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Existence of solution and asymptotic behavior for a two-phase singular elliptic free boundary problem

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Abstract. We study a superlinear two-phase equation related to the diffusion of a magnetic field in a plasma, which is ruled by a power term in the subcritical Sobolev exponent range plus a singular inverse of a power. We obtain the existence of a two-parameter family of solutions coming from critical points of mountain-pass type. We also address the regularity of solutions and the regularity of free boundaries, which are smooth hypersurfaces outside a set of Hausdorff co-dimension at most 3. Furthermore, we study the asymptotic behavior of the family with respect to the two parameters, separately. This leads to approximated solutions of the problem without the singular term, and to a concentration phenomena of solutions around a point of maximum distance to the boundary of the domain.

1. Introduction

Let Ω be a bounded smooth domain of $\mathbb{R}^N$ with $N \geq 2$. This work is devoted to the study of nontrivial solutions of

$$\begin{cases} -\Delta u = (-(1-\beta)(u-1)^{-\beta} + \lambda(u-1)^{p-1})\chi_{\{u>1\}} & \text{in } \Omega \setminus \partial\{u>1\}, \\ u = 0 & \text{on } \partial\Omega, \\ |u_\nu^+|^2 - |u_\nu^-|^2 = 2 & \text{on } \partial\{u>1\}, \end{cases} \tag{1}$$

where $0 < \beta < 1$ and $\lambda > 0$ are constants; $p > 2$ if $N = 2$ and $2 < p < 2^* = 2N/(N-2)$ if $N \geq 3$; $\chi_{\{u>1\}}$ denotes the characteristic function corresponding to the set $\{u > 1\}$; $\partial\{u > 1\}$ is the free boundary of u;

$$u_\nu^+ = \lim_{\delta \to 0^+} \nabla u \cdot \nu^+ \quad \text{and} \quad u_\nu^- = \lim_{\delta \to 0^+} \nabla u \cdot \nu^-,$$

where ν^+ is the normal to $\partial\{u > 1+\delta\}$ pointing inwards of $\{u > 1+\delta\}$; and ν^- is the normal to $\partial\{u > 1-\delta\}$ pointing to the interior of $\{u \leq 1-\delta\}$, denoted by $\{u \leq 1-\delta\}^\circ$, for every $\delta > 0$. The condition on $\partial\{u > 1\}$ is read in the weak sense as

$$\lim_{\delta \to 0^+} \int_{\{u=1+\delta\}} (|\nabla u|^2 - 2)\phi \cdot \nu^+ \, d\mathcal{H}^{N-1} - \lim_{\delta \to 0^+} \int_{\{u=1-\delta\}} |\nabla u|^2 \phi \cdot \nu^- \, d\mathcal{H}^{N-1} = 0$$

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for every $\phi \in C^1(\Omega, \mathbb{R}^N)$ with $u \neq 1$ on the support of ϕ, where $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure.

Solutions of two-phase problems are often obtained by minimizing the associated energy functional. In recent advances [14, 15, 28, 41, 51] authors have obtained solutions arising from general critical points that are not minima. Specifically, Jerison and Perera [28] extended the theory known for local minimizers [2, 7, 48, 49] of the functional $J\colon H_0^1(\Omega) \to \mathbb{R}$ defined by

$$J(u) = \int_\Omega \Big[\frac{1}{2}|\nabla u|^2 + \chi_{\{u>1\}} - \frac{\lambda}{p}(u-1)^p \chi_{\{u>1\}}\Big] dx. \tag{2}$$

The above functional J is associated to the well-known superlinear plasma problem [12, 25, 45, 46]

$$\begin{cases} -\Delta u = \lambda (u-1)^{p-1} \chi_{\{u>1\}} & \text{in } \Omega \setminus \partial\{u>1\}, \\ u = 0 & \text{on } \partial\Omega, \\ |u_\nu^+|^2 - |u_\nu^-|^2 = 2 & \text{on } \partial\{u>1\}. \end{cases} \tag{3}$$

In [28], the authors proved the existence of a nontrivial Lipschitz continuous solution of (3) arising as a mountain-pass critical point of J and satisfying properties known for minimizers, such as nondegeneracy and free boundary regularity, meaning the finiteness of the $(N-1)$-dimensional Hausdorff measure and that in fact it is a smooth hypersurface outside a set of Hausdorff dimension at most $N-3$, that is, for $N=2$ the free boundary is a smooth curve. The measure theory issues of the present paper can be recalled from the book [23].

It is worth noting that the regularity for minimizers, as in the minimal surfaces theory, is less restrictive: in this case, the singular set has at most Hausdorff dimension $N-5$; see, for instance, [20, 21, 33]. This is in strict relation with the geometry of free boundaries [7, 47]. Article [28] still relies on the usual methods introduced by Caffarelli in the seminal works [8–11] toward the study of the free boundary regularity: "flat implies Lipschitz" and "Lipschitz implies smooth" in dimension 2, along with the monotonicity formulas of Weiss in [48, 49] for higher dimensions. The framework built in [28] was further generalized in [41] to achieve results for (3) with a larger class of nonlinearities with growth of at most a power in the subcritical Sobolev range. In the present note, we use a such framework to obtain solutions to singular equations.

Problem (1) belongs to the class of singular elliptic equations with Van der Waals–type interactions, which appear in steady states modeling several fluid dynamics—for example, non-Newtonian fluids, plasma allocation, stability of thin liquid films, and electrostatic actuation, among others [29, 30, 32, 39, 40]. Solutions to the one-phase version are well known and extensively studied [13, 16–19, 35, 37] and in particular, in [14], solutions were obtained for a similar two-phase free boundary problem ruled by a Prandtl–Batchelor-type equation with a term $u^{-\beta}$, but singular near the boundary of the domain.

Critical points of the functional $E\colon H_0^1(\Omega) \to \mathbb{R}$ given by

$$E(u) = \int_\Omega \Big[\frac{1}{2}|\nabla u|^2 + \chi_{\{u>1\}} + (u-1)_+^{1-\beta} - \frac{\lambda}{p}(u-1)_+^p\Big]dx,$$

where $(u-1)_+ = \max\{u-1,0\}$, are indeed weak solutions of (1). Due to the term $\chi_{\{u>1\}}$, the functional E is discontinuous and therefore classical variational methods do not apply directly. We consider approximations like in [19, 35], induced by the perturbations $f\colon\mathbb{R} \to [0,2]$, where f is a C^∞-function with support in $(0,1)$ and satisfying $\int_0^1 f(s)ds = 1$, and $g_\varepsilon\colon \mathbb{R} \to \mathbb{R}$, $0 < \varepsilon < 1$, given by

$$g_\epsilon(t) = \begin{cases} \dfrac{t}{(t+\varepsilon)^{1+\beta}} & \text{if } t \geq 0, \\ \qquad 0 & \text{if } t < 0. \end{cases}$$

Thus, we have a family of perturbed functionals $E_\varepsilon\colon H_0^1(\Omega) \to \mathbb{R}$, $0 < \varepsilon < 1$, defined by

$$E_\varepsilon(u) = \int_\Omega \Big[\frac{1}{2}|\nabla u|^2 + F\Big(\frac{u-1}{\varepsilon}\Big) + (1-\beta)G_\varepsilon(u-1) - \frac{\lambda}{p}(u-1)_+^p\Big]\,dx,$$

where $G_\varepsilon(s) = \int_0^s g_\varepsilon(t)\,dt$ and $F(s) = \int_0^s f(t)\,dt$ are both nonnegative and nondecreasing, and F satisfies $F(t) = 0$ if $t \leq 0$, $0 < F(t) \leq 1$ if $0 < t < 1$, and $F(t) = 1$ if $t \geq 1$. Note that each E_ε is now a C^1-functional with $E_\varepsilon'\colon H_0^1(\Omega) \to (H_0^1(\Omega))^*$ given by

$$E_\varepsilon'(u)v = \int_\Omega \Big[\nabla u \nabla v + \Big(\frac{1}{\varepsilon}f\Big(\frac{u-1}{\varepsilon}\Big) + (1-\beta)g_\varepsilon(u-1) - \lambda(u-1)_+^{p-1}\Big)v\Big]\,dx.$$

Critical points u_ε of each E_ε are weak solutions of

$$\begin{cases} -\Delta u = -\dfrac{1}{\varepsilon}f\Big(\dfrac{u-1}{\varepsilon}\Big) - (1-\beta)g_\varepsilon(u-1) + \lambda(u-1)_+^{p-1} & \text{in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u = 0 & \text{on } \partial\Omega \end{cases} \tag{4}$$

and actually, by standard elliptic regularity theory, u_ε are $C^{2,\alpha}$-solutions in $\{u_\varepsilon > 1\}$, where the right-hand side of (4) is Hölder continuous, and in $\{u_\varepsilon \leq 1\}^\circ$, where u_ε is harmonic. Moreover, a nontrivial u_ε satisfies

$$\begin{cases} -\Delta u_\varepsilon = 0 & \text{in } \{u_\varepsilon \leq 1\}, \\ \quad u_\varepsilon = 1 & \text{on } \partial\{u_\varepsilon \geq 1\}, \\ \quad u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

and thus the maximum principle guarantees that $u_\varepsilon > 0$ in Ω. If $\{u_\varepsilon > 1\} = \emptyset$, then $\{u_\varepsilon \leq 1\} = \Omega$ and thus u_ε must be trivial, which is a contradiction. Hence, we must have $u_\varepsilon > 1$ in a nonempty open set.

In view of [19, 35, 37], due to the sign behavior of the nonlinearity, solutions generally exist for sufficiently large λ. In Section 2, we prove the existence of a nontrivial mountain-pass critical point of each E_ε for a sufficiently large λ. The proof relies on a deformation lemma and a mountain-pass theorem adapted to this case. We prove some uniform estimates in Section 2.1, which are necessary to obtain in Section 3 a limit candidate u for a solution of (1) as $\varepsilon \to 0^+$. We show that this candidate is actually a critical point of the original functional E and we verify that it is a minimizer when restricted to the associated Nehari manifold given by

$$\begin{aligned}\mathcal{N} = \Big\{u \in H_0^1(\Omega) \mid (u-1)_+ \not\equiv 0, \int_{\{u>1\}} |\nabla u|^2\, dx \\ = \int_{\{u>1\}} [-(1-\beta)(u-1)^{1-\beta} + \lambda(u-1)^p]\, dx\Big\}.\end{aligned}$$

Recall that $u \in H_0^1(\Omega)$ is a mountain-pass point of E if, given a neighborhood $U \subset H_0^1(\Omega)$ of u, the set $\{v \in U \mid E(v) < E(u)\}$ is neither empty nor path connected.

We can state our results as follows. We denote by $\mathcal{L}^N$ the N-dimensional Lebesgue measure.

Theorem 1.1. *There exists a $\lambda_0 > 0$ such that, for each $\lambda > \lambda_0$, there is a sequence u_{ε_j} in $H_0^1(\Omega)$ of critical points of E_{ε_j} converging to a function $u \in H_0^1(\Omega) \cap C^2(\overline{\Omega} \setminus \partial\{u > 1\})$ that is Lipschitz continuous on $\overline{\Omega}$ and a solution of (1) in $\Omega \setminus \partial\{u > 1\}$, as $\varepsilon_j \to 0^+$, and satisfying*

$$u_{\varepsilon_j} \to u \quad \textit{uniformly in } \overline{\Omega}, \tag{5}$$

$$u_{\varepsilon_j} \to u \quad \textit{in } H_0^1(\Omega), \tag{6}$$

$$E(u) \le \liminf_{j\to\infty} c^*_{\varepsilon_j} \le \limsup_{j\to\infty} c^*_{\varepsilon_j} \le E(u) + \mathcal{L}^N(u^{-1}(1)), \tag{7}$$

where

$$c^*_{\varepsilon_j} = \inf_{\gamma\in\Gamma_{\varepsilon_j}} \max_{t\in[0,1]} E_{\varepsilon_j}(\gamma(t)) \tag{8}$$

with

$$\Gamma_{\varepsilon_j} = \big\{\gamma \in C^0([0,1], H_0^1(\Omega)) \mid \gamma(0) = 0,\ E_{\varepsilon_j}(\gamma(1)) < 0\big\}. \tag{9}$$

Furthermore, $u \in \mathcal{N}$, $E(u) > 0$, $c^* = E(u) = \inf_{\mathcal{N}} E$, u is a mountain-pass point of E, and u satisfies

$$\begin{aligned}\int_\Omega \Big[\Big(\frac{1}{2}|\nabla u|^2 + \chi_{\{u>1\}} + (u-1)_+^{1-\beta} - \frac{\lambda}{p}(u-1)_+^p\Big) \operatorname{div}\Phi - \nabla u(D\Phi)\cdot\nabla u\Big]\, dx = 0, \\ \forall \Phi \in C_0^1(\Omega, \mathbb{R}^N).\end{aligned}$$

To verify that the limit candidate u is actually a solution, we prove in Section 4 that it is nondegenerate in the following sense, which is essential to obtain the regularity properties of the free boundary.

Definition 1.2. *A function u that is Lipschitz continuous on $\overline{\Omega}$ with Lipschitz constant L is said to be nondegenerate if there exist constants $\rho_0, C > 0$ depending on N, p, β, and L such that*

$$u(x) \ge 1 + C\operatorname{dist}(x, \{u \le 1\}) \quad \textit{for every } x \in \{u > 1\} \textit{ with } \operatorname{dist}(x, \{u \le 1\}) \le \rho_0.$$

Furthermore, a property that implies the nondegeneracy of the phase $\{u < 1\}$ is defined as follows.

Definition 1.3. *We say that a function u that is Lipschitz continuous on $\overline{\Omega}$ with Lipschitz constant L satisfies the positive density property if there exist constants $\rho_0, C > 0$ depending on N, p, β, and L such that*

$$C \leq \frac{\mathcal{L}^N(\{u > 1\} \cap B_\rho(x))}{\mathcal{L}^N(B_\rho(x))} \leq 1 - C,$$

for every $x \in \partial\{u > 1\}$ and $0 < \rho \leq \rho_0$.

In possession of the nondegeneracy properties, we also are able to assert the satisfiability of the free boundary condition in the following sense.

Definition 1.4. *We say that a function u that is continuous on $\overline{\Omega}$ satisfies the free boundary condition of (3) in the viscosity sense if u is of the form*

$$u(x) = 1 + \alpha((x - x_0)\cdot\nu)_+ - \gamma((x - x_0)\cdot\nu)_- + o(|x - x_0|) \quad \textit{when } x \to x_0, \tag{10}$$

around each tangent point x_0, the one where a ball $B \subset \{u > 1\})$ or $B \subset \{u \leq 1\}^\circ$ is tangent to $\partial\{u > 1\}$ at x_0, for some constants $\alpha > 0$ and $\gamma \geq 0$ satisfying $\alpha^2 - \gamma^2 = 2$. Here ν is the unit normal to ∂B at x_0, pointing inwards of B if $B \subset \{u > 1\})$ and outwards of B if $B \subset \{u \leq 1\}^\circ$.

We state the regularity results of the free boundary obtained in Sections 4 and 5. It is worth mentioning that except for the free boundary condition in the viscosity sense, the results are valid for any solution satisfying the properties of u in Theorem 1.1, and do not depend now on the behavior of the critical points of regularized functional E_ε when $\varepsilon \to 0^+$.

Theorem 1.5. *If u is the limiting function of Theorem 1.1, then u is nondegenerate, $\mathcal{H}^{N-1}(\partial\{u > 1\}) < \infty$, and u satisfies the positive density property and the free boundary condition of (3) in the viscosity sense. Moreover, $\partial\{u > 1\}$ is a $C^{1,\alpha}$-hypersurface outside a singular set of Hausdorff dimension at most $N - 3$.*

Note that the results above guarantee two families u_β and u_λ of solutions of (1), one with respect to the parameter $0 < \beta < 1$ and the other one with respect to $\lambda \geq \lambda_0$, where $\lambda_0 > 0$ is given by Theorem 1.1, respectively.

In [36] the following one-phase problem was studied:

$$\begin{cases} -\Delta u = (-(1-\beta)u^{-\beta} + \lambda u^q)\chi_{\{u>0\}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < q < 1$. The authors obtained a family of solutions u_β in $C^{1,\gamma}_{\text{loc}}(\Omega)$, with $\gamma = (1-\beta)/(1+\beta)$, admitting a subsequence u_{β_j} that converges to a minimizer of

$$\mathcal{J}(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \chi_{\{u>0\}} - \frac{\lambda}{q+1}u^{q+1}\right) dx$$

when $\beta_j \to 1^-$. Motivated by their approach to recover solutions of a so-called Alt–Caffarelli-type functional [1], in Section 6 we obtain an approximation to a critical point of the functional J defined in (2). In this respect, we establish the result below.

Theorem 1.6. *If u_β are the limiting functions of Theorem 1.1 for each fixed β, then there is a subsequence u_{β_j} converging, as $\beta_j \to 1^-$, to a critical point of the functional*

$$u \mapsto \mathcal{L}^N(\{u > 1\}) + J(u).$$

The volume of the phase $\{u > 1\}$ appears here due to the convergence

$$\int_{\{u_\beta > 1\}} (u_\beta - 1)^{1-\beta}\, dx \to \mathcal{L}^N(\{u > 1\})$$

and because the functional E carries a term $\mathcal{L}^N(\{u > 1\})$.

Furthermore, the one-phase free boundary problem

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + \lambda u^p & \text{in } \Omega, \\ \hfill u \geq 0 & \text{in } \Omega, \\ \hfill u = 0 & \text{on } \partial\Omega \end{cases}$$

was studied in [19], where the authors obtained a family of nontrivial solutions $u_\lambda \in C^1(\overline{\Omega}) \cap C^\infty(\Omega)$ for all $\lambda > 0$, which for sufficiently large λ exhibit a radial profile of least energy around some $x_\lambda \in \Omega$. They also showed that this sequence of points converges to a point $x_{\max}$ of maximum distance from the boundary $\partial\Omega$ as $\lambda \to \infty$. In Section 7, we show that this concentration phenomena also holds for a family u_λ obtained in Theorem 1.1. We mention [24, 44] where such themes were also studied.

Theorem 1.7. *Let u_λ be limit solutions of (1) obtained by Theorem 1.1. Then a seequence x_λ of maximum points of u_λ converges to a point $x_{\max}$ of maximum distance from the boundary $\partial\Omega$ when $\lambda \to \infty$.*

By the maximum principle, the points x_λ must be inside the cores $\{u_\lambda > 1\}$ and therefore as $\lambda \to \infty$, the location of $\{u_\lambda > 1\}$ tends to be as equidistant from the boundary as possible. We remark that this result was possible due to the strategy in [19], which only relies on localized energy expansions.

Throughout the work, we write the positive and the negative parts of a function w by

$$w_+ = \max\{w, 0\} \quad \text{and} \quad w_- = \max\{-w, 0\},$$

respectively.

2. Existence of critical points of E_ε

In this section, we show that each E_ε possesses a critical point of mountain-pass-type when the parameter λ is sufficiently large.

Proposition 2.1. *There exists a $\lambda_0 > 0$ independent of ε such that E_ε has a nontrivial critical point for every $\lambda > \lambda_0$.*

The hypothesis on λ appears when we guarantee in Lemma 2.3 below that the class Γ_ε of continuous paths from the origin to $\{u \in H_0^1(\Omega) \mid E_\varepsilon(u) < 0\}$ is nonempty. The key ingredient to prove Proposition 2.1 is the first deformation lemma, providing a homotopy of paths in Γ_ε and leading to a mountain-pass-type theorem (see [4]) in Lemma 2.5.

We will break the proof in some lemmas. Recall that a Palais–Smale sequence for each E_ε is a sequence of functions u_j in $H_0^1(\Omega)$ such that $E_\varepsilon(u_j)$ is bounded and $E'_\varepsilon(u_j) \to 0$ as $j \to \infty$. If every Palais–Smale sequence for E_ε possesses a convergent subsequence in $H_0^1(\Omega)$, we say that E_ε satisfies the Palais–Smale condition.

Lemma 2.2. *The energy functional E_ε satisfies the Palais–Smale condition.*

Proof. Let u_j be a Palais–Smale sequence for E_ε. We first show that u_j is bounded in $H_0^1(\Omega)$. Fix a sequence $0 < \rho_j < 1$ with $\rho_j \to 0^+$ as $j \to \infty$. By the definition of a Palais–Smale sequence, there exists a $j_0 \in \mathbb{N}$ such that if $j \geq j_0$, then

$$\begin{aligned} E_\varepsilon(u_j) &= \int_\Omega \Big[\frac{1}{2}|\nabla u_j|^2 + F\Big(\frac{u_j - 1}{\varepsilon}\Big) + (1-\beta)G_\varepsilon(u_j - 1) - \frac{\lambda}{p}(u_j - 1)_+^p\Big]\,dx \\ &\leq C \end{aligned} \tag{11}$$

for some constant $C > 0$ and

$$\begin{aligned} &|E'_\varepsilon(u_j)v| \\ &\quad = \Big|\int_\Omega \Big[\nabla u_j \nabla v + \frac{1}{\varepsilon} f\Big(\frac{u_j - 1}{\varepsilon}\Big)v + (1-\beta)g_\varepsilon(u_j - 1)v - \lambda(u_j - 1)_+^{p-1}v\Big]\,dx\Big| \\ &\quad \leq \rho_j \|v\|_{H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega). \end{aligned} \tag{12}$$

Write u_j as

$$u_j = (u_j - 1)_+ + (1 - (u_j - 1)_-).$$

Since F, and G_ε are nonnegative, (11) and the Sobolev embedding theorem yield

$$\int_\Omega \Big[\frac{1}{2}|\nabla (u_j - 1)_+|^2 + \frac{1}{2}|\nabla (u_j - 1)_-|^2\Big]\,dx \leq C + \frac{\lambda}{p}\int_\Omega ((u_j - 1)_+)^p\,dx. \tag{13}$$

From (12) with $v = (u_j - 1)_+/p$ and the facts that $f(t)t \leq 2$ and $g_\varepsilon(t)t \leq |t|^{1-\beta}$ for all $t > 0$, we obtain

$$\begin{aligned} &\lambda \int_\Omega ((u_j - 1)_+)^p\,dx \\ &\qquad \leq \int_\Omega [|\nabla (u_j - 1)_+|^2 + 2 + (1-\beta)|(u_j - 1)_+|^{1-\beta}]\,dx \\ &\qquad\quad + \rho_j \|(u_j - 1)_+\|_{H_0^1(\Omega)}. \end{aligned} \tag{14}$$

Combining (13) with (14) gives

$$
\begin{aligned}
&\int_\Omega \Big[\Big(\frac{1}{2}-\frac{1}{p}\Big)|\nabla(u_j-1)_+|^2+\frac{1}{2}|\nabla(u_j-1)_-|^2\Big]\,dx\\
&\qquad \le C+\frac{\rho_j}{p}\|(u_j-1)_+\|_{H_0^1(\Omega)}+\int_\Omega\Big[\frac{2}{p}+\frac{(1-\beta)}{p}|(u_j-1)_+|^{1-\beta}\Big]\,dx\\
&\qquad \le C+\frac{(3-\beta)}{p}\mathcal{L}^N(\Omega)+\frac{1}{p}\|(u_j-1)_+\|_{H_0^1(\Omega)}\\
&\qquad\qquad +\frac{(1-\beta)}{p}\int_\Omega |(u_j-1)_+|\,dx.
\end{aligned}
$$

Since $p>2$, the Sobolev embedding theorem shows that

$$
\begin{aligned}
\Big(\frac{1}{2}-\frac{1}{p}\Big)\|(u_j-1)_+\|^2_{H_0^1(\Omega)} &\le \Big(\frac{1}{2}-\frac{1}{p}\Big)\|(u_j-1)_+\|^2_{H_0^1(\Omega)}+\frac{1}{2}\|(u_j-1)_+\|^2_{H_0^1(\Omega)}\\
&\le C_1+C_2\|(u_j-1)_+\|_{H_0^1(\Omega)}
\end{aligned}
$$

for some constants $C_1, C_2>0$ independent of u_j. This implies that both $\|(u_j-1)_+\|_{H_0^1(\Omega)}$ and $\|(u_j-1)_-\|_{H_0^1(\Omega)}$ are bounded, and hence u_j is bounded in $H_0^1(\Omega)$.

Now there exists a subsequence, still denoted by u_j, such that

$$
\begin{cases}
u_j \rightharpoonup u & \text{weakly in } H_0^1(\Omega),\\
u_j \to u & \text{in } L^q(\Omega) \text{ for every } 1\le q<2^*,\\
u_j \to u & \text{a.e. in } \Omega \quad \text{as } j\to\infty
\end{cases}
\tag{15}
$$

for some $u\in H_0^1(\Omega)$. Since $f(t)\le 2$ and $g_\varepsilon(t)\le g_\varepsilon(\varepsilon/\beta)$ for all $t\in\mathbb{R}$, (15) implies that

$$
\Big|\int_\Omega f\Big(\frac{u_j-1}{\varepsilon}\Big)u_j\,dx-\int_\Omega f\Big(\frac{u_j-1}{\varepsilon}\Big)u\,dx\Big|\to 0,
\tag{16}
$$

$$
*\Big|\int_\Omega g_\varepsilon(u_j-1)u_j\,dx-\int_\Omega g_\varepsilon(u_j-1)u\,dx\Big|\to 0
$$

and, using Hölder inequality, that

$$
\Big|\int_\Omega (u_j-1)_+^{p-1}u_j\,dx-\int_\Omega (u_j-1)_+^{p-1}u\,dx\Big|\to 0.
\tag{17}
$$

Since $E'_\varepsilon(u_j)u_j\to 0$ and $E'_\varepsilon(u_j)u\to 0$, from (12) and (16)–(17), we obtain

$$
\begin{aligned}
&\lim_{j\to\infty}\|u_j\|^2_{H_0^1(\Omega)}\\
&\quad=\lim_{j\to\infty}-\int_\Omega\Big[-\frac{1}{\varepsilon}f\Big(\frac{u_j-1}{\varepsilon}\Big)u_j-(1-\beta)g_\varepsilon(u_j-1)u_j+\lambda(u_j-1)_+^{p-1}u_j\Big]\,dx\\
&\quad=\lim_{j\to\infty}\int_\Omega\Big[-\frac{1}{\varepsilon}f\Big(\frac{u_j-1}{\varepsilon}\Big)u-(1-\beta)g_\varepsilon(u_j-1)u+\lambda(u_j-1)_+^{p-1}u\Big]\,dx\\
&\quad=\lim_{j\to\infty}\int_\Omega \nabla u_j\nabla u\,dx\\
&\quad=\|u\|^2_{H_0^1(\Omega)}.
\end{aligned}
$$

It follows that u_j converges strongly to u in $H_0^1(\Omega)$ and the proof is concluded. ∎

We now check that E_ε has a functional geometry of mountain-pass-type with local minimum at 0.

Lemma 2.3. *There exists a $\lambda_0 > 0$ independent of ε such that if $\lambda > \lambda_0$, then Γ_ε defined in (9) is nonempty. Furthermore,*

$$E_\varepsilon(u) \geq \frac{1}{4}\|u\|^2_{H_0^1(\Omega)}, \quad \forall u \in H_0^1(\Omega) \text{ with } \|u\|_{H_0^1(\Omega)} \leq \rho \tag{18}$$

and the levels c_ε^* defined in (8) are positive.

Proof. Let $\phi \in H_0^1(\Omega)$ be a function with $\{\phi > 1\} \neq \emptyset$ and $\|(\phi-1)_+\|_{H_0^1(\Omega)} \geq 1$. For instance, let $x_0 \in \Omega$ be such that there exists an r satisfying $0 < r^{N-1} < \min\{1, 4^N/(\mathcal{L}^N(B_1(0))(3^N - 2^N))\}$ with $B_r(x_0) \subset \Omega$ and take ϕ a smooth function on Ω which is symmetric with respect to x_0 such that $\phi = 0$ in $\Omega \setminus B_r(x_0)$,

$$\phi = \frac{3r}{\sqrt{\mathcal{L}^N(B_{3r/4}(x_0)) - \mathcal{L}^N(B_{r/2}(x_0))}} \quad \text{in } B_{r/4}(x_0)$$

and

$$\phi(x) = \frac{4(r - |x - x_0|)}{\sqrt{\mathcal{L}^N(B_{3r/4}(x_0)) - \mathcal{L}^N(B_{r/2}(x_0))}} \quad \text{in } B_{3r/4}(x_0) \setminus B_{r/2}(x_0).$$

Since $F(t) \leq 1$, $(1-\beta)G_\varepsilon(t) \leq |t|^{1-\beta}$ for all $t \in \mathbb{R}$ and

$$\begin{aligned}\int_\Omega (\phi-1)_+^{1-\beta}\,dx &= \int_{\{(\phi-1)_+\leq 1\}} (\phi-1)_+^{1-\beta}\,dx + \int_{\{(\phi-1)_+>1\}} (\phi-1)_+^{1-\beta}\,dx \\ &\leq \mathcal{L}^N(\Omega) + \int_{\{(\phi-1)_+>1\}} |\phi-1|\,dx \leq \mathcal{L}^N(\Omega) + \|\phi\|_{L^1(\Omega)},\end{aligned}$$

we have that

$$\begin{aligned}E_\varepsilon(\phi) &\leq \frac{1}{2}\|\phi\|^2_{H_0^1(\Omega)} + \mathcal{L}^N(\Omega) + \int_\Omega (\phi-1)_+^{1-\beta}\,dx - \frac{\lambda}{p}\|(\phi-1)_+\|^p_{L^p(\Omega)} \\ &< 0, \quad \forall \lambda > \lambda_0,\end{aligned}$$

where

$$\lambda_0 = \frac{p(\|\phi\|^2_{H_0^1(\Omega)} + 2\mathcal{L}^N(\Omega) + \|\phi\|_{L^1(\Omega)})}{\|(\phi-1)_+\|^p_{L^p(\Omega)}}.$$

The path $\gamma_\phi \in C^0([0,1], H_0^1(\Omega))$ given by $\gamma_\phi(t) = t\phi$ implies that $\Gamma_\varepsilon \neq \emptyset$.

Now, since F and G are nonnegative, $(t-1)_+ \leq |t|$ for every $t \in \mathbb{R}$, and $2 < p < 2^*$, we have

$$E_\varepsilon(u) \geq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - \mathcal{C}_p^p \frac{\lambda}{p}\|u\|^p_{H_0^1(\Omega)}, \quad \forall u \in H_0^1(\Omega),$$

where $\mathcal{C}_p$ is the best constant of the Sobolev embedding into $L^p(\Omega)$; see [22]. Taking

$$\rho < \min\Big\{1, \Big(\frac{p}{4\lambda \mathcal{C}_p^p}\Big)^{\frac{1}{p-2}}\Big\},$$

we deduce (18). Hence, since $E_\varepsilon(0) = 0$, any path $\gamma \in \Gamma_\varepsilon$ must pass through the region $\|u\|^2_{H_0^1(\Omega)} \le \rho$ where $E_\varepsilon > 0$ before $E_\varepsilon|_\gamma$ becomes negative (note that $\|\phi\|^2_{H_0^1(\Omega)} > \rho$). It follows that

$$c_\varepsilon^* \ge \frac{1}{4}\rho^2 > 0. \qquad \blacksquare$$

Corollary 2.4. *Let λ_0 and c_ε^* be given by Lemma 2.3. If*

$$c^* = \inf_{\gamma \in \Gamma_0} \max_{t \in [0,1]} E(\gamma(t)),$$

where

$$\Gamma_0 = \big\{\gamma \in C^0([0,1], H_0^1(\Omega)) \mid \gamma(0) = 0,\ E(\gamma(1)) < 0\big\},$$

then $c^* \ge c_\varepsilon^* > 0$.

Proof. If $u \in H_0^1(\Omega)$, then $F((u-1)/\varepsilon) \le \chi_{\{u>1\}}$ and $(1-\beta)G_\varepsilon(u-1) \le (u-1)_+^{1-\beta}$. It follows that $E_\varepsilon(u) \le E(u)$, which implies that Γ_0 is a subset of Γ_ε. Hence, if $\gamma \in \Gamma_0$, we obtain

$$c_\varepsilon^* \le \max_{u \in \gamma([0,1])} E_\varepsilon \le \max_{u \in \gamma([0,1])} E$$

and the result follows from Lemma 2.3. $\blacksquare$

We now conclude the proof of Proposition 2.1.

Lemma 2.5. *Let λ_0 be given by Lemma 2.3. If $\lambda > \lambda_0$, then there exists a nontrivial critical point of E_ε at level c_ε^* defined in (8).*

Proof. Suppose that there are no critical points at level c_ε^*. Since E_ε satisfies the Palais–Smale condition by Lemma 2.2, $\Gamma_\varepsilon \ne \emptyset$, and $c_\varepsilon^* > 0$ by Lemma 2.3, it follows from [42, Lemma 1.3.3] that there exist a constant $0 < \delta < c_\varepsilon^*/2$ and a continuous mapping $\eta_\delta \in C^0(H_0^1(\Omega) \times [0,1], H_0^1(\Omega))$ such that

$$\eta_\delta(\cdot, t) \quad \text{is a homeomorphism for all } 0 \le t \le 1,$$
$$\eta_\delta\big(\big\{u \in H_0^1 \mid E_\varepsilon(u) \le c_\varepsilon^* + \delta\big\} \times \{1\}\big) \subset \big\{u \in H_0^1 \mid E_\varepsilon(u) \le c_\varepsilon^* - \delta\big\}$$

and

$$\eta_\delta|_{\{u \in H_0^1 \mid E_\varepsilon(u) \le 0\} \times \{t\}} \quad \text{is the identity mapping for all } 0 \le t \le 1.$$

By the definition of infimum, there exists a path $\gamma \in \Gamma_\varepsilon$ with

$$\max_{u \in \gamma([0,1])} E_\varepsilon(u) \le c_\varepsilon^* + \delta.$$

But η_δ takes γ to a path $\widetilde{\gamma} \in \Gamma_\varepsilon$ with

$$\max_{u\in\widetilde{\gamma}([0,1])} E_\varepsilon(u) \le c_\varepsilon^* - \delta,$$

which contradicts the definition of infimum in c_ε^*. ∎

2.1. Uniform estimates

Aiming a limit solution when $\varepsilon_j \to 0^+$ for some sequence, we need uniform estimates to guarantee sufficient regularity and growth rate so that convergence happens. Here we assume that $\lambda > \lambda_0$ in each E_ε, $0 < \varepsilon < 1$, where λ_0 is given by Lemma 2.3 and we denote by u_ε a nontrivial critical point of E_ε from Lemma 2.5. As noted in the introduction, recall that each u_ε is a solution of (4), which is $C^{2,\alpha}$ in $\{u_\varepsilon > 1\}$ and $\{u_\varepsilon < 1\}$ and strictly positive in Ω.

Proposition 2.6. *There exists a constant $C > 0$ independent of ε such that*

$$\|u_\varepsilon\|_{H_0^1(\Omega)} \le C, \tag{19}$$

$$\max_{\Omega} u_\varepsilon \le C \tag{20}$$

and

$$u_\varepsilon \le \varphi_0, \quad \textit{in particular with } \{u_\varepsilon \ge 1\} \subset \{\varphi_0 \ge 1\} \subset\subset \Omega, \tag{21}$$

where φ_0 is the solution of

$$\begin{cases} -\Delta\varphi_0 = \Lambda & \textit{in } \Omega,\\ \quad\ \varphi_0 = 0 & \textit{on } \partial\Omega, \end{cases}$$

and here $\Lambda > 0$ is a constant independent of ε such that $\lambda(u_\varepsilon - 1)_+^{p-1} \le \Lambda$, which exists by (20). Moreover,

$$\max_{B_{r/2}(x)} |\nabla u_\varepsilon| \le \frac{C}{r} \quad \textit{for all } x \in \Omega \textit{ and } r > 0 \textit{ with } B_r(x) \subset \Omega. \tag{22}$$

Proof. First we prove (19). By Lemma 2.5 and Corollary 2.4, it follows that

$$E_\varepsilon(u_\varepsilon) - E_\varepsilon'(u_\varepsilon)\frac{(u_\varepsilon - 1)_+}{p} = c_\varepsilon^* \le c^*. \tag{23}$$

Write u_ε as

$$u_\varepsilon = (u_\varepsilon - 1)_+ + (1 - (u_\varepsilon - 1)_-).$$

Using the facts that $F, G_\varepsilon \ge 0$, $f(t)t \le 2$, and $g_\varepsilon(t)t \le |t|^{1-\beta}$ for all $t > 0$ and the Sobolev embedding in (23), we obtain

$$\begin{aligned}\left(\frac{1}{2} - \frac{1}{p}\right)\|(u_\varepsilon - 1)_+\|^2_{H_0^1(\Omega)} &+ \frac{1}{2}\|(u_\varepsilon - 1)_-\|^2_{H_0^1(\Omega)}\\ &\le c^* + \int_\Omega \left[\frac{2}{p} + \frac{(1-\beta)}{p}|(u_\varepsilon - 1)_+|^{1-\beta}\right] dx\\ &\le C_1 + C_2\|(u_\varepsilon - 1)_+\|_{H_0^1(\Omega)},\end{aligned}$$

for some constants $C_1, C_2 > 0$ independent of ε. This implies that both $\|(u_\varepsilon - 1)_+\|_{H_0^1(\Omega)}$ and $\|(u_\varepsilon - 1)_-\|_{H_0^1(\Omega)}$ are bounded by a constant independent of ε, and so is $\|u_\varepsilon\|_{H_0^1(\Omega)}$.

Now, we deal with estimate (20). Since $f, g_\varepsilon \geq 0$, we have

$$-\Delta u_\varepsilon \leq \lambda (u_\varepsilon - 1)_+^{p-1} \leq \lambda u_\varepsilon^{p-1}. \tag{24}$$

The fact that $2 < p < 2^*$ implies that $N(p-2)/2 < 2^*$, and hence we can fix an exponent q satisfying

$$\max\{1, N(p-2)/2\} < q < 2^*.$$

Thus, by [5, Theorem 3.1] and the Sobolev embedding, we obtain

$$\max_{\Omega} u_\varepsilon \leq C\|u_\varepsilon\|_{L^q(\Omega)}^{2q/(2q-N(p-2))} \leq C_3\|u_\varepsilon\|_{H_0^1(\Omega)}^{2q/(2q-N(p-2))},$$

for some constant $C_3 > 0$ independent of ε. The proof of (20) follows immediately from estimate (19).

To check (21), we observe from (24) that

$$-\Delta u_\varepsilon \leq \Lambda = -\Delta \varphi_0.$$

The estimate is then direct from the comparison principle.

Finally, we prove (22). Let $x \in \Omega$ and $r > 0$ such that $B_r(x) \subset \Omega$. We have that v_ε defined by

$$v_\varepsilon(y) = u_\varepsilon(ry + x) - 1 \quad \text{for } y \in B_1(0)$$

satisfies

$$-\Delta v_\varepsilon + r^2(1-\beta)\chi_{\{v_\varepsilon>0\}}\frac{v_\varepsilon}{(v_\varepsilon+\varepsilon)^{1+\beta}} = -\frac{r^2}{\varepsilon} f\left(\frac{v_\varepsilon}{\varepsilon}\right) + r^2\lambda v_\varepsilon^{p-1}\chi_{\{v_\varepsilon>0\}} \quad \text{in } B_1(0).$$

Since $f \leq 2$ and v_ε is bounded independently of ε by estimate (20), then [19, Proposition 2.2] implies that

$$|\nabla v_\varepsilon|^2 \leq C_4 v_\varepsilon^{1-\beta} \quad \text{in } B_{1/2}(0),$$

where $C_4 > 0$ is a constant independent of ε. Using (20), we conclude that

$$\max_{B_{r/2}(x)} |\nabla u_\varepsilon| \leq \frac{1}{r} \max_{B_{1/2}(0)} |\nabla v_\varepsilon| \leq \frac{C_5}{r}$$

for some constant $C_5 > 0$ independent of ε. ∎

3. Nonminimizing solution

We are going to show that (5), (6), and (7) in Theorem 1.1 hold, concerning the limit of a sequence u_ε of critical points of E_ε when $\varepsilon \to 0^+$. In what follows, given a $\delta > 0$, we denote the δ-neighborhood of $\partial\Omega$ in Ω by

$$N_\delta(\partial\Omega) = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \leq \delta\}. \tag{25}$$

Proposition 3.1. *Let λ_0 be given by Lemma 2.3 and $0 < \varepsilon_j < 1$ be a sequence such that $\varepsilon_j \to 0^+$ as $j \to \infty$. If $\lambda > \lambda_0$ and u_{ε_j} is a critical point of E_{ε_j} at level $c^*_{\varepsilon_j}$ from Lemma 2.5, then there exists a subsequence, still denoted by ε_j, and a function $u \in H^1_0(\Omega) \cap C^2(\overline{\Omega} \setminus \partial\{u > 1\})$ that is Lipschitz continuous on $\overline{\Omega}$ such that*

$$u_{\varepsilon_j} \to u \quad \text{uniformly in } \overline{\Omega}, \tag{26}$$

$$-\Delta u = (-(1-\beta)(u-1)^{-\beta} + \lambda(u-1)^{p-1})\chi_{\{u>1\}} \quad \text{in } \Omega \setminus \partial\{u > 1\}, \tag{27}$$

$$u_{\varepsilon_j} \to u \quad \text{in } H^1_0(\Omega) \tag{28}$$

and

$$E(u) \le \liminf_{j\to\infty} c^*_{\varepsilon_j} \le \limsup_{j\to\infty} c^*_{\varepsilon_j} \le E(u) + \mathcal{L}^N(u^{-1}(1)). \tag{29}$$

Proof. We start by showing (26). By estimate (19), the family u_{ε_j} is uniformly bounded in $H^1_0(\Omega)$ and hence, by the Sobolev embedding, there exists a subsequence, still denoted by u_{ε_j}, such that u_{ε_j} converges weakly in $H^1_0(\Omega)$ and strongly in $L^q(\Omega)$, for every $1 \le q < 2^*$, to some $u \in H^1_0(\Omega)$. Estimate (21) implies that there exists a $\delta_0 > 0$ independent of ε_j such that $u_{\varepsilon_j} < 1$ in $N_{\delta_0}(\partial\Omega)$ defined by (25), for every ε_j. It follows that each u_{ε_j} is harmonic in $N_{\delta_0}(\partial\Omega)$ and hence, by elliptic regularity theory, the family u_{ε_j} is uniformly bounded in $C^3(N_{\delta_0/2}(\partial\Omega))$. Thus there is a subsequence, still denoted by ε_j, such that u_{ε_j} converges to u in $C^2(N_{\delta_0/2}(\partial\Omega))$. Taking a finite covering of $\Omega \setminus N_{\delta_0/2}(\partial\Omega)$ by balls of radius $\delta_0/4$ so that each ball of radius $\delta_0/2$ is contained in Ω, estimate (22) and the mean value inequality imply that the family u_{ε_j} is uniformly Lipschitz continuous in $\Omega \setminus N_{\delta_0/2}(\partial\Omega)$. Notice that the mean value inequality also implies that u_{ε_j} is Lipschitz continuous in $N_{\delta_0/2}(\partial\Omega)$, hence on $\overline{\Omega}$. Therefore, by estimate (20) and the Arzelà–Ascoli theorem, there exists a further subsequence, still denoted by ε_j, such that u_{ε_j} converges uniformly to u on $\overline{\Omega}$ and u is Lipschitz continuous on $\overline{\Omega}$. Summing up, we have

$$\begin{cases} u_{\varepsilon_j} \rightharpoonup u & \text{weakly in } H^1_0(\Omega), \\ u_{\varepsilon_j} \to u & \text{in } L^q(\Omega) \text{ for every } 1 \le q < 2^*, \\ u_{\varepsilon_j} \to u & \text{in } C^2(N_{\delta_0/2}(\partial\Omega)), \\ u_{\varepsilon_j} \to u & \text{uniformly on } \overline{\Omega} \quad \text{as } j \to \infty. \end{cases} \tag{30}$$

Now we prove (27) with a similar approach to the proof in [37]. First, we claim that

$$(u-1)^{-\beta}\chi_{\{u>1\}} \in L^1_{\mathrm{loc}}(\Omega). \tag{31}$$

In fact, if $K \subset \Omega$ is a compact set and $\varsigma \in C^\infty_0(\Omega)$ is a function such that $0 \le \varsigma \le 1$ and $\varsigma = 1$ in K, then by (30), we have

$$\begin{aligned}(1-\beta)\int_\Omega g_{\varepsilon_j}(u_{\varepsilon_j}-1)\varsigma\,dx &\le \lambda\int_\Omega (u_{\varepsilon_j}-1)_+^{p-1}\varsigma\,dx - \int_\Omega \nabla u_{\varepsilon_j}\nabla\varsigma\,dx \\ &\to \lambda\int_\Omega (u-1)_+^{p-1}\varsigma\,dx - \int_\Omega \nabla u\nabla\varsigma\,dx \quad \text{as } j \to \infty.\end{aligned}$$

Consequently, if $\rho > 0$, applying Fatou's lemma gives

$$\int_K (u-1)^{-\beta}\chi_{\{u\geq 1+\rho\}}\,dx \leq \liminf_{j\to\infty}\int_\Omega g_{\varepsilon_j}(u_{\varepsilon_j}-1)\varsigma\,dx < \infty.$$

Letting $\rho \to 0^+$, from Fatou's lemma we conclude that $(u-1)^{-\beta}\chi_{\{u>1\}} \in L^1(K)$ and the claim is proved.

Let $\varphi \in C_0^\infty(\{u>1\})$ and $\eta \in C^\infty(\mathbb{R})$ with $0 \leq \eta \leq 1$, $\eta(s) = 0$ if $s \leq 1/2$ and $\eta(s) = 1$ if $s \geq 1$. Moreover, let Ω_1 be an open set with $\overline{\Omega_1} \subset \{u > 1\}$ and $\overline{\text{support}(\varphi)} \subset \Omega_1$. By (30), for every $0 < m < 1$ there exists a $j_0 > 0$ such that $|u_{\varepsilon_j} - u| \leq m/8$ whenever $j > j_0$ with $\varepsilon_j < m/8$. Recalling that u_{ε_j} is a critical point of E_{ε_j}, $E'_{\varepsilon_j}(u_{\varepsilon_j})\varphi\eta((u_{\varepsilon_j}-1)/m) = 0$ implies that

$$\int_\Omega \nabla u_{\varepsilon_j}\nabla\Big(\varphi\eta\Big(\frac{u_{\varepsilon_j}-1}{m}\Big)\Big)\,dx = I_{\varepsilon_j}, \tag{32}$$

where

$$I_{\varepsilon_j} = \int_{\Omega_1}\Big(-\frac{1}{\varepsilon_j}f\Big(\frac{u_{\varepsilon_j}-1}{\varepsilon_j}\Big) - (1-\beta)g_{\varepsilon_j}(u_{\varepsilon_j}-1) + \lambda(u_{\varepsilon_j}-1)_+^{p-1}\Big)\varphi\eta\Big(\frac{u_{\varepsilon_j}-1}{m}\Big)\,dx.$$

Note that

$$I_{\varepsilon_j} \to \int_{\Omega_1}\Big(-(1-\beta)(u-1)^{-\beta} + \lambda(u-1)^{p-1}\Big)\varphi\eta\Big(\frac{u-1}{m}\Big)\,dx \quad \text{as } j\to\infty. \tag{33}$$

In fact, let $j > j_0$. In $\{u < 1 + m/4\}$, we have that $u_{\varepsilon_j} < 1 + m/2$, and hence $\eta((u_{\varepsilon_j} - 1)/m) = 0$. Moreover, in $\{u \geq 1 + m/4\}$, it holds that $u_{\varepsilon_j} \geq 1 + m/8 > 1 + \varepsilon_j$. Thus

$$I_{\varepsilon_j} = \int_{\Omega_1\cap\{u\geq 1+m/4\}}(-(1-\beta)g_{\varepsilon_j}(u_{\varepsilon_j}-1) + \lambda(u_{\varepsilon_j}-1)^{p-1})\varphi\eta\Big(\frac{u_{\varepsilon_j}-1}{m}\Big)\,dx$$

and (33) follows directly from (30) and the dominated convergence theorem. Since $\eta \leq 1$, (31) and the dominated convergence theorem once more imply that

$$\begin{aligned}&\int_{\Omega_1}(-(1-\beta)(u-1)^{-\beta} + \lambda(u-1)^{p-1})\varphi\eta\Big(\frac{u-1}{m}\Big)\,dx\\ &\qquad\to \int_{\Omega_1}(-(1-\beta)(u-1)^{-\beta} + \lambda(u-1)^{p-1})\varphi\,dx \quad \text{as } m\to 0^+.\end{aligned}$$

Hence

$$\begin{aligned}I_{\varepsilon_j} \to &\int_{\Omega_1}(-(1-\beta)(u-1)^{-\beta} + \lambda(u-1)^{p-1})\varphi\,dx\\ &\text{as } j\to\infty \text{ and then as } m\to 0^+.\end{aligned}\tag{34}$$

Now we deal with the left-hand side of (32). Since $\overline{\text{support}(\varphi)} \subset \Omega_1$, we can write

$$\int_\Omega \nabla u_{\varepsilon_j}\cdot\nabla\Big(\varphi\eta\Big(\frac{u_{\varepsilon_j}-1}{m}\Big)\Big)\,dx = I_1 + I_2,$$

where

$$I_1 = \int_{\Omega_1} (\nabla u_{\varepsilon_j} \cdot \nabla\varphi)\eta\Big(\frac{u_{\varepsilon_j}-1}{m}\Big)\,dx$$

and

$$I_2 = \int_{\Omega_1} \frac{|\nabla u_{\varepsilon_j}|^2}{m}\eta'\Big(\frac{u_{\varepsilon_j}-1}{m}\Big)\varphi\,dx.$$

Letting $j \to \infty$ and then $m \to 0^+$, (30) and the dominated convergence theorem give

$$I_1 \to \int_{\Omega_1} \nabla u \cdot \nabla\varphi\,dx.$$

Note that there exists a finite covering of $\overline{\Omega_1}$, say $B^i_{\frac{\rho}{2}}$, by balls of radius

$$\frac{\rho}{2} = \frac{\operatorname{dist}(\overline{\Omega_1}, \partial\Omega)}{2}.$$

By estimate (22), (30), and the dominated convergence theorem, it follows that

$$\begin{aligned}\limsup_{j\to\infty}|I_2| &\le \sum_i \frac{C^2}{m\rho^2}\lim_{j\to\infty}\int_{B^i_{\frac{\rho}{2}}\cap\{\frac{m}{2}\le u_{\varepsilon_j}-1\le m\}}\Big|\eta'\Big(\frac{u_{\varepsilon_j}-1}{m}\Big)\varphi\Big|\,dx\\ &= \sum_i \frac{C^2}{m\rho^2}\int_{B^i_{\frac{\rho}{2}}\cap\{\frac{m}{2}\le u-1\le m\}}\Big|\eta'\Big(\frac{u-1}{m}\Big)\varphi\Big|\,dx. \end{aligned} \tag{35}$$

Finally, by the fact that $\eta'((u-1)/m)$ is uniformly bounded and the dominated convergence theorem, we have that each integral in $B^i_{\frac{\rho}{2}}$ on the right-hand side of (35) tends to 0 as $m \to 0^+$. This implies that $I_2 \to 0$ and thus

$$\int_{\Omega} \nabla u_{\varepsilon_j} \cdot \nabla\Big(\varphi\eta\Big(\frac{u_{\varepsilon_j}-1}{m}\Big)\Big)\,dx \to \int_{\Omega_1}\nabla u\cdot\nabla\varphi\,dx$$
$$\text{as } j \to \infty \text{ and then as } m \to 0^+. \tag{36}$$

Hence, by (32), (34), and (36) we obtain

$$\int_\Omega \nabla u\cdot\nabla\varphi\,dx = \int_\Omega\Big[-(1-\beta)(u-1)^{-\beta} + \lambda(u-1)^{p-1}\Big]\varphi\,dx. \tag{37}$$

Therefore, u is a weak solution (and thus C^2) of (27) in $\{u > 1\}$.

In $\{u < 1\}$, to see that u is harmonic in $\{u < 1\}$, let $\varphi \in C_0^\infty(\{u < 1\})$. Note that there exists an $\varepsilon_0 > 0$ such that $u \le 1 - 2\varepsilon_0$ in support(φ), which implies that $u_{\varepsilon_j} \le 1 - \varepsilon_j$ in support(φ) for j sufficiently large with $\varepsilon_j < \varepsilon_0$. Consequently, $E'(u_{\varepsilon_j})\varphi = 0$ implies that

$$\int_\Omega \nabla u_{\varepsilon_j}\cdot\nabla\varphi\,dx = 0.$$

By (30) and the dominated convergence theorem, we have that u is harmonic in the weak sense. We remark that u is harmonic in the larger set $\{u \leq 1\}^{\circ}$, interior of $\{u \leq 1\}$. Since f and g_{ε_j} are nonnegative, from $E'(u_{\varepsilon_j}) = 0$ we have

$$\int_{\Omega} \nabla u_{\varepsilon_j} \cdot \nabla \varphi \, dx \leq \int_{\Omega} \lambda (u_{\varepsilon_j} - 1)_+^{p-1} \varphi \, dx, \quad \forall \varphi \in C_0^{\infty}(\Omega).$$

Letting $j \to \infty$, it follows that u satisfies

$$-\Delta u \leq \lambda (u-1)_+^{p-1} \quad \text{in } \Omega \tag{38}$$

in the weak sense. Thus, the same argument as in the proof of [28, Lemma 3.1] applies: since u is harmonic in $\{u < 1\}$, we deduce that $\min\{u, 1\}$ satisfies the super-mean-value property and thus $\Delta \min\{u, 1\} \leq 0$ in the weak sense. This implies, along with (38), that u is harmonic in the weak sense (and thus strongly) in $\{u \leq 1\}^{\circ}$.

To prove (28), we only need to verify that

$$\limsup_{j \to \infty} \|u_{\varepsilon_j}\|_{H_0^1(\Omega)} \leq \|u\|_{H_0^1(\Omega)}, \tag{39}$$

because weak convergence already implies that $\|u\|_{H_0^1(\Omega)} \leq \liminf_{j\to\infty} \|u_{\varepsilon_j}\|_{H_0^1(\Omega)}$ (see [6, Propositions 3.5 and 3.32]). Testing (4) with $(u_{\varepsilon_j} - 1)$ and using that $f \geq 0$, we obtain

$$\begin{aligned}\int_{\Omega} |\nabla u_{\varepsilon_j}|^2 \, dx \leq & \int_{\Omega} \Big[-(1-\beta) g_{\varepsilon_j}(u_{\varepsilon_j} - 1)(u_{\varepsilon_j} - 1) + \lambda (u_{\varepsilon_j} - 1)_+^p \Big] \, dx \\ & - \int_{\partial\Omega} \frac{\partial u_{\varepsilon_j}}{\partial \nu} \, d\mathcal{H}^{N-1},\end{aligned}$$

where ν is the outer unit normal to $\partial\Omega$. By (30), it follows that

$$\begin{aligned}&\int_{\Omega} \lambda (u_{\varepsilon_j} - 1)_+^p \, dx - \int_{\partial\Omega} \frac{\partial u_{\varepsilon_j}}{\partial \nu} \, d\mathcal{H}^{N-1} \\ &\quad \to \int_{\Omega} \lambda (u-1)_+^p \, dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, d\mathcal{H}^{N-1} \quad \text{as } j \to \infty,\end{aligned}$$

and thus

$$\int_{\Omega} |\nabla u_{\varepsilon_j}|^2 \, dx \leq \int_{\Omega} \Big[-(1-\beta)(u-1)_+^{1-\beta} + \lambda (u-1)_+^p \Big] \, dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, d\mathcal{H}^{N-1}. \tag{40}$$

Now, fix a sequence $0 < \rho_i < 1$ with $\rho_i \to 0$ as $i \to \infty$. For every ρ_i, if we input $\varphi = (u - 1 - \rho_i)_+$ in (37), it holds that

$$\begin{aligned}\int_{\{u > 1+\rho_i\}} |\nabla u|^2 \, dx = \int_{\{u>1+\rho_i\}} \Big[& -(1-\beta)(u-1)^{-\beta}(u-1-\rho_i)_+ \\ & + \lambda (u-1)_+^{p-1}(u-1-\rho_i)_+ \Big] \, dx \end{aligned} \tag{41}$$

and if we test $\Delta u = 0$ with $(u - 1 + \rho_i)_-$ over Ω, we have

$$\int_{\{u<1-\rho_i\}} |\nabla u|^2\, dx = -(1-\rho_i)\int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, d\mathcal{H}^{N-1}. \tag{42}$$

Thus, since $\int_{u^{-1}(1)} |\nabla u|^2\, dx = 0$, adding (41) and (42) and letting $i \to \infty$ yields

$$\int_\Omega |\nabla u|^2\, dx = \int_\Omega \Big[-(1-\beta)(u-1)_+^{1-\beta} + \lambda(u-1)_+^p\Big]\, dx \; - \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, d\mathcal{H}^{N-1}.$$

This implies, along with (40), that

$$\limsup_{j\to\infty} \int_\Omega |\nabla u_{\varepsilon_j}|^2\, dx \le \int_\Omega |\nabla u|^2\, dx$$

and thus (39) is verified.

Finally, we prove (29). Consider the decomposition

$$\begin{aligned} E_{\varepsilon_j}(u_{\varepsilon_j}) = \int_\Omega \Big[\frac{1}{2}|\nabla u_{\varepsilon_j}|^2 + F\Big(\frac{u_{\varepsilon_j}-1}{\varepsilon_j}\Big)\chi_{\{u\neq 1\}} + (1-\beta)G_{\varepsilon_j}(u_{\varepsilon_j}-1)\chi_{\{u\neq 1\}} \\ - \frac{\lambda}{p}(u_{\varepsilon_j}-1)_+^p\Big]dx \\ + \int_{u^{-1}(1)} \Big[F\Big(\frac{u_{\varepsilon_j}-1}{\varepsilon_j}\Big) + (1-\beta)G_{\varepsilon_j}(u_{\varepsilon_j}-1)\Big]dx. \end{aligned} \tag{43}$$

Since $F((u_{\varepsilon_j}-1)/\varepsilon_j)\chi_{\{u\neq 1\}}$ converges almost everywhere to $\chi_{\{u>1\}}$, $G_{\varepsilon_j}(u_{\varepsilon_j}-1)\chi_{\{u\neq 1\}}$ converges almost everywhere to $(u-1)^{1-\beta}\chi_{\{u>1\}}$, by (30) it follows that the first integral in (43) converges to $E(u)$. Furthermore, since $0 \le F \le 1$ and $0 \le (1-\beta)G_\varepsilon(t) \le |t|^{1-\beta}$ for all $t \in \mathbb{R}$, it follows that

$$\begin{aligned} 0 &\le \int_{u^{-1}(1)} \Big[F\Big(\frac{u_{\varepsilon_j}-1}{\varepsilon_j}\Big) + (1-\beta)G_{\varepsilon_j}(u_{\varepsilon_j}-1)\Big]\, dx \\ &\le \mathcal{L}^N(u^{-1}(1)) + \int_{u^{-1}(1)} |u_{\varepsilon_j}-1|^{1-\beta}\, dx \to \mathcal{L}^N(u^{-1}(1)) \quad \text{as } j \to \infty, \end{aligned}$$

which concludes the proof of (29). ∎

We verify that u obtained in Proposition 3.1 is a critical point of mountain-pass-type on the Nehari manifold $\mathcal{N}$.

Proposition 3.2. *The limit solution u given by Proposition 3.1 satisfies*

$$u \in \mathcal{N}, \tag{44}$$

$$E(u) > 0, \tag{45}$$

$$c^* = E(u) = \inf_{\mathcal{N}} E, \tag{46}$$

$$u \textit{ is a mountain-pass point of } E \tag{47}$$

and

$$\int_\Omega \Big[\Big(\frac{1}{2}|\nabla u|^2 + \chi_{\{u>1\}} + (u-1)_+^{1-\beta} - \frac{\lambda}{p}(u-1)_+^p\Big)\operatorname{div}\Phi - (\nabla u)D\Phi\cdot\nabla u\Big]\,dx = 0,$$
$$\forall \Phi \in C_0^1(\Omega,\mathbb{R}^N). \tag{48}$$

Proof. First, we prove (44). Since u satisfies (27), testing this equation with $(u-1)_+$ over Ω yields

$$\int_{\{u>1\}} |\nabla u|^2\,dx = \int_{\{u>1\}} \Big[-(1-\beta)(u-1)^{1-\beta} + \lambda(u-1)^p\Big]\,dx \tag{49}$$

and thus $u \in \mathcal{N}$.

To check (45), using (49) in $E(u)$ gives

$$E(u) = \int_{\{u<1\}} \frac{1}{2}|\nabla u|^2\,dx + \mathcal{L}^N(\{u>1\})$$
$$+ \int_{\{u>1\}} \Big[\Big(\frac{1}{2}-\frac{1}{p}\Big)|\nabla u|^2 + \Big(1-\frac{(1-\beta)}{p}\Big)(u-1)^{1-\beta}\Big]\,dx. \tag{50}$$

Since $p > 2 > 1-\beta$, it follows that $E(u) > 0$.

Now we prove (46). The minimality of u in the restriction $E|_{\mathcal{N}}$ follows from the claim: $c^* \le \inf_{\mathcal{N}} E$. In fact, since $u \in \mathcal{N}$, we have that $\inf_{\mathcal{N}} E \le E(u)$, and by (29), Corollary 2.4, and the claim, it follows that

$$E(u) \le \liminf_{j\to\infty} c^*_{\varepsilon_j} \le \limsup_{j\to\infty} c^*_{\varepsilon_j} \le c^* \le \inf_{\mathcal{N}} E \le E(u). \tag{51}$$

We will prove the claim next. Define

$$W = \{u \in H_0^1(\Omega) \mid (u-1)_+ \not\equiv 0\}.$$

Note that $\mathcal{N} \subset W$ and since

$$u = (u-1)_+ + (1-(u-1)_-),$$

we can take the curve passing through u, given by

$$\zeta_u(t) = \begin{cases} (1+t)(1-(u-1)_-) & \text{if } -1 \le t \le 0,\\ (1-(u-1)_-) + t(u-1)_+ & \text{if } t > 0. \end{cases} \tag{52}$$

Along the curve, if $-1 \le t \le 0$, it holds that

$$E(\zeta_u(t)) = \frac{(1+t)^2}{2}\int_{\{u<1\}} |\nabla u|^2\,dx$$

and

$$\frac{dE(\zeta_u)}{dt}(t) = (1+t)\int_{\{u<1\}} |\nabla u|^2\,dx.$$

If $t > 0$, we have

$$E(\zeta_u(t)) = \frac{1}{2}\int_{\{u<1\}} |\nabla u|^2\, dx + \frac{t^2}{2}\int_{\{u>1\}} |\nabla u|^2\, dx + \mathcal{L}^N(\{u > 1\}) \\ + t^{1-\beta}\int_{\{u>1\}} (u-1)^{1-\beta}\, dx - \frac{\lambda t^p}{p}\int_{\{u>1\}} (u-1)^p\, dx$$

and

$$\frac{dE(\zeta_u)}{dt}(t) = t\int_{\{u>1\}} |\nabla u|^2\, dx + (1-\beta)t^{-\beta}\int_{\{u>1\}} (u-1)^{1-\beta}\, dx \\ - \lambda t^{p-1}\int_{\{u>1\}} (u-1)^p\, dx.$$

Note that $E|_{\zeta_u}$ is not smooth. In fact, it is discontinuous at $t = 0$ and

$$\lim_{t\to 0^+} E(\zeta_u(t)) = \frac{1}{2}\int_{\{u<1\}} |\nabla u|^2\, dx + \mathcal{L}^N(\{u > 1\}) > \frac{1}{2}\int_{\{u<1\}} |\nabla u|^2\, dx = E(\zeta_u(0)).$$

Also, there exists a $t_u > 0$ satisfying

$$\int_{\{u>1\}} |\nabla u|^2\, dx + (1-\beta)(t_u)^{-\beta-1}\int_{\{u>1\}} (u-1)^{1-\beta}\, dx \\ - \lambda(t_u)^{p-2}\int_{\{u>1\}} (u-1)^p\, dx = 0, \tag{53}$$

since $E|_{\zeta_u}$ is positive for t close to 0 and negative for t sufficiently large, so that $E|_{\zeta_u}$ is increasing in $[1, t_u)$ (it is always increasing in $[-1, 0]$) and decreasing in (t_u, ∞) with $E(\zeta_u(t)) \to -\infty$ when $t \to \infty$.

This last property implies in particular that there exists a $\overline{t} > 1$ sufficiently large so that $E(\zeta_u(\overline{t})) < 0$. We have a path $\gamma_u \in \Gamma_0$ given by

$$\gamma_u(t) = \zeta_u((\overline{t} + 1)t - 1) \tag{54}$$

satisfying

$$c^* \le \max_{v\in\gamma_u([0,1])} E(v) = E(\zeta_u(t_u)) = E(u),$$

and therefore the claim is proved.

Now, to show (47), suppose that U is a neighborhood of u. Using continuity of the path γ_u defined in (54), there are points $0 < t^a < \overline{t} < t^b < 1$ such that $\gamma_u(t^a), \gamma_u(t^b) \in U$ and $\{w \in U \mid E(w) < c^*\}$ is nonempty. If U is path connected, then there exists a path $\widetilde{\gamma}$ joining $\gamma_u(t^a)$ and $\gamma_u(t^b)$. But $\gamma_u|_{[0,t^a]} \circ \widetilde{\gamma} \circ \gamma_u|_{[t^b,1]}$ is a path in Γ_0 with E restricted to this path being less than c^*, which is a contradiction. Hence u is a mountain-pass point of E.

Finally, we deal with (48). Note that the variational equation corresponding to variations Φ to each E_{ε_j} is

$$\int_\Omega \Big[\Big(\frac{1}{2}|\nabla u_{\varepsilon_j}|^2 + F\Big(\frac{u_{\varepsilon_j}-1}{\varepsilon_j}\Big) + (1-\beta)G_{\varepsilon_j}(u_{\varepsilon_j}-1) - \frac{\lambda}{p}(u_{\varepsilon_j}-1)_+^p\Big)\operatorname{div}\Phi \\ - \nabla u_{\varepsilon_j}(D\Phi)\cdot\nabla u_{\varepsilon_j}\Big]\,dx = 0. \tag{55}$$

In fact, if $\Phi \in C_0^1(\Omega, \mathbb{R}^N)$, then

$$\begin{aligned}\int_\Omega \operatorname{div}\Big[\Big(\frac{1}{2}|\nabla u_{\varepsilon_j}|^2 + F\Big(\frac{u_{\varepsilon_j}-1}{\varepsilon_j}\Big) + (1-\beta)G_{\varepsilon_j}(u_{\varepsilon_j}-1) - \frac{\lambda}{p}(u_{\varepsilon_j}-1)_+^p\Big)\Phi \\ - (\nabla u_{\varepsilon_j}\Phi)\cdot\nabla u_{\varepsilon_j}\Big]\,dx \\ = \int_\Omega \Big[\Big(\frac{1}{2}|\nabla u_{\varepsilon_j}|^2 + F\Big(\frac{u_{\varepsilon_j}-1}{\varepsilon_j}\Big) + (1-\beta)G_{\varepsilon_j}(u_{\varepsilon_j}-1) - \frac{\lambda}{p}(u_{\varepsilon_j}-1)_+^p\Big)\operatorname{div}\Phi \\ - \nabla u_{\varepsilon_j}(D\Phi)\cdot\nabla u_{\varepsilon_j}\Big]\,dx \\ + \int_\Omega \Big(-\Delta u_{\varepsilon_j} + \frac{1}{\varepsilon} f\Big(\frac{u_{\varepsilon_j}-1}{\varepsilon}\Big) + (1-\beta)g_\varepsilon(u_{\varepsilon_j}-1) - \lambda(u_{\varepsilon_j}-1)_+^{p-1}\Big)\,dx.\end{aligned}$$

Since the left-hand side vanishes by the divergence theorem and u_{ε_j} satisfies (4), we obtain (55).

Note that by (51), we have $c^*_{\varepsilon_j} \to c^*$ as $j \to \infty$, which implies also that $E_{\varepsilon_j}(u_{\varepsilon_j}) \to E(u)$. Since Proposition 3.1 guarantees that u_{ε_j} converges uniformly to u and strongly in $H_0^1(\Omega)$, the same computation in [28] (see paragraphs after its Definition 4.3) yields

$$\lim_{j\to\infty}\int_\Omega F\Big(\frac{u_{\varepsilon_j}-1}{\varepsilon_j}\Big)\operatorname{div}\Phi\,dx = \lim_{j\to\infty}\int_{\{u>1\}} F\Big(\frac{u_{\varepsilon_j}-1}{\varepsilon_j}\Big)\operatorname{div}\Phi\,dx = \int_{\{u>1\}}\operatorname{div}\Phi\,dx$$

and we can let $j \to \infty$ in (55) and apply dominated convergence theorem in the other terms to obtain (48). ■

4. Nondegeneracy property

We now verify that u is nondegenerate in the sense of Definition 1.2, meaning that around the free boundary $\partial\{u > 1\}$, the function u is not equal to 1, but instead, it grows in $\{u > 1\}$ above the distance function when close to the free boundary.

Proposition 4.1. *The limit solution u given by Proposition 3.1 is nondegenerate.*

Proof. From the proof of Proposition 3.2, we take for every $v \in W$ the path ζ_v defined in (52), which intersects $\mathcal{N}$ at t_v, the point given in (53). The key ingredient is the use of the continuous projection $\pi: W \to \mathcal{N}$ defined by

$$\pi(v) = \zeta_v(t_v) = (1-(v-1)_-) + t_v(v-1)_+.$$

Note that by (50), if $v \in W$, then

$$E(\pi(v)) = \frac{1}{2}\int_{\{v<1\}} |\nabla v|^2\, dx + \mathcal{L}^N(\{v > 1\}) + \Big(\frac{1}{2} - \frac{1}{p}\Big)t_v^2 \int_{\{v>1\}} |\nabla v|^2\, dx$$
$$+ \Big(1 - \frac{(1-\beta)}{p}\Big)t_v^{1-\beta}\int_{\{v>1\}} (v-1)^{1-\beta}\, dx, \tag{56}$$

and if $v \in \mathcal{N}$,

$$E(v) = \frac{1}{2}\int_{\{v<1\}} |\nabla v|^2\, dx + \mathcal{L}^N(\{v > 1\})$$
$$+ \Big(\frac{1}{2} - \frac{1}{p}\Big)\int_{\{v>1\}} |\nabla v|^2\, dx + \Big(1 - \frac{(1-\beta)}{p}\Big)\int_{\{v>1\}} (v-1)^{1-\beta}\, dx. \tag{57}$$

Let $x_0 \in \{u > 1\}$. By continuity, there exists a $0 < \rho < 1$ such that $B_\rho(x_0) \subset \{u > 1\}$ with some $\overline{x_0} \in \partial B_\rho(x_0)$ satisfying $u(\overline{x_0}) = 1$. We need to show that v given by

$$v(y) = \frac{u(x_0 + \rho y) - 1}{\rho}, \quad \forall y \in B_1(0)$$

satisfies $v(0) \geq k > 0$, for some constant $k > 0$.

Since u is Lipschitz by Proposition 3.1 with Lipschitz constant $L > 0$ in $\{u \geq 1\}$, we have

$$0 < v(y) = \rho^{-1}(u(x_0 + \rho y) - u(\overline{x_0})) \leq L\rho^{-1}|x_0 - \overline{x_0} + \rho y|$$
$$\leq L\rho^{-1}|x_0 - \overline{x_0}| + L\rho^{-1}|\rho y| \leq 2L, \tag{58}$$

for every $y \in B_1(0)$. Also, note that

$$-\Delta v(y) = -\rho\Delta u(x_0 + \rho y) = -\rho^{1-\beta}(1-\beta)v(y)^{-\beta} + \rho^p\lambda v(y)^{p-1}.$$

Let h be the solution of

$$\begin{cases} -\Delta h = -\rho^{1-\beta}(1-\beta)v^{-\beta} + \rho^p\lambda v^{p-1} & \text{in } B_1(0),\\ \hfill h = 0 & \text{on } \partial B_1(0), \end{cases}$$

which exists by [27, Corollary 6.14] and satisfies

$$\max_{\overline{\Omega}} |h| \leq C\lambda\rho^p L^{p-1},$$

by [27, Proposition 2.15]. By the Harnack inequality on $v - h + C\lambda\rho^p L^{p-1}$ and estimate (58), it holds that

$$v(y) \leq C(v(0) + \rho^p), \quad \forall y \in B_{2/3}(0)$$

for some constant $C > 0$ depending only on n and L. Let $\psi\colon B_1(0) \to [0,1]$ be a smooth cutoff function vanishing in $\overline{B_{1/3}(0)}$, positive in $B_{2/3}(0) \setminus \overline{B_{1/3}(0)}$, and equal to 1 in $B_1(0) \setminus B_{2/3}(0)$. Moreover, define

$$z(x) = 1 + \rho w\Big(\frac{x - x_0}{\rho}\Big), \quad \forall x \in B_\rho(x_0),$$

where $w\colon B_1(0) \to \mathbb{R}$ is given by

$$w(y) = \begin{cases} \min\{v(y), C(v(0) + \rho^p)\psi(y)\} & \text{if } y \in B_{2/3}(0), \\ v(y) & \text{if } y \in B_1(0) \setminus B_{2/3}(0) \end{cases}$$

and set

$$\mathcal{D} = \big\{x \in B_{2\rho/3}(x_0) \ \mid \ v(y) > C(v(0) + \rho^p)\psi(y), \ y = \rho^{-1}(x - x_0)\big\}.$$

Note that $(z-1)_- = (u-1)_-$, $z|_{\overline{B_{\rho/3}(x_0)}} = 1$, $\{z > 1\} = \{u > 1\} \setminus \overline{B_{\rho/3}(x_0)}$, and that outside $\mathcal{D}$, z is equal to u. Since u is a minimizer of $E|_{\mathcal{N}}$ by Proposition 3.2, (56) and (57) imply that

$$\begin{aligned} &\Big(\frac{1}{2} - \frac{1}{p}\Big)t_z^2 \int_{\mathcal{D}} |\nabla z|^2 \, dx + \Big(\frac{1}{2} - \frac{1}{p}\Big)(t_z^2 - 1) \int_{\{u>1\}\setminus\mathcal{D}} |\nabla u|^2 \, dx \\ &\quad + \Big(1 - \frac{(1-\beta)}{p}\Big)(t_z^{1-\beta} - 1) \int_{\{u>1\}\setminus\mathcal{D}} (u-1)^{1-\beta} \, dx \\ &\quad + \Big(1 - \frac{(1-\beta)}{p}\Big)t_z^{1-\beta} \int_{\mathcal{D}} (z-1)^{1-\beta} \, dx \\ &\geq \mathcal{L}^N(B_{1/3}(0))\rho^N. \end{aligned} \tag{59}$$

Furthermore, since $u, \pi(z) \in \mathcal{N}$, it follows that

$$\int_{\{u>1\}} |\nabla u|^2 \, dx = -(1-\beta) \int_{\{u>1\}} (u-1)^{1-\beta} \, dx + \lambda \int_{\{u>1\}} (u-1)^p \, dx = A$$

and

$$\int_{\{z>1\}} t_z^2 |\nabla (z-1)_+|^2 \, dx = -(1-\beta) \int_{\{z>1\}} t_z^{1-\beta}(z-1)^{1-\beta} \, dx + \lambda \int_{\{z>1\}} t_z^p (z-1)^p dx,$$

which implies that $t_z > 1$, because from (59) if $t_z \leq 1$ by contradiction, we would obtain

$$\begin{aligned} \mathcal{L}^N(B_{1/3}(0))\rho^N &\leq \Big(1 - \frac{(1-\beta)}{2}\Big) \int_{\{z>1\}} \Big[(z-1)^{1-\beta} - (u-1)^{1-\beta}\Big] \, dx \\ &\quad + \lambda\Big(\frac{1}{2} - \frac{1}{p}\Big) \int_{\{z>1\}} \Big[(z-1)^p - (u-1)^p\Big] \, dx \\ &\quad - \Big(1 - \frac{(1-\beta)}{2}\Big) \int_{\overline{B_{\rho/3}(x_0)}} (u-1)^{1-\beta} \, dx \\ &\quad - \lambda\Big(\frac{1}{2} - \frac{1}{p}\Big) \int_{\overline{B_{\rho/3}(x_0)}} (u-1)^p \, dx \leq 0. \end{aligned}$$

By definition (53) of t_z, we also have

$$\begin{aligned}\int_{\{u>1\}\setminus\mathcal{D}} |\nabla u|^2\,dx + \int_{\mathcal{D}} |\nabla z|^2\,dx + (1-\beta)t_z^{-\beta-1}\int_{\{u>1\}\setminus\mathcal{D}} (u-1)^{1-\beta}\,dx \\ + (1-\beta)t_z^{-\beta-1}\int_{\mathcal{D}} (z-1)^{1-\beta}\,dx \\ = \lambda t_z^{p-2}\int_{\{u>1\}\setminus\mathcal{D}} (u-1)^p\,dx + \lambda t_z^{p-2}\int_{\mathcal{D}} (u-1)^p\,dx. \qquad (60)\end{aligned}$$

Moreover, since $u-1<2L\rho$ in $\mathcal{D}$ for ρ sufficiently small, we deduce that

$$\int_{\mathcal{D}} |\nabla z|^2\,dx = C^2(v(0)+\rho^p)^2\rho^N \int_{\{y\mid x\in\mathcal{D}\}} |\nabla\psi|^2\,dy \le C_1\rho^N, \qquad (61)$$

$$\mathcal{L}^N(\mathcal{D}) \le C_2\rho^N,$$

$$\int_{\mathcal{D}} (u-1)^p\,dx \le C_3\rho^{p+N}. \qquad (62)$$

and

$$\int_{\mathcal{D}} (u-1)^{1-\beta}\,dx \le C_4\rho^{1-\beta+N}.$$

From (60)–(62) and using that $t_z>1$, we obtain

$$\begin{aligned} t_z^{p-2} &\le \frac{\int_{\{u>1\}} |\nabla u|^2\,dx + \int_{\mathcal{D}} |\nabla z|^2\,dx + (1-\beta)\int_{\{u>1\}}(u-1)^{1-\beta}\,dx}{\lambda\int_{\{u>1\}}(u-1)^p\,dx - \lambda\int_{\mathcal{D}}(u-1)^p\,dx} \\ &= \frac{B+\int_{\mathcal{D}}|\nabla z|^2\,dx}{B-\lambda\int_{\mathcal{D}}(u-1)^p\,dx} \le 1+\frac{1}{B}\int_{\mathcal{D}}|\nabla z|^2\,dx + C_5\rho^{q+N},\end{aligned}$$

where $B=A+(1-\beta)\int_{\{u>1\}}(u-1)^{1-\beta}$ and $C_5>0$ is a constant depending only on L and N, and $q=\min\{p,N\}$. Then, by (59), it holds that

$$\int_{\mathcal{D}} |\nabla z|^2\,dx + C_6\rho^{q+N} \ge 2\mathcal{L}^N(B_{1/3}(0))\rho^N.$$

Therefore, for ρ sufficiently small, it follows from (61) that $v(0)$ is bounded from below by a positive constant k, and the proof is concluded. ■

5. Free boundary regularity

This section is devoted to the proof of Theorem 1.5, concerning the free boundary regularity for u given by Proposition 3.1.

5.1. Finiteness of the Hausdorff measure

First, we show that the free boundary does not develop large sets of singularities.

Proposition 5.1. *If $x_0 \in \partial\{u > 1\}$, then there exists a universal constant $C > 0$ such that*

$$\mathcal{H}^{N-1}(\partial\{u > 1\} \cap B_\rho(x_0)) \leq C\rho^{N-1}$$

for every $\rho \leq \operatorname{dist}(\{u > 1\}, \partial\Omega)/2$.

Proof. Let $x_j \in \partial\{u > 1\}$ and $0 < \rho_j < \min\{\rho/2, 1/2\}$ such that $B_{\rho_j}(x_j)$ is a finite covering of $\partial\{u > 1\} \cap B_\rho(x_0)$. Let $\delta = \min_j \rho_j/2$ and, for each $1 \leq j \leq N$, take points $x_j^k \in \partial\{u > 1\}$ such that $B_\delta(x_j^k)$ is a finite covering of $\partial\{u > 1\} \cap B_{\rho_j}(x_j)$. Moreover, let $\mathcal{H}_\delta^{N-1}$ denote the $(N-1)$-Hausdorff measure of step δ and

$$V_\delta(\partial\{u > 1\}) = \big\{x \in \Omega \mid \operatorname{dist}(x, \partial\{u > 1\}) \leq \delta\big\}$$

be the δ-neighborhood of $\partial\{u > 1\}$. Once we prove that

$$\mathcal{L}^N(V_\delta(\partial\{u > 1\} \cap B_{\rho+\delta}(x_0)) \leq C\delta(\rho+\delta)^{N-1}, \tag{63}$$

then we have

$$\begin{aligned}\mathcal{H}_\delta^{N-1}(\partial\{u > 1\} \cap B_\rho(x_0)) &\leq \sum_j \sum_k \mathcal{L}^N(B_\delta(x_j^k)) \\ &\leq C\mathcal{L}^N(V_\delta(\partial\{u > 1\}) \cap B_{\rho+\delta}(x_0)) \\ &\leq C\delta(\rho+\delta)^{N-1} \leq C(\rho+\delta)^{N-1}\end{aligned}$$

and the proof is concluded as $\delta \to 0^+$. We note that (63) follows from Lemmas 5.2 and 5.3 below, which imply that

$$\begin{aligned}\mathcal{L}^N(V_\delta(\partial\{u > 1\}) \cap B_{\rho+\delta}(x_0)) &\leq \sum_j \sum_k \mathcal{L}^N(B_{2\delta}(x_j^k)) \leq C\sum_j \sum_k \mathcal{L}^N(B_\delta(x_j^k)) \\ &\leq C\sum_j \sum_k \int_{B_\delta(x_j^k)} |\nabla u|^2 dx \leq C\int_{\cup_{j,k} B_\delta(x_j^k)} |\nabla u|^2 dx \\ &\leq C\int_{B_{\rho+\delta}(x_0)\cap\{|u-1|\leq\delta\}} |\nabla u|^2\, dx \leq C\delta(\rho+\delta)^{N-1},\end{aligned}$$

and we now conclude the proof. ∎

Lemma 5.2. *If $\Omega_0 \subset \Omega$ is compact, then*

$$\int_{B_r(x_0)} |\nabla u|^2\, dx \geq C\mathcal{L}^N(B_r(x_0)),$$

for some constant $C > 0$ and all $r \leq r_0$, where r_0 is a radius depending on Ω_0.

Proof. We just need to follow the proof of [38, Proposition 7.3], using that u is Lipschitz continuous on $\overline{\Omega}$ by Proposition 3.1 and nondegenerate by Proposition 4.1. ∎

Lemma 5.3. *If* $r \leq \operatorname{dist}(\{u > 1\}, \partial\Omega)$ *and* $\tau > 0$, *then*

$$\int_{B_r(x_0)\cap\{|u-1|\leq\tau\}} |\nabla u|^2 \, dx \leq C\tau r^{N-1},$$

for some constant $C > 0$.

Proof. Let $0 < s < \tau$ and define $u_s = \min\{(u-1-s)_+, \tau - s\}$. It follows that

$$-u_s \Delta u = \left(-(1-\beta)(u-1)^{-\beta}\chi_{\{u>1\}} + \lambda(u-1)_+^{p-1}\right)u_s \quad \text{in } \Omega.$$

Integrating it over $B_r(x_0)$, we have

$$\begin{aligned}\int_{B_r(x_0)} \nabla u \nabla u_s \, dx &= \int_{\partial B_r(x_0)} \frac{\partial u}{\partial \nu} u_s \, d\mathcal{H}^{N-1} \\ &\quad + \int_{B_r(x_0)} \left(-(1-\beta)(u-1)^{-\beta}\chi_{\{u>1\}} + \lambda(u-1)_+^{p-1}\right)u_s \, dx,\end{aligned}$$

where ν points out of $\partial B_r(x_0)$. Since u is Lipschitz continuous, we obtain

$$\left|\int_{\partial B_r(x_0)} \frac{\partial u}{\partial \nu} u_s \, d\sigma\right| \leq L\tau\mathcal{H}^{N-1}(\partial B_r(x_0)) \leq C\tau r^{N-1}.$$

Thus, if $M = \max_{\overline{\Omega}} u$, then

$$\begin{aligned}\int_{B_r(x_0)} \left(-(1-\beta)(u-1)^{-\beta}\chi_{\{u>1\}} + \lambda(u-1)_+^{p-1}\right)u_s \, dx &\leq C\tau M^{p-1}\mathcal{L}^N(B_r(x_0)) \\ &\leq C\tau r^N.\end{aligned}$$

Since

$$\int_{B_r(x_0)} \nabla u \, \nabla u_s \, dx = \int_{B_r(x_0)\cap\{1+s<u<1+\tau\}} |\nabla u|^2 \, dx,$$

the result follows as $s \to 0^+$. ∎

Taking a finite cover of $\partial\{u > 1\}$ as the restriction of balls of radii $\rho \leq \operatorname{dist}(\{u > 1\}, \partial\Omega)/2$, Proposition 5.1 readily implies that $\mathcal{H}^{N-1}(\partial\{u > 1\}) < \infty$.

5.2. Topological boundary equals measure-theoretic boundary

We will show that $\partial\{u > 1\} \subset \partial_*\{u > 1\}$, since the other inclusion holds trivially by definition of the measure-theoretic boundary ∂_*; see [23]. This is a consequence of the positive density property (see Definition 1.3).

Proposition 5.4. *The solution u satisfies the positive density property.*

The proof of this result makes use of the superlinear growth of u around the free boundary and some estimates concerning the behavior of u also around the free boundary from the phase $\{u \leq 1\}$, which we clarify below.

Lemma 5.5. *Given* $x_0 \in \partial\{u > 1\}$, *there exist constants* $0 < \rho_0 \leq \operatorname{dist}(\{u > 1\}, \partial\Omega)$ *and* $\gamma > 0$ *such that*

$$\sup_{B_\rho(x_0)} (u - 1) \geq \gamma\rho$$

holds for every $0 < \rho \leq \rho_0$. *In particular, for each* ρ, *the supremum is achieved at least at a point in* $B_\rho(x_0) \setminus B_{\rho/2}(x_0)$.

Proof. We claim that there exist constants $0 < \rho_0 \leq \operatorname{dist}(\{u > 1\}, \partial\Omega)$ and $C > 0$ such that if $x \in \{u > 1\}$ with $\operatorname{dist}(x, \{u > 1\}) \leq \rho_0$, then there holds

$$u(\overline{x}) - 1 \geq (1 + C)(u(x) - 1)$$

for some $\overline{x} \in B_{\operatorname{dist}(x,\{u>1\})}(x)$. Let $x_0 \in \partial\{u > 1\}$ and fix a $0 < \rho \leq \rho_0$. Then by continuity, there exists an $x_1 \in B_{\rho/4}(x_0)$ such that $u(x_1) > 1$ and it follows that

$$r_1 = \operatorname{dist}(x, \{u > 1\}) \leq |x_1 - x_0| < \frac{\rho}{4} < \rho_0.$$

Then by the claim, there exists a $x_2 \in \partial B_{r_1}(x_1)$ such that

$$u(x_2) - 1 \geq (1 + C)(u(x_1) - 1).$$

Iteratively, we have $x_{j+1} \in \partial B_{\rho_j}(x_j)$ with $\rho_j = \operatorname{dist}(x_j, \{u > 1\})$ such that

$$u(x_{j+1}) - 1 \geq (1 + C)(u(x_j) - 1)$$

for $1 \leq j \leq k$ and $x_2, \ldots, x_k \in B_\rho(x_0)$. By induction we deduce that

$$u(x_{j+1}) - 1 \geq (1 + C)^k(u(x_1) - 1)$$

and since u is bounded, we obtain $x_{k+1} \notin B_\rho(x_0)$. Since u is nondegenerate, we have

$$u(x_{j+1}) - u(x_j) \geq C(u(x_j) - 1) \geq C\rho_j$$

for $1 \leq j \leq k - 1$, which implies, by the triangle inequality, that

$$u(x_k) \geq u(x_1) + C\sum_{j=1}^{k-1}\rho_j > 1 + C\sum_{j=1}^{k-1}|x_{j+1} - x_j| \geq 1 + \lambda c|x_k - x_1|. \tag{64}$$

Suppose that $x_k \in B_{\rho/2}(x_0)$. We have

$$|x_{k+1} - x_k| \leq |x_k - x_0| < \frac{\rho}{2}$$

and so

$$|x_{k+1} - x_0| < \rho,$$

which cannot occur because $x_{k+1} \notin B_\rho(x_0)$. Thus, we have $x_k \notin B_{\rho/2}(x_0)$. Since $x_1 \in B_{\rho/4}(x_0)$, it follows that

$$|x_k - x_1| > \frac{\rho}{4},$$

which implies that

$$u(x_k) - 1 \geq \frac{C\rho}{4}$$

by (64).

It remains to prove the claim. We argue by contradiction to obtain sequences $x_j \in \{u > 1\}$ and $C_j, \rho_j \to 0^+$ as $j \to \infty$, where $\rho_j = \operatorname{dist}(x_j, \{u \leq 1\})$, satisfying

$$u(x) - 1 < (1 + C_j)(u(x_j) - 1) \tag{65}$$

for all $x \in \partial B_{\rho_j}(x_j)$. By continuity, there exist $\overline{x_j} \in \partial B_{\rho_j}(x_j)$ such that $u(\overline{x_j}) = 1$. Define then the rescalings

$$v_j(y) = \frac{u(x_j + \rho_j y) - 1}{\rho_j}$$

and

$$y_j = \frac{\overline{x_j} - x_j}{\rho_j}.$$

So Proposition 3.1 guarantees that v_j is a solution in $C(\overline{B_1(0)}) \cap C^2(B_1(0))$ of

$$-\Delta v_j = \big(-(1-\beta)\rho_j^{1-\beta} v_j^{-\beta} + \lambda \rho_j^p v_j^{p-1}\big)\chi_{\{v_j > 0\}} \tag{66}$$

in $B_1(0)$ with $v_j(y_j) = 0$, and by (65),

$$v_j(y) - 1 < (1 + C_j) v_j(0) \tag{67}$$

for all $y \in \partial B_1(0)$. By the nondegeneracy of u, we have $u(x_j) \geq 1 + C\rho_j$ and this also implies that

$$v_j(0) \geq C. \tag{68}$$

By Lipschitz continuity of u, we have that v_j is also Lipschitz continuous in $\overline{B_1(0)}$,

$$v_j(y) \leq \frac{L}{\rho_j} |x_j - \overline{x_j} + \rho_j y| \leq 2L,$$

for all $y \in \overline{B_1(0)}$ and we obtain

$$\int_{B_1(0)} |\nabla v_j|^2 \, dy = \rho_j^{-N} \int_{B_{\rho_j}(x_j} |\nabla u|^2 \, dx \leq C L^2.$$

Thus, by the Sobolev embedding and the dominated convergence theorem, there exists a subsequence, which we still denote by v_j, converging to a Lipschitz continuous function v, uniformly in $\overline{B_1(0)}$ and weakly in $H_0^1(B_1(0))$. Also, y_j converges to some $y_0 \in \partial B_1(0)$. Multiplying equation (66) by $\varphi \in C_0^\infty(B_1(0))$ and integrating over $B_1(0)$, we obtain

$$\int_{B_1(0)} \nabla v_j \, \nabla \varphi \, dx = \int_{B_1(0)} \left(-(1-\beta)\rho_j^{1-\beta} v_j^{-\beta} + \lambda \rho_j^p v_j^{p-1} \right) \chi_{\{v_j>0\}} \varphi \, dx.$$

Letting $\rho_j \to 0^+$, we have that v is harmonic in $B_1(0)$ with $v(y) \le v(0)$ for all $y \in \partial B_1(0)$, by (67). By the maximum principle, v is constant, but (68) implies that $v(0) \ge C > 0 = v(y_0)$, which is a contradiction and the claim is proved. ∎

Lemma 5.6. *Given $x_0 \in \partial\{u > 1\}$, there exist constants $0 < \rho_0 \le \operatorname{dist}(\{u > 1\}, \partial\Omega)$, $c_0, c_1 > 0$ such that*

$$u(x_1) - 1 \ge c_1 \rho \tag{69}$$

for some $x_1 \in \partial B_\rho(x_0)$ and

$$\frac{1}{|\partial B_\rho(x_0)|} \int_{\partial B_\rho(x_0)} (u-1)_+ \, d\mathcal{H}^{N-1} \ge c_0 \rho. \tag{70}$$

Proof. By contradiction, suppose (69) does not hold. Then there exist sequences $\rho_j, c_j \to 0^+$ and $x_j \in \partial\{u > 1\}$ such that

$$u(x) - 1 < c_j \rho_j$$

for all $x \in \partial B_{\rho_j}(x_j)$. Since u is Lipschitz, we obtain

$$(u(x) - 1)_+ \le |u(x) - u(x_j)| \le L|x - x_j| < L\rho_j,$$

for all $x \in B_{\rho_j}(x_j)$. It follows that

$$-\Delta(u-1) \le -(1-\beta)(u-1)_+^{-\beta} + \lambda(u-1)_+^{p-1} \le \lambda L^{p-1} \rho_j^{p-1}.$$

Consider the sequence v_j of solutions of

$$\begin{cases} -\Delta v_j = \lambda L^{p-1} \rho_j^{p-1} & \text{in } B_{\rho_j}(x_j), \\ \qquad v_j = c_j \rho_j & \text{on } \partial B_{\rho_j}(x_j). \end{cases}$$

We have

$$v_j \le c_j \rho_j + c_2 \rho_j^{p-1}$$

for some $c_2 > 0$. By Lemma 5.5, there exists $\gamma > 0$ and for j sufficiently large, a point $y_j \in B_{\rho_j}(x_j) \setminus B_{\rho_j/2}(x_j)$ such that

$$u(y_j) - 1 \ge \gamma \rho_j.$$

Since

$$u(y_j) - 1 \le v_j(y_j),$$

we have

$$\gamma \leq c_j + c_2 \rho_j^{p-2},$$

which is a contradiction since $c_j + c_2 \rho_j^{p-2} \to 0$.

Now, for $x_1 \in \partial B_\rho(x_0)$, we deduce by (69) and the Lipschitz continuity of u that

$$u(x) - 1 > \frac{c_1 \rho}{2}$$

for all $x \in B_{c_1\rho/(2L)}(x_1)$. This readily implies (70) by some c_0. ∎

Lemma 5.7. *Let $\lambda > \lambda_0$ for λ_0 given by Proposition 2.1 and u be the limit solution given by Proposition 3.1. Given $x_0 \in \partial\{u > 1\}$, there exist constants $0 < \rho_0 \leq \operatorname{dist}(\{u > 1\}, \partial\Omega)$, $\varepsilon_0, c, C > 0$ such that if*

$$\mathcal{H}^{N-1}(\{u \leq 1\} \cap \partial B_\rho(x_0)) \leq \varepsilon \rho^{N-1}, \tag{71}$$

for every $0 < \rho < \rho_0$ and some $0 < \varepsilon < \varepsilon_0$, then the following estimates hold:

$$\int_{B_\rho(x_0)} |\nabla v|^2\, dx \leq \int_{B_\rho(x_0)} |\nabla u|^2\, dx - c\rho^N, \tag{72}$$

$$\int_{B_\rho(x_0)} |\nabla (v-1)_-|^2\, dx \leq C\varepsilon^{1/N}\rho^N$$

and

$$\frac{\lambda}{p}\int_{B_\rho(x_0)} \left[(u-1)_+^p + (v-1)_+^p\right] dx - \int_{B_\rho(x_0)} \left[(u-1)_+^{1-\beta} - (v-1)_+^{1-\beta}\right] dx \leq C\rho^{p+N}, \tag{73}$$

where v is the solution of

$$\begin{cases} -\Delta v = 0 & \text{in } B_\rho(x_0), \\ v = u & \text{on } \partial B_\rho(x_0). \end{cases}$$

Proof. Estimates (71) and (72) are proved in exactly the same way as estimates 7.4 and 7.5 of [28, Lemma 7.3]. Let ρ_0 be from Lemma 5.6. Since u is Lipschitz, we have

$$|u(x) - 1| = |u(x) - u(x_0)| \leq L|x - x_0| < L\rho,$$

for all $x \in B_\rho(x_0)$. Since v is harmonic in $B_\rho(x_0)$ and it satisfies the Dirichlet condition $v = u$ on $\partial B_\rho(x_0)$ where $|u - 1| \leq L\rho$, we obtain by the maximum principle that

$$|v(x) - 1| \leq L\rho$$

for all $x \in B_\rho(x_0)$. Since

$$-(1-\beta)(u-1)_+^{1-\beta} + \lambda(u-1)_+^p \le \lambda L^p \rho_j^p$$

in $B_\rho(x_0)$, estimate (73) follows. ■

We are going to prove Proposition 5.4 next.

Proof. Let $x_0 \in \partial\{u > 1\}$. By Lemma 5.5, there exist constants $0 < \widetilde{\rho_0} \le \operatorname{dist}(\{u > 1\}, \partial\Omega)$ and $\gamma > 0$ such that if $0 < \rho \in \le \widetilde{\rho_0}$, then we have

$$u(x_1) - 1 \ge \frac{\gamma\rho}{2}, \tag{74}$$

for some $x_1 \in B_{\rho/2}(x_0) \setminus B_{\rho/4}(x_0)$. Let ρ be the minimum between $\widetilde{\rho_0}$ and the constant ρ_0 in Lemma 5.7. Since u is Lipschitz, taking $k = \min\{1/2, \gamma/(2L)\}$, we have

$$u(x) \ge u(x_1) - L|x - x_1| > 1 + \frac{\gamma\rho}{2} - Lk\rho \ge 1$$

for all $x \in B_{k\rho}(x_1)$, by (74) and $k \le \gamma/(2L)$. This implies that

$$\frac{\mathcal{L}^N(\{u > 1\} \cap B_\rho(x_0))}{\mathcal{L}^N(B_\rho(x_0))} \ge k^N,$$

which proves the first inequality.

Now suppose by contradiction that the second inequality does not occur, that is, the quantity is not bounded away from 1. We have that there exists a $x_0 \in \partial\{u > 1\}$ such that

$$\mathcal{L}^N(\{u \le 1\} \cap B_{\overline{\rho}}(x_0)) < \theta\overline{\rho}^N \tag{75}$$

for some $\theta, \overline{\rho} > 0$. By the coarea formula [23], we have

$$\mathcal{H}^{N-1}(\{u \le 1\} \cap \partial B_\rho(x_0)) \le \varepsilon\rho^{N-1}$$

for some $\overline{\rho}/2 < \rho < \overline{\rho}$, where $\varepsilon = 2^N\theta$. Take v as in Lemma 5.5 and a function w satisfying $w = v$ in $B_\rho(x_0)$ and $w = u$ in $\Omega \setminus B_\rho(x_0)$. We claim that $E(\pi(w)) < E(u)$ for ρ and ε sufficiently small. But this is a contradiction, since $\pi(w) \in \mathcal{N}$ and by Proposition 3.2, u is a minimizer of $E|_{\mathcal{N}}$. So, we will now prove the claim. By Lemma 5.5, we obtain

$$\int_\Omega |\nabla w|^2\, dx \le \int_\Omega |\nabla u|^2\, dx - c\rho^N, \tag{76}$$

$$\int_{\{w \le 1\}} |\nabla w|^2\, dx \le \int_{\{u \le 1\}} |\nabla u|^2\, dx + C\varepsilon^{1/N}\rho^N \tag{77}$$

and

$$\left|\frac{\lambda}{p}\int_\Omega \left[(w-1)_+^p - (u-1)_+^p\right] dx - \int_\Omega \left[(w-1)_+^{1-\beta} + (u-1)_+^{1-\beta}\right] dx\right| \le C\rho^{p+N}. \tag{78}$$

By (75), we have

$$\int_{B_\rho(x_0)} |\nabla(u-1)_-|^2\,dx \le \int_{B_{\overline{\rho}}(x_0)} |\nabla(u-1)_-|^2\,dx < L^2\lambda\overline{\rho}^N \le L^2\varepsilon\rho^N.$$

By (76), we deduce that

$$\int_\Omega |\nabla(w-1)_+|^2\,dx \le \int_\Omega |\nabla(u-1)_+|^2\,dx - c\rho^N, \tag{79}$$

for ε sufficiently small and

$$\begin{aligned}\mathcal{L}^N(\{w>1\}) &\le \mathcal{L}^N(\{u>1\}\setminus B_\rho(x_0)) + \mathcal{L}^N(B_\rho(x_0))\\ &= \mathcal{L}^N(\{u>1\}) + \mathcal{L}^N(\{u\le 1\}\cap B_\rho(x_0))\\ &\le \mathcal{L}^N(\{u>1\}) + \lambda\overline{\rho}^N \le \mathcal{L}^N(\{u>1\}) + \varepsilon\rho^N.\end{aligned} \tag{80}$$

Since $u, \pi(w) \in \mathcal{N}$, we have

$$\int_{\{u>1\}} |\nabla u|^2\,dx = \int_{\{u>1\}} \Big(-(1-\beta)(u-1)^{1-\beta} + \lambda(u-1)^p\Big)\,dx \tag{81}$$

and by (53),

$$\int_{\{w>1\}} |\nabla w|^2\,dx + (1-\beta)t_w^{-\beta-1}\int_{\{w>1\}} (w-1)^{1-\beta}\,dx - \lambda t_w^{p-2}\int_{\{w>1\}} (w-1)^p\,dx = 0.$$

By (79) and (81), we obtain

$$\begin{aligned}\int_\Omega |\nabla(w-1)_+|^2\,dx &\le \int_\Omega \big(-(1-\beta)(u-1)_+^{1-\beta} + \lambda(u-1)_+^p\big)\,dx - \frac{c}{2}\rho^N\\ &\le \int_\Omega \big(-(1-\beta)(w-1)_+^{1-\beta} + \lambda(w-1)_+^p\big)\,dx - \frac{c}{2}\rho^N + C\rho^{p+N}\\ &\le \int_\Omega \big(-(1-\beta)(w-1)_+^{1-\beta} + \lambda(w-1)_+^p\big)\,dx,\end{aligned}$$

for ρ sufficiently small. Since $t_w \le 1$ by (76), (81), and (78), using (57) and (81), it follows that

$$\begin{aligned}E(\pi(w)) &= \frac{1}{2}\int_{\{w<1\}} |\nabla w|^2\,dx + \mathcal{L}^N(\{w>1\}) + t_w^2\Big(\frac{1}{2}-\frac{1}{p}\Big)\int_{\{w>1\}} |\nabla w|^2\,dx\\ &\quad + \Big(1-\frac{(1-\beta)}{p}\Big)t_w^{1-\beta}\int_{\{w>1\}} (w-1)^{1-\beta}\,dx\\ &\le \frac{1}{2}\int_{\{w<1\}} |\nabla w|^2\,dx + \mathcal{L}^N(\{w>1\}) + \Big(\frac{1}{2}-\frac{1}{p}\Big)\int_{\{w>1\}} |\nabla w|^2\,dx\\ &\quad + \Big(1-\frac{(1-\beta)}{p}\Big)\int_{\{w>1\}} (w-1)^{1-\beta}\,dx.\end{aligned} \tag{82}$$

Finally, using (82) with (77), (78), (79), (80), and (81), we have

$$\begin{aligned} E(\pi(w)) &\le \frac{1}{2}\int_{\{u<1\}} |\nabla u|^2\,dx + \mathcal{L}^N(\{u>1\}) + \Big(\frac{1}{2}-\frac{1}{p}\Big)\int_{\{u>1\}} |\nabla u|^2\,dx \\ &\quad + \Big(1-\frac{(1-\beta)}{p}\Big)\int_{\{u>1\}} (u-1)^{1-\beta}\,dx \\ &\quad + \frac{C}{2}\varepsilon^{1/N}\rho^N - \Big(\frac{1}{2}-\frac{1}{p}\Big)\frac{c}{2}\rho^N + \varepsilon\rho^N + C\rho^{p+N} \\ &= E(u) - \Big[-\frac{C}{2}\varepsilon^{1/N} + \Big(\frac{1}{2}-\frac{1}{p}\Big)\frac{c}{2} - \varepsilon - C\rho^p\Big]\rho^N < E(u), \end{aligned}$$

for ε sufficiently small and the claim is proved. ∎

5.3. Free boundary condition

Once the nondegeneracy and the positive density properties are proved, we are able to verify that the limit solution u satisfies the free boundary condition of (3) in the weak sense (1.4).

Proposition 5.8. *If $\lambda > \lambda_0$ for λ_0 given by Proposition 2.1, then the limit solution u given by Proposition 3.1 satisfies the free boundary condition in the viscosity sense (10) of Definition 1.4.*

Proof. Let u_{ε_j} be the sequence of solutions of (4) given by Proposition 3.1 converging to u. Since

$$(1-\beta)g_\varepsilon(u_{\varepsilon_j}-1) - \lambda(u_{\varepsilon_j}-1)_+^{p-1} \to (1-\beta)(u-1)_+^{-\beta} - (u-1)_+^{p-1}$$

uniformly when $\varepsilon_j \to 0$ and u is nondegenerate, then [31, Corollaries 7.1 and 7.2] imply that u satisfies the free boundary condition in the viscosity sense. We note that since around $\partial\{u>1\}$ the sides $\{u>1\}$ and $\{u \le 1\}$ are nondegenerate by Proposition 5.4, it follows that $\gamma < 0$ indeed does not happen in (10). ∎

5.4. The free boundary as a smooth hypersurface

Now we conclude the free boundary regularity analysis. This can be obtained along the same lines from [28, Section 8] and [41, Theorem 2.8] and we highlight it for completeness. Here we assume that u is the limit solution given by Proposition 3.1. Instead of the inhomogeneous modification in [28, 41] of the theory developed for zero right-hand side in [9, 11], we now deal with $-\Delta u = -(u-1)^{-\beta}$ in $\{u>1\}$ near the free boundary and we can rely on an inhomogeneous modification of the methods in [3, 34, 43].

Proposition 5.9. *The free boundary $\partial\{u>1\}$ is a $C^{1,\alpha}$-hypersurface outside a singular set of Hausdorff dimension at most $N-3$.*

The idea to prove Proposition 5.9 is to argue by induction on dimension and follow the framework introduced by Caffarelli in the series of seminal works [8–11] to the study of regularity of free boundaries. We start with the base case $N = 2$.

To this aim, we perform a blow-up analysis for all $N \geq 2$, by considering the sequence of rescaled blow-up limits around each $x \in \partial\{u > 1\}$, given by

$$w_j(y) = \frac{u(x + r_j y) - 1}{r_j}. \tag{83}$$

By the Arzelà–Ascoli theorem, this sequence converges uniformly to a Lipschitz function w_∞ on compact sets of $\mathbb{R}^N$. This limit inherits the regular properties of u, translated as follows.

Proposition 5.10. *The limiting function w_∞ of the rescaled w_j defined in (83) satisfies:*

$$w_\infty \textit{ is Lipschitz continuous in } \mathbb{R}^N \textit{ and harmonic in } \{w_\infty > 0\} \textit{ and in } \{w_\infty \leq 0\}^\circ; \tag{84}$$

$$w_\infty \textit{ is nondegenerate}, \tag{85}$$

that is, there exists a constant $C > 0$ such that $w_\infty(y) \geq Cr$, for every $r > 0$ and y with $B_r(y) \subset \{w_\infty > 0\}$;

$$\partial\{w_\infty > 0\} \textit{ has locally finite perimeter,} \tag{86}$$

that is, there exists a constant $C > 0$ such that

$$\mathcal{H}^{N-1}(B_r(x_0) \cap \partial\{w_\infty > 0\}) \leq C r^{N-1}$$

for every $x_0 \in \partial\{w_\infty > 0\}$ and $r > 0$;

$$w_\infty \textit{ satisfies the positive density property for open sets,} \tag{87}$$

that is, there exists a constant $C > 0$ such that

$$\mathcal{L}^N(B_r(x) \cap \{w_\infty \leq 0\}^\circ) \geq C r^N$$

and

$$\mathcal{L}^N(B_r(x) \cap \{w_\infty > 0\}) \geq C r^N$$

for every $x \in \partial\{w_\infty > 0\}$ and $r > 0$;

$$w_\infty \textit{ satisfies the free boundary condition in the viscosity sense,} \tag{88}$$

that is, it holds that

$$w_\infty(y) = \alpha\big((y - y_0) \cdot \nu\big)_+ - \gamma\big((y - y_0) \cdot \nu\big)_- + o(|y - y_0|) \quad \textit{when } y \to y_0,$$

for every tangent ball B to the free boundary and $y_0 \in \partial\{w_\infty > 0\} \cap \partial B$, and some $\alpha, \gamma > 0$ satisfying $\alpha^2 - \gamma^2 = 2$, where ν is the unit normal to ∂B pointing inwards at x_0 if $B \subset \{w_\infty > 0\}$, and outwards if $B \subset \{w_\infty \le 0\}^\circ$;

$$\int_{\mathbb{R}^N}\Big[\Big(\frac{1}{2}|\nabla w_\infty|^2+\chi_{\{w_\infty>0\}}\Big)\operatorname{div}\Phi-\nabla w_\infty(D\Phi)\cdot\nabla w_\infty\Big]dx=0,$$
$$\forall \Phi \in C_0^\infty(\mathbb{R}^N,\mathbb{R}^N); \tag{89}$$

and

$$w_\infty \textit{ is homogeneous of degree } 1, \tag{90}$$

that is,

$$w_\infty(ry) = r w_\infty(y), \quad \forall y \in \mathbb{R}^N.$$

Proof. The proof follows exactly the one of [28, Lemma 8.1], and for convenience, we outline it here with the appropriate modifications.

First, we prove (84). Let $x_0 \in \partial\{u > 1\}$ and consider its blow-up w_j defined in (83). On every compact $K \subset \{w_\infty > 0\}$, we have $w_j > 0$ for j sufficiently large in a small neighborhood of K and it holds that

$$-\Delta w_j = \big(-r_j^{1-\beta}(1-\beta)w_j^{-\beta} + r_j^p \lambda w_j^{p-1}\big)\chi_{\{w_j>0\}}.$$

By standard elliptic regularity theory, we have that $w_j \in C^{2,\alpha}(K)$, which implies that $w_j \to w_\infty$ in the C^2-norm as $r_j \to 0^+$. To conclude that

$$-\Delta w_\infty = 0$$

both in $\{w_\infty > 0\}$ and in $\{w_\infty \le 0\}^\circ$, it suffices to show that

$$\operatorname{dist}(\{w_j > 0\} \cap B_\rho(y), \{w_\infty > 0\} \cap B_\rho(y)) \to 0$$

for every ρ and $y \in \mathbb{R}^N$, where dist here is the Hausdorff distance. We can verify at one phase and the other will follow in exactly the same way. It suffices to show that if $x \notin \{w_\infty > 0\}$, then there exists a ball $B_\delta(x)$ sufficiently small such that $w_j(y) \le 0$ for all $y \in B_\delta(x)$ and all j sufficiently large. Since w_∞ is continuous, there exists a $\delta > 0$ sufficiently small such that $B_{2\delta} \subset\subset \{w_\infty \le 0\}^\circ$. By the uniform convergence of w_j to w_∞, we have $w_j \le \widetilde{\delta}$ in $B_{2\delta}(x)$, for j sufficiently large and $0 < \widetilde{\delta} < \delta$ sufficiently small. Suppose by contradiction that there exists a $y_1 \in B_\delta(x)$ with $w_j(y_1) > 0$. Since w_j is nondegenerate by Proposition 4.1, it follows that there exists a y_2 with $|y_2 - y_1| \le C\widetilde{\delta}$ such that $w_j(y_2) \le 0$, which implies by continuity that there exists a $y_3 \in \partial\{w_j > 0\}$. This is a contradiction with the nondegeneracy of w_j, because this gives us $w_j > \widetilde{\delta}$ in $B_\delta(x)$. This proves (84).

Now (85) follows directly from the nondegeneracy of w_j by Proposition 4.1 and (86) is a consequence of Proposition 5.1 and the fact that $\{w_j > 0\} \cap B_\rho(y) \to \{w_\infty > 0\} \cap B_\rho(y)$ in Hausdorff distance for every ρ and $y \in \mathbb{R}^N$ as in the proof of (84).

To prove (87), note that since the positive density property holds for w_j by Proposition 5.4, we only have to verify that the sets $\{w_j > 0\}$ and $\{w_\infty > 0\}$ only differ locally by a set of zero Lebesgue measure. But by the uniform convergence of w_j to w_∞, it follows that

$$\mathcal{L}^N((\{w_\infty > 0\} \setminus \{w_j > 0\}) \cap B) = o(\operatorname{dist}(\{w_j > 0\} \cap B \to \{w_\infty > 0\} \cap B)) \quad \text{as } r_j \to 0^+$$

for every ball $B \subset \mathbb{R}^N$. The same applies to $\{w_j \leq 0\}^\circ$ and $\{w_\infty \leq 0\}^\circ$. This proves (87).

Now, since w_∞ is a solution of the classical free boundary harmonic problem by (84) and by (85) it is nondegenerate, we have that (88) is a consequence of the works [7, 11, 31].

By Proposition 3.2, w_j satisfies

$$\int_B \Big[\Big(\frac{1}{2} |\nabla w_j|^2 + \chi_{\{w_j > 0\}} + r_j^{1-\beta} w_j^{1-\beta} \chi_{\{w_j > 0\}} - r_j^p \frac{\lambda}{p} w_j^p \chi_{\{w_j > 0\}} \Big) \operatorname{div} \Phi \\ - \nabla w_j (D\Phi) \cdot \nabla w_j \Big] \, dx = 0, \quad \forall \Phi \in C_0^1(B, \mathbb{R}^N).$$

By the proof of (87), we have that $\chi_{\{w_j > 0\}} \to \chi_{\{w_\infty > 0\}}$ in $L^1(B)$ and since ∇w_j converges pointwise up to a subsequence to ∇w_∞, (89) follows from the dominated convergence theorem as $r_j \to 0^+$.

Finally, (90) is a consequence of [28, Corollary 9.2] and the proof is concluded. ∎

By the properties of w_∞, we can guarantee that the free boundary of u is flat around each of its points.

Lemma 5.11. *If $N = 2$, u is locally flat around each point in $\partial\{u > 1\}$.*

Proof. By Proposition 5.10, w_∞ is harmonic in $\{w_\infty > 0\}$ and in $\{w_\infty \leq 0\}^\circ$ and since it is nondegenerate and homogeneous of degree 1, then w_∞ should be a plane in both $\{w_\infty > 0\}$ and $\{w_\infty \leq 0\}^\circ$. Since w_∞ satisfies the boundary condition in the viscosity sense and the positive density property, then w_∞ must be of type

$$w_\infty(y) = \alpha(y \cdot \nu)_+ - \gamma(y \cdot \nu)_-,$$

for all $y \in \mathbb{R}^N$ and for some $\alpha > 0$, $\gamma \geq 0$, and $\nu \in S^1$. ∎

In dimension 2, the regularity of the free boundary follows in the literature ([8–11]) as the “flat implies Lipschitz” and the “Lipschitz implies smooth” programs to conclude that in dimension 2, the free boundary is locally a $C^{1,\alpha}$ curve. As in [28, 41] where the inhomogeneous modification of the theory developed for zero right-hand side in [9, 11] is considered to guarantee that regularity, since in (1) the right-hand side is also zero at the free boundary and behaves like $-\Delta u = (u-1)^{-\beta}$ in $\{u > 1\}$, we can rely on the methods in [3, 34, 43], which can also be modified to our inhomogeneous version.

Finally, we are ready to prove Proposition 5.9. The exact same inductive argument on dimension as in the proof of [28, Theorem 1.5] follows. It relies on [49, Section 4], which is an adaptation of the classical dimension reduction technique used to prove measure regularity of minimal surfaces arising as minimizers of the area functional (see [26, Chapter 11]).

We proceed to the proof of Proposition 5.9 in what follows.

Proof. Suppose $N = 3$, fix a $x_0 \in \partial\{u > 1\}$ and consider w_∞ defined in (83). For every $y_0 \in \{w_\infty > 0\} \setminus \{0\}$, since w_∞ is Lipschitz by Proposition 5.10, there exists a sequence $\{\rho_j\} \subset \mathbb{R}_+$ with $\rho_j \to 0^+$ such that $\rho_j^{-1} w_\infty(y_0 + \rho_j z) \to \overline{w_\infty}(z)$ for some function $\overline{w_\infty}$. We have that this limit is two-dimensional, since $y_0\ \nabla\overline{w_\infty}(z) = 0$ for all $z \in \mathbb{R}^N \setminus \partial\{\overline{w_\infty} > 0\}$, by the vanishing of the radial derivative of w_∞. By Proposition 5.10, it is also homogeneous. Hence $\overline{w_\infty}$ must be flat and by the same argument in dimension 2, the free boundary $\partial\{w_\infty > 0\}$ is locally smooth around y_0. Thus u is flat locally around x_0, except for possibly x_0 itself. Taking a finite cover, we have that $\partial\{u > 1\}$ is smooth except for finitely many points, which proves the case $N = 3$. [49, Theorem 4.5] proves that $\mathcal{H}^{N-2+d}(\partial\{u > 1\}) = 0$ for all $d > 0$, and therefore the proof is concluded. ∎

6. Approximation to the functional $\mathcal{L}^N(\{u > 1\}) + J(u)$

We prove the asymptotic behavior $\beta \to 1^-$ in Theorem 1.6.

Proof. Fix a sequence $0 < \beta_j < 1$ with $\beta_j \to 1^-$ as $j \to \infty$. From Proposition 3.1, for every β_j there exists a solution $u_{\beta_j} \in H_0^1(\Omega) \cap C^2(\overline{\Omega} \setminus \partial\{u_\beta > 1\})$ of (1) that is Lipschitz continuous on $\overline{\Omega}$. Moreover, Proposition 3.2 guarantees that u_{β_j} is a minimizer of $E|_{\mathcal{N}}$ and also satisfies

$$\int_\Omega \Big[\Big(\frac{1}{2}|\nabla u_{\beta_j}|^2 + \chi_{\{u_{\beta_j}>1\}} + (u_{\beta_j} - 1)_+^{1-\beta_j} - \frac{\lambda}{p}(u_{\beta_j} - 1)_+^p\Big) \operatorname{div}\Phi \\ - \nabla u_{\beta_j}(D\Phi)\cdot\nabla u_{\beta_j}\Big]\, dx = 0, \quad \forall \Phi \in C_0^1(\Omega, \mathbb{R}^N). \tag{91}$$

Write

$$u_{\beta_j} = (u_{\beta_j} - 1)_+ + (1 - (u_{\beta_j} - 1)_-).$$

Since $u_{\beta_j} \in \mathcal{N}$, we obtain

$$\begin{aligned}
&\|(u_{\beta_j} - 1)_+\|^2_{H_0^1(\Omega)} + \|(u_{\beta_j} - 1)_-\|^2_{H_0^1(\Omega)} \\
&\quad \le C\|(u_{\beta_j} - 1)_+\|_{H_0^1(\Omega)} + \int_\Omega \Big(-(1-\beta_j)((u_{\beta_j} - 1)_+)^{1-\beta_j} + \lambda((u_{\beta_j} - 1)_+)^p\Big)dx \\
&\quad \le C_1 + C_2\|(u_{\beta_j} - 1)_+\|^p_{H_0^1(\Omega)}
\end{aligned}$$

for some constants $C_1, C_2 > 0$ independent of β_j. This implies that $\|(u_{\beta_j}-1)_+\|_{H_0^1(\Omega)}$ and $\|(u_{\beta_j}-1)_-\|_{H_0^1(\Omega)}$ are bounded by a constant independent of β_j, and so is $\|u_{\beta_j}\|_{H_0^1(\Omega)}$. By the Sobolev embedding there exists a subsequence, still denoted by u_{β_j}, such that

$$\begin{cases} u_{\beta_j} \rightharpoonup u & \text{weakly in } H_0^1(\Omega), \\ u_{\beta_j} \to u & \text{in } L^q(\Omega) \text{ for every } 1 \le q < 2^*, \\ u_{\beta_j} \to u & \text{a.e. in } \Omega \quad \text{as } j \to \infty \end{cases}$$

for some $u \in H_0^1(\Omega)$. Proceeding analogously as in the proof of Proposition 3.1, we further obtain that up to a subsequence, still denoting it by β_j,

$$\begin{cases} u_{\beta_j} \to u & \text{in } C^2(N_{\delta_0/2}(\partial\Omega)), \\ u_{\beta_j} \to u & \text{uniformly on } \overline{\Omega} \quad \text{as } j \to \infty. \end{cases}$$

We shall prove that u satisfies

$$\int_\Omega \Big[\Big(\frac{1}{2}|\nabla u|^2 + 2\chi_{\{u>1\}} - \frac{\lambda}{p}(u-1)_+^p\Big) \operatorname{div}\Phi - \nabla u(D\Phi)\cdot\nabla u\Big]\, dx = 0,$$
$$\forall \Phi \in C_0^1(\Omega, \mathbb{R}^N).$$

It suffices to note that

$$\lim_{j\to\infty} \int_\Omega (u_{\beta_j}-1)_+^{1-\beta_j}\, dx = \lim_{j\to\infty} \int_{\{u_{\beta_j}>1\}} (u_{\beta_j}-1)^{1-\beta_j}\, dx = \mathcal{L}^N(\{u>1\})$$

and hence, the result follows by letting $j \to \infty$ in (91) and applying the dominated convergence theorem. Then, we have the following energy bounds:

$$\mathcal{L}^N(\{u>1\}) + J(u) \le \liminf_{j\to\infty} c_{\beta_j}^* \le \limsup_{j\to\infty} c_{\beta_j}^* \le \mathcal{L}^N(\{u>1\}) + c_J^*$$
$$\le \mathcal{L}^N(\{u>1\}) + J(u),$$

where c_J^* is the critical level associated to the mountain-pass level of $J(u)$ as in [28, Theorem 1.2]. ∎

7. Concentration as $\lambda \to \infty$

Finally, we deal with Theorem 1.7. Its proof follows some ideas related to the ones of [19, Theorem 1.2].

Proposition 7.1. *Let u_λ be the family of limit solutions of (1) obtained by Proposition 3.1 with each $\lambda > \lambda_0$ for λ_0 given by Proposition 2.1. Then there exists a sequence x_λ of maximum points of u_λ converging to a point $x_{\max}$ of maximum distance from the boundary $\partial\Omega$, when $\lambda_j \to \infty$.*

Proof. Note that each x_λ occurs in $\{u_\lambda > 1\}$, since each u_λ is nondegenerate by Proposition 4.1. The proof is concluded once we show the claim below.

Claim 7.2. *There exists a constant* $C > 0$ *such that*

$$\operatorname{dist}(x_\lambda, \partial\Omega) \geq \max_{x\in\Omega} \operatorname{dist}(x, \partial\Omega) - C\lambda^{-(1+\beta)/2(p+\beta-1)}.$$

For each λ, consider the sequence $u_{\lambda,\varepsilon}$ in Proposition 3.1 converging to u_λ. Let $x_{\lambda,\varepsilon} \in \Omega$ be a point satisfying

$$u_{\lambda,\varepsilon}(x_{\lambda,\varepsilon}) = \max_{x\in\{u_{\lambda,\varepsilon}>1\}} u_{\lambda,\varepsilon}(x)$$

and the rescaling

$$v_{\lambda,\varepsilon}(y) = \lambda^{1/(p-1+\beta)}(u_{\lambda,\varepsilon}(x_{\lambda,\varepsilon} + \lambda^{-(1+\beta)/2(p-1+\beta)}y) - 1),$$

defined for all

$$y \in \Omega_{\lambda,\varepsilon} = \left\{\lambda^{(1+\beta)/2(p+\beta-1)}(x - x_{\lambda,\varepsilon}) \mid x \in \Omega\right\},$$

which satisfies

$$\begin{cases} -\Delta v_{\lambda,\varepsilon} = -\dfrac{1}{\varepsilon\lambda^{\beta/(p+\beta-1)}} f\left(\dfrac{v_{\lambda,\varepsilon}}{\varepsilon\lambda^{1/(p+\beta-1)}}\right) & \\ \qquad\qquad - (1-\beta)g_{\varepsilon\lambda^{1/(p+\beta-1)}}(v_{\lambda,\varepsilon}) + (v_{\lambda,\varepsilon})_+^{p-1} & \text{in } \Omega_{\lambda,\varepsilon}, \\ v_{\lambda,\varepsilon} = 0 & \text{on } \partial\Omega_{\lambda,\varepsilon}. \end{cases}$$

The associated energy to this equation is given by

$$\begin{aligned} E_{\lambda,\varepsilon}(v) = \int_{\Omega_{\lambda,\varepsilon}} \Bigg[\frac{1}{2}|\nabla v|^2 &+ \frac{1}{\lambda^{(\beta-1)/(p+\beta-1)}} F\left(\frac{v}{\varepsilon\lambda^{1/(p+\beta-1)}}\right) \\ &+ (1-\beta)G_{\varepsilon\lambda^{1/(p+\beta-1)}}(v) - \frac{1}{p}v_+^p\Bigg]dy, \quad \forall v \in H_0^1(\Omega_{\lambda,\varepsilon}). \end{aligned}$$

The mountain-pass level $c^*_{\lambda,\varepsilon}$ defined in (8) is equivalent to

$$c^*_{\lambda,\varepsilon} = \inf_{v\in H_0^1(\Omega_{\lambda,\varepsilon})} \max_{\{t\geq 0\}} E_{\lambda,\varepsilon}(tv);$$

see, for instance, [50, Theorem 4.2]. Claim 7.2 is proved from the fact that

$$\lim_{\varepsilon\to 0^+} \operatorname{dist}(x_{\lambda,\varepsilon}, \partial\Omega) \geq \max_{x\in\Omega} \operatorname{dist}(x, \partial\Omega) - C\lambda^{-(1+\beta)/2(p+\beta-1)},$$

for some $C > 0$, which is a consequence of the estimates

$$\begin{aligned} &\log(c^*_{\lambda,\varepsilon} - c^*_{\lambda,\varepsilon\lambda^{1/(p+\beta-1)}}) \\ &\quad \leq -2\big(\varepsilon^{-(1+\beta)/2}\lambda^{-(1+\beta)/2(p+\beta-1)} - C\big)\big(\lambda^{(1+\beta)/2(p+\beta-1)} \max_{x\in\Omega} \operatorname{dist}(x, \partial\Omega) - C_1\big) \\ &\qquad + C\log(\lambda) \end{aligned} \tag{92}$$

and

$$\begin{aligned}
&\log(c^*_{\lambda,\varepsilon} - c^*_{\lambda,\varepsilon\lambda^{1/(p+\beta-1)}}) \\
&\quad \geq -2\big(\varepsilon^{-(1+\beta)/2}\lambda^{-(1+\beta)/2(p+\beta-1)} + C\big)\big(\mathrm{dist}(0,\partial\Omega_{\lambda,\varepsilon}) + o(1) - C_2\big) \\
&\qquad - C\log(\lambda) \qquad (93)
\end{aligned}$$

for some constants C_1, C_2, and C, and that

$$\mathrm{dist}(0,\partial\Omega_{\lambda,\varepsilon}) = \lambda^{(1+\beta)/2(p+\beta-1)}\,\mathrm{dist}(x_{\lambda,\varepsilon},\partial\Omega).$$

We will prove (92) and (93) based on the following reasoning concerning local energy expansions: consider critical points of the functionals defined on a ball $B_\rho(0) \subset \mathbb{R}^N$, for all $\rho > 0$ and $\widetilde{\varepsilon} > 0$, by

$$\begin{aligned}
&v \in H^1_0(B_\rho(0)) \\
&\quad \mapsto E_{\rho,\widetilde{\varepsilon}}(v) = \int_{B_\rho(0)} \Big[\frac{1}{2}|\nabla v|^2 + \frac{1}{\lambda^{(\beta-1)/(p+\beta-1)}} F\Big(\frac{v}{\widetilde{\varepsilon}}\Big) + (1-\beta)G_{\widetilde{\varepsilon}}(v) - \frac{1}{p}v_+^p\Big]dy.
\end{aligned}$$

They satisfy

$$\begin{cases}
-\Delta v = -\dfrac{1}{\widetilde{\varepsilon}\lambda^{(\beta-1)/(p+\beta-1)}} f\Big(\dfrac{v}{\widetilde{\varepsilon}}\Big) - (1-\beta)g_{\widetilde{\varepsilon}}(v) + v_+^{p-1} & \text{in } B_\rho(0), \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad v = 0 & \text{on } \partial B_\rho(0),
\end{cases}$$

with critical level

$$c^*_{\rho,\widetilde{\varepsilon}} = \inf_{v\in H^1_0(B_\rho(0))} \max_{\{t\geq 0\}} E_{\rho,\widetilde{\varepsilon}}(tv).$$

When $\rho \to \infty$, we can consider the full energy

$$\begin{aligned}
&v \in H^1_0(R^N) \\
&\quad \mapsto E_{\mathbb{R}^N,\widetilde{\varepsilon}}(v) = \int_{\mathbb{R}^N} \Big[\frac{1}{2}|\nabla v|^2 + \frac{1}{\lambda^{(\beta-1)/(p+\beta-1)}} F\Big(\frac{v}{\widetilde{\varepsilon}}\Big) + (1-\beta)G_{\widetilde{\varepsilon}}(v) - \frac{1}{p}v_+^p\Big]dy,
\end{aligned}$$

with critical level given by

$$c^*_{\mathbb{R}^N,\widetilde{\varepsilon}} = \inf_{v\in H^1_0(\mathbb{R}^N)} \max_{\{t\geq 0\}} E_{\mathbb{R}^N,\widetilde{\varepsilon}}(tv),$$

We need the following estimates from [19, Proposition 6.1], which are still valid with the terms f or F added, with no relevant changes in its proof:

Lemma 7.3. *There exist constants $C_1, C_2, C, \widetilde{\varepsilon}_0 > 0$ such that*

$$\log(c^*_{\rho,\widetilde{\varepsilon}} - c^*_{\mathbb{R}^N,\widetilde{\varepsilon}}) \geq -2(\varepsilon^{-(1+\beta)/2} - C)(\rho - C_2) - C\log(\rho) \qquad (94)$$

and

$$\log(c^*_{\rho,\varepsilon} - c^*_{\mathbb{R}^N,\varepsilon}) \leq -2(\varepsilon^{-(1+\beta)/2} + C)(\rho - C_1) + C\log(\rho), \qquad (95)$$

for all $\rho > C_1 + 3$ and $\widetilde{\varepsilon} \leq \widetilde{\varepsilon}_0$.

Since $\Omega_{\lambda,\varepsilon} \subset B_\rho(\widetilde{x}_{\lambda,\varepsilon})$, for $\rho = \lambda^{(1+\beta)/2(p+\beta-1)} \max_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)$ and $\widetilde{x}_{\lambda,\varepsilon} \in \Omega_{\lambda,\varepsilon}$, the point of maximum distance from $\partial\Omega_{\lambda,\varepsilon}$, we obtain

$$c^*_{\lambda,\varepsilon} \le c^*_{\rho,\widetilde{\varepsilon}}$$

with $\widetilde{\varepsilon} = \varepsilon\lambda^{1/(p+\beta-1)}$, and thus estimate (92) follows from (95).

To prove estimate (93), denote $v = v_{\lambda,\varepsilon}$ and assume without loss of generality that $x_{\lambda,\varepsilon} \to x_\lambda$ in $\overline{\Omega}$ for the same subsequence $\varepsilon \to 0^+$ such that $u_{\lambda,\varepsilon} \to u_\lambda$. Denote $R_{\lambda,\varepsilon} = \operatorname{dist}(x_{\lambda,\varepsilon}, \partial\Omega)$ and $R_0 = \operatorname{dist}(x_\lambda, \partial\Omega)$. For each $\delta > 0$, take $R_0' > 0$ such that $\mathcal{L}^N(B_{R_0'}(x_\lambda)) = \mathcal{L}^N(\Omega \cap B_{R_0+\delta}(x_\lambda))$ and $\delta' \in (R_0' - R_0, \delta)$. Take also a smooth cut-off η with uniformly bounded gradient and

$$\eta(s) = \begin{cases} 1 & \text{if } \ 0 \le s \le R_{\lambda,\varepsilon} + \delta', \\ 0 & \text{if } \ s \ge R_{\lambda,\varepsilon} + \delta, \end{cases}$$

and set $\eta_\lambda(s) = \eta(\lambda^{-(1+\beta)/2(p+\beta-1)} s)$ and $\widetilde{v}(y) = v(y)\eta_\lambda(|y|)$, for all $y \in \Omega_{\lambda,\varepsilon}$. For each $0 \le t \le t_0$, for t_0 sufficiently large, it holds that

$$\begin{aligned} E_{\lambda,\varepsilon}(t\widetilde{v}) \le E_{\lambda,\varepsilon}(tv) &+ \int_{\Omega_{\lambda,\varepsilon}} t^2 \eta_\lambda(|y|)|\eta_\lambda'(|y|)|v|\nabla v| + t^2 \frac{v^2 |\eta_\lambda'(|y|)|^2}{2} \, dy \\ &+ \frac{1}{\lambda^{(\beta-1)/(p-\beta-1)}} \int_{\Omega_{\lambda,\varepsilon}} F\Big(\frac{tv}{\varepsilon\lambda^{1/(p+\beta-1)}}\Big) - F\Big(\frac{tv\eta_\lambda(|x|)}{\varepsilon\lambda^{1/(p+\beta-1)}}\Big) \, dy \\ &+ (1-\beta) \int_{\Omega_{\lambda,\varepsilon}} G_{\varepsilon\lambda^{1/(p+\beta-1)}}(tv) - G_{\varepsilon\lambda^{1/(p+\beta-1)}}(tv\eta_\lambda(|x|)) \, dy \\ &+ \frac{t^p}{p} \int_{\Omega_{\lambda,\varepsilon}} v^p (1 - \eta_\lambda(|x|)^p) \, dy. \end{aligned} \tag{96}$$

Again, since $\Omega_{\lambda,\varepsilon} \subset B_\rho(\widetilde{x}_{\lambda,\varepsilon})$, for $\rho = \lambda^{(1+\beta)/2(p+\beta-1)} \max_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)$, then we can consider radial solutions. By the fact that f and F are non-negative, [19, Lemma 6.3] still holds and it asserts that there exist constants $C_1, \varepsilon_0 > 0$ such that

$$v(y) \le \exp\big(-((\varepsilon\lambda^{1/(p+\beta-1)}) - 1)^{1/2}(|y| - C_1)\big)$$

and

$$|\nabla v(y)| \le \exp\big(-((\varepsilon\lambda^{1/(p+\beta-1)}) - 1)^{1/2}(|y| - C_1) + C \log(1/\varepsilon) + C \log(\lambda)\big)$$

for all $y \in \Omega_{\lambda,\varepsilon}$ with $|y| > C_1$ and $\varepsilon \le \varepsilon_0$. Expanding $G_{\varepsilon\lambda^{1/(p+\beta-1}}$, we have

$$G_{\varepsilon\lambda^{1/(p+\beta-1}}(v) = \frac{v^2}{2\varepsilon^{1+\beta}\lambda^{\frac{1+\beta}{p+\beta-1}}} + O\Big(\frac{v^3}{\varepsilon^{2+\beta}\lambda^{\frac{2+\beta}{p+\beta-1}}}\Big),$$

given that $v \leq 2^{-1}\varepsilon\lambda^{1/(p+\beta-1)}$, which holds in $\Omega_{\lambda,\varepsilon} \setminus B_{r(\lambda,\varepsilon)}(0)$, where $r(\lambda,\varepsilon) = (R_{\lambda,\varepsilon} + \delta')\lambda^{\frac{1+\beta}{2(p+\beta-1)}}$. Therefore, it holds that

$$\begin{aligned}\int_{\Omega_{\lambda,\varepsilon}} G_{\varepsilon\lambda^{1/(p+\beta-1)}}(tv) - G_{\varepsilon\lambda^{1/(p+\beta-1)}}(tv\eta_\lambda(|x|))\, dy &\leq \frac{Ct^3}{\varepsilon^{2+\beta}\lambda^{\frac{2+\beta}{p+b-1}}}\int_{\Omega_{\lambda,\varepsilon}\setminus B_{r(\lambda,\varepsilon)}(0)} v^3 \\ &\leq \exp\bigl(-3((\varepsilon\lambda^{1/(p+\beta-1)}) - 1)^{1/2}(r(\lambda,\varepsilon) - C_1) + C\log(\varepsilon^{-1}) + C\log(\lambda)\bigr). \end{aligned} \tag{97}$$

Analogously, we have

$$\begin{aligned}&\int_{\Omega_{\lambda,\varepsilon}} F\Bigl(\frac{tv}{\varepsilon\lambda^{1/(p+\beta-1)}}\Bigr) - F\Bigl(\frac{tv\eta_\lambda(|y|)}{\varepsilon\lambda^{1/(p+\beta-1)}}\Bigr)\, dy \\ &\qquad \leq \exp\bigl(-3((\varepsilon\lambda^{1/(p+\beta-1)}) - 1)^{1/2}(r(\lambda,\varepsilon) - C_1) + C\log(\varepsilon^{-1}) + C\log(\lambda)\bigr), \\ &\int_{\Omega_{\lambda,\varepsilon}} \frac{t^p}{p} v^p(1 - \eta_\lambda(|y|)^p)\, dy \\ &\qquad \leq \exp\bigl(-(p+1)((\varepsilon\lambda^{1/(p+\beta-1)}) - 1)^{1/2}(r(\lambda,\varepsilon) - C_1) + C\log(\varepsilon^{-1}) + C\log(\lambda)\bigr), \\ &\int_{\Omega_{\lambda,\varepsilon}} t^2\eta_\lambda(|y|)|\eta'_\lambda(|y|)v||\nabla v|\, dy \\ &\qquad \leq \exp\bigl(-2((\varepsilon\lambda^{1/(p+\beta-1)}) - 1)^{1/2}(r(\lambda,\varepsilon) - C_1) + C\log(\varepsilon^{-1}) + C\log(\lambda)\bigr),\end{aligned}$$

and

$$\begin{aligned}&\int_{\Omega_{\lambda,\varepsilon}} \frac{t^2}{2} v^2|\eta_\lambda(|x|)|^2\, dy \\ &\qquad \leq \exp\bigl(-2((\varepsilon\lambda^{1/(p+\beta-1)}) - 1)^{1/2}(r(\lambda,\varepsilon) - C_1) + C\log(\varepsilon^{-1}) + C\log(\lambda)\bigr).\end{aligned} \tag{98}$$

Hence, using estimates (97) and (98) with the aid of (96), it holds that

$$\begin{aligned}E_{\lambda,\varepsilon}(t\widetilde{v}) \leq E_{\lambda,\varepsilon}(tv) &+ \exp\bigl(-2((\varepsilon\lambda^{1/(p+\beta-1)}) - 1)^{1/2}(r(\lambda,\varepsilon) - C_1) + C\log(\varepsilon^{-1}) \\ &+ C\log(\lambda)\bigr).\end{aligned} \tag{99}$$

Now the rest of the proof follows exactly as in the one of [19, Theorem 1.2]. We choose $R'_{\lambda,\varepsilon} > 0$ satisfying $\mathcal{L}^N(B_{R'_{\lambda,\varepsilon}}(x_\lambda)) = \mathcal{L}^N(\Omega \cap B_{R_{\lambda,\varepsilon}+\delta}(x_\lambda))$. Considering the symmetric decreasing rearrangement $\widetilde{v}^*$ of $\widetilde{v}$ given by

$$\widetilde{v}^*(y) = \int_0^\infty \chi_{\{\widetilde{v}(y)>t\}^*}\, dt,$$

where

$$\{\widetilde{v}(y) > t\}^* = \bigl\{y \in \mathbb{R}^N \mid \mathcal{L}^N(B_1(0))|y|^N < \mathcal{L}^N\bigl(\{y \in \mathbb{R}^N \mid \widetilde{v}(y) > t\}\bigr)\bigr\},$$

we obtain

$$E_{\lambda,\varepsilon}(t\widetilde{v}) \geq E_{\rho,\varepsilon\lambda^{1/(p+\beta-1)}}(tv^*),$$

with $\rho = R'_{\lambda,\varepsilon}\lambda^{\frac{1+\beta}{2(p+\beta-1)}}$. Let t be such that

$$E_{\rho,\varepsilon\lambda^{1/(p+\beta-1)}}(t\widetilde{v}^*) = \max_{s\in[0,t_0]} E_{\rho,\varepsilon\lambda^{1/(p+\beta-1)}}(s\widetilde{v}^*).$$

By (99), we have

$$\begin{aligned} &c^*_{\lambda,\varepsilon} - c^*_{\lambda,\varepsilon\lambda^{1/(p+\beta-1)}} \\ &\quad \geq -\exp\big(-2\big(\varepsilon^{-(1+\beta)}\lambda^{-(1+\beta)/(p+\beta-1)} - 1\big)^{1/2}\big(r(\lambda,\varepsilon) - C_1\big) + C\log(\varepsilon^{-1}) \\ &\qquad + C\log(\lambda)\big). \end{aligned} \tag{100}$$

Using (94) in (100), we obtain (93) and the proof of the Claim 7.2 is concluded. ■

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