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The $\mathbb{Z}_2$-valued spectral flow of a symmetric family of Toeplitz operators

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Abstract. We consider families $A(t)$ of self-adjoint operators with symmetry that causes the spectral flow of the family to vanish. We study the secondary $\mathbb{Z}_2$-valued spectral flow of such families. We prove an analogue of the Atiyah–Patodi–Singer–Robbin–Salamon theorem, showing that this secondary spectral flow of $A(t)$ is equal to the secondary $\mathbb{Z}_2$-valued index of the suspension operator $\frac{d}{dt} + A(t)$.

Applying this result, we show that the graded secondary spectral flow of a symmetric family of Toeplitz operators on a complete Riemannian manifold equals the secondary index of a certain Callias-type operator. In the case of a pseudo-convex domain, this leads to an odd version of the secondary Boutet de Monvel's index theorem for Toeplitz operators. When this domain is simply a unit disc in the complex plane, we recover the bulk-edge correspondence for the Graf–Porta module for 2D topological insulators of type AII.

1. Introduction

Atiyah, Patodi, and Singer [3] showed that the spectral flow of a periodic path $A = A(t)$ of self-adjoint Fredholm operators on a Hilbert space H is equal to the index of the "suspension" operator $D_A := \frac{d}{dt} + A(t)$. This result was extended to non-periodic families by Robbin and Salamon [33].

In this paper, we assume that there is an anti-linear, anti-unitary, anti-involution $\alpha: H \to H$, $\alpha^2 = -1$, $\alpha^* = \alpha$, and define a transformation τ of paths of operators by $\tau: A(t) \to \alpha A(-t)\alpha^{-1}$. If the path $A(t)$ is τ-invariant, then its spectral flow vanishes. However, the secondary $\mathbb{Z}_2$-valued invariant called the *half-spectral flow* is defined, cf. [20, 21]. In this situation, the suspension operator D_A commutes with τ. Such an operator is called *odd symmetric* and its index vanishes because of the symmetry. But a secondary $\mathbb{Z}_2$-valued index $\operatorname{ind}_\tau D_A$, called the *$\tau$-index*, is defined, [12, 20, 21, 34].

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The first main result of our paper is the analogue of the Atiyah–Patodi–Singer–Robbin–Salamon theorem for half-spectral flow:

$$\operatorname{sf}_\tau A = \operatorname{ind}_\tau D_A. \tag{1.1}$$

Note that recently a different version of the Robin–Salamon-type theorem for the $\mathbb{Z}_2$-valued spectral flow was obtained in [8]. In this paper, the authors considered paths of skew-adjoint Fredholm operators on a real vector space. The main difference with our result is that in [8] every operator $A(t)$ possesses a symmetry (it is skew-adjoint), while we consider the case when the symmetry relates $A(t)$ with $A(-t)$. We discuss the relationship between our result and [8] in Section 4.6.

Next, we consider a complete Riemannian manifold M edowed with an involution $\theta^M : M \to M$. Let $E = E^+ \oplus E^-$ be a graded Clifford bundle over M endowed with an anti-unitary anti-involution θ^E covering θ^M. We say that E is a *graded quaternionic Clifford bundle*. Then the corresponding Dirac operator D is odd symmetric. We denote by $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ the kernel of D and by P the orthogonal projection onto $\mathcal{H}$. Typically, it is an infinite-dimensional space.

Given a family f_t of self-adjoint matrix-valued functions on M, parametrized by a circle $t \in S^1$, we denote by M_{f_t} the operator of multiplication by f_t and define a family of Toeplitz operators

$$T_{f_t} := P M_{f_t} P : \mathcal{H} \to \mathcal{H}, \quad t \in S^1.$$

Let $T^{\pm}_{f_t}$ denote the restriction of T_{f_t} to $\mathcal{H}^{\pm}$. Under suitable assumptions on D and f_t, cf. Section 7, the operators T_{f_t} are Fredholm. We assume that the family f_t is odd symmetric. Then so is the family T_{f_t}.

The Dirac operator D induces an ungraded Dirac operator $\mathcal{D}$ on the manifold $\mathcal{M} := S^1 \times M$. For large enough constant c, the operator

$$\mathcal{B}_{cf} := \mathcal{D} + ic\mathcal{M}_{f_t}$$

is an odd symmetric Callias-type operator. The τ-index of such operators was investigated in [12]. We show that the *graded half-spectral flow* is equal to the index of the operator $\mathcal{B}_{cf}$:

$$\operatorname{sf}_\tau T^+_{f_t} - \operatorname{sf}_\tau T^-_{f_t} = \operatorname{ind}_\tau \mathcal{B}_{cf}. \tag{1.2}$$

This is a $\mathbb{Z}_2$-valued analogue of the main result of [10].

Remark 1.1. One can extend (1.2) to a family f_t parametrized by $t \in \mathbb{R}$, similar to (1.1), but more assumptions on the behavior of f_t is needed. The interested reader can easily reconstruct the details.

In the case when M is a strongly pseudo-convex domain in $\mathbb{C}^n$ with smooth boundary N, it follows from [22, §5] that $\mathcal{H}^- = \{0\}$. Hence, the above formula computes the half-spectral flow of $T^+_{f_t}$. Applying the Callias index theorem for τ-index, which was obtained in our previous paper [12], we obtain an explicit formula for a graded Dirac operator $\mathcal{D}_\mathcal{N}$ on $\mathcal{N} := S^1 \times N$, such that

$$\mathrm{sf}_\tau\, T^+_{f_t} = \mathrm{ind}_\tau\, \mathcal{D}^+_\mathcal{N}. \tag{1.3}$$

(Note that $\mathrm{sf}_\tau\, T^-_{f_t} = 0$ since $\mathcal{H}^- = \{0\}$). This is a spectral flow version of the $\mathbb{Z}_2$-valued analogue of the Boutet de Monvel's index theorem for Toeplitz operators, [9] (see also [16, 26]) established in [12, §8]

We call (1.3) the *generalized bulk-edge correspondence* because if $M = \{z \in \mathbb{C} : |z| \leq 1\}$ is the unit disc in $\mathbb{C}$ this is equivalent to the bulk-edge correspondence for the Graf–Porta model for 2D topological insulators with time-reversal symmetry, [25]. In Section 9, we use (1.3) to obtain a new proof of the bulk-edge correspondence in the Graf–Porta model.

The paper is organized as follows.

In Section 2, we recall the main facts about the τ-index of an odd symmetric operator from [12, 20, 21, 34].

In Section 3, we recall the main steps of the Robbin–Salamon proof of the equality between the spectral flow of $A(t)$ and the index of D_A. The notation introduced in this section is used in the subsequent sections.

In Section 4, we study τ-invariant paths $A(t)$ of self-adjoint Fredholm operators parametrized by $t \in \mathbb{R}$. We define their half-spectral flow and show that it has many properties similar to the usual $\mathbb{Z}$-valued spectral flow. The main result of this section is the equality (1.1) between the half-spectral flow of $A(t)$ and the τ-index of D_A. The proofs of many statements in this section are postponed to Section 5.

In Section 6, we show that the equality (1.1) also holds for periodic paths, i.e., for paths parametrized by a circle.

In Section 7, we study families of Toeplitz operators on complete Riemannian manifolds M associated with a τ-invariant family of functions $f_t \colon M \to \mathrm{Herm}(k)$ with values in the space $\mathrm{Herm}(k)$ of Hermitian $k \times k$-matrices. We compute the graded spectral flow of such families, (1.1). The proof of the main result is postponed to Section 10.

In Section 8, we consider the case when the dimension of M is even and give a more explicit formula for the right-hand side of (1.1) using the $\mathbb{Z}_2$-version of the Callias index theorem. When M is a pseudo-convex domain in $\mathbb{C}^n$ the formula is especially nice and we call it the *generalized bulk-edge correspondence*.

In Section 9, we recall the Graf–Porta model for topological insulators of type AII and show that the result of the previous section implies the equality between the bulk and the edge index in this model.

In Section 10, we prove the main result of Section 7.

2. The τ-index

In this section, we recall the construction of the secondary $\mathbb{Z}_2$-valued index of odd symmetric operators [20, 21, 34]. We work with unbounded operators and recall some results about the $\mathbb{Z}_2$-valued index of an odd symmetric differential operator on a non-compact involutive manifold [12].

2.1. An anti-linear involution

Let H be a complex Hilbert space. We denote the inner product on H by $\langle\cdot,\cdot\rangle_H$. An anti-linear map $\tau\colon H \to H$ is called an *anti-involution* if $\tau^* = -\tau = \tau^{-1}$. Here we denote by τ^* the unique anti–linear operator satisfying

$$\langle \tau x, y\rangle_H = \overline{\langle x, \tau^* y\rangle_H} \quad \text{for all } x, y \in H.$$

We say that τ is an *anti-unitary anti-involution.*

The following version of *Kramers' degeneracy* [29] is proven in [12, Lemma 2.2].

Lemma 2.1. *An anti-linear anti-involution τ does not have non-zero fixed vectors, i.e., $\tau x = x$ if and only if $x = 0$. Further, if τ acts on a finite-dimensional space H, then* dim H *is even. Moreover, there exists a subspace $L \in H$ such that $H = L \oplus \tau L$.*

2.2. Odd symmetric operators

Let $W \subset H$ be a dense subspace and let $D\colon H \to H$ be a closed unbounded linear operator on H whose domain is W. We endow W with the inner product

$$\langle x, y\rangle_W = \langle x, y\rangle_H + \langle Dx, Dy\rangle_W.$$

Then W is a Hilbert space and the operator $D\colon W \to H$ is a bounded.

Definition 2.2. A closed operator $D\colon H \to H$ with a dense domain W is called *odd symmetric* (or τ-symmetric) if

$$\tau D \tau^{-1} = D^*. \tag{2.1}$$

We denote by $\mathcal{B}_\tau(W, H)$ the space of τ-symmetric operators on H whose domain is W. This is a Banach space with the norm defined as the operator norm of $D\colon W \to H$.

2.3. Graded odd symmetric operators

If $H = H^+ \oplus H^-$ is a graded Hilbert space, we say that D is *odd with respect to the grading* if $D\colon H^\pm \to H^\mp$. We denote $D^\pm$ the restriction of D to $H^\pm$ and write $D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$.

Let W be the domain of D. Then $W^\pm := W \cap H^\pm$ is the domain of $D^\pm$ and $W = W^+ \oplus W^-$. If, in addition, D is self-adjoint, then $(D^\pm)^* = D^\mp$.

Assume that the anti-unitary anti-involution τ and the D are odd with respect to the grading, $\tau, D\colon H^\pm \to H^\mp$. Then we say that D is a *graded odd symmetric operator*.

We denote the space of graded odd symmetric self-adjoint operators by $\widehat{\mathcal{B}}_\tau(W, H)$. If $D \in \widehat{\mathcal{B}}_\tau(W, H)$, then $\tau D^\pm \tau^{-1} = D^\mp = (D^\pm)^*$. Then *the operators $D^\pm$ are odd symmetric*. Notice that, in this case, $\tau\colon W^\pm \to W^\mp$.

2.4. The $\mathbb{Z}_2$-valued index

If D is an odd symmetric operator, then $\operatorname{ind} D = 0$ by (2.1). Following [34], we define the *secondary* $\mathbb{Z}_2$-valued invariant – the *τ-index of D*.

Definition 2.3. If D is a Fredholm odd symmetric operator on H its *τ-index* is

$$\operatorname{ind}_\tau D := \dim \ker D \mod 2. \tag{2.2}$$

If $D \in \widehat{\mathcal{B}}_\tau(W, H)$ is a graded odd symmetric self-adjoint operator on a graded Hilbert space H, we define

$$\operatorname{ind}_\tau D^+ := \dim \ker D^+ \mod 2.$$

The following version of the homotopy invariance of the index is proven in [12, Theorem 2.5].

Theorem 2.4. *The τ-index $\operatorname{ind}_\tau D$ is constant on the connected components of $\mathcal{B}_\tau(W, H)$. The τ-index $\operatorname{ind}_\tau D^+$ is constant on the connected components of $\widehat{\mathcal{B}}_\tau(W, H)$.*

We now discuss some geometric examples of odd symmetric operators, [12].

2.5. Quaternionic vector bundles

An *involutive manifold* (M, θ^M) is a Riemannian manifold M together with a metric preserving involution $\theta^M\colon M \to M$. A *quaternionic vector bundle* over an involutive manifold M is hermitian vector bundle E endowed with an anti-linear unitary bundle map $\theta^E\colon E \to \theta^* E$, such that $(\theta^E)^2 = -1$, cf. [19,23,28]. This means that for every

$x \in M$ there exists an anti-linear map $\theta_x^E \colon E_x \to E_{\theta(x)}$, which depends smoothly on x and satisfies $\theta^E_{\theta^M(x)}\theta^E_x = -1$, $(\theta^E_x)^*\theta^E_x = 1$.

If $E = E^+ \oplus E^-$ is a graded vector bundle and θ^E is *odd with respect to the grading* (i.e., $\theta^E(E^\pm) = E^\mp$), we say that (E, θ^E) is a *graded quaternionic bundle*.

Given a quaternionic vector bundle (E, θ^E) over an involutive manifold (M, θ^M) we define an anti-unitary anti-involution τ on the space $\Gamma(M, E)$ of sections of E by

$$\tau \colon f(x) \mapsto \theta^E f(\theta^M x), \quad f \in \Gamma(M, E). \tag{2.3}$$

2.6. Odd symmetric elliptic operators

Let $(E = E^+ \oplus E^-, \theta^E)$ be a graded quaternionic vector bundle over an involutive manifold (M, θ^M). Let $D \colon \Gamma(M, E) \to \Gamma(M, E)$ be a self-adjoint odd symmetric elliptic differential operator on M, which is odd with respect to the grading, i.e., $D \colon \Gamma(M, E^\pm) \to \Gamma(M, E^\mp)$. We set $D^\pm := D|_{\Gamma(M,E^\pm)}$. Then $\tau D^\pm \tau^{-1} = (D^\pm)^* = D^\mp$ and

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$

with respect to the decomposition $E = E^+ \oplus E^-$.

Suppose now that D is Fredholm. In particular, this is always the case when M is compact. Then the τ-index $\operatorname{ind}_\tau D^+ \in \mathbb{Z}_2$ is defined by (2.2). In [12], we computed this index in several examples and proved the $\mathbb{Z}_2$-valued analogues of the relative index theorem, the Callias index theorem, and the Boutet de Monvel's index theorem for Toeplitz operators.

3. The spectral flow and the Robbin–Salamon theorem

This section recalls some basic definitions and results from [33]. We also fix the notation which will be used in the next section.

3.1. Some spaces of operators

Let $A \colon H \to H$ be a closed operator with dense domain W. As in Section 2.2, we define the scalar product on W by

$$\langle x, y\rangle_W = \langle x, y\rangle_H + \langle Ax, Ay\rangle_H$$

and view A as a bounded operator $A \colon W \to H$. We denote by $\|A\|$ its norm. This is just the operator norm of $A(1 + A^*A)^{-1/2}$.

We now make a basic assumption.

Assumption 3.1. The embedding $W \hookrightarrow H$ is compact.

Let $\mathcal{S}(W, H)$ denote the space of self-adjoint (possibly unbounded) operators on H whose domain is W and *which have a non-zero resolvent set*. If Assumption 3.1 is satisfied, then every $A \in \mathcal{S}$ has a compact resolvent and, hence, A has a discrete spectrum. In particular, it is Fredholm.

From this point on, we always assume Assumption 3.1.

Next, we consider families of unbounded operators $A(t)\colon H \to H$ $(t \in \mathbb{R})$ with the same domain W. We fix the norm on W, for example, defined by the scalar product

$$\langle x, y\rangle_H + \langle A(0)x, A(0)y\rangle_H,$$

and we denote by $\mathcal{S}(W, H)$ the space of bounded linear operators $W \to H$. Then for each t, we have $A(t) \in \mathcal{S}(W, H)$.

Let $\mathcal{A}(\mathbb{R}, W, H)$ denote the space of norm continuous maps $A\colon \mathbb{R} \to \mathcal{S}(W, H)$ which have limits

$$A^{\pm} = \lim_{t\to\pm\infty} A(t) \tag{3.1}$$

and the operators $A^{\pm}\colon W \to H$ are bijective. This is an open subspace in the Banach space of all maps satisfying (3.1) with the norm

$$\|A\|_{\mathcal{A}} := \sup_{t\in\mathbb{R}} \|A(t)\|.$$

Denote by $\mathcal{A}^1(\mathbb{R}, W, H)$ the set of those $A \in \mathcal{A}$ which are continuously differentiable in the norm topology and satisfy

$$\|A\|_{\mathcal{A}^1} := \sup_{t\in\mathbb{R}}(\|A(t)\| + \|\dot{A}(t)\|) < \infty.$$

The space $\mathcal{A}^1$ is dense in $\mathcal{A}$.

The following theorem is proven in [33, Theorem 4.23].

Theorem 3.2. *There is a unique map $\mu\colon \mathcal{A}(\mathbb{R}, W, H) \to \mathbb{Z}$ satisfying the following axioms.*

- (i) Homotopy. *μ is constant on the connected components (in norm topology) of $\mathcal{A}(\mathbb{R}, W, H)$.*
- (ii) Constant. *If $A(t)$ is independent of t, then $\mu(A) = 0$.*
- (iii) Direct sum. $\mu(A_1 \oplus A_2) = \mu(A_1) + \mu(A_2)$.
- (iv) Normalization. *For $W = H = \mathbb{R}$ and $A(t) = \arctan(t)$, we have $\mu(A) = 1$.*

Remark 3.3. Robbin and Salamon include an extra *catenation* axiom in their theorem. But then, in [33, Proposition 4.26] they show that this axiom follows from the others four axioms and, hence, can be omitted. Since for our main object of study – the half-spectral flow – the catenation axiom does not make sense, we prefer to formulate Theorem 3.2 without it.

3.2. The spectral flow

For $A(t) \in \mathcal{A}$, let $\operatorname{sf} A$ denote the spectral flow of $A(t)$. There are many equivalent ways to define $\operatorname{sf} A$. Let us recall the definition given in [33].

We say that $t \in \mathbb{R}$ is a *crossing* for A if $\ker A(t) \neq \{0\}$. First, consider the case when $A \in \mathcal{A}^1$. If t is a crossing for A, we define the *crossing operator*

$$\Gamma(A,t) := P(A,t)\dot{A}(t)P(A,t)|_{\ker A(t)},$$

where $P(A,t)\colon H \to H$ denotes the orthogonal projection onto the kernel of $A(t)$. We say that the crossing t is *regular* if the crossing operator $\Gamma(A,t)$ is non-degenerate. A regular crossing is *simple* if $\dim \ker A(t) = 1$.

If t is a regular crossing for A, we denote by $\operatorname{sign} \Gamma(A,t)$ the signature (number of positive minus the number of negative eigenvalues) of $\Gamma(A,t)$.

If t is a simple crossing, there is a unique real-valued function $\lambda(s)$, defined near t, such that $\lambda(s)$ is an eigenvalue of $A(s)$ for all s and $\lambda(t) = 0$. We call λ the *crossing eigenvalue*. Then

$$\operatorname{sign} \Gamma(A,t) = \operatorname{sign} \dot{\lambda}(t).$$

If $A \in \mathcal{A}^1$ has only regular crossings, then we define

$$\operatorname{sf} A = \sum_t \operatorname{sign} \Gamma(A,t), \tag{3.2}$$

where the sum is taking over all crossings of A. Thus, if $A(t)$ has only simple crossings

$$\operatorname{sf} A = \sum_t \operatorname{sign} \dot{\lambda}(t).$$

Remark 3.4. A much more general definition and a deep study of the spectral flow can be found in [7, 31].

The following result is a combination of several theorems from [33].

Proposition 3.5. *The following statements hold.*

(i) *In every connected component of $\mathcal{A}(\mathbb{R}, W, H)$, there is an $A \in \mathcal{A}^1$ which has only simple crossings.*

(ii) *If $A_0, A_1 \in \mathcal{A}^1$ belong to the same connected component of $\mathcal{A}(\mathbb{R}, W, H)$ and have only regular crossings, then* $\operatorname{sf} A_0 = \operatorname{sf} A_1$.

We now define the spectral flow of $A \in \mathcal{A}(\mathbb{R}, W, H)$ as the spectral flow of any of the operators $A_1 \in \mathcal{A}^1(\mathbb{R}, W, H)$ which lies in the same connected component of $\mathcal{A}(\mathbb{R}, W, H)$ as A and has only regular crossings. It follows from Proposition 3.5 (ii) that sf A is well defined and is constant on connected components of $\mathcal{A}(\mathbb{R}, W, H)$. Moreover, it is easy to see that it satisfies the other hypothesis of Theorem 3.2. Thus, we obtain the following theorem (cf. [33, Lemma 4.27]).

Theorem 3.6. *The unique function $\mu(A)$ defined in Theorem 3.2 is equal to the spectral flow of A*

3.3. The index and the spectral flow

Consider the spaces

$$\mathcal{H} := L^2(\mathbb{R}, H), \quad \mathcal{W} := L^2(\mathbb{R}, W) \cap W^{1,2}(\mathbb{R}, H)$$

equipped with the norms

$$\|\xi\|_{\mathcal{H}} := \int_{-\infty}^{\infty} \|\xi(t)\|_H \, dt,$$

$$\|\xi\|_{\mathcal{W}} := \int_{-\infty}^{\infty} (\|\xi(t)\|_W + \|\dot{\xi}(t)\|_H) dt,$$

Given $A \in \mathcal{A}^1(\mathbb{R}, W, H)$, consider the differential operator

$$D_A : \xi(t) \mapsto \frac{d}{dt}\xi(t) + A(t)\xi(t). \tag{3.3}$$

This is a bounded operator $\mathcal{W} \to \mathcal{H}$. Robbin and Salamon show that D_A is Fredholm ([33, Theorem 3.12]) and that ind D_A satisfies the conditions of Theorem 3.2 ([33, p. 20]). Thus, it must be equal to the unique function μ of Theorem 3.2. Hence, by Theorem 3.6, we have the following result.

Theorem 3.7. *For $A \in \mathcal{A}(\mathbb{R}, W, H)$, we have*

$$\operatorname{ind}\Big(\frac{d}{dt} + A(t)\Big) = \operatorname{sf} A.$$

3.4. A periodic family of operators

Consider an element $A \in \mathcal{A}(\mathbb{R}, W, H)$, such that there exists a bijective $B \in \mathcal{S}(W, H)$ such that

$$A(t) = B \quad \text{for all } t \notin [-\pi, \pi].$$

Let

$$S^1 = \{e^{it} : -\pi \le t \le \pi\}$$

be the unit circle, and define $\widetilde{A}: S^1 \to \mathcal{S}$ by

$$\widetilde{A}(e^{it}) := A(t).$$

We denote the space of such maps $\widetilde{A}$ by $\Omega\mathcal{S}(W, H)$. This is a version of a loop space of self-adjoint operators on H, namely, we restrict ourselves to loops with certain analytic properties. Then

$$\mathrm{sf}(\widetilde{A}) := \mathrm{sf}(A) = \sum_{-\pi<t<\pi} \mathrm{sign}\,\Gamma(A,t)$$

is the spectral flow of $\widetilde{A}$ as it is defined in [3].

It is shown in [3] that the isomorphism $\pi_1(\mathcal{S}) = \pi_0(\Omega\mathcal{S}) = \mathbb{Z}$ is detected by the spectral flow. In other words, if the spectral flow of a loop $\widetilde{A}$ vanishes, then $\widetilde{A}$ is homotopic to a constant path in $\Omega\mathcal{S}$.

Consider the operator

$$D_{\widetilde{A}}: L^2(S^1, W) \cap W^{1,2}(S^1, H) \to L^2(\mathbb{R}, H),$$

defined by

$$D_{\widetilde{A}}: \xi(e^{it}) \mapsto \frac{d}{dt}\xi(e^{it}) - A(e^{it})\xi(e^{it}).$$

This is just the operator (3.3) with periodic boundary conditions. Thus, cf. [3, p. 95]

$$\mathrm{ind}\, A = \mathrm{ind}\, \widetilde{A}.$$

Hence, we recover from Theorem 3.7 the following result, originally proven in [3]:

$$\mathrm{ind}\, D_{\widetilde{A}} = -\,\mathrm{sf}\, \widetilde{A}.$$

Equivalently,

$$\mathrm{ind}\Big(\frac{d}{dt} + A\Big) = \mathrm{sf}\, \widetilde{A}.$$

4. The half-spectral flow of a family of operators on real line

In this section, we consider τ-symmetric paths of self-adjoint operators. Because of the symmetry, the usual spectral flow vanishes. We define a secondary invariant *half-spectral flow*, with values in $\mathbb{Z}_2$. In [21] this invariant is called the *odd half-spectral flow*. This part of the section roughly follows [20], but we consider a slightly different setting because of our future applications.

Theorem 4.1 states that several properties of half-spectral flow define it uniquely. This is an analogue of Theorem 3.2. If A is a τ-symmetric path, then the operator D_A, cf. (3.3) is τ-symmetric (in the sense of Section 2) with respect to some anti-unitary anti-involution τ. We use the uniqueness of the spectral flow to show that $\operatorname{ind}_\tau D_A$ is equal to the half-spectral flow of A. This is an analogue of the Robbin–Salamon Theorem 3.7 for our secondary invariants. This result should be compared to [8, §8], though we consider different paths of operators. We discuss the relationship of our result with [8] in Section 4.6.

4.1. An anti-linear anti-involution

As in Section 3.1, we assume that $W \hookrightarrow H$ is a compact embedding of a Hilbert space W into a Hilbert space H. We identify W with its image in H. Let $\alpha\colon H \to H$ be an *anti-linear anti-involution* and assume that $\alpha(W) = W$.

The map α defies anti-linear maps on the spaces of functions $\mathcal{W}$ and $\mathcal{H}$ of Section 3.3, which we also denote by α. Define the maps $\tau\colon \mathcal{W} \to \mathcal{W}$ and $\tau\colon \mathcal{H} \to \mathcal{H}$ by

$$\tau\colon \xi(t) \mapsto \alpha\xi(-t). \tag{4.1}$$

By a slight abuse of the language, we denote by the same letter the conjugation $\tau\colon \mathcal{A}(\mathbb{R}, W, H) \to \mathcal{A}(\mathbb{R}, W, H)$,

$$\tau\colon A(t) \mapsto \tau A(t)\tau^{-1} = \alpha A(-t)\alpha^{-1}. \tag{4.2}$$

Let

$$\mathcal{A}_\tau(\mathbb{R}, W, H) := \{A \in \mathcal{A}(\mathbb{R}, W, H) : \tau(A) = A\}.$$

denote the set of τ-invariants in $\mathcal{A}$ and set $\mathcal{A}^1_\tau := \mathcal{A}_\tau \cap \mathcal{A}^1$. Recall that if $A \in \mathcal{A}$, then $A(t)$ has a discrete spectrum and compact resolvent for all $t \in \mathbb{R}$.

4.2. The half-spectral flow of a τ-invariant path of operators

Suppose $A \in \mathcal{A}_\tau$. If $t \in \mathbb{R}$ is a crossing for A, then $-t$ is also a crossing for A and

$$\Gamma(A, t) = -\alpha\Gamma(A, -t)\alpha^{-1}.$$

Hence, $\operatorname{sign}\Gamma(A, t) = -\operatorname{sign}\Gamma(A, -t)$ and, if 0 is a crossing, $\operatorname{sign}\Gamma(A, 0) = 0$. We conclude that *the spectral flow of any $A \in \mathcal{A}_\tau$ is equal to zero.* In this section, we define a *secondary invariant* of A – the $\mathbb{Z}_2$-valued half-spectral flow.

Before giving the definition and discussing the properties of the half-spectral flow, let us put it in a perspective of the works of Atiayh and Singer [4] and Carey, Phillips, and Schulz-Baldes [17].

Recall that the loop space $\Omega\mathcal{S}(W,H)$ was introduced in Section 3.4. Define the map $\tau\colon \Omega\mathcal{S} \to \Omega\mathcal{S}$ by

$$\tau(\widetilde{A}(e^{it})) := \alpha \widetilde{A}(e^{-it})\alpha,$$

and let

$$\Omega_\tau\mathcal{S}(W,H) := \{\widetilde{A} \in \Omega\mathcal{S}(W,H) : \tau(\widetilde{A}) = \widetilde{A}\},$$

be the space of τ-invariant loops.

Since the spectral flow of $\widetilde{A}$ vanishes, it follows from [4] that is $\widetilde{A}$ is homotopic in the space $\Omega\mathcal{S}$ to a constant path. However, if the half-spectral flow does not vanish, this homotopy cannot be chosen inside the space $\Omega_\tau\mathcal{S}$.

4.3. The uniqueness of half-spectral flow

The first main result of this section is the following analogue of Theorem 4.1.

Theorem 4.1. *There is a unique map $\mu_\tau\colon \mathcal{A}_\tau(\mathbb{R},W,H) \to \mathbb{Z}_2$ satisfying the following axioms.*

- (i) Homotopy. *μ_τ is constant on the connected components (in norm topology) of $\mathcal{A}_\tau(\mathbb{R},W,H)$.*
- (ii) Constant. *If $A(t)$ is independent of t, then $\mu_\tau(A) = 0$.*
- (iii) Direct sum. $\mu_{\tau_1\oplus\tau_2}(A_1 \oplus A_2) = \mu_{\tau_1}(A_1) + \mu_{\tau_2}(A_2)$.
- (iv) Normalization. *For $W = H = \mathbb{C}^2$,*

$$\alpha\colon \mathbb{C}^2 \to \mathbb{C}^2, \quad (z_1,z_2) \mapsto (\bar{z}_2, -\bar{z}_1).$$

and

$$A(t) = \begin{pmatrix} \arctan(t) & 0 \\ 0 & -\arctan(t) \end{pmatrix},$$

we have $\mu_\tau(A) = 1$.

The number $\mu_\tau(A)$ is called the half-spectral flow *(or the τ-*spectral flow*) of A.*

The proof of the theorem is postponed to Section 5.6.

4.4. The half-spectral flow

We present an explicit construction of the half-spectral flow in terms of the crossing operators. Recall that $t \in \mathbb{R}$ is called a crossing for $A(t)$ if $\ker A(t) \neq \{0\}$. We say that a crossing t is *isolated* if the exists $\delta > 0$ such that t is the only crossing for $A(t)$ on the interval $(t-\delta, t+\delta)$. Note that if $A \in A^1_\tau$ and t is a regular crossing (cf. Section 3.2), then t is an isolated crossing.

First, assume that $A \in \mathcal{A}_\tau$ has only isolated crossings. Since we are only interested in mod 2 value of the flow, we can replace the signature of the crossing operator with the rank of the spectral projection $P(A,t)$:

$$\operatorname{sign}\Gamma(A,t) = \operatorname{rank} P(A,t) \mod 2. \tag{4.3}$$

The space $\ker P(A,0)$ is α-invariant. It follows from Lemma 2.1 that $\operatorname{rank} P(A,0)$ is even. As opposed to the crossing operators $\Gamma(A,t)$, the projections $P(A,t)$ are defined for all $A \in \mathcal{A}_\tau$ (and not only for those in $\mathcal{A}_\tau^1$). If $A \in \mathcal{A}_\tau$ has only isolated crossings, we define

$$\operatorname{sf}_\tau(A) := \sum_{t<0} \operatorname{rank} P(A,t) + \frac{1}{2} \operatorname{rank} P(A,0) \mod 2. \tag{4.4}$$

We have the following analogue of Proposition 3.5.

Proposition 4.2. *If $A_0, A_1 \in \mathcal{A}_\tau(\mathbb{R}, W, H)$ belong to the same connected component and have only isolated crossings, then $\operatorname{sf}_\tau A_0 = \operatorname{sf}_\tau A_1$.*

The proof is given in Section 5.1.

Lemma 4.3. *In every connected component of $\mathcal{A}_\tau(\mathbb{R}, W, H)$, there exists a path that has only isolated crossings.*

Proof. Let $A \in \mathcal{A}_\tau(\mathbb{R}, W, H)$. By [33, Theorem 4.22], the path $A + \delta$ has only isolated crossings for almost all δ. Clearly, this path is homotopic to A in $\mathcal{A}_\tau(\mathbb{R}, W, H)$. ∎

We now define the half-spectral flow of $A \in \mathcal{A}_\tau(\mathbb{R}, W, H)$ as the spectral flow of any of the operators A_1 with isolated crossings lying in the same connected components of $\mathcal{A}_\tau$ as A. By Proposition 4.2, this definition is independent of the choice of A_1.

Theorem 4.4. *The map $\operatorname{sf}_\tau \colon \mathcal{A}_\tau(\mathbb{R}, W, H) \to \mathbb{Z}_2$ satisfies the conditions of Theorem 4.1 and, hence, is the half-spectral flow.*

The proof of the theorem is given in Section 5.4.

4.5. The τ–index of D_A

Let $A \in \mathcal{A}_\tau$ and let D_A be as in (3.3). Then

$$\tau D_A \tau^{-1} = D_A^*,$$

where τ is defined in (4.1). Hence, D_A is τ-invariant and its τ-index is defined by (2.2),

$$\operatorname{ind}_\tau D_A = \dim\ker D_A \mod 2.$$

We have the following analogue of the Robbin–Salamon Theorem 3.7.

Theorem 4.5. *The map* $\mathrm{ind}_\tau\colon \mathcal{A}_\tau(\mathbb{R}, W, H) \to \mathbb{Z}_2$ *satisfies the conditions of Theorem* 4.1 *and, hence, is equal to the half-spectral flow:*

$$\mathrm{ind}_\tau\Big(\frac{d}{dt} + A(t)\Big) = \mathrm{sf}_\tau\, A \mod 2.$$

The proof of Theorem 4.5 is presented in Section 5.5.

4.6. Comparison with the spectral flow of a path of skew-adjoint operators

In a seminal paper [8], the authors proved an analogue of the Robin–Salamon theorem for a path of skew-adjoint operators. This is closely related to our result, but the significant difference is that in [8] each operator $A(t)$ is skew-adjoint, while our assumption gives a relationship between $A(t)$ and $A(-t)$. As we explain below, our $\mathbb{Z}_2$-valued spectral flow can be easily computed as a spectral flow of a special path of skew-adjoint operators. Thus, the result of [8] expresses this spectral flow as a $\mathbb{Z}_2$-valued index of a certain Fredholm operator. Unfortunately, this operator is not the same as $\frac{d}{dt} + A(t)$. It is possible – but not easy to show by a direct computation – that the two operators have the same index, thus deducing our result from [8]. Below, we provide some details and indicate how this program can be carried through.

For $A(t) \in \mathcal{A}_\tau(\mathbb{R}, W, H)$, suppose, for simplicity, that the operator $A(0)$ is invertible. Set

$$A_{<0}(t) := \begin{cases} A(t) & \text{for } t \le 0, \\ A(0) & \text{for } t > 0, \end{cases} \qquad A_{>0}(t) := \begin{cases} A(t) & \text{for } t \ge 0, \\ A(0)d & \text{for } t < 0. \end{cases}$$

Consider the “doubling” of the path A:

$$\widehat{A}(t) := \begin{pmatrix} 0 & A_{<0}(t) \\ -\alpha A_{>0}(-t)\alpha^{-1} & 0 \end{pmatrix}.$$

This is a path of skew adjoint operators. The $\mathbb{Z}_2$-valued spectral flow of $\widehat{A}$ considered in [8] is exactly equal to $\mathrm{sf}_\tau\, A$. [8, Theorem 8.1] expresses this spectral flow as a $\mathbb{Z}_2$-valued index of a certain “suspension” operator of the form similar to $\frac{d}{dt} + A$. Using the gluing formula for the τ-index, [13], one can show that the latter index is equal to $\mathrm{ind}_\tau\big(\frac{d}{dt} + A(t)\big)$. Thus, the half-spectral flow of A is equal to the τ-index of $\frac{d}{dt} + A$. Combining it with [8, Theorem 8.1], we see that the $\mathbb{Z}_2$-valued indexes of $\frac{d}{dt} + A$ and of the operator in [8] coincide. This can also be shown by a direct, but non-trivial argument.

In conclusion, our Theorem 4.5 is closely related to [8, Theorem 8.1], but the relationship is not very straightforward. This is why we preferred to give a direct proof of our result in the next section, by adopting the original argument of Robbin and Salamon [33] to our $\mathbb{Z}_2$-valued invariants.

5. Proof of $\mathbb{Z}_2$-valued version of the Robbin–Salamon theorem

This section is dedicated to the proofs of Proposition 4.2 and Theorems 4.4, 4.5, and 4.1.

5.1. Proof of Proposition 4.2

Suppose $A_0, A_1 \in \mathcal{A}_\tau(\mathbb{R}, W, H)$ be two paths which have only isolated crossings.

Lemma 5.1. *There exists a continuous path A_s ($s \in [0,1]$) in $\mathcal{A}_\tau$ connecting A_0 and A_1 such that all A_s have only isolated crossings.*

Proof. Recall that both $A_0(0)$ and $A_1(0)$ are odd symmetric. Choose a continuous path R_s of odd symmetric operators connecting $A_0(0)$ and $A_1(0)$ (for example, one can take a linear path).

In the proof of [33, Theorem 4.23 on p. 16], Robbin and Salamon show that there exists a continuous path $A'_s \in \mathcal{A}(\mathbb{R}, W, H)$ connecting A_0 and A_1 such that every A'_s has only isolated crossings. The same argument shows that this path can be chosen so that $A'_s(0) = R_s$. Then the path

$$A_s(t) := \begin{cases} A'_s(t) & \text{for } t \le 0, \\ \tau A'_s(-t)\tau^{-1} & \text{for } t > 0 \end{cases}$$

is continuous, lies in $\mathcal{A}_\tau(\mathbb{R}, W, H)$, has only isolated crossings, and connects A_0 with A_1. ■

Let A_s ($s \in [0,1]$) be a as in the lemma. We need to prove that $\mathrm{sf}_\tau(A_s)$ is independent of s. If $t = 0$ is not a crossing point for some A_{s_0}, then there exists $\epsilon > 0$ such that 0 is not a crossing for all A_s, $s \in (s_0 - \epsilon, s_0 + \epsilon)$. Then the path

$$\widetilde{A}_s(t) := \begin{cases} A_s(t) & \text{for } t \le 0, \\ A_s(0) & \text{for } t > 0 \end{cases}$$

is in $\mathcal{A}(\mathbb{R}, W, H)$ for $s \in (s_0 - \epsilon, s_0 + \epsilon)$. The τ-spectral flow of A_s is equal mod 2 to the usual $\mathbb{Z}$-valued spectral flow $\mathrm{sf}(\widetilde{A}_s)$. Hence, it is independent of $s \in (s_0 - \epsilon, s_0 + \epsilon)$ by the stability of the usual spectral flow under homotopy.

If 0 is a crossing for A_{s_0}, then there exist $\epsilon, \delta > 0$ such that

(i) there are no non-zero eigenvalues of $A_{s_0}(0)$ in the interval $(-\delta, \delta)$;

(ii) there are no crossings t of A_{s_0} with $0 < |t| \le \epsilon$;

(iii) $\pm\delta$ are not in the spectrum of $A_s(t)$ for all $s \in (s_0-\epsilon, s_0+\epsilon)$, $t \in (-\epsilon,\epsilon)$.

Then the paths

$$\widetilde{A}_s(t) := \begin{cases} A_s(t) & \text{for } t \le -\epsilon, \\ A_s(-\epsilon) & \text{for } t > -\epsilon, \end{cases}$$

and

$$\widehat{A}_s(t) := \begin{cases} A_s(-\epsilon) & \text{for } t < -\epsilon, \\ A_s(t) & \text{for } -\epsilon \le t \le \epsilon, \\ A_s(\epsilon) & \text{for } t > \epsilon \end{cases}$$

are in $\mathcal{A}(\mathbb{R}, W, H)$ for all $s \in (s_0-\epsilon, s_0+\epsilon)$, $t \in (-\epsilon,\epsilon)$. By (4.4) , we have

$$\mathrm{sf}_\tau(A_s) = \mathrm{sf}(\widetilde{A}_s) + \sum_{-\epsilon<t<0} \operatorname{rank} P(A_s,t) + \frac{1}{2} \operatorname{rank} P(A_s,0), \mod 2. \tag{5.1}$$

Since $t=0$ is not a crossing for $\widetilde{A}_s$ ($s \in (s_0-\epsilon, s_0+\epsilon)$, $\mathrm{sf}(\widetilde{A}_s)$ is independent of s on this interval.

From (3.2) and (4.3), we see that the mod 2 reduction of usual $\mathbb{Z}$-valued spectral of $\widehat{A}_s$ is given by

$$\mathrm{sf}(\widehat{A}_s) = \sum_{-\epsilon<t<0} \operatorname{rank} P(A_s,t) + \operatorname{rank} P(A_s,0) + \sum_{0<t<\epsilon} \operatorname{rank} P(A_s,t) \mod 2.$$

By (4.2), $P(A_s,t) = \alpha P(A_s,-t)\alpha^{-1}$. It follows that

$$\sum_{-\epsilon<t<0} \operatorname{rank} P(A_s,t) = \sum_{0<t<\epsilon} \operatorname{rank} P(A_s,t)$$

and $\mathrm{sf}(\widehat{A}_s) = \operatorname{rank} P(A_s,0) \mod 2$. From the stability of the usual spectral flow under homotopy, we have $\mathrm{sf}(\widehat{A}_{s_0}) = \mathrm{sf}(\widehat{A}_s)$ and, hence,

$$\operatorname{rank} P(A_s,0) = \operatorname{rank} P(A_{s_0},0) \mod 2.$$

Substituting this equality into (5.1) and using that $\mathrm{sf}(\widetilde{A}_s)$ is independent of $s \in (s_0-\epsilon, s_0+\epsilon)$, we conclude that

$$\mathrm{sf}_\tau(A_s) = \mathrm{sf}(\widetilde{A}_{s_0}) + \frac{1}{2} \operatorname{rank} P(A_{s_0},0) = \mathrm{sf}_\tau(A_{s_0}) \mod 2.$$

We now pass to the proof of Theorems 4.4 and 4.5.

5.2. The normalization axiom

Let $W = H = \mathbb{C}^2$ and let α and A be as in Theorem 4.1 (iv). Then the eigenvalues of $A(t)$ are $\pm\arctan(t)$. By (4.4), $\mathrm{sf}_\tau(A) = 1$. Hence, we obtain the following lemma.

Lemma 5.2. sf_τ *satisfies the normalization condition of Theorem* 4.1.

Also, by [12, Lemma 2.7] we have the following lemma.

Lemma 5.3. *The map* $A \mapsto \mathrm{ind}_\tau D_A$ *satisfies the normalization condition of Theorem* 4.1.

5.3. A constant path

Let us consider the case when $A(t) = A_0 \in \mathcal{A}_\tau$ is independent of t. Then A_0 is bijective. It follows that $\mathrm{sf}_\tau(A) = 0$, i.e., sf_τ satisfy axiom (ii) of Theorem 4.1.

Further, if $D_A\xi(t) = 0$, then $\xi(t) = e^{-A_0 t}\xi(0)$, which is not square-integrable, unless $\xi(0) = 0$. Hence, $\ker D_A = \{0\}$. The following lemma summarizes this subsection.

Lemma 5.4. *The maps* $A \mapsto \mathrm{sf}_\tau(A)$ *and* $A \to \mathrm{ind}_\tau D_A$ *satisfy axiom* (ii) *of Theorem* 4.1.

5.4. Proof of Theorem 4.4

By Proposition 4.2, sf_τ satisfies the homotopy axiom (i) of Theorem 4.1. By Lemmas 5.2 and 5.4, sf_τ satisfies axioms (ii) and (iv) of Theorem 4.1. It is clear that it also satisfies the direct sum axiom (iii).

5.5. Proof of Theorem 4.5

By Lemmas 5.3 and 5.4, $\mathrm{ind}_\tau D_A$ satisfies axioms (ii) and (iv) of Theorem 4.1. It is clear, that it also satisfies the direct sum axiom (iii). By Theorem 2.4 it also satisfies the homotopy axiom. Thus, it coincides with the half-spectral flow by Theorem 4.1.

5.6. The proof of Theorem 4.1

Theorem 4.4 states that sf_τ satisfies the axioms of Theorem 4.1. Hence, the existence of μ_τ is proven.

To prove the uniqueness, we need to show that any μ_2 satisfying the axioms of Theorem 4.1 is equal to sf_τ.

A verbatim repetition of the Robbin–Salamon proof of Theorem 3.2, cf. [33, §4], shows that there exists a unique map $\mu_2: \mathcal{A}(\mathbb{R}, W, H) \to \mathbb{Z}_2$, satisfying axioms (i)–(iv) of Theorem 3.2. This μ_2 is just the mod 2 reduction of μ.

For any $B \in \mathcal{A}$, the *symmetrization* of B is the path

$$\widetilde{B}(t) := \begin{cases} B(t) \oplus \alpha B(-t)\alpha^{-1} & \text{for } t \le 0, \\ \alpha B(-t)\alpha^{-1} \oplus B & \text{for } t > 0 \end{cases} \in \mathcal{A}_\tau(\mathbb{R}, W \oplus W, H \oplus H).$$

We claim that the map $B \mapsto \mu_\tau(\widetilde{B})$ satisfies all axioms for μ_2. Indeed, if B_1 is homotopic to B_2 in $\mathcal{A}$, then $\widetilde{B}_1$ is homotopic to $\widetilde{B}_2$ in $\mathcal{A}_\tau$, which implies the homotopy axiom. The rest of the axioms follow immediately from the corresponding axioms for μ_τ. It follows now from the uniqueness of μ_2 that

$$\mu_2(B) = \mu_\tau(\widetilde{B}). \tag{5.2}$$

We want to compute $\mu_\tau(A)$ in terms of μ_2 for any $A \in \mathcal{A}_\tau$. First, consider a path $A \in \mathcal{A}_\tau$, which is bijective at 0 and set

$$A_{\le 0}(t) := \begin{cases} A(t) & \text{for } t \le 0, \\ A(0) & \text{for } t > 0. \end{cases}$$

Then $A_{\le 0} \in \mathcal{A}(\mathbb{R}, W, H)$ and the symmetrization $\widetilde{A}_{\le 0}$ is the direct sum of A and the constant path $A(0)$:

$$\widetilde{A}_{\le 0} = A \oplus A(0).$$

Thus, from the direct sum and the constant axiom for μ_τ we conclude that $\mu_\tau(A) = \mu_\tau(\widetilde{A}_{\le 0})$. Using (5.2), we get

$$\mu_\tau(A) = \mu_\tau(\widetilde{A}_{\le 0}) = \mu_2(A_{\le 0}) \equiv \sum_{t<0} \operatorname{rank} P(A,t) = \operatorname{sf}_\tau(A). \tag{5.3}$$

Here, the third equality holds modulo 2 and follows from Theorem 3.6 and (4.3).

Consider a general $A \in \mathcal{A}_\tau$. Then, by Lemma 2.1, the kernel of $A(0)$ is even dimensional, and there is a subspace L in it such that $\ker A(0) = L \oplus \alpha L$. Then there exists $\epsilon > 0$ and a path $A' \in \mathcal{A}_\tau$ homotopic to A, such that $A'(t)$ is bijective for all $t \in (-\epsilon, \epsilon)$, $\ker A(-\epsilon) = L$, and $\ker A(\epsilon) = \tau(L)$. Then,

$$\operatorname{rank} P(A', -\epsilon) = \dim L = \frac{1}{2} \operatorname{rank} P(A, 0).$$

It now follows from (5.3), that $\mu_\tau(A)$ is given by (4.4).

6. The $\mathbb{Z}_2$-valued index and the $\mathbb{Z}_2$-valued spectral flow on a circle

In this section, we consider a τ-invariant family of self-adjoint operators $A(t)$, parametrized by $t \in S^1$ and prove an analogue of Theorem 4.5 for it. For the usual $\mathbb{Z}$-valued spectral flow this result basically follows from the APS index theorem, cf. [3, §7]. This theorem is unavailable for the τ-index. Instead, we deduce the result for the periodic families from Theorem 4.5.

Throughout this section, the Hilbert spaces W and H are as in Section 3.1.

6.1. Families of operators on an interval and on a circle

Let $\mathcal{A}([-\pi,\pi], W, H)$ be the space of norm continuous maps $A\colon [-\pi,\pi] \to \mathcal{S}(W,H)$. For $A \in \mathcal{A}([-\pi,\pi], W, H)$, set

$$\widetilde{A}(t) := \begin{cases} A(-\pi) & \text{for } t < -\pi, \\ A(t) & \text{for } -\pi \geq t \leq \pi, \\ A(\pi) & \text{for } t > \pi. \end{cases} \tag{6.1}$$

Then $\widetilde{A} \in \mathcal{A}(\mathbb{R}, W, H)$ if and only if $A(\pi)$ and $A(-\pi)$ are bijective.

Let

$$\mathcal{A}(S^1, W, H) := \{A \in \mathcal{A}([-\pi,\pi], W, H) : A(-\pi) = A(\pi)\}.$$

We view elements of $\mathcal{A}(S^1, W, H)$ as families of operators on a circle.

Suppose that $\alpha\colon H \to H$ is an anti-unitary anti-involution. As in Section 4.1, we define the anti-linear anti-involution τ on the spaces $\mathcal{A}([-\pi,\pi], W, H)$ and $\mathcal{A}(S^1, W, H)$. Let $\mathcal{A}_\tau([-\pi,\pi], W, H)$ and $\mathcal{A}_\tau(S^1, W, H)$ denote the subspaces of τ-invariant elements in $\mathcal{A}([-\pi,\pi], W, H)$ and $\mathcal{A}(S^1, W, H)$ respectively.

Note that, if $A \in \mathcal{A}_\tau(S^1, W, H)$, then

$$A(-\pi) = A(\pi) = \alpha A(\pi)\alpha^{-1},$$

i.e., the operator $A(\pi) = A(-\pi)$ is odd symmetric.

6.2. The half-spectral flow on the circle

For $A \in \mathcal{A}_\tau(S^1, W, H)$ which have only isolated crossings, we define the half-spectral flow by

$$\mathrm{sf}_\tau(A) := \sum_{-\pi<t<0} \operatorname{rank} P(A,t) + \frac{1}{2}\operatorname{rank} P(A,0) + \frac{1}{2}\operatorname{rank} P(A,-\pi) \mod 2.$$

A verbatim repetition of the arguments in the proof of Proposition 4.2 shows that if A_0 and A_1 lie in the same connected component of $\mathcal{A}_\tau(S^1, W, H)$ and have isolated crossings, then $\mathrm{sf}_\tau(A_0) = \mathrm{sf}_\tau(A_1)$. As in Section 4.4, for a general $A \in \mathcal{A}_\tau(S^1, W, H)$, we define the half-spectral flow $\mathrm{sf}_\tau(A)$ as the half-spectral flow of any A_1 which has isolated crossing and lies in the same connected component of $\mathcal{A}_\tau(S^1, W, H)$ as A.

6.3. The operator D_A on an interval and on a circle

Let $\mathcal{H}$ and $\mathcal{W}$ be as in Section 3.3 and let $\mathcal{H}_{[-\pi,\pi]}$ and $\mathcal{W}_{[-\pi,\pi]}$ be similar spaces with $\mathbb{R}$ replaced by $[-\pi, \pi]$ (thus $\mathcal{H}_{[-\pi,\pi]} := L^2([-\pi, \pi], H)$, etc.). Let

$$\mathcal{W}_{S^1} := \{\xi \in \mathcal{W}_{[-\pi,\pi]} : \xi(-\pi) = \xi(\pi)\}.$$

We view $\mathcal{W}_{S^1}$ as a space of H-valued functions on S^1. We also view $\mathcal{H}_{[-\pi,\pi]}$ as the space of square-integrable H-valued functions on S^1 and use the notations $\mathcal{H}_{[-\pi,\pi]}$ and $\mathcal{H}_{S^1}$ for this space interchangeably.

Suppose $A \in \mathcal{A}_\tau(S^1, W, H)$ and consider the operator

$$D_{A,S^1}\colon \xi(t) \mapsto \frac{d}{dt}\xi(t) + A(t)\xi(t), \quad \xi \in \mathcal{W}_{S^1}.$$

It is a bounded odd symmetric operator $\mathcal{W}_{S^1} \to \mathcal{H}_{S^1}$. If we view $\mathcal{W}_{S^1}$ as a subspace of $\mathcal{W}_{[-\pi,\pi]}$, then D_A becomes an operator on the interval $[-\pi, \pi]$ with periodic boundary conditions. It is well known that this operator is Fredholm. When H is a space of square-integrable sections of a bundle over a compact manifold, this follows from the standard description of elliptic boundary conditions for PDE, cf., for example, [6, §7.3] (in [6] the periodic boundary conditions are called "transmission conditions", cf. [6, Example 7.28]). The abstract description of elliptic boundary conditions is very similar, but simpler, cf. [5, Appendix A]).

It follows from Theorem 2.4 that the map $A \to \mathrm{ind}_\tau(D_A)$ is constant on the connected components of $\mathcal{A}_\tau(S^1, W, H)$.

The main result of this section is the following analogue of Theorem 3.7.

Theorem 6.1. *Suppose that $A \in \mathcal{A}_\tau(S^1, W, H)$. Then, the τ-index of the operator $D_{A,S^1}\colon \mathcal{W}_{S^1} \to \mathcal{H}_{S^1}$ is equal to the half-spectral flow* $\mathrm{sf}_\tau(A)$.

The rest of this section is devoted to the proof of this theorem.

6.4. The case when $A(-\pi) = A(\pi)$ is invertible

Suppose, first, that $A(-\pi) = A(\pi)$ is invertible. Then, the operator (6.1) is in $\mathcal{A}_\tau(\mathbb{R}, W, H)$ and $\mathrm{sf}_\tau(\widetilde{A}) = \mathrm{sf}_\tau(A)$. Hence, from Theorem 3.7, we conclude that

$$\mathrm{ind}_\tau D_{\widetilde{A}} = \mathrm{sf}_\tau A. \tag{6.2}$$

We now recall the abstract Atiyah–Patodi–Singer (APS) boundary conditions, cf. [36, §4]. Let $\mathcal{H}_{<0} \subset H$ (resp. $\mathcal{H}_{>0} \subset H$) denote the span of eigenvectors of $A(-\pi) = A(\pi)$ with negative (resp. positive) eigenvalues. Set

$$\mathcal{W}_{\mathrm{APS}} := \{f \in \mathcal{W}_{[-\pi,\pi]} : f(-\pi) \in \mathcal{H}_{<0}, f(\pi) \in \mathcal{H}_{>0}\},$$
$$\mathcal{W}^{\dagger}_{\mathrm{APS}} := \{f \in \mathcal{W}_{[-\pi,\pi]} : f(-\pi) \in \mathcal{H}_{>0}, f(\pi) \in \mathcal{H}_{<0}\}.$$

Then, $\tau\colon \mathcal{W}_{\mathrm{APS}} \to \mathcal{W}^{\dagger}_{\mathrm{APS}}$.

The operator $D_{A,\mathrm{APS}}\colon \mathcal{W}_{\mathrm{APS}} \to \mathcal{H}$ is called the operator D_A with the *APS boundary conditions*. The adjoint of this operator is the operator

$$D^*_{A,\mathrm{APS}} = -\frac{d}{dt} + A(t)\colon \mathcal{W}^{\dagger}_{\mathrm{APS}} \to \mathcal{H}.$$

It follows that $\tau D_{A,\mathrm{APS}}\tau^{-1} = D^*_{A,\mathrm{APS}}$, i.e., *the operator $D_{A,\mathrm{APS}}$ is odd symmetric*.

Recall that the operator $\widetilde{A}$ is defined in (6.1). By [36, Proposition 4.5], the dimension of the kernel of $D_{A,\mathrm{APS}}$ is equal to the dimension of the kernel of the operator $D_{\widetilde{A}}$. Hence, the τ-indices of these operators are equal

$$\operatorname{ind}_\tau D_{A,\mathrm{APS}} = \operatorname{ind}_\tau D_{\widetilde{A}}.$$

Using (6.2), we obtain

$$\operatorname{ind}_\tau D_{A,\mathrm{APS}} = \operatorname{sf}_\tau A.$$

Theorem 6.1 for the case when $A(\pi)$ is invertible follows now from the following

Proposition 6.2. *If $A(\pi) = A(-\pi)$ is invertible, then* $\operatorname{ind}_\tau D_{A,\mathrm{APS}} = \operatorname{ind}_\tau D_{A,S^1}$.

Proof. In the proof of [6, Theorem 8.17] (see also the proof of [14, Theorem 5.11]), the similar statement is proven for the usual index in the case when H is the space of square-integrable sections over a manifold so that D_A is a PDE. The proof goes by constructing a continuous family of elliptic boundary conditions B_s which interpolates between the periodic and the APS boundary conditions: B_0 is the APS boundary conditions and B_1 is the periodic boundary conditions. This construction also works in our case, where the proof of the fact that the intermediate boundary conditions B_s are Fredholm is exactly the same as in the PDE case. (The details can be found in [5, Appendix A].) Then, one obtains a family of operators D_{A,B_s} with different domains. One then shows that there is a unitary equivalent continuous family of operators $D_{A,s}$ of operators with domain $\mathcal{W}_{S^1}$. By construction, all the operators D_{A,B_s} are odd symmetric. Thus, the proposition follows from the stability of the τ-index (Theorem 2.4). ∎

6.5. The case when $A(\pi)$ is not invertible

Suppose $A(-\pi) = A(\pi)$ is not invertible. Since $A(\pi)$ has a discrete spectrum, for small enough $\epsilon > 0$, the operator $A(\pi) + \epsilon$ is invertible. The path $A + \epsilon \in \mathcal{A}_\tau(S^1, W, H)$ is in the same connected component of $\mathcal{A}_\tau(S^1, W, H)$ as A. Hence,

$$\mathrm{sf}_\tau(A + \epsilon) = \mathrm{sf}_\tau(A), \quad \mathrm{ind}_\tau(D_A) = \mathrm{ind}_\tau(D_{A+\epsilon}). \tag{6.3}$$

It is shown in Section 6.4 that $\mathrm{sf}_\tau(A + \epsilon) = \mathrm{ind}(D_{A+\epsilon})$. Theorem 6.1 follows now from (6.3).

7. The half-spectral flow of a family of Toeplitz operators

This section studies a family of odd symmetric Toeplitz operators on a complete Riemannian manifold. Our main result is that the graded half-spectral flow of such a family is equal to the τ-index of a certain Callias-type operator. This is a $\mathbb{Z}_2$-valued analogue of the main result of [10]. In the next section, we apply it to obtain a new proof and a generalization of the bulk-edge correspondence in the Graf–Porta model of topological insulators with time-reversal symmetry [25].

7.1. A (generalized) Dirac operator

Let M be a complete Riemannian manifold. Recall from [30, §II.5] that a graded bundle $E = E^+ \oplus E^-$ over M is called a *Dirac bundle* if it is a Hermitian vector bundle endowed with the Clifford action

$$c\colon T^*M \to \mathrm{End}(E), \quad (c(v)\colon E^\pm \to E^\mp,\ c(v)^2 = -|v|^2,\ c(v)^* = -c(v)),$$

and a Hermitian connection $\nabla^E = \nabla^{E^+} \oplus \nabla^{E^-}$ compatible with the Clifford action.

We extend the Clifford action to the product $E \otimes \mathbb{C}^k$ and denote by D the associated (generalized) Dirac operator. In local coordinates, it can be written in the form $D = \sum_j c(dx^j)\nabla^E_{\partial_j}$. We view D as an unbounded self-adjoint operator $D\colon L^2(M, E \otimes \mathbb{C}^k) \to L^2(M, E \otimes \mathbb{C}^k)$.

From this point on we make the following.

Assumption 7.1. Zero is an isolated point of the spectrum of D.

Let $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- \subset L^2(M, E \otimes \mathbb{C}^k)$ denote the kernel of D and let $P\colon L^2(M, E \otimes \mathbb{C}^k) \to \mathcal{H}$ be the orthogonal projection. Here $\mathcal{H}^\pm$ is the intersection of $\mathcal{H}$ with $L^2(M, E^\pm \otimes \mathbb{C}^k)$. We denote by $P^\pm\colon L^2(M, E^\pm \otimes \mathbb{C}^k) \to \mathcal{H}^\pm$ the restriction of P to $L^2(M, E^\pm \otimes \mathbb{C}^k)$.

7.2. Matrix valued functions

Let $BC(M,k)$ denote the Banach algebra of continuous bounded functions $f(x)$ on M with values in the space $\mathrm{Herm}(k)$ of Hermitian complex-valued $k \times k$-matrices.

Let $C_0(M,k)$ denote the closure of the subalgebra of functions with compact support in $BC(M,k)$.

Let $C_g^\infty(M,k) \subset BC(M,k)$ denote the space of smooth functions with values in $\mathrm{Herm}(k)$ which are bounded and such that df vanishes at infinity. Then, $C_0(M,k) \subset C_g(M,k)$.

For $f \in C_g^\infty(M;k)$, we denote by $M_f\colon L^2(M, E \otimes \mathbb{C}^k) \to L^2(M, E \otimes \mathbb{C}^k)$ the multiplication by $1 \otimes f$ and by $M_f^\pm$ the restriction of M_f to $L^2(M, E^\pm \otimes \mathbb{C}^k)$.

7.3. The Toeplitz operator

Recall that $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ denotes the kernel of D and $P\colon L^2(M, E \otimes \mathbb{C}^k) \to \mathcal{H}$ is the orthogonal projection.

Definition 7.2. The operator

$$T_f := PM_f P\colon \mathcal{H} \to \mathcal{H}$$

is called the *Toeplitz operator defined by* f. We denote by $T_f^\pm$ the restriction of T_f to $\mathcal{H}^\pm$. This is a self-adjoint operator which is even with respect to the grading $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$.

Definition 7.3. We say that a matrix-valued function $f \in C_g^\infty(M;k)$ is *invertible at infinity* if there exists a compact set $K \subset M$ and $C_1 > 0$ such that $f(x)$ is an invertible matrix for all $x \notin K$ and

$$\|f(x)^{-1}\| < C_1 \quad \text{for all } x \notin K. \tag{7.1}$$

The following result is proven in [16, Lemma 2.6].

Proposition 7.4. *If D satisfies Assumption 7.1 and $f \in C_g^\infty(M;k)$ is invertible at infinity, then the Toeplitz operator T_f is Fredholm.*

7.4. A quaternionic bundle structure

Assume now that $\theta^M\colon M \to M$ is an involution. Let $\theta^E\colon E \to \theta^* E$ be an anti-linear *involution* (not anti-involution!), which preserves the Hermitian metric on E. We assume that θ^E has an odd grading degree, i.e., $\theta^E\colon E^\pm \to E^\mp$. Let $\theta^{\mathbb{C}^k}\colon \mathbb{C}^k \to \mathbb{C}^k$ be an anti-unitary anti-involution. By Lemma 2.1, it implies that k is even. The tensor product

$$\theta^{E\otimes\mathbb{C}^k} := \theta^E \otimes \theta^{\mathbb{C}^k} \tag{7.2}$$

is an anti-unitary anti-involution of odd grading degree. As in (2.3), $\theta^{E\otimes\mathbb{C}^k}$ induces an anti-unitary anti-involution $\tau\colon\Gamma(M,E\otimes\mathbb{C}^k)\to\Gamma(M,E\otimes\mathbb{C}^k)$.

Assumption 7.5. The generalized Dirac operator D on $E\otimes\mathbb{C}^k$ is graded odd symmetric, $\tau D\tau^{-1}=D$.

7.5. An odd symmetric family of matrix-valued functions

Let $S^1=\{e^{it}:t\in[-\pi,\pi]\}$ be the unit circle. Consider the involution $\theta^{S^1}\colon t\mapsto -t$ on S^1. Then (S^1,θ^{S^1}) is an involutive manifold.

Set $\mathcal{M}=S^1\times M$. We write the points of $\mathcal{M}$ as (t,x), $t\in S^1$, $x\in M$, and view a smooth matrix-valued function $\mathfrak{f}\colon\mathcal{M}\to\mathrm{Herm}(k)$ as a smooth family $f_t\colon M\to\mathrm{Herm}(k)$, $t\in S^1$. Thus, we usually write this function as $f_t(x)$. We are interested in the functions with $f_t\in C^\infty_g(M;k)$.

The involutions θ^M and θ^{S^1} define an involution $\theta^{\mathcal{M}}:=\theta^{S^1}\times\theta^M$ on $\mathcal{M}$. As usual, we define the anti-unitary anti-involution τ on the spaces of matrix-valued functions on $\mathcal{M}$ by

$$(\tau f)_t(x)=\theta^{\mathbb{C}^k}f_{\theta^{S^1}t}(\theta^M x)(\theta^{\mathbb{C}^k})^{-1}.$$

Definition 7.6. A smooth family $\mathfrak{f}\colon S^1\to C^\infty_g(M;k)$, $t\mapsto f_t$, of matrix-valued functions is called *odd symmetric* if $\tau f=f$.

We denote by $\mathcal{F}_\tau(M,k)$ the set of smooth families $\mathfrak{f}=\{f_t\}$ of odd symmetric matrix-valued functions, such that f_t is invertible at infinity of M for all $t\in S^1$ and there exists a constant $C_2>0$ such that

$$\left\|\frac{\partial}{\partial t}f_t(x)\right\|<C_2,\quad\text{for all } t\in S^1, x\in M. \tag{7.3}$$

Suppose $\mathfrak{f}=\{f_t\}\in\mathcal{F}_\tau(M;k)$. It follows from (7.1) and (7.3) that there exists a compact set $K\subset M$ and a large enough constant $\alpha>0$ such that

$$\frac{\partial}{\partial t}f_t(x)<\frac{\alpha}{2}f_t(x)^2\quad\text{for all } x\notin K. \tag{7.4}$$

Here the inequality $A<B$ between two self-adjoint matrices means that for any vector $v\neq 0$, we have $\langle Av,v\rangle<\langle Bv,v\rangle$.

7.6. The graded half-spectral flow of a family of self-adjoint Toeplitz operators

If $\mathfrak{f}\in\mathcal{F}_\tau(M;k)$, then $T_{\mathfrak{f}}$ $(t\in S^1)$ is a periodic family of odd symmetric self-adjoint Fredholm operators (the Fredholmness follows from Proposition 7.4). Recall that we denote by $T^\pm_{\mathfrak{f}}$ the restriction of $T_{\mathfrak{f}}$ to $\mathcal{H}^\pm$. Our goal is to compute the graded half-spectral flow

$$\mathrm{sf}_\tau\,T^+_{\mathfrak{f}}-\mathrm{sf}_\tau\,T^-_{\mathfrak{f}}$$

of this family.

7.7. A Callias-type operator on $S^1 \times M$

Recall that $\mathcal{M} = S^1 \times M$ is an involutive manifold with $\theta^{\mathcal{M}} = \theta^{S^1} \times \theta^M$. Denote by $\pi_1\colon S^1 \times M \to S^1$ and $\pi_2\colon S^1 \times M \to M$ the natural projections. By a slight abuse of notation, we denote the pull-backs $\pi_1^* dt, \pi_2^* dx \in T^*\mathcal{M}$ by dt and dx respectively. Set

$$\mathcal{E} := \pi_2^* E.$$

Then $\mathcal{E}$ is naturally an ungraded quaternionic Dirac bundle with Clifford action $\mathcal{c}\colon T^*\mathcal{M} \to \mathrm{End}(\mathcal{E})$ such that $\mathcal{c}(dx) = c(dx)$ and $\mathcal{c}(dt)$ is given, with respect to the decomposition $\mathcal{E} = \pi_2^* E^+ \oplus \pi_2^* E^-$, by the matrix

$$\mathcal{c}(dt) = \begin{pmatrix} i \cdot \mathrm{Id} & 0 \\ 0 & -i \cdot \mathrm{Id} \end{pmatrix}.$$

Let $\mathcal{D}$ be the corresponding Dirac operator. With respect to the decomposition

$$L^2(\mathcal{M}, \mathcal{E} \otimes \mathbb{C}^k) = L^2(S^1) \otimes L^2(M, E \otimes \mathbb{C}^k),$$

it takes the form

$$\mathcal{D} = \mathcal{c}(dt)\frac{\partial}{\partial t} \otimes 1 + 1 \otimes D.$$

It follows that $\mathcal{D}$ is odd symmetric. We remark that, as opposed to D, the operator $\mathcal{D}$ is not graded.

Let now $\mathcal{f} = \{f_t\} \in \mathcal{F}_\tau(M; k)$. We view it as a smooth function on $\mathcal{M}$. Let $\mathcal{M}_{\mathcal{f}}\colon L^2(\mathcal{M}, \mathcal{E} \otimes \mathbb{C}^k) \to L^2(\mathcal{M}, \mathcal{E} \otimes \mathbb{C}^k)$ denote the multiplication by $\mathcal{f}$. It is an odd symmetric operator. The commutator

$$[\mathcal{D}, \mathcal{M}_{\mathcal{f}}] := \mathcal{D} \circ \mathcal{M}_{\mathcal{f}} - \mathcal{M}_{\mathcal{f}} \circ \mathcal{D}$$

is a zero-order differential operator, i.e., a bundle map $\mathcal{E} \otimes \mathbb{C}^k \to \mathcal{E} \otimes \mathbb{C}^k$.

From (7.4) and our assumption that df_t vanishes at infinity, we conclude that there exist constants $c, d > 0$ and a compact set $\mathcal{K} \subset \mathcal{M}$, called *an essential support* of $\mathcal{B}_{c\mathcal{f}}$, such that

$$[\mathcal{D}, c\mathcal{M}_{\mathcal{f}}](t,x) < c^2 \mathcal{M}_{\mathcal{f}}(t,x)^2 - d \quad \text{for all } (t,x) \notin \mathcal{K}. \tag{7.5}$$

It follows that

$$\mathcal{B}_{c\mathcal{f}} := \mathcal{D} + ic\mathcal{M}_{\mathcal{f}}$$

is a *Callias-type operator* in the sense of [1, 15] (see also [11, §2.5]). In particular, it is Fredholm. It is also odd symmetric. The τ-index of odd symmetric Callias-type operators was studied in [12, §5].

The main result of this section is the following.

Theorem 7.7. *Let $\mathfrak{f} \in \mathcal{F}_\tau(M;k)$ be a smooth periodic family of invertible at infinity matrix-valued functions. Suppose that Assumptions 7.1 and 7.5 are satisfied. Then,*

$$\mathrm{sf}_\tau(T_{\mathfrak{f}}^+) - \mathrm{sf}_\tau(T_{\mathfrak{f}}^-) = \mathrm{ind}_\tau \mathcal{B}_{c\mathfrak{f}}. \tag{7.6}$$

Remark 7.8. Since both sides of (7.6) live in $\mathbb{Z}_2$, we can replace the minus sign on the left-hand side with the plus sign. We prefer to write the minus sum to stress the analogy with the $\mathbb{Z}$-valued case [10, Theorem 2.1] and to emphasize that this expression gives the *graded half-spectral flow* of the family $T_{\mathfrak{f}}$.

Remark 7.9. Note that, as opposed to df, the differential $d\mathfrak{f}$ does not vanish at infinity. Because of this $\mathcal{B}_{c\mathfrak{f}}$, does not satisfy the conditions of [16, Corollary 2.7] and its index does not vanish in general.

Remark 7.10. In many interesting applications, e.g., in the generalized Graf–Porta model, $H^- = \{0\}$. Hence, $\mathrm{sf}(T_{\mathfrak{f}}^-) = 0$ and (7.6) computes $\mathrm{sf}(T_{\mathfrak{f}}^+)$.

Before giving the proof of Theorem 7.7 in Section 10, we present some special cases and applications of this theorem.

8. The even-dimensional case: the generalized bulk-edge correspondence

Suppose that the dimension of M is even. Then the dimension of $\mathcal{M}$ is odd and by the $\mathbb{Z}_2$-valued version of the Callias-type index theorem [12, §5] the τ-index of $\mathcal{B}_{c\mathfrak{f}}$ is equal to the index of a certain Dirac operator on a compact hypersurface $\mathcal{N} \subset \mathcal{M}$. It follows from Theorem 7.7 that the half-spectral flow of a family of Toeplitz operators on M is equal to the τ-index of a Dirac-type operator on $\mathcal{N}$. We refer to this equality as a *generalized bulk-edge correspondence,* since, as we explain in the next section, in the special case when M is the unit disc in $\mathbb{C}$, this reduces to the bulk-edge correspondence for the Graf–Porta model of topological insulators with time-reversal symmetry [25, 28].

8.1. The $\mathbb{Z}_2$-valued Callias-type theorem

We recall the $\mathbb{Z}_2$-valued version of the Callias-type theorem from [12].

Let $N \subset M$ be a θ^M-invariant hypersurface such that there is an essential support $\mathcal{K} \subset \mathcal{M}$ of $\mathcal{B}_{c\mathfrak{f}}$ whose boundary $\partial\mathcal{K} = \mathcal{N} := S^1 \times N$. In particular, the restriction of $\mathfrak{f}$ to $\mathcal{N}$ is invertible and satisfies (7.5). Then, there are quaternionic vector bundles $\mathcal{F}_{\mathcal{N}\pm}$ over $\mathcal{N}$ such that

$$\mathcal{M} \times \mathbb{C}^k = \mathcal{F}_{\mathcal{N}+} \oplus \mathcal{F}_{\mathcal{N}-},$$

and the restriction of $\not{f}$ to $\mathcal{F}_{\mathcal{N}+}$ (respectively, $\mathcal{F}_{N-}$) is positive definite (respectively, negative definite). Since $\not{f}$ is a τ-invariant function, the restriction of $\theta^{\mathbb{C}^k}$ to $\mathcal{F}_{\mathcal{N}\pm}$ defines a quaternionic structure on these vector bundles.

Let E_N denote the restriction of the bundle E to N. Then $\mathcal{E}_{\mathcal{N}} := \pi_2^* E_N$ is the restriction of $\mathcal{E}$ to $\mathcal{N}$. It is naturally a Dirac bundle over $\mathcal{N}$. The restriction of $\theta^{\mathcal{E}} \otimes \theta^{\mathbb{C}^k}$ to $\mathcal{E}_{\mathcal{N}} \otimes \mathcal{F}_{N+}$ defines a structure of a quaternionic bundle on $\mathcal{E}_{\mathcal{N}} \otimes \mathcal{F}_{N+}$. The induced Dirac-type operator $\mathcal{D}_{\mathcal{N}}$ on $\mathcal{E}_{\mathcal{N}} \otimes \mathcal{F}_{\mathcal{N}+}$ is odd symmetric.

Let v denote the unit normal vector to $\mathcal{N}$ pointing towards $\mathcal{K}$. Then $i\,c(v)\colon \mathcal{E}_{\mathcal{N}} \to \mathcal{E}_{\mathcal{N}}$ is an involution. We denote by $\mathcal{E}_{\mathcal{N}}^{\pm 1}$ the eigenspace of $i\,c(v)$ with eigenvalue ± 1. This defines a grading $\mathcal{E}_{\mathcal{N}} = \mathcal{E}_{\mathcal{N}}^{+1} \oplus \mathcal{E}_{\mathcal{N}}^{-1}$ on the Dirac bundle $\mathcal{E}_N$. The operator $\mathcal{D}_{\mathcal{N}}$ is odd with respect to the induced grading on $\mathcal{E}_{\mathcal{N}} \otimes \mathcal{F}_{\mathcal{N}+}$. (This grading is different from the one induced by the grading on E. Note that the operator $\mathcal{D}_{\mathcal{N}}$ is not an odd operator with respect to the grading induced by the grading on E.) We denote by $\mathcal{D}_{\mathcal{N}}^{\pm}$ the restriction of $\mathcal{D}_{\mathcal{N}}$ to $\mathcal{E}_{\mathcal{N}}^{\pm 1} \otimes \mathcal{F}_{\mathcal{N}+}$.

Theorem 5.3 of [12] states that

$$\operatorname{ind}_\tau \mathcal{B}_{c\not{f}} = \operatorname{ind}_\tau \mathcal{D}_{\mathcal{N}}^{+}.$$

8.2. Computation of the graded spectral flow in even-dimensional case

We now get the following corollary of Theorem 7.7.

Corollary 8.1. *Under the conditions of Theorem 7.7, assume that* $\dim M$ *is even. Let* $j\colon N \hookrightarrow M$ *be a hypersurface such that there is an essential support* $\mathcal{K} \subset \mathcal{M}$ *of* $\mathcal{B}_{c\not{f}}$ *whose boundary* $\partial\mathcal{K} = \mathcal{N} := S^1 \times N$*. Then,*

$$\operatorname{sf}_\tau(T_{\not{f}}^+) - \operatorname{sf}_\tau(T_{\not{f}}^-) = \operatorname{ind}_\tau \mathcal{D}_{\mathcal{N}}^{+}.$$

8.3. The pseudo-convex domain

Let M be a bounded strongly pseudo-convex domain in $\mathbb{C}^n$ with smooth boundary $N := \partial\bar{M}$. Let g^M be the Bergman metric on M, cf. [35, §7]. We define $E = \Lambda^{n,*}(T^*M)$ and set $E^+ = \Lambda^{n,\mathrm{even}}(T^*M)$, $E^- = \Lambda^{n,\mathrm{odd}}(T^*M)$. Then E is naturally a Dirac bundle over M whose space of smooth section coincides with the Dolbeault complex $\Omega^{n,*}(M)$ of M with coefficients in the canonical bundle $K = \Lambda^{n,0}(T^*M)$. Moreover, the corresponding Dirac operator is given by

$$D = \bar{\partial} + \bar{\partial}^*,$$

where $\bar{\partial}$ is the Dolbeault differential and $\bar{\partial}^*$ its adjoint with respect to the L^2-metric induced by the Bergman metric on M. Then (M, g^M) is a complete Kähler manifold.

By [22, §5], zero is an isolated point of the spectrum of D and $\mathscr{H} := \ker D$ is a subset of the space of $(n,0)$-forms

$$\mathscr{H} := \ker D \subset \Omega^{n,0}(M). \tag{8.1}$$

In particular, Assumption 7.1 is satisfied and $\mathscr{H}^- = \{0\}$.

Let $\theta^M : M \to M$ be an anti-holomorphic metric preserving involution. We define an anti-unitary involution (not anti-involution!) on E by

$$\theta^E : \omega \mapsto \overline{(\theta^M)^* \omega}, \quad \omega \in \Lambda^{n,*}(T^*M),$$

where the bar denotes the complex conjugation and $(\theta^M)^*$ is the pull-back. Note that, since θ^M is anti-holomorphic, $(\theta^M)^*$ sends $(n,*)$-forms to $(*,n)$-forms and the complex conjugation sends them back to $(n,*)$-forms.

Let $\theta^{\mathbb{C}^k} : \mathbb{C}^k \to \mathbb{C}^k$ be an anti-unitary anti-involution and let $\theta^{E\otimes\mathbb{C}^k}$ be given by (7.2). Then $E \otimes \mathbb{C}^k$ is a quaternionic bundle and $D \otimes 1$ is an odd symmetric operator on it. In view of (8.1), Corollary 8.1 implies the following.

Corollary 8.2. *Let $\bar{\mathscr{f}}(t,x)$ be a smooth τ-invariant* $\mathrm{Herm}(k)$*-valued function on* $\mathcal{N} := S^1 \times N$ *such that $\bar{\mathscr{f}}(t,x)$ is invertible for all $t \in S^1$, $x \in N$. Let $\mathscr{f} : \mathcal{M} \to$* $\mathrm{Herm}(k)$ *be a smooth τ-invariant extension of $\bar{\mathscr{f}}$ to $\mathcal{M}$ (i.e., $\mathscr{f}|_{\mathcal{N}} = \bar{\mathscr{f}}$). Then, $\mathscr{f} \in \mathcal{F}_\tau(M,k)$ and*

$$\mathrm{sf}_\tau\, T_{\mathscr{f}} = \mathrm{ind}_\tau\, \mathcal{D}^+_{\mathcal{N}}.$$

We refer to this result as the *generalized bulk-edge correspondence* for the reason explained in the next section.

9. The Graf–Porta model of topological insulators with time-reversal symmetry

In this section, we briefly review a tight-binding model for two-dimensional topological insulators, see, for example, [2,24,27,32]. We are mostly interested in insulators with fermionic time-reversal symmetry. Our exposition follows the description in [28] (see also [25]). We show that the bulk-edge correspondence for this model follows immediately from our Corollary 8.2.

9.1. The bulk Hamiltonian

We consider the following *tight binding model.* The "bulk" state space is the space $l^2(\mathbb{Z}\times\mathbb{Z}, \mathbb{C}^k)$ of square-integrable sequences $\phi = \{\phi_{mn}\}_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}$, where $\phi_{mn} \in \mathbb{C}^k$.

The *bulk Hamiltonian* $H: l^2(\mathbb{Z}\times\mathbb{Z},\mathbb{C}^k)\to l^2(\mathbb{Z}\times\mathbb{Z},\mathbb{C}^k)$ is defined as follows. Consider a sequence of matrices $A_{p,q}:\mathbb{C}^k\to\mathbb{C}^k$ such that $\sum_{pq}\|A_{p,q}\|<\infty$ and define H by

$$(H\phi)_{mn}=\sum_{p,q}A_{p,q}\phi_{m-p,n-q}.$$

We assume that H is self-adjoint, which is equivalent to the equality

$$A_{-p,-q}=A^*_{p,q}. \tag{9.1}$$

The Fourier transform of H is a family of self-adjoint $k\times k$-matrices

$$H(t,s):=\sum_{pq}A_{p,q}e^{ipt}e^{iqs},\quad (t,s)\in[-\pi,\pi]\times[-\pi,\pi]\simeq S^1\times S^1.$$

depending smoothly on s and t.

We assume that the bulk Hamiltonian has a spectral gap at Fermi level $\mu\in\mathbb{R}$, i.e., there exists $\epsilon>0$ such that the spectrum of $H(s,t)$ does not intersect the interval $(\mu-\epsilon,\mu+\epsilon)$ for all $(s,t)\in S^1\times S^1$. In particular, the operator $H(s,t)-\mu$ is invertible for all $(s,t)\in S^1\times S^1$. Thus, the trivial bundle $(S^1\times S^1)\times\mathbb{C}^k$ over the torus $S^1\times S^1$ decomposes into the direct sum of subbundles

$$(S^1\times S^1)\times\mathbb{C}^k=\mathcal{F}_+\oplus\mathcal{F}_-$$

such that the restriction of $H(s,t)-\mu$ to $\mathcal{F}_+$ is positive definite and the restriction of $H(s,t)-\mu$ to $\mathcal{F}_-$ is negative define. The bundle $\mathcal{F}_+$ is called the *Bloch bundle*.

9.2. The time-reversal symmetry

Define the involution $\theta^{S^1\times S^1}: S^1\times S^1\to S^1\times S^1$ by $\theta^{S^1\times S^1}(t,s)=(-t,-s)$ $((s,t)\in[-\pi,\pi]\times[-\pi,\pi])$. Let

$$\theta^{\mathbb{C}^k}:\mathbb{C}^k\to\mathbb{C}^k$$

be an anti-unitary anti-involution (hence, k is even).

The anti-unitary anti-involution

$$\tau:\phi(t,s)\to\theta^{\mathbb{C}^k}\phi(-t,-s)$$

on the space $L^2(S^1\times S^1,\mathbb{C}^k)$ of square-integrable $\mathbb{C}^k$-valued functions on $S^1\times S^1$ is called a *time-reversal symmetry.* We assume that $H(s,t)$ is τ-invariant. Equivalently, this means that

$$\theta^{\mathbb{C}^k}A_{p,q}(\theta^{\mathbb{C}^k})^{-1}=A^*_{-p,-q}=A_{p,q}, \tag{9.2}$$

where in the last equality we used (9.1).

Remark 9.1. In [25], the authors consider the operators $H_q(t) := \sum_p A_{p,q}e^{ipt}$. Then, the time-reversal symmetry condition (9.2) becomes

$$\theta^{\mathbb{C}^k} H_q(t)(\theta^{\mathbb{C}^k})^{-1} = H_q(-t),$$

which agrees with [25, Definition 2.4].

Notice that τ preserves the subbundles $\mathcal{F}_\pm$ of the trivial bundle over $S^1 \times S^1$. Hence, $\mathcal{F}_\pm$ a quaternionic bundles over $S^1 \times S^1$ (in particular, their ranks are even).

9.3. The $\mathbb{Z}_2$-valued bulk index

Let $M = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in $\mathbb{C}$. It is a pseudo-convex domain with boundary $\partial M = S^1$. As usual, we set $\mathcal{M} := S^1 \times M$ so that $\mathcal{N} := \partial\mathcal{M} = S^1 \times S^1$. The function

$$\mathfrak{f}(t,z) := \sum_{pq} A_{p,q}e^{ipt}z^q, \quad t \in [-\pi,\pi], z \in M, \tag{9.3}$$

is an extension of $f(t,s) := H(t,s)$ from $\partial\mathcal{M}$ to $\mathcal{M}$ as in Corollary 8.2. Let $T_{\mathfrak{f}}$ and $\mathcal{D}_{\mathcal{N}}$ be as in Section 8.3. Then $\mathcal{D}_{\mathcal{N}}$ is an odd symmetric Dirac-type operator acting on the sections of the bundle $\mathcal{F}_+$ over $\partial\mathcal{M} = S^1 \times S^1$.

Definition 9.2. The $\mathbb{Z}_2$*-valued bulk index* of the Hamiltonian H is

$$\mathcal{I}_{\mathrm{Bulk},\tau} := \mathrm{ind}_\tau\, \mathcal{D}_{\mathcal{N}}^+. \tag{9.4}$$

Graf and Porta, [25, §4], give a different, more combinatorial, definition of the bulk index.

Lemma 9.3. *Our definition of the bulk index is equivalent to* [25, Definition 4.8].

Proof. Since the set of Dirac-type operators on the space of differential forms $\Omega^{n,*}(M,\mathcal{F}_+)$ is connected, the bulk index depends only on the pair $(\mathcal{F}_+,\tau)$.

More generally, for any quaternionic bundle $\mathcal{F}$ over $\mathcal{N} = S^1 \times S^1$, there exists a complementary quaternionic bundle $\mathcal{F}'$ such that $\mathcal{F} \oplus \mathcal{F}' \simeq \mathcal{N} \times \mathbb{C}^k$ and a function $H\colon \mathcal{M} \to \mathrm{Herm}(k)$ such that $\mathcal{F} = \mathcal{F}_+$ (where $\mathcal{F}_+$ is as in Section 9.1). Thus, the bulk index can be viewed as a map from the set of quaternionic bundles over $S^1 \times S^1$ to $\mathbb{Z}_2$.

The Grothendieck group of quaternionic vector bundles over an involutive manifold X is the symplectic K-theory $KSp^0(X)$, see [28, §2] and references therein. In particular,

$$KSp^0(S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z}_2,$$

where the $\mathbb{Z}$ summand corresponds to the product bundles $E = (S^1 \times S^1) \times \mathbb{C}^n$, cf. [28, §7.1].

Recall that, by Lemma 2.1, the rank of any quaternionic bundle over $S^1 \times S^1$ is even. It follows that for a product bundle, the kernel of the Dirac operator on $\Omega^{n,*}(M, \mathbb{C}^n) = \Omega^{n,*}(M) \otimes \mathbb{C}^n$ is even dimensional. Thus, the bulk index vanishes on product bundles. Hence, the bulk index defines a map

$$\mathcal{I}_{\mathrm{Bulk},\tau}\colon KSp^0(S^1 \times S^1)/\mathbb{Z} \simeq \mathbb{Z}_2 \to \mathbb{Z}_2.$$

There is only one non-trivial map like this. [12, Example 3] shows that the bulk index is not always equal to zero and thus defines a unique non-trivial map from $KSp^0(S^1 \times S^1)/\mathbb{Z}$ to $\mathbb{Z}_2$.

The Graf–Porta construction of the bulk index, [25, Definition 4.8], also defines a non-trivial map $KSp^0(S^1 \times S^1)/\mathbb{Z} \to \mathbb{Z}_2$ (see also [28, §7.1], where the Graf–Porta bulk index is described as a map from $KSp^0(S^1 \times S^1)/\mathbb{Z}$). Hence, this index must coincide with $\mathcal{I}_{\mathrm{Bulk},\tau}$. ∎

9.4. The edge Hamiltonian

We now take the boundary into account. The "edge" state space is the space $l^2(\mathbb{Z}_{\geq 0} \times \mathbb{Z}, \mathbb{C}^k)$ of square integrable sequences of vectors in $\mathbb{C}^k$ on the half-lattice $\mathbb{Z}_{\geq 0} \times \mathbb{Z}$.

The Fourier transform of the bulk Hamiltonian $H\colon l^2(\mathbb{Z} \times \mathbb{Z}, \mathbb{C}^k) \to l^2(\mathbb{Z} \times \mathbb{Z}, \mathbb{C}^k)$ in the direction "along the edge"

$$H_q(t) := \sum_p A_{p,q} e^{ipt}, \quad t \in [-\pi, \pi],$$

transforms H into a family of self-adjoint translation-invariant operators

$$H(t)\colon l^2(\mathbb{Z}, \mathbb{C}^k) \to l^2(\mathbb{Z}, \mathbb{C}^k), \quad (H(t)\phi)_n := \sum_p H_p(t)\phi_{n-p}.$$

Then, $H(t)$ depends smoothly on $t \in S^1$. It follows from (9.1) that $H(-t) = H(t)^*$ and (9.2) is equivalent to

$$\theta^{\mathbb{C}^k} H(t) (\theta^{\mathbb{C}^k})^{-1} = H(-t). \tag{9.5}$$

(compare to [25, Definition 2.4]).

We view $S^1 \simeq [-\pi, \pi]$ as an involutive manifold with involution $\theta^{S^1}\colon t \to -t$. The anti-involution $\theta^{\mathbb{C}^k}$ defines a structure of a quaternionic bundle on the trivial bundle $S^1 \times \mathbb{C}^k$ over S^1. As usual, we define an anti-involution

$$\tau\colon L^2(S^1, l^2(\mathbb{Z}, \mathbb{C}^k)) \to L^2(S^1, l^2(\mathbb{Z}, \mathbb{C}^k)), \quad \tau\colon f(t) \mapsto \theta^{\mathbb{C}^k} f(-t).$$

Then $H(t)$ is τ-invariant by (9.5).

Let

$$\Pi : l^2(\mathbb{Z}, \mathbb{C}^k) \to l^2(\mathbb{Z}_{\geq 0}, \mathbb{C}^k)$$

denote the projection. Clearly, $\theta^{\mathbb{C}^k} \Pi (\theta^{\mathbb{C}^k})^{-1} = \Pi$.

Definition 9.4. The *edge Hamiltonian* is the family of Toeplitz operators

$$H^{\#}(t) := \Pi \circ H(t) \circ \Pi : l^2(\mathbb{Z}_{\geq 0}, \mathbb{C}^k) \to l^2(\mathbb{Z}_{\geq 0}, \mathbb{C}^k), \quad t \in S^1.$$

Then,

$$\theta^{\mathbb{C}^k} H^{\#}(t) (\theta^{\mathbb{C}^k})^{-1} = H^{\#}(-t).$$

Definition 9.5. The $\mathbb{Z}_2$*-valued edge index* $\mathcal{I}_{\mathrm{Edge},\tau}$ of the Hamiltonian H is the half-spectral flow of the edge Hamiltonian

$$\mathcal{I}_{\mathrm{Edge},\tau} := \mathrm{sf}_\tau(H^{\#}(t)). \tag{9.6}$$

Theorem 9.6 (Bulk-Edge Correspondence). $\mathcal{I}_{\mathrm{Bulk},\tau} = \mathcal{I}_{\mathrm{Edge},\tau}$.

Proof. Consider the unit disc $B := \{z \in \mathbb{C} : |z| \leq 1\}$. This is a strongly pseudo-convex domain in $\mathbb{C}$. We view the bulk Hamiltonian $H(s,t)$ as a function on $\partial B \times S^1$ with values in the set $\mathrm{Herm}(k)$ of invertible Hermitian $k \times k$-matrices.

We endow B with the Bergman metric and consider the Dolbeault-Dirac operator $D = \bar\partial + \bar\partial^*$ on the space $L^2\Omega^{1,*}(B, \mathbb{C}^k) = L^2\Omega^{1,0}(B, \mathbb{C}^k) \oplus L^2\Omega^{1,1}(B, \mathbb{C}^k)$ of square-integrable $(1,*)$-forms on B. Let $\mathcal{H} := \ker D$. Then $\mathcal{H} \subset L^2\Omega^{1,0}(B, \mathbb{C}^k)$, cf. [22, §5]. Let $P : L^2\Omega^{1,*}(B, \mathbb{C}^k) \to \mathcal{H}$ denote the orthogonal projection.

As in (9.3), we define a family of functions on the unit disc by

$$f_t(z) := \sum_{pq} A_{p,q} e^{ipt} z^q, \quad t \in [-\pi, \pi], z \in M,$$

and consider the family of Toeplitz operators

$$T_{f_t} := P \circ M_f \circ P : \mathcal{H} \to \mathcal{H}, \quad t \in [-\pi, \pi],$$

This family is closely related to the family $H^{\sharp}(t)$ as we shall now explain.

The map $\varphi : l^2(\mathbb{Z}_{\geq 0}) \to \mathcal{H}$ which sends $u = \{u_j\}_{j \geq 0} \in l^2(\mathbb{Z}_{\geq 0}, \mathbb{C}^k)$ to the 1-form

$$\phi(u) := \sum_{j \geq 0} u_j z^j \, dz \in \mathcal{H},$$

is an isomorphism of vector spaces. By [18], the difference

$$T_{f_t} - \phi \circ H^{\#}(t) \circ \phi^{-1} : \mathcal{H} \to \mathcal{H}, \quad t \in [-\partial, \pi].$$

is a continuous family of compact operators. By Proposition 4.2,

$$\mathrm{sf}_\tau(T_{f_t}) = \mathrm{sf}_\tau(\phi \circ H^\#(t) \circ \phi^{-1}) = \mathrm{sf}_\tau(H^\#(t)).$$

The theorem follows now from definitions of bulk and edge indexes, (9.4), (9.6), and Corollary 8.2. ∎

10. The proof of Theorem 7.7

As in [10, §5], the proof of Theorem 7.7 consists of two steps. First (Lemma 10.1) we apply Theorem 6.1 to conclude that the half-spectral flow $\mathrm{sf}_\tau(T_{f_t})$ is equal to the τ-index of a certain odd symmetric operator on $\mathcal{M}$. Then, using the argument similar to [16], we show that the latter index is equal to the τ-index of $\mathcal{B}_{c\ell}$.

10.1. The spectral flow as an index

Recall that the space $L^2(\mathcal{M}, \mathcal{E} \otimes \mathbb{C}^k)$ has an involution τ. We view the space $L^2(S^1, \mathcal{H})$ of square integrable functions with values in $\mathcal{H} \subset L^2(M, \mathcal{E} \otimes \mathbb{C}^k)$ as a subspace of $L^2(\mathcal{M}, \mathcal{E} \otimes \mathbb{C}^k)$. Then τ induces an anti-involution on $L^2(S^1, \mathcal{H})$ which we still denote by τ. The family T_{f_t} naturally induces an operator $L^2(S^1, \mathcal{H}) \to L^2(S^1, \mathcal{H})$ which we still denote by T_{f_t}. Let $T^\pm_{f_t}$ denote the restriction of T_{f_t} to $L^2(S^1, \mathcal{H}^\pm)$. These operators are odd symmetric.

Theorem 6.1 implies the following.

Lemma 10.1. *Under the assumptions of Theorem 7.7, we have*

$$\mathrm{sf}_\tau(T^\pm_{f_t}) = \mathrm{ind}_\tau\Big(\frac{\partial}{\partial t} + T^\pm_{f_t}\Big)\Big|_{L^2(S^1,\mathcal{H}^\pm)}. \tag{10.1}$$

Since $\mathrm{sf}_\tau(T^\pm_{f_t}) = -\mathrm{sf}_\tau(-T^\pm_{f_t})$, the spectral flows $\mathrm{sf}_\tau(T^\pm_{f_t})$ and $\mathrm{sf}_\tau(-T^\pm_{f_t})$ are equal mod 2. Hence, (10.1) is equivalent to

$$\mathrm{sf}_\tau(T^\pm_{f_t}) = \mathrm{ind}_\tau\Big(\frac{\partial}{\partial t} - T^\pm_{f_t}\Big)\Big|_{L^2(S^1,\mathcal{H}^\pm)}. \tag{10.2}$$

Lemma 10.2. *Under the assumptions of Theorem 7.7, we have*

$$\mathrm{ind}_\tau \mathcal{B}_{c\ell} = \mathrm{ind}_\tau\Big(\frac{\partial}{\partial t} - T^+_{f_t}\Big)\Big|_{L^2(S^1,\mathcal{H}^+)} + \mathrm{ind}_\tau\Big(\frac{\partial}{\partial t} + T^-_{f_t}\Big)\Big|_{L^2(S^1,\mathcal{H}^-)}. \tag{10.3}$$

Proof. Recall that we denote by P the orthogonal projection $L^2(M, E \otimes \mathbb{C}^k) \to \mathcal{H}$. To simplify the notation, we write P also for the operator $1 \otimes P$ on $L^2(\mathcal{M}, \mathcal{E} \otimes \mathbb{C}^k)$ and set $Q := 1 - P$. Then, $L^2(S^1, \mathcal{H})$ coincides with the image of the projection

$$P : L^2(\mathcal{M}, \mathcal{E} \otimes \mathbb{C}^k) \to L^2(\mathcal{M}, \mathcal{E} \otimes \mathbb{C}^k).$$

It is shown in the proof of [10, Lemma 5.3] that the operator

$$\mathcal{B}_{c\ell} - P \circ \mathcal{B}_{c\ell} \circ P|_{\mathrm{Im} P} - Q \circ \mathcal{B}_{c\ell} \circ Q|_{\mathrm{Im} Q}$$

is compact. Hence,

$$\operatorname{ind}_\tau \mathcal{B}_{c\ell} = \operatorname{ind}_\tau P \circ \mathcal{B}_{c\ell} \circ P|_{\mathrm{Im} P} + \operatorname{ind}_\tau Q \circ \mathcal{B}_{c\ell} \circ Q|_{\mathrm{Im} Q}. \tag{10.4}$$

Let us first compute the second term on the right-hand side. Set

$$A := c(dt)\frac{\partial}{\partial t} + ic\mathcal{M}_\ell.$$

Then, $\mathcal{B}_{c\ell} = 1 \otimes D + A$. Consider a one-parameter family of operators

$$\mathcal{B}_{c\ell,u} := 1 \otimes D + uA, \quad 0 \le u \le 1.$$

As in Section 7.7, from (7.4) and our assumption that df_t vanishes at infinity, we conclude that, for all $u > 0$, the operator $\mathcal{B}_{c\ell,u}$ is of Callias type and, hence, Fredholm.

For $\phi \in L^2(\mathcal{M}, \mathcal{E} \otimes \mathbb{C}^k)$, define a new norm $|||\phi|||$ by

$$|||\phi|||^2 = \|\phi\|^2 + \|(1 \otimes D)\phi\|^2 + \left\|\frac{\partial}{\partial t}\phi\right\|^2 + \|\mathcal{M}_\ell \phi\|^2 \tag{10.5}$$

and let

$$W := \{\phi \in L^2(\mathcal{M}, \mathcal{E} \otimes \mathbb{C}^k) : |||\phi||| < \infty\}.$$

Then W is a dense subspace of $L^2(\mathcal{M}, \mathcal{E} \otimes \mathbb{C}^k)$. We consider it as a Hilbert space with norm (10.5). Then $\mathcal{B}_{c\ell,u}$ $(0 \le u \le 1)$ is a norm continuous family of operators $W \to L^2(\mathcal{M}, \mathcal{E} \otimes \mathbb{C}^k)$. One checks that $P, Q \colon W \to W$.

The operator $Q \circ \mathcal{B}_{c\ell,0} \circ Q|_{\mathrm{Im} Q} = Q \circ D \circ Q|_{\mathrm{Im} Q}$ is invertible. Hence, it is Fredholm and its τ-index is equal to 0. Thus, $Q \circ \mathcal{B}_{c\ell,u} \circ Q|_{\mathrm{Im} Q}$ is a norm-continuous family of operators from $Q(W)$ to $\mathrm{Im} Q \subset L^2(\mathcal{M}, \mathcal{E} \otimes \mathbb{C}^k)$ and these operators are Fredholm for all $u \in [0, 1]$. We conclude that the index of these operators is independent of u and is equal to 0.

From (10.4), we now obtain

$$\begin{aligned}
\operatorname{ind}_\tau \mathcal{B}_{c\ell} &= \operatorname{ind} \mathcal{B}_{c\ell,1} = \operatorname{ind}_\tau P \circ \mathcal{B}_{c\ell,1} \circ P|_{\mathrm{Im} P} \\
&= \operatorname{ind}_\tau P^+ \circ \Big(i\frac{\partial}{\partial t} - ic\mathcal{M}_\ell\Big) \circ P^+\Big|_{\mathrm{Im} P^+} \\
&\quad + \operatorname{ind}_\tau P^- \circ \Big(-i\frac{\partial}{\partial t} - ic\mathcal{M}_\ell\Big) \circ P^-\Big|_{\mathrm{Im} P^-} \\
&= \operatorname{ind}_\tau \Big(\frac{\partial}{\partial t} - T_{f_t}^+\Big)\Big|_{L^2(S^1, \mathcal{H}^+)} + \operatorname{ind}_\tau \Big(\frac{\partial}{\partial t} + T_{f_t}^-\Big)\Big|_{L^2(S^1, \mathcal{H}^-)}.
\end{aligned}$$

■

10.2. Proof of Theorem 7.7

Theorem 7.7 follows now from (10.1), (10.2), and (10.3).

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