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A microlocal Cauchy problem through a crossing point of Hamiltonian flows

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Abstract. In this paper, we consider 2×2 matrix-valued pseudodifferential equations in which the two characteristic sets intersect with finite contact order. We show that the asymptotic behavior of its solution changes dramatically before and after the crossing point, and provide a precise asymptotic formula. The proof relies on a normal form reduction and a detailed analysis of a simple first-order system.

Contents

1. Introduction

1.1. Main results

In this paper, we discuss the existence, uniqueness, and asymptotic behavior of solutions of a system of pseudodifferential equations:

$$\mathcal{P}u \equiv 0 \quad \text{m.l. on } \Omega, \tag{1.1}$$

where

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- $\Omega \subset \mathbb{R}^2 = T^*\mathbb{R}$ is a connected open neighborhood of $(0,0)$;
- $\mathcal{P}$ is a 2×2-matrix-valued semiclassical pseudodifferential operator of the form

$$\mathcal{P} = \begin{pmatrix} P_1 & hQ_1 \\ hQ_2 & P_2 \end{pmatrix} \quad \text{acting on } L^2(\mathbb{R}, \mathbb{C}^2),$$

defined as the Weyl quantization (see (2.1)) of the matrix-valued symbol[1]

$$p(x,\xi) := \begin{pmatrix} p_1(x,\xi) & hq_1(x,\xi) \\ hq_2(x,\xi) & p_2(x,\xi) \end{pmatrix} \in \mathcal{M}_2(C_b^\infty). \tag{1.2}$$

The motivations and the generality of the model are discussed in details in Section 1.2 and Appendix C. We send the reader unfamiliar with the vocabulary of microlocal analysis to Section 2 for precise definitions (see also Section 1.3 for a simpler, non-microlocal example). Throughout the article, we assume that the following two conditions hold.

Condition A (Real principal type). For each $j = 1, 2$, the function p_j is of real principal type at $(0,0)$, that is, p_j is real-valued and

$$p_j(0,0) = 0, \quad \partial p_j(0,0) \neq 0.$$

Condition B (Finite contact order). There exists $m \geq 1$ such that $H_{p_1}^k p_2(0,0) = 0$ for all $k \in \{0, \dots, m-1\}$ and

$$H_{p_1}^m p_2(0,0) \neq 0.$$

Here, we write $H_{p_1} := \partial_\xi p_1 \partial_x - \partial_x p_1 \partial_\xi$. In the following, the *transversal case* refers to the situation where $m = 1$ and the *tangential case* to situations where $m \geq 2$.

The domain Ω is chosen to be sufficiently small so that the following conditions hold.

- The characteristic sets

$$\Lambda_j := \{(x,\xi) \in \mathbb{R}^2 \mid p_j(x,\xi) = 0\} \quad (j = 1, 2)$$

are submanifolds and intersect only at $(0,0)$ inside Ω. Geometrically, the intersection is of finite contact order m by Condition B. We refer to m as *the contact order* of Λ_1 and Λ_2;

[1]Here, we assume that the symbol p is bounded on $\mathbb{R}^2$. However, we can still apply our main theorem for unbounded symbols such as $p_j(x,\xi) = \xi^2 + V_j(x)$ or $p_j \in S(m)$ for an order function m (see [22, Section 4.4]) since all the statements below are microlocalised on a compact subset of the phase space $T^*\mathbb{R}$.

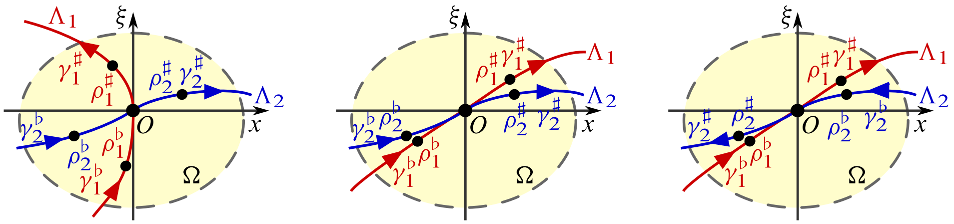

Figure 1. Hamiltonian flows and notations.

- For $j = 1, 2$, the portion $(\Omega \cap \Lambda_j) \setminus \{(0,0)\}$ is divided into two connected components $\gamma_j^\flat$ and $\gamma_j^\sharp$, where the Hamiltonian flow generated by H_{p_j} is incoming to the crossing point $(0,0)$ on $\gamma_j^\flat$ while it is outgoing from $(0,0)$ on $\gamma_j^\sharp$. See Figure 1.

In the setting above, the uniqueness, existence, and asymptotic behavior of microlocal solutions *away from the crossing point* $(0,0)$ are well known and can be analysed using the usual WKB method. In particular, it is known that microlocal solutions of (1.1) are microlocalised on $\Lambda_1 \cup \Lambda_2$ – meaning that their wavefront set is included in the characteristic sets, see Definition 2.2. On each connected component $\gamma_j^\bullet$ ($j = 1, 2$, $\bullet \in \{\flat, \sharp\}$), one has the following.

- *Uniqueness.* The space of microlocal solutions near $\gamma_j^\bullet$ is *one-dimensional.* More precisely, $u \equiv 0$ microlocally near a point of $\gamma_j^\bullet$ implies that $u \equiv 0$ microlocally on all $\gamma_j^\bullet$.
- *Existence.* There exists a non-trivial solution $w_j^\bullet$ of (1.1) microlocally on $\gamma_j^\bullet$, uniquely determined up to a multiplicative constant.
- *Asymptotic behavior.* The solution $w_j^\bullet$ is a WKB state (or Lagrangian distribution) associated with the Lagrangian submanifold Λ_j. Generically for p_1 and p_2, we can write $w_j^\bullet(x) = \sigma_j^\bullet(x) e^{i\phi_j(x)/h}$, where ϕ_j is a solution to the eikonal equation $p_j(x, \phi_j'(x)) = 0$ and the amplitude $\sigma_j^\bullet$ satisfies a transport equation.

According to the uniqueness and existence written above, for each solution u of (1.1) microlocally on Ω, there exist four complex numbers $\alpha_1^\flat, \alpha_2^\flat, \alpha_1^\sharp, \alpha_2^\sharp \in \mathbb{C}$ such that

$$u \equiv \alpha_j^\bullet w_j^\bullet \quad \text{m.l. on } \gamma_j^\bullet \quad (j = 1, 2,\ \bullet \in \{\flat, \sharp\}). \tag{1.3}$$

See Proposition 2.16 for the more rigorous statements, proved by standard WKB analysis.

The purpose of this paper is to study the equation (1.1) *near the crossing point* $(0,0)$. More precisely, we consider the following problems.

Problem 1. How many microlocal solutions are there for (1.1)?

Problem 2. What is the relationship between the coefficients $\alpha_j^\bullet$?

Problem 1 is solved by the following theorem.

Theorem 1.1 (Existence and uniqueness of the microlocal Cauchy problem). *In the above setting, the following holds.*

(i) Uniqueness. *Let u be a microlocal solution of $\mathcal{P}u = 0$ on Ω. If there exist two points $\rho_1^\flat \in \gamma_1^\flat$ and $\rho_2^\flat \in \gamma_2^\flat$ such that $u \equiv 0$ microlocally near $\rho_1^\flat$ and $\rho_2^\flat$, then*

$$u \equiv 0 \quad \textit{m.l. on } \Omega.$$

(ii) Existence *Fix two WKB states $w_j^\flat$ ($j = 1, 2$), which are tempered and satisfy* (1.1) *microlocally on $\gamma_j^\flat$. Then, for each $\alpha_1^\flat, \alpha_2^\flat \in \mathbb{C}$, there exists a tempered u such that*

$$\mathcal{P}u \equiv 0 \quad \textit{m.l. on } \Omega, \qquad u \equiv \alpha_j^\flat w_j^\flat \quad \textit{m.l. on } \gamma_j^\flat \quad (j = 1, 2). \tag{1.4}$$

We refer to the constants $\alpha_1^\flat$ and $\alpha_2^\flat$ appearing in the second identity (1.4) *as the* microlocal Cauchy data of the system (1.1).

In particular, the space of microlocal solutions on Ω of (1.1) is *two-dimensional*. The term “microlocal Cauchy data” is taken from [5], where a microlocal Cauchy problem involving a scalar Hamiltonian with a hyperbolic fixed point is studied.

We now turn to Problem 2. Fix two WKB states $w_j^\bullet$ ($j = 1, 2$), which are tempered and satisfy (1.1) microlocally on $\gamma_j^\bullet$. Thanks to Theorem 1.1 (ii), for given $\alpha_1^\flat, \alpha_2^\flat \in \mathbb{C}$, we can find a solution u of (1.4). As stated above in (1.3), there exist $\alpha_1^\sharp, \alpha_2^\sharp \in \mathbb{C}$ such that

$$u \equiv \alpha_j^\sharp w_j^\sharp \quad \text{m.l. on } \gamma_j^\sharp \quad (j = 1, 2). \tag{1.5}$$

By virtue of Theorem 1.1 (i), the constants $\alpha_1^\sharp, \alpha_2^\sharp \in \mathbb{C}$ are unique modulo $\mathcal{O}(h^\infty)$. Since the equation (1.4) is linear, we can find a 2×2 matrix T, called *the transfer matrix,* such that

$$\begin{pmatrix} \alpha_1^\sharp \\ \alpha_2^\sharp \end{pmatrix} = T \begin{pmatrix} \alpha_1^\flat \\ \alpha_2^\flat \end{pmatrix},$$

which can be defined modulo $\mathcal{O}(h^\infty)$. Our next theorem describes the asymptotic behavior of the transfer matrix. Here, we remark that the transfer matrix $T = T(w_1^\flat, w_2^\flat, w_1^\sharp, w_2^\sharp)$ depends on the choice of the WKB states $w_j^\bullet$ ($j = 1, 2$, $\bullet \in \{\flat, \sharp\}$), as described in (1.18) in the example below. To simplify the connection formula, we will choose these WKB states as in the definition below.

Definition 1.2. (i) Let v_j be a microlocal solution to the scalar equation $Pv_j \equiv 0$ m.l. on Ω. We say that v_j is *normalised* if

$$\frac{e^{-\frac{i\Theta(p_j)}{2}}}{\sqrt{2\pi h}}\int_{\mathbb{R}} e^{-\frac{x^2}{2h}} v_j(x)\,dx = 1 + \mathcal{O}(h), \tag{1.6}$$

where we put

$$\Theta(p_j) := \begin{cases} -\arctan\left(\dfrac{\partial_x p_j(0,0)}{\partial_\xi p_j(0,0)}\right) & (\partial_\xi p_j(0,0) \neq 0), \\ -\dfrac{\pi}{2} & (\partial_\xi p_j(0,0) = 0). \end{cases} \tag{1.7}$$

(ii) Let $w_j^\bullet$ ($j = 1, 2, \bullet \in \{\flat, \sharp\}$) be a microlocal solution to equation (1.1) microlocally on $\gamma_j^\bullet$. The solution $w_j^\bullet$ is said to be *normalised* if

$$w_j^\bullet \equiv v_j \mathbf{e}_j + \mathcal{O}_{L^2(\mathbb{R};\mathbb{C}^2)}(h) \quad \text{m.l. on } \gamma_j^\bullet, \tag{1.8}$$

where the pair of $\mathbf{e}_1 = {}^\mathsf{T}(1,0)$ and $\mathbf{e}_2 = {}^\mathsf{T}(0,1)$ is the canonical basis of $\mathbb{C}^2$, and v_j is a normalised microlocal solution to the scalar equation $P_j v_j \equiv 0$ m.l. on Ω.

Remark 1.3. Normalised solutions indeed exist (Propositions 2.13 and 2.16) and are unique up to $\mathcal{O}(h)$. The ambiguity involving an $\mathcal{O}(h)$-term comes from that in (1.6) and (1.8).

We now state our second result which deals with Problem 2. The symbol $\delta_{m,1}$ stands for Kronecker's delta: $\delta_{m,1} = 1$ for $m = 1$ and $\delta_{m,1} = 0$ for $m \geq 2$.

Theorem 1.4 (Microlocal connection formula). *Suppose that the WKB solutions $w_j^\bullet$ are normalised in the sense of Definition 1.2. Then, the transfer matrix $T = T(w_1^\flat, w_2^\flat, w_1^\sharp, w_2^\sharp)$ satisfies the asymptotic formula*

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i h^{\frac{1}{m+1}} \begin{pmatrix} 0 & \omega_1 q_1(0,0) \\ \omega_2 q_2(0,0) & 0 \end{pmatrix} + \mathcal{O}\left(h^{\frac{2}{m+1}}\left(\log\frac{1}{h}\right)^{\delta_{m,1}}\right), \tag{1.9}$$

where the q_j are the anti-diagonal terms of p defined in (1.2) and the ω_j are given by

$$\begin{aligned} \omega_1 &= 2\mu_m\left(\frac{-\operatorname{sgn}(sH_{p_1}^m p_2(0,0))\pi}{2(m+1)}\right)\mathbf{\Gamma}\left(\frac{m+2}{m+1}\right) \\ &\quad \times \left(\frac{|\partial_{(x,\xi)} p_2(0,0)|}{|\partial_{(x,\xi)} p_1(0,0)|}\frac{(m+1)!}{|H_{p_1}^m p_2(0,0)|}\right)^{\frac{1}{m+1}}, \\ \omega_2 &= 2\mu_m\left(\frac{-\operatorname{sgn}(sH_{p_2}^m p_1(0,0))\pi}{2(m+1)}\right)\mathbf{\Gamma}\left(\frac{m+2}{m+1}\right) \\ &\quad \times \left(\frac{|\partial_{(x,\xi)} p_1(0,0)|}{|\partial_{(x,\xi)} p_2(0,0)|}\frac{(m+1)!}{|H_{p_2}^m p_1(0,0)|}\right)^{\frac{1}{m+1}}. \end{aligned}$$

Here, the constant s and the function $\mu_m = \mu_m(\theta)$ are defined by

$$\begin{aligned} s &:= \operatorname{sgn} \prod_{j=1}^{2} (H_{p_j}(0,0), \tau(\Theta(p_j)))_{\mathbb{R}^2}, \quad \tau(\theta) = (\cos\theta, \sin\theta), \\ \mu_m(\theta) &:= \frac{e^{i\theta} + e^{i(-1)^{m+1}\theta}}{2}. \end{aligned} \tag{1.10}$$

where $(\cdot,\cdot)_{\mathbb{R}^2}$ stands for the usual (Euclidean) inner product in $\mathbb{R}^2$.

Remark 1.5. The ambiguity of the choices of normalised solutions $w_j^{\bullet}$ is not explicit in this formula since the estimate of the remainder $\mathcal{O}(h^{\frac{2}{m+1}}(\log h^{-1})^{\delta_{m,1}})$ is rougher than $\mathcal{O}(h)$.

Remark 1.6. By the definition (1.7) of $\Theta(p_j)$, the vector $\tau(\theta)$ defined by (1.10) is the unit vector parallel to the Hamiltonian vector $H_{p_j}(0,0)$, that is,

$$H_{p_j}(0,0) = \pm|H_{p_j}(0,0)|\tau(\Theta(p_j)), \quad (H_{p_j}(0,0), \tau(\Theta(p_j)))_{\mathbb{R}^2} = \pm|H_{p_j}(0,0)|,$$

where the identities hold true for the same sign. The sign is positive if and only if either $\partial_\xi p_j(0,0) > 0$ or $\partial_\xi p_j(0,0) = 0$ and $\partial_x p_j(0,0) > 0$ holds (see Figures 2 and 3). In the tangential case, one has $\Theta(p_1) = \Theta(p_2)$. This implies the identity

$$s = \operatorname{sgn}(H_{p_1}(0,0), H_{p_2}(0,0))_{\mathbb{R}^2}.$$

Remark 1.7. The formulae of ω_1 and ω_2 in Theorem 1.4 are written symmetrically. We can also write them using only either one of $H_{p_1}^m p_2(0,0)$ or $H_{p_2}^m p_1(0,0)$ since one has the identity

$$H_{p_1}^m p_2(0,0) = -\Big(\frac{|\partial_{(x,\xi)} p_1(0,0)|}{|\partial_{(x,\xi)} p_2(0,0)|}\Big)^{m-1} H_{p_2}^m p_1(0,0).$$

This identity will be proven in Lemma 5.6.

1.2. Motivations and context

The main motivation to computing the transfer matrix (1.9) is to provide a general method for computing spectral asymptotics, namely the eigenvalue splitting ([2]) and the distribution of resonances ([3, 4, 11–13, 16]) of matrix-valued Schrödinger operators in the semiclassical limit $h \to +0$. Essentially, such distribution is obtained by studying the associated eigenstates and resonant states, which are microlocalised on the characteristics Λ_j and behave as microlocal solutions of the eigenvalue equation (1.1). The behavior of such solutions on the characteristics is known outside crossing points, therefore establishing a microlocal connection formula at those

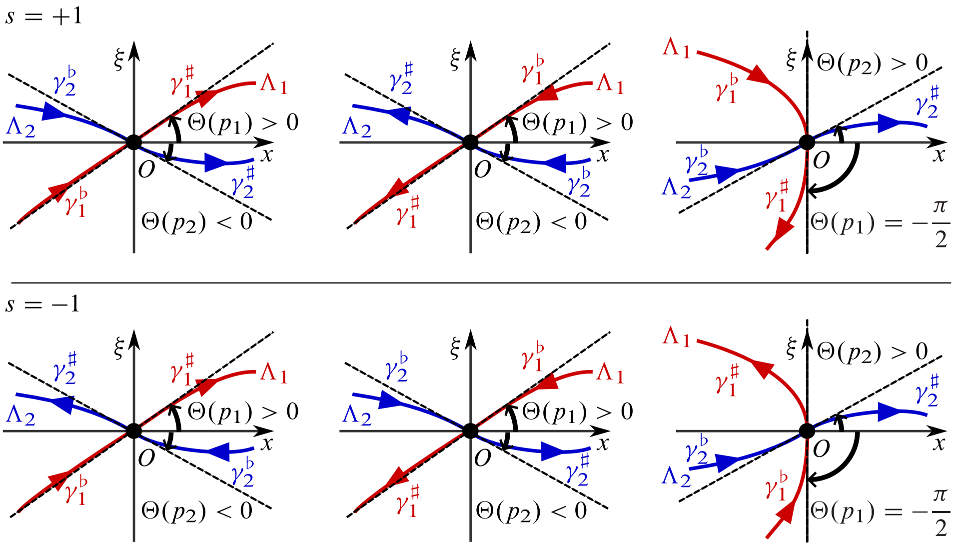

Figure 2. Hamiltonian flows, the sign of s introduced by (1.10), and $\Theta(p_j)$ introduced by (1.7) for the transversal case $m = 1$.

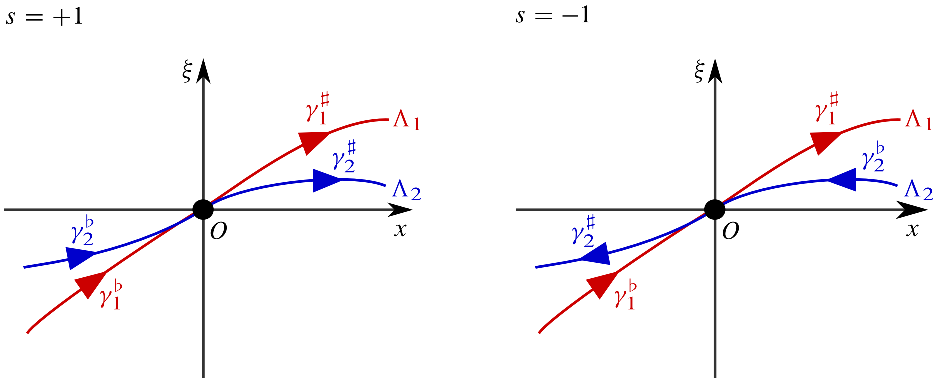

Figure 3. Hamiltonian flows and the sign of s introduced by (1.10) for the tangential case $m \geq 2$.

crossing points is enough to describe resonant and eigenstates.[2] Such an approach is employed in [2–4, 13, 16]. While [2, 13] used a normal form reduction to obtain a microlocal connection formula for transversal case, [3, 4, 16] used a non-microlocal method to obtain the formula for tangential case. Our goal is to unify those different methods using a normal form reduction which applies to both situations.

[2] We also find a similar approach for the asymptotics of the scattering matrix of matrix-valued Schrödinger operators with a transversal crossing of Hamiltonian flows in [1].

The study of a microlocal problem via a normal form reduction is not new, and the idea traces back to the works of Landau and Zener on avoided crossings. We distinguish two types of normal form reductions of matrix-valued operators.

- The first essentially consists in reducing a matrix-valued symbol of principal type to a scalar-valued symbol of the form $x\xi$, which has a hyperbolic fixed point at $(0,0)$ ([13] or [7]). Here the interaction term is assumed to be elliptic at the crossing point.
- The second method consists in reducing the matrix-valued symbol into another simpler matrix-valued symbol without ellipticity of the interaction term. Such method was developed for codimension 3 crossings in [6], where the normal form is given by $\begin{pmatrix} hD_{x_1} & x_2 \\ x_2 & x_1 \end{pmatrix}$ on $\mathbb{R}^2$. This includes not only the $x_2 = \mathcal{O}(h)$-regime of that operator (which corresponds to our operator) but also the cases where the non-diagonal entries are not necessary small. The result in [6] was applied in [21] for 1-dimensional, transversal crossings.

For both methods, tangential crossings were excluded up until now because of the absence of stable/unstable theorems of degenerate fixed points. In his work on transversal crossings, Colin de Verdière [6] adapted Sternberg's linearization theorem for Hamiltonian systems, which is regarded as a counterpart of stable/unstable theorems. However, the technique there cannot be applied to tangential crossings, at least not directly. To handle tangential crossings without a normal form (but with $\mathcal{O}(h)$ non-diagonal entries), the first named author extended in [3] the construction of [11] of exact solutions near the crossing point. The second named author improved this method in [16] under a relaxed elliptic assumption on the interaction term.

We also mention some studies concerning the closely related problem of the time propagation of semiclassical wave packets. Such a problem gathered interest in the late 1990s with the works of G. A. Hagedorn and others (see e.g. [14]). Among many works on the subject, we mention in particular [10] which deals with time propagation through tangential crossings in codimension 1. They provide a microlocal normal form reduction but only for the transversal case.

Our normal form result, Theorem 3.1, accounts for both transversal and tangential crossings and holds without any ellipticity assumption. It relies essentially on the Darboux theorem and the Egorov theorem, and makes for a much simpler proof than that of [21] for the transversal case. Moreover, we apply this normal form reduction to recover the results of [3,11–13] all at once, through the microlocal connection formula (Theorem 1.4). In practice, since P is a 2×2 matrix-valued differential operator of order 1 at the principal level, we end up computing 2×2 change-of-basis matrices (see (4.7)), whereas for example the direct computation of [3, Proposition 3.12] for the Schrödinger operator (which is of order 2 at the principal level) involves 4×4 change-of-basis matrices.

In the rest of the section, we sketch an outline of the connection formula through a simple example and we apply Theorem 1.4 to recover the connection formulae for Schrödinger-type operators. In Section 2, we lay out the background of microlocal analysis needed for this paper. The organization of Sections 3, 4, and 5 will be detailed in Section 2.8.

1.3. Simple case (non-microlocal)

Let us describe our method for a simple case with

$$P_1 = hD_x, \quad P_2 = hD_x - f(x), \quad Q_1 = Q_2 = q(x)$$

where $f \in C^\infty(\mathbb{R})$ and $q \in C_c^\infty(\mathbb{R})$ are real-valued functions satisfying

$$\begin{aligned} f^{(k)}(0) &= 0 \quad \text{for } k = 0, 1, \dots, m-1, \quad f^{(m)}(0) \neq 0, \\ f(x) &\neq 0 \quad \text{for } x \in \operatorname{supp} q \setminus \{0\}. \end{aligned}$$

Although Theorems 1.1 and 1.4 deal with microlocal solutions, we focus on the exact solutions here.

The equation $\mathcal{P}u = 0$ with $u(x) = {}^\mathsf{T}(u_1(x), u_2(x))$ is written as the following system of differential equations:

$$\begin{cases} hD_x u_1(x) = -hq(x)u_2(x), \\ (hD_x - f(x))u_2(x) = -hq(x)u_1(x). \end{cases} \tag{1.11}$$

First, we observe that this equation can be written by the Duhamel formula as

$$u_1(x) = \alpha_1^\flat - i\Gamma_+(e^{-\frac{i}{h}F(\cdot)}u_2), \quad u_2(x) = e^{\frac{i}{h}F(x)}(\alpha_2^\flat - i\Gamma_-(u_1)), \tag{1.12}$$

with

$$\alpha_1^\flat = u_1(-1) \quad \text{and} \quad \alpha_2^\flat = u_2(-1)\exp\Big(\frac{i}{h}\int_{-1}^{0} f(y)\,dy\Big),$$

where we define

$$\Gamma_\pm v(x) := \int_{-1}^{x} e^{\pm F(y)} q(y)v(y)\,dy, \quad F(x) := \int_0^x f(y)\,dy.$$

Existence and uniqueness of exact solutions

Equation (1.11) is a first order differential equation and the space of the solutions is two-dimensional. Moreover, the solution is uniquely determined by the initial data

$\alpha_1^\flat, \alpha_2^\flat \in \mathbb{C}$ and can be written explicitly as

$$\begin{aligned} u_1 &= (I + \Gamma_+\Gamma_-)^{-1}(\alpha_1^\flat - i\alpha_2^\flat \Gamma_+(\mathbb{1})), \\ u_2 &= e^{\frac{i}{h}F}(I + \Gamma_-\Gamma_+)^{-1}(\alpha_2^\flat - i\alpha_1^\flat \Gamma_-(\mathbb{1})), \end{aligned} \tag{1.13}$$

due to (1.12), where $\mathbb{1}$ is the constant function valued at 1. Here, the operator $I + \Gamma_+\Gamma_-$ is invertible for sufficiently small $h > 0$, in fact, we have $\|\Gamma_\pm\Gamma_\mp\|_{L^\infty \to L^\infty} = \mathcal{O}(h^{\frac{1}{m+1}})$ by the degenerate stationary phase formula (see Proposition 4.4).

Asymptotic behavior of the solution

To describe the asymptotic behavior of the solution, we need more precise estimates for $\Gamma_\pm$. We have $\|\Gamma_\pm\Gamma_\mp(\mathbb{1})\|_{L^\infty \to L^\infty} = \mathcal{O}(h^{\frac{2}{m+1}}(\log \frac{1}{h})^{\delta_{m,1}})$ and

$$\Gamma_\pm(\mathbb{1})(x) = \begin{cases} \mathcal{O}(h) & (-1 \le x \le -\epsilon), \\ h^{\frac{1}{m+1}} q(0)\omega_\pm + \mathcal{O}\left(h^{\frac{2}{m+1}}\left(\log \frac{1}{h}\right)^{\delta_{m,1}}\right) & (\epsilon \le x \le 1), \end{cases}$$

for fixed $0 < \epsilon < 1$ with $\omega_+ = \omega, \omega_- = \bar{\omega}$ and ω defined by (4.8). This is a consequence of the degenerate stationary phase formula (see Proposition 4.4), where we note that the critical point of F on $\operatorname{supp} q$ is $x = 0$ only. By (1.13) and the Neumann series expansion $(I + \Gamma_\pm\Gamma_\mp)^{-1} = \sum_{k=0}^\infty (-1)^k (\Gamma_\pm\Gamma_\mp)^k$, we conclude

$$\begin{pmatrix} u_1(x) \\ e^{-\frac{i}{h}F(x)} u_2(x) \end{pmatrix} = \begin{cases} \begin{pmatrix} \alpha_1^\flat \\ \alpha_2^\flat \end{pmatrix} & (-1 \le x \le -\epsilon), \\ \begin{pmatrix} \alpha_1^\flat - i h^{\frac{1}{m+1}} q(0)\omega\alpha_2^\flat \\ \alpha_2^\flat - i h^{\frac{1}{m+1}} q(0)\bar{\omega}\alpha_1^\flat \end{pmatrix} & (\epsilon \le x \le 1), \end{cases} \tag{1.14}$$

modulo $\mathcal{O}(h^{\frac{2}{m+1}}\delta)$. In particular, we see that the asymptotic behavior of the solution u dramatically changes around $x = 0$.

Transfer matrix

Now, we describe the asymptotic behavior of the transfer matrix T with respect to the basis $\mathtt{W}_1^\flat, \mathtt{W}_2^\flat$ and $\mathtt{W}_1^\sharp, \mathtt{W}_2^\sharp$ that are WKB solutions for $-1 \le x \le -\epsilon$ and $\epsilon \le x \le 1$ respectively of the form $\mathtt{W}_1^\bullet = {}^{\mathsf{T}}(1, 0) + \mathcal{O}(h)$ and $\mathtt{W}_2^\bullet = {}^{\mathsf{T}}(0, e^{\frac{i}{h}F(x)}) + \mathcal{O}(h)$ for $\bullet \in \{\flat, \sharp\}$. Note here that we are using the functions $\mathtt{W}_2^\flat$, $\mathtt{W}_2^\sharp$ not normalised in the sense of Definition 1.2 for a notational simplicity (the normalization requires a multiplication by a constant depending on $f'(0)$). The asymptotic expansion (1.14) yields

$$u(x) = \begin{cases} \alpha_1^\flat \mathtt{W}_1^\flat + \alpha_2^\flat \mathtt{W}_2^\flat & (-1 \le x \le -\epsilon), \\ (\alpha_1^\flat - i h^{\frac{1}{m+1}} q(0)\omega\alpha_2^\flat)\mathtt{W}_1^\sharp + (\alpha_2^\flat - i h^{\frac{1}{m+1}} q(0)\bar{\omega}\alpha_1^\flat)\mathtt{W}_2^\sharp & (\epsilon \le x \le 1), \end{cases}$$

modulo $\mathcal{O}\big(h^{\frac{2}{m+1}}(\log\frac{1}{h})^{\delta_{m,1}}\big)$. Then, the transfer matrix T can be calculated as

$$\begin{pmatrix}\alpha_1^\flat\\ \alpha_2^\flat\end{pmatrix} \mapsto \begin{pmatrix}\alpha_1^\flat - ih^{\frac{1}{m+1}}q(0)\omega\alpha_2^\flat\\ \alpha_2^\flat - ih^{\frac{1}{m+1}}q(0)\bar{\omega}\alpha_1^\flat\end{pmatrix} = \left(I_2 - ih^{\frac{1}{m+1}}\begin{pmatrix}0 & q(0)\omega\\ q(0)\bar{\omega} & 0\end{pmatrix}\right)\begin{pmatrix}\alpha_1^\flat\\ \alpha_2^\flat\end{pmatrix}$$

modulo $\mathcal{O}\big(h^{\frac{2}{m+1}}(\log\frac{1}{h})^{\delta_{m,1}}\big)$.

1.4. The example of the matrix-valued Schrödinger operator

We now turn to a microlocal example. We consider a matrix-valued Schrödinger operator

$$\begin{pmatrix}P_1 - E_0 & hQ\\ hQ & P_2 - E_0\end{pmatrix}$$

with $P_j := (hD_x)^2 + V_j(x)$ $(j = 1, 2)$, $E_0 \in \mathbb{R}$ and Q is the multiplication operator by a smooth, real-valued function $q(x)$. Below, we apply Theorem 1.4 and recover the transfer matrix of P_{Sc} at the crossing points as obtained in [3, Theorem 3.4] and [4, Theorem 2].

Assume that the function $V_2 - V_1$ vanishes only at $x = 0$, at a finite order $n \in \mathbb{N}\setminus\{0\}$:

$$(V_2 - V_1)^{(j)}(0) = 0 = \quad \text{for all } j \in \{0, \dots, n-1\},$$

and

$$(V_2 - V_1)^{(n)}(0) \neq 0.$$

For simplicity, assume that $V_1(0) = 0$ and $|x^{-1}V_j'(x)| > 0$ near 0. The characteristic sets $\Lambda_j := \{\xi^2 + V_j(x) = E_0\}$ associated with $P_j - E_0$, $j = 1, 2$, are assumed to satisfy the following:

(i) if $E_0 > 0$ then $\Lambda_1 \cap \Lambda_2 = \{(0, \xi_0), (0, -\xi_0)\}$. This is the situation considered in [3];

(ii) if $E_0 = 0$ then $\Lambda_1 \cap \Lambda_2 = \{(0, 0)\}$. This is the situation considered in [4].

where $\xi_0 = \sqrt{E_0}$. The crossing order m of Λ_1 and Λ_2 is $m = n$ for (i) and $m = 2n$ for (ii). In (ii), the crossing point $(0, 0)$ is also a turning point (or caustic), meaning that $\partial_\xi p_j(0, 0) = 0$.

Transfer matrix for (i)

The computation of the transfer matrix for both crossing points is similar (see Remark 1.10), therefore we only detail computations for the transfer matrix at the crossing point $(0, \xi_0)$. Since $(0, \xi_0)$ is not a turning point, we may solve the eikonal and transport equations and fix a basis of WKB solutions $w_1^\flat$, $w_2^\flat$, $w_2^\sharp$ and $w_2^\sharp$ for P_{Sc} of the form

$$w_j^\bullet(x,h) = e^{\frac{i}{h}\int_0^x \sqrt{E_0 - V_j(y)}\,dy}\sigma_j(x)(\mathbf{e}_j + \mathcal{O}(h)) \tag{1.15}$$

where

$$\sigma_j(x) := c_j\Big(1 - \frac{V_j(x)}{E_0}\Big)^{-\frac{1}{4}} \text{ and } c_j := \sqrt{\frac{|\partial_{(x,\xi)} p_j(0,\xi_0)|}{|\partial_\xi p_j(0,\xi_0)|}} = \Big(1 + \frac{V_j'(0)^2}{4E_0}\Big)^{\frac{1}{4}}.$$

According to Proposition 2.13, the identity $\sigma_j(0) = c_j$ ensures that those solutions are normalised in the sense of Definition 1.2. Now, fix a neighborhood Ω of $(0, \xi_0)$. Theorem 1.1 ensures there exist constants $\alpha_j^\bullet = \alpha_j^\bullet(u) \in \mathbb{C}$ such that, for any microlocal solution u of $P_{\mathrm{Sc}} u \equiv 0$ near Ω,

$$u \equiv \alpha_j^\bullet w_j^\bullet \quad \text{microlocally near } \rho_j^\bullet.$$

The transfer matrix $T_{\mathrm{Sc}}(0, \xi_0)$ of P_{Sc} at $(0, \xi_0)$ is uniquely defined by ${}^\mathsf{T}(\alpha_1^\sharp, \alpha_2^\sharp) = T_{\mathrm{Sc}}(0,\xi_0){}^\mathsf{T}(\alpha_1^\flat, \alpha_2^\flat)$. We recall that the crossing order is $m = n$ and that

$$\partial_{x,\xi} p_j(0,\xi_0) = (V_j'(0), 2\xi_0), \quad H_{p_1}^n p_2(0,\xi_0) = 2^n \xi_0^n (V_2 - V_1)^{(n)}(0), \quad \text{and } s = 1.$$

We then apply Theorem 1.4 and recover the result of Assal, Fujiie, and Higuchi [3] on the asymptotics of $T_{\mathrm{Sc}}(0, \xi_0)$.

Proposition 1.8. *In case (i), the transfer matrix $T_{Sc}(0, \xi_0)$ satisfies*

$$T_{Sc}(0,\xi_0) = I_2 - ih^{\frac{1}{n+1}}\begin{pmatrix} 0 & \omega q(0) \\ \bar{\omega} q(0) & 0 \end{pmatrix} + \mathcal{O}_{\mathbb{C}^2 \to \mathbb{C}^2}\Big(h^{\frac{2}{n+1}}\Big(\log\frac{1}{h}\Big)^{\delta_{n,1}}\Big),$$

as $h \to +0$ with

$$\omega = \mu_n\Big(\frac{-\operatorname{sgn}((V_2 - V_1)^{(n)}(0))}{2(n+1)}\pi\Big)\mathbf{\Gamma}\Big(\frac{n+2}{n+1}\Big)\Big(\frac{2(n+1)!}{|(V_2 - V_1)^{(n)}(0)|}\Big)^{\frac{1}{n+1}}.$$

Remark 1.9. Rigorously speaking, one should first translate the crossing point to $(0, 0)$ by conjugating P_{Sc} by the Fourier Integral Operator $e^{\frac{i}{h}x\xi_0}$ generated by the symplectomorphism $\kappa(x, \xi) := (x, \xi - \xi_0)$. Then, Theorem 1.4 applied to

$$\mathcal{P} := e^{-\frac{i}{h}x\xi_0}\, P_{\mathrm{Sc}}\, e^{\frac{i}{h}x\xi_0}$$

yields the transfer matrix of $\mathcal{P}$. Deducing the transfer matrix P_{Sc} from that of $\mathcal{P}$ is straightforward (see Proposition 2.10).

Remark 1.10. Note that $H_{p_1}^n p_2(0, -\xi_0) = (-1)^n H_{p_1}^n p_2(0, \xi_0)$ so that

$$T_{\mathrm{Sc}}(0,-\xi_0) = I_2 - ih^{\frac{1}{n+1}}\begin{pmatrix} 0 & \bar{\omega} q(0) \\ \omega q(0) & 0 \end{pmatrix} + \mathcal{O}_{\mathbb{C}^2 \to \mathbb{C}^2}\Big(h^{\frac{2}{n+1}}\Big(\log\frac{1}{h}\Big)^{\delta_{n,1}}\Big).$$

Transfer matrix for (ii): turning point

Since $(0,0)$ is a turning point, it is not possible to directly construct WKB solutions as above because $E_0 - V_j = -V_j$ changes sign at $x = 0$. Instead, we remove the turning point by rotating the phase space with the symplectomorphism $\kappa(y,\eta) := (\eta, -y)$ in order to have $\partial_\eta(\kappa^* p_j)(0,0) = V_j'(0) \neq 0$. The associated unitary metaplectic operator is $M_{\frac{\pi}{2}} := e^{\frac{i\pi}{4}} \mathscr{F}_h$ where $\mathscr{F}_h$ is the semiclassical Fourier transform defined in (2.11). First, we construct WKB solutions for the rotated operator $M_{\frac{\pi}{2}}^{-1} P_{\mathrm{Sc}} M_{\frac{\pi}{2}}$ by solving the eikonal and transport equations with a stationary phase argument, then we normalise them using Proposition 2.13 as above. We obtain

$$v_j^\bullet(y) := e^{\frac{i}{h}\phi_j(y)} \sigma_j(y)(\mathbf{e}_j + \mathcal{O}(h)), \tag{1.16}$$

where $\phi_j(y) := \int_0^x V_j^{-1}(-y^2)\,dy$ and $\sigma_j(y) := |V_j'(0)|^{\frac{1}{2}} |V_j'(V_j^{-1}(-y^2))|^{-\frac{1}{2}}$. Next, we bring back the $v_j^\bullet$ to normalised WKB[3] solutions $w_j^\bullet$ of the original problem by setting

$$w_j^\bullet(x) := -i M_{\frac{\pi}{2}} v_j^\bullet(x) := -i M_{\frac{\pi}{2}}[e^{\frac{i}{h}\phi_j(\cdot)}\sigma_j(\cdot)](x)(\mathbf{e}_j + \mathcal{O}(h)). \tag{1.17}$$

The normalization constant $-i$ comes from the identities $\Theta(p_j) = -\frac{\pi}{2}$, $\Theta(\kappa^* p_j) = 0$, and

$$\begin{aligned} \frac{e^{-\frac{i\Theta(p_j)}{2}}}{\sqrt{2\pi h}} \int_{\mathbb{R}} e^{-\frac{x^2}{2h}} M_{\frac{\pi}{2}} v_j^\bullet(x)\,dx &= \frac{e^{\frac{i\pi}{4}}}{\sqrt{2\pi h}} \int_{\mathbb{R}} M_{\frac{\pi}{2}}(e^{-\frac{x^2}{2h}}) v_j^\bullet(x)\,dx \\ &= i(1 + \mathcal{O}(h)). \end{aligned}$$

We finally apply Theorem 1.4 after recalling that $m = 2n$ as well as

$$\partial_{x,\xi} p_j(0,0) = (V_j'(0), 0), \quad H_{p_1}^n p_2(0,\xi_0) = \frac{(-1)^n (2n)!}{n!} (V_1'(0))^n (V_2 - V_1)^{(n)}(0),$$

and $s = 1$, and recover the result of [4] on the asymptotics of $T_{\mathrm{Sc}}(0,0)$.

Proposition 1.11. *In case (ii), the transfer matrix $T_{Sc}(0,0)$ satisfies*

$$T_{Sc}(0,0) = I_2 - i h^{\frac{1}{2n+1}} \begin{pmatrix} 0 & \omega_1 q(0) \\ \omega_2 q(0) & 0 \end{pmatrix} + \mathcal{O}_{\mathbb{C}^2 \to \mathbb{C}^2}(h^{\frac{2}{2n+1}}),$$

as $h \to +0$ with

$$\omega_j = 2\Big(\frac{|V_{\hat{j}}'(0)|}{|V_j'(0)|} \frac{(2n+1)n!}{|V_j'(0)|^n |(V_2 - V_1)^{(n)}(0)|}\Big)^{\frac{1}{2n+1}} \boldsymbol{\Gamma}\Big(\frac{2n+2}{2n+1}\Big) \cos\Big(\frac{\pi}{2(2n+1)}\Big) \in \mathbb{R}.$$

where $\hat{j} = 1,2$ is such that $\{j,\hat{j}\} = \{1,2\}$.

[3]Rigorously, for (1.17) to rigorously define a WKB solution, we must replace $\sigma_j(y)$ by $\chi_\bullet(y)\sigma_j(y)$ where $\chi_\bullet(y)$ is a smooth cutoff supported in $(-\infty, 0]$ for $\bullet = \flat$ and $[0, +\infty)$ for $\bullet = \sharp$.

Comparison of transfer matrices

Let us denote $w_j^\bullet$ the WKB solutions we defined respectively in (1.15) and (1.16) on one hand, and $v_j^\bullet$ those of respectively [3] and [4] on the other hand. From Theorem 1.1, we know there exist $C_j^\bullet \in \mathbb{C}$, $j = 1, 2$, $\bullet \in \{\flat, \sharp\}$, such that $v_j^\bullet \equiv C_j^\bullet w_j^\bullet$ near $C_j^\bullet$. A straightforward computation (somewhat similar to (5.3)) gives

$$T(w_1^\flat, w_2^\flat, w_1^\sharp, w_2^\sharp) = \operatorname{diag}(C_1^\sharp\ C_2^\sharp) T(v_1^\flat, v_2^\flat, v_1^\sharp, v_2^\sharp) \operatorname{diag}(C_1^\flat\ C_2^\flat)^{-1}. \tag{1.18}$$

In case (i), it is easy to see that $C_j^\flat = E_0^{-\frac{1}{4}} c_j$ so that $(C_1^\sharp)^{-1} C_2^\flat$ cancels out with the term $|\partial_{(x,\xi)} p_2(0,0)|/|\partial_{(x,\xi)} p_1(0,0)|$ appearing in ω_1. Therefore, our transfer matrix is the same as in [3]. In the case (ii), a careful computation yields $C_j^\flat = \sqrt{2}|V_j'(0)|^{-\frac{1}{2}}$ and $C_j^\sharp = i\sqrt{2}|V_j'(0)|^{-\frac{1}{2}}$. This explains the factor i appearing between our transfer matrix and that of [4].

1.5. Notation

We fix some notations. We denote the set of natural numbers by $\mathbb{N} = \{0, 1, 2, \ldots\}$. We write $D_{x_j} = i^{-1}\partial_{x_j}$. The function space $\mathcal{S}(\mathbb{R}^n)$ denotes the set of all smooth, rapidly decreasing scalar functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ denotes the set of all tempered distributions on $\mathbb{R}^n$. Let $C_b^\infty(\mathbb{R}^n)$ denote the symbol class of smooth functions with bounded derivatives on $\mathbb{R}^n$. For a function space $X \subset \mathcal{S}'(\mathbb{R}^n)$, let $\mathcal{M}_N(X)$ with $N \in \mathbb{N} \setminus \{0\}$ be the set of $N \times N$-matrix valued functions whose entries belong to X. For simplicity, for an h-dependent family of functions $a = (a_h)_{0<h\le h_0}$ with $0 < h_0 \le 1$ and a Fréchet space X, we write $a \in X$ if all the seminorms of a_h in X are uniformly bounded with respect to $0 < h \le h_0$. For Banach spaces X and Y, $B(X,Y)$ denotes the set of all linear bounded operators from X to Y. For a Banach space X, we denote the norm of X by $\|\cdot\|_X$. We say that a set $S \subset \mathbb{R}^2$ is *compactly contained in* another set $T \subset \mathbb{R}^2$ and write $S \Subset T$ if the closure of S is compact and contained in T.

2. Preliminaries

2.1. Quantization and symbolic calculus

We briefly recall some basic properties of semiclassical and microlocal analysis to fix the notation. Please refer to the textbooks [9, 17, 22] for more details.

The Weyl quantization of a matrix-valued function $a = a(x,\xi) \in \mathcal{M}_N(C_b^\infty)$ is the semiclassical pseudodifferential operator $a^w(x, hD_x)$ defined by

$$a^w(x, hD_x)u(x) := \frac{1}{(2\pi h)^n} \iint\limits_{T^*\mathbb{R}^n} e^{\frac{i}{h}(x-y)\cdot\xi} a\Big(\frac{x+y}{2}, \xi\Big) u(y)\, dy\, d\xi, \tag{2.1}$$

for vector-valued functions $u \in \mathcal{S}(\mathbb{R}^n;\mathbb{C}^N)$. The function a is called the *symbol* of the operator $a^w(x,hD_x)$. Let Ψ_h denote the set of semiclassical pseudodifferential operators with scalar symbols in $C_b^\infty(\mathbb{R}^n;\mathbb{C})$. By the Calderón–Vaillancourt theorem ([22, Theorem 4.23]), each element of Ψ_h extends to a bounded operator on $L^2(\mathbb{R}^n;\mathbb{C}^N)$. We also define the set Ψ_h^{comp} that consists of operators $A \in \Psi_h$ whose integral kernels are compactly supported and whose wave front sets (defined in the next subsection) are compact.

For scalar-valued symbols $a, b \in C_b^\infty(T^*\mathbb{R}^n;\mathbb{C})$, there exists a symbol $a\#b \in C_b^\infty(T^*\mathbb{R}^n;\mathbb{C})$ such that $a^w(x,hD_x)b^w(x,hD_x) = (a\#b)^w(x,hD_x)$ and $a\#b = ab + \frac{h}{2i}\{a,b\} + \mathcal{O}_{C_b^\infty}(h^2)$, where $\{a,b\} = \partial_\xi a \cdot \partial_x b - \partial_x a \cdot \partial_\xi b$.

We now state some useful boundedness properties of pseudodifferential operators on L^∞, used to study the integral operators $\Gamma_\pm$ that appear in Section 1.3 (see also Proposition 4.4).

Lemma 2.1. *Let $A \in \Psi_h^{\mathrm{comp}}$. Then there exists $C > 0$ (independent of $0 < h \le 1$) such that $\|A\|_{L^\infty \to L^\infty}, \|A\|_{W^{1,\infty} \to W^{1,\infty}} \le C$ for $0 < h \le 1$. Here $W^{1,\infty}$ is the Sobolev space with the norm $\|u\|_{W^{1,\infty}} := \|u\|_{L^\infty} + \sum_{j=1}^n \|D_{x_j} u\|_{L^\infty}$.*

Proof. We write

$$A = a^w(x,hD_x) \quad \text{with } a \in \mathcal{S}(\mathbb{R}^n).$$

First, we prove the uniform boundedness of $\|A\|_{L^\infty \to L^\infty}$. The proof is essentially the same as [22, Theorem 7.15] that deals with Fourier multipliers. The integral kernel of A is given by

$$K_h(x,y) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} a\Big(\frac{x+y}{2}, \xi\Big) e^{\frac{i}{h}(x-y)\cdot\xi} \, d\xi$$

An integration by parts shows $|K_h(x,y)| \le Ch^{-n}\big(1 + \frac{1}{h}|x-y|\big)^{-n-1}$ with a constant $C > 0$ independent of x, y, h. By Young's inequality, we have $\|Au\|_{L^\infty} \le Ch^{-n}\|(1+\frac{1}{h}|x|)^{-n-1}\|_{L^1}\|u\|_{L^\infty} = C\|(1+|x|)^{-n-1}\|_{L^1}\|u\|_{L^\infty}$, which proves the uniform boundedness of $\|A\|_{L^\infty \to L^\infty}$. Since $[D_{x_j}, A] = -i(\partial_{\xi_j} a)^w(x,hD_x)$ and $\partial_{\xi_j} a \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}
\|D_{x_j} Au\|_{L^\infty} &\le \|AD_{x_j} u\|_{L^\infty} + \|[D_{x_j}, A]u\|_{L^\infty} \\
&\le (\|A\|_{L^\infty \to L^\infty} + \|(\partial_{x_j} a)^w(x,hD_x)\|_{L^\infty \to L^\infty})\|u\|_{W^{1,\infty}} \\
&\le C\|u\|_{W^{1,\infty}},
\end{aligned}$$

which proves the uniform boundedness of $\|A\|_{W^{1,\infty} \to W^{1,\infty}}$. ∎

2.2. Microlocalization and wavefront set

Definition 2.2. Let a family $f = \{f_h\}_{h\in(0,h_0]} \subset L^2(\mathbb{R}^n; \mathbb{C}^N)$ be *tempered in h,* that is, there exists $k \geq 0$ such that $f_h = \mathcal{O}_{L^2}(h^{-k})$ as $h \to 0$.

(i) We say f is *microlocally infinitely small* near a point $(x_0, \xi_0) \in T^*\mathbb{R}^n$ and write

$$f \equiv 0 \quad \text{m.l. near } (x_0, \xi_0),$$

if there exists $\chi \in \mathcal{M}_N(C_b^\infty)$ such that $\chi(x_0, \xi_0) = 1$ and $\|\chi^w(x, hD_x) f(h)\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h^\infty)$. We also write $f \equiv 0$ m.l. on Ω if it is microlocally infinitely small near each point of Ω. The *semiclassical wavefront set* $\mathrm{WF}_h(f) \subset T^*\mathbb{R}^n$ is defined as the set of points where f is not microlocally infinitely small. We say that f is *microlocalised* on a set U whenever $\mathrm{WF}_h(f) \subset U$.

(ii) Let $A = a^w(x, hD_x)$ be a pseudodifferential operator with a symbol $a \in \mathcal{M}_N(C_b^\infty)$. We say f is a *microlocal solution* to $Af = 0$ on $\Omega \subset \mathbb{R}^{2n}$ if $Af \equiv 0$ m.l. on Ω.

(iii) f is *compactly microlocalised* whenever there exists $\chi \in \mathcal{M}_N(C_c^\infty)$ independent of h such that $(1 - \chi^w(x, hD_x)) f = \mathcal{O}_{\mathcal{S}(\mathbb{R}^n)}(h^\infty)$.

Remark 2.3. If $f \equiv 0$ m.l. near $(x_0, \xi_0) \in T^*\mathbb{R}^n$, then there exists a neighborhood $(x_0, \xi_0) \in U \subset T^*\mathbb{R}^n$ such that $\|\chi^w(x, hD_x) f(h)\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h^\infty)$ for any $\chi \in \mathcal{M}_N(C_c^\infty)$ whose support is contained in U (see [22, Theorem 8.13]).

Definition 2.4. (i) We say a family $T = \{T_h\}_{h\in(0,h_0]}$ of continuous linear operators $T_h : \mathcal{S} \to \mathcal{S}'$ is *tempered* if there exists $N \geq 0$ such that

$$\|\mathcal{L}_{-N} T_h \mathcal{L}_{-N}\| = \mathcal{O}_{L^2\to L^2}(h^{-N}) \quad \text{as } h \to 0,$$

where $\mathcal{L}_{-N} := (1 + |x|^2)^{-N}(1 + |hD_x|^2)^{-N}$.

(ii) Let $T = \{T_h\}_{h\in(0,h_0]}$ and $S = \{S_h\}_{h\in(0,h_0]}$ be tempered families of operators and z and w in $T^*\mathbb{R}^n$. We write $T \equiv S$ m.l. near (z, w) if there exist neighborhoods U and V of z and w such that

$$B(T - S)A = \mathcal{O}_{\mathcal{S}\to\mathcal{S}}(h^\infty).$$

for any pair of pseudodifferential operators $A = a^w(x, hD_x)$ and $B = b^w(x, hD_x)$ with $\operatorname{supp} a \subset U'$ and $\operatorname{supp} b \subset V'$.

(iii) We say that a tempered family $T = \{T(h)\}_{h\in(0,h_0]}$ of operators is *microlocally invertible* near z and w in $T^*\mathbb{R}^n$ if there exist neighborhoods U and V of z and w and a tempered family $S = \{S_h\}_{(0,h_0]}$ of operators such that

$$ST \equiv I \quad \text{m.l. near } (z, z) \quad \text{and} \quad TS \equiv I \quad \text{m.l. near } (w, w).$$

Then S is called *a microlocal inverse* near (z, w) of T. We denote S simply by T^{-1}.

Lemma 2.5. *For any $A \in \Psi_h$ and any compact set $K \subset T^*\mathbb{R}^n$, there exists $A' \in \Psi_h^{\mathrm{comp}}$ such that $A \equiv A'$ m.l. on $K \times K$.*

Proof. This follows from the pseudolocal property of the pseudodifferential operators. ∎

Proposition 2.6 (Elliptic parametrix). *Let $A = a^w(x, hD_x)$ be a pseudodifferential operator with a matrix-valued symbol $a \in \mathcal{M}_N(C_b^\infty)$.*

(i) *If the inequality $|\det a| \geq c$ holds on U with an h-independent $c > 0$, then A is microlocally invertible near $U \times U$. Moreover, the symbol of each microlocal inverse of A coincides modulo $\mathcal{O}_{\mathcal{M}_N(C_b^\infty)}(h)$ with a^{-1} on U.*

(ii) *Let u be a microlocal solution near $U \subset T^*\mathbb{R}^n$ to the pseudodifferential equation $Au(x) = 0$. The first statement implies that one has*

$$\mathrm{WF}_h(u) \cap U \subset \{\det a = \mathcal{O}(h^\infty)\}.$$

Proof. This follows from the usual parametrix construction (for example, see [22, Theorem 4.29]). We omit the details. ∎

The following example is given in [22, Section 8.4, Examples].

Example 2.7. For $\sigma = \sigma(x) \in C^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $\phi = \phi(x) \in C^\infty(\mathbb{R}^n; \mathbb{R})$ (real-valued), one has $\mathrm{WF}_h(\sigma(x)e^{\frac{i}{h}\phi(x)}) \subset \{(x, \partial_x\phi(x));\ x \in \operatorname{supp}\sigma\}$.

2.3. Fourier integral operators

We review some properties of semiclassical Fourier integral operators. We refer to [18, 22] for proofs, details, and related topics.

Definition 2.8. Let κ be a *symplectomorphism* near $(0,0)$, that is to say a diffeomorphism between neighborhoods of $(0,0) \in T^*\mathbb{R}^n$ preserving the standard symplectic form $d\xi \wedge dx$ and satisfying $\kappa(0,0) = (0,0)$.

(i) Let F be a bounded operator on $L^2(\mathbb{R}^n)$. We say that the operator F *quantises* κ near $(0,0)$ if for all $a \in C_b^\infty(T^*\mathbb{R}^n)$, we have

$$a^w(x, hD_x)F = Fb^w(x, hD_x) \quad \text{m.l. near } ((0,0),(0,0))$$

where $b \in C_b^\infty(T^*\mathbb{R}^n)$ satisfies $b = \kappa^* a + \mathcal{O}_{C_b^\infty}(h)$.

(ii) Assume that the projection

$$\{(x,\xi,y,\eta) \in T^*\mathbb{R}^n \times T^*\mathbb{R}^n; \kappa(y,\eta) = (x,\xi)\} \ni (x,\xi,y,\eta) \mapsto (x,\eta) \in \mathbb{R}^n \times \mathbb{R}^n \tag{2.2}$$

is a diffeomorphism[4] near $(0,0)$. A smooth function ψ defined near $(0,0) \in \mathbb{R}^n \times \mathbb{R}^n$ is called a *generating function*[5] of κ if for (x,η) near $(0,0)$,

$$\kappa(\partial_\eta \psi(x,\eta),\eta) = (x, \partial_x \psi(x,\eta)), \quad \det \partial_x \partial_\eta \psi(x,\eta) \neq 0, \quad \psi(0,0) = 0. \tag{2.3}$$

(iii) Under the same assumption as (ii), a bounded operator $F = F_h$ on $L^2(\mathbb{R}^n)$ is called a *Fourier integral operator*[6] associated with κ if F is of the form

$$Fu(x) = \frac{1}{(2\pi h)^n} \iint_{\mathbb{R}^{2n}} e^{\frac{i}{h}(\psi(x,\eta) - y\cdot\eta)} \alpha(x,\eta) u(y) dy d\eta,$$

with a function $\alpha(x,\eta) \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ supported near $(0,0)$ where (2.3) holds and $\alpha \sim \sum_{j=0}^\infty h^j \alpha_j$. We say that F is *microlocally unitary* near $((0,0),(0,0))$ whenever

$$\alpha_0(0,0) = |\partial_x \partial_\eta \psi(0,0)|^{\frac{1}{2}} \tag{2.4}$$

holds.[7]

Proposition 2.9. *Let κ be a symplectomorphism near $(0,0)$. Assume that the projection (2.2) is a diffeomorphism near $(0,0)$. Then, the following holds.*

(i) Egorov's theorem. *Each Fourier integral operator F associated with κ and satisfying $\alpha_0(0,0) \neq 0$ quantises κ near $(0,0)$. Conversely, if F is a bounded operator on $L^2(\mathbb{R}^n)$ quantizing κ near $(0,0)$, then F is a Fourier integral operator associated with κ.*

(ii) *If a Fourier integral operator F associated with κ is microlocally unitary near $((0,0),(0,0))$ and $A = a^w(x, hD_x) \in \Psi_h$ satisfies $a(0,0) = 1$, then we can find a Fourier integral operator F' such that $FA \equiv F'$ m.l. near $((0,0),(0,0))$ and F' is a Fourier integral operator associated with κ which is microlocally unitary near $((0,0),(0,0))$.*

Proof. The proof is standard and the first statement is given around [18, (4.6)] (see also [19, Lemma 3.4]). We omit the details. ∎

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[4]This condition is equivalent to the $n \times n$ block $\frac{\partial x}{\partial y}$ in the derivative $d\kappa(0,0)$ being invertible, where $(x,\xi) = \kappa(y,\eta)$.

[5]A generating function always exists due to the assumption that the projection (2.2) is a diffeomorphism near $(0,0)$ due to the implicit function theorem and Poincaré's lemma.

[6]We can still define the notion of Fourier integral operators without the assumption on the projection (2.2). However, the general definition is a bit complicated to state and not needed in this paper.

[7]As stated in [18, (4.5)], if $F^*F \equiv I$ m.l. near $((0,0),(0,0))$, then $|\alpha_0(0,0)| = |\partial_x \partial_\eta \psi(0,0)|^{\frac{1}{2}}$ holds. Here, we impose the stronger condition (2.4) just for simplicity.

The following proposition is a modified version of [18, Lemma 4.1], and provides an analogue of a composition formula for Fourier integral operators.

Proposition 2.10. *Suppose $n = 1$. Let $p, \phi, \tilde{\phi} \in C^\infty(T^*\mathbb{R})$ be real-valued functions and κ be a symplectomorphism near $(0,0)$ such that*

- $\phi(0) = \tilde{\phi}(0) = \phi'(0) = \tilde{\phi}'(0) = p(0,0) = 0$ *and* $\partial_{(x,\xi)} p(0,0) \neq 0$;
- $p(x, \phi'(x)) = (\kappa^* p)(x, \tilde{\phi}'(x)) = 0$ *around* $x = 0$;[8]
- *the projection* (2.2) *is a diffeomorphism near* $(0,0)$.

Suppose that a bounded operator F on $L^2(\mathbb{R}^n)$ is a Fourier integral operator associated with κ and is microlocally unitary near $((0,0),(0,0))$ in the sense of Definition 2.8 (iii). Then, for $\tilde{\sigma} = \tilde{\sigma}_h \in C_b^\infty(\mathbb{R})$ satisfying $\tilde{\sigma}(x) = \tilde{\sigma}_0(x) + \mathcal{O}_{C_b^\infty}(h)$ with $\tilde{\sigma}_0 \in C_b^\infty(\mathbb{R})$ independent of h, we have

$$F(e^{\frac{i}{h}\tilde{\phi}(x)}\tilde{\sigma})(x) = e^{\frac{i}{h}\phi(x)}\sigma(x) \tag{2.5}$$

with a function $\sigma = \sigma_h \in C_b^\infty(\mathbb{R})$ satisfying $\sigma(x) = \sigma_0(x) + \mathcal{O}_{C_b^\infty}(h)$ with $\sigma_0 \in C_b^\infty(\mathbb{R})$ independent of h. Moreover,

$$\sigma_0(0) = e^{\frac{\pi i}{4}\upsilon} \left| \frac{\partial_\xi(\kappa^* p)(0,0)}{\partial_\xi p(0,0)} \right|^{\frac{1}{2}} \tilde{\sigma}_0(0). \tag{2.6}$$

Here, the index υ is given by

$$\upsilon = \operatorname{sgn} \partial^2_{(y,\eta)}(\tilde{\phi}(y) - y \cdot \eta + \psi(0,\eta))|_{(y,\eta)=(0,0)},$$

where ψ is a generating function of κ.

Proof. By the assumption, we can find neighborhoods $\Omega, \tilde{\Omega}$ of $0 \in \mathbb{R}$ such that $\kappa(\tilde{\Lambda}) = \Lambda$, where $\Lambda = \{(x, \phi'(x)) \mid x \in \Omega\}$ and $\tilde{\Lambda} = \{(x, \tilde{\phi}'(x)) \mid x \in \tilde{\Omega}\}$. By virtue of [18, Lemma 4.1], we can find $\sigma = \sigma_0 + \mathcal{O}_{C_b^\infty}(h)$ satisfying (2.5) and

$$\sigma_0(x) = e^{\frac{\pi i}{4}\upsilon} \frac{\alpha_0(x, \phi'(g(x)))}{|\partial_x \partial_\eta \psi(x, \phi'(g(x)))|^{\frac{1}{2}}} |g'(x)|^{\frac{1}{2}} \tilde{\sigma}_0(g(x)), \tag{2.7}$$

where, $g(x)$ denotes the function defined near $x = 0$ such that $\kappa(g(x), \tilde{\phi}'(g(x))) = (x, \phi'(x))$. The precise value of υ is not calculated in [18, Lemma 4.1]. We do not

[8] These first two conditions with $\partial_{(x,\xi)} p(0,0) \neq 0$ imply $\partial_\xi p(0,0) \neq 0$ and $\partial_\xi(\kappa^* p)(0,0) \neq 0$. Conversely, if $\partial_\xi p(0,0) \neq 0$ and $\partial_\xi(\kappa^* p)(0,0) \neq 0$ are satisfied, then such phase functions $\phi, \tilde{\phi}$ exist due to the implicit function theorem and Poincaré's lemma.

include this calculation here, as it can be easily done using a stationary phase argument. Next, we claim that

$$g'(0) = \frac{\partial_\xi(\kappa^* p)(0,0)}{\partial_\xi p(0,0)} \tag{2.8}$$

holds. Let $q = q(x,\xi)$ be a function defined near $(0,0)$ such that $\kappa^* q(x,\xi) = x$ and $q(0,0) = 0$. Then, one has[9] $g(x) = q(x, \tilde{\phi}'(x))$, and

$$g'(x) = (\partial_x q + \tilde{\phi}''(x)\partial_\xi q)(x, \phi'(x)) = \frac{\{p,q\}}{\partial_\xi p}(x, \tilde{\phi}'(x)),$$

where we use $(\partial_x p + \tilde{\phi}''(x)\partial_\xi p)(x, \tilde{\phi}'(x)) = 0$, which follows from $p(x, \tilde{\phi}'(x)) = 0$. Then $\tilde{\phi}'(0) = 0$ and $\{p,q\} = \{\kappa^* p, \kappa^* q\} = \partial_\xi(\kappa^* p)$ imply (2.8).

Now, we observe that $g(0) = 0$ holds by $\kappa(0,0) = (0,0)$. Since we assumed F is microlocally unitary near $((0,0),(0,0))$, we have $\alpha_0(0,0) = |\partial_x \partial_\eta \psi(0,0))|^{\frac{1}{2}}$ (see Definition 2.8). Combining these with (2.7) and (2.8), we obtain (2.6). ∎

2.4. Metaplectic operators associated with rotations

Fourier integral operators associated with linear symplectic maps can be defined as unitary operators on L^2 and are called *metaplectic operators*. We recall some properties of a metaplectic operator $M_\theta \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ associated with a one-dimensional ($n = 1$) linear symplectic map $\kappa_\theta \colon \mathbb{R}^2 \to \mathbb{R}^2$ corresponding to a rotation:

$$\kappa_\theta(x,\xi) = ((\cos\theta)x + (\sin\theta)\xi, -(\sin\theta)x + (\cos\theta)\xi). \tag{2.9}$$

The operator M_θ is uniquely characterised up to a multiplication by a constant of modulus one by its unitarity and by the exact form of the Egorov's theorem:

$$M_\theta^{-1} a^w(x, hD_x) M_\theta = (\kappa_\theta^* a)^w(x, hD_x) \quad \text{for any } a \in C_b^\infty(\mathbb{R}^2). \tag{2.10}$$

In this manuscript, we fix it as an integral operator

$$M_\theta u(x) = \frac{|\cos\theta|^{-\frac{1}{2}}}{2\pi h} \int_{\mathbb{R}^2} \exp\Big(\frac{i}{h}\Big(-\frac{\tan\theta}{2}\Big(x^2 - 2\frac{x\eta}{\sin\theta} + \eta^2\Big) - y\eta\Big)\Big) u(y)\, dy\, d\eta,$$

for $|\theta| < \frac{\pi}{2}$, and $M_{\frac{\pi}{2}} = e^{\frac{i\pi}{4}} \mathscr{F}_h$, where $\mathscr{F}_h$ is the semiclassical Fourier transform defined by

$$\mathscr{F}_h u(x) = \frac{1}{\sqrt{2\pi h}} \int_{\mathbb{R}} e^{-\frac{i}{h}xy} u(y)\, dy. \tag{2.11}$$

[9]By the definition of g, we have $q(x, \tilde{\phi}'(x)) = q(\kappa(g(x), *)) = (\kappa^* q)(g(x), *) = g(x)$.

Lemma 2.11. *For $-\frac{\pi}{2} < \theta \leq \frac{\pi}{2}$, one has*

$$M_\theta e^{-\frac{\cdot^2}{2h}}(x) = e^{\frac{i\theta}{2}} e^{-\frac{x^2}{2h}}.$$

Proof. First, we recall that one has $\mathcal{F}_h e^{-\frac{\cdot^2}{2h}}(\eta) = e^{-\frac{\eta^2}{2h}}$. This implies

$$\begin{aligned} M_\theta e^{-\frac{\cdot^2}{2h}}(x) &= \frac{|\cos\theta|^{-\frac{1}{2}}}{\sqrt{2\pi h}} \int_{\mathbb{R}} \exp\Big(\frac{-e^{i\theta}\eta^2 + i2x\eta - i(\sin\theta)x^2}{2h\cos\theta}\Big)\, d\eta \\ &= e^{-\frac{x^2}{2h}} \frac{|\cos\theta|^{-\frac{1}{2}}}{\sqrt{2\pi h}} \int_{\mathbb{R}} \exp\Big(\frac{-e^{i\theta}(\eta - ie^{-i\theta}x)^2}{2h\cos\theta}\Big)\, d\eta. \end{aligned}$$

Then by the change of variable $\xi = e^{\frac{i\theta}{2}}|\cos\theta|^{-\frac{1}{2}}(\eta - ie^{-i\theta}x)$, and by the Cauchy theorem, we obtain the required formula. ■

2.5. Translation only in ξ-variable

As another special case of symplectic maps, consider

$$\kappa_\phi(x,\xi) = (x, \xi + \partial_x\phi(x))$$

for a real-valued function $\phi \in C_c^\infty(\mathbb{R}^n;\mathbb{R})$. In this case, the multiplication by $e^{\frac{i\phi}{h}}$ is a Fourier integral operator which quantises κ_ϕ.

Proposition 2.12. *Let $A = a^w(x, hD_x)$ be a pseudodifferential operator with $a \in \mathcal{S}(T^*\mathbb{R}^n)$. One has*

$$e^{-\frac{i\phi}{h}} A e^{\frac{i\phi}{h}} = a_\phi^w(x, hD_x) + \mathcal{O}_{\mathcal{S}'\to\mathcal{S}}(h^\infty),$$

with $a_\phi = \kappa_\phi^* a + \mathcal{O}_{C_b^\infty}(h)$. Moreover, one has

$$\operatorname{supp} a_\phi \subset \kappa_\phi^{-1}(\operatorname{supp} a).$$

According to the above proposition, a WKB state remains a WKB state under the action of a pseudodifferential operator that is, for a polynomially bounded $\sigma \in C_b^\infty(\mathbb{R}^n)$ and a symbol $a \in \mathcal{S}(T^*\mathbb{R}^n)$, one has

$$a^w(x, hD_x)\Big(\sigma(x)e^{\frac{i\phi(x)}{h}}\Big) = \Big(a_\phi^w(x, hD_x)\sigma(x)\Big)e^{\frac{i\phi(x)}{h}} + \mathcal{O}_{\mathcal{S}}(h^\infty). \tag{2.12}$$

2.6. Microlocal solutions for the scalar equations away from the crossing point

In this subsection, we review some basic properties of the scalar equations

$$P_j v_j \equiv 0 \tag{2.13}$$

where $P_j = p_j^w(x, hD_x)$ and p_j satisfies Condition A.

The first and second statements of the following proposition is a version of our main theorem for the scalar equation. The third statement tells how we choose a constant to get a normalised solution in the sense of Definition 1.2 under the condition $\partial_\xi p_j(0,0) \neq 0$. The fourth one ensures that the normalization condition is invariant under the action of metaplectic transforms with positive small angles, which will be useful in deducing the connection formula when $\partial_\xi p_j(0,0) = 0$.

Proposition 2.13. *Fix $j = 1, 2$.*

(i) *The space of microlocal solutions to the equation (2.13) on each connected small neighborhood of $(0,0)$ is one-dimensional.*[10]

(ii) *If $\partial_\xi p_j(0,0) \neq 0$, then for each $c_j \in \mathbb{R}$ independent of h, the unique microlocal solution v_j (m.l. near $(0,0)$) to (2.13) with $v_j(0) = c_j + \mathcal{O}(h^\infty)$ is of the form*

$$v_j(x) = e^{\frac{i}{h}\phi_j(x)}\sigma_j(x,h), \quad \sigma_j(0,h) = c_j, \tag{2.14}$$

where ϕ_j is the unique solution to the eikonal equation

$$p_j(x, \phi_j'(x)) = 0, \quad \phi_j(0) = 0 \tag{2.15}$$

and $\sigma_j(x,h) \in C_b^\infty(\mathbb{R})$ can be constructed by solving the transport equations and admits the asymptotic expansion $\sigma_j(x,h) \sim \sum_{l\geq 0} h^l \sigma_{j,l}(x)$.

(iii) *The microlocal solution v_j given in (2.14) is normalised in the sense of Definition 1.2 (i) if and only if*

$$c_j = \sqrt{\frac{|\partial_{(x,\xi)} p_j(0,0)|}{|\partial_\xi p_j(0,0)|}}. \tag{2.16}$$

(iv) *The metaplectic operator M_θ maps a normalised microlocal solution $v_{j,\theta}$ near $(0,0)$ of $M_\theta^{-1} P_j M_\theta v = 0$ to that of (2.13), for small $\theta > 0$.*

[10] More precisely, there exists a microlocal solution v_j to (2.13) such that $v_j \not\equiv 0$ near $(0,0)$, and for any microlocal solution $\tilde{v}_j$, there exists a constant k such that $\tilde{v}_j \equiv k v_j$ m.l. near $(0,0)$.

Remark 2.14. The function

$$-\arctan\Big(\frac{\partial_x p_j(\exp(tH_{p_j})(0,0))}{\partial_\xi p_j(\exp(tH_{p_j})(0,0))}\Big)$$

has a discontinuity at $t=0$ if $t=0$ is a simple zero of $(\partial_\xi p_j)(\exp(tH_{p_j})(0,0))=0$. Then, the normalization constants $c_{j,\pm\varepsilon}$ for the WKB solution v_j at $\exp(\pm\varepsilon H_{p_j})(0,0)$ differ by $\frac{\pi}{2}$ in the limit $\varepsilon\to+0$:

$$\lim_{\varepsilon\to+0}|c_{j,+\varepsilon}-c_{j,-\varepsilon}|=\frac{\pi}{2}.$$

This gives an elementary explanation how the Maslov index appears.

Remark 2.15. In general, the normalization is not invariant by the action of M_θ. If $v_{j,\theta}$ is a normalised microlocal solution to $M_\theta^{-1}P_jM_\theta v_{j,\theta}=0$ with $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, then the function

$$v_j:=e^{-\frac{i\pi}{2}\nu}M_\theta v_{j,\theta}$$

is a normalised microlocal solution of $P_jv_j=0$, where ν is given by (2.19).

The proof of (i) and (ii) is well known and we omit to write it here (for the uniqueness statement, see [8, Lemme 18]). In the following, we prove the parts (iii) and (iv).

Proof of Proposition 2.13 (iii). Let $v_j(x)=e^{\frac{i}{h}\phi_j(x)}\sigma_j(x)$ be the microlocal solution of the form (2.14). For any h-independent $\delta>0$, the integral outside a δ-neighborhood of $x=0$ is exponentially small with respect to $h\to+0$:

$$\Big|\int_{|x|\geq\delta}e^{-\frac{x^2}{2h}}e^{\frac{i}{h}\phi_j(x)}\sigma_j(x)dx\Big|\leq Ce^{-\frac{\delta^2}{2h}}. \tag{2.17}$$

On the other hand, if $\delta>0$ is sufficiently small, the change of the variable $x^2-2i\phi_j(x)=y^2$ is valid for $x\in[-\delta,+\delta]$. Note that one has $\phi_j(x)=\frac{1}{2}\phi_j''(0)x^2+\mathcal{O}(x^3)$ as $x\to0$, and $y=(1-i\phi_j''(0))^{\frac12}x(1+\mathcal{O}(x))$. Then it follows from the expansion $\sigma_j(x)=c_j+\mathcal{O}(x)+\mathcal{O}(h)$ near $x=0$ that

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi h}}\int_{|x|\leq\delta}e^{-\frac{x^2}{2h}}e^{\frac{i}{h}\phi_j(x)}\sigma_j(x)\,dx\\
&=\frac{1}{\sqrt{2\pi h}}\int_{|x|\leq\delta}e^{-\frac{x^2}{2h}}e^{\frac{i}{h}\phi_j(x)}(c_j+\mathcal{O}(x)+\mathcal{O}(h))\,dx\\
&=\frac{(1-i\phi_j''(0))^{-\frac12}}{\sqrt{2\pi h}}\int_{\gamma_\delta}e^{-\frac{y^2}{2h}}(c_j+\mathcal{O}(y)+\mathcal{O}(h))=\frac{c_j+\mathcal{O}(h)}{(1-i\phi_j''(0))^{\frac12}}\,dy\\
&=\Big|\frac{\partial_\xi p_j(0,0)}{\partial_{(x,\xi)}p_j(0,0)}\Big|^{\frac12}c_je^{\frac{i\Theta(p_j)}{2}}+\mathcal{O}(h),
\end{aligned} \tag{2.18}$$

with a contour γ_δ in the complex plane, where we have applied the Cauchy theorem and the identities

$$|1 - i\phi_j''(0)|^2 = 1 + |\phi_j''(0)|^2 = \left|\frac{\partial_{(x,\xi)} p_j(0,0)}{\partial_\xi p_j(0,0)}\right|^2,$$
$$\arg((1 - i\phi_j''(0))^{-\frac{1}{2}}) = \frac{1}{2}\arctan\phi_j''(0) = \frac{1}{2}\arctan\left(\frac{\partial_x p_j(0,0)}{\partial_\xi p_j(0,0)}\right) = -\frac{\Theta(p_j)}{2},$$

deduced from (2.14). Combining (2.18) with (2.17), we obtain the claim in (iii). ■

Proof of Proposition 2.13 (iv). Let $v_{j,\theta}$ satisfy the normalization condition (1.6) and $M_\theta^{-1} P_j M_\theta v_{j,\theta} \equiv 0$ microlocally near the origin. Then Lemma 2.11 together with the unitarity of M_θ and $M_\theta^* = M_{-\theta}$ shows for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ the identity

$$\frac{e^{-\frac{i}{2}\Theta(p_j)}}{\sqrt{2\pi h}} \int_{\mathbb{R}} e^{-\frac{x^2}{2h}} M_\theta v_{j,\theta}(x)\,dx = \frac{e^{-\frac{i}{2}\Theta(p_j)}}{\sqrt{2\pi h}} \int_{\mathbb{R}} M_{-\theta} e^{-\frac{\cdot^2}{2h}}(x) v_{j,\theta}(x)\,dx$$

which is equal to $e^{\frac{i}{2}(\Theta(p_{j,\theta}) - \Theta(p_j) - \theta)}$. By definition of Θ, one has the identity (see Figure 4)

$$\Theta(p_{j,\theta}) = \Theta(p_j) + \theta + \pi\nu,$$

with the index $\nu = \nu(p_j, \theta) \in \{0, \pm 1\}$ determined by

$$\Theta(p_j) + \theta + \pi\nu \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{2.19}$$

The proposition follows from the left-continuity with respect to θ of $\Theta(p_{j,\theta})$. ■

2.7. Normalised microlocal solutions to the original equation

In this subsection, we discuss the behavior of microlocal solutions to the original equation

$$\mathcal{P}u \equiv 0 \tag{2.20}$$

away from the crossing point $(0,0) \in T^*\mathbb{R}$. The following proposition is an analogue for the matrix-valued equation of Proposition 2.13.

Proposition 2.16. *The following statements hold.*

(i) *For each $j = 1, 2$ and $\bullet \in \{\flat, \sharp\}$, the space of microlocal solutions to the equation* (2.20) *on $\gamma_j^\bullet$ is one-dimensional, where $\gamma_j^\bullet$ is introduced in Condition* B.

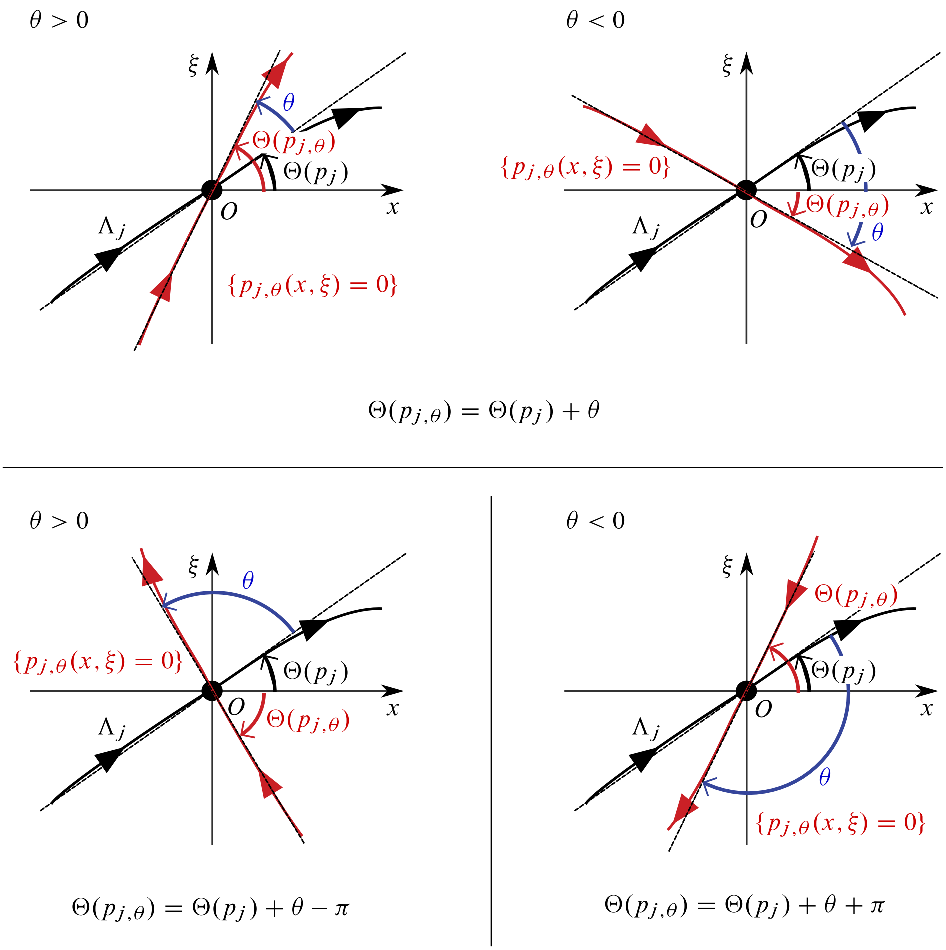

Figure 4. Rotation of the symbol p_j by κ_θ for various θ and the relation between $\Theta(p_j)$, $\Theta(p_{j,\theta})$, and θ for each case (see (2.19)).

(ii) *Suppose that*

$$\partial_\xi p_1(0,0) \neq 0 \quad \textit{and} \quad \partial_\xi p_2(0,0) \neq 0 \tag{2.21}$$

hold. Each microlocal solution $w_j^\bullet$ *to* (2.20) *m.l. on* $\gamma_j^\bullet$ *is given by*

$$w_j^\bullet = e^{\frac{i\phi_j}{h}} \begin{pmatrix} \sigma_{j,1} \\ \sigma_{j,2} \end{pmatrix}, \quad \mathrm{WF}_h(w_j^\bullet) \subset \gamma_j^\bullet, \tag{2.22}$$

where ϕ_j *is the same as* (2.15) *and* $\sigma_{j,k}(x,h) \in C_b^\infty(\mathbb{R} \setminus \{0\})$ *can be constructed by solving the transport equations and admits the asymptotic expansion* $\sigma_{j,k}(x,h) \sim \sum_{l \geq 0} h^l \sigma_{j,k,l}(x)$ *with* $\sigma_{j,3-j,0} = 0$*. Moreover, the principal part* $\sigma_{j,j,0}$ *is smooth at* $x = 0$ *(see* [2, Remark 5.2]*).*

(iii) *The microlocal solution w_j given in* (2.22) *is normalised in the sense of Definition* 1.2 (ii) *if and only if we choose $\sigma_{j,j,0}(0)$ as* (2.16).

(iv) *The metaplectic operator M_θ maps a normalised microlocal solution $w_{j,\theta}$ near* $(0,0)$ *of $M_\theta^{-1}\mathcal{P}M_\theta u = 0$ to that of* (2.20)*, for small $\theta > 0$.*

Remark 2.17. The amplitudes $\sigma_{j,k}$ are singular at $x = 0$, which corresponds to the crossing point of p_1 and p_2. In other words, the usual WKB construction does not work at $x = 0$.

The proof of (i) and (ii) is more or less well known. In fact, both (i) and (ii) are consequences of [2, Appendix B.2 and Proposition 5.1] whenever (2.21) holds. When condition (2.21) is violated, we choose $\theta \in \mathbb{R}$ such that $\kappa_\theta^* p_1$ and $\kappa_\theta^* p_2$ satisfy (2.21). By virtue of (2.9) and (2.10), the problem can be reduced to the case where (2.21) holds. We omit the details. Part (iii) directly follows from Proposition 2.13 (iii).

Let us consider the case where the operator $\mathcal{P}$ is of the following form:

$$\mathcal{P} = P_{\mathrm{nf}} := \begin{pmatrix} hD_x & hR_1 \\ hR_2 & hD_x - f(x) \end{pmatrix},$$

where $f \in C_b^\infty(\mathbb{R};\mathbb{R})$ be a smooth, real-valued function vanishing at $x = 0$, and R_1, $R_2 \in \Psi_h^{\mathrm{comp}}$ are pseudodifferential operators. In this case, the functions ϕ_j and v_j are explicitly written as

$$\phi_1(x) = 0, \quad \phi_2(x) = F(x) = \int_0^x f(y)\,dy,$$

$$v_1(x) = 1, \quad v_2(x) = e^{\frac{i}{h}F(x)}(1 + if'(0))^{\frac{1}{2}}.$$

However, for simplicity, we use the WKB solutions $\tilde{v}_j(x)$ normalised by $|\tilde{v}_j(0)| = 1$ instead of $v_j(x)$. The normalised solutions are given by

$$\tilde{v}_1(x) = v_1(x) = 1, \quad \tilde{v}_2(x) = e^{\frac{i}{h}F(x)}.$$

Then the WKB solutions $\widetilde{w}_j^\bullet$ to the equation $\mathcal{P}w = 0$ associated with $\tilde{v}_j$ admit the asymptotic behavior

$$\widetilde{w}_1^\bullet(x) = \begin{pmatrix} 1 + \mathcal{O}_{C_b^\infty}(h) \\ \mathcal{O}_{C_b^\infty}(h) \end{pmatrix}, \quad \widetilde{w}_2^\bullet(x) = e^{\frac{i}{h}F(x)}\begin{pmatrix} \mathcal{O}_{C_b^\infty}(h) \\ 1 + \mathcal{O}_{C_b^\infty}(h) \end{pmatrix} \quad (\bullet = \flat, \sharp). \tag{2.23}$$

The construction of such microlocal solutions is detailed in Appendix B.

2.8. Outline of the proof of Theorems 1.1 and 1.4

In this manuscript, we prove our main results in the following way. In Section 3, we show the existence of a Fourier integral operator $\mathcal{F}$ which reduces the original equation $\mathcal{P}u = 0$ to the reduced equation $P_{\mathrm{nf}}\mathcal{F}^*u = 0$ microlocally near $(0,0)$. Then in Section 4, we prove our main theorems for the reduced operator. Finally, in Section 5, we study the correspondence between the transfer matrices for the original operator and for the reduced operator.

In more detail, the normal form reduction of Section 3 is done by a classical argument. The symplectomorphism κ is taken so that $\kappa^* p_1(x,\xi) = \xi$. Then we can factorise $\kappa^* p_2 = a_0(x,\xi)(\xi - f(x))$ with an elliptic factor a_0 by the implicit function theorem, which allows us to regard its quantization as a simple first order operator. By the standard quantization procedure ([22, Section 11.2]), we can find Fourier integral operators F_1 and F_2 which both quantise κ such that $F_j^* P_j F_j$ for $j = 1,2$ are of simple forms. The operator $\mathcal{F}$ is defined by $\mathcal{F} = \mathrm{diag}(F_1\ F_2)$.

The proof in Section 4 of the existence and uniqueness of the microlocal solutions for given microlocal Cauchy data and the asymptotics of the transfer matrix for the reduced operator is done by a microlocal method. This differs from the use of exact solutions in previous studies, especially for the tangential case [3, 15]. The main ingredient of this section is the estimate of an integral operator where the phase function of the integrand has a degenerate stationary point.

To compare the transfer matrices, we study in Section 5 the linear relationship between u and $\mathcal{F}v$, where u is a microlocal solution to the original equation $\mathcal{P}u = 0$ and v is that to the reduced equation $P_{\mathrm{nf}}v = 0$. To see that, we study the action of the Fourier integral operator on WKB states. We also write the computation of the asymptotic behavior shown in Theorem 1.4 at the end of this section.

3. Reduction to a matrix normal form

In this section, we construct a normal form of our original operator $\mathcal{P}$ introduced in Section 1.1.

Theorem 3.1. *Assume that Conditions A and B are true. Then, there exist*

- *a symplectomorphism κ near $(0,0)$ (see Definition 2.8);*
- *pseudodifferential operators $A = a^w(x, hD_x) \in \Psi_h$ and $R_j = r_j^w(x, hD_x) \in \Psi_h^{\mathrm{comp}}$ for $j = 1,2$ with $a, r_1, r_2 \in C_b^\infty(\mathbb{R}^2)$;*
- *a real-valued function $f \in C_b^\infty(\mathbb{R};\mathbb{R})$;*
- *a family $\mathcal{F} = \{\mathcal{F}_h\}_{h\in(0,h_0]} = \mathrm{diag}(F_1\ F_2)$ of L^2-bounded operators microlocally invertible near $((0,0),(0,0))$,*

such that the following holds.

(i) *We have*

$$\mathcal{F}^{-1}\mathcal{P}\mathcal{F} \equiv \operatorname{diag}(1\ A)\begin{pmatrix} hD_x & hR_1 \\ hR_2 & hD_x - f(x) \end{pmatrix} \quad \textit{m.l. near } ((0,0),(0,0)).$$

(ii) *The operator A is microlocally invertible near $((0,0),(0,0))$. Both operators F_1 and F_2 quantise κ and are microlocally unitary near $((0,0),(0,0))$ in the sense of Definition 2.8.*

(iii) $(\kappa^* p_1)(x,\xi) = \xi$ *around* $(0,0)$, $\partial_\xi(\kappa^* p_2)(0,0) \neq 0$ *and*

$$\partial_\xi p_j(0,0) \neq 0 \quad (j = 1,2) \Rightarrow \frac{\partial_\xi p_1(0,0)\partial_\xi p_2(0,0)}{\partial_\xi(\kappa^* p_1)(0,0)\partial_\xi(\kappa^* p_2)(0,0)} > 0. \tag{3.1}$$

(iv) *Setting $c = (\partial_\xi(\kappa^* p_2)(0,0))^{-1}$, we have*

$$r_1(0,0) = q_1(0,0) + \mathcal{O}(h), \quad r_2(0,0) = cq_2(0,0) + \mathcal{O}(h)$$

and

$$f^{(k)}(0) = 0 \quad (0 \leq k \leq m-1), \quad f^{(m)}(0) = -c(H_{p_1}^m p_2)(0,0). \tag{3.2}$$

(v) *In the tangential case $m \geq 2$, the constant c is independent of κ and calculated explicitly as*

$$c = s\frac{|\partial_{(x,\xi)} p_1(0,0)|}{|\partial_{(x,\xi)} p_2(0,0)|},$$
$$s = \operatorname{sgn}(\partial_{(x,\xi)} p_1(0,0), \partial_{(x,\xi)} p_2(0,0))_{\mathbb{R}^2} = \operatorname{sgn}(H_{p_1}(0,0), H_{p_2}(0,0))_{\mathbb{R}^2}.$$

Remark 3.2. In the transversal case, the constant c does depend on the choice of κ.

This theorem is proven in two steps. We first find a symplectomorphism κ which reduces the symbols into a normal form (Lemma 3.3). We then construct a suitable quantization $\mathcal{F}$ of κ in the second step. Lemma 3.4 is the crucial part in the second step.

Lemma 3.3. *There exist a locally elliptic real-valued symbol $a_0 \in C_b^\infty(\mathbb{R}^2)$, a symplectomorphism κ near $(0,0)$, and a real-valued function $f \in C_b^\infty(\mathbb{R};\mathbb{R})$ satisfying (3.2) and the properties in Theorem 3.1 (iii) such that*

$$\begin{cases} (\kappa^* p_1)(x,\xi) = \xi, \\ (\kappa^* p_2)(x,\xi) = a_0(x,\xi)(\xi - f(x)). \end{cases}$$

Moreover, we have $a_0(0,0) = \partial_\xi(\kappa^* p_2)(0,0)$ and the constant c satisfies the property stated in Theorem 3.1 (v).

Proof. We proceed in four steps.

Step 1. First, we construct a symplectomorphism κ near $(0,0)$ satisfying the properties in Theorem 3.1 (iii). By virtue of Darboux's theorem ([22, Theorem 12.1] with $A=\{1\}$ and $B=\emptyset$), there exists a symplectomorphism κ_0 preserving $(0,0)$ such that $(\kappa_0^* p_1)(x,\xi)=\xi$.

(1) In the tangential case, Condition B implies $\{p_1,p_2\}(0,0)=0$ and hence

$$\partial_x(\kappa_0^* p_2)(0,0)=\{(\kappa_0^* p_1),(\kappa_0^* p_2)\}(0,0)=0,$$

where we use the fact that κ_0 preserves the Poisson bracket (see Jacobi's formula in [22, Theorem 2.10]). Then we obtain $\partial_\xi(\kappa_0^* p_2)(0,0)\neq 0$ from Condition A. Moreover, the condition (3.1) is automatically satisfied in the tangential case. Thus, we can take $\kappa := \kappa_0$.

(2) In the transversal case, it may happen that $\partial_\xi(\kappa_0^* p_2)(0,0)=0$ (for instance, one can consider $(\kappa_0^* p_2)(x,\xi)=x$) or (3.1) may not be satisfied. In this case, we obtain the desired κ by composing κ_0 with a linear symplectomorphism κ_λ. More precisely, we define $\kappa_\lambda(x,\xi)=(x+\lambda\xi,\xi)$ and put $\kappa := \kappa_0\circ\kappa_\lambda$, where $\lambda\in\mathbb{R}$ is determined below. It is easy to check that the mapping κ is a symplectomorphism preserving $(0,0)$, satisfying $(\kappa^* p_1)(x,\xi)=\xi$ and

$$\begin{aligned}\partial_\xi(\kappa^* p_2)(0,0)&=\partial_x(\kappa_0^* p_2)(0,0)+\partial_\xi(\kappa_0^* p_2)(0,0)\\&=\lambda\{p_1,p_2\}(0,0)+\partial_\xi(\kappa_0^* p_2)(0,0).\end{aligned}$$

Since $\{p_1,p_2\}(0,0)\neq 0$, we can find $\lambda\in\mathbb{R}$ such that $\partial_\xi(\kappa^* p_2)(0,0)\neq 0$ and (3.1) (by choosing a sufficiently large λ with the appropriate sign).

Step 2. According to the implicit function theorem with $\partial_\xi(\kappa^* p_2)(0,0)\neq 0$, there exists a globally defined, real-valued, smooth function f such that, in a neighborhood of $(0,0)$,

$$(x,\xi)\in\{\kappa^* p_2=0\}\iff \xi=f(x).$$

On the other hand, Taylor's theorem at $(x,\xi)=(x,f(x))$ implies

$$(\kappa^* p_2)(x,\xi)=a_0(x,\xi)(\xi-f(x)),$$

where

$$a_0(x,\xi):=\int_0^1 \partial_\xi(\kappa^* p_2)(x,f(x)+t(\xi-f(x)))\,dt.$$

Since $(0,0)\in\Lambda_2$, we have $f(0)=0$ and hence $a_0(0,0)=\partial_\xi(\kappa^* p_2)(0,0)\neq 0$.

Step 3. We prove that the function f satisfies (3.2). To this end, we observe $H_{p_1}^k p_2 = \partial_x^k(\kappa^* p_2)$ due to Jacobi's formula [22, Theorem 2.10] with $(\kappa^* p_1)(x,\xi) = \xi$. Then Condition B implies

$$\begin{aligned} &(\partial_x^k(\kappa^* p_2))(0,0) = 0 \quad (k \in \{1,\dots,m-1\}), \\ &(\partial_x^m(\kappa^* p_2))(0,0) = (H_{p_1}^m p_2)(0,0) \neq 0. \end{aligned} \tag{3.3}$$

Differentiating the implicit equation $(\kappa^* p_2)(x, f(x)) = 0$, we obtain

$$f'(x) = -\frac{(\partial_x(\kappa^* p_2))(x, f(x))}{(\partial_\xi(\kappa^* p_2))(x, f(x))}.$$

Here we note that the denominator is a smooth function that does not vanish around $x = 0$ by Step 1. For the higher order derivatives, we inductively obtain

$$f^{(k)}(x) = -\frac{(\partial_x^k(\kappa^* p_2))(x, f(x))}{(\partial_\xi(\kappa^* p_2))(x, f(x))} + g_k(x), \tag{3.4}$$

where g_k is a sum of terms having a factor of $(\partial_x^l(\kappa^* p_2))(x, f(x))$ with $l \in \{0,1,\dots,k-1\}$ and hence $g_k(0) = 0$ for $k \in \{1,\dots,m\}$ by (3.3). In fact, this formula is true for $k = 1$ so that

$$\begin{aligned} &\frac{\partial}{\partial x}((\partial_x^k(\kappa^* p_2))(x, f(x))) \\ &= (\partial_x^{k+1}(\kappa^* p_2))(x, f(x)) - \frac{(\partial_x(\kappa^* p_2))(x, f(x))(\partial_\xi \partial_x^k(\kappa^* p_2))(x, f(x))}{(\partial_\xi(\kappa^* p_2))(x, f(x))}, \end{aligned}$$

which proves (3.4). Now, the identity (3.4) with (3.3) and $g_k(0) = 0$ for $k \in \{1,\dots,m\}$ shows (3.2).

Step 4. Finally, we see that $c = (\partial_\xi(\kappa^* p_2)(0,0))^{-1}$ satisfies the property in Theorem 3.1 (v). Suppose $m \geq 2$. In this case, one has $H_{p_1} p_2(0,0) = \{p_1, p_2\}(0,0) = 0$ and hence $p_2(x,\xi) = \tilde{c} p_1(x,\xi) + \mathcal{O}(|(x,\xi)|^2)$ near $(x,\xi) = (0,0)$ with a constant $\tilde{c} \in \mathbb{R}$. The condition $\partial_{(x,\xi)} p_2(0,0) \neq 0$ implies that

$$\tilde{c} = \frac{\langle \partial_{(x,\xi)} p_1(0,0), \partial_{(x,\xi)} p_2(0,0) \rangle}{|\partial_{(x,\xi)} p_1(0,0)|^2} \neq 0.$$

Then, we have $\kappa^* p_2(x,\xi) = \tilde{c}\kappa^* p_1(x,\xi) + \mathcal{O}(|(x,\xi)|^2) = \tilde{c}\xi + \mathcal{O}(|(x,\xi)|^2)$ and hence $c^{-1} = \partial_\xi(\kappa^* p_2)(0,0) = \tilde{c}$. ■

Lemma 3.4. *Let κ be a symplectomorphism as in Lemma 3.3. Then there exist an operator F_0 that quantises κ and is microlocally invertible near $((0,0),(0,0))$ in the* *sense of Definition 2.8* (i) *and three pseudodifferential operators* $A = a^w(x, hD_x)$, $B_1 = b_1^w(x, hD_x)$, $B_2 = b_2^w(x, hD_x)$ *with* $a, b_1, b_2 \in C_b^\infty(\mathbb{R}^2)$ *such that*

$$(F_0B_1)^{-1}P_1(F_0B_1) \equiv hD_x, \quad (F_0B_2)^{-1}P_2(F_0B_2) \equiv A(hD_x - f(x)),$$

microlocally near $((0,0),(0,0))$, $a(0,0) = a_0(0,0) + \mathcal{O}(h)$, *and* $b_j(0,0) = 1$ $(j = 1,2)$. *In particular, A is microlocally invertible near* $((0,0),(0,0))$.

Proof. By Lemma 3.3 and [22, Theorem 11.6], there exists a unitary operator F_0 that quantises κ microlocally near $((0,0),(0,0))$. In particular, we have

$$F_0^{-1}P_1F_0 \equiv hD_x + h\varepsilon_1^w, \quad F_0^{-1}P_2F_0 \equiv A_0(hD_x - f(x)) + h\varepsilon_2^w,$$

microlocally near $((0,0),(0,0))$, with $\varepsilon_1, \varepsilon_2 \in C_b^\infty$, where we put $A_0 = a_0^w(x, hD_x)$ with a_0 defined in Lemma 3.3. The proof of [22, Theorem 12.3] (see [22, (12.2.11)]) shows the existence of pseudodifferential operators $B_1 = b_1^w$ and $B_2 = b_2^w$ such that $b_1(0,0) = b_2(0,0) = 1$ and

$$B_1^{-1}(hD_x + h\varepsilon_1^w)B_1 \equiv hD_x, \quad B_2^{-1}(hD_x - f(x) + hA_0^{-1}\varepsilon_2^w)B_2 \equiv hD_x - f(x)$$

microlocally near $((0,0),(0,0))$. We used the fact that A_0, B_1 and B_2 are microlocally invertible near $((0,0),(0,0))$ thanks to Proposition 2.6. Now, the lemma is proven with $A = B_2^{-1}A_0B_2$. ■

Now, we prove Theorem 3.1.

Proof of Theorem 3.1. We define $\mathcal{F} := \mathrm{diag}(F_1\ F_2)$ with $F_1 = F_0B_1$ and $F_2 = F_0B_2$. Since F_0 quantises κ, the operators F_1 and F_2 quantise κ as well by the composition rule of the pseudodifferential operators (Proposition 2.9 (ii)). This proves (ii). Lemma 3.4 yields

$$\mathcal{F}^{-1}\mathcal{P}\mathcal{F} \equiv \begin{pmatrix} hD_x & hF_1^{-1}Q_1F_2 \\ hF_2^{-1}Q_2F_1 & A(hD_x - f(x)) \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}\begin{pmatrix} hD_x & hR_1 \\ hR_2 & hD_x - f(x) \end{pmatrix}$$

m.l. near $((0,0),(0,0))$, where we set $R_1 = F_1^{-1}Q_1F_2$ and $R_2 = A^{-1}F_2^{-1}Q_2F_1$. Thus, (i) follows.

Since F_0 quantises κ, its conjugation $F_0^{-1}Q_jF_0$ is a pseudodifferential operator with symbol $\kappa^*q_j + \mathcal{O}_{C_b^\infty}(h)$ near $(0,0)$ (see Proposition 2.9). Recall that we have put $F_j = F_0B_j$ with pseudodifferential operators $B_j = b_j^w$ with $b_j(0,0) = 1$. Therefore, R_1 and R_2 are pseudodifferential operators whose symbols $r_1, r_2 \in C_b^\infty$ satisfy $r_1 = b_1^{-1}b_2\kappa^*q_1 + \mathcal{O}_{C_b^\infty}(h)$ and $r_2 = a^{-1}b_1b_2^{-1}\kappa^*q_2 + \mathcal{O}_{C_b^\infty}(h)$ near $(0,0)$ by the composition rule. Next, we observe that $a(0,0) = a_0(0,0) + \mathcal{O}(h) = \partial_\xi(\kappa^*p_2)(0,0)$ $(\neq 0) + \mathcal{O}(h)$. Hence, $r_2(0,0) = a(0,0)^{-1}q_2(0,0) = cq_2(0,0) + \mathcal{O}(h)$, where we recall $\kappa(0,0) = (0,0)$ and that we have defined $c = (\partial_\xi(\kappa^*p_2)(0,0))^{-1}$. The relation

$r_1(0,0) = q_1(0,0) + \mathcal{O}(h)$ is proved similarly. This concludes the proof of the first statement of (iv).

(v) and the second statement of (iv) follow from Lemma 3.3. Moreover, we may assume that both F_1 and F_2 are microlocally unitary near $((0,0),(0,0))$ thanks to Proposition 2.9 (ii). This with the microlocal invertibility of A (Lemma 3.4) proves (iii), which concludes the proof. ∎

4. Microlocal connection formula for the reduced equation

In this section, we work with the reduced operator

$$P = P_{\mathrm{nf}} = \begin{pmatrix} hD_x & hR_1 \\ hR_2 & hD_x - f(x) \end{pmatrix}$$

where $R_1, R_2 \in \Psi_h^{\mathrm{comp}}$ and $f \in C_b^\infty(\mathbb{R})$ is a real-valued function such that

$$f^{(j)}(0) = 0 \quad \text{for } j = 0, 1, \dots, m-1, \qquad f^{(m)}(0) \neq 0. \tag{4.1}$$

Note that one has $\gamma_1^\flat = \{\xi = 0,\ x < 0\} \cap \Omega$, $\gamma_1^\sharp = \{\xi = 0,\ x > 0\} \cap \Omega$, $\gamma_2^\flat = \{\xi = f(x),\ x < 0\} \cap \Omega$, and $\gamma_2^\sharp = \{\xi = f(x),\ x > 0\} \cap \Omega$. We fix $I^\flat \Subset (-\infty, 0)$, $I^\sharp \Subset (0, +\infty)$ and two microlocal solutions $\mathtt{W}_j^\bullet$ of $Pu = 0$ near each $\rho_j^\bullet \in \gamma_j^\bullet$ for $\bullet = \flat, \sharp$ and $j = 1, 2$ such that locally for $x \in I^\bullet$,

$$\mathtt{W}_1^\bullet(x) = \begin{pmatrix} 1 + \mathcal{O}_{C_b^\infty}(h) \\ \mathcal{O}_{C_b^\infty}(h) \end{pmatrix} \quad \text{and} \quad \mathtt{W}_2^\bullet(x) = e^{\frac{i}{h}F(x)} \begin{pmatrix} \mathcal{O}_{C_b^\infty}(h) \\ 1 + \mathcal{O}_{C_b^\infty}(h) \end{pmatrix}. \tag{4.2}$$

Such solutions exist as stated in Proposition B.1. For $R, R' \in \Psi_h^{\mathrm{comp}}$ and fixed $x_0 < 0$, we define

$$F(x) := \int_0^x f(y)dy, \quad \Gamma_{0,R} v(x) := \int_{x_0}^x Rv(y)\,dy,$$

$$\Gamma_{\pm,R} v(x) := \int_{x_0}^x e^{\pm\frac{i}{h}F(y)} Rv(y)\,dy.$$

Moreover, we define

$$\Gamma_+ := \Gamma_{+,(R_1)_f}, \quad \Gamma_- = \Gamma_{-,R_2}, \quad K_1 := -\Gamma_+ \circ \Gamma_-, \quad K_2 := -\Gamma_+ \circ \Gamma_-, \tag{4.3}$$

where we set $R_f := e^{-\frac{i}{h}F(x)} R e^{\frac{i}{h}F(x)} \in \Psi_h^{\mathrm{comp}}$. From (4.1), it is easy to see

$$F^{(j)}(0) = 0 \quad \text{for } j = 0, 1, \dots, m, \qquad F^{(m+1)}(0) = f^{(m)}(0) \neq 0. \tag{4.4}$$

We consider the following microlocal Cauchy problem:

$$Pu \equiv 0 \quad \text{m.l. on } \Omega, \qquad u \equiv \alpha_j^\flat \mathtt{W}_j^\flat \quad \text{m.l. near } \rho_j^\flat, \tag{4.5}$$

where $\Omega \Subset \mathbb{R}^2$ is an open neighborhood of $(0,0)$, $\alpha_j^\flat \in \mathbb{C}$, and $\rho_j^\flat \in \gamma_j^\flat \cap \Omega$ ($j = 1,2$).

The next propositions are the main results of this section. The first one states the uniqueness and the existence of the microlocal Cauchy problem (4.5) and the second one provides the asymptotic expansion of the transfer matrix.

Proposition 4.1 (Existence and uniqueness for the reduced problem). *There exist $h_0 \in (0,1]$ and an open neighborhood $\Omega \Subset \mathbb{R}^2$ of $(0,0)$ such that the following holds for $h \in (0,h_0]$.*

(i) *The solution of the microlocal Cauchy problem (4.5) is unique. Namely, if a tempered family u satisfies (4.5) with $\alpha_1^\flat = \alpha_2^\flat = 0$ for some $\rho_j^\flat \in \gamma_j^\flat \cap \Omega$ ($j = 1,2$), then*

$$u \equiv 0 \quad \textit{m.l. on } \Omega.$$

(ii) *Let $\rho_j^\flat \in \gamma_j^\flat \cap \Omega$. Then, for each $(\alpha_1^\flat, \alpha_2^\flat) \in \mathbb{C}^2$, there exists a tempered family $u = {}^\mathsf{T}(u_1,u_2)$ such that (4.5) is true. Moreover, if $\alpha_j^\flat$ is depending on $h > 0$ and $|\alpha_j^\flat| = \mathcal{O}(1)$ as $h \to 0$, then*

$$u_1(x) \equiv \alpha_1^\flat - i\alpha_2^\flat \Gamma_+(\mathbb{1})(x) + \mathcal{O}_{L^\infty}\left(h^{\frac{2}{m+1}}\left(\log\frac{1}{h}\right)^{\delta_{m,1}}\right) \tag{4.6}$$

$$u_2(x) \equiv e^{\frac{i}{h}F(x)}(\alpha_2^\flat - i\alpha_1^\flat \Gamma_-(\mathbb{1})(x)) + \mathcal{O}_{L^\infty}\left(h^{\frac{2}{m+1}}\left(\log\frac{1}{h}\right)^{\delta_{m,1}}\right) \tag{4.7}$$

hold m.l. on Ω. Here, $\mathbb{1}$ denotes the constant function equal to one.

Now, we can define the transfer matrix T with respect to the basis which we have fixed in (4.2). In fact, from Proposition 4.1, we can define a linear map $(\alpha_1^\flat, \alpha_2^\flat) \in \mathbb{C}^2 \mapsto u$ whose range is included in the set of tempered families u satisfying (4.5). From Proposition 2.16, for $j = 1,2$ there exists $\alpha_j^\sharp \in \mathbb{C}$ such that $u = \alpha_j^\sharp \mathtt{W}_j^\sharp$ m.l. near $\rho_j^\sharp$. Then, we have a linear map $u \mapsto (\alpha_1^\sharp, \alpha_2^\sharp) \in \mathbb{C}^2$. We let $T\colon (\alpha_1^\flat, \alpha_2^\flat) \mapsto (\alpha_1^\sharp, \alpha_2^\sharp)$ be the composition of these two linear maps.

Proposition 4.2 (Microlocal connection formula for the reduced operator). *The matrix T admits the asymptotic formula*

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - ih^{\frac{1}{m+1}} \begin{pmatrix} 0 & r_1(0,0)\omega \\ r_2(0,0)\bar{\omega} & 0 \end{pmatrix} + \mathcal{O}\left(h^{\frac{2}{m+1}}\left(\log\frac{1}{h}\right)^{\delta_{m,1}}\right),$$

where r_j are symbols of R_j,

$$R_j = r_j^w(x, hD_x),$$

and

$$\omega = 2\mu_m\Big(\frac{\operatorname{sgn}(f^{(m)}(0))}{2(m+1)}\pi\Big)\Gamma\Big(\frac{m+2}{m+1}\Big)\Big(\frac{(m+1)!}{|f^{(m)}(0)|}\Big)^{\frac{1}{m+1}}. \tag{4.8}$$

with $\mu_m(\theta) := 2^{-1}(e^{i\theta} + e^{i(-1)^{m+1}\theta})$.

We prove those two propositions in the Sections 4.4 and 4.5 after a few preliminary lemmas. By Lemma 2.5, we may assume that for a compact neighborhood $S \subset \mathbb{R}$ of 0,

$$\operatorname{supp} E_{R_1} \cup \operatorname{supp} E_{R_2} \subset S \times S, \quad f'(x) \neq 0 \quad \text{for } x \in S \setminus \{0\}, \tag{4.9}$$

where E_{R_j} denotes the integral kernel of R_j.

4.1. Microlocal Duhamel principle of the scalar equation

Lemma 4.3 (Microlocal Duhamel formula for the scalar equation). *Let u and v be tempered families of L^2 functions, $R \in \Psi_h^{\mathrm{comp}}$, and $\Omega \subset \mathbb{R}^2$ be an open neighborhood of $(0,0)$ such that its intersection with $\{\xi = f(x)\}$ is connected. Suppose that u and v satisfy*

$$(hD_x - f(x))u + hRv \equiv 0 \quad \textit{m.l. on } \Omega. \tag{4.10}$$

Then, for each $x_0 \in \mathbb{R}$, there exists a constant $C \in \mathbb{C}$ such that

$$u(x) \equiv e^{\frac{i}{h}F(x)}\Big(C - i\int_{x_0}^{x} e^{-\frac{i}{h}F(y)}Rv(y)\,dy\Big) \quad \textit{m.l. on } \Omega. \tag{4.11}$$

Proof. Put $w(x) := -ie^{\frac{i}{h}F(x)}\int_{x_0}^{x} e^{-\frac{i}{h}F(y)}Rv(y)\,dy$. By (4.10), one has

$$(hD_x - f(x))(u(x) - w(x)) \equiv 0 \quad \text{m.l. on } \Omega.$$

On the other hand, we clearly have $(hD_x - f(x))(e^{\frac{i}{h}F(x)}) = 0$. By the uniqueness of the microlocal solutions (Proposition 2.13 i), they are linearly dependent. This proves (4.11). ∎

4.2. Estimates for integral operators

The purpose of this subsection is to derive some estimates for the integral operators related to the Duhamel formula (4.10). The following proposition corresponds to [11, Proposition 3.1] or [3, Proposition 3.8].

Proposition 4.4. *Fix $i = 1, 2$. There exists an h-independent constant $C > 0$ such that the following holds.*

(i) *We have*

$$\|\Gamma_\pm\|_{L^\infty\to L^\infty} \le C, \quad \|\Gamma_\pm(\mathbb{1})\|_{L^\infty} \le Ch^{\frac{1}{m+1}}, \tag{4.12}$$

$$\|K_j\|_{L^\infty\to L^\infty} \le Ch^{\frac{1}{m+1}}, \quad \|K_j(\mathbb{1})\|_{L^\infty} \le Ch^{\frac{2}{m+1}}\Big(\log\frac{1}{h}\Big)^{\delta_{m,1}}, \tag{4.13}$$

for $0 < h \le 1$ and $j = 1, 2$.

(ii) *Let $I_\flat \Subset (-\infty, 0)$ and $I_\sharp \Subset (0, \infty)$. Then,*

$$\|\Gamma_\pm(\mathbb{1})\|_{L^\infty(I_\flat)} = \mathcal{O}(h), \quad \|K_i(\mathbb{1})\|_{L^\infty(I_\flat)} = \mathcal{O}(h),$$
$$\|\Gamma_\pm(\mathbb{1}) - \omega_\pm r_\pm(0,0)h^{\frac{1}{m+1}}\|_{L^\infty(I_\sharp)} = \mathcal{O}(h^{\frac{2}{m+1}}),$$

as $h \to 0$, where $\omega_+ = \omega$, $\omega_- = \bar{\omega}$, $r_+ = r_1$, and $r_- = r_2$. We recall that ω is given in (4.8).

Remark 4.5. In general, $\|\Gamma_j u\|_{L^\infty} = \mathcal{O}(h^{\frac{1}{m+1}})$ and $\|K_j u\|_{L^\infty} \le Ch^{\frac{2}{m+1}}\big(\log\frac{1}{h}\big)^{\delta_{m,1}}$ do not hold as $h \to 0$ even if $u = u_h$ is a uniformly bounded family in L^∞ and its optimal bound of the first estimate is $\mathcal{O}(1)$. The second inequalities in (4.12) and (4.13) mean that these bounds can be improved if u is the constant function $\mathbb{1}$.

Proof. First, we observe that conditions (4.4) and (4.9) allow us to use the degenerate stationary phase theorem (Lemma A.1) in the following proof.

(i) First, we claim that there exists $C > 0$ such that

$$\|\Gamma_\pm\|_{L^\infty\to W^{1,\infty}} \le C, \quad \|\Gamma_\pm\|_{W^{1,\infty}\to L^\infty} \le Ch^{\frac{1}{m+1}} \tag{4.14}$$

for $0 < h \le 1$, where $\|u\|_{W^{1,\infty}} = \|u\|_{L^\infty} + \|D_x u\|_{L^\infty}$. We prove it for Γ_+ and the proof for Γ_- is similar. By the definition of Ψ_h^{comp}, there exists a compact interval $J \subset \mathbb{R}$ (independent of $0 < h \le 1$) such that $(R_1)_f = \chi_J(R_1)_f$, where χ_J is the characteristic function of the subset J. We also note that $(R_1)_f \in \Psi_h^{\mathrm{comp}}$ by Proposition 2.12 and $(R_1)_f = \chi_J(R_1)_f$. By Hölder's inequality and the definition of Γ_+,

$$\|\Gamma_+ u\|_{L^\infty} \le \|\chi_J\|_{L^1}\|(R_1)_f\|_{L^\infty\to L^\infty}\|u\|_{L^\infty} \le C\|u\|_{L^\infty},$$

where we use Lemma 2.1 for the uniform boundedness of $(R_1)_f$ on L^∞. Moreover,

$$D_x \circ \Gamma_+ u(x) = -ie^{\frac{i}{h}F(x)}\chi_J(x)(R_1)_f u(x),$$

which implies $\|D_x \circ \Gamma_+\|_{L^\infty \to L^\infty} \leq \|\chi_J\|_{L^\infty}\|(R_1)_f\|_{L^\infty \to L^\infty} \leq C$. These prove the first inequality in (4.14). The second inequality in (4.14) follows from the degenerate stationary phase estimate (A.1):

$$\left| \int_{x_0}^{x} e^{\frac{i}{h}F(x)} \chi_J(y)(R_1)_f u(y)\, dy \right| \leq C h^{\frac{1}{m+1}} \|\chi_J(R_1)_f u\|_{W^{1,\infty}} \leq C'\|u\|_{W^{1,\infty}},$$

where we use the trivial bound $\sup_{0<h\leq 1} h^{\frac{1}{m+1}} \log\left(\frac{1}{h}\right) < \infty$ and the uniform boundedness of $(R_1)_f = \chi_J(R_1)_f$ between $W^{1,\infty}$ stated in Lemma 2.1.

The first estimate in (4.12) directly follows from (4.14). Using (4.14) again, we obtain

$$\|K_1\|_{L^\infty \to L^\infty} \leq \|\Gamma_1\|_{W^{1,\infty} \to L^\infty}\|\Gamma_2\|_{L^\infty \to W^{1,\infty}} \leq C' h^{\frac{1}{m+1}},$$

which shows the first bound in (4.13) regarding K_1. The proof for K_2 is similar.

Next, we deal with the second inequality in (4.12). We note

$$\Gamma_+(\mathbb{1})(x) = \int_{x_0}^{x} e^{+\frac{i}{h}F(y)} a_h(y)\, dy,$$

where $a_h(x) := \chi_J(x)(R_1)_f(\mathbb{1})(x)$. Clearly, for each $\alpha \in \mathbb{N}$, $|\partial_x^\alpha a_h(x)|$ are uniformly bounded with respect to $0 < h \leq 1$ and $x \in \mathbb{R}$ by (2.12). Then, the inequality

$$\|\Gamma_+(\mathbb{1})\|_{L^\infty} \leq C h^{\frac{1}{m+1}}$$

follows from the degenerate stationary phase estimate (A.1). The proof for Γ_- is similar.

Finally, we prove the second estimate in (4.13). We deal with K_1 only. Set $g(x) = (R_1)_f \Gamma_-(\mathbb{1})(x)$. Since $(R_1)_f \in \Psi_h^{\mathrm{comp}}$, the function w is supported in a compact subset which is independent of $0 < h \leq 1$. By the degenerate stationary phase estimate (A.1), we have

$$\begin{aligned} |K_1(\mathbb{1})(x)| &= \left| \int_{x_0}^{x} e^{\frac{i}{h}F(y)} g(y)\, dy \right| \\ &\leq C h^{\frac{1}{m+1}} \|g\|_{L^\infty} + C h^{\frac{2}{m+1}} \left(\log \frac{1}{h}\right)^{\delta_{m,1}} \|D_x g\|_{L^\infty}, \end{aligned}$$

where $C > 0$ is independent of $x \in \mathbb{R}$ and $0 < h \leq 1$. On the other hand, $\|g\|_{L^\infty} = \mathcal{O}(h^{\frac{1}{m+1}})$ and $\|D_x g\|_{L^\infty} = \mathcal{O}(1)$ due to (4.12), (4.14), and Lemma 2.1. This completes the proof.

(ii) The derivative of the phase F of $e^{\frac{i}{h}F(x)}$ does not vanish on S appearing in (4.9), so an integration by parts and the $W^{1,\infty}$-boundedness of $(R_1)_f$ and R_2 (Lemma 2.1) give $\|\Gamma_\pm(\mathbb{1})\|_{L^\infty(I_\flat)} = \mathcal{O}(h)$ as well as $\|K_1(1)\|_{L^\infty(I_\flat)} = \mathcal{O}(h)$.

The last estimate comes from the degenerate stationary expansion (A.2) due to the conditions (4.4) and (4.9). ∎

We need one more technical lemma.

Lemma 4.6. *Let $I_\flat \Subset (-\infty, 0)$ and $i = 1, 2$. If $r \in L^\infty(\mathbb{R})$ satisfies $\|r\|_{L^\infty(I_\flat)} = \mathcal{O}(h^\infty)$ for all $I_\flat \Subset (-\infty, 0)$, then*

$$\|\Gamma_\pm r\|_{L^\infty(I_\flat)} = \mathcal{O}(h^\infty), \quad \|K_i r\|_{L^\infty(I_\flat)} = \mathcal{O}(h^\infty).$$

Proof. We only consider Γ_+ and K_1. We write

$$\widetilde{I_\flat} := \mathrm{Conv}(I_\flat, \{x_0\}),$$

where $\mathrm{Conv}(A, B)$ denotes the convex hull of the sets A, B. Since $x_0 < 0$, we have $\|r\|_{L^\infty(\widetilde{I_\flat})} = \mathcal{O}(h^\infty)$. Moreover, we have $\|\Gamma_+ r\|_{L^\infty(I_\flat)} \le |\widetilde{I_\flat}| \|(R_1)_f r\|_{L^\infty(\widetilde{I_\flat})}$ and $\|K_1 r\|_{L^\infty(I_\flat)} \le |\widetilde{I_\flat}| \|(R_1)_f \Gamma_- r\|_{L^\infty(\widetilde{I_\flat})}$. Since both $(R_1)_f$ and R_2 are pseudolocal, the estimate follows. ∎

4.3. Existence of basis of the microlocal solution space

In this subsection, we construct a basis of microlocal solutions near the crossing point via Neumann series.

Lemma 4.7. *There exist tempered families $v, \tilde{v} \in L^\infty(\mathbb{R}; \mathbb{C}^2)$ such that the following holds as $h \to +0$: Let $\Omega \Subset \mathbb{R}^2$ be an open neighborhood of $(0,0)$ and $I_\flat \Subset (-\infty, 0)$.*

(i) $\|v\|_{L^\infty(\mathbb{R};\mathbb{C}^2)}, \|\tilde{v}\|_{L^\infty(\mathbb{R};\mathbb{C}^2)} = \mathcal{O}(1)$;

(ii) *The functions $v, \tilde{v}$ are exact solutions: $Pv = 0$ and $P\tilde{v} = 0$;*

(iii) *There exists an h-dependent matrix $A = (A_{ij}) \in M_2(\mathbb{C})$ such that $A = I + \mathcal{O}(h)$ and*

$$v \equiv A_{11}\mathtt{W}_1^\flat + A_{12}\mathtt{W}_2^\flat, \quad \tilde{v} \equiv A_{21}\mathtt{W}_1^\flat + A_{22}\mathtt{W}_2^\flat \quad \textit{m.l. on } \Omega \cap (I_\flat \times \mathbb{R}); \tag{4.15}$$

(iv) *We have*

$$v(x) = \begin{pmatrix} 1 \\ -ie^{(\frac{i}{h}F)(x)}\Gamma_{-,R_2}(\mathbb{1})(x) \end{pmatrix} + \mathcal{O}_{L^\infty(\mathbb{R};\mathbb{C}^2)}\Big(h^{\frac{2}{m+1}}\Big(\log\frac{1}{h}\Big)^{\delta_{m,1}}\Big),$$

$$\tilde{v}(x) = \begin{pmatrix} -i\Gamma_{+,(R_1)_f}(\mathbb{1})(x) \\ e^{\frac{i}{h}F(x)} \end{pmatrix} + \mathcal{O}_{L^\infty(\mathbb{R};\mathbb{C}^2)}\Big(h^{\frac{2}{m+1}}\Big(\log\frac{1}{h}\Big)^{\delta_{m,1}}\Big).$$

Proof. We observe that $\|K_j\|_{L^\infty\to L^\infty} = \mathcal{O}(h^{\frac{1}{m+1}})$ by Proposition 4.4, and hence the Neumann series $(I - K_j)^{-1} = \sum_{\ell=0}^{\infty} K_j^\ell$ converges in the operator topology in $L^\infty(\mathbb{R})$ for sufficiently small $h > 0$.

We observe that the exact equation $Pu = 0$ for $u = {}^{\mathsf{T}}(u_1, u_2)$ can be written as in (1.13) (with the notation (4.3)). Let us define $v = {}^{\mathsf{T}}(v_1, v_2)$ and $\tilde{v} = {}^{\mathsf{T}}(\tilde{v}_1, \tilde{v}_2)$ as (1.13) with $(\alpha_1^\flat, \alpha_2^\flat) = (1, 0)$ and $(\alpha_1^\flat, \alpha_2^\flat) = (0, 1)$ respectively:

$$v_1 := \sum_{\ell=0}^{\infty} K_1^\ell(\mathbb{1}), \qquad v_2 := -i e^{\frac{i}{h}F(x)} \Gamma_-(v_1),$$
$$\tilde{v}_2 := e^{\frac{i}{h}F(x)} \sum_{\ell=0}^{\infty} K_2^\ell(\mathbb{1}), \quad \tilde{v}_1 := -i\Gamma_+(\tilde{v}_2).$$

Then it is easy to check property (i) because $\|K_j\|_{L^\infty\to L^\infty} = \mathcal{O}(h^{\frac{1}{m+1}})$. Property (ii) is clearly satisfied by the construction of v and $\tilde{v}$.

We observe that $\|\sum_{\ell=1}^{\infty} K_j^\ell(\mathbb{1})\|_{L^\infty} = \mathcal{O}(h^{\frac{2}{m+1}}(\log\frac{1}{h})^{\delta_{m,1}})$ holds for $j = 1, 2$ by Proposition 4.4. Then, property (iv) can be deduced from the definition of $\tilde{h}$ and $v, \tilde{v}$.

Finally, we prove (iii). By the uniqueness result of the microlocal solution away from the crossing point (Proposition 2.16 (i)), we can find a (h-dependent) matrix $A = (A_{ij}) \in M_2(\mathbb{C})$ such that (4.15) holds. Thus, we only need to prove $A = I + \mathcal{O}(h)$ as $h \to 0$. We observe that

$$\|\Gamma_\pm(\mathbb{1})\|_{L^\infty(I_\flat)} = \mathcal{O}(h), \quad \|K_j^\ell(\mathbb{1})\|_{L^\infty(I_\flat)} = \mathcal{O}(h),$$

for $j = 1, 2$ and $\ell \geq 1$ by Proposition 4.4 (ii). Using these estimates and the identity (4.2), we obtain

$$v(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathcal{O}_{L^\infty(I_\flat;\mathbb{C}^2)}(h) = \mathtt{W}_1^\flat(x) + \mathcal{O}_{L^\infty(I_\flat;\mathbb{C}^2)}(h),$$
$$\tilde{v}(x) = e^{\frac{i}{h}F(x)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathcal{O}_{L^\infty(I_\flat;\mathbb{C}^2)}(h) = \mathtt{W}_2^\flat(x) + \mathcal{O}_{L^\infty(I_\flat;\mathbb{C}^2)}(h).$$

uniformly in $x \in I_\flat$. Therefore, the identity $A = I + \mathcal{O}(h)$ follows. ∎

4.4. Proof of the existence and the uniqueness of the microlocal solution

Proof of Proposition 4.1. (i) Let u be a tempered family satisfying (4.5) with $\alpha_j^\flat = 0$ for $j = 1, 2$. By the uniqueness result away from the crossing point (Proposition 2.16 (i)), it suffices to prove $u \equiv 0$ m.l. near $(0, 0)$. Taking Ω sufficiently small and cutting u off appropriately in the phase space, we may assume[11]

$$u \text{ is compactly microlocalised and } \mathrm{WF}_h(u_1) \cup \mathrm{WF}_h(u_2) \subset \Lambda_1 \cup \Lambda_2. \tag{4.16}$$

First, we claim

$$\|u_j\|_{L^\infty(I)} = \mathcal{O}(h^\infty) \quad \text{for all } I \Subset (-\infty, 0) \text{ and } j = 1, 2. \tag{4.17}$$

Using $\alpha_j^\flat = 0$ for $j = 1, 2$, we have $\mathrm{WF}_h(u_j) \cap ((-\infty, 0) \times \mathbb{R}) = \emptyset$ by Proposition 2.16 and (4.16). Since $u = {}^{\mathsf{T}}(u_1, u_2)$ is compactly microlocalised, we obtain equation (4.17).

Next, we show

$$u_1(x) \equiv K_1^N u_1(x) \quad \text{m.l. on } \Omega \tag{4.18}$$

for all $N \in \mathbb{N}$. By Lemma 4.3, there exists $C_{2,1} \in \mathbb{C}$ such that

$$u_2(x) \equiv e^{\frac{i}{h}F(x)}(C_{2,1} - i\Gamma_{-,R_2} u_1(x)) \quad \text{m.l. on } \Omega.$$

By Lemma 4.6 and (4.17), $\|u_2\|_{L^\infty(I)}, \|\Gamma_- u_1\|_{L^\infty(I)} = \mathcal{O}(h^\infty)$ for all $I \Subset (-\infty, 0)$. These estimates imply $C_{2,1} = O(h^\infty)$ and hence

$$u_2(x) \equiv -ie^{\frac{i}{h}F(x)}\Gamma_- u_1(x) \quad \text{m.l. on } \Omega.$$

Substituting this identity to the first entry of the system in (4.5), we have

$$hD_x u_1 - ihR_1(e^{\frac{i}{h}F(x)}\Gamma_- u_1(x)) \equiv 0 \quad \text{m.l. on } \Omega.$$

By using Lemma 4.3 (with $f = 0$) again, we find $C_{1,1} \in \mathbb{C}$ satisfying

$$u_1(x) \equiv C_{1,1} + K_1 u_1 \quad \text{m.l. on } \Omega,$$

where we use $R_1 e^{\frac{i}{h}F(x)} = e^{\frac{i}{h}F(x)}(R_1)_f$. A similar argument shows $C_{1,1} = O(h^\infty)$ and hence

$$u_1(x) \equiv K_1 u_1(x) \quad \text{m.l. on } \Omega.$$

This proves (4.18) with $N = 1$. Continuing this procedure, we obtain[12] (4.18) for all $N \in \mathbb{N}$.

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[11](4.5) just implies $(\mathrm{WF}_h(u_1) \cup \mathrm{WF}_h(u_2)) \cap \Omega \subset \Lambda_1 \cup \Lambda_2$.

[12]The identity $v_1 \equiv v_2$ m.l. on Ω does not imply $K_1 v_1 \equiv K_1 v_2$ m.l. on Ω in general, since K_1 is not pseudolocal. Therefore, we need to repeat the procedure used in the proof for $N = 1$.

Since u is tempered, $\|u_j\|_{L^\infty} = \mathcal{O}(h^{-M})$ as $h \to 0$ for some $M > 0$. By Lemma 4.4, we have $\|K_1^N u_1\|_{L^\infty} = \mathcal{O}(h^{\frac{N}{m+1}-M})$ for all $N \in \mathbb{N}$ and hence

$$u_1 \equiv \mathcal{O}_{L^\infty}(h^{\frac{N}{m+1}-M}) \quad \text{m.l. on } \Omega$$

by (4.18). This shows $u_1 \equiv 0$ m.l. on Ω. The proof for $u_2 \equiv 0$ m.l. on Ω is similar.

(ii) We use the symbols $v, \tilde{v}, A$ as in Lemma 4.7. Since $A = I + \mathcal{O}(h)$ as $h \to 0$, A is invertible when h is sufficiently small. Then $B := A^{-1} = (B_{ij})$ that exists and $B = I + \mathcal{O}(h)$ holds for such $h > 0$. By the definition of B and Lemma 4.7 (iii), we have

$$\mathtt{w}_1^\flat \equiv B_{11}v + B_{12}\tilde{v}, \quad \mathtt{w}_2^\flat \equiv B_{21}v + B_{22}\tilde{v} \quad \text{m.l. on } \Omega \cap (I_\flat \cap \mathbb{R}).$$

Now, we define

$$u := \alpha_1^\flat(B_{11}v + B_{12}\tilde{v}) + \alpha_2^\flat(B_{21}v + B_{22}\tilde{v}).$$

It is easy to see $u \equiv \alpha_1^\flat \mathtt{w}_1^\flat + \alpha_2^\flat \mathtt{w}_2^\flat$ m.l. on $\Omega \cap (I_\flat \times \mathbb{R})$ and hence

$$u \equiv \alpha_j^\flat \mathtt{w}_j^\flat \quad \text{m.l. on } \rho_j^\flat$$

since $\mathtt{w}_1^\flat \equiv 0$ and $\mathtt{w}_2^\flat \equiv 0$ m.l. on $\rho_2^\flat$ and $\rho_1^\flat$ respectively due to Proposition 2.16 (ii). Since $Pv = 0$ and $P\tilde{v} = 0$, we have $Pu \equiv 0$ m.l. on Ω. The asymptotic expansions (4.6) and (4.7) directly follows from Lemma 4.7 (iv). This completes the proof. ∎

4.5. Proof of the connection formula

Proof of Proposition 4.2. We write $T = (t_{ij})_{(i,j)}$. We calculate the asymptotic formula of the coefficients t_{11} and t_{21}. The computation of t_{12}, t_{22} is similar.[13] By Proposition 4.1 (ii), we can find a microlocal solution $u = {}^\mathsf{T}(u_1, u_2)$ of $Pu \equiv 0$ near $(0,0)$ such that for $\bullet \in \{\flat, \sharp\}$ and $j = 1, 2$,

$$u \equiv \alpha_j^\bullet \mathtt{w}_j^\bullet \quad \text{m.l. on } \gamma_j^\bullet \cap \Omega \qquad \text{with} \quad (\alpha_1^\flat, \alpha_2^\flat) = (1, 0), \tag{4.19}$$

and $\|u_j\|_{L^\infty} = \mathcal{O}(1)$ as $h \to 0$ for $j = 1, 2$. By the definition of the transfer matrix T, we have $\alpha_1^\sharp = t_{11}$ and $\alpha_2^\sharp = t_{21}$. Thus, our task is to calculate the asymptotic formula for $\alpha_j^\sharp$ for $j = 1, 2$.

[13]To do this, one has to calculate the symbol of $(R_1)_f$ at $(0,0)$. Proposition 2.12 shows that the symbol of $(R_1)_f$ is $r_1(x, x + f(x))$ modulo $\mathcal{O}(h)$ terms. Since $f(0) = 0$, its value at $(0,0)$ is $r_1(0,0) + \mathcal{O}(h)$. Then a similar calculation to the one below leads to the asymptotic expansion of t_{12}, t_{22}.

By Proposition 4.1 (ii) with $(\alpha_1^\flat, \alpha_2^\flat) = (1, 0)$, we have

$$\begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} \equiv \begin{pmatrix} 1 \\ -ie^{\frac{i}{h}F(x)}\Gamma_-(1)(x) \end{pmatrix} + \mathcal{O}_{L^\infty}\left(h^{\frac{2}{m+1}}\left(\log\frac{1}{h}\right)^{\delta_{m,1}}\right)$$

m.l. on Ω. Combining this with Proposition 4.4 (ii), we see that for each $I_\sharp \Subset (0, \infty)$,

$$\begin{aligned}\begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} &\equiv \begin{pmatrix} 1 \\ -i\omega_- r_2(0,0)h^{\frac{1}{m+1}}e^{\frac{i}{h}F(x)} \end{pmatrix} + \mathcal{O}_{L^\infty}\left(h^{\frac{2}{m+1}}\left(\log\frac{1}{h}\right)^{\delta_{m,1}}\right) \\ &\equiv \mathtt{W}_1^\sharp(x) - i\bar{\omega} r_2(0,0)h^{\frac{1}{m+1}}\mathtt{W}_2^\sharp(x) + \mathcal{O}_{L^\infty}\left(h^{\frac{2}{m+1}}\left(\log\frac{1}{h}\right)^{\delta_{m,1}}\right)\end{aligned}$$

m.l. on $\Omega \cap (I_\sharp \times \mathbb{R})$, where we use

$$\mathtt{W}_1^\sharp = {}^\mathsf{T}(1, 0) + \mathcal{O}_{L^\infty}(h) \quad \text{and} \quad \mathtt{W}_2^\sharp = {}^\mathsf{T}(0, e^{\frac{i}{h}F(x)}) + \mathcal{O}_{L^\infty}(h).$$

Consequently, we obtain

$$\begin{aligned} t_{11} &= \alpha_1^\sharp = 1 + \mathcal{O}_{L^\infty}\left(h^{\frac{2}{m+1}}\left(\log\frac{1}{h}\right)^{\delta_{m,1}}\right) \\ t_{21} &= \alpha_2^\sharp - i\bar{\omega} r_2(0,0)h^{\frac{1}{m+1}} + \mathcal{O}_{L^\infty}\left(h^{\frac{2}{m+1}}\left(\log\frac{1}{h}\right)^{\delta_{m,1}}\right)\end{aligned}$$

from (4.19). This completes the proof. ∎

5. Correspondence of transfer matrices

According to Theorem 3.1, there exist an L^2-bounded operator $\mathcal{F}$ and an elliptic pseudodifferential operator A such that

$$\mathcal{F}^{-1}\mathcal{P}\mathcal{F} \equiv \operatorname{diag}(1\ A)P_{\mathrm{nf}}, \quad P_{\mathrm{nf}} = \begin{pmatrix} hD_x & hR_1 \\ hR_2 & hD_x - f(x) \end{pmatrix},$$

microlocally near $((0,0),(0,0))$. Let $T = T(w_1^\flat, w_2^\flat, w_1^\sharp, w_2^\sharp)$ be the transfer matrix for the original operator $\mathcal{P}$ and let $T_{\mathrm{nf}} = T_{\mathrm{nf}}(\mathtt{W}_1^\flat, \mathtt{W}_2^\flat, \mathtt{W}_1^\sharp, \mathtt{W}_2^\sharp)$ be that of the reduced operator P_{nf}. Here, $w_j^\bullet$ denotes the WKB solutions for $\mathcal{P}$ given in Proposition 2.16, and $\mathtt{W}_j^\bullet$ denotes that of P_{nf} in (2.23) constructed in Proposition B.1. Note that the functions $\mathtt{W}_2^\flat$, $\mathtt{W}_2^\sharp$ are not normalised in the sense of Definition 1.2 for a notational simplicity. One has the following relation between these matrices.

Proposition 5.1. *Assume* (2.21). *Then one has*

$$T = I + \begin{pmatrix} 0 & \left|\frac{c}{c'}\right|^{\frac{1}{2}} t_{12} \\ (\operatorname{sgn} c)\left|\frac{c}{c'}\right|^{-\frac{1}{2}} t_{21} & 0 \end{pmatrix} + \mathcal{O}\left(h^{\frac{2}{m+1}}\left(\log\frac{1}{h}\right)^{\delta_{m,1}}\right),$$

where t_{jk} *stands for the* (j,k)*-entry of* T_{nf}*,* $c = (\partial_\xi(\kappa^* p_2)(0,0))^{-1}$ *appearing in Theorem* 3.1*, and*

$$c' = \frac{|\partial_{(x,\xi)} p_1(0,0)|}{|\partial_{(x,\xi)} p_2(0,0)|}.$$

Once this proposition is proved, Theorem 1.4 follows immediately from this with the asymptotic behavior of T_{nf} given in Proposition 4.2. See Section 5.3 for the detail of the computation.

5.1. Proof of Proposition 5.1 when c is positive

In this subsection, we first prove Proposition 5.1 in the case of positive c. The proof for the case of negative c is provided in Section 5.2.

Lemma 5.2. *Assume that* (2.21) *holds and that the constant* c *introduced in Theorem* 3.1 (iv) *is positive. Then, there exist constants* $\beta_j^\bullet \in \mathbb{C}$ *such that*

$$\mathscr{F}\mathtt{W}_j^\bullet \equiv \beta_j^\bullet w_j^\bullet \quad \textit{microlocally near } \gamma_j^\bullet, \tag{5.1}$$

for each $(j,\bullet) \in \{1,2\} \times \{\flat, \sharp\}$*. They admit the asymptotic formula*

$$\beta_1^\bullet = |\partial_{(x,\xi)} p_1(0,0)|^{-\frac{1}{2}} + \mathcal{O}(h), \quad \beta_2^\bullet = |c\,\partial_{(x,\xi)} p_2(0,0)|^{-\frac{1}{2}} + \mathcal{O}(h), \tag{5.2}$$

as $h \to +0$*.*

Proof of Proposition 5.1 *when* $c > 0$ *using Lemma* 5.2*.* By the definition (5.1) of $\beta_j^\bullet$, we have the identity

$$T = \operatorname{diag}(\beta_1^\sharp\ \beta_2^\sharp) T_{\mathrm{nf}} \operatorname{diag}(\beta_1^\flat\ \beta_2^\flat)^{-1}. \tag{5.3}$$

According to Proposition 4.2, one has

$$T_{\mathrm{nf}} = I + \begin{pmatrix} 0 & t_{12} \\ t_{21} & 0 \end{pmatrix} + \mathcal{O}\left(h^{\frac{2}{m+1}} \left(\log \frac{1}{h}\right)^{\delta_{m,1}}\right).$$

We obtain Proposition 5.1 by substituting this formula and (5.2) into (5.3). ∎

We will apply Proposition 2.10 to prove Lemma 5.2. To do this, we need the following lemmas.

Lemma 5.3. *Assume that* (2.21) *holds. Then, the projection* (2.2) *is a diffeomorphism near* $(0,0)$*.*

Proof. It suffices to prove that the 1×1 block $\frac{\partial x}{\partial y}(0,0)$ in the derivative $d\kappa(0,0)$ is invertible. If $\frac{\partial \xi}{\partial y}(0,0) = 0$, then we have $\frac{\partial x}{\partial y}(0,0) \neq 0$ since $d\kappa(0,0)$ is invertible. Thus, we may assume $\frac{\partial \xi}{\partial y}(0,0) \neq 0$. Differentiating the equation $(\kappa^* p_1)(y,\eta) = \eta$ with respect to y, we have

$$0 = \frac{\partial x}{\partial y}(0,0)\partial_x p_1(0,0) + \frac{\partial \xi}{\partial y}(0,0)\partial_\xi p_1(0,0).$$

Since $\partial_\xi p_1(0,0) \neq 0$ by (2.21), we conclude $\frac{\partial x}{\partial y}(0,0) \neq 0$. ∎

In the next lemma, we compute the index υ appearing in Proposition 2.10 in our situation.

Lemma 5.4. *Put $F(y) = \int_0^y f(t)\,dt$. Under* (2.21) *and* (3.1), *one has*

$$\operatorname{sgn} \partial^2_{(y,\eta)}(-y \cdot \eta + \psi(0,\eta))|_{(y,\eta)=(0,0)} = 0, \tag{5.4}$$

$$\operatorname{sgn} \partial^2_{(y,\eta)}(F(y) - y \cdot \eta + \psi(0,\eta))|_{(y,\eta)=(0,0)} = 0. \tag{5.5}$$

Proof. We see that the Hessian matrix is computed as

$$\partial^2_{(y,\eta)}(F(y) - y \cdot \eta + \psi(0,\eta))|_{(y,\eta)=(0,0)} = \begin{pmatrix} f'(0) & -1 \\ -1 & \partial_\eta^2 \psi(0,0) \end{pmatrix}. \tag{5.6}$$

The determinant of this matrix is -1 if F vanishes identically. This proves (5.4). For (5.5), we will calculate the determinant of (5.6) as

$$\begin{aligned} &\det(\partial^2_{(y,\eta)}(F(y) - y \cdot \eta + \psi(0,\eta)))|_{(y,\eta)=(0,0)} \\ &\quad = -\frac{\partial_\xi(\kappa^* p_1)(0,0)}{\partial_\xi p_1(0,0)} \frac{\partial_\xi p_2(0,0)}{\partial_\xi(\kappa^* p_2)(0,0)}. \end{aligned} \tag{5.7}$$

We observe that (5.5) is equivalent to this determinant being negative. Then, (5.5) follows from (3.1), (5.7), and the elementary identity $\operatorname{sgn}\left(\frac{a}{b} \cdot \frac{c}{d}\right) = \operatorname{sgn}\left(\frac{b}{a} \cdot \frac{c}{d}\right)$.

In the following, we prove the identity (5.7). Let $g_j = g_j(x)$ be the function defined near $x = 0$ by

$$\kappa(g_1(x), 0) = (x, \phi_1'(x)), \quad \kappa(g_2(x), f(g_2(x))) = (x, \phi_2'(x)).$$

We observe that $g_1(0) = g_2(0) = 0$ holds because $\kappa(0,0) = (0,0)$ and (2.15). We claim that

$$\det \partial^2_{(y,\eta)}(F(y) - y \cdot \eta + \psi(0,\eta)) = -\frac{g_1'(0)}{g_2'(0)} \tag{5.8}$$

holds. The definition of the generating function, $\kappa(\partial_\eta\psi(x,\eta),\eta) = (x, \partial_x\psi(x,\eta))$, implies that one has $g_1(x) = \partial_\eta\psi(x,0)$ and $g_2(x) = \partial_\eta\psi(x, f(g_2(x))$. By differentiating in x, we obtain

$$\begin{aligned} g_1'(x) &= \partial_x\partial_\eta\psi(x,0),\\ g_2'(x) &= \partial_x\partial_\eta\psi(x, f(g_2(x))) + g_2'(x) f'(g_2(x))\partial_\eta^2\psi(x, f(g_2(x))). \end{aligned}$$

We deduce, in particular, that $g_1'(0) = \partial_x\partial_\eta\psi(0,0)$ from the first identity, and

$$f'(0)\partial_\eta^2\psi(0,0) - 1 = -\frac{g_1'(0)}{g_2'(0)}$$

where we use $g_1(0) = g_2(0) = 0$ and $f(0) = 0$. This with (5.6) proves (5.8).

By the same calculation as (2.8),

$$g_j'(0) = \frac{\partial_\xi(\kappa^* p_j)(0,0)}{\partial_\xi p_j(0,0)} \quad \text{for } j = 1,2. \tag{5.9}$$

Combining (5.8) with (5.9), we obtain (5.7). ■

Proof of Lemma 5.2. Let $\tilde{\gamma}_j^\bullet$ ($(j,\bullet) \in \{1,2\}\times\{\flat,\sharp\}$) be the four connected components of $(\{\xi = 0\}\cap\{\xi = f(x)\})\setminus\{0\}$. Our assumption $c = (\partial_\xi(\kappa^* p_2)(0,0))^{-1} > 0$ implies $\kappa(\tilde{\gamma}_j^\bullet) = \gamma_j^\bullet$ in a neighborhood of $(0,0)$. Since $\mathtt{W}_j^\bullet$ and $w_j^\bullet$ are WKB solutions associated with $\tilde{\gamma}_j^\bullet$ and $\gamma_j^\bullet$ respectively, the uniqueness of the microlocal solutions on each $\gamma_j^\bullet$ ($(j,\bullet) \in \{1,2\}\times\{\flat,\sharp\}$) (Proposition 2.16 (i)) implies that we can find constants $\beta_j^\bullet$ satisfying (5.1). Therefore, it suffices to prove the asymptotic formula (5.2) of $\beta_j^\bullet$.

By Proposition 2.16 (ii) and (iii), we have

$$w_1^\bullet(x) = e^{\frac{i}{h}\phi_1(x)}\begin{pmatrix} c_1^\bullet \\ 0\end{pmatrix} + \mathcal{O}(h), \quad w_2^\bullet(x) = e^{\frac{i}{h}\phi_2(x)}\begin{pmatrix} 0 \\ c_2^\bullet\end{pmatrix} + \mathcal{O}(h) \quad \text{with} \tag{5.10}$$

$$c_j^\bullet = |\partial_{(x,\xi)} p_j(0,0)|^{\frac12}|\partial_\xi p_j(0,0)|^{-\frac12}. \tag{5.11}$$

Since $\mathtt{W}_1^\bullet = {}^{\mathsf{T}}(\mathbb{1},0) + \mathcal{O}(h)$ and $\mathtt{W}_2^\bullet = e^{\frac{i}{h}F(\cdot)}\cdot{}^{\mathsf{T}}(0,\mathbb{1}) + \mathcal{O}(h)$, Proposition 2.10 with Lemma 5.4 implies

$$\mathcal{F}\mathtt{W}_1^\bullet \equiv \begin{pmatrix} F_1(\mathbb{1}) \\ 0\end{pmatrix} + \mathcal{O}(h), \quad \mathcal{F}\mathtt{W}_2^\bullet \equiv \begin{pmatrix} 0 \\ F_2(e^{\frac{i}{h}F(\cdot)})\end{pmatrix} + \mathcal{O}(h) \tag{5.12}$$

for $\bullet = \flat,\sharp$. Since F_1 and F_2 are microlocally unitary Fourier integral operators in the sense of Definition 2.8 (iii), Proposition 2.10 (with $\tilde{\phi}_1(x) = 0$ and $\tilde{\phi}_2(x) = F(x)$) implies

$$F_1(\mathbb{1})(x) = \sigma_1(x,h)e^{\frac{i}{h}\phi_1(x)}, \quad F_2(e^{\frac{i}{h}F(\cdot)}) = \sigma_2(x,h)e^{\frac{i}{h}\phi_2(x)} \tag{5.13}$$

with

$$\sigma_j(0,h) = \left|\frac{\partial_\xi(\kappa^* p_j)(0,0)}{\partial_\xi p_j(0,0)}\right|^{\frac{1}{2}} + \mathcal{O}(h) \quad (j = 1,2), \tag{5.14}$$

where the index υ appearing in Proposition 2.10 is equal to 0 by virtue of Lemma 5.4. By (5.1), (5.10), (5.12), and (5.13), it must hold that

$$\beta_j^\bullet c_j^\bullet = \sigma_j(0,h) + \mathcal{O}(h^\infty).$$

Combining this with (5.11) and (5.14), we obtain

$$\beta_j^\bullet = |\partial_\xi(\kappa^* p_j)(0,0)|^{\frac{1}{2}} |\partial_{(x,\xi)} p_j(0,0)|^{-\frac{1}{2}} + \mathcal{O}(h).$$

Finally, we recall that we have $\partial_\xi(\kappa^* p_1)(0,0) = 1$ and $c = (\partial_\xi(\kappa^* p_2)(0,0))^{-1}$, which shows (5.2). ∎

5.2. Proof of Proposition 5.1 when c is negative

Let us consider the case with negative c. The major difference is the fact that the Hamiltonian flow induced by the Hamiltonian $\xi - f(x)$ has the inverse direction to that by $\kappa^* p_2(x,\xi) = a_0(x,\xi)(\xi - f(x))$ when $c = a_0(0,0)^{-1}$ is negative.

Lemma 5.5. *Assume $c < 0$. There exist constants $\beta_1^\flat$, $\tilde{\beta}_2^\flat$, $\beta_1^\sharp$, and $\tilde{\beta}_2^\sharp$ such that*

$$\mathcal{F}\mathtt{W}_1^\bullet \equiv \beta_1^\bullet w_1^\bullet \quad \textit{microlocally near } \gamma_1^\bullet \quad (\bullet = \flat, \sharp), \tag{5.15}$$

and

$$\begin{aligned} \mathcal{F}\mathtt{W}_2^\flat &\equiv \tilde{\beta}_2^\sharp w_2^\sharp \quad \textit{microlocally near } \gamma_2^\sharp, \\ \mathcal{F}\mathtt{W}_2^\sharp &\equiv \tilde{\beta}_2^\flat w_2^\flat \quad \textit{microlocally near } \gamma_2^\flat. \end{aligned} \tag{5.16}$$

They admit the asymptotic formula

$$\beta_1^\bullet = |\partial_{(x,\xi)} p_1(0,0)|^{-\frac{1}{2}} + \mathcal{O}(h), \quad \tilde{\beta}_2^\bullet = |c\,\partial_{(x,\xi)} p_2(0,0)|^{-\frac{1}{2}} + \mathcal{O}(h), \tag{5.17}$$

as $h \to +0$.

Proof. Let $\tilde{\gamma}_j^\bullet$ $((j,\bullet) \in \{1,2\} \times \{\flat,\sharp\})$ be four connected components of $(\{\xi = 0\} \cap \{\xi = f(x)\}) \setminus \{0\}$. Our assumption $c = (\partial_\xi(\kappa^* p_2)(0,0))^{-1} < 0$ implies

$$\kappa(\tilde{\gamma}_1^\bullet) = \gamma_1^\bullet, \quad \kappa(\tilde{\gamma}_2^\flat) = \gamma_2^\sharp, \quad \kappa(\tilde{\gamma}_2^\sharp) = \gamma_2^\flat$$

in a neighborhood of $(0,0)$. This is the main difference from the case $c > 0$. From these relations, we obtain (5.15) and (5.16) with constants $\beta_1^\bullet$ and $\tilde{\beta}_2^\bullet$. The asymptotic formula (5.17) of $\beta_j^\bullet$ and $\tilde{\beta}_2^\bullet$ can be proved similarly to the proof of Lemma 5.2. ∎

Proof of Proposition 5.1 *when* $c < 0$. Let w satisfy the microlocal Cauchy problem (1.4) for the original operator $\mathcal{P}$ with the initial condition given by $\alpha_1^\flat$, $\alpha_2^\flat$, and let $\alpha_1^\sharp$ and $\alpha_2^\sharp$ be the outgoing data (1.5) of w. By the definition (5.1) of $\beta_j^\bullet$ and the transfer matrix, we have shown the identity

$$\begin{pmatrix}\alpha_1^\sharp\\ \alpha_2^\flat\end{pmatrix} = \operatorname{diag}(\beta_1^\sharp\tilde{\beta}_2^\flat)T_{\mathrm{nf}}\operatorname{diag}(\beta_1^\flat\tilde{\beta}_2^\sharp)^{-1}\begin{pmatrix}\alpha_1^\flat\\ \alpha_2^\sharp\end{pmatrix}, \quad \begin{pmatrix}\alpha_1^\sharp\\ \alpha_2^\sharp\end{pmatrix} = T\begin{pmatrix}\alpha_1^\flat\\ \alpha_2^\flat\end{pmatrix}.$$

An algebraic computation[14] shows

$$T = \operatorname{diag}(\beta_1^\sharp\tilde{\beta}_2^\sharp)\frac{1}{1-t^{21}t_{12}}\begin{pmatrix}t_{11} & t_{12}t^{22}\\ t^{21}t_{11} & t^{22}\end{pmatrix}\operatorname{diag}(\beta_1^\flat\tilde{\beta}_2^\flat)^{-1}, \tag{5.18}$$

where t_{jk} are the (j,k)-entry of T_{nf}, and t^{jk} are those of T_{nf}^{-1}. Now, we observe that

$$\frac{1}{1-t^{21}t_{12}}\begin{pmatrix}t_{11} & t_{12}t^{22}\\ t^{21}t_{11} & t^{22}\end{pmatrix} = I + \begin{pmatrix}0 & t_{12}\\ t_{21} & 0\end{pmatrix} + \mathcal{O}\left(h^{\frac{2}{m+1}}\left(\log\frac{1}{h}\right)^{\delta_{m,1}}\right)$$

hold since one has

$$T_{\mathrm{nf}} = I + \begin{pmatrix}0 & t_{12}\\ t_{21} & 0\end{pmatrix} + \mathcal{O}\left(h^{\frac{2}{m+1}}\left(\log\frac{1}{h}\right)^{\delta_{m,1}}\right),$$

$$T_{\mathrm{nf}}^{-1} = I - \begin{pmatrix}0 & t_{12}\\ t_{21} & 0\end{pmatrix} + \mathcal{O}\left(h^{\frac{2}{m+1}}\left(\log\frac{1}{h}\right)^{\delta_{m,1}}\right)$$

and $t_{12}, t_{21} = \mathcal{O}(h^{\frac{1}{m+1}})$ according to Proposition 4.2 and the Neumann series expansion. We obtain Proposition 5.1 by substituting these formulae and (5.17) into (5.18).

■

[14] The identity

$$\begin{pmatrix}x\\ w\end{pmatrix} = \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}y\\ z\end{pmatrix} \iff \begin{pmatrix}y\\ z\end{pmatrix} = \begin{pmatrix}a^{11} & a^{12}\\ a^{21} & a^{22}\end{pmatrix}\begin{pmatrix}x\\ w\end{pmatrix}$$

with $(a^{jk})_{j,k} = (a_{jk})_{j,k}^{-1} \in \mathcal{M}_2(\mathbb{C})$ implies $z = a^{21}x + a^{22}w = a^{21}(a_{11}y + a_{12}z) + a^{22}w$. This gives $z = (1 - a_{12}a^{21})^{-1}(a^{21}a_{11}y + a^{22}w)$. By substituting this identity into $x = a_{11}y + a_{12}z$, x is also written as a linear combination of y and w. Then we obtain

$$\begin{pmatrix}x\\ z\end{pmatrix} = \frac{1}{1-a_{12}a^{21}}\begin{pmatrix}a_{11} & a_{12}a^{22}\\ a^{21}a_{11} & a^{22}\end{pmatrix}\begin{pmatrix}y\\ w\end{pmatrix}.$$

5.3. Computation of the asymptotics

We provide an outline of the computation of the asymptotics of the transfer matrix given in Theorem 1.4. It is deduced by combining the formulae written in Theorem 3.1, Propositions 4.2, 5.1, and the following lemma.

Lemma 5.6. *One has*

$$s = \operatorname{sgn} c, \tag{5.19}$$

where s and c are defined in (1.10) *and Theorem* 3.1 (iv)*, respectively. One also has*

$$H_{p_1}^m p_2(0,0) = -c^{m-1} H_{p_2}^m p_1(0,0). \tag{5.20}$$

Proof. We prove the two equation separately.

Proof of (5.19). According to the condition (3.1), the sign of c coincides with that of the product $(\partial_\xi p_1(0,0))(\partial_\xi p_2(0,0))$ under the condition (2.21) (the product does not vanish). By definition of $\Theta(p_j) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the vector $\tau(\Theta(p_j))$ is the unit vector parallel to the Hamiltonian vector $H_{p_j}(0,0)$. This implies that the sign of $(H_{p_j}(0,0), \tau(\Theta(p_j)))_{\mathbb{R}^2}$ coincides with that of $(\cos(\Theta(p_j)))\partial_\xi p_j(0,0)$ if the latter does not vanish. Note that the condition (2.21): $\partial_\xi p_j(0,0) \neq 0$ is equivalent to $\Theta(p_j) \neq -\frac{\pi}{2}$. This condition also implies $\cos(\Theta(p_j)) > 0$, and consequently (5.19).

Otherwise, we use the rotation map κ_θ as in Proposition 2.16. Recall that the value

$$(H_{p_{j,\theta}}(0,0), \tau(\Theta(p_{j,\theta})))_{\mathbb{R}^2} = (H_{p_j}(0,0), \tau(\Theta(p_j)))_{\mathbb{R}^2}, \quad (p_{j,\theta} := \kappa_\theta^* p_j)$$

is invariant for small $\theta \geq 0$. For small $\theta > 0$, the equality $\Theta(p_{j,\theta}) = \Theta(p_j) + \theta$ implies $\partial_\xi p_{j,\theta}(0,0) \neq 0$. This reduces the problem to the case which is already treated above.

Proof of (5.20). Since the identity (5.20) is trivial for the transversal case ($m = 1$), we only prove it for $m > 1$. By employing κ such that $\kappa^* p_1(x,\xi) = \xi$, one can easily check for each $k \geq 1$ that

$$\begin{aligned} &H_{p_1}^k p_2(0,0) = \partial_x^k(\kappa^* p_2)(0,0), \\ &H_{p_2}^k p_1(0,0) = -H_{\kappa^* p_2}^{k-1} \partial_x(\kappa^* p_2)(0,0) = -[(\partial_\xi(\kappa^* p_2))\partial_x]^{k-1}\partial_x(\kappa^* p_2)(0,0). \end{aligned} \tag{5.21}$$

By the definition of the contact order, $H_{p_1}^k p_2(0,0) = \partial_x^k(\kappa^* p_2)(0,0) = 0$ holds for $1 \leq k \leq m-1$. Plugging this into the second equality of (5.21), one obtains

$$H_{p_2}^m p_1(0,0) = -c^{-(m-1)}\partial_x^m(\kappa^* p_2)(0,0) = -c^{-(m-1)} H_{p_1}^m p_2(0,0). \qquad \blacksquare$$

Proof of Theorem 1.4. Due to Proposition 2.13 (iv) and the definition, the transfer matrix T is invariant under the conjugation of the metaplectic operator M_θ for sufficiently small $\theta > 0$. Moreover, we can find sufficiently small $\theta > 0$ such that the symbol $p_{j,\theta} := \kappa_\theta^* p_j$ of $M_\theta^{-1} P_j M_\theta$ satisfies (2.21) by Condition A, where κ_θ is the rotation defined in (2.9). Thus, we may assume (2.21).

As in Theorem 3.1 (iv), $r_1(0,0)$ and $r_2(0,0)$ coincide with $q_1(0,0)$ and $cq_2(0,0)$ modulo $\mathcal{O}(h)$. Then, by combining Propositions 4.2 and 5.1, the anti-diagonal entries of the transfer matrix T are given by $q_1(0,0)\omega_1$ and $q_2(0,0)\omega_2$ modulo $\mathcal{O}(h)$, where ω_1 and ω_2 are defined by

$$\omega_1 = \left|\frac{c}{c'}\right|^{\frac{1}{2}} \omega, \quad \omega_2 = |cc'|^{\frac{1}{2}} \bar{\omega}. \tag{5.22}$$

Here, ω is defined by (4.8) in terms of the m-th derivative at $x = 0$ of the function $f(x)$ appearing in the normal form. According to the formula (3.2) in Theorem 3.1, this function f satisfies $f^{(m)}(0) = -cH_{p_1}^m p_2(0,0)$. Then ω is rewritten as

$$\omega = 2|c|^{-\frac{1}{m+1}} \mu_m\Big(\frac{-\operatorname{sgn}(sH_{p_1}^m p_2(0,0))\pi}{2(m+1)}\Big)\mathbf{\Gamma}\Big(\frac{m+2}{m+1}\Big)\Big(\frac{(m+1)!}{|H_{p_1}^m p_2(0,0)|}\Big)^{\frac{1}{m+1}}.$$

Here, we also applied (5.19). The asymptotic formula for ω_1 given in Theorem 1.4 follows from this with (5.22) and the definition of c and c'. More concretely, one obtains the identity $\left|\frac{c}{c'}\right|^{\frac{1}{2}}|c|^{-\frac{1}{m+1}} = |c'|^{-\frac{1}{m+1}}$ with $c' = |\partial_{(x,\xi)} p_1(0,0)|/|\partial_{(x,\xi)} p_2(0,0)|$, which is obvious for $m = 1$, and is implied by $|c| = c'$ in the tangential case $m \geq 2$ (see Theorem 3.1 (v)).

For the formula of ω_2, we also use (5.20). The difference between ω_1 and ω_2 caused by the complex conjugate only appears in μ_m and $\overline{\mu_m}$ since the other factors are real. Moreover, μ_m is real when m is even. When m is odd, one has $\overline{\mu_m(\theta)} = \mu_m(-\theta)$ on one hand, and

$$H_{p_1}^m p_2(0,0) = -c^{m-1} H_{p_2} p_1(0,0) = -|c|^{m-1} H_{p_2} p_1(0,0),$$

on the other hand. We conclude here that

$$\overline{\mu_m\Big(\frac{-\operatorname{sgn}(sH_{p_1}^m p_2(0,0))\pi}{2(m+1)}\Big)} = \mu_m\Big(\frac{-\operatorname{sgn}(sH_{p_2}^m p_1(0,0))\pi}{2(m+1)}\Big).$$

Concerning the absolute value, (5.22) implies

$$|\omega_2| = |c'||\omega_1|.$$

The equality (5.20) shows

$$|c'|\Big(\frac{|\partial_{(x,\xi)} p_2(0,0)|}{|\partial_{(x,\xi)} p_1(0,0)|}\frac{(m+1)!}{|H_{p_1}^m p_2(0,0)|}\Big)^{\frac{1}{m+1}} = \Big(\frac{|\partial_{(x,\xi)} p_1(0,0)|}{|\partial_{(x,\xi)} p_2(0,0)|}\frac{(m+1)!}{|H_{p_2}^m p_1(0,0)|}\Big)^{\frac{1}{m+1}}.$$

This is the only factor of ω_2 different from ω_1 other than μ_m. ∎

A. The stationary phase method for a degenerate phase

Lemma A.1. *Let $I \subset \mathbb{R}$ be a bounded, h-independent interval. Let $F \in C^\infty(I, \mathbb{R})$ be a function whose derivative vanishes nowhere but at $x = 0$, at a finite order $m \geq 1$. Let $a \in C^\infty(I, \mathbb{C})$. Then*

$$\int_I a(y) e^{\frac{i}{h} F(y)}\, dy \leq C\Big[h^{\frac{1}{m+1}} \|a\|_{L^\infty(I)} + h^{\frac{2}{m+1}} \log\Big(\frac{1}{h}\Big)^{\delta_{m,1}} \|a'\|_{L^\infty(I)} \Big], \tag{A.1}$$

where $\delta_{m,1}$ is the Kronecker delta. Moreover, if $J \Subset I \cap (0, +\infty)$ is h-independent, then for all $I \ni x_0 < 0$ and $x \in J$,

$$\begin{aligned}
&\int_{x_0}^{x} a(y) e^{\frac{i}{h}(F(y) - F(0))}\, dy \\
&\quad = 2\mu_m\Big(\frac{\operatorname{sgn}(F^{(m+1)}(0))\pi}{2(m+1)}\Big) \mathbf{\Gamma}\Big(\frac{m+2}{m+1}\Big)\Big(\frac{m+1}{|F^{(m+1)}(0)|}\Big)^{\frac{1}{m+1}} a(0) h^{\frac{1}{m+1}} \\
&\qquad + \mathcal{O}_{L^\infty(J)}(h^{\frac{2}{m+1}}),
\end{aligned} \tag{A.2}$$

where $\mathbf{\Gamma}$ is Euler's Gamma function and μ_m is defined by

$$\mu_m(\theta) := 2^{-1}(e^{i\theta} + e^{i(-1)^{m+1}\theta})$$

for θ in $\mathbb{R}$.

Proof. The estimate (A.1) comes from an integration by parts. For the expansion (A.2), perform an h-independent change of variable on $\int e^{\frac{i}{h} y^{m+1}}\, dy$. The last integral is computed with the residue theorem (Fresnel-type integral). We send the reader to [20, Chap. VIII] for more details. ∎

B. Construction of WKB solutions for the reduced equation

We consider the reduced operator of Section 4 defined by

$$P = P_{\mathrm{nf}} = \begin{pmatrix} hD_x & hR_1 \\ hR_2 & hD_x - f(x) \end{pmatrix}.$$

Let Ω be a neighborhood of $(0,0)$. Let $\rho_j^\bullet \in \gamma_j^\bullet$ for $\bullet = \flat, \sharp$ and $j = 1, 2$. We define WKB solutions near each of these four points. Here, $\gamma_j^\flat$ denotes $\Gamma_j \cap \{\xi < 0\}$ and $\gamma_j^\sharp$ denotes $\Gamma_j \cap \{\xi > 0\}$ as illustrated on Figure 5.

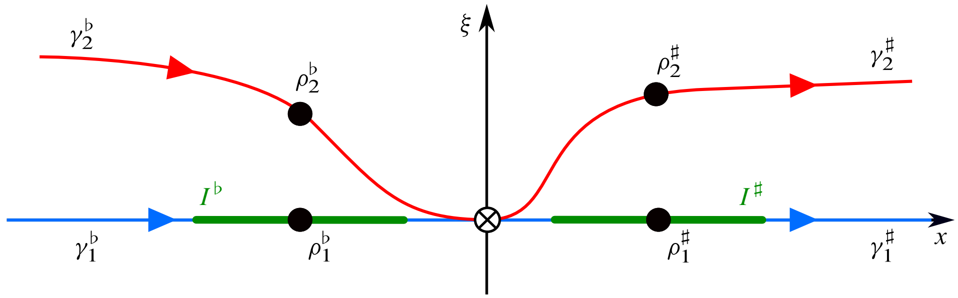

Figure 5. Crossing of Hamiltonian flows for the reduced case.

Proposition B.1. *Let $I^\flat \Subset (-\infty, 0)$ and $I^\sharp \Subset (0, +\infty)$ be intervals such that $\rho_j^\bullet \in I^\bullet \times \mathbb{R}_\xi$. For each $j = 1, 2$ and $\bullet = \flat, \sharp$ there exist a microlocal WKB solution $\mathtt{W}_j^\bullet$ of $Pu = 0$ near $\rho_j^\bullet$ satisfying*

$$\mathtt{W}_1^\bullet(x) = \begin{pmatrix} 1 + \mathcal{O}_{C_b^\infty}(h) \\ \mathcal{O}_{C_b^\infty}(h) \end{pmatrix}$$

and

$$\mathtt{W}_2^\bullet(x) = e^{\frac{i}{h}F(x)} \begin{pmatrix} \mathcal{O}_{C_b^\infty}(h) \\ 1 + \mathcal{O}_{C_b^\infty}(h) \end{pmatrix} \quad \textit{for } x \in I^\bullet,$$

where $F(x) = \int_0^x f(y)\,dy$. Moreover,

$$\mathrm{WF}_h(\mathtt{W}_1^\bullet(x)) \subset \{(x, 0) \in T^*\mathbb{R} \mid x \in \mathbb{R}\}$$

and

$$\mathrm{WF}_h(\mathtt{W}_2^\bullet(x)) \subset \{(x, f(x)) \mid x \in \mathbb{R}\}$$

by Example 2.7.

Proof. The proof is more or less standard and is known as the WKB construction. The construction of the four solutions ($\bullet = \flat, \sharp$ and $j = 1, 2$) is very similar, so we only construct $\mathtt{W}_2^\flat$. Without loss of generality, we may assume that the integral kernel k_j of R_j satisfies $\operatorname{supp}(k_j) \Subset I^\flat \times I^\flat$. We look for $\mathtt{W}_2^\flat$ under the WKB form $\mathtt{W}_2^\flat(x) \sim e^{\frac{i}{h}\phi(x)} \sum_{k=0}^\infty \mathtt{W}_k(x) h^k$ where the amplitude $\mathtt{W}_k(x) = (\mathtt{W}_{k,1}(x), \mathtt{W}_{k,2}(x))$ is vector-valued. For a fixed $N \in \mathbb{N}$, the function $e^{-\frac{i}{h}\phi(x)} P e^{\frac{i}{h}\phi(x)} \sum_{k=0}^N h^k \mathtt{W}_k(x)$ is calculated as follows:

$$e^{-\frac{i}{h}\phi(x)} P e^{\frac{i}{h}\phi(x)} \sum_{k=0}^{N} h^k \mathtt{W}_k(x) = \sum_{k=0}^{N+1} h^k \mathtt{T}_k, \tag{B.1}$$

with

$$\mathtt{T}_0 = \begin{pmatrix} \phi'(x)\mathtt{W}_{0,1}(x) \\ (\phi'(x) - f(x))\mathtt{W}_{0,2}(x) \end{pmatrix}, \quad \mathtt{T}_{N+1} = \begin{pmatrix} -i\mathtt{W}'_{N,1}(x) + (R_1)_f \mathtt{W}_{N,2}(x) \\ -i\mathtt{W}'_{N,2}(x)(x) + (R_2)_f \mathtt{W}_{N,1}(x) \end{pmatrix},$$

and

$$\mathtt{T}_k = \begin{pmatrix} \phi'(x)\mathtt{W}_{k,1}(x) - i\mathtt{W}'_{k-1,1}(x) + (R_1)_f \mathtt{W}_{k-1,2}(x) \\ (\phi'(x) - f(x))\mathtt{W}_{k,2} - i\mathtt{W}'_{k-1,2}(x)(x) + (R_2)_f \mathtt{W}_{k-1,1}(x) \end{pmatrix} \quad (1 \le k \le N).$$

Here, $(R_j)_f$ is the pseudodifferential operator obtained from the conjugation of R_j by $e^{\frac{i}{h}\phi(x)}$ and given by Lemma 2.12. Remark that the integral kernel of $(R_j)_f$ is $e^{\frac{i}{h}(\phi(y)-\phi(x))}k_j(x,y)$, which is also compactly supported in $I^\flat \times I^\flat$. For $\mathtt{W}_2^\flat$ to be a microlocal solution, we require the terms for $0 \le k \le N$ of the above to vanish. We may assume $\mathtt{W}_0 \neq 0$. A possible choice of ϕ and $\mathtt{W}_0$ for the first term to vanish is

$$\phi(x) := \int_0^x f(y)\,dy \quad \text{and} \quad (\mathtt{W}_{0,1}(x), \mathtt{W}_{0,2}(x)) := (0,1) \text{ for all } x \in \mathbb{R}.$$

The equation $\phi' - f = 0$ satisfied by ϕ is called the *eikonal equation* associated with[15] Γ_2. The system of *transport equations* is obtained by requiring the terms for $1 \le k \le N$ in the sum of (B.1) to vanish. It is solved by amplitudes $\mathtt{W}_k \in C^\infty(I^\flat)$ defined inductively by

$$\begin{cases} \mathtt{W}_{k+1,1}(x) := f(x)^{-1}(i\mathtt{W}'_{k,1}(x) - (R_1)_f \mathtt{W}_{k,2}(x)), \\ \mathtt{W}_{k+1,2}(x) := -i\displaystyle\int_{x_0}^x (R_2)_f \mathtt{W}_{k+1,1}(y)\,dy, \end{cases} \qquad \text{for all } x \in I^\flat,$$

where $x_0 \in I^\flat$ is any point in $I^\flat$. We now define via a Borel summation (a slightly modified version of [22, Theorem 4.15]) a smooth function

$$\mathtt{W}_2^\flat \sim e^{\frac{i}{h}\phi(x)} \sum_{k=0}^{\infty} \mathtt{W}_k(x) h^k.$$

Finally, using the support properties of $\mathtt{W}_k$ combined with [22, Theorem 8.13], we show that $\mathtt{W}_2^\flat$ is indeed a microlocal solution of $Pu = 0$ near $\rho_2^\flat$. ∎

[15] Considering the eikonal equation $\phi' = 0$ associated with Γ_1 would yield a construction of $\mathtt{W}_1^\flat$.

C. Generality of the model

In this section, we show how more general models involving non-diagonal matrix-valued operators can be reduced to the diagonal framework considered in this manuscript. Let $L(x,\xi)$ be a 2×2 Hermitian matrix-valued C^∞ function, not necessarily diagonal.[16] If[17] in a small neighborhood of $(x,\xi)=(0,0)$, the eigenvalues and an orthonormal pair of associated eigenvectors of $L(x,\xi)$ are given by real-valued C^∞ functions $\lambda_1(x,\xi)$ and $\lambda_2(x,\xi)$ and by vector-valued C^∞ functions $\mathrm{v}_1(x,\xi)$ and $\mathrm{v}_2(x,\xi)$, then the microlocal equation $L^w(x,hD_x)\tilde{u}\equiv 0$ near $(0,0)$ is reduced to

$$p^w(x,hD_x)u\equiv 0,\quad p(x,\xi)=\begin{pmatrix}\lambda_1(x,\xi) & 0\\ 0 & \lambda_2(x,\xi)\end{pmatrix}+\mathcal{O}(h)\quad \text{near } (x,\xi)=(0,0),$$

with the change of the unknown function $u(x)=V^w(x,hD_x)\tilde{u}(x)$, where we write $V(x,\xi)=(\mathrm{v}_1(x,\xi),\ \mathrm{v}_2(x,\xi))$.

In general, the two eigenvalues of $L(x,\xi)$ cross at $(x,\xi)=(0,0)$ if and only if $L(0,0)=\frac{1}{2}(\operatorname{tr}L(0,0))I_2$. In other words, the trace-free part of $L(x,\xi)$, given by

$$M(x,\xi):=L(x,\xi)-\frac{1}{2}(\operatorname{tr}L(x,\xi))I_2,$$

vanishes at $(0,0)$.

Assumption 1. Let $L\in\mathcal{M}_2(C_b^\infty)$ be a Hermitian matrix-valued symbol whose trace free part $M(x,\xi)$ vanishes on a codimension-one submanifold of $T^*\mathbb{R}$ satisfying the following conditions:

- there exists a function $y\in C^\infty(T^*\mathbb{R};\mathbb{R})$ such that $y(0,0)=0$, $\partial y(0,0)\neq 0$, and such that
$$\{(x,\xi)\in\Omega;\ M(x,\xi)=O_{2\times 2}\}=\{(x,\xi)\in\Omega;\ y(x,\xi)=0\},$$
holds with some neighborhood $\Omega\subset T^*\mathbb{R}$ of $(0,0)$;
- there exists a positive integer n such that $\partial_{x,\xi}^\alpha M(0,0)=O_{2\times 2}$ holds for any multi-index $\alpha\in\mathbb{N}^2$ with $|\alpha|\leq n-1$, but $\partial_{x,\xi}^{\alpha_0}M(0,0)\neq O_{2\times 2}$ holds for some $\alpha_0\in\mathbb{N}^2$ with $|\alpha_0|=n$;
- there exists a positive integer ℓ such that $H_{\operatorname{tr}L}^k y(0,0)=0$ holds for $0\leq k\leq \ell-1$ and $H_{\operatorname{tr}L}^\ell y(0,0)\neq 0$.

[16]Adding an $\mathcal{O}(h)$ non-Hermitian term does not affect the result.

[17]This smooth diagonalization is not always possible. There are many counterexamples when some eigenvalues cross with each other. We show a sufficient condition and examples for this diagonalization to be smooth.

Proposition C.1. *Under Assumption 1, we have the following.*

(1) *There exist real-valued C^∞ functions $\lambda_1(x,\xi)$ and $\lambda_2(x,\xi)$ such that they are the eigenvalues of $L(x,\xi)$ in a small neighborhood $\Omega_0 \subset \Omega$ of $(0,0)$, and that $\lambda_1(x,\xi) \neq \lambda_2(x,\xi)$ holds on $\{(x,\xi) \in \Omega_0;\ y(x,\xi) \neq 0\}$.*

(2) *Put $m := n\ell$. The eigenvalues λ_1 and λ_2 satisfy*

$$\begin{aligned}&\lambda_1(0,0) = \lambda_2(0,0),\\ &H_{\lambda_1}^k \lambda_2(0,0) = 0 \quad \text{for } 1 \le k \le m-1, \quad H_{\lambda_1}^m \lambda_2(0,0) \neq 0.\end{aligned}$$

(3) *There exist two $\mathbb{C}^2$-valued C^∞ functions $\mathrm{v}_1(x,\xi)$ and $\mathrm{v}_2(x,\xi)$ forming an orthonormal pair of normalised eigenvectors associated respectively with $\lambda_1(x,\xi)$ and $\lambda_2(x,\xi)$ of $L(x,\xi)$ in Ω_0.*

Proof. Under Assumption 1, the matrix $M(x,\xi)$ is factorised by $y(x,\xi)^n$. More precisely, there exists a 2×2 Hermitian, trace-free, matrix-valued C^∞ function $\widetilde{M}(x,\xi)$ such that

$$M(x,\xi) = y(x,\xi)^n \widetilde{M}(x,\xi),$$

holds in Ω and $\widetilde{M}(0,0) \neq O_{2\times 2}$. This follows from the Taylor expansion in a coordinate system including y.

Since $\widetilde{M}(x,\xi)$ is a trace-free Hermitian matrix, there exist two smooth functions $g_1(x,\xi)$ and $g_2(x,\xi)$ such that g_1 is real and

$$\widetilde{M}(x,\xi) = \begin{pmatrix} g_1(x,\xi) & g_2(x,\xi) \\ \overline{g_2(x,\xi)} & -g_1(x,\xi) \end{pmatrix}.$$

The eigenvalues $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ of $\widetilde{M}(x,\xi)$ are explicitly given by

$$\begin{aligned}&r(x,\xi) := \sqrt{g_1(x,\xi)^2 + |g_2(x,\xi)|^2},\\ &\tilde{\lambda}_1(x,\xi) := r(x,\xi), \quad \tilde{\lambda}_2(x,\xi) := -r(x,\xi).\end{aligned}$$

Put $t_\pm(x,\xi) := \sqrt{2r(x,\xi)(r(x,\xi) \pm g_1(x,\xi))}$. The fact $\widetilde{M}(0,0) \neq O_{2\times 2}$ implies that $g_1(0,0)^2 + |g_2(0,0)|^2$ is positive, and hence (by continuity) $g_1(x,\xi)^2 + |g_2(x,\xi)|^2$ is positive near $(0,0)$. Thus, $r(x,\xi)$ is C^∞ there. Either $t_+(0,0) \neq 0$ or $t_-(0,0) \neq 0$ holds, so (again by continuity) the corresponding $t_\pm$ does not vanish in a neighborhood of $(0,0)$. More precisely, $t_\pm(0,0) \neq 0$ provided that either $g_2(0,0) \neq 0$ or $\pm g_1(0,0) > 0$ holds. In each case, the normalised eigenvectors v_1 and v_2 associated with $\tilde{\lambda}_1$ and with $\tilde{\lambda}_2$ are given by

$$\mathrm{v}_1(x,\xi) := \frac{1}{t_+(x,\xi)} \begin{pmatrix} r(x,\xi) + g_1(x,\xi) \\ \overline{g_2(x,\xi)} \end{pmatrix},$$

$$\mathrm{v}_2(x,\xi) := \frac{1}{t_+(x,\xi)}\begin{pmatrix} -g_2(x,\xi) \\ r(x,\xi) + g_1(x,\xi) \end{pmatrix},$$

when $t_+ \neq 0$, and

$$\mathrm{v}_1(x,\xi) := \frac{1}{t_-(x,\xi)}\begin{pmatrix} g_2(x,\xi) \\ -r(x,\xi) + g_1(x,\xi) \end{pmatrix},$$

$$\mathrm{v}_2(x,\xi) := \frac{1}{t_-(x,\xi)}\begin{pmatrix} r(x,\xi) - g_1(x,\xi) \\ \overline{g_2(x,\xi)} \end{pmatrix},$$

when $t_- \neq 0$.

With the unitary matrix-valued function $V(x,\xi) := (\mathrm{v}_1(x,\xi), \mathrm{v}_2(x,\xi))$, the matrix $L(x,\xi)$ is also diagonalised as

$$V(x,\xi)^* L(x,\xi) V(x,\xi) = \begin{pmatrix} \lambda_1(x,\xi) & 0 \\ 0 & \lambda_2(x,\xi) \end{pmatrix},$$

with the eigenvalues $\lambda_1(x,\xi)$ and $\lambda_2(x,\xi)$ of $L(x,\xi)$ given by

$$\lambda_1(x,\xi) := \frac{1}{2}\mathrm{tr}L(x,\xi) + y(x,\xi)^n \tilde{\lambda}_1(x,\xi),$$

$$\lambda_2(x,\xi) := \frac{1}{2}\mathrm{tr}L(x,\xi) + y(x,\xi)^n \tilde{\lambda}_2(x,\xi).$$

We now compute the contact order. It follows essentially from the standard properties[18] of the Poisson bracket $\{f,g\} = H_f g = \partial_\xi f \partial_x g - \partial_x f \partial_\xi g$. Each term of $H_{\lambda_1}^k \lambda_2$ ($k = 1, 2, \dots, m-1$) is factorised by one of $\{H_{\mathrm{tr}L}^j y;\ j = 0, 1, \dots, \ell - 1\}$ where it vanishes at $(0,0)$. One also has

$$H_{\lambda_1}^m \lambda_2(0,0) = -2^{1-\ell}\frac{m!}{(\ell!)^n}(H_{\mathrm{tr}L}^\ell y(0,0))^n r(0,0) \neq 0. \qquad \blacksquare$$

Example C.2. Let $L(x,\xi)$ be a differential operator of the form

$$L(x,\xi) = \xi^\ell I_2 + \mathcal{V}(x),$$

where ℓ is a positive integer and $\mathcal{V}(x)$ is a 2×2 Hermitian matrix-valued function, not necessarily diagonal. Note that $\ell = 2$ corresponds to the Schrödinger operator. Put $V_0(x) := \frac{1}{2}\mathrm{tr}\mathcal{V}(x)$ and $M_{\mathcal{V}}(x) := \mathcal{V}(x) - V_0(x) I_2$. Suppose that $M_{\mathcal{V}}(x)$ vanishes at

[18] $\{f,g\} = -\{g,f\}$, $\{f, g+r\} = \{f,g\} + \{f,r\}$, and $\{f, gr\} = r\{f,g\} + g\{f,r\}$ hold for any smooth functions f, g, r.

$x = 0$ with a finite order n (i.e., $y(x,\xi) = x$ and $M_{\mathcal{V}}(x)$ is factorised by x^n). Then the eigenvalues $\lambda_1(x,\xi)$ and $\lambda_2(x,\xi)$ of $L(x,\xi)$ are given by

$$\lambda_1(x,\xi) = \xi^\ell + V_0(x) + x^n r(x), \quad \lambda_2(x,\xi) = \xi^\ell + V_0(x) - x^n r(x),$$

where $r(x)$ is defined by $r(x) := \sqrt{g_1(x)^2 + |g_2(x)|^2}$ with

$$g_1(x) = \frac{V_{11}(x) - V_{22}(x)}{2x^n}, \quad g_2(x) = \frac{V_{12}(x)}{x^n}, \quad \mathcal{V}(x) = \begin{pmatrix} V_{11}(x) & V_{12}(x) \\ V_{21}(x) & V_{22}(x) \end{pmatrix}.$$

When ℓ is even, the sets $\Lambda_1 = \{\lambda_1(x,\xi) = 0\}$ and $\Lambda_2 = \{\lambda_2(x,\xi) = 0\}$ cross at the points $(0, \pm|V_0(0)|^{\frac{1}{\ell}})$ if $V_0(0) \le 0$, and the contact order there is $m = n$ when $V_0(0) < 0$, $m = n\ell$ when $V_0(0) = 0$. When ℓ is odd, they cross at $(0, -\operatorname{sgn}(V_0(0))|V_0(0)|^{\frac{1}{\ell}})$, and the contact order there is $m = n$ when $V_0(0) \neq 0$, $m = n\ell$ when $V_0(0) = 0$.

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