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Non-existence of cusps for degenerate Alt–Caffarelli functionals

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Abstract. We study a class of free-boundary problems for *degenerate* one-phase Alt–Caffarelli functionals $J_Q(v,\Omega) := \int_\Omega |\nabla v|^2 + Q^2(x)\chi_{\{v>0\}}dx$. More specifically, we consider $Q(x) = \mathrm{dist}(x,\Gamma)^\gamma$ for an affine k-plane Γ and $\gamma > 0$. Because Q vanishes on Γ, the techniques of Alt and Caffarelli (1981) for proving non-degeneracy of local minimizers u and weak geometric regularity of their positivity sets $\{u > 0\}$ fail near Γ. In this note we prove that despite the degeneration of Q local minimizers, u and $\{u > 0\}$ still satisfy an analogous non-degeneracy condition near Γ. This non-degeneracy is sufficient to eliminate the existence of cusps in a wider class of cases than previously known.

1. Introduction

In this note, we improve upon a non-degeneracy result of [2] for local minimizers of one-phase Alt–Caffarelli functionals in (1.1) in the case that the functionals are *degenerate* and the positivity set intersects $\{Q = 0\}$; see Theorem 1.2. As an application, we eliminate the existence of cusps for these free-boundary problems.

Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be an open set with Lipschitz boundary with $\mathcal{H}^{n-1}(\partial\Omega) > 0$. The one-phase Alt–Caffarelli functional is defined as follows:

$$J_Q(v,\Omega) := \int_\Omega |\nabla v(x)|^2 + Q^2(x)\chi_{\{v>0\}}dx. \tag{1.1}$$

For a $u_0 \in W^{1,2}(\Omega)$ satisfying $u_0 \geq 0$, we define the class

$$\mathcal{K}_{u_0,\Omega} := \left\{u \in W^{1,2}(\Omega) : u - u_0 \in W^{1,2}_0(\Omega)\right\}.$$

A function $u \in \mathcal{K}_{u_0,\Omega}$ is called a *local minimizer* of (1.1) in $\mathcal{K}_{u_0,\Omega}$ if there exists an $\varepsilon_0 > 0$ such that $J_Q(u,\Omega) \leq J_Q(v,\Omega)$ for every $v \in K_{u_0,\Omega}$ satisfying

$$\|\nabla(u-v)\|^2_{L^2(\Omega)} + \|\chi_{\{u>0\}} - \chi_{\{v>0\}}\|_{L^1(\Omega)} < \varepsilon_0. \tag{1.2}$$

Where quantification is necessary, we shall refer to such a function u as an ε_0-*local mini-*

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mizer. Since any local minimizer u in $\mathcal{K}_{u_0,\Omega}$ is a local minimizer in $\mathcal{K}_{u,\Omega}$, we shall often suppress the class in our notation. For any function u that is a local minimizer of (1.1), the set $\partial\{u > 0\} \cap \Omega$ is called the *free-boundary* of the *positivity set* $\{u > 0\}$.

The study of the partial regularity of the free boundary for local minimizers of functionals like (1.1) has been an active area of research since Alt and Caffarelli's seminal paper [2], which proved a regularity theory for the free boundary in analogy with the regularity theory for minimal surfaces and obstacle problems (see [6, 12, 17, 30]). Their work has formed the foundation of the study of many related free-boundary problems (see [3, 7–11, 18, 27], just to name a few).

However, virtually all work on the geometry of the free-boundary for local minimizers of J_Q (and its generalizations) has continued to assume that the function Q is *non-degenerate* in the sense that

$$0 < Q_{\min} \leq Q \leq Q_{\max} < \infty.$$

This assumption plays an important role in the non-degeneracy of the locally minimizing function u and the regularity of the free-boundary. This relationship is highlighted in the following lemma:

Lemma 1.1 ([2, Lemma 3.4]). *Let u be an ε_0-local minimizer of $J_Q(\cdot, B_2(0))$ in the class $K_{u_0,B_2(0)}$. Let $s \in (0, 1)$ be fixed. Then, there exist constants $C_{\min} = C(n, s)$ and $c_0(n, Q_{\max}, \varepsilon_0) > 0$ such that for all $0 < r \leq c_0$ and all $x_0 \in B_1(0)$ such that if $\{u > 0\} \cap B_{sr}(x_0) \neq \emptyset$, then*

$$\fint_{\partial B_r(x_0)} u\, d\sigma > C_{\min} r \min_{y \in B_{sr}(x_0)} \{Q(y)\}.$$

Thus, the assumption that $0 < Q_{\min}$ implies that if $x_0 \in \partial\{u > 0\} \cap \Omega$, rescalings $\widetilde{u}_{x_0,r}(y) := \frac{1}{r} u(ry + x_0)$ are uniformly non-degenerate for all sufficiently small r. This guarantees that limit functions—as might arise in compactness or blow-up arguments, for example—exist and are non-trivial. It is also instrumental in proving that the positivity set of local minimizers satisfies both the interior and exterior corkscrew condition (a kind of weak geometric regularity; see Lemma 2.9). Among other things, this immediately implies that the set

$$\Sigma := \left\{x_0 \in \partial\{u > 0\} \cap \Omega : \Theta^n_{\{u>0\}}(x_0) = 0\right\} \tag{1.3}$$

is empty in regions where $Q_{\min} > 0$. Note that we use

$$\Theta^n_E(x_0) := \lim_{r \to 0} \frac{\mathcal{H}^n(E \cap B_r(x_0))}{\omega_n r^n}$$

to denote the density of the set E.

And yet, there are natural situations in which Q may vanish. Let $n = 2$, write $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, $\Omega = [0, 1] \times [0, +\infty)$, and consider (1.1) with $Q(x, y) = \sqrt{(h - y)_+}$ for some

$0 < h < \infty$. Two-dimensional periodic waves of incompressible, irrotational fluids of finite depth may be modeled as *critical points* of (1.1) for such choices (see [19, 20, 23, 26]). Moreover, there are interesting situations in which the resulting free-boundary actually touches $\{Q = 0\}$. In the case of two-dimensional periodic water waves, this is the famous Stokes wave, the wave of extreme height (see [4, 24, 25, 28]).

In [5], Arama and Leoni began to study the extent to which two-dimensional periodic water waves and the Stokes wave may be modeled using *minimizers* of (1.1). By imposing appropriate boundary conditions, and combined with recent work characterizing smooth critical points in [13], Arama and Leoni proved an analog of [19] for the crest of the wave in the minimizer context. Subsequent investigations [14, 15] have extended this work to the non-physical cases $0 < \gamma$ and $2 \leq n$. However, because these works focused upon modeling periodic water waves, they assumed that the boundary values u_0 were symmetric. Under this assumption, a trivial symmetrization argument implies $\Sigma = \emptyset$. The question of whether or not one can model the Stokes wave using minimizers of (1.1) remains open.

Inspired by [5, 15], the first author studied $\partial\{u > 0\} \cap \{Q = 0\}$ for a more general class of Q without assumptions of symmetry [22]. However, because Lemma 1.1 becomes vacuous near $\{Q = 0\}$, the problem of u degenerating as it approaches $\{Q = 0\}$ could not be eliminated at that time. As a result, it was proved (among other things) that for $n \geq 2$ and $Q(x) = \operatorname{dist}(x, \Gamma)^\gamma$, where Γ is a $C^{1,\alpha}$-submanifold of dimension $0 \leq k \leq n-1$ and $\gamma > 0$, if u is a local minimizer of (1.1) then

$$\partial\{u > 0\} \cap \Gamma \cap \Omega = \mathcal{S} \cup \Sigma \tag{1.4}$$

where Σ is as in (1.3) ($\Theta^n_{\{u>0\}}(x_0) = 0$ if and only if $\Theta^n_{Q,\{u>0\}}(x_0) = 0$) and the weighted Q^2-density

$$\Theta^n_{Q,\{u>0\}}(x) := \lim_{r \to 0} \frac{\int_{B_r(x)} Q^2(y)\chi_{\{u>0\}}(y)\,d\mathcal{H}^n(y)}{\int_{B_r(x)} Q^2(y)\,d\mathcal{H}^n(y)}$$

satisfies $\Theta^n_{Q,\{u>0\}}(x) \in [c(n,\gamma), C(n,\gamma)] \subset (0,1)$ for all $x \in \mathcal{S}$.

Using techniques from the study of sets of finite perimeter, the first author was able to show $\mathcal{H}^{n-1}(\Sigma) = 0$, but neither stronger estimates on Σ nor elimination of Σ was obtained at that time.

To further aid the study of local minimizers of (1.1) for which Q is allowed to vanish, we prove the following improvement on Lemma 1.1:

Theorem 1.2 (Weak non-degeneracy of local minimizers). *Let $n \geq 2$ and $1 \leq k \leq n-1$ be integers. Let $\Omega = B_2(0) \subset \mathbb{R}^n$, $\Gamma = \{(x, 0) \in \mathbb{R}^k \times \mathbb{R}^{n-k}\}$, $\gamma > 0$, and $Q(x) = \operatorname{dist}(x, \Gamma)^\gamma$ in the definition of (1.1). There is a constant $c_0(n, \gamma, \varepsilon_0) > 0$ such that if u is a local minimizer of (1.1) in the class $\mathcal{K}_{u_0,B_2(0)}$, then for all $s \in (0,1)$ there exist constants $C(n,\gamma,s) > 0$, and for all radii $0 < r \leq c_0$ and $x_0 \in B_1(0)$ the following statement holds: if $\partial\{u > 0\} \cap \Gamma \cap B_{sr}(x_0) \neq \emptyset$ then*

$$\fint_{\partial B_r(x_0)} u\,d\sigma > rC(n,\gamma,s) \max_{y \in B_r(x_0)} \{Q(y)\}. \tag{1.5}$$

Moreover, the constant $C(n, \gamma, s)$ remains bounded as $\gamma \to 0^+$. (See the definitions of $C_1(n, \gamma), C_2(n, \gamma), C_3(n, \gamma)$ in Lemmas 2.8, 2.9, and 4.2, respectively.

The main improvement is that we are able to use the local *maximum* of Q in Theorem 1.2, rather than the local *minimum* of Q as in Lemma 1.1. The constant c_0 may be taken to be the same as in Lemma 1.1 with our specific choice of Q. Moreover, the scalar s in the condition $\partial\{u > 0\} \cap \Gamma \cap B_{sr}(x_0) \neq \emptyset$ only functions to ensure that there is a ball $B_{1-s}(x) \subset B_r(x_0)$, centered on $\partial\{u > 0\} \cap \Gamma$, which will necessarily also intersect $\{u > 0\}$.

1.1. The underlying technical result

The techniques used in [2] to prove Lemma 1.1 are based upon comparing u in balls $B_r(0)$ with two specific functions. However, there is no naive way to modify the comparison arguments of [2] in order to obtain a result like Theorem 1.2. Nevertheless, one can hope to use comparison techniques (in the form of Lemma 1.1) if one has more information on the way in which the free-boundary approaches $\Gamma = \{Q = 0\}$. Consider the following example:

Example 1.3. By checking the Euler–Lagrange equation for (1.1), one can verify that the function

$$u(r, \theta) = \begin{cases} \frac{\sqrt{2}}{3}\rho^{\frac{3}{2}}\cos(\frac{3}{2}\theta) & \theta \in [-\pi/3, \pi/3], \\ 0 & \theta \in [-\pi, -\pi/3) \cup (\pi/3, \pi] \end{cases} \tag{1.6}$$

is a critical point of (1.1) for $Q(r, \theta) = |r\sin(\theta)|^{1/2}$ (in Cartesian coordinates, $Q(x, y) = |y|^{\frac{1}{2}}$). See [29] or [5] for a more extensive discussion of this example and the Stokes wave. However, for our purposes, the important thing is to observe that for any radius $r > 0$,

$$B_{r/4}(r/2, \pi/3) \subset B_r(0,0) \setminus B_{r/8}(\{Q = 0\}).$$

Hence, one may apply Lemma 1.1 (and its concomitants like Lemma 2.9, below) in $B_{r/4}(r/2, \pi/3)$ to observe that $\fint_{\partial B_r(0)} u\,d\sigma \geq C_{\min} r/4 \min_{B_{\frac{r}{4}}(r/2,\pi/3)}\{Q\}$. Since by our specific choice of Q, $\min_{B_{\frac{r}{4}}(r/2,\pi/3)}\{Q\} \sim_\gamma \max_{B_r(0,0)}\{Q\}$, we see that (1.5) nonetheless holds for the function u in (1.6).

This simple example shows that in order to prove Theorem 1.2 it suffices, roughly speaking, to prove that there exists a constant $c(n, \gamma) > 0$ such that for any local minimizer u if $x_0 \in \partial\{u > 0\} \cap \{Q = 0\}$ and $r > 0$ is sufficiently small then there exists a point $x' \in \partial\{u > 0\} \cap B_r(x_0)$ with

$$0 < c(n, \gamma) \leq \frac{\operatorname{dist}(x', \Gamma)}{r},$$

that is, one needs to lower bound the "slope" at which $\{u > 0\}$ may approach or lift off from Γ independent of u and x_0 at all sufficiently small scales (see Lemma 4.1). This amounts to eliminating cusps. All points in $\mathcal{S}$, including potentially flat wave profiles, naturally already satisfy Theorem 1.2.

In fact, we prove a more general result. In order to state it, we make the following definitions, which describes the geometry we wish to capture. We henceforth write $(x, y) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, $B_r^n(x, y) \subset \mathbb{R}^n$, $B_r^k(x) \subset \mathbb{R}^k$, and $B_r^{n-k}(y) \subset \mathbb{R}^{n-k}$. Because the weak geometry we wish describe concerns the "height" of the positivity set over Γ, we make the following definition:

Definition 1.4. Let $u \in C^0(B_2^n(0, 0))$ and $\mathcal{O} \subset \mathbb{R}^n$ be a component of $\{u > 0\}$. For $x \in B_1^k(0)$ and $r > 0$, $R < \infty$, we define

$$\begin{aligned} \mathrm{Height}(r, \mathcal{O}, R) &:= \sup\bigl\{|y'| : (x', y') \in \bigl(\partial B_r^k(0) \times B_R^{n-k}(0)\bigr) \cap \mathcal{O}\bigr\}, \\ M(r, u, \mathcal{O}, R) &:= \sup\bigl\{u(x', y') : (x', y') \in \bigl(\partial B_r^k(0) \times B_R^{n-k}(0)\bigr) \cap \mathcal{O}\bigr\}. \end{aligned}$$

Because we always consider $\partial B_r^k(0) \times B_R^{n-k}(0)$ in the definitions above, we shall translate the set $\mathcal{O} - (x, 0)$ when we need to consider the geometry of $\mathcal{O}$ near another point $(x, 0)$.

With this notation, we introduce the following definition:

Definition 1.5. (Property P with constant N) Let $u : \mathbb{R}^n \to \mathbb{R}$ be a non-negative function in the class $\mathcal{K}_{u_0, B_2^n(0,0)}$. For $N \in \mathbb{N}$, a ball $B_r^n(x, 0) \subset B_2^n(0, 0)$ is said to satisfy *Property P with constant N* if

$$\bigl(B_r^k(x) \times B_{\frac{5r}{N}}^{n-k}(0)\bigr) \cap \{u > 0\}$$

has a component $\mathcal{O}$ that satisfies the following conditions:

(1) (Intersection) $\mathcal{O} \cap (\{x\} \times B_{5r/N}^{n-k}(0)) \neq \emptyset$.

(2) (Upper Height Bound) $\displaystyle\sup_{0 \le \rho \le r} \mathrm{Height}(\rho, \mathcal{O}, 5r/N) \le \frac{4r}{N}$.

(3) (Lower Height Bound) $\displaystyle\inf_{\frac{r}{16N} \le \rho \le r} \mathrm{Height}(\rho, \mathcal{O}, 5r/N) \ge \frac{r}{8NC_3(n, \gamma)^2}$,

where $C_3(n, \gamma)$ is a constant to be defined in Lemma 4.2.

Remark 1.6. Observe that Property P with constant N is geometric in nature in the sense that it is invariant under rescaling. To be precise, if $\mathcal{O} \subset B_r^n(x, 0)$ is the set that satisfies the conditions in Property P with constant N, then $\frac{1}{r}(\mathcal{O} - (x, 0))$ satisfies these conditions in $B_1^n(0, 0)$.

With this definition, we can state our main technical result.

Theorem 1.7 (Technical theorem). *Let n, k, Γ, γ, $\Omega = B_2^n(0, 0)$, and $Q(x, y) = |y|^\gamma$ as in Theorem 1.2. Let u be an ε_0-local minimizer of (1.1) in the class $\mathcal{K}_{u_0, B_2^n(0,0)}$, and let $c_0 > 0$ as in Lemma 1.1. There is an integer $N_0 = N_0(n, \gamma)$ such that for all $0 < r \le c_0$ and all balls $B_r^n(x, 0) \subset B_1^n(0, 0)$, the ball $B_r^n(x, 0)$ does not have Property P with constant $N \ge N_0(n, \gamma)$.*

Remark 1.8. Note that Property P and therefore Theorem 1.7 concerns behavior at macroscopic scales, not just infinitesimal scales. Its main applications are to prove Theorem 1.2 and to eliminate density zero points (see Theorem 1.10, below). However, it may be of independent interest, since it begins to describe the macroscopic geometry of the positivity set $\{u > 0\}$ near Γ.

1.2. Applications

Without a result like Theorem 1.2, one of the central problems in the study of the geometry of the free boundary for local minimizers of (1.1) for which Q is allowed to vanish is the existence or non-existence of cusps. To address this, we introduce the following definition:

Definition 1.9 (Cusps). Let $\Omega \subset \mathbb{R}^n$ and $u : \Omega \to \mathbb{R}_+$. A point $(x, 0) \in \partial\{u > 0\} \cap \Omega$ is a *cusp point* if there exists a neighborhood V of $(x, 0)$ such that there exists a component $\mathcal{O} \subset \{u > 0\} \cap V$ for which both

(1) $(x, 0) \in \overline{\mathcal{O}}$.

(2) $\Theta^n_{Q,\mathcal{O}}((x, 0)) = 0$.

The component $\mathcal{O} \cap V$ will be called the *cusp*. We denote the set of cusp points Σ_{cusp}.

In a recent paper [21], the first author proved that for $n = 2$ and $Q(x, y) = |y|^\gamma$ for $0 < \gamma$, if one assumes that $\{u > 0\} \cap \{(x, 0) : x \in \mathbb{R}\} = \emptyset$, then $\Sigma = \emptyset$. This was done by carefully considering the geometry of a cusp and computing explicit estimates on a special domain variation. Since the argument only depends upon the geometry of the cusp (or, indeed, by Remark 2.7) it naturally eliminates Σ_{cusp} as well. This case covers the physically relevant two-dimensional variational formulation of the Stokes wave in [5, 15] (with symmetry assumptions removed). However, the techniques used in [21] fail in higher dimensions. Thus, the question of cusps was open for $n > 2$, co-dimensions $1 \leq k \leq n-1$, and even the case that $\{u > 0\} \cap \{(x, 0) \in \mathbb{R}^k \times \mathbb{R}^{n-k} : x \in \mathbb{R}^k\} \neq \emptyset$ in $n = 2$. Eliminating cusps in these circumstances served as the inspiration for proving Theorem 1.2.

As an immediate consequence of Theorem 1.2 and Theorem 1.7, we have the following result:

Theorem 1.10 (Eliminating cusps). *Let n, k, $0 < \gamma$, and $Q(x, y) = |y|^\gamma$. Let $\Omega \subset \mathbb{R}^n$ be an open set with Lipschitz boundary such that $\mathcal{H}^{n-1}(\partial\Omega) > 0$. If u is a local minimizer of* (1.1)*, then $\Sigma_{cusp} = \emptyset$.*

Moreover, for any $(x, 0) \in \partial\{u > 0\} \cap \Gamma \cap \Omega$ and any $c \in (0, 1)$, there are at most finitely many components of $B_{\text{dist}((x,0),\partial\Omega)}(x, 0) \cap \{u > 0\}$ that intersect

$$B_{c\,\text{dist}((x,0),\partial\Omega)}(x, 0).$$

This immediately implies that under the assumptions of Theorem 1.2, the results of [22] characterize the geometry of the full intersection set $\partial\{u > 0\} \cap \Omega$. We state them in their local form below.

Corollary 1.11 (Applying [22]). *Under the assumptions of Theorem 1.2, if u is an ε_0-local minimizer of (1.1) in the class $\mathcal{K}_{u_0,B_2^n(0,0)}$ and $k = n-1$ then $\partial\{u > 0\} \cap \Gamma = \mathcal{S}$ for $\mathcal{S}$ as in (1.4). In particular, if $k = n-1$ then $\partial\{u > 0\} \cap \Gamma$ is countably $(n-2)$-rectifiable and*

$$\mathrm{Vol}(B_r(\partial\{u > 0\} \cap \Gamma \cap B_1(0))) \leq C(n, \gamma, \varepsilon_0) r^2.$$

See [22] for further details and estimates on the quantitative strata of $\partial\{u > 0\} \cap \Gamma$.

1.3. Outline

In Section 2, we recount some basic results from [2] concerning local minimizers. Most importantly, we state Lemma 2.6, which lets us consider restrictions $u|_{\mathcal{O}}$ of local minimizers u to components of their positivity set $\mathcal{O} \subset \{u > 0\}$ without loss of generality. Additionally, standard growth estimates and interior ball conditions are stated.

In Section 3, we prove Theorem 1.7. Roughly speaking, Theorem 1.7 rests upon the contradiction of two growth estimates (Lemmas 3.3 and 3.4). Lemma 3.3 gives lower bounds for the growth of u based upon standard estimates for subharmonic functions. Lemma 3.4 gives upper bounds on the growth of u based upon satisfying the Lipschitz bounds (Lemma 2.8) and the geometry of $\mathcal{O}$ in balls with Property P with constant N. Obtaining a contradiction between these two estimates proves Theorem 1.7.

In Section 4, we provide the proofs of Theorem 1.2 and Theorem 1.10. The key lemma is Lemma 4.1, which gives sufficient conditions for proving the existence of balls with Property P with constant N and represents the main technical contribution of this paper. See Section 4 for further discussion. Since by Theorem 1.7, there exists an N_0 such that no ball can have Property P with constant $N \geq N_0$, the sufficient conditions in the hypotheses of Lemma 4.1 must fail. Roughly speaking, this gives a lower bound on the "slope" at which the free boundary approaches Γ. Thus, the proof of Theorem 1.2 essentially consists in generalizing the argument given for the example given by (1.6), that is, we may simply apply Lemma 1.1.

The proof of Theorem 1.10 essentially consists in showing that near a cusp point we can always apply Lemma 4.1 and find a ball with Property P with constant N for N arbitrarily large. This is done in Lemma 5.1. The proofs of Lemma 5.1 and Theorem 1.10 are given in Section 5.

2. Preliminaries

For the remainder of this paper, let $n \geq 2$ and $1 \leq k \leq n-1$ be integers. Recall that for $z \in \mathbb{R}^n$, we write $z = (x, y) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$. Similarly, we write $B_r^n(x, y) \subset \mathbb{R}^n$ and $B_r^k(x) \subset \mathbb{R}^k$. Our motivation for this notation comes from the definitions of the set $\Gamma = \{(x, 0) \in \mathbb{R}^k \times \mathbb{R}^{n-k}\}$ and the function $Q(x, y) = |y|^\gamma$.

2.1. Basic properties of local minimizers

First, we note that local minimizers behave well under rescaling.

Definition 2.1 ([2, Remark 3.1]). Let $0 < \gamma$, and u be an ε_0-local minimizer of (1.1) in the class $\mathcal{K}_{u_0,B^n_2(0,0)}$. Then, for any $B^n_r(x,0) \subset B^n_2(0,0)$, we define the rescalings

$$u_{(x,0),r}(x',y') := \frac{u(r(x',y')+(x,0))}{r^{\gamma+1}}.$$

The function $u_{(x,0),r}$ is an ε'-local minimizer of (1.1) in the class $K_{u_{(x,0),r},B^n_{\frac{1}{r}}((0,0))}$, where by (1.2), we have

$$\varepsilon' = \varepsilon_0 \max\{r^{-n}, r^{-n+2\gamma}\}.$$

One of the most fundamental aspects of local minimizers of (1.1) is that they are subharmonic.

Lemma 2.2 ([2, Lemmas 2.2–2.4]). *Let u be a local minimizer of* (1.1) *in* $\mathcal{K}_{u_0,B^n_2(0,0)}$. *The function u is subharmonic in $B^n_2(0,0)$ and harmonic in $\{u>0\} \cap B^n_2(0,0)$. Moreover,* $0 \le u \le \operatorname{ess\,sup}_{\partial B^n_2(0,0)} u_0$.

Following the definition of Almgren's frequency function in [1] and its subsequent use in the study of harmonic functions, Lemma 2.2 immediately gives weak growth estimates on local minimizers in Lemma 2.4, below.

Definition 2.3. For any continuous $v : \mathbb{R}^n \to \mathbb{R}$ and any $(x,y) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, we define the quantity

$$H(r,(x,y),v) := \int_{\partial B^n_r(x,y)} v^2 d\sigma.$$

Lemma 2.4. (Preliminary growth estimate) *Let u be an ε_0-local minimizer of* (1.1) *for* $\Omega = B^n_2(0,0)$. *Then for any* $B^n_r(x,y) \subset B^n_R(x,y) \subset\subset B^n_2(0,0)$,

$$H(R,(x,y),u) \ge \left(\frac{R}{r}\right)^{n-1} H(r,(x,y),u).$$

Proof. Since $u \ge 0$ and both u and $g(w) = |w|^2$ are subharmonic, the composition u^2 is also subharmonic. The desired result follows from the fact that the averages of subharmonic functions are non-decreasing. ∎

Remark 2.5. We observe that somewhat stronger growth conditions are obtainable. Using a mollification argument, one can follow the standard arguments of, for example, [16, Corollaries 2.2.5 and 2.2.6] (phrased for harmonic functions) to obtain a slightly larger exponent than $n-1$. However, in the subsequent work, the power $n-1$ is sufficient.

One of the central features of functional (1.1) is that it is non-linear. This is easily seen from the fact that for $u_1, u_2 \ge 0$, it is possible that $\chi_{\{u_1+u_2>0\}}(x) \neq \chi_{\{u_1>0\}}(x) + \chi_{\{u_2>0\}}(x)$, $\mathcal{H}^n$-almost everywhere. Indeed, even if one has boundary data $u_{0,1}, u_{0,2}$

$\in C^0(\partial\Omega)$ that satisfies $\operatorname{spt}(u_{0,1}) \cap \operatorname{spt}(u_{0,2}) = \emptyset$, it is not the case that $u_1 + u_2$ is necessarily a critical point in $\mathcal{K}_{u_{0,1}+u_{0,2},\Omega}$ for local minimizers u_1, u_2 in $\mathcal{K}_{u_{0,1},\Omega}$, $\mathcal{K}_{u_{0,2},\Omega}$, respectively. However, a limited sort of linearity holds for local minimizers u when we consider the components of their positivity sets. We make this rigorous in Lemma 2.6, below.

Lemma 2.6. *Let u be an ε_0-local minimizer of* (1.1) *in $\mathcal{K}_{u_0,B_2^n(0,0)}$. If $(x,0) \in \partial\{u > 0\} \cap B_1^n(0,0)$, $0 < r \le 1$, and $\mathcal{O}$ is a connected component of $B_r^n(x,0) \cap \{u > 0\}$, then $u|_{\mathcal{O}}$ is an ε_0-local minimizer of* (1.1) *in $\mathcal{K}_{u|_{\mathcal{O}},B_r^n(x,0)}$.*

Proof. If $\mathcal{O}$ is the only component of $B_r^n(x,0) \cap \{u > 0\}$, the lemma is trivial. Therefore, without loss of generality, we suppose that

$$B_r^n(x,0) \cap \{u > 0\} = \mathcal{O} \cup \mathcal{O}'$$

for $\mathcal{O}, \mathcal{O}'$ mutually disjoint, non-empty connected components. We argue by contradiction: suppose that $u|_{\mathcal{O}}$ is not an ε_0-local minimizer of (1.1) in $\mathcal{K}_{u|_{\mathcal{O}},B_r^n(x,0)}$. Then there must exist a $v \in \mathcal{K}_{u|_{\mathcal{O}},B_r^n(x,0)}$ such that (1.2) holds and $J_Q(v, B_r^n(x,0)) < J_Q(u|_{\mathcal{O}}, B_r^n(x,0))$.

We claim that the function $w := \max\{u|_{\mathcal{O}'}, v\}$ produces a contradiction. First, observe that

$$\begin{aligned} J_Q(w, B_r^n(0,0)) &\le J_Q(u|_{\mathcal{O}'}, B_r^n(0,0)) + J_Q(v, B_r^n(0,0)) \\ &< J_Q(u|_{\mathcal{O}'}, B_r^n(0,0)) + J_Q(u|_{\mathcal{O}}, B_r^n(0,0)) = J_Q(u, B_r^n(x,0)). \end{aligned}$$

However, we also have that

$$\begin{aligned} &\|\nabla(w-u)\|^2_{L^2(B_r^n(0,0))} + \|\chi_{\{w>0\}} - \chi_{\{u>0\}}\| \\ &\quad= \int_{B_r^n(0,0)\cap\{w=v\}} |\nabla(v-u)|^2 dx + \int_{B_r^n(0,0)\cap\mathcal{O}\setminus\{w=v\}} |\nabla(v-u)|^2 dx \\ &\qquad + \|\chi_{\{v>0\}\setminus\{\mathcal{O}'\}} - \chi_{\mathcal{O}}\| \\ &\quad\le \|\nabla(v-u|_{\mathcal{O}})\|^2_{L^2(B_r^n(0,0))} + \|\chi_{\{v>0\}} - \chi_{\{u|_{\mathcal{O}}>0\}}\| \le \varepsilon_0. \end{aligned}$$

This contradicts u being an ε_0-local minimizer and thus proves the lemma. ∎

Remark 2.7. As a result of Lemma 2.6, we are able to restrict local minimizers to their components without loss of generality. In particular, this implies that the statement $\Sigma = \emptyset$ is equivalent to $\Sigma_{\text{cusp}} = \emptyset$.

2.2. Estimates and the constant c_0

As touched upon in Section 1.1, the techniques used in [2] to establish the non-degeneracy of a local minimizer u rely upon comparing u with other functions in balls. More specifically, [2] compares u with two other functions:

(1) The harmonic extension $h_{(x_0,y_0),r}$ of u in the ball $B_r^n(x_0,y_0) \subset B_2^n(0,0)$.

(2) The function $w = \min\{u, v\}$ in $B^n_r(x_0, y_0) \subset B^n_2(0,0)$ for

$$v(x, y) = \Big(\sup_{(x',y')\in B^n_{r\sqrt{s}}(x_0,y_0)} \{u(x', y')\}\Big) \max\Big\{1 - \frac{|(x, y)|^{2-n} - r^{2-n}}{(sr)^{2-n} - r^{2-n}}, 0\Big\}.$$

Thus, the fact that $\|\nabla v\|^2_{L^2(B^n_r(x_0,0))} \leq C(s, n) \sup_{(x',y')\in B^n_{r\sqrt{s}}(x_0,y_0)}\{u^2(x', y')\}r^{n-2}$ and harmonic functions are energy minimizers gives rise to a *local scale* $\tilde{r}_{x,y} > 0$ defined by the conditions

$$\begin{cases} \omega_n r^n & < \varepsilon_0/3, \\ \|\nabla u\|^2_{L^2(B^n_r(x,y))} & < \varepsilon_0/3, \\ C(s, n) \sup_{(x',y')\in B^n_r(0,0)}\{u^2(x', y')\}r^{n-2} & < \varepsilon_0/3 \end{cases} \tag{2.1}$$

for all $0 < r < \tilde{r}_{(x,y)}$. Hence, it follows that for all $0 < r < \tilde{r}_{(x,y)}$, (1.2) holds with $v = h_{(x,y),r}$ and $v = w$ as in (2), above. It is using this local scale that Alt and Caffarelli initially employ their comparison arguments. However, through an implicit use of Harnack's inequality, they are able to obtain the following lemma, which is independent of the local scale:

Lemma 2.8 ([2, Corollary 3.3]; [22, Corollary 3.6]). *If $\partial\{u > 0\} \cap B^n_1(0,0) \neq \emptyset$, then for all $(x, y) \in \{u > 0\} \cap B^n_1(0,0)$,*

$$|\nabla u(x, y)| \leq C_1(n, \gamma) \max\big\{\mathrm{dist}((x, y), \partial\{u > 0\}), |y|\big\}^{\gamma}.$$

Moreover, we may take the constant $C_1(n, \gamma) = C(n)2^{\gamma}$.

This shows that if $\partial\{u > 0\} \cap B^n_1(0,0) \neq \emptyset$, then $\|\nabla u\|^2_{L^2(B^n_r(x,y))} \lesssim_n r^n 2^{\gamma}$ and

$$\sup_{(x',y')\in B^n_2(0,0)} \{u^2(x', y')\} \leq 4^{2+2\gamma}.$$

Hence, there is a *uniform scale* $c_0(n, \gamma, \varepsilon_0) > 0$ such that (2.1) is satisfied for all $0 < r \leq c_0$. For general Q, we write this scale as $c_0(n, Q_{\max}, \varepsilon_0)$ as in Lemma 1.1. But, since in this paper we only consider $Q(x, y) = |y|^{\gamma}$, we shall write $c_0(n, \gamma, \varepsilon_0)$ as in Theorems 1.2 and 1.7.

Using this $c_0(n, \gamma, \varepsilon_0) > 0$, one applies the results of [2] to our choice of Q, as above, to obtain the following lemma:

Lemma 2.9 (Interior corkscrew condition, [22, Lemma 3.9]; cf. [2, Theorem 3.2]). *Let $\Omega = B^n_2(0,0)$, $u_0 \in W^{1,2}(B^n_2(0,0))$ satisfy $u_0 \geq 0$, and let u be a ε_0-local minimizer of (1.1) in the class $\mathcal{K}_{u_0,B^n_2(0,0)}$. Let $0 < r \leq c_0(n, \gamma, \varepsilon_0)$ and $(x, y) \in \partial\{u > 0\}$ satisfy $B^n_r(x, y) \subset B^n_2(0,0) \setminus \Gamma$. Then there exists a point $(x', y') \in \{u > 0\} \cap \partial B^n_{\frac{1}{2}r}(x, y)$ and pair of positive constants $C_2(n, \gamma)$ and $c = c(n, \gamma)$ such that*

$$u \geq C_2(n, \gamma) r \min_{B^n_{r/2}(x,y)} Q$$

in $B^n_{\frac{c}{2}r}(x', y')$, *where* $C_2(n, \gamma) = C_{\max}(n) - C_1(n, \gamma)c(n, \gamma)$ *and* $C_{\max}(n)$ *is a dimensional constant given by* [2, Theorem 3.2].

Remark 2.10. By Lemma 2.6, we may assume without loss of generality that the ball guaranteed by Lemma 2.9 is contained within the component $\mathcal{O} \subset \{u > 0\}$ such that $(x, y) \in \partial\mathcal{O}$.

3. Growth estimates and the proof of Theorem 1.7

The proof of Theorem 1.7 rests upon Lemmas 3.3 and 3.4, below. In order to state them, we make the following auxiliary definition of certain cylindrical neighborhoods:

Definition 3.1. For any $(x, 0) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, we define

$$A_j(x) := \big(B^k_{2j}(x) \setminus B^k_{2(j-1)}(x)\big) \times B^{n-k}_1(0).$$

Observe that for all $j \in \mathbb{N}$ and any $(x, 0) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$,

$$\mathcal{H}^n(A_j(x)) \approx_n j^{k-1}.$$

Lemma 3.2. *Let* $(x, 0) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$. *Then, for all* $j \in \mathbb{N}$

$$1 \leq \mathcal{H}^1\big(\{0 < r : \partial B^n_r(x, 0) \cap B^n_1(\Gamma) \subset A_j(x)\}\big).$$

Proof. A point (x', y') is contained in $A_j(x)$ if

$$2(j-1) \leq \sqrt{(x'_1)^2 + \ldots + (x'_k)^2} \leq 2j$$

and

$$0 \leq \sqrt{(y'_1)^2 + \ldots + (y'_{n-k})^2} \leq 1.$$

Thus, $\partial B^n_r(x, y) \cap B^n_1(\Gamma) \subset A_j(x)$ when $(2j-2)^2 + 1 \leq r^2 \leq (2j)^2$. Observing that $\min_{j \in \mathbb{N}} |2j - \sqrt{(2(j-1))^2 + 1}| = 1$ finishes the proof. ∎

The first growth estimate, stated below, is a direct consequence of the growth from Lemma 2.4 and the geometry assumed by Property P with constant N.

Lemma 3.3. *Let u be an ε_0-local minimizer of* (1.1) *for* $\Omega = B^n_2(0, 0)$. *Let* $c_0 > 0$ *as in Section* 2.2. *Suppose that* $0 < \rho \leq c_0$, $N \in \mathbb{N}$ *and* $B^n_\rho(x, 0) \subset B^n_2(0, 0)$ *satisfies Property P with constant* N. *Then, if we let* $\mathcal{O}' \subset \{u > 0\}$ *be the component guaranteed by Property P with constant* N *and* $\mathcal{O} = \frac{N}{8\rho}[\mathcal{O}' - (x, 0)]$, *then for all integers* $1 \leq j \leq N/8$,

$$\int_{A_j(0) \cap \mathcal{O}} u^2_{(x,0), \frac{8\rho}{N}} \, dV \geq c'(n, \gamma) j^{n-1}.$$

Proof. Let $B^n_\rho(x,0)$, $\mathcal{O}'$, and u be as in the hypotheses. Recall that as given in Definition 1.5, $\mathcal{O}'$ is a component of

$$\big(B^k_r(x) \times B^{n-k}_{\frac{5r}{N}}(0)\big) \cap \{u > 0\}.$$

By Lemma 2.6, it suffices to assume that $u = u|_{\mathcal{O}'}$. Consider $u_{(x,0),\frac{8\rho}{N}}$. By the assumption that $B^n_\rho(x,0)$ satisfies Property P with constant N, by Definition 1.5 (2),

$$\mathrm{Height}\Big(\frac{N}{8}, \mathcal{O}, \frac{5}{8}\Big) \leq \frac{1}{2},$$

and therefore $\mathcal{O} \subset \cup_{j=1}^{\lfloor N/8 \rfloor} A_j(0)$. Hence, we see that $u_{(x,0),\frac{8\rho}{N}}$ satisfies the hypotheses of Lemma 2.4 in $B^n_{N/8}(0,0)$. Therefore, we compute as follows. For ease of notation we define $R_j = 2j$ and $r_j = \sqrt{(2(j-1))^2 + 1}$ for $j \in \mathbb{N}$.

$$\begin{aligned}
\int_{A_j(0)\cap\mathcal{O}} u^2_{(x,0),\frac{8\rho}{N}} dV &\geq \int_{r_j}^{R_j} \int_{\partial B^n_\rho(0,0)\cap\mathcal{O}} u^2_{(x,0),\frac{8\rho}{N}} d\sigma d\rho \\
&= \int_{r_j}^{R_j} H(\rho, (0,0), u_{(x,0),\frac{8\rho}{N}}) d\rho \\
&\geq \int_{r_j}^{R_j} \frac{\rho^{n-1}}{r_1^{n-1}} H(r_1, (0,0), u_{(x,0),\frac{8\rho}{N}}) d\rho \\
&\geq 2^{n-1} j^{n-1} H(1, (0,0), u_{(x,0),\frac{8\rho}{N}}).
\end{aligned}$$

Now, we estimate $H(1,(0,0),u_{(x,0),\frac{8\rho}{N}})$. By assumption, Property P with constant N tells us that

$$\inf_{\frac{1}{128} \leq \rho \leq \frac{N}{8}} \mathrm{Height}\Big(\rho, \mathcal{O}, \frac{5}{8}\Big) \geq \frac{1}{64 C_3(n,\gamma)^2},$$

$$\sup_{x' \in \mathbb{R}^k \text{ s.t. } |x'| = 1/64} \sup\big\{|y'| : (x',y') \in \mathcal{O}\big\} \in \Big[\frac{1}{64 C_3(n,\gamma)^2}, \frac{1}{2}\Big].$$

Hence, by Lemma 2.9 applied to $u_{(x,0),r}$ at the ball $B_{|y'|}(x',y')$ for $(x',y') \in \partial\mathcal{O}$ satisfying $|x - x'| = \frac{1}{32 C_3(n,\gamma)^2}$ and $|y'| \geq \frac{1}{64 C_3(n,\gamma)^2}$, the Maximum Principle, and Remark 2.10, there must be a point $(v,w) \in \partial B^n_1(0,0) \cap \mathcal{O}$ such that

$$u_{(x,0),r}(v,w) = \max_{\partial B^n_1(0,0)} u_{(x,0),\frac{8\rho}{N}} \geq \frac{C_2(n,\gamma)}{128^{1+\gamma} C_3(n,\gamma)^{2(1+\gamma)}}.$$

By Lemma 2.8, there is a $c(n,\gamma)$ such that

$$\min_{B^n_{\frac{c(n,\gamma)}{2\cdot 128^{1+\gamma} C_3(n,\gamma)^{2(1+\gamma)}}(v,w)} u_{(x,0),\frac{8\rho}{N}} \geq \frac{C_1(n,\gamma)}{2 \cdot 128^{1+\gamma} C_3(n,\gamma)^{2(1+\gamma)}}.$$

Hence, we have $H(1,(0,0),u_{(x,0),\frac{8\rho}{N}}) \geq c'(n,\gamma)$. This proves the lemma. ■

Our second growth condition follows from the Lipschitz bound in Lemma 2.8 and the geometric bounds imposed by Property P with constant N. Consequently, it does not depend upon the uniform scale $c_0 > 0$.

Lemma 3.4. *Let u be an ε_0-local minimizer of (1.1) in $\mathcal{K}_{u_0, B_2^n(0,0)}$. For all $0 < \rho \le 1$ and $8 < N \in \mathbb{N}$, if $B_\rho^n(x,0) \subset B_2^n(0,0)$ satisfies the Property P with constant N then, if $\mathcal{O}' \subset \{u > 0\}$ is the component guaranteed by Property P with constant N and $\mathcal{O} = \frac{N}{8\rho}[\mathcal{O}' - (x,0)]$, then for all integers $1 \le j \le N/8$,*

$$\int_{A_j(0)\cap\mathcal{O}} u^2_{(x,0),\frac{8\rho}{N}}\,dV \le C(n)\,j^{k-1}.$$

Proof. The hypothesis that $B_\rho^n(x,0)$ satisfies Property P with constant N implies that $\partial\{u > 0\} \cap B_\rho^n(x,0) \neq \emptyset$. Therefore, by Lemma 2.8 we may estimate $u_{(x,0),\frac{8\rho}{N}} \le C_1(n,\gamma)$ on $A_j(0) \cap \mathcal{O}$ for all integers $0 \le j \le N/8$. Therefore,

$$\int_{A_j(0)\cap\mathcal{O}} u^2_{(x,0),\frac{8\rho}{N}}\,dV \lesssim_n \int_{A_j(0)\cap\mathcal{O}} 1\,dV \le \int_{A_j(0)} 1\,dV \le C(n)\,j^{k-1}. \qquad \blacksquare$$

3.1. Proof of Theorem 1.7

Let u be an ε_0-local minimizer in the class $\mathcal{K}_{u_0, B_2^n(0,0)}$, and let $c_0(n,\gamma,\varepsilon_0) > 0$ be as in Section 2.2. Suppose that $0 < \rho \le c_0$, $8 < N \in \mathbb{N}$, and $B_\rho^n(x,0) \subset B_2^n(0,0)$ satisfies Property P with constant N. We consider $u_{(x,0),\frac{8\rho}{N}}$ and $\mathcal{O}$, the component of $\{u_{(x,0),\frac{8\rho}{N}} > 0\}$ that is guaranteed by Property P with constant N.

By Lemma 3.3 and Definition 3.1, we have that

$$\int_{A_j(0)\cap\mathcal{O}} u^2_{(x,0),\frac{8\rho}{N}}\,dx \gtrsim_{n,\gamma} j^{n-1}$$

for all integers $1 \le j \le N/8$. By Lemma 3.4 and Remark 3.1 we have that for all $1 \le j \le N/8$,

$$\int_{A_j(0)\cap\mathcal{O}} u^2_{(x,0),\frac{8\rho}{N}}\,dx \lesssim_{n,\gamma} j^{k-1}.$$

Therefore, if N is sufficiently large (depending only upon n,γ) we may take j sufficiently large and obtain a contradiction. $\blacksquare$

4. Local geometry of the positivity set and the proof of Theorem 1.2

The proof of Theorem 1.2 relies upon Lemma 4.1, below, which proves sufficient conditions for the existence of a ball with Property P with constant N.

Lemma 4.1. *Suppose that n, k, Γ, γ, $Q(x,y) = |y|^\gamma$, and u are as in Theorem 1.2. Let $c_0(n,\gamma,\varepsilon_0) > 0$ as in Section 2.2. Let $N \in \mathbb{N}$ satisfy*

$$N > C_3(n,\gamma) + 2$$

for a constant $C_3(n,\gamma)$ to be defined in Lemma 4.2. If $(x,0) \in \partial\{u > 0\}$ and there exists a component of $\mathcal{O} \subset \{u > 0\}$ and a radius $0 < r$ such that $0 < r < c_0$ and

$$\mathcal{O}_{(x,0),r} \cap B_1^n(0,0) \subset B_1^k(x) \times B^{n-k}_{\frac{1}{4N}}(0), \tag{4.1}$$

then there exists a ball $B_\rho^n(x',0)$ with Property P with constant $N' = \frac{N-2}{2C_3(n,\gamma)} - \frac{1}{2} > 0$.

The proof of Lemma 4.1 is deferred until Section 4.1. In preparation for this, we prove Lemma 4.2, which provides sufficient conditions for obtaining weak geometric control on the positivity set. Lemma 4.2 relies essentially upon the subharmonicity of local minimizers and Lemmas 2.8 and 2.9.

Because we always consider $\partial B_r^k(0) \times B_R^{n-k}(0)$ in the definitions above, we shall translate the set $\mathcal{O} - (x,0)$ when we need to consider the geometry of $\mathcal{O}$ near $(x,0)$.

Lemma 4.2 (Height bound). *Let u be an ε_0-local minimizer of (1.1) in $\mathcal{K}_{u_0,B_2^n(0,0)}$. Let $c_0(n,\gamma,\varepsilon_0) > 0$ as in Section 2.2, and let $0 < \rho \le c_0$. Suppose that $\mathcal{O}$ is a component of $\{u > 0\}$. Let $x \in B_1^k(0)$, $0 < a < A \le 2$, and let $0 \le r < R < \infty$ be radii such that $\mathcal{O}_{(x,0),\rho}$ satisfies*

$$a \le \text{Height}(r, \mathcal{O}_{(x,0),\rho}, 5), \quad \text{Height}(R, \mathcal{O}_{(x,0),\rho}, 5) \le A.$$

Then there exists a constant $C_3(n,\gamma)$ such that if $2AC_3(n,\gamma) \le 5$, then

$$\frac{C_3(n,\gamma)^{-1}a}{2} < \inf_{r' \in [r+\frac{a}{4}, R]} \text{Height}(r', \mathcal{O}_{(x,0),\rho}, 5),$$

$$\sup_{r' \in [r, R-1]} \text{Height}(r', \mathcal{O}_{(x,0),\rho}, 5) < C_3(n,\gamma) 2A,$$

and

$$\begin{aligned} C_2(n,\gamma)(\frac{a}{2})^{1+\gamma} &\le \min_{r' \in [r+\frac{a}{2}, R]} M(r', u_{(x,0),\rho}, \mathcal{O}_{(x,0),\rho}, 5) \\ &\le \max_{r' \in [r,R]} M(r', u_{(x,0),\rho}, \mathcal{O}_{(x,0),\rho}, 5) \le C_1(n,\gamma) A^{1+\gamma}. \end{aligned}$$

Moreover, $C_1(n,\gamma)$ is as in Lemma 2.8, $C_2(n,\gamma)$ is as in Lemma 2.9, and

$$C_3(n,\gamma) = \Big(\frac{C_1(n,\gamma)}{C_2(n,\gamma)}\Big)^{\frac{1}{1+\gamma}}.$$

Proof. First, we note that by the Maximum Principle, for all $r \le r_1 < r_2 \le R$, we may estimate $M(r_1, u_{(x,0),\rho}, \mathcal{O}_{(x,0),\rho}, 5) \le M(r_2, u_{(x,0),\rho}, \mathcal{O}_{(x,0),\rho}, 5)$.

Second, we observe that by our choice of $0 < r < c_0(n,\gamma,\varepsilon_0)$, we may apply Lemma 2.9 to $u_{(x,0),\rho}$ at scale 1. Thus, if $\text{Height}(r, \mathcal{O}_{(x,0),\rho}, 5) = \alpha \le 2$, then

$$C_2(n,\gamma)\Big(\frac{\alpha}{2}\Big)^{1+\gamma} \le M\Big(r + \frac{\alpha}{2}, u_{(x,0),\rho}, \mathcal{O}_{(x,0),\rho}, 5\Big).$$

Moreover, if $\mathrm{Height}(r, \mathcal{O}_{(x,0),\rho}, 5) = \alpha \le 2$, then by Lemma 2.8

$$M(r, u_{(x,0),\rho}, \mathcal{O}_{(x,0),\rho}, 5) \le C_1(n,\gamma)\alpha^{1+\gamma}.$$

Therefore, since by assumption $\mathrm{Height}(R, \mathcal{O}_{(x,0),\rho}, 5) \le A$, we have

$$M(R, u_{(x,0),\rho}, \mathcal{O}_{(x,0),\rho}, 5) \le C_1(n,\gamma)A^{1+\gamma},$$

which implies that $\mathrm{Height}(r', \mathcal{O}_{(x,0),\rho}, 5) \le 2C_3(n,\gamma)A$ for all $r < r' \le R-1$, where

$$C_3(n,\gamma) = \Big(\frac{C_1(n,\gamma)}{C_2(n,\gamma)}\Big)^{\frac{1}{1+\gamma}}.$$

Similarly, assuming that $\mathrm{Height}(r, 0, \mathcal{O}_{(x,0),\rho}, 5) \ge a$, by Theorem 2.9

$$M\Big(r + \frac{a}{2}, u_{(x,0),\rho}, \mathcal{O}_{(x,0),\rho}, 5\Big) \ge C_2(n,\gamma)\Big(\frac{a}{2}\Big)^{1+\gamma},$$

from which we obtain that $\mathrm{Height}(r', \mathcal{O}_{(x,0),\rho}, 5) \ge C_3(n,\gamma)^{-1}a/2$ for all $r + a/2 \le r' \le R$. ■

We now prove sufficient conditions under which we can find a region that satisfies the hypotheses of Lemma 4.2. Note that Lemma 4.3 does not depend upon the radii $c_0 > 0$. The lemma and its proof are adaptations of [21, Lemma 3.4].

Lemma 4.3. *Let u be a local minimizer of (1.1). Suppose that for some $x \in B_1^k(0)$ and some component $\mathcal{O} \subset \{u > 0\}$, there exists a radius $0 < r \le 1/5$ and $8 < N < \infty$ such that the following conditions hold:*

(1) *(Containment)*

$$\mathcal{O}_{(x,0),r} \cap B_1^n(0,0) \subset B_1^k(x) \times B_{1/4N}^{n-k}(0).$$

(2) *(Intersection)*

$$\partial B_1^k(0) \times B_1^{n-k}(0) \cap \mathcal{O}_{(x,0),r} \ne \emptyset,$$
$$\partial B_{1/4}^k(0) \times B_1^{n-k}(0) \cap \mathcal{O}_{(x,0),r} \ne \emptyset.$$

Then, there exists a radius $r/2 < \rho \le r$ and an $i \in \mathbb{N}$ such that the rescaling $\mathcal{O}_{(x,0),\frac{\rho}{2^i \cdot 4N}}$ satisfies the following conditions:

(i) *There exists a large radius $1 \ll N_0 < \infty$ satisfying $4N \le N_0$ such that*

$$\mathrm{Height}(N_0, \mathcal{O}_{(x,0),\frac{\rho}{2^i 4N}}, 5) \le 1.$$

(ii) *For the radius $N_0 - N$,*

$$\mathrm{Height}(N_0 - N, \mathcal{O}_{(x,0),\frac{\rho}{2^i 4N}}, 5) \ge \frac{1}{2}.$$

Proof. Let $8 < N < \infty$, $0 < r$, and $u_{(x,0),r}$ be given as in the hypotheses. Let $r_0 = 1$, $r_1 = 1 - \frac{1}{2}2^{-1}$, and inductively $r_{i+1} = r_i - \frac{1}{2}2^{-(i+1)}$. Note then, that $r_i = \frac{2^i+1}{2^{i+1}}$.

By assumption, $\text{Height}(1, \mathcal{O}_{(x,0),r}, 5) \leq \frac{1}{4N}$. If $\text{Height}(r_1, \mathcal{O}_{(x,0),r}, 5) \geq \frac{1}{2}(\frac{1}{4N})$, then the rescaling $u_{(x,0),\frac{r_0}{4N}}$ proves the lemma with $N_0 = 4N$. If not, we proceed inductively. We claim that there exists an $i \in \mathbb{N}$ for which

$$\text{Height}(r_i, \mathcal{O}_{(x,0),r,5}) \leq \frac{1}{2^i}\frac{1}{4N}, \quad \text{Height}(r_{i+1}, \mathcal{O}_{(x,0),r,5}) \geq \frac{1}{2^{i+1}}\frac{1}{4N}.$$

Then, $u_{(x,0),\frac{r}{2^i 4N}}$ satisfies the claims of the lemma with $N_0 = r_i \cdot 2^i 4N = 2(2^{i+1}+1)N$, since $(r_i - r_{i+1}) \cdot 2^i 4N = \frac{1}{2}2^{-i+1} \cdot 2^i 4N = N$.

Thus, we reduce the proof to demonstrating that such a scale must exist. Suppose for the sake of contradiction that $\text{Height}(r_i, \mathcal{O}_{(x,0),r}, 5) < \frac{1}{2^i}(\frac{1}{4N})$ for all $i \in \mathbb{N}$. If this were so, then first we observe that

$$\lim_{i\to\infty} r_i = \lim_{i\to\infty} 1 - \frac{1}{2}\sum_{k=1}^{i}\frac{1}{2^i} = \frac{1}{2}.$$

The fact that $u_{(x,0),r}$ is continuous, having a sequence of scales $r_i \searrow 1/2$ such that the height satisfies $\lim_{i\to\infty} \text{Height}(r_i, \mathcal{O}_{(x,0),\rho}, 5) = 0$, implies

$$\text{M}(1/2, u_{(x,0),r}, \mathcal{O}_{(x,0),r}, 5) = 0.$$

Combined with the weak containment property, this forces $u_{(x,0),r} = 0$ in the window $\mathcal{O}_{(x,0),r} \cap (B_{\frac{1}{2}}(0)^k \times [-5,5]^{n-k})$ by the Maximum Principle. This contradicts the intersection property in the hypotheses. Therefore, there must be an $i \in \mathbb{N}$ such that i is the first integer such that $\text{Height}(r_{i+1}, \mathcal{O}_{(x,0),r}, 5) \geq \frac{1}{2^{i+1}}(\frac{1}{4N})$. Thus, for this i the function $u_{(x,0),\frac{r}{2^i \cdot 4N}}$ satisfies the lemma, by the previous paragraph. ∎

Remark 4.4. Note that the second intersection hypothesis is essential to the contradiction argument.

Corollary 4.5. *Let u be an ε_0-local minimizer of (1.1). Suppose that u satisfies the hypotheses of Lemma 4.3 with $0 < r \leq c_0(n,\gamma,\varepsilon_0)$ and $8 < N < \infty$, and let $\frac{r}{2} < \rho < r$ be the promised scale. Then, Lemma 4.2 implies that there exists a $0 < \rho' = C_3(n,\gamma)\rho$ such that*

$$\begin{aligned}\frac{1}{4C_3(n,\gamma)^2} &< \inf_{s\in\left[\frac{N_0-N+\frac{1}{8}}{C_3(n,\gamma)}, \frac{N_0}{C_3(n,\gamma)}\right]} \text{Height}(s, \mathcal{O}_{(x,0),\rho'}, 5),\\ &\sup_{s\in\left[\frac{N_0-N}{C_3(n,\gamma)}, \frac{N_0}{C_3(n,\gamma)}-1\right]} \text{Height}(s, \mathcal{O}_{(x,0),\rho'}, 5) \leq 2,\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}\frac{C_2(n,\gamma)^2}{C_1(n,\gamma)}\left(\frac{1}{4}\right)^{1+\gamma} &\leq \min_{s\in\left[\frac{N_0-N+1/8}{C_3(n,\gamma)},\frac{N_0}{C_3(n,\gamma)}\right]} M(s,0,u_{(x,0),\rho'},\mathcal{O}_{(x,0),\rho'},5)\\ &\leq \max_{s\in\left[\frac{N_0-N}{C_3(n,\gamma)},\frac{N_0}{C_3(n,\gamma)}\right]} M(s,0,u_{(x,0),\rho'},\mathcal{O}_{(x,0),\rho'},2)\leq C_2(n,\gamma).\end{aligned}$$

4.1. Proof of Lemma 4.1

Note that Lemma 4.5 essentially obtains the kind of "height" control we wish to prove. However, it obtains this in an annular region in which the inner radius $N_0 - N$ is uncontrolled. The proof of Lemma 4.1 essentially consists in considering a disc contained within the annulus.

Let $(x,0)\in\partial\{u>0\}$ and suppose that $\mathcal{O}$ and $0<r<c_0$ satisfy (4.1) and the hypotheses of Lemma 4.1 for $C_3(n,\gamma)+2<N\in\mathbb{N}$. Then, the intersection hypotheses of Lemma 4.3 is satisfied, and Lemmas 4.3 and 4.2 and Corollary 4.5 imply that for ρ' as in Corollary 4.5, the estimate in (4.2) holds for $\mathcal{O}_{(x,0),\rho'}$ for some $N_0\geq 4N$.

Now, let $x'\in\mathbb{R}^k\times\mathbb{R}^{n-k}$ be chosen such that $|x'|=\frac{1}{2}[(\frac{N_0}{C_3(n,\gamma)}-1)+(\frac{N_0-N+1/8}{C_3(n,\gamma)})]$ and $\{(x',y):y\in\mathbb{R}\}\cap\overline{\mathcal{O}_{(x,0),\rho'}}\neq\emptyset$. For any such x',

$$B^k_{\frac{N-2}{2C_3(n,\gamma)}-\frac{1}{2}}(x')\subset\subset\left(B^k_{\frac{N_0}{C_3(n,\gamma)}-1}(x')\setminus B^k_{\frac{N_0-N+1/8}{C_3(n,\gamma)}}(x')\right)$$

and by Corollary 4.5, $\mathcal{O}_{(x',0),\rho'}\cap B^n_{\frac{N-2}{2C_3(n,\gamma)}-\frac{1}{2}}(0,0)$ satisfies

$$\begin{aligned}\frac{1}{4C_3(n,\gamma)^2} &< \inf_{s\in[0,\frac{N-2}{2C_3(n,\gamma)}-\frac{1}{2}]}\mathrm{Height}(s,\mathcal{O}_{(x',0),\rho'},5)\\ &\leq \sup_{s\in[0,\frac{N-2}{2C_3(n,\gamma)}-\frac{1}{2}]}\mathrm{Height}(s,\mathcal{O}_{(x',0),\rho'},5)\leq 2.\end{aligned}$$

Thus, $B_{\frac{N-2}{2C_3(n,\gamma)}-\frac{1}{2}}(x',0)$ has Property P with constant $N'=\frac{N-2}{2C_3}-\frac{1}{2}$. Since Property P is scale invariant (in the sense of Remark 1.6), this proves Lemma 4.1. ■

4.2. Proof of Theorem 1.2

Let u be an ε_0-local minimizer of (1.1) in $\mathcal{K}_{u_0,B^n_2(0,0)}$, and suppose that $(x,0)\in\partial\{u>0\}\cap B^n_1(0,0)$. Let $c_0(n,\gamma,\varepsilon_0)>0$ be as in Section 2.2, $N_0(n,\gamma)$ as in Theorem 1.7, and $N(n,\gamma)=2N_0+1$. Then, by Theorem 1.7 for all radii $0<r\leq c_0$, $B^n_r(x,0)$ cannot satisfy the hypotheses of Lemma 4.1. Therefore, for all radii $0<\rho\leq c_0$ there must exist a point $(x',y')\in B^n_\rho(x,0)\cap\{u>0\}$ with $|y'|\geq\frac{\rho}{4N}$. Therefore, by Lemmas 1.1 and 2.9 applied to $B^n_{\frac{|y'|}{2}}(x', y')$, we see that

$$\fint_{\partial B^n_{\frac{|y'|}{2}}(x',y')} u d\sigma > C_{\min}\Big(\frac{\rho}{32N(n,\gamma)}\Big)^{1+\gamma} \geq \Big(\frac{C_{\min}(n)}{32N(n,\gamma)}\Big)^{1+\gamma} \rho \max_{y \in B^n_{\frac{|y'|}{2}}(x',y')}\{Q(y)\}$$

$$\geq \Big(\frac{C_{\min}(n)}{32N(n,\gamma)}\Big)^{1+\gamma} \rho \Big(\frac{\rho}{32N(n,\gamma)}\Big)^{\gamma} = C(n,\gamma)\rho^{1+\gamma}.$$

Thus, for any $0 < r < c_0$ and any $0 < s < 1$, if $(x, 0) \in B^n_{sr}(x_0, y_0) \cap \partial\{u > 0\} \cap \Gamma$ we can take $\rho = \frac{(1-s)}{2} r$ so that $B^n_{2\rho}(x, 0) \subset B^n_r(x_0, y_0)$. The foregoing calculation shows that we can find a $(x', y') \in B^n_{\frac{(1-s)}{2}r}(x, 0) \cap \{u > 0\}$ such that

$$\fint_{\partial B^n_{\frac{|y'|}{2}}(x',y')} u d\sigma \geq C(n,\gamma)\Big(\frac{(1-s)}{2} r\Big)^{1+\gamma}.$$

Thus, there must exist a point $(v', w') \in \partial B^n_{\frac{|y'|}{2}}(x', y')$ such that

$$u(v', w') \geq C(n,\gamma)(\frac{1-s}{2})^{1+\gamma} r^{1+\gamma}.$$

Since $B^n_{\frac{|y'|}{2}}(x', y') \subset B_{\frac{3}{2}(1-s)r}(x, 0) \subset B^n_r(x_0, y_0)$, by the Maximum Principle we infer that

$$\sup_{\partial B^n_r(x_0,y_0)} u \geq \sup_{\partial B^n_{(1-s)r}(x,0)} u \geq C(n,\gamma)\Big(\frac{1-s}{2} r\Big)^{1+\gamma}.$$

Thus, by the Lipschitz bound from Lemma 2.8, it follows that there must exist a point $(v, w) \in \partial B^n_r(x_0, y_0)$ such that $u > C(n,\gamma)2^{-1-\gamma}(1-s)^{1+\gamma} r^{1+\gamma}/2$ on

$$B^n_{\frac{C(n,\gamma)(1-s)^{1+\gamma}}{2^{1+2\gamma} C_1(n,\gamma)} r}(v, w).$$

Hence,

$$\fint_{\partial B^n_r(x_0,y_0)} u d\sigma \geq C'(n,\gamma,s) r^{1+\gamma} \geq C(n,\gamma,s) r \max_{(v,w) \in B_r(x_0,y_0)}\{Q(v,w)\}.$$

5. Proof of Theorem 1.10

Theorem 1.10 follows from Theorem 1.7 and the following lemma:

Lemma 5.1 (Attenuation radius). *Suppose that u is an ε_0-local minimizer of* (1.1) *in $B^n_2(0,0)$ for $Q(x, y) = |y|^\gamma$, as above. Let $(0,0) \in \Sigma_{\text{cusp}}$, and let $\mathcal{O}$ be a cusp with $(0,0)$ its cusp point. For any $\eta > 0$, we can find a radius $0 < r(\eta) < c_0$ (as in Section 2.2) such that*

$$\mathcal{O}_{(0,0),r} \cap \big(B^k_1(0) \times B^{n-k}_1(0)\big) \subset \big\{(x, y) \in \mathbb{R}^k \times \mathbb{R}^{n-k} : |y| \leq \eta |x|\big\}.$$

Proof. Suppose that the claim is false. Then, there exists an $0 < \eta$ for which there is a sequence of radii $r_j \to 0$ such that there exists a $(x_j, y_j) \in \partial\mathcal{O}_{(0,0),r_j} \cap B_1^n(0,0)$ for which $|y_j| \geq \eta|x_j|$. For sufficiently small $0 < r < c_0$, then we may apply Lemmas 2.8 and 2.9 to obtain

$$\frac{\mathcal{H}^n(B_1^n(0,0) \cap \mathcal{O}_{(0,0),r_j})}{\omega_n} > c > 0.$$

This contradicts $\mathcal{O}$ being a cusp. ■

5.1. Proof of Theorem 1.10

We argue by contradiction. Let Ω as in the hypothesis and assume that $(x,0) \subset B_{2r}^n(x,0) \subset \Omega$. Then, if u is a local minimizer in $\mathcal{K}_{u_0,\Omega}$, u is an ε_0-local minimizer in $\mathcal{K}_{u,B_{2r}^n(x,0)}$ for some $\varepsilon_0 > 0$. Lemma 5.1 implies that for any $N \in \mathbb{N}$, we may find a neighborhood at scale $0 < r(N) < c_0(n,\gamma,\varepsilon_0)$ that satisfies (4.1). Hence, for $N = 3N_0$ in particular, Lemma 4.1 implies that there exists a ball with Property P with constant $\frac{N-1}{2}$. However, this contradicts Theorem 1.7, which states that there is an $N_0(n,\gamma)$ such that no such ball $B_{r(N)}^n(x,0)$ can have Property P with constant N for $N \geq N_0$. Therefore, no cusp points can exist.

To see the claim that there are only finitely many components of $B_{\mathrm{dist}((x,0),\partial\Omega)}(x,0) \cap \{u > 0\}$ that intersect $B_{c\,\mathrm{dist}((x,0),\partial\Omega)}(x,0)$, we note that by Lemma 2.6, we may apply Theorem 1.2 to each component. Then, the Interior Corkscrew Condition given by Lemma 2.9 applied at $B_{(1-c)\,\mathrm{dist}((x,0),\partial\Omega)}(x,0)$ gives that there must exist a disjoint collection of balls $\{B_{(1-c)c(n,\gamma)\,\mathrm{dist}((x,0),\partial\Omega)}\}(x_i', y_i')\}_i \subset B_{\mathrm{dist}((x,0),\partial\Omega)}(x,0)$. Since there can only be finitely many such disjoint balls, the claim is proven.

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