a cylinder [59] (which have recently been made mathematically rigorous [19]).

Another class of norm growth results is based on shearing. The first example of "generic" norm growth was demonstrated by Nadirashvili [62], who demonstrated examples of small perturbations of the Couette flow in a periodic channel and the corresponding Taylor–Couette in an annulus, which lead to unbounded norm growth like $\|\nabla\omega(t)\|_{L^\infty} \gtrsim t$. The recent results of [28] greatly expand on this idea to produce many more examples of "wandering" solutions and unbounded gradient growth on open sets of initial data in a variety of settings (again near stable stationary states). Using the mechanism of inviscid damping, a stronger characterization of small-scale creation – the loss of L^2 pre-compactness of vorticity – has been proved in a few examples for open sets of small perturbations of special stationary states in sufficiently smooth topologies. This was first done for the Couette flow in $\mathbb{T} \times \mathbb{R}$ [7] followed by results near a point vortex in the plane in [43], and strictly monotone shear flows in a channel satisfying a suitable spectral condition (no embedded eigenvalues) [42, 44, 60].

1.4.2. Invariant measures of Euler. If a Gaussian measure μ on H^σ is known to be invariant for the dynamics of a partial differential equation, for which a standard local Cauchy theory[4] is known, then the global solutions $\Omega^\omega(t)$ starting at time zero from Ω_0^ω (which has law μ) almost-surely satisfy the bound

$$\|\Omega^\omega(t)\|_{H^\sigma} \leqslant C_\omega \sqrt{\log(t+2)}, \tag{1.9}$$

where the random constant C_ω satisfies $\mathbb{E}[e^{cC_\omega^2}] < \infty$. This is known as the Bourgain globalization argument [13]. Such a bound for the growth of the Sobolev norms is in sharp contrast with (1.8) and seems at odds with the existing norm growth results discussed in Section 1.4.1 and the observed dynamics of generic initial conditions on the torus or the sphere (see numerical simulations in, e.g., [26,61,70,72] and the references therein and the discussion in Section 1.4.3 below). Note that if the invariant measure μ only has weaker moment bounds, or the measure μ is only quasi-invariant (in a quantitative way), then an upper bound for the growth of Sobolev norms similar to (1.9) is known to hold with $\sqrt{\log(t+2)}$ replaced by a different function of time (usually a polynomial) depending on the moments, the PDE, or the quasi-invariance parameters.

For all $\sigma \geqslant 1$, in [54, 55], invariant measures of the 2D Euler equations were constructed in H^σ from suitable limits of regularized problems using compactness methods. See also [35, 71] for the SQG model or the septic Schrödinger equations cases. On the support of these measures, one can show growth estimates of the type

$$\|\Omega(t)\|_{H^\sigma} \leqslant C_\omega(1+t)^\alpha, \quad \alpha > 0,$$

but an important question remains: What is the support of such invariant measures? Very little is currently known (e.g., we do not know whether all such measures are in

[4] That is, if $\|\Omega_0\|_{H^s} \sim \lambda$ then $\Omega(t)$ exists in H^s on times $[0,\tau]$, $\tau \sim c\lambda^{-1}$ and for every $t \in [0,\tau]$ there holds $\|\Omega(t)\|_{H^s} \leqslant 2\lambda$.

fact supported only on stationary solutions). The identification and characterization of "fluctuation-dissipation measures" (i.e., limits obtained by adding stochastic noise balanced against viscosity and taking inviscid limits, as in for example [35, 54]) and similar limiting measures is mostly open for non-linear problems, especially PDEs. We do not know whether these are Gaussian measures or not. See [4–6] for the much simpler problem of passive scalars where the corresponding measures can be classified, sometimes exactly.

1.4.3. Random initial data and 2D turbulence. The *direct enstrophy cascade*, i.e., the tendency for enstrophy to be transferred from large to small scales, is central to the modern understanding of 2D turbulence. Turbulence primarily concerns the inviscid limit of the Navier–Stokes equations (rather than directly on Euler) and the 2D turbulence problem is greatly complicated by the presence of the inverse energy cascade. See [3, 33, 52, 56] where predictions about this dual cascade were first made and [8, 10, 11, 29, 53, 72] for more discussion. For empirical evidence of 2D turbulence see the surveys [11, 16, 50], for observations from atmospheric data see [18, 58], and for a few of the many experiments see, e.g., [38, 49, 64, 66, 68] and the references therein.

We remark that physicists and engineers often consider turbulence in a statistically stationary state, i.e., where external driving produces a continuous input of enstrophy, either from the boundaries or through body forcing, and the $t \to \infty$ limit is taken before the inviscid limit (see discussions in [11]), which is somewhat different from a random initial data problem. All classical predictions, such as the analogue of the Kolmogorov 4/5 law (see [8, 9, 17, 32, 58, 78, 80]) and the Batchelor–Kraichnan power spectrum are most easily interpreted in this regime. Nevertheless, the random initial data problem is observed to lead to closely related dynamics referred to as "decaying turbulence", where one sees a transient enstrophy cascade, often generically with a decaying Batchelor–Kraichnan spectrum (see, e.g., [15, 40, 67] and the references therein). The inverse cascade, at least on $\mathbb{T}^2$ or the disk, is observed to lead to behavior sometimes referred to as "condensation" or "vortex crystallization" in which most of the energy congregates into certain large-scale coherent structures; see, e.g., [37] and [69].

In any case, the existing norm growth results discussed above, and especially Šverák's conjecture, are intimately connected to the ability of 2D Euler to send enstrophy to small scales, as observed in 2D turbulence (Shnirelman's conjecture [27, Conjecture 4] is also intimately related to the observed condensation effect). The observed turbulent dynamics strongly suggest that the Euler equations initialized from data where the spatial and frequency scales are sufficiently independent are likely to create smaller scales provided the initial conditions are more regular than at least H^1 (where the non-linear interactions of small scales could formally start to dominate) but more likely the result holds at least all the way down to the regularity of the Batchelor–Kraichnan spectrum (corresponding to $H^{-\varepsilon}$ regularity, i.e., $a_n = \langle n \rangle^{-2}$). It even seems plausible that anything more regular than the Gibbs measure should not be invariant unless it is extremely degenerate. These ideas motivate both Conjecture 1.10 and Conjecture 1.15.

1.5. Plan of the paper

In Section 2 we introduce the notation used in this paper and gather some standard facts about the bilinear form of the Euler equations for the convenience of the reader. In Section 3 we provide the reader with the main idea underlying the proof of Theorem A, which is then reduced to proving two key statements: Proposition 3.1, which is the goal of Section 4 and Proposition 3.2 which is the goal of Section 5. Then Section 6 is devoted to proving Theorem C, Corollary 1.9, and Theorem B.

2. Preliminaries

2.1. Notation used

We recall that in this article $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

We write $\langle n\rangle = \sqrt{1+|n|^2}$, where $|n|^2 = n_1^2 + n_2^2$ for $n = (n_1, n_2) \in \mathbb{Z}^2$ and $n^\perp = (-n_2, n_1)$.

We recall that the Fourier transform of a regular-enough $f\colon \mathbb{T}^2 \to \mathbb{R}^N$ is the function $\hat{f}\colon \mathbb{Z}^2 \to \mathbb{R}^N$ defined by

$$\hat{f}(n) = \int_{\mathbb{T}^2} f(x) e^{-in\cdot x}\, \mathrm{d}x,$$

which we simply refer to as $\hat{f}_n$ for notational simplicity. We also write $\mathcal{F}(u)$ in place of $\hat{u}$. The Fourier inversion formula therefore reads

$$f(x) = \frac{1}{(2\pi)^2} \sum_{n\in\mathbb{Z}^2} \hat{f}_n e^{in\cdot x}.$$

We define the space $H^\sigma(\mathbb{T}^2)$ as the set of functions f such that $\|f\|_{H^\sigma} < \infty$ where

$$\|f\|_{H^\sigma} := \left(\sum_{n\in\mathbb{Z}^2} \langle n\rangle^{2\sigma} |\hat{f}_n|^2 \right)^{\frac{1}{2}}.$$

2.2. The bilinear form

Let us denote by $U[\Omega]$ the vector field $\mathbb{T}^2 \to \mathbb{R}^2$ defined by $U[\Omega] = \nabla^\perp(-\Delta)^{-1}\Omega$, which is uniquely defined in the class of divergence-free vector fields. We recall that thanks to Calderón–Zygmund estimates, the linear operator $f \mapsto \nabla U[f]$ is continuous $L^p \to L^p$ for any $p \in (1,\infty)$.

We define the bilinear form for any smooth enough functions $f, g\colon \mathbb{T}^2 \to \mathbb{R}$ by

$$B(f,g) = \frac{1}{2} U[f]\cdot\nabla g + \frac{1}{2} U[g]\cdot\nabla f,$$

and use the notation $B_1(\Omega) := B(\Omega,\Omega) = U[\Omega]\cdot\nabla\Omega$.

We will also use the following notation:

$$\begin{aligned} B_2(\Omega) &= B(\Omega, B_1(\Omega)), \\ B_3(\Omega) &= B(B_1(\Omega), B_1(\Omega)) + 2B(\Omega, B_2(\Omega)), \\ B_3'(\Omega) &= B(B_1(\Omega), B_2(\Omega)), \quad \tilde{B}_3(\Omega) = B(B_2(\Omega), B_2(\Omega)). \end{aligned}$$

Lemma 2.1 (Properties of the B_i). *Let $f, g, \Omega: \mathbb{T}^2 \to \mathbb{R}$. Then*

(i) $(U[g] \cdot \nabla f, f)_{L^2} = 0$;

(ii) *for any $s > 0$, and any $0 < \varepsilon < s$ there holds*

$$|\langle B(f, f), f \rangle_{H^s}| \leqslant C(s, \varepsilon) \| f \|_{H^{1+\varepsilon}} \| f \|_{H^s}^2, \tag{2.1}$$

and for any $s > 1$ there holds

$$|\langle B(f, g), f \rangle_{H^s}| \leqslant C(s, \varepsilon) \| g \|_{H^{s+1+\varepsilon}} \| f \|_{H^s}^2; \tag{2.2}$$

(iii) *similarly, for any $0 < \varepsilon < s$ there hold*

$$\begin{aligned} &\| B_1(\Omega) \|_{H^s} \leqslant C(s, \varepsilon) \| \Omega \|_{H^{s+1}}^2, \\ &\| B_2(\Omega) \|_{H^s} \leqslant C(s, \varepsilon) \| \Omega \|_{H^{s+2}}^3, \\ &\| B_3(\Omega) \|_{H^s} \leqslant C(s, \varepsilon) \| \Omega \|_{H^{s+3}}^4, \\ &\| B_3'(\Omega) \|_{H^s} \leqslant C(s, \varepsilon) \| \Omega \|_{H^{s+3}}^5, \quad \| \tilde{B}_3(\Omega) \|_{H^s} \leqslant C(s, \varepsilon) \| \Omega \|_{H^{s+3}}^6. \end{aligned}$$

Proof. Part (i) is a consequence of the fact that $U[b]$ is divergence-free.

For (ii), we rely on the following Kato–Ponce commutator estimate:

$$\begin{aligned} \| [\langle \nabla \rangle^s, U[f] \cdot \nabla] g \|_{L^2(\mathbb{T}^2)} \lesssim \| U[f] \|_{W^{1,\infty}(\mathbb{T}^2)} \| g \|_{H^s(\mathbb{T}^2)} \\ + \| f \|_{H^s(\mathbb{T}^2)} \| g \|_{W^{1,\infty}(\mathbb{T}^d)}. \end{aligned} \tag{2.3}$$

This is direct consequence of the Kato–Ponce estimate [48] in the Euclidean setting, and can be transferred to the case $\mathbb{T}^2$ as was done in [57, Lemma 2.2]. See also [73] for more general considerations. Then estimate (2.1) is a consequence of (2.3), (i), and the Sobolev embedding $H^{1+\varepsilon} \hookrightarrow L^\infty$, as well as the fact that $f \mapsto \nabla U[f]$ is continuous as a linear map $H^s \to H^s$. Estimate (2.2) follows from similar computations and product estimates on $U[f] \cdot \nabla g$ in H^s.

Finally, (iii) follows from product laws in Sobolev spaces. ∎

2.3. Invariant measures bounds

We recall the following almost-sure upper bounds for the growth of Sobolev norms for solutions to (E) with initial data sampled on the support of an invariant Gaussian measure.

Proposition 2.2 ([13]). *Let $\sigma > 2$ and μ be a Gaussian measure on $H^\sigma(\mathbb{T}^2)$ which is invariant under the flow of (E). Then there exist a deterministic constant $c > 0$ and a random constant C_ω such that for almost every ω there holds for any $t \geqslant 0$,*

$$\|\Omega^\omega(t)\|_{H^\sigma} \leqslant C_\omega \sqrt{\log(2+t)},$$

and where $\mathbb{E}[e^{cC_\omega^2}] < \infty$.

3. Proof of Theorem A

The main idea in the proof of Theorem A is that in short time we expect the following expansion:

$$\Omega^\omega(t) \simeq \Omega_0^\omega - tB_1(\Omega_0^\omega) + t^2 B_2(\Omega_0^\omega), \tag{3.1}$$

and that this ansatz is responsible for some change of the Sobolev norms.

More precisely the crucial claim is the following.

Proposition 3.1. *Let $s > 0$. Then there holds*

$$\|\mathfrak{m}(\nabla)(\Omega_0^\omega - tB_1(\Omega_0^\omega) + t^2 B_2(\Omega_0^\omega))\|^2_{L^2_\omega L^2_x} = \|\mathfrak{m}(\nabla)\Omega_0^\omega\|^2_{L^2_\omega L^2_x} + \gamma^{\mathfrak{m}} t^2 + \delta^{\mathfrak{m}} t^4,$$

where $\delta^{\mathfrak{m}} := \|\mathfrak{m}(\nabla)B_3(\Omega_0^\omega)\|^2_{L^2_\omega L^2}$ and

$$\gamma^{\mathfrak{m}} := \|\mathfrak{m}(\nabla)B_1(\Omega_0^\omega)\|^2_{L^2_\omega L^2} + 2\mathbb{E}[\langle \mathfrak{m}(\nabla)\Omega_0^\omega, \mathfrak{m}(\nabla)B_2(\Omega_0^\omega)\rangle_{L^2}]$$

is the constant given by (1.5).

The ansatz (3.1) is not an equality and the remainder is given by

$$w^\omega(t) := \Omega^\omega(t) - \Omega_0^\omega + tB_1(\Omega_0^\omega) - t^2 B_2(\Omega_0^\omega). \tag{3.2}$$

We next aim to show that this remainder is small for $s \in (0, \sigma - 3)$ (the loss of three derivatives comes from the number of derivatives in the ansatz, i.e., in B_2). Given the rapid non-linear instabilities inherent in the Euler equations, it seems difficult to directly justify this ansatz in H^s. However, the assumption of invariance (quantitative quasi-invariance would also yield similar results) implies significant a priori estimates coming from globalization (Proposition 2.2). These are leveraged in H^s energy estimates to yield the following stability estimate.

Proposition 3.2. *Assume that the law of Ω_0^ω is given by $\mu_{(a_n)} \in \mathcal{M}^\sigma$ for some $\sigma > 3$, and let $s \in (0, \sigma - 3)$. Suppose that $\mu_{(a_n)}$ is invariant under the flow of (E). Then there exist a deterministic time $t_1 := t_1(s, \sigma, (a_n)) > 0$ and a deterministic constant $C := C(s, \sigma, (a_n)) > 0$ such that*

$$\|\mathfrak{m}(\nabla)w^\omega(t)\|_{L^2_\omega L^2_x} \leqslant Ct^3 \quad \text{for all } t \in [0, t_1],$$

where w^ω is defined by (3.2).

Once this a priori estimate on the remainder is obtained, one can reach a contradiction with invariance using the fact that under the assumption of invariance it holds that

$$\mathbb{E}[\|\mathfrak{m}(\nabla)\Omega^{\omega}(t)\|_{L^2_x}^2] = \mathbb{E}[\|\mathfrak{m}(\nabla)\Omega_0^{\omega}\|_{L^2_x}^2].$$

We postpone the proof of Proposition 3.1 to Section 4 and the proof of Proposition 3.2 to Section 5, and first finish the proof of the main theorem.

Proof of Theorem A. We can assume $\mathrm{supp}\{a_n\} \cap \mathrm{supp}\, \mathfrak{m} \neq \varnothing$, otherwise $\gamma^{\mathfrak{m}} = 0$. Then, with positive probability, $\mathfrak{m}(\nabla)\Omega_0^{\omega} \neq 0$ because $\mathrm{supp}\{a_n\} \cap \mathrm{supp}\, \mathfrak{m} \neq \varnothing$, so that it follows $\|\mathfrak{m}(\nabla)\Omega_0^{\omega}\|_{L^2_\omega L^2_x} > 0$. We use Proposition 3.1 and a Taylor expansion to write

$$\begin{aligned}\|\mathfrak{m}(\nabla)(\Omega^{\omega}(t) - w^{\omega}(t))\|_{L^2_\omega L^2_x} &= \sqrt{\|\mathfrak{m}(\nabla)\Omega_0^{\omega}\|^2_{L^2_\omega L^2_x} + \gamma^{\mathfrak{m}} t^2 + \delta^{\mathfrak{m}} t^4} \\ &= \|\mathfrak{m}(\nabla)\Omega_0^{\omega}\|_{L^2_\omega L^2_x} + \frac{\gamma^{\mathfrak{m}}}{2\|\mathfrak{m}(\nabla)\Omega_0\|_{L^2_\omega L^2_x}} t^2 + \mathcal{O}(t^4),\end{aligned}$$

so that by the triangle inequality and Proposition 3.2 there holds

$$\begin{aligned}\frac{\gamma^{\mathfrak{m}}}{2\|\mathfrak{m}(\nabla)\Omega_0^{\omega}\|_{L^2_\omega L^2_x}} t^2 + \mathcal{O}(t^3) &\leqslant \|\mathfrak{m}(\nabla)\Omega^{\omega}(t)\|_{L^2_\omega L^2_x} - \|\mathfrak{m}(\nabla)\Omega_0^{\omega}\|_{L^2_\omega L^2_x} \\ &\leqslant \frac{\gamma^{\mathfrak{m}}}{2\|\mathfrak{m}(\nabla)\Omega_0^{\omega}\|_{L^2_\omega L^2_x}} t^2 + \mathcal{O}(t^3),\end{aligned}$$

for $0 \leqslant t \leqslant t_1$, so that for times $0 \leqslant t \leqslant t_2 = t_2(s, \sigma, (a_n)) \ll t_1$ we can write

$$\begin{aligned}c_1 \frac{\gamma^{\mathfrak{m}}}{\|\mathfrak{m}(\nabla)\Omega_0^{\omega}\|_{L^2_\omega L^2_x}} t^2 &\leqslant \|\mathfrak{m}(\nabla)\Omega^{\omega}(t)\|_{L^2_\omega L^2_x} - \|\mathfrak{m}(\nabla)\Omega_0^{\omega}\|_{L^2_\omega L^2_x} \\ &\leqslant c_2 \frac{\gamma^{\mathfrak{m}}}{\|\mathfrak{m}(\nabla)\Omega_0^{\omega}\|_{L^2_\omega L^2_x}} t^2 \qquad (3.3)\end{aligned}$$

with $(c_1, c_2) = (3/4, 1/4)$ if $\gamma^{\mathfrak{m}} < 0$ and $(1/4, 3/4)$ if $\gamma^{\mathfrak{m}} > 0$. Because of invariance there holds

$$\|\mathfrak{m}(\nabla)\Omega^{\omega}(t)\|_{L^2_\omega L^2_x} - \|\mathfrak{m}(\nabla)\Omega_0^{\omega}\|_{L^2_\omega L^2_x} = 0,$$

which contradicts (3.3). ∎

4. The orthogonality argument

The goal of this section is to prove Proposition 3.1.

For the sake of simplicity we write Ω in place of Ω_0^{ω}. This will not lead to any confusion since the arguments in this section do not involve time evolution. We write

$$\Omega = \sum_{n \in \mathbb{Z}^2} a_n g_n^{\omega} e^{in\cdot x} = \frac{1}{(2\pi)^2} \sum_{n \in \mathbb{Z}^2} \widehat{\Omega}_n e^{in\cdot x}.$$

Similarly, let us write $U[\Omega] = (U^1, U^2)$ where $U^j = \sum_{n\in\mathbb{Z}^2} \widehat{U}^j_n e^{in\cdot x}$. With this notation, the relationship $\nabla \wedge U[\Omega] = \Omega$ reads $\widehat{\Omega}_n = i(n_1\widehat{U}^2_n - n_2\widehat{U}^1_n)$ and the relation $\nabla \cdot U[\Omega] = 0$ reads $n_1\widehat{U}^1_n + n_2\widehat{U}^2_n = 0$, so that we obtain

$$\widehat{U}^1_n = -\frac{n_2}{i|n|^2}\widehat{\Omega}_n \quad \text{and} \quad \widehat{U}^2_n = \frac{n_1}{i|n|^2}\widehat{\Omega}_n.$$

Writing that $B_1(\Omega) = U^1\partial_1\Omega + U^2\partial_2\Omega$ we obtain

$$\widehat{B_1(\Omega)}(n) = \sum_{k+m=n} \frac{m\cdot k^\perp}{|k|^2}\widehat{\Omega}_k\widehat{\Omega}_m = \frac{1}{2}\sum_{k+m=n} m\cdot k^\perp\Big(\frac{1}{|k|^2} - \frac{1}{|m|^2}\Big)\widehat{\Omega}_k\widehat{\Omega}_m,$$

and more generally

$$\widehat{B(\alpha,\beta)}(n) = \frac{1}{2}\sum_{k+m=n} m\cdot k^\perp\Big(\frac{1}{|k|^2} - \frac{1}{|m|^2}\Big)\hat{\alpha}_k\hat{\beta}_m.$$

The following computation is a key step in the proof of Proposition 3.1.

Lemma 4.1. *Assume that $(a_n)_{n\in\mathbb{Z}^2} \in h^{s+1}$ for some $s \geqslant 0$. Let us recall that $B_2(\Omega) = B(\Omega, B_1(\Omega,\Omega))$. Then there holds*

$$\begin{aligned}\mathbb{E}[\langle \mathfrak{m}(\nabla)\Omega, \mathfrak{m}(\nabla)B_2(\Omega)\rangle_{L^2}] \\ = \frac{1}{2(2\pi)^4}\sum_{n,q\in\mathbb{Z}^2} |\mathfrak{m}(n)|^2|a_n|^2|a_q|^2(q\cdot n^\perp)^2\Big(\frac{1}{|n|^2} - \frac{1}{|q|^2}\Big)\Big(\frac{1}{|q|^2} - \frac{1}{|q+n|^2}\Big).\end{aligned}$$

Proof. We will decompose $B_1(\Omega)$ to obtain the claimed result. More precisely, we write

$$\begin{aligned}\mathbb{E}[\langle \mathfrak{m}(\nabla)\Omega, \mathfrak{m}(\nabla)B_2(\Omega)\rangle_{H^s}] &= \frac{1}{2}\mathbb{E}[\langle \mathfrak{m}(\nabla)\Omega, \mathfrak{m}(\nabla)(U[\Omega]\cdot\nabla B(\Omega,\Omega))\rangle_{L^2}] \\ &\quad + \frac{1}{2}\mathbb{E}[\langle \mathfrak{m}(\nabla)\Omega, \mathfrak{m}(\nabla)(U[B(\Omega,\Omega)]\cdot\nabla\Omega)\rangle_{L^2}] \\ &=: \frac{1}{2}(E_1 + E_2).\end{aligned}$$

Writing

$$\begin{aligned}\widehat{U[\Omega]\cdot\nabla B(\Omega,\Omega)}(n) &= \sum_{k+m=n} \frac{m\cdot k^\perp}{|k|^2}a_k g^\omega_k \widehat{B(\Omega,\Omega)}(m) \\ &= \sum_{\substack{k+m=n\\ p+q=m}} \frac{m\cdot k^\perp q\cdot p^\perp}{|k|^2|p|^2} g^\omega_k g^\omega_p g^\omega_q a_k a_p a_q,\end{aligned}$$

and using Plancherel's theorem, we find

$$(2\pi)^4 E_1 = \sum_{n\in\mathbb{Z}^2} |\mathfrak{m}(n)|^2 \sum_{\substack{k+m=n\\ p+q=m}} \frac{m\cdot k^\perp q\cdot p^\perp}{|k|^2|p|^2}\bar{a}_n a_k a_p a_q \mathbb{E}[\bar{g}^\omega_n g^\omega_k g^\omega_p g^\omega_q].$$

Similarly,

$$\begin{aligned}\mathcal{F}(U[B(\Omega,\Omega)]\cdot\nabla\Omega)(n) &= \sum_{k+m=n}\frac{m\cdot k^\perp}{|k|^2}\widehat{B(\Omega,\Omega)}(k)a_m g_m^\omega \\ &= \sum_{\substack{k+m=n\\ p+q=k}}\frac{m\cdot k^\perp}{|k|^2}\frac{q\cdot p^\perp}{|p|^2}g_m^\omega g_p^\omega g_q^\omega a_m a_p a_q\end{aligned}$$

so that

$$(2\pi)^4 E_2 = \sum_{n\in\mathbb{Z}^2}|\mathfrak{m}(n)|^2\sum_{\substack{k+m=n\\ p+q=k}}\frac{m\cdot k^\perp}{|k|^2}\frac{q\cdot p^\perp}{|p|^2}\bar{a}_n a_m a_p a_q\mathbb{E}[\bar{g}_n^\omega g_m^\omega g_p^\omega g_q^\omega].$$

Now we make the following crucial observation: because the g_j^ω are complex Gaussian random variables, there holds

$$(q\cdot p^\perp)\mathbb{E}[\bar{g}_n^\omega g_k^\omega g_p^\omega g_q^\omega] = 0$$

unless the elements of $\{n,k,p,q\}$ are paired two by two (note that the cases $|n| = |k| = |p| = |q|$ are such that $q\cdot p^\perp = 0$). There are two possible pairings: the first case is $(n,k) = (p,-q)$, $n\neq\pm k$, and the second case is $(n,k) = (q,-p)$, $n\neq\pm k$. Remark that the pairing $(n,p) = (k,-q)$ is such that $q\cdot p^\perp = 0$, therefore giving a zero contribution.

Let us write $(2\pi)^4 E_1 = (2\pi)^4\sum_{n\in\mathbb{Z}^2}\langle n\rangle^{2s}(E_{11}(n)+E_{12}(n))$, where $E_{11}(n)$ (resp. $E_{12}(n)$) corresponds to the first (resp. second) pairing case. For the first contribution we find

$$E_{11}(n) = \sum_{q\in\mathbb{Z}^2}\frac{(q\cdot n^\perp)^2}{|n|^2|q|^2}|a_n|^2|a_q|^2,$$

where we have used that $m\cdot k^\perp = q\cdot n^\perp$.

Similarly,

$$E_{12}(n) = -\sum_{p\in\mathbb{Z}^2}\frac{(p\cdot n^\perp)^2}{|p|^4}|a_p|^2|a_n|^2,$$

so that finally (relabeling p to q and summing both contributions) we obtain

$$(2\pi)^4 E_1 = -\sum_{n,q\in\mathbb{Z}^2}|\mathfrak{m}(n)|^2(q\cdot n^\perp)^2|a_n|^2|a_q|^2\Big(\frac{1}{|q|^4}-\frac{1}{|q|^2|n|^2}\Big).$$

Writing similarly $(2\pi)^4 E_2 = (2\pi)^4\sum_{n\in\mathbb{Z}^2}|\mathfrak{m}(n)|^{2s}(E_{21}(n)+E_{22}(n))$ we find

$$E_{21}(n) = -\sum_{q\in\mathbb{Z}^2}\frac{(q\cdot n^\perp)^2}{|n+q|^2|n|^2}|a_n|^2|a_q|^2$$

and

$$E_{22}(n) = \sum_{p\in\mathbb{Z}^2} \frac{(p\cdot n^\perp)^2}{|p+n|^2|p|^2}|a_n|^2|a_p|^2,$$

so that after the change of variables $(n, p) \mapsto (q, n)$ in A_{22}, we obtain

$$E_2 = -\sum_{n,q\in\mathbb{Z}^2} |\mathfrak{m}(n)|^{2s}(q\cdot n^\perp)^2|a_n|^2|a_q|^2\Big(\frac{1}{|q+n|^2|n|^2} - \frac{1}{|q+n|^2|q|^2}\Big),$$

which is enough to conclude the proof. ∎

We also need the following simpler computation.

Lemma 4.2. *Assuming $(a_n)_{n\in\mathbb{Z}^2} \in h^{s+1}$ for some $s \geqslant 0$, there holds*

$$\mathbb{E}[\|\mathfrak{m}(\nabla)B_1(\Omega)\|_{L^2}^2] = \frac{1}{(2\pi)^4}\sum_{n,q\in\mathbb{Z}^2} |\mathfrak{m}(n+q)|^2\frac{(q\cdot n^\perp)^2}{2}\Big(\frac{1}{|q|^2} - \frac{1}{|n|^2}\Big)^2|a_q|^2|a_n|^2.$$

Proof. First we write

$$\|\mathfrak{m}(\nabla)B_1(\Omega)\|_{L^2}^2 = \frac{1}{(2\pi)^4}\sum_{n\in\mathbb{Z}^2}|\mathfrak{m}(n)|^2 \sum_{\substack{k+m=n\\ k'+m'=n}} c_{k,m}c_{k',m'}a_k a_m \bar{a}_{k'}\bar{a}_{m'} g_k^\omega g_m^\omega \bar{g}_{k'}^\omega \bar{g}_{m'}^\omega,$$

where $c_{k,m} = \frac{m\cdot k^\perp}{2}(\frac{1}{|k|^2} - \frac{1}{|m|^2})$.

Taking the expectation in the above and using that $c_{k,m} = 0$ if $|k| = |m|$, there are two possible pairings of $\{k, m, k', m'\}$ to consider: $(k, m) = (k', m')$ and $(k, m) = (m', k')$ both leading to the same contribution, therefore

$$\begin{aligned}
&(2\pi)^4\mathbb{E}[\|\mathfrak{m}(\nabla)B_1(\Omega)\|_{L^2}^2]\\
&\quad= 2\sum_{n\in\mathbb{Z}^2}|\mathfrak{m}(n)|^2\sum_{k+m=n}\frac{(k\cdot m^\perp)^2}{4}\Big(\frac{1}{|k|^2} - \frac{1}{|m|^2}\Big)^2|a_m|^2|a_k|^2\\
&\quad= \sum_{n,q\in\mathbb{Z}^2}|\mathfrak{m}(n+q)|^2\frac{(q\cdot n^\perp)^2}{2}\Big(\frac{1}{|q|^2} - \frac{1}{|n|^2}\Big)^2|a_q|^2|a_n|^2,
\end{aligned}$$

where in the last step we just relabeled: $(n, k, m) \mapsto (n + q, q, n)$. ∎

We go back to the notation Ω_0^ω in the following proof.

Proof of Proposition 3.1. Let $s > 0$. Expanding the squared H^s norm and taking the expectation we compute

$$\begin{aligned}
&\|\mathfrak{m}(\nabla)(\Omega_0^\omega - tB_1(\Omega_0^\omega) + t^2B_2(\Omega_0^\omega))\|_{L^2_\omega L^2}^2\\
&= \|\mathfrak{m}(\nabla)\Omega_0^\omega\|_{L^2_\omega L^2}^2 + \gamma^{\mathfrak{m}}t^2 + \delta^{\mathfrak{m}}t^4 - 2t\mathbb{E}[\langle\mathfrak{m}(\nabla)\Omega_0^\omega, \mathfrak{m}(\nabla)B_1(\omega_0^\omega)\rangle_{L^2}]\\
&\quad - 2t^3\mathbb{E}[\langle\mathfrak{m}(\nabla)B_1(\Omega_0^\omega), \mathfrak{m}(\nabla)B_2(\omega_0^\omega)\rangle_{L^2}],
\end{aligned}$$

where $\delta^{\mathfrak{m}} = \|\mathfrak{m}(\nabla)B_2(\Omega_0^\omega)\|^2_{L^2_\omega L^2}$ and

$$\gamma^{\mathfrak{m}} = \mathbb{E}[\|\mathfrak{m}(\nabla)B_1(\Omega_0^\omega)\|^2_{L^2} + 2\langle \mathfrak{m}(\nabla)\Omega_0^\omega, \mathfrak{m}(\nabla)B_2(\Omega_0^\omega)\rangle_{L^2}].$$

Using Lemmas 4.1 and 4.2 we have that

$$\gamma^{\mathfrak{m}} = \sum_{n,q\in\mathbb{Z}^2} b_{n,q}\Big(|\mathfrak{m}(n)|^2\Big(\frac{1}{|n|^2}-\frac{1}{|q|^2}\Big)\Big(\frac{1}{|q|^2}-\frac{1}{|q+n|^2}\Big) + \frac{|\mathfrak{m}(n+q)|^2}{2}\Big(\frac{1}{|q|^2}-\frac{1}{|n|^2}\Big)^2\Big),$$

where $b_{n,q} = \frac{1}{(2\pi)^4}(|a_n||a_q|(q\cdot n^\perp))^2(\frac{1}{|n|^2}-\frac{1}{|q|^2})$. Observe now that $b_{q,n} = -b_{n,q}$ so that using the symmetry between n and q, we finally find

$$\gamma^{\mathfrak{m}} = \sum_{n,q\in\mathbb{Z}^2} b_{n,q}\Big(\frac{|\mathfrak{m}(n)|^2}{2}\Big(\frac{1}{|q|^2}-\frac{1}{|q+n|^2}\Big) + \frac{|\mathfrak{m}(q)|^2}{2}\Big(\frac{1}{|q+n|^2}-\frac{1}{|n|^2}\Big) + \frac{|\mathfrak{m}(n+q)|^2}{2}\Big(\frac{1}{|n|^2}-\frac{1}{|q|^2}\Big)\Big),$$

which is (1.5), as claimed.

It remains to observe that

$$\mathbb{E}[\langle \mathfrak{m}(\nabla)\Omega_0^\omega, \mathfrak{m}(\nabla)B_1(\omega_0^\omega)\rangle_{L^2}] = \mathbb{E}[\langle \mathfrak{m}(\nabla)B_1(\Omega_0^\omega), \mathfrak{m}(\nabla)B_2(\omega_0^\omega)\rangle_{L^2}] = 0.$$

In order to do so, observe that computations used in the proof of Lemma 4.1 imply

$$\mathbb{E}[\langle \mathfrak{m}(\nabla)\Omega_0^\omega, \mathfrak{m}(\nabla)B_1(\Omega_0^\omega)\rangle_{L^2}] = \sum_{\substack{n\in\mathbb{Z}^2\\ k+m=n}} |\mathfrak{m}(n)|^{2s}\frac{m\cdot k^\perp}{|k|^2}a_k a_m \bar{a}_n \mathbb{E}[g_k^\omega g_m^\omega \bar{g}_n^\omega] = 0,$$

since $\mathbb{E}[g_k^\omega g_m^\omega \bar{g}_n^\omega] = 0$ for all values of $k, m, n \in \mathbb{Z}^2$. It follows that

$$\mathbb{E}[\langle \mathfrak{m}(\nabla)\Omega_0^\omega, \mathfrak{m}(\nabla)B_1(\Omega_0^\omega)\rangle_{L^2}] = 0.$$

For the same reason, it also holds that $\mathbb{E}[\langle \mathfrak{m}(\nabla)B_1(\Omega_0^\omega), \mathfrak{m}(\nabla)B_2(\Omega_0^\omega)\rangle_{L^2}] = 0$. ∎

5. The perturbative argument

Let us recall that $w^\omega(t)$, defined by

$$\Omega^\omega(t) = \Omega_0^\omega - tB_1(\Omega_0^\omega) + t^2 B_2(\Omega_0^\omega) + w^\omega(t),$$

is the remainder in the ansatz (3.1). In particular, $w^\omega(0) = 0$. First, by definition of $w^\omega(t)$, we have

$$\partial_t \Omega^\omega(t) = -B_1(\Omega_0^\omega) + 2tB_2(\Omega_0^\omega) + \partial_t w^\omega(t).$$

Expanding $B(\Omega^\omega(t), \Omega^\omega(t))$, using that $\Omega^\omega(t)$ solves (E) we find that $w^\omega(t)$ (which will be denoted w, omitting the ω superscripts) satisfies the following evolution equation:

$$\partial_t w(t) = -B(w(t), w(t)) \tag{5.1}$$
$$- 2B(\Omega_0, w(t)) + 2tB(B_1(\Omega_0), w(t)) \tag{5.2}$$
$$- t^2 \underbrace{\big(B(B_1(\Omega_0), B_1(\Omega_0)) + 2B(\Omega_0, B_2(\Omega_0))\big)}_{B_3(\Omega_0)} - 2t^2 B(B_2(\Omega_0), w(t)) \tag{5.3}$$
$$+ 2t^3 \underbrace{B(B_1(\Omega_0), B_2(\Omega_0))}_{B_3'(\Omega_0)} - t^4 \underbrace{B(B_2(\Omega_0), B_2(\Omega_0))}_{\tilde{B}_3(\Omega_0)}. \tag{5.4}$$

Let us make a few comments:

- Our proof proceeds by performing energy estimates in H^s, targeting an estimate of the form $\|w(t)\|_{H^s} \leqslant Ct^3 e^{ct}$ in short time (the exponential accounts for the use of the Grönwall inequality). One major threat to obtain such estimates will come from the random constants appearing in the estimates. For example, we need to avoid factors like $e^{\|\Omega_0\|_{H^\sigma}^3}$ appearing in our final estimate (which would not be integrable with respect to μ). In particular, we have to pay attention when estimating (5.2). We want an estimate of the form $\|w(t)\|_{H^s} \leqslant Ct^3$; we expect that (5.3) will be part of the main term in the estimate, but that (5.4) will contribute negligibly.
- We point out that the usual energy estimate on the term $B(w(t), w(t))$ would bring a potential double-exponential bound, which in short time is bounded. In our case we must take advantage of $w(0) = 0$, so that the non-linear term $B(w(t), w(t))$ should be viewed as a small contribution.
- Because the method we employ is to use the ansatz (3.1) for $\Omega^\omega(t)$, we will need to estimate $B_3(\Omega_0)$, $\tilde{B}_3(\Omega_0)$, $B_3'(\Omega_0)$, and it is the reason why we require $\sigma > s + 3$.

Proof of Proposition 3.2. In this proof we only work with $t \leqslant 1$, and we will later restrict to shorter intervals of time.

We start by deriving a first a priori bound for $w^\omega(t)$ by using the triangle inequality, Lemma 2.1, and Proposition 2.2;

$$\begin{aligned}\|w(t)\|_{H^{s'}} &\leqslant \|\Omega^\omega(t)\|_{H^{s'}} + \|\Omega_0^\omega\|_{H^{s'}} + t\|B_1(\Omega_0^\omega)\|_{H^{s'}} + t^2\|B_2(\Omega_0^\omega)\|_{H^{s'}} \\ &\leqslant C_\omega\sqrt{\log(2+t)} + C(1+t+t^2)(\|\Omega_0^\omega\|_{H^{s'}} + \|\Omega_0^\omega\|_{H^{s'+1}}^2 + \|\Omega_0^\omega\|_{H^{s'+2}}^3) \\ &\leqslant C(C_\omega + (1 + \|\Omega_0^\omega\|_{H^{s'+2}}^3)) =: D_\omega\end{aligned} \tag{5.5}$$

for all $0 \leqslant t \leqslant 1$ and all $s' \leqslant \sigma - 2$.

We will later use that the random constant D_ω lies in L_ω^p for all $1 \leqslant p < \infty$. However, note that it lacks exponential moments, therefore directly plugging this estimate into (5.1) and performing the usual energy estimates would yield a bound on $\|\mathfrak{m}(\nabla)w(t)\|_{L^2}$, which would not lie in L_ω^2.

Let us now move to the energy estimate. We first note that $\|\mathfrak{m}(\nabla)w(t)\|_{L^2} \lesssim \|w(t)\|_{H^s}$.

We compute (using (5.1) to decompose $\partial_t w(t)$), $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_{H^s}^2 = (\partial_t w(t), w(t))_{H^s}$. First, we use Lemma 2.1 to estimate (5.1) by

$$|(B(w(t), w(t)), w(t))_{L^2}| \leqslant C(s,\varepsilon)\|w(t)\|_{H^s}^2 \|w(t)\|_{H^{1+\varepsilon}},$$

where $C(s,\varepsilon)$ is a global deterministic constant.

For the linear terms in (5.2) we rely on Lemma 2.1 again to bound

$$\big(B(\Omega_0, w(t)), w(t)\big)_{H^s} \leqslant C(s,\varepsilon)\|\Omega_0\|_{H^{s+1+\varepsilon}}\|w(t)\|_{H^s}^2$$

and

$$\big(B(B_1(\Omega_0), w(t)), w(t)\big)_{H^s} \leqslant C(s,\varepsilon)\|B_1(\Omega_0)\|_{H^{s+1+\varepsilon}}\|w(t)\|_{H^s}^2,$$

which together imply

$$\begin{aligned}\big(-2B(\Omega_0, w(t)) + 2tB(B_1(\Omega_0), w(t)), w(t)\big)_{H^s} \\ \leqslant C(s,\varepsilon)(1 + \|\Omega_0\|_{H^{s+2+\varepsilon}}^2)\|w(t)\|_{H^s}^2. \end{aligned} \tag{5.6}$$

To estimate (5.3) we use the Cauchy–Schwarz inequality and Lemma 2.1 to obtain

$$(B_3(\Omega_0), w(t))_{H^s} \leqslant \|B_3(\Omega_0)\|_{H^s}\|w(t)\|_{H^s} \leqslant C(s,\varepsilon)\|\Omega_0\|_{H^{s+3}}^4\|w(t)\|_{H^s}.$$

Another application of Lemma 2.1 yields

$$\begin{aligned}\big(B(B_2(\Omega_0), w(t)), w(t)\big)_{H^s} &\leqslant \|B_2(\Omega_0)\|_{H^{s+1+\varepsilon}}\|w(t)\|_{H^s} \\ &\leqslant C(s,\varepsilon)\|\Omega_0\|_{H^{s+3+\varepsilon}}^3\|w(t)\|_{H^s}^2 \\ &\leqslant D_\omega\|\Omega_0\|_{H^{s+3+\varepsilon}}^3\|w(t)\|_{H^s},\end{aligned}$$

where in the last inequality we have used (5.5). Therefore,

$$\begin{aligned}\big(-t^2B_3(\Omega_0) - 2t^2B(B_2(\Omega_0), w(t)), w(t)\big)_{H^s} \\ \leqslant D_\omega t^2(1 + \|\Omega_0\|_{H^{s+3+\varepsilon}}^4)\|w(t)\|_{H^s}. \end{aligned} \tag{5.7}$$

Finally, also using Lemma 2.1 we can bound the remainder terms (5.4) and obtain

$$(2t^3B_3'(\Omega_0) - t^4\tilde{B}_3(\Omega_0), w(t))_{H^s} \leqslant C(s,\varepsilon)t^3(1 + \|\Omega_0\|_{H^{s+3}}^6)\|w(t)\|_{H^s}. \tag{5.8}$$

Combining (5.6), (5.7), and (5.8) we finally obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_{H^s}^2 \leqslant \|w(t)\|_{H^s}^2\|w(t)\|_{H^{1+\varepsilon}} + C(s,\varepsilon)(1 + \|\Omega_0\|_{H^{s+2+\varepsilon}}^2)\|w(t)\|_{H^s}^2 \\ + D_\omega t^2(1 + \|\Omega_0\|_{H^{s+3+\varepsilon}}^4)\|w(t)\|_{H^s} \\ + C(s,\varepsilon)t^3(1 + \|\Omega_0\|_{H^{s+3}}^6)\|w(t)\|_{H^s}.\end{aligned}$$

Using $t \leqslant 1$, dividing by $\|w(t)\|_{H^s}$, and collecting the constants we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_{H^s} \leqslant \|w(t)\|_{H^s}\|w(t)\|_{H^{1+\varepsilon}} + k_\omega\|w(t)\|_{H^s} + K_\omega t^2, \tag{5.9}$$

where $\mathbb{E}[e^{ck_\omega}] < \infty$ for c small enough and $K_\omega \in L_\omega^p$ for all finite p.

Let us now use (5.5) for $0 \leqslant t \leqslant 1$ in (5.9) to bound

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_{H^s} \leqslant D'_\omega,$$

where $D'_\omega \in L^p_\omega$ for all finite p. Note that we have used that $1+\varepsilon \leqslant \sigma - 2$ in order to use (5.5) and Lemma 2.1. Integrating this differential bound yields $\|w(t)\|_{H^s} \leqslant D'_\omega t$ for $0 \leqslant t \leqslant 1$. Using the latter inequality (also used with $s = 1+\varepsilon$) in (5.9), we obtain the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_{H^s} \leqslant k_\omega \|w(t)\|_{H^s} + K'_\omega t^2,$$

where $K'_\omega := (D'_\omega)^2 + K_\omega$. Therefore, the Grönwall inequality implies

$$\|w(t)\|_{H^s} \leqslant \frac{1}{3}K'_\omega t^3 \exp(k_\omega t) \leqslant \frac{1}{3}K'_\omega t^3 \exp\Big(\frac{c}{2}k_\omega\Big)$$

if $0 \leqslant t \leqslant \frac{c}{2}$. Therefore $\|w(t)\|_{L^2_\omega H^s} \leqslant \frac{1}{3}\mathbb{E}[K_\omega^2 \exp(ck_\omega)]^{\frac{1}{2}} t^3$, and the claim is proven with $C = \frac{1}{3}\mathbb{E}[(K'_\omega)^2 \exp(ck_\omega)]^{\frac{1}{2}} < \infty$. ∎

6. The condition $\gamma \neq 0$

6.1. First examples of non-invariant Gaussian measures

We start by providing a very simple example of $(a_n)_{n\in\mathbb{Z}^2}$ for which $\gamma_{s,(a_n)} \neq 0$.

Lemma 6.1. *Let $a_n = 1$ if $n \in \{(1,0),(-1,0),(0,2),(0,-2)\}$, and $a_n = 0$ otherwise. If $s > 1$ then $\gamma_{s,(a_n)} \neq 0$. Therefore, the associated measure is not invariant under the flow of* (E).

Proof. We can compute $\gamma_{s,(a_n)}$ explicitly. Using the notation of (1.5) we find

$$\gamma = 3\Big(\frac{6^s}{2} + \frac{3\cdot 2^s}{10} - \frac{4\cdot 3^s}{5}\Big) = \frac{3}{20}\Big(15\cdot 6^s - 16\cdot 5^s + 2^s\Big),$$

and we need to check that this does not vanish. Note that $s > 1$ and we observe that $(\frac{6}{5})^s > \frac{6}{5} > \frac{16}{15}$, which implies $15\cdot 6^s - 16\cdot 5^s > 0$ so that $\gamma > 0$. ∎

As a corollary for all $\sigma > 3$, one can construct a sequence $(a_n)_{n\in\mathbb{Z}^2}$ such that the associated measure $\mu_{(a_n)}$ is "exactly" supported on H^σ and $\gamma_{s,(a_n)} \neq 0$ for all $s \in (0,\sigma]$.

Corollary 6.2. *Let $\sigma > 3$. There exists $\mu \in \mathcal{M}^\sigma$ that is not invariant under the flow of* (E) *and which satisfies $\mu(H^\sigma) = 1$, and $\mu(H^{\sigma+\varepsilon}) = 0$ for all $\varepsilon > 0$.*

Proof. We first remark that if $(a_n)_{n\in\mathbb{Z}^2} \in h^\sigma_{\mathrm{rad}} \setminus \bigcup_{\varepsilon>0} h^{\sigma+\varepsilon}_{\mathrm{rad}}$ with non-vanishing a_n, then the associate measure $\mu_{(a_n)}$ satisfies $\mu(H^\sigma) = 1$, and $\mu(H^{\sigma+\varepsilon}) = 0$. That $\mu(H^\sigma) = 1$

holds follows from (1.3). The fact that for any $\varepsilon > 0$ there holds $\mu(H^{\sigma+\varepsilon}) = 0$ is a more difficult but standard fact; we refer to the appendix of [14] for a detailed proof.

Let $b_n = \frac{1}{\langle n\rangle^{\sigma+1}\log(\langle n\rangle)}$, which belongs to $h^\sigma_{\mathrm{rad}} \setminus \bigcup_{\varepsilon>0} h^{\sigma+\varepsilon}_{\mathrm{rad}}$. Also let $(a_n)_{n\in\mathbb{Z}^2}$ be the sequence from Lemma 6.1 and let us consider $c_n = a_n + \varepsilon b_n$. Then by expanding $\gamma_{s,(c_n)}$ we can check that $\gamma_{s,(c_n)} = \gamma_{s,(a_n)} + \mathcal{O}(\varepsilon)$, and since $\gamma_{s,(a_n)} > 0$, we can take ε small enough so that $\gamma_{s,(c_n)} > 0$. ∎

6.2. Proof of Theorem C

The proof essentially follows from the following continuity lemma.

Lemma 6.3. *Let $s \geqslant 0$ and $\sigma \geqslant s+1$. The map $\gamma\colon (a_n)_{n\in\mathbb{Z}^2} \mapsto \gamma_{s,(a_n)}$, where $\gamma_{s,(a_n)}$ is defined by (1.5) and is continuous as an operator $h^\sigma(\mathbb{Z}^2) \to \mathbb{R}$.*

Proof. Let us prove the continuity using the sequential characterization: we take $a^k = (a^k_n)_{\in\mathbb{Z}^2}$ converging to $a = (a_n)_{\in\mathbb{Z}^2}$ in $h^\sigma(\mathbb{Z}^2)$. Then let us write

$$\gamma_{a^k} = \sum_{n,q\in\mathbb{Z}^2} |a^k_n|^2 |a^k_q|^2 \alpha_{n,q},$$

where $\alpha_{n,q} = (q\cdot n^\perp)^2(\frac{1}{|n|^2} - \frac{1}{|q|^2})\beta_{n,q} = \mathcal{O}(\langle n\rangle^{2(s+1)}\langle q\rangle^{2(s+1)})$, where $\beta_{n,q}$ is defined in (1.6). This crude bound will be enough for our purpose. Using the identity

$$|a^k_n|^2|a^k_q|^2 - |a_n|^2|a_q|^2 = (|a^k_n||a^k_q| - |a_n||a_q|)(|a^k_n||a^k_q| + |a_n||a_q|),$$

and using the Cauchy–Schwarz inequality we infer

$$\begin{aligned}|\gamma_{a^k} - \gamma_a|^2 \leqslant &\sum_{n,q\in\mathbb{Z}^2} (|a^k_n||a^k_q| - |a_n||a_q|)^2 \langle n\rangle^{2(s+1)}\langle q\rangle^{2(s+1)} \\ &\times \sum_{n,q\in\mathbb{Z}^2} (|a^k_n||a^k_q| + |a_n||a_q|)^2 \langle n\rangle^{2(s+1)}\langle q\rangle^{2(s+1)}.\end{aligned}$$

First, we bound

$$\sum_{n,q\in\mathbb{Z}^2} (|a^k_n||a^k_q| + |a_n||a_q|)^2 \langle n\rangle^{2(s+1)}\langle q\rangle^{2(s+1)} \leqslant \|a^k\|^4_{h^{s+1}} + \|a\|^4_{h^{s+1}}$$

which is bounded (in k) because $s+1 \leqslant \sigma$. For the other term we write

$$(|a^k_n||a^k_q| - |a_n||a_q|)^2 \lesssim |a_q|^2|a^k_n - a_n|^2 + |a^k_n|^2|a^k_q - a_q|^2,$$

so that we obtain

$$\begin{aligned}&\sum_{n,q\in\mathbb{Z}^2} (|a^k_n||a^k_q| - |a_n||a_q|)^2 \langle n\rangle^{2(s+1)}\langle q\rangle^{2(s+1)} \\ &\qquad\lesssim \|a\|^2_{h^{s+1}}\|a^k - a\|^2_{h^{s+1}} + \|a_k\|^2_{h^{s+1}}\|a^k - a\|^2_{h^{s+1}} = \mathcal{O}(\|a^k - a\|^2_{h^\sigma}),\end{aligned}$$

which goes to zero because $s+1 \leqslant \sigma$, therefore proving the continuity. ∎

That K^σ is open as stated in Theorem C (i) now follows from Lemma 6.3 and the fact that $K^\sigma = \gamma^{-1}(\mathbb{R} \setminus \{0\})$, therefore open as the reciprocal image of an open set by a continuous function. Moreover, Lemma 6.3 holds with h^σ_{rad} in place of h^σ.

Finally, let us explain why K^σ is dense in h^σ (from which the corresponding result on $\mathcal{K}^\sigma$ will follow). Let $(a_n)_{n\in\mathbb{Z}^2}$ be such that $\gamma_{(a_n)} = \gamma_{s,(a_n)} = 0$, and let $(b_n)_{n\in\mathbb{Z}^2}$ be radially symmetric such that $\gamma_{(b_n)} = \gamma_{s,(b_n)} \neq 0$, such as constructed in Lemma 6.1. One can therefore consider $c_n = a_n + \varepsilon b_n$ for $\varepsilon \ll 1$. There holds $\|c - a\|_{h^\sigma} = \mathcal{O}(\varepsilon)$, and expanding $\gamma_{(c_n)}$ in terms of ε_n we find

$$\gamma_{(c_n)} = \gamma_{(a_n)} + \kappa_1\varepsilon + \kappa_2\varepsilon^2 + \kappa_3\varepsilon^3 + \varepsilon^4\gamma_{(b_n)}$$

for some real constants κ_1, κ_2, and κ_3. By assumption, there holds $\gamma_{(a_n)} = 0$. If $\kappa_1 \neq 0$ then one just needs to take ε small enough so that $|\kappa_2\varepsilon^2 + \kappa_3\varepsilon^3 + \varepsilon^4\gamma_{(b_n)}| \leqslant \frac{\kappa_1\varepsilon}{2}$ and therefore $\gamma_{(c_n)} \neq 0$. If $\kappa_1 = 0$ we can do the same analysis on κ_2 and so on, so that the only remaining case is $\kappa_1 = \kappa_2 = \kappa_3 = 0$, in which case we use that $\gamma_{(b_n)} \neq 0$.

Theorem C (ii) will follow from the continuity in the Wasserstein-1 distance of the map $(a_n) \mapsto \mu_{(a_n)}$. This continuity can be seen from (1.7): let $a^k = (a^k_n)_{n\in\mathbb{Z}^2}$ which converges to $a = (a_n)_{n\in\mathbb{Z}^2}$ in h^σ (resp. h^σ_{rad}) and let $\mu_k := \mu_{a^k}$, $\mu := \mu_a$. For any 1-Lipschitz function φ, we compute

$$\begin{aligned}
\left|\int_{H^\sigma} \varphi(u)\,\mathrm{d}(\mu-\mu_k)(u)\right| &= \left|\mathbb{E}\left[\varphi\Big(\sum_{n\in\mathbb{Z}^2} a^k_n g^\omega_n e^{in\cdot x}\Big) - \varphi\Big(\sum_{n\in\mathbb{Z}^2} a_n g^\omega_n e^{in\cdot x}\Big)\right]\right| \\
&\leqslant \mathbb{E}\left[\left\|\sum_{n\in\mathbb{Z}^2} a^k_n g^\omega_n e^{in\cdot x} - \sum_{n\in\mathbb{Z}^2} a_n g^\omega_n e^{in\cdot x}\right\|_{H^\sigma(\mathbb{T}^2)}\right] \\
&\leqslant \left\|\sum_{n\in\mathbb{Z}^2} a^k_n g^\omega_n e^{in\cdot x} - \sum_{n\in\mathbb{Z}^2} a_n g^\omega_n e^{in\cdot x}\right\|_{L^2_\omega H^\sigma},
\end{aligned}$$

where in the first line of the above computation we have used the push-forward definition of the measures μ_k and μ. Taking the supremum over such functions φ yields

$$W_1(\mu,\mu_k) \leqslant \|a^k - a\|_{H^\sigma},$$

which goes to zero as $k \to \infty$.

6.3. Proof of Theorem 1.9

Our main tool is a generalization of the Kakutani criterion. Kakutani [46] characterized the equivalence between tensor products of equivalent measures. For Gaussian measures in Hilbert space, we use the following generalization due to Feldman and Hájek:

Theorem 6.4 (Feldman–Hájek [12, Theorem 2.7.2]). *Let μ and ν be Gaussian random variables on a Hilbert space X with means m_μ, m_ν and covariance operators C_μ, C_ν. Then μ and ν are either equivalent or mutually singular, and equivalence holds if and only if the three following conditions are met:*

(1) *μ and ν have the same Cameron–Martin space* $H = C_\mu^{\frac{1}{2}}(X) = C_\nu^{\frac{1}{2}}(X)$.

(2) $m_\mu - m_\nu \in H$.

(3) $(C_\mu^{-\frac{1}{2}} C_\nu^{\frac{1}{2}})(C_\mu^{-\frac{1}{2}} C_\nu^{\frac{1}{2}})^* - I$ *is a Hilbert–Schmidt operator.*

We will only use criterion (3) to prove that two given Gaussian measures are mutually singular. In particular, as we consider very simple Gaussian measures on the Hilbert space $H^\sigma(\mathbb{T}^2)$ with diagonal covariance operators, we obtain the following useful corollary.

Corollary 6.5. *Write $e_n(x) = e^{in\cdot x}$, and let us consider the two Gaussian measures*

$$\mu = \sum_{n\in\mathbb{Z}^2} h_n e_n \quad \text{and} \quad \nu = \sum_{n\in\mathbb{Z}^2} \tilde{h}_n e_n$$

on $H^\sigma(\mathbb{T}^2)$, where $h_k, \tilde{h}_k$ are centered Gaussian variables of variances σ_k, $\tilde{\sigma}_k$. If

$$\sum_{n\in\mathbb{Z}^2} \Big(\frac{\tilde{\sigma}_n^2}{\sigma_n^2} - 1\Big)^2 = \infty,$$

then μ and ν are mutually singular.

In order to prove Corollary 1.9, let us fix a sequence $(a_n)_{n\in\mathbb{Z}^2}$ such that its associated measure belongs to $\mathcal{M}^\sigma_{\mathrm{rad}}$ and such that $\gamma_{s,(a_n)} \neq 0$. Let us consider a sequence $(\varepsilon_n^\xi)_{n\in\mathbb{Z}^2}$ of independent and identically distributed Bernoulli random variables satisfying $\mathbb{P}(\xi : \varepsilon_n^\xi = 1) = \mathbb{P}(\xi : \varepsilon_n^\xi = -1) = \frac{1}{2}$. We introduce

$$b_n^\xi = a_n\Big(1 + \frac{\varepsilon_n^\xi}{2\langle n\rangle}\Big)^{\frac{1}{2}},$$

and consider the associated measures $\mu_\xi := \mu_{(b_n^\xi)}$. We check that there is a set Ξ_1 such that $\mathbb{P}(\xi \in \Xi_1) > 0$ and for all $\xi \in \Xi_1$, there holds $\gamma_{(b_n^\xi)} \neq 0$. Indeed, there holds

$$\mathbb{E}[\gamma_{s,(b_n^\xi)}] = \sum_{n,q\in\mathbb{Z}^2} \mathbb{E}[|b_n^\xi|^2 |b_q^\xi|^2]\beta_{n,q}.$$

Since $\beta_{n,q} = 0$ when $n = q$, it follows that

$$\begin{aligned}\mathbb{E}[|b_n^\xi|^2|b_q^\xi|^2] &= |a_n|^2|a_q|^2\mathbb{E}\Big[1 + \frac{\varepsilon_n^\xi}{\langle n\rangle}\Big]\mathbb{E}\Big[1 + \frac{\varepsilon_q^\xi}{\langle q\rangle}\Big] \\ &= |a_n|^2|a_q|^2\end{aligned}$$

so that $\mathbb{E}[\gamma_{s,(b_n^\xi)}] = \gamma_{s,(a_n)} \neq 0$, by assumption, therefore $\gamma_{s,(b_n^\xi)} \neq 0$ for $\xi \in \Xi_1$ of positive measure.

Let us fix $\xi_1 \in \Xi_1$, and apply Corollary 6.5 to μ_{ξ_1} and μ_ξ for some $\xi \in \Xi$ (the probability space). We write that

$$\Big(\frac{1 + \frac{\varepsilon_n^{\xi_1}}{2\langle n\rangle}}{1 + \frac{\varepsilon_n^{\xi}}{2\langle n\rangle}} - 1\Big)^2 = \frac{1 - \varepsilon_n^{\xi_1}\varepsilon_n}{2\langle n\rangle^2} + \mathcal{O}(|n|^{-3}) = \frac{1}{2\langle n\rangle^2} - \frac{\varepsilon_n^{\xi_1}\varepsilon_n^{\xi}}{2\langle n\rangle^2} + \mathcal{O}(|n|^{-3}). \tag{6.1}$$

The series $\sum_{n\in\mathbb{Z}^2}\frac{1}{\langle n\rangle^2}$ diverges, and the series of terms $\mathcal{O}(|n|^{-3})$ converges. Finally, the random (in ξ) series $\sum_{n\in\mathbb{Z}^2}\frac{\varepsilon_n^{\xi_1}\varepsilon_n^{\xi}}{\langle n\rangle^2}$ is convergent for almost every $\xi\in\Xi$ (let Ξ_* be this set of ξ) by application of Kolmogorov's three series theorem (we refer, e.g., to [30, Section 1.8]). Therefore, we obtain that the series of general terms (6.1) is divergent for all $\xi\in\Xi_*$, which is of probability 1. This yields uncountably many measures μ_ξ that are mutually absolutely singular with respect to μ_{ξ_1}. Note that this does not yield a set of measures which are pairwise mutually singular. The end of the proof proceeds by contradiction: assume that there are only a countable number of measures $\{\mu_k\}_{k\geqslant 1}$ which are pairwise mutually singular and such that for each μ_k, $\gamma\neq 0$. Let $\Xi_{(k)}$ be the set of ξ such that $\mu_\xi\perp\mu_k$. We have shown that $\mathbb{P}(\xi\in\Xi_1\cap\Xi_{(k)})=\mathbb{P}(\xi\in\Xi_1)>0$. Therefore, $\mathbb{P}(\xi\in\Xi_1\cap\bigcap_{k\geqslant 1}\Xi_{(k)})=\mathbb{P}(\xi\in\Xi_1)>0$. Observe now that for all $\xi\in\Xi_1\cap\bigcap_{k\geqslant 1}\Xi_{(k)}$ there holds $\mu_\xi\perp\mu_k$ for all $k\geqslant 1$ and all $\gamma_{(b_n^\xi)}\neq 0$, therefore a contradiction.

6.4. Proof of Theorem B

First, if $\mathrm{supp}(a_n)$ is either contained in a line passing through the origin or a circle centered at the origin, then thanks to [31], $\mu_{(a_n)}$ is supported on stationary solutions of (E), and therefore invariant under its flow.

For the converse, let us start with the case where μ has a finite Fourier support, i.e.,

$$S=\{n\in\mathbb{Z}^2,\ a_n\neq 0\}\text{ is finite.}$$

Let us follow the terminology in [31] and say that a pair (n,q) is degenerate if n and q are collinear (i.e., the line passing through n and q passes through the origin) or belong to the same circle centered at 0. In other words, (n,q) is degenerate if and only if $(n\cdot q^\perp)(\frac{1}{|n|^2}-\frac{1}{|q|^2})=0$.

If there exists $s\in(0,\sigma]$ such that $\gamma_s\neq 0$, there is nothing to prove. Otherwise, let us assume that $\gamma_s=0$ for all $s\in(s_0-\varepsilon,s_0+\varepsilon)$ for some $s_0\in(0,\sigma)$. Differentiating the equality $\gamma_s=0$ with respect to s and summing linear combinations of iterated derivatives of this equality, we find that for any polynomial $P\in\mathbb{R}[X]$ there holds

$$\begin{aligned}0=\sum_{|n|,|q|}|a_n|^2|a_q|^2(n\cdot q^\perp)^2\Big(\frac{1}{|n|^2}-\frac{1}{|q|^2}\Big)\\ \times\Big(P(2\log(\langle n+q\rangle))\langle n+q\rangle^{2s}\Big(\frac{1}{|n|^2}-\frac{1}{|q|^2}\Big)\\ +P(2\log(\langle n\rangle))\langle n\rangle^{2s}\Big(\frac{1}{|q|^2}-\frac{1}{|n+q|^2}\Big)\\ +P(2\log(\langle q\rangle))\langle q\rangle^{2s}\Big(\frac{1}{|n+q|^2}-\frac{1}{|n|^2}\Big)\Big).\end{aligned}\tag{6.2}$$

Let $M=\max\{|n|,a_n\neq 0\}$ and let $C_M:=\{n\in\mathbb{Z}^2,|n|=M,\text{ and }a_n\neq 0\}$. Let P be a polynomial such that $P(2\log(\langle k\rangle))=0$ for any $k\leqslant M$, and $P(2\log(\langle k\rangle))=1$ if

$M < k \leqslant 2M$. Using this polynomial in (6.2) yields

$$\sum_{n,q:|n+q|>M} \langle n+q\rangle^{2s}|a_n|^2|a_q|^2(n\cdot q^{\perp})^2\Big(\frac{1}{|n|^2}-\frac{1}{|q|^2}\Big)^2 = 0, \tag{6.3}$$

from which we infer that if $|n+q| > M$ and $a_n, a_q \neq 0$, then (n,q) is degenerate.

We distinguish between the case $|C_M| > 2$ and the case $|C_M| \leqslant 2$.

Assume that $|C_M| > 2$. We claim that all the modes n such that $a_n \neq 0$ are included in the circle $\{|n| = M\}$. In order to obtain this result, let us observe that since $|C_M| > 2$ and that C_M is symmetric (by assumption on the a_n) it means that there exist at least $p_1, p_2 \in C_M$ such that p_2 does not belong to the line passing through 0 and p_1. Let $q \neq p_1, p_2$ be such that $a_q \neq 0$. Since $a_{-q} = a_q \neq 0$, and either $|p_i + q| > M$ or $|p_i - q| > M$, there is no loss in generality in assuming $|p_i + q| > M$, so that for $i = 1, 2$, (p_i, q) is degenerate in view of (6.3). Since this holds for p_1 and p_2 not belonging to the same line passing through 0, this implies that q belongs to the circle of radius M centered at 0.

Let us assume $|C_M| \leqslant 2$, which actually means $|C_M| = 2$ and $C_M = \{p, -p\}$. Let us prove that any q such that $a_q \neq 0$ belongs to the line passing through 0 and p. Let q be such that $|q| < M$ and $a_q \neq 0$. Without loss of generality we can assume $|p + q| > M$ (otherwise there holds $|-p + q| > M$ and replace p by $-p$). Using (6.3) it follows that (p, q) is degenerate, and since $|q| < M = |p|$ it follows that q belongs to the line passing through 0 and p as claimed.

Remark 6.6. Combining this result with Theorem C, it holds that for any sequence $(a_n)_{n\in\mathbb{Z}^2}$ with compact support which is not degenerate, there exists an open neighborhood of $\mu_{(a_n)}$ in $\mathcal{M}^{\sigma}$ such that any measure in this neighborhood is non-invariant under the flow of (E).

Now, let us assume that there exist infinitely many activated Fourier modes $a_n \neq 0$. By assumption one can fix $n_k \in \mathrm{supp}(a_n)$, such that $|n_{k+1}| \geqslant |n_k| + 1$ and $\frac{a_{n_{k+1}}}{a_{n_k}} \to 0$.

Our argument proceeds in two steps: first we show that there is a large angular sector of the plane that asymptotically contains no activated Fourier mode, and then we rely on this non-activated region to propagate this non-activation information in order to arrive at the desired conclusion. Note that in our argument the strong decay assumption is only used in the first step.

Fix some n_0 for which $a_{n_0} \neq 0$ and $|n_0| \geqslant 2$. We use the following notation: B_M denotes the ball of $\mathbb{R}^2$ centered at 0 of radius M, and $C(p_0, \theta)$ the set of $n \in \mathbb{Z}^2$ for which the angle $\theta(n, p_0)$ between n and p_0 is smaller than $\theta(n, p_0)$ in absolute value.

Lemma 6.7. *Under the above assumptions, there exist $M = M(n_0)$, $\theta_0 = \theta_0(n_0, M)$ and $\theta_0^{\perp} = \theta_0^{\perp}(n_0, M)$ such that the following hold:*

(i) *If $|p| > M$ and $p \notin C(n_0^{\perp}, \theta_0^{\perp}) \cup C(n_0, \theta_0)$ then $a_p = 0$.*

(ii) $C(n_0, \theta_0) \cap C_M \subset \mathbb{R}n_0$.

Proof. Let $M > 0$ be chosen later and let $p \in \mathbb{Z}^2$ be such that $|p| > M$. Because either $|p + n_0| > |p|$ or $|p - n_0| > p$ we can assume without loss of generality that $|p + n_0| > |p|$. We apply the condition $\gamma = 0$ with the family of $\beta^{p_0}_{n,q}$ defined by (1.6) with $p_0 = p + n_0$ (and $\mathfrak{m}(n) = \mathbf{1}_{n=p_0}$). Exploiting the symmetry in q and n yields

$$\sum_{n+q=p_0} |a_n|^2|a_q|^2(n\cdot q^\perp)^2\Big(\frac{1}{|n|^2}-\frac{1}{|q|^2}\Big)^2 = 2\sum_{q\in\mathbb{Z}^2}|a_{p_0}|^2|a_q|^2(p_0\cdot q^\perp)^2\Big(\frac{1}{|p_0|^2}-\frac{1}{|q|^2}\Big)\Big(\frac{1}{|p_0+q|^2}-\frac{1}{|q|^2}\Big).$$

The left-hand side can be lower bounded by the (n_0, p) contribution, and the right-hand side can be upper bounded using the triangle inequality, so that

$$|a_{n_0}|^2|a_p|^2(n_0\cdot p^\perp)^2\Big(\frac{1}{|n_0|^2}-\frac{1}{|p|^2}\Big)^2 \leqslant 8|a_{p_0}|^2|p_0|^2\sum_{q\in\mathbb{Z}^2}|q|^2|a_q|^2.$$

Imposing $M \geqslant 2|n_0|$ it follows that

$$\sin^2(\theta(n_0,p))\frac{|a_{n_0}|^2}{|n_0|^2}|p|^2|a_p|^2 \leqslant \frac{16\times 8}{9}|a_{n_0+p}|^2|n_0+p|^2\sum_{q\in\mathbb{Z}^2}|q|^2|a_q|^2,$$

where $\theta(n_0, p)$ denotes the angle between n_0 and p. We can rewrite it as

$$\begin{aligned}|\sin(\theta(n_0,p))| &\leqslant \frac{4\sqrt{8}|n_0||a_{p+n_0}||n_0+p|}{3|a_{n_0}||p||a_p|}\sqrt{\sum_{q\in\mathbb{Z}^2}|q|^2|a_q|^2}\\ &\leqslant C(n_0)\Big(1+\frac{|n_0|}{M}\Big)\frac{|a_{n_0+p}|}{|a_p|} \leqslant C(n_0)\frac{|a_{n_0+p}|}{|a_p|}. \qquad (6.4)\end{aligned}$$

At this point we are going to restrict p outside the angular sector $C(n_0^\perp, \theta_0)$ in order to ensure that $|n_0 + p| \geqslant |p| + 1$, which happens if

$$2\cos(\theta(n_0,p)) \geqslant \frac{2}{|n_0|}-\frac{|n_0|-1/|n_0|}{|p|} = \frac{2}{|n_0|}-\mathcal{O}\Big(\frac{1}{M}\Big).$$

Therefore, we can restrict to $p \notin C(n_0^\perp, \theta_0)$ where $\cos\theta_0 = \frac{1}{|n_0|}$. Note that we use $|n_0| \geqslant 2$. It follows that $|n_0 + p| \geqslant |p| + 1$, and therefore by the assumption we can choose $M = M(n_0)$ large enough such that $C(n_0)\frac{|a_{n_0+p}|}{|a_p|} \leqslant \frac{1}{4}$. It then suffices to take $\theta_0 < \arcsin\frac{1}{4}$ in order to ensure that (i) holds (otherwise (6.4) is contradicted). Condition (ii) is obtained by imposing further $\theta_0 \lesssim \frac{1}{M}$. ∎

Let us now fix $M = M(n_0)$, $\theta_0 = \theta_0(n_0, M)$, and $\theta_0^\perp = \theta_0^\perp(n_0, M)$ as in the previous lemma. Let us prove that $\mathrm{supp}(a_n)$ is included in the strip $S_{n_0,M(n_0)} := \{p = (p_1, p_2) \in \mathbb{Z}^2 : |p\cdot n_0^\perp| \leqslant |n_0|MK\}$, where $K > \frac{1}{\cos\theta_0-\sin\theta_0^\perp}$. Indeed, take $p \notin S_{n_0,M(n_0)}$. Let us start

with the case $p \in C(n_0^\perp, \theta_0^\perp)$. Fix $k \in \mathbb{Z}$ such that $p + kn_0 \notin C(n_0^\perp, \theta_0^\perp) \cup C(n_0, \theta_0)$. This can be done because we have $K(\cos\theta_0 - \sin\theta_0^\perp) > 1$, which ensures that such a k exists. Then the previous lemma implies $a_{p+kn_0} = 0$. Applying the condition $\gamma = 0$ with the $\beta_{n,q}^{p+kn_0}$ we find that the couple $(n_0, p + (k-1)n_0)$ is degenerate. Because $|p + (k-1)n_0| > |n_0|$ and $p \notin \mathbb{R}n_0$, this forces $a_{p+(k-1)n_0} = 0$. We can iterate this argument and finally obtain $a_p = 0$.

Assume now that there exists an activated mode $n_1 \notin \mathbb{R}n_0$. Then the previous argument can be applied to n_1 and finally the activated modes are necessarily contained in the intersection of two non-parallel strips of $\mathbb{Z}^2$, i.e., in a finite set, so that the case of compact support implies that the set is contained in a circle centered at the origin.

6.5. Proof of Theorem D

In this section we prove that the Gaussian measure on $H^4(\mathbb{T}^2)$ induced by

$$\sum_{n\in\mathbb{Z}^2\setminus\{0\}} \frac{g_n^\omega}{\langle n\rangle^5 \log(3+\langle n\rangle^2)} e^{in\cdot x}$$

satisfies $\gamma_{\frac{1}{2}} = \gamma > 0$ (in this section we choose $s = \frac{1}{2}$ and henceforth drop the subscript from γ). Such an example is a typical example one might consider to produce a measure whose support is in H^4 but not in $\bigcup_{\varepsilon>0} H^{4+\varepsilon}$.

Our proof that $\gamma > 0$ is computer assisted, making use of interval arithmetic to enclose the true value of γ in an interval whose endpoints are computed numerically. This method allows us to fully account for CPU rounding errors; see [74] for an introduction to mathematically rigorous computation using interval arithmetic. We will use interval arithmetic to give a strictly positive lower bound on γ and then use analytical estimates to show that the error of the truncation is sufficiently small to deduce $\gamma > 0$.

We first compute for some $N > 0$ the partial sum

$$\gamma_N := \sum_{|n|,|q|>N} \alpha_{n,q} := \sum_{|n|,|q|<N} |a_n|^2|a_q|^2(q\cdot n^\perp)^2\Big(\frac{1}{|n|^2} - \frac{1}{|q|^2}\Big)\beta_{q,n},$$

where $\beta_{q,n} = \beta_{q,n,s}$ is given by (1.6). Then we separately bound the error $|\gamma - \gamma_N|$ analytically. The two estimates we prove for $N = 30$ are as follows:

$$\begin{cases} \frac{1}{2}\gamma_N \in [0.00011184535610465990373, 0.00011184535613147070557], \\ \frac{1}{2}|\gamma - \gamma_N| \in [0, 0.000105349479423897216787]. \end{cases}$$

The interval for γ_N is obtained using interval arithmetic implemented in Python with the `mpmath` package.[5] See Appendix A for the code used to carry out the computation.

[5] https://mpmath.org

Computations were carried out with python 3.10.6 and the CPU is 11th Gen Intel(R) Core(TM) i5-1135G7 @ 2.40 GHz.

Next, let us consider the analytic estimate on $|\gamma - \gamma_N|$. First, note that as the summation in γ is symmetric in n and q, there holds

$$\gamma_N = \sum_{|n|,|q|<N} \alpha_{n,q} = 2 \sum_{|n|,|q|<N} \alpha_{n,q} \mathbf{1}_{|n|<|q|},$$

so that

$$|\gamma - \gamma_N| \leqslant 2 \sum_{|q|\geqslant 1} \sum_{|n|\geqslant N} \alpha_{n,q} \mathbf{1}_{|n|<|q|} + 2 \sum_{|n|\geqslant 1} \sum_{|q|\geqslant N} \alpha_{n,q} \mathbf{1}_{|n|<|q|}.$$

Using $|n| < |q|$ we have $\langle n\rangle^{2s} \leqslant 2^s |q|^{2s}$ and $\langle q\rangle^{2s} \leqslant 2^s |q|^{2s}$, as well as $\langle n+q\rangle^{2s} \leqslant 5^s |q|^{2s}$. Therefore a crude bound on $\beta_{n,q}$ reads

$$\alpha_{n,q} \mathbf{1}_{|n|<|q|} \leqslant |q|^2 |n|^2 \frac{1}{|n|^2} (2^{s+2} + 5^s) |q|^{2s} \frac{1}{|n|^{10}|q|^{10}} = \frac{2^{3/2} + 3^{1/2}}{|n|^{10}|q|^7},$$

where we have used $s = \frac{1}{2} > 0$. We have also used $\frac{1}{\log(3+\langle n\rangle^2)} \leqslant 1$. We will use the following bounds:

- $2^{5/2} + 5^{1/2} < 8$, which is obtained by direct computations (by hand).
- For any $\tau \geqslant 4$ we bound

$$\sum_{n\in\mathbb{Z}^2\setminus\{0\}} \frac{1}{|n|^\tau} \leqslant \sum_{n\in\mathbb{Z}^2\setminus\{0\}} \frac{1}{|n|^4} = \sum_{k\geqslant 1} |\{n\in\mathbb{Z}^2,\ |n| = k\}| \frac{1}{k^4}.$$

  We use the very simple fact that

$$\{n\in\mathbb{Z}^2,\ |n| = k\} \subset \{n\in\mathbb{Z}^2 : \max_{|n_1|,|n_2|} \leqslant k\} \setminus \{0\},$$

  so that $|\{n\in\mathbb{Z}^2,\ |n| = k\}| \leqslant (2k+1)^2 - 1 = 4k^2 + 4k \leqslant 8k^2$ and therefore

$$\sum_{n\in\mathbb{Z}^2\setminus\{0\}} \frac{1}{|n|^\tau} \leqslant \sum_{k\geqslant 1} \frac{8k^2}{k^4} \leqslant \frac{8\pi^2}{6}.$$

  We can also write

$$\sum_{|q|\geqslant N} \frac{1}{|q|^K} = \sum_{k\geqslant 0} \sum_{\substack{2^k(N+1) \\ \leqslant |q| < 2^{k+1}N}} \frac{1}{|q|^K} \leqslant 12 N^{-K+2} \sum_{k\geqslant 0} 2^{-(K-2)k},$$

therefore since $\sum_{k\geqslant 0} 2^{-(K-2)k} \leqslant 2$ we have the crude bounds

$$\sum_{\substack{|n|<|q| \\ |q|\geqslant N, |n|\geqslant 1}} \alpha_{n,q} \leqslant 8 \times \frac{8\pi^2}{6} \times \frac{24}{N^5},$$

and

$$\sum_{\substack{|n|<|q| \\ |q|\geqslant 1, |n|\geqslant N}} \alpha_{n,q} \leqslant 8 \times \frac{24^2}{N^{13}},$$

so that finally

$$\begin{aligned}\frac{1}{2}|\gamma - \gamma_N| &\leqslant \frac{1536}{N^5}\left(\frac{\pi^2}{6} + \frac{3}{N^8}\right) \\ &< \frac{1536}{N^5}\left(\frac{10}{6} + \frac{3}{N^8}\right) =: \varepsilon,\end{aligned}$$

where we have used $\pi^2 < 10$. An interval for the value of ε is also computed in Appendix A.

A. Python code

```
import numpy as np
from mpmath import iv #this is the interval arithmetic package
from mpmath import *
sig=4 #value of sigma
s=1 #value of s
N=30 #cutoff in the partial sum
def norm(n,m): #defines the square of the norm
    return iv.mpf(n**2+m**2)

def bkt(n,m): #defines the square of Japanese bracket
    return iv.mpf(1+n**2+m**2)

def ps(n,m,p,q):
    return iv.mpf((n*q-m*p)**2)

def a2(n,m): #defines |a_n|^2
    return iv.mpf(1/((bkt(n,m)**(sig+1))*(iv.log(3+bkt(n,m))**2)
        ))
S=iv.mpf(0)
for n in range(-N+1,N,1):
    print('step '+str(n))
    for m in range(-N+1,N,1):
        for p in range(-N+1,N,1):
            for q in range(-N+1,N,1):
                scalar=n*q-m*p
                result=0
                if(scalar !=0):
                    ps=iv.mpf((n*q-m*p)**2)
                    Nn=n**2+m**2
                    Nq=p**2+q**2
                    Nnq=(n+p)**2 + (m+q)**2
                    if((Nn < Nq) and (Nn != Nnq) and (Nnq != Nq)
                        and (Nn < N**2) and (Nq < N**2)):
                        result=a2(n,m)*a2(p,q)*ps*(1/norm(n,m)
                            -1/norm(p,q))*((bkt(n+p,m+q)**s)*(1/
```

```
                                norm(n,m)-1/norm(p,q))+(bkt(n,m)**s)
                                *(1/norm(p,q)-1/norm(n+p,m+q))+(bkt(
                                p,q)**s)*(1/norm(n+p,m+q)-1/norm(n,m
                                )))
33                    S=S+result
34  print(S)
35  print(iv.mpf((1536/(N**5))*(10/6+3/(N**8))))
```

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