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Corrigendum to “Classification of global solutions of a free boundary problem in the plane”

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Abstract. This note corrects the statement and the proof of point (1) in Theorem 1.1 of [*Interfaces Free Bound.* **25** (2023), no. 3, 455–490].

The aim of this note is to correct the claim in (1.8) of point (1) of Theorem 1.1 of [2]. The correction is needed due to a mistake in a technical calculation which alters the classification presented there. The computational error in [2] occurs in the proof of Lemma 7.3. The error takes place on p. 485, two lines below (7.34). That inequality is incorrect, since $a > 1$. This in turn invalidates the claim in (7.25), which was used to prove (1.8). It is a pleasure to thank Daniel Restrepo Montoya for having pointed out to us this mistake.

We will adopt here the same notation as in [2]. Moreover, given $\rho \geqslant 0$, we denote by $\langle \rho \rangle$ the greatest integer strictly less than ρ.

Then, the correct version of Theorem 1.1 of [2] needs to replace (1) there (we follow here the notation of [2], according to which $a = \frac{2}{2-\gamma}$) with the following:

(1) *We have*

$$\gamma \in (0, 2).$$

Also, up to a rotation,

- *either* $\gamma = 1$ *and one of the following situations holds:*
  - *we have that*
  $$u(x) = C_1 x_1^2 + C_2 x_2^2,$$
  with C_1, $C_2 \geqslant 0$ *and* $C_1 + C_2 = \frac{1}{2}$,
  - *up to a rotation,*
  $$u(x) = \frac{(x_2)_+^2}{2},$$
- *or* $\gamma \in (1, 2)$, *in which case* u *is given by one of the following:*
  - $u(x) = C_a |x|^a$, *with* $C_a := \frac{(2(a-1))^{a/2}}{a^{3a/2}}$,

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- *u has a polar representation of the form* $r^a g(\theta)$, *with* $g\colon \mathbb{S}^1 \to [0,+\infty)$ *non-constant, with* $\langle \frac{2}{\sqrt{2-\gamma}} \rangle$ *different possible choices of* g, *where two of these solutions are the ones presented in* (2) *and* (3) *and the other* $\langle \frac{2}{\sqrt{2-\gamma}} \rangle - 2$ *choices of* g *are strictly positive,*

• *or* $\gamma \in (0,1)$.

Proof. The proof presented in [2] that $\gamma \in (0,2)$ was correct, so we focus on the remaining claims. If $\gamma = 1$, by [2, p. 464], we write u in polar form as $r^2 g(\theta)$, with $4g + g'' = 1$. The general solution of this ordinary differential equation is

$$g(\theta) = \frac{1}{4} + c_1 \cos(2\theta) + c_2 \sin(2\theta),$$

with $c_1, c_2 \in \mathbb{R}$.

Up to a rotation, one can suppose that g has a minimum at $\theta = 0$, whence $g'(0) = 0$ and $g''(0) \geqslant 0$. From this, one obtains that $c_2 = 0$ and $c_1 \leqslant 0$. Since $0 \leqslant g(0) = \frac{1}{4} + c_1$, we conclude that $c_1 \in [-\frac{1}{4}, 0]$.

Correspondingly, we find that

$$u(x) = \Big(\frac{1}{4} + c_1\Big)x_1^2 + \Big(\frac{1}{4} - c_1\Big)x_2^2,$$

whenever $u(x) > 0$. Hence, by setting $C_1 := \frac{1}{4} + c_1$ and $C_2 := \frac{1}{4} - c_1$, we find that

$$u(x) = C_1 x_1^2 + C_2 x_2^2,$$

as long as $u(x) > 0$.

The case of u vanishing on a positive set occurs when either $C_1 = 0$ or $C_2 = 0$. Up to a rotation, suppose that $C_1 = 0$, then $u(x) = \frac{x_2^2}{2}$ whenever $u(x) > 0$, which produces, again up to a rotation, also the solution $u(x) = \frac{(x_2)_+^2}{2}$.

Hence, it remains to discuss the case $\gamma \in (1,2)$. For this, one distinguishes the case in which g never vanishes (corresponding to [2, (8.1)]) and the case in which g vanishes somewhere (corresponding to [2, (8.2)]). The latter case is addressed in [2, p. 487] and produces the two solutions

$$u(x) = \frac{2^{a/2}}{a^a}(x_2)_+^a \quad \text{and} \quad u(x) = \frac{2^{a/2}}{a^a}|x_2|^a, \tag{1}$$

which are listed in points (2) and (3) of [2].

Therefore, we can suppose that g never vanishes (this requires a correction, since it used [2, Lemma 7.3]). In this case, one can use [1, Theorem 5.1]. Indeed, following Section 5 there, one considers solutions in the polar form $r^a g(\theta)$, finding the following possibilities:

(a) the null solution,

(b) g constantly equal to $(\frac{\gamma}{a^2})^{1/(2-\gamma)} = C_a$ (this solution is called x_1 in [1]),

(c) two nonnegative (but not strictly positive) solutions (corresponding to ψ_2 and ψ_3 in the notation of [1]) that can be written in the form

$$g_1(\theta) := \left(\frac{2}{a^2}\right)^{1/(2-\gamma)} |\cos\theta|^a$$

and

$$g_2(\theta) := \begin{cases} g_1(\theta) & \text{if } \theta \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right), \\ 0 & \text{if } \theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \end{cases}$$

(d) and $\langle \frac{2}{\sqrt{2-\gamma}} \rangle - 2$ additional positive solutions (according to [1, (5.11)]).

We can disregard case (a), since [2, Theorem 1.1] was considering nontrivial solutions. The solution in (b) corresponds to $u(x) = C_a|x|^a$.

Regarding (c), we see that

$$u_1(r,\theta) := r^a g_1(\theta) = \left(\frac{2}{a^2}\right)^{1/(2-\gamma)} |x_1|^a = \frac{2^{a/2}}{a^a}|x_1|^a.$$

This solution is included in [2, Theorem 1.1 (3)] (up to a rotation, see (1)).

Also,

$$u_2(r,\theta) := r^a g_2(\theta) = \frac{2^{a/2}}{a^a}(x_1)_+^a,$$

which is included in [2, Theorem 1.1 (2)] (up to a rotation, see (1)). ■

It is worth observing that the classification of solutions when $\gamma \in (0,1)$ is still to be exhaustively described.

We stress that [2, Theorem 1.1 (1)–(4)] (with point (1) corrected here) lists solutions in a non-mutually-exclusive way.

For the facility of the reader, we now state the revised version of Theorem 1.1 of [2] in a mutually exclusive way.

Theorem. *Let $a > 0$ and $\gamma \neq 0$. Assume that $u \in C(\mathbb{R}^2)$ is a nontrivial, nonnegative, positively homogeneous solution of degree a of the equation*

$$\Delta u = \gamma u^{\gamma-1}$$

in a connected component of $\mathbb{R}^2 \cap \{u > 0\}$.

Moreover, if $\gamma \neq -2$, suppose that for each connected component S of $(B_2 \setminus B_{1/2}) \cap \{u > 0\}$ we have that

$$u^{1/a} \in C^{\xi}(\bar{S}) \quad \textit{for some } \xi > \begin{cases} 3-2a & \textit{if } a \in (0,1), \\ \frac{1}{a} & \textit{if } a > 1. \end{cases} \tag{I}$$

Then,

$$\gamma \in (0,2) \cup \{-2\} \quad \textit{and} \quad a = \frac{2}{2-\gamma}.$$

Additionally, the following classification holds true:

(i) *If* $\gamma \in (1,2)$*, then* u *has a polar representation of the form* $r^a g(\theta)$*, with* $g\colon \mathbb{S}^1 \to [0,+\infty)$ *non-constant, with* $\langle \frac{2}{\sqrt{2-\gamma}} \rangle + 1$ *different possible choices of* g*, where three of these solutions are, up to a rotation,*

$$u(x) = \frac{(2(a-1))^{a/2}}{a^{3a/2}}|x|^a, \quad u(x) = \frac{2^{a/2}}{a^a}(x_2)_+^a, \quad u(x) = \frac{2^{a/2}}{a^a}|x_2|^a, \tag{II}$$

and the other $\langle \frac{2}{\sqrt{2-\gamma}} \rangle - 2$ *choices of* g *are strictly positive.*

(ii) *If* $\gamma = 1$*, then either*

$$u(x) = C_1 x_1^2 + C_2 x_2^2, \tag{III}$$

with $C_1, C_2 \geqslant 0$ *and* $C_1 + C_2 = \frac{1}{2}$*, or, up to a rotation,*

$$u(x) = \frac{(x_2)_+^2}{2}. \tag{IV}$$

(iii) *If* $\gamma = -2$*, given* $c \in \mathbb{R} \setminus \{0\}$*, up to a rotation and a reflection, the positivity set of* u *contains the cone*

$$\mathcal{C}_c := \{(r\cos\theta, r\sin\theta),\ r > 0 \text{ and } \theta \in (0, T_c)\},$$

with

$$T_c := \begin{cases} 2\pi - 2\arctan\left(\dfrac{1}{c}\right) & \textit{when } c > 0, \\ -2\arctan\left(\dfrac{1}{c}\right) & \textit{when } c < 0, \end{cases}$$

$u = 0$ *on* $\partial\mathcal{C}_c$*, and, for every* $x \in \mathcal{C}_c$*,*

$$u(x) = 2^{3/4}\sqrt{x_2 - cx_1 + c|x|}.$$

Proof. By [2, Theorem 1.1], as corrected on the first page of this corrigendum, we know that $\gamma \in (-\infty, 2)$, with the description for the cases $\gamma \in (0,2)$ as in (i) and (ii) here above.

The case (iii) here above corresponds to [2, Theorem 1.1 (4)]. It remains to rule out the situation $\gamma \in (-\infty, 0) \setminus \{-2\}$. This case does not produce solutions satisfying (I), since, up to a rotation, in view of [2, Theorem 1.1 (2), (3)], the solutions produced would be proportional to $(x_2)_+^a$ and $|x_2|^a$, which do not satisfy (I). ∎

Some final remarks. The case in which (I) is not assumed is discussed in [2, Theorem 1.2]. Also, the functions in (II) are always solutions; when $\gamma = 1$, they fall into the classification (III)–(IV), but (III) contains other solutions as well (this is the reason for which $\gamma = 1$ needs to be treated as a separate case).

References

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