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# Short second moment bound and subconvexity for GL(3) $L$-functions

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**Abstract.** Let $\pi$ be a Hecke cusp form for $\mathrm{SL}(3,\mathbb{Z})$. We bound the second moment average of $L(s,\pi)$ over a short interval to obtain the subconvexity estimate

$$L(1/2+it,\pi) \ll_{\pi,\varepsilon} (1+|t|)^{3/4-1/8+\varepsilon}.$$

This is the first time a short second moment has been used to obtain a subconvexity bound in a higher rank group.

*Keywords:* subconvexity, delta method, short second moment, Voronoi summation.

## Contents



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## 1. Introduction and statement of result

The problem of bounding automorphic $L$-functions on the central line $\mathrm{Re}(s) = 1/2$ has enthralled researchers for over a century. The first subconvexity bound was announced by Littlewood in proceedings of the London Mathematical Society in 1921. Adopting a method of Weyl for bounding exponential sums, Hardy and Littlewood proved that

$$\zeta(1/2 + it) \ll_\varepsilon (1 + |t|)^{1/6+\varepsilon},$$

for any $\varepsilon > 0$. In contrast, the trivial (convexity) bound, a simple consequence of the functional equation of the zeta function, yields the exponent $1/4$ in place of $1/6$. Complete details of the proof of Hardy–Littlewood's result appeared only later in a paper of Landau [23]. The exponent $1/6$, which is two-thirds of the convexity bound, is 'fundamental', and can be achieved using other methods, e.g. short moment computation of Iwaniec [18] and high moment computation of Heath-Brown [14]. To go beyond $1/6$, the so-called Weyl bound, one needs deeper understanding of short exponential sums, as achieved systematically by van der Corput and others [13]. Accordingly the Weyl bound has been breached, albeit mildly, dozens of times in the last hundred years, and the present record due to Bourgain [9] states that

$$\zeta(1/2+it) \ll_\varepsilon (1+|t|)^{1/6-1/84+\varepsilon}.$$

The interest in such improvements lies not in the bound itself, but in the method that leads to it. However, the strength of the bound documents the strength of the method, and as such is an important marker.

The classical result of Hardy–Littlewood–Weyl was first extended to degree 2 $L$-functions by Good [12] in the 1980s. For any holomorphic form $f$ of full level, Good established the bound

$$L(1/2+it, f) \ll_{f,\varepsilon} (1+|t|)^{1/3+\varepsilon}.$$

Since this $L$-function can be compared with the square of the Riemann zeta function, one notices that the above exponent matches the Weyl exponent. Later the same bound was established for Maass forms by Meurmann [27]. Around the same time Jutila [20] proved the same result using more elementary means based on his study of exponential sums twisted by GL(2) Fourier coefficients. More recently, Booker et al. [8] established the Weyl bound for $L$-functions of holomorphic cusp forms for $\Gamma_1(N)$ for any level $N$. This was extended to degree 2 $L$-functions, of any level, nebentypus and spectral parameter by the first author [2]. However, any improvement over this bound is yet to be achieved. The recent joint work [15] of the third author with Holowinsky and Qi makes some progress towards this important problem by establishing cancellation in very short sums (beyond the Weyl range). Unfortunately, that work does not yield anything new in the generic length (square root of the conductor), and hence one does not get a sub-Weyl bound for the $L$-function.

The $t$-aspect subconvexity problem for degree 3 $L$-functions remained an important open problem for a long time, till the breakthrough work of Li [24], who adopted the method of Conrey and Iwaniec [10] to establish a subconvex bound for symmetric square (self-dual) $L$-functions,

$$L(1/2+it, \mathrm{Sym}^2 f) \ll_{f,\varepsilon} (1+|t|)^{11/16+\varepsilon}.$$

Since non-negativity of central $L$-values plays a crucial role in the Conrey–Iwaniec approach, it seems that Li's method cannot be extended beyond self-dual forms. However, the bound has been improved substantially. First McKee, Sun and Ye [26] established the exponent $2/3$, using the techniques of Li, but applying finer tools to deal with the exponential integrals. This was further improved by Nunes [33] to $5/8$, which doubles the initial saving given by Li. Qi [34] extended the result to arbitrary number fields with the same exponent of $5/8$. Lin, Nunes and Qi [25] established the much stronger exponent $3/5$ over the rationals, which is most likely the limit of the method in this set-up, but still falls short of the Weyl exponent $1/2$. Hybrid bounds have been obtained by Young [35] and Khan–Young [21] by estimating the second moment of $\mathrm{GL}(3) \times \mathrm{GL}(2)$ $L$-functions.

In the case of general non-self-dual degree 3 $L$-functions $L(s,\pi)$, the first subconvex bound was established by the third author [30] using the delta method,

$$L(1/2+it,\pi) \ll_{\pi,\varepsilon} (1+|t|)^{11/16+\varepsilon}.$$

This was improved by the first author [3], who established the exponent $27/40$ by applying the delta method with more careful analysis of the integral transforms. His exponent is the limit of the delta method recipe in this context. There has been progress in proving a $t$-aspect subconvexity bound for Rankin–Selberg convolutions of GL(2) and GL(3) $L$-functions. The third author [31] used the delta method to prove a subconvexity bound for $\mathrm{GL}(3)\times\mathrm{GL}(2)$ $L$-functions. Recently, Blomer–Jana–Nelson [6] proved a $t$-aspect Weyl-type bound for $\mathrm{GL}(2)\times\mathrm{GL}(2)$ $L$-functions by modifying the ideas of Bernstein–Reznikov [4] and using the Kuznetsov trace formula.

In a recent preprint, Nelson [32] has announced a solution of the $t$-aspect subconvexity problem for standard $L$-functions, regardless of its degree. This breakthrough work is based on the period approach introduced in earlier landmark papers of Bernstein–Reznikov [4] and Michel–Venkatesh [28]. For low degree $L$-functions, Nelson's bound is worse than what is already known. It is however understood that the method has the potential to produce strong bounds even in the case of low degrees.

### *1.1. Statement of results and methodology*

We now state our results and discuss the main ideas. Let $\pi$ be a Hecke cusp form of type $(\nu_1,\nu_2)$ for $\mathrm{SL}(3,\mathbb{Z})$. Let the normalized Fourier coefficients be given by $\lambda(m_1,m_2)$ (so that $\lambda(1,1)=1$). The $L$-series associated with $\pi$ is given by

$$L(s,\pi)=\sum_{n\geq 1}\lambda(1,n)n^{-s} \quad \text{for } \mathrm{Re}(s)>1.$$

Our main result is the following theorem.

**Theorem 1.1.** *Let $\pi$ be a Hecke–Maass cusp form for* $\mathrm{SL}(3,\mathbb{Z})$. *Then for any $\varepsilon>0$ and $t^{1/2}<M<t^{1-\varepsilon}$, we have*

$$\int_{t-M}^{t+M}|L(1/2+iv,\pi)|^2\,dv \ll_{\pi,\varepsilon} t^{\varepsilon}\left(\frac{t^{9/4}}{M^{3/2}}+\frac{M^3}{t^{21/20}}+M^{7/4}t^{3/40}+M^{15/14}t^{15/28}\right).$$

*In particular, when $M=t^{2/3}$, we have*

$$\int_{t-t^{2/3}}^{t+t^{2/3}}|L(1/2+iv,\pi)|^2\,dv \ll_{\pi,\varepsilon} t^{5/4+\varepsilon}.$$

The second moment average can be used to bound the $L$-function by modifying Good's arguments [12] (Lemma 3.5 below). We find that for any $\log t<M<t^{1-\varepsilon}$,

$$|L(1/2+it,\pi)|^2 \ll (\log t)\left(1+\int_{\mathbb{R}}U(v/M)|L(1/2+it+iv,\pi)|^2\,dv\right), \qquad (1.1)$$

where $U \in C_c^\infty([1,2])$ is an appropriate bump function. This yields the following subconvexity bound.

**Theorem 1.2.** *Let $\pi$ be a Hecke–Maass cusp form for* $\mathrm{SL}(3,\mathbb{Z})$. *Then for any $\varepsilon > 0$,*

$$L(1/2+it,\pi) \ll_{\pi,\varepsilon} t^{3/4-1/8+\varepsilon}.$$

Studying moments of $L$-functions is an important theme in modern analytic number theory. The bound in Theorem 1.1 compares with the best known unconditional estimate for the sixth moment of the Riemann zeta function, which follows from the work of Ivić [17]:

$$\int_T^{2T} |\zeta(1/2+it)|^6\,dt \ll T^{5/4+\varepsilon}.$$

Various results have been obtained by estimating the first and second moment of $L$-functions in the last twenty years. This is the first time a short second moment has been used to obtain a subconvex bound in higher rank.

Li [24] and subsequent works improving her subconvexity exponent consider the short first moment of a family of $\mathrm{GL}(3)\times\mathrm{GL}(2)$ $L$-functions

$$\sum_{|t_j-T|<M} L(1/2,\pi\times f_j) + \int_{T-M}^{T+M} |L(1/2+it,\pi)|^2\,dt.$$

Here the GL(3) form $\pi$ is self-dual and fixed, and the average is taken over a family $\{f_j\}$ of GL(2) forms with eigenvalues $t_j$. In comparison, we consider a short second moment of the $L$-function of a fixed GL(3) Hecke–Maass cusp form. The clear advantage is our ability to take $\pi$ not necessarily self-dual.

We briefly discuss the main ideas to highlight the novelty of our proof. We start by using an approximate functional equation in (1.1) and execute the $v$-integral to obtain a shifted sum twisted with an additive character,

$$S_{M,H}(N) \sim M \sum_{h\sim H}\sum_{n\sim N} \lambda(1,n)\overline{\lambda(1,n+h)}e\left(\frac{th}{2\pi n}\right),$$

with $H$ going up to $N^{1+\varepsilon}/M$. Next, we apply Duke–Friedlander–Iwaniec's delta method (Lemma 4.1) to separate the oscillations of the Fourier coefficients,

$$S_{M,H}(N) \sim M \sum_{h\sim H}\sum_{n\sim N} \lambda(1,n)e\left(\frac{th}{2\pi n}\right) \sum_{m\sim N} \overline{\lambda(1,m)}\delta(m=n+h).$$

The application of the delta method contributes $Q^2$ additive frequencies of size $1 \le q \le Q$, where $Q$ is a parameter to be chosen later. This is followed by applying dual summation formulas to the $n$-, $m$- and $h$-sums. However, at this point, we diverge from the 'routine' and apply a combination of the Cauchy–Schwarz inequality and the duality principle. This allows us to change the lengths of summations inside the absolute-value squared obtained after applying the Cauchy–Schwarz inequality and may be considered the first novelty of our proof.

The next step is to apply dual summation formulas to the new $m$- and $n$-sums followed by stationary phase analysis of the various integral transforms. Although it is not novel, one needs extreme care in this analysis. This brings us close to the convexity barrier.

The previous application of Cauchy–Schwarz and the duality principle had created two copies of the $h$- and $q$-sums. The final steps involve applying the Poisson summation formula to the new $h_1, h_2$-sum and the $q_1, q_2$-sum. However, we are faced with a unique issue due to the duality principle. The $q_1, q_2$-sum has a structure of the form

$$\sup_{\|\beta\|_2=1} \sum\sum_{q_1,q_2\sim Q} \beta(q_1)\overline{\beta(q_2)}e\left(\frac{aq_2}{q_1}+\frac{bq_1}{q_2}\right)G(q_1,q_2),$$

where $a, b$ are some integers, $G$ is some function and $\beta$ is a sequence of complex numbers coming from the duality principle. While we would like to extract savings from the $q_1, q_2$-sums, the presence of $\beta$ is an obstacle. To overcome this, we apply the Cauchy–Schwarz inequality to take out the $q_1$-sum. This allows us to get rid of $\beta(q_1)$ and hence perform Poisson summation in the $q_1$-sum to obtain cancellations. If we stop the analysis here, the character sum at this stage is a Kloosterman sum. Application of the Weil–Deligne bound yields

$$\int_{t-M}^{t+M} |L(1/2+it,\pi)|^2\,dt \ll_{\pi,\varepsilon} t^{\varepsilon}\left(\frac{t^{9/4}}{M^{3/2}}+\frac{M^3}{t^{21/20}}+M^{7/4}t^{3/40}+M^{9/8}t^{9/16}\right).$$

The optimal choice of the length of the second moment is $M = t^{27/42}$, giving the subconvexity bound

$$L(1/2+it,\pi) \ll_{\pi,\varepsilon} t^{9/14+\varepsilon}.$$

This already improves upon the best known exponent of $27/40$ for a not necessarily self-dual form obtained in [3], but we can do better. Since the above process did not make use of the $q_2$-sum, the previous Cauchy–Schwarz inequality created two copies of it, say $q_2$- and $q_3$-sums. The same analysis is done for the $q_2$-sum without using the $q_3$-sum. We can thus iterate the above process ad infinitum to get further savings. This may be considered the second novelty of our proof.

We highlight that doing so results in increasingly complicated character sums. We bound these exponential sums by using the methods due to Adolphson and Sperber [1] obtained by their extension of Dwork's cohomology theory from smooth projective hypersurfaces in characteristic $p$ to a general class of exponential sums. The associated Newton polyhedron after the first such application is depicted in Figure 1.

We also highlight that we need a careful application of the sophisticated tools developed by Adolphson–Sperber because the size of the Newton polyhedron grows with each application of the Cauchy–Schwarz inequality.

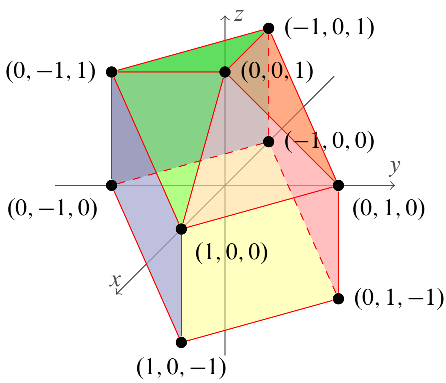


**Fig. 1.** Newton polyhedron for the character sum in the base case

*Notations*

In the rest of the paper, we use the notation $e(x) = e^{2\pi i x}$. For any real numbers $a$ and $b$, we write $a \sim b$ to mean $k_1 < |a/b| < k_2$ for some absolute constants $k_1, k_2 > 0$. For $j \geq 0$, $a \sim_j b$ means $k_1 < a/b < k_2$ but $k_1, k_2$ may depend on $j$. In certain places, we will need to choose $k_i$ very close to 1 so that stationary phase analysis is simplified later. Specifically, when we choose $k_1 = 1$ and $k_2 = 1 + 10^{-10}$, we write $a \sim_0 b$. We use $\varepsilon$ to denote an arbitrarily small positive constant that can change depending on the context. For brevity, we assume $t > 0$. Indeed, the same analysis holds for $t < 0$ by replacing $t$ with $-t$ appropriately.

We will denote by $U(x)$ and $\varphi(x)$ two compactly supported smooth weight functions whose definitions can be revised based on the context. For $\alpha > 1$, $C \in \mathbb{R}$ and $k \in \mathbb{Z}$, we refer to the intervals $[\alpha^k C, \alpha^{k+1} C]$ as dyadic intervals even though $\alpha$ may not equal 2.

We say a function $f$ is $Z$-*inert* if $x^{-j} \frac{d^j}{dx^j} f(x) \ll_j Z^j$ for any $j \geq 0$. We say $f$ is *flat* if it is 1-inert.

## 2. Sketch of proof

We expand upon the above discussion via a sketch of proof for Theorem 1.1. Let $\log t < M < t^{1-\varepsilon}$ be a parameter and $U \in C_c^\infty([-2, 2])$ be such that $U(x) = 1$ for $-1 \leq x \leq 1$.

### 2.1. Short second moment average and approximate functional equation

By the approximate functional equation (3.1), it suffices to bound

$$S(N) := \int_{\mathbb{R}} U\left(\frac{v}{M}\right) \left| \sum_{n=1}^{\infty} \lambda(1, n)\, n^{-i(t+v)} V\left(\frac{n}{N}\right) \right|^2 dv,$$

where $N \ll t^{3/2+\varepsilon}$ and $V \in C_c^\infty([1,2])$ may depend on $t$ and satisfies $V^{(j)}(x) \ll_j 1$ for $j \geq 0$. Opening the square and integrating yields a shifted sum (Lemma 6.1),

$$S(N) \ll Mt^\varepsilon + \sup_{H \ll Nt^\varepsilon/M} \frac{|S_{M,H}(N)|}{N},$$

where the shifted sum is

$$S_{M,H}(N) \sim M \sum_{h\sim H}\sum_{n\sim N} \lambda(1,n)\overline{\lambda(1,n+h)}e\left(\frac{th}{2\pi n}\right).$$

For this sketch, we focus on the generic cases $H = N/M$, i.e. on $S_{M,N/M}(N)$.

We note that Lemma 3.6 gives us the trivial bound

$$S_{M,N/M}(N) \ll N^2 t^\varepsilon. \tag{2.1}$$

### 2.2. *Delta method*

Next we apply the DFI delta method of Lemma 4.1 to detect $m = n + h$,

$$\begin{aligned} S_{M,N/M}(N) \sim \frac{M}{Q} \sum_{1\leq q\leq Q} \frac{1}{q} \sum_{h\sim N/M}\sum_{n\sim N} \lambda(1,n)e\left(\frac{th}{2\pi n}\right) \sum_{m\sim N} \overline{\lambda(1,m)} \\ \times \sideset{}{^\star}\sum_{\alpha \bmod q} e\left(\frac{\alpha(n+h-m)}{q}\right) \int_{\mathbb{R}} g(q,x)e\left(\frac{(n+h-m)x}{qQ}\right) dx. \end{aligned}$$

For the purpose of this sketch, we focus on the generic case $x \sim 1$.

### 2.3. *Dual summation formulas*

After separating the variables with the delta method, we apply dual summation formulas. Applying Poisson summation to the $h$-sum, and Voronoi summation (Lemma 3.3) to the $m$- and $n$-sums, we roughly get

$$\begin{aligned} S_{M,N/M}(N) \sim \frac{MN^{4/3}}{Q^2 t^{1-\varepsilon}} \sum_{q\sim Q}\sum_{h\sim qt/N}\sum_{m\ll N^2/Q^3} \frac{\overline{\lambda(m)}}{m^{1/3}} \sum_{n \ll \frac{N^2}{Q^3}+\frac{Q^3t^3}{M^3N}} \frac{\lambda(n)}{n^{1/3}} \\ \times S(\bar{h},m;q)S(\bar{h},n;q) \int_{|u|\ll 1} e\big(f_1(m,h,q,u) + f_2(n,h,q,u)\big)\, du, \end{aligned}$$

with the analytic oscillations being of size $f_1(m,h,q,u) \sim N/(qQ)$ and $f_2(n,h,q,u) \sim (nN)^{1/3}/q$. The variable $u$ is a secondary oscillatory term that is created in the Voronoi summation of the $n$-sum. It does not have any significance in the generic case when $N \sim t^{3/2}$, but we get a little extra saving when $N$ is smaller and the main oscillatory terms cancel out in the diagonal case. In the rigorous proof, we do not utilize the cancellations in the integrals until later stages due to the fact that $x$ can be smaller than 1. We choose

$Q \ll \sqrt{MN/t}$. Indeed to match the oscillation, we want to choose $Q = \sqrt{MN/t}$, but we will choose $Q$ smaller when $N$ is small. Focusing on the generic case

$$m \sim \frac{N^2}{Q^3} \quad \text{and} \quad n \sim \frac{N^2}{Q^3} + \frac{Q^3 t^3}{M^3 N} \sim \frac{N^2}{Q^3},$$

we roughly have

$$S_{M,N/M}(N) \sim \frac{M}{t^{1-\varepsilon}} \sum_{q \sim Q} \sum_{h \sim qt/N} \sum_{m \sim N^2/Q^3} \overline{\lambda(m)} \sum_{n \sim N^2/Q^3} \lambda(n) \\ \times S(\overline{h}, m; q) S(\overline{h}, n; q) \int_{|u| \ll 1} e\big(f_1(m,h,q,u) + f_2(n,h,q,u)\big)\, du,$$

with the analytic oscillations being of size $f_1(m,h,q)$, $f_2(n,h,q) \sim \frac{N}{qQ}$.

## 2.4. *Cauchy–Schwarz inequality*

Next we apply the Cauchy–Schwarz inequality to take out the $q, h$-sums and the $u$-integral, giving us

$$S_{M,N/M}(N) \ll \sqrt{S_1 S_2},$$

where roughly

$$S_j \sim \frac{M}{t^{1-\varepsilon}} \sum_{q \sim Q} \sum_{h \sim qt/N} \int_{|u| \ll 1} \Big| \sum_{m \sim N^2/Q^3} \lambda(m) S(\overline{h}, m; q) e(f_j(m,h,q,u)) \Big|^2 du.$$

**Remark 2.1.** Note that the $q$- and $h$-sums are shorter than the $n$- and $m$-sums. Therefore it would have been beneficial to instead take out the $n$- and $m$-sums out of the absolute value squared in the above step. However, our approach benefits by allowing us to separate the $n$- and $m$-sums.

## 2.5. *Duality principle*

We apply the duality principle (Lemma 4.2) together with Lemma 3.6 to take the $n$- and $m$-sums outside the absolute value squared and obtain

$$S_j \ll \frac{MN^2}{Q^3 t^{1-\varepsilon}} \sup_{\|\beta\|_2 = 1} \sum_{m \sim N^2/Q^3} \Big| \sum_{q \sim Q} \sum_{h \sim qt/N} S(\overline{h}, m; q) \\ \times \int_{|u| \ll 1} \beta(q,h,u) e(f_j(m,h,q,u))\, du \Big|^2.$$

We thus also get rid of the GL(3) Fourier coefficients, decreasing the complexity of the problem at hand.

*2.6. Poisson summation formula*

Opening the square and applying Poisson summation to the $m$-sum gives us roughly

$$S_j \ll \frac{MN^4}{Q^8 t^{1-\varepsilon}} \sup_{\|\beta\|_2=1} \sum\sum_{q_1,q_2\sim Q} \sum\sum_{h_i\sim q_i t/N} \sum_{|m|\ll Q^2\sqrt{q_1q_2}/N} \sum_{\gamma \bmod q_1q_2} S(\overline{h_1},\gamma;q_1)S(\overline{h_2},\gamma;q_2)$$
$$\times e\left(\frac{m\gamma}{q_1q_2}\right)\int_{|u_1|\ll 1}\int_{|u_2|\ll 1} \beta(q_1,h_1,u_1)\overline{\beta(q_2,h_2,u_2)}$$
$$\times \int_{\mathbb{R}} V(y) e\left(f\left(\frac{N^2}{Q^3}y,h_1,q_1,u_1\right) - f\left(\frac{N^2}{Q^3}y,h_2,q_2,u_2\right) - \frac{mN^2y}{q_1q_2Q^3}\right) dy\, du_1\, du_2.$$

Now we have a spectrum of cases depending on the size of $m$. For the sketch, we shall focus on two extreme cases, the diagonal corresponding to $m=0$, and the off-diagonal where $m\sim Q^3/N$ is as large as possible. The other cases will yield bounds in between these two cases.

*2.7. Diagonal terms*

For the diagonal $m=0$, we essentially have $q_1=q_2\ (=q)$, $h_1=h_2$ and $|u_1-u_2|\ll \frac{q^2t}{MN}$. In such a case, one can evaluate the $\gamma$-sum and obtain the bound

$$\frac{MN^4}{Q^8t^{1-\varepsilon}} Q^3 \frac{Q^2t}{MN} \ll \frac{N^3t^\varepsilon}{Q^3}.$$

Here is the only place where we extract an extra saving coming from the secondary oscillatory terms having $u_1,u_2$. This is possible as the main oscillatory terms cancel out when $q_1=q_2, h_1=h_2$. Moreover, we will see that $\frac{Q^2t}{MN}$ is less than 1 only when $N$ is smaller than the generic size $t^{3/2}$ by our choice of $Q$.

*2.8. Off-diagonal terms*

For the off-diagonal where $m\sim Q^3/N$, evaluating the $\gamma$-sum and applying stationary phase analysis to the $y$-integral gives us roughly

$$\frac{MN^{7/2}}{Q^5t^{1-\varepsilon}} \sup_{\|\beta\|_2=1} \sum_{m\sim Q^3/N} \sum\sum_{q_1,q_2\sim Q} \sum\sum_{h_1,h_2\sim qt/N} \beta(q_1,h_1)\overline{\beta(q_2,h_2)}$$
$$\times e\left(-\frac{\overline{mh_1}q_2}{q_1} - \frac{\overline{mh_2}q_1}{q_2}\right) e(F(m,h_1,h_2,q_1,q_2))$$

with $F(m,h_1,h_2,q_1,q_2)\sim N/Q^2$. Here we dropped the $u_1,u_2$-integral in the sketch as we will just bound it trivially with $\|\beta\|_2=1$. The bound at this point is

$$\frac{MN^{7/2}}{Q^5t^{1-\varepsilon}} \frac{Q^3}{N} Q \frac{Qt}{N} \ll MN^{3/2}t^\varepsilon.$$

We want to get a saving from the $q_j$-sums. Assuming we are in the generic case $(q_1, q_2) = 1$ in this sketch, reciprocity gives us

$$\frac{MN^{7/2}}{Q^5 t^{1-\varepsilon}} \sup_{\|\beta\|_2=1} \sum_{m\sim Q^3/N} \sum_{q_1,q_2\sim Q}\sum \sum_{h_1,h_2\sim q_i t/N}\sum \beta(q_1,h_1)\overline{\beta(q_2,h_2)}$$
$$\times e\bigg(\frac{\overline{q_1}q_2}{mh_1} + \frac{\overline{q_2}q_1}{mh_2}\bigg) e\bigg(-\frac{q_2}{mh_1q_1} - \frac{q_1}{mh_2q_2} + F(m,h_1,h_2,q_1,q_2)\bigg).$$

## 2.9. *Infinite Cauchy–Schwarz and Poisson summation*

We want to apply Poisson summation to the $q_1, q_2$-sums, but the $\beta$ coefficients obtained from the duality principle prevent us from doing so. To deal with the $\beta$ coefficients, we have to apply the Cauchy–Schwarz inequality to take out the $q_1$-sum while leaving the $q_2$-sum inside the square. Applying Poisson summation afterwards to the $q_1$-sum would give us a saving of

$$\bigg(\sqrt{\frac{\text{old length}}{\text{new length}}}\bigg)^{1/2} \sim \bigg(\frac{N^2}{Qt^2}\bigg)^{1/4}$$

in $S_1(N)$. The Cauchy–Schwarz process creates two copies of $q_2$-sum, say $q_2, q_3$-sums. We then repeat the process of taking out the $q_2$-sum, leaving the $q_3$-sum inside the square and applying the Poisson summation to the $q_2$-sum. This gives us a saving of

$$\bigg(\frac{N^2}{Qt^2}\bigg)^{1/8}.$$

**Remark 2.2.** The above saving is a result of careful analysis of complicated exponential sums, and we need to apply the Weil–Deligne bound via the result of Adolphson–Sperber obtained by their extension of Dwork's cohomology theory from smooth projective hypersurfaces in characteristic $p$ to a general class of exponential sums.

Iterating this process $\nu$ times for $\nu$ sufficiently large yields a total saving of

$$\bigg(\frac{N^2}{Qt^2}\bigg)^{1/4} \times \bigg(\frac{N^2}{Qt^2}\bigg)^{1/8} \times \cdots \times \bigg(\frac{N^2}{Qt^2}\bigg)^{1/2^\nu} \sim \bigg(\frac{N^2}{Qt^2}\bigg)^{1/2} t^{-\varepsilon}$$

and hence the contribution of $m \sim Q^3/N$ is bounded by

$$MN^{3/2}t^{\varepsilon}\bigg(\frac{Qt^2}{N^2}\bigg)^{1/2} \ll M\sqrt{QN}t^{1+\varepsilon}.$$

## 2.10. *Final calculations*

As a result, assuming all other cases lie between the two cases sketched, we have, for $Q \ll \sqrt{MN/t}$,

$$S_j \ll \frac{N^3 t^{\varepsilon}}{Q^3} + M\sqrt{QN}t^{1+\varepsilon},$$

and hence

$$\frac{S_{M,N/M}(N)}{N} \ll \frac{N^2 t^{\varepsilon}}{Q^3} + \frac{\sqrt{Q} M t^{1+\varepsilon}}{\sqrt{N}}.$$

We choose $Q = \sqrt{M} N^{2/3}/t^{3/4} \ll \sqrt{MN/t}$, giving us

$$\frac{S_{M,N/M}(N)}{N} \ll \frac{t^{9/4+\varepsilon}}{M^{3/2}} + \frac{M^{5/4} t^{5/8+\varepsilon}}{N^{1/6}}.$$

Together with the trivial bound in (2.1), we have

$$\frac{S_{M,N/M}(N)}{N} \ll \frac{t^{9/4+\varepsilon}}{M^{3/2}} + \min\left\{ N t^{\varepsilon}, \frac{M^{5/4} t^{5/8+\varepsilon}}{N^{1/6}} \right\} \ll \frac{t^{9/4+\varepsilon}}{M^{3/2}} + M^{15/14} t^{15/28+\varepsilon}.$$

The optimal choice is $M = t^{2/3}$, which gives us the bound

$$\frac{S_{M,N/M}(N)}{N} \ll t^{5/4+\varepsilon},$$

and

$$\begin{aligned} |L(1/2+it,\pi)|^2 &\ll (\log t)\left(1 + \int_{\mathbb{R}} U\left(\frac{v}{t^{2/3}}\right) |L(1/2+it+iv,\pi)|^2 \, dv\right) \\ &\ll_{\pi,\varepsilon} t^{5/4+\varepsilon}. \end{aligned}$$

## 3. Preliminaries on automorphic forms

Let $\pi$ be a Maass form for $\mathrm{SL}(3,\mathbb{Z})$, which is an eigenfunction for all the Hecke operators. Let the Fourier coefficients be $\lambda(n_1,n_2)$, normalized so that $\lambda(1,1) = 1$. The Langlands parameter $(\alpha_1,\alpha_2,\alpha_3)$ associated with $\pi$ satisfies $\alpha_1+\alpha_2+\alpha_3 = 0$. The dual cusp form $\tilde{\pi}$ has Langlands parameters $(-\alpha_3,-\alpha_2,-\alpha_1)$. The $L$-function $L(s,\pi)$ satisfies a functional equation

$$\gamma(s,\pi)L(s,\pi) = \gamma(1-s,\pi)L(1-s,\tilde{\pi}),$$

where $\gamma(s,\pi)$ and $\gamma(s,\tilde{\pi})$ are the associated gamma factors. We refer the reader to Goldfeld's book on automorphic forms for $\mathrm{GL}(n,\mathbb{Z})$ [11] for the theory of automorphic forms on higher rank groups.

### *3.1. Approximate functional equation and Voronoi summation formula*

We are interested in bounding $L(s,\pi)$ on the critical line $\mathrm{Re}(s) = 1/2$. For that, we approximate $L(1/2+it,\pi)$ by a smoothed sum of length $t^{3/2+\varepsilon}$. This is known as the approximate functional equation and is proved by applying the Mellin transformation to $f$ followed by using the above functional equation.

**Lemma 3.1** ([19, Theorem 5.3]). *Let $G(u)$ be an even, holomorphic function bounded in the strip $-4 \le \mathrm{Re}(u) \le 4$ and normalized by $G(0) = 1$. Then for $s$ in the strip $0 \le \sigma \le 1$,*

$$L(s,\pi) = \sum_{n\ge 1} \lambda(1,n) n^{-s} V_s(n) + \frac{\gamma(1-s,\tilde{\pi})}{\gamma(s,\pi)} \sum_{n\ge 1} \overline{\lambda(1,n)} n^{-(1-s)} V_{1-s}(n),$$

*where*

$$V_s(y) = \frac{1}{2\pi i}\int_{(3)} y^{-u} G(u) \frac{\gamma(s+u,\tilde{\pi})}{\gamma(s,\pi)} \frac{du}{u},$$

*and $\gamma(s,\pi)$ is a product of certain $\Gamma$-functions appearing in the functional equation of $L(s,\pi)$.*

**Remark 3.2.** On the critical line, Stirling's approximation to $\gamma(s,\pi)$ followed by integration by parts to the integral representation of $V_{1/2\pm it}(n)$ gives arbitrary savings for $n \gg (1+|t|)^{3/2+\varepsilon}$.

One of the main tools in our proof is a Voronoi type formula for $\mathrm{GL}(3,\mathbb{Z})$. Let $h$ be a compactly supported smooth function on $(0,\infty)$, and let $\tilde{h}(s) = \int_0^\infty h(x)x^{s-1}\,dx$ be its Mellin transform. For $\sigma > -1 + \max\{-\mathrm{Re}(\alpha_1), -\mathrm{Re}(\alpha_2), -\mathrm{Re}(\alpha_3)\}$ and $a = 0, 1$, define

$$\gamma_a(s) = \frac{\pi^{-3s-3/2}}{2} \prod_{i=1}^{3} \frac{\Gamma\left(\frac{1+s+\alpha_i+a}{2}\right)}{\Gamma\left(\frac{-s-\alpha_i+a}{2}\right)}.$$

Further set $\gamma_\pm(s) = \gamma_0(s) \mp i\gamma_1(s)$ and let

$$H_\pm(y) = \frac{1}{2\pi i}\int_{(\sigma)} y^{-s}\gamma_\pm(s)\tilde{h}(-s)\,ds. \tag{3.1}$$

We need the following Voronoi type formula (see [5, 24, 29]).

**Lemma 3.3.** *Let $h$ be a compactly supported smooth function on $(0,\infty)$. Then*

$$\sum_{n=1}^{\infty} \lambda(1,n) e(an/q) h(n) = q \sum_{\pm} \sum_{n_0 | q} \sum_{n=1}^{\infty} \frac{\lambda(n,n_0)}{n n_0} S(\overline{a}, \pm n; q/n_0) H_\pm(n_0^2 n/q^3),$$

*where $(a,q) = 1$ and $a\overline{a} \equiv 1 \bmod q$.*

Stirling's approximation of $\gamma_\pm(s)$ gives $\gamma_\pm(\sigma+i\tau) \ll 1 + |\tau|^{3\sigma+3/2}$. Moreover, on $\mathrm{Re}(s) = -1/2$,

$$\gamma_\pm(-1/2+i\tau) = (|\tau|/(e\pi))^{3i\tau}\Phi_\pm(\tau), \qquad \text{where } \Phi_\pm'(\tau) \ll |\tau|^{-1}.$$

**Lemma 3.4** ([5, Lemma 6]). *Let $h$ be a compactly supported smooth function on $[a,b] \subset (0,\infty)$ and $H_\pm$ be defined as in (3.1). Then there exist constants $\gamma_\ell$ depending only on the Langlands parameters $(\alpha_1,\alpha_2,\alpha_3)$ such that*

$$H_\pm(x) = x\int_0^\infty h(y) \sum_{\ell=1}^{L} \frac{\gamma_\ell}{(xy)^{\ell/3}} e(\pm 3(xy)^{1/3})\,dy + R(x),$$

*where*

$$x^k \frac{d^k}{dx^k} R(x) \ll \|h\|_\infty x^{1+(k-L)/3}.$$

*The implied constant above depends on* $a, b, \alpha_1, \alpha_2, \alpha_3, k$ *and* $L$.

### 3.2. *Short second moment average*

We bound the square of the $L$-function by a second moment average over a short interval by following Good's arguments [12].

**Lemma 3.5.** *Let* $T/2 \le t \le T$. *Then*

$$|L(1/2+it,\pi)|^2 \ll (\log t)\left(1 + \int_{-\log T}^{\log T} |L(1/2+it+iv,\pi)|^2 e^{-v^2/2}\, dv\right).$$

*Proof.* Let $c = 1/\log t$ for $t \ge 10$. By the residue theorem,

$$\begin{aligned} &L^2(1/2+it,\pi) \\ &= \frac{1}{2\pi i}\int_{(c)} L^2(1/2+it+s,\pi)\frac{e^{s^2}}{s}\, ds - \frac{1}{2\pi i}\int_{(-c)} L^2(1/2+it+s,\pi)\frac{e^{s^2}}{s}\, ds. \end{aligned}$$

The functional equation for $L(s,\pi)$ and Stirling's approximation give

$$L(1/2-c+i(t+v)) \ll (1+|t+v|^{3c})|L(1/2+c+i(t+v))|.$$

Therefore

$$|L(1/2+it,\pi)|^2 \ll \int_{-\infty}^{\infty} |L(1/2+c+i(t+v))|^2(1+|t+v|^{6c})\frac{e^{-v^2}}{(v^2+c^2)^{1/2}}\, dv. \tag{3.2}$$

Similarly, for $1/2 < \sigma < 1$,

$$\begin{aligned} &L^2(\sigma+i\tau_1,\pi) \\ &\quad = \frac{1}{2\pi i}\left(\int_{(1)} L^2(\sigma+i\tau_1+s,\pi)\frac{e^{s^2}}{s}\, ds - \int_{(1/2-\sigma)} L^2(\sigma+i\tau_1+s,\pi)\frac{e^{s^2}}{s}\, ds\right) \\ &\quad \ll 1 + \int_{-\infty}^{\infty} |L(1/2+i(\tau_1+\tau_2),\pi)|^2 \frac{e^{-\tau_2^2}}{(\tau_2^2+(1/2-\sigma)^2)^{1/2}}\, d\tau_2. \end{aligned}$$

Putting $\sigma = 1/2+c$, $\tau_1 = t+v$, substituting the above estimate into (3.2) and trivially estimating the $\tau_2$-integral, we obtain

$$|L(1/2+it,\pi)|^2 \ll (\log t)\left(1 + \int_{-\infty}^{\infty} |L(1/2+it+iv,\pi)|^2 e^{-v^2/2}\, dv\right).$$

Since $L(1/2+it+iv,\pi)$ is polynomially bounded, the integral can be cut off at $[-\log T, \log T]$ with a negligible error term. ∎

We will also use the Ramanujan bound on average which follows from Rankin–Selberg theory.

**Lemma 3.6** (Ramanujan bound on average). *We have*

$$\sum_{n_1^2 n_2 \le x} \sum |\lambda(n_2, n_1)|^2 \ll_{\pi,\varepsilon} x^{1+\varepsilon}.$$

## 4. Preliminary lemmas

**Lemma 4.1** (DFI delta method). *Let $Q > 1$. Then*

$$\delta(n = 0) = \frac{1}{Q} \sum_{1 \le q \le Q} \frac{1}{q} \sideset{}{^\star}\sum_{a \bmod q} e\left(\frac{an}{q}\right) \int_{\mathbb{R}} g(q, x) e\left(\frac{nx}{qQ}\right) dx,$$

*where $g(q, x)$ is a smooth function satisfying*

$$g(q, x) = 1 + O_A\left(\frac{Q}{q}\left(\frac{q}{Q} + |x|\right)^A\right), \quad g(q, x) \ll |x|^{-A}, \quad \text{for any } A \ge 1, \tag{4.1}$$

*and*

$$\frac{\partial^j}{\partial x^j} g(q, x) \ll |x|^{-j} \min(|x|^{-1}, Q/q) \log Q \quad \text{for } j \ge 1.$$

*Moreover,*

$$g(q, x) \ll Q^{\varepsilon}. \tag{4.2}$$

*Proof.* See [19, Section 20.5], and [16, Lemma 15] for minor corrections in the original estimates of the derivatives of $g(q, x)$. To prove (4.2), we observe that when $q > Q^{1-\varepsilon}$, the first property in (4.1) gives us $g(q, x) \ll Q^{\varepsilon}$ by taking $A = 1$. In the case $q \le Q^{1-\varepsilon}$ and $x > Q^{\varepsilon}$, the second property in (4.1) gives us $g(q, x) \ll Q^{\varepsilon}$ by taking $A = 1$. In the remaining case where $q, x \le Q^{1-\varepsilon}$, the first property gives us $g(q, x) \ll 1$ by taking $A$ sufficiently large. Combining all the cases, we get the result. ∎

Note that the properties $g(q, x)$ stated in Lemma 4.1 give us

$$\begin{cases} g(q, x) = 1 + O(t^{-A}) \text{ for any } A > 0 & \text{if } x < t^{-\varepsilon/10} \text{ and } q < Qt^{-\varepsilon/10}, \\ g(q, x) \text{ is } t^{\varepsilon/10}\text{-inert with respect to } x & \text{otherwise.} \end{cases} \tag{4.3}$$

We will also use the duality principle (see [19, Chapter 7]).

**Lemma 4.2** (Duality principle). *Let $\phi : \mathbb{Z}^2 \to \mathbb{C}$. For any complex numbers $a_m$,*

$$\sum_n \left|\sum_m a_m \phi(m, n)\right|^2 \ll \left(\sum_m |a_m|^2\right) \sup_{\|\beta\|_2 = 1} \sum_m \left|\sum_n \beta(n) \phi(m, n)\right|^2,$$

*where the supremum is taken over all sequences of complex numbers $\beta(n)$ such that*

$$\|\beta\|_2 = \sqrt{\sum_n |\beta(n)|^2} = 1.$$

## 5. Stationary phase analysis

We need to use stationary phase analysis for oscillatory integrals. Let $\mathcal{I}$ be an integral of the form

$$\mathcal{I} = \int_a^b w(t)e^{i\phi(t)}\,dt, \tag{5.1}$$

where $w$ and $\phi$ are real valued smooth functions on $\mathbb{R}$. The fundamental estimate for integrals of the form (5.1) is the $r$th-derivative test

$$\mathcal{I} \ll \Big(\operatorname*{Var}_{[a,b]} w(t)\Big) \Big/ \Big(\min_{[a,b]} |\phi^{(r)}(t)|^{1/r}\Big).$$

We will however need sharper estimates and will use the stationary phase analysis of Blomer–Khan–Young [7] to analyze $\mathcal{I}$ and state the estimates in the language of inert functions as given by Kiral–Petrow–Young [22].

**Definition 5.1.** Let $\mathcal{F}$ be an indexing set and $X : \mathcal{F} \to [1,\infty)$ be a function of $T \in \mathcal{F}$, so that $X_T \in [1,\infty)$. A family $\{w_T\}_{T\in\mathcal{F}}$ of smooth functions supported on a product of dyadic intervals in $\mathbb{R}^d_{>0}$ is called $X$-*inert* if for each $j = (j_1,\ldots,j_d) \in \mathbb{Z}^d_{\geq 0}$, we have

$$\sup_{T\in\mathcal{F}} \sup_{(x_1,\ldots,x_d)\in\mathbb{R}^d_{>0}} X_T^{-(j_1+\cdots+j_d)} |x_1^{j_1}\ldots x_d^{j_d}\, w_T^{(j_1,\ldots,j_d)}(x_1,\ldots,x_d)| \ll_{j_1,\ldots j_d} 1.$$

**Lemma 5.2.** *Suppose that $w = w_T(t)$ is a family of $X$-inert functions with compact support in $[Z,2Z]$ such that $w^{(j)}(t) \ll (X/Z)^j$. Also suppose that $\phi$ is smooth and satisfies $\phi^{(j)}(t) \ll Y/Z^j$ for some $Y/X^2 \geq R \geq 1$ for all $t$ in the support of $w$. Let*

$$\mathcal{I} = \int_{-\infty}^{\infty} w(t)e^{i\phi(t)}\,dt.$$

(1) *If $|\phi'(t)| \gg Y/Z$ in the support of $w$, then $\mathcal{I} \ll_A ZR^{-A}$ for arbitrarily large $A$. Moreover, the statement also holds under the weaker condition $Y/X \geq R$.*

(2) *If $\phi''(t) \gg Y/Z^2$ in the support of $w$, and there exists some not necessarily unique $t_0 \in \mathbb{R}$ such that $\phi'(t_0) = 0$, then*

$$\mathcal{I} = \frac{e^{i\phi(t_0)}}{\sqrt{\phi''(t_0)}} F_T(t_0) + O_A(ZR^{-A})$$

*for any $A \geq 0$, where $F_T$ is a family of $X$-inert functions (possibly depending on $A$) supported on $t_0 \sim Z$.*

(3) *In the setting of (2), let $U \gg R^{\varepsilon}(\phi''(t_0))^{-1/2}$ and let $W_0 \in C_c^{\infty}([-2,2])$ be such that $W_0(x) = 1$ for $|x| \leq 1$. Then*

$$\mathcal{I} = \int_{-\infty}^{\infty} w(t)W_0\Big(\frac{t-t_0}{U}\Big)e^{i\phi(t)}\,dt + O_A(R^{-A})$$

*for any $A \geq 0$.*

## 6. Main calculations

### *6.1. A short second moment*

We first apply Lemma 3.5 to bound the square of $L(1/2+it,\pi)$ by a short second moment. Let $T/2 \le t \le T$. Then

$$|L(1/2+it,\pi)|^2 \ll (\log t)\bigg(1+\int_{-\log T}^{\log T} |L(1/2+it+iv,\pi)|^2 e^{-v^2/2}\,dv\bigg). \quad (6.1)$$

Let $\varepsilon>0$ and let $\log t < M < t^{1-\varepsilon}$ be a parameter. Let $U \in C_c^\infty(\mathbb{R})$ be a fixed function supported in $[-2,2]$ and satisfying $U(x)=1$ for $-1\le x\le 1$. Then (6.1) gives us

$$|L(1/2+it,\pi)|^2 \ll (\log t)\bigg(1+\int_{\mathbb{R}} U\Big(\frac{v}{M}\Big)|L(1/2+it+iv,\pi)|^2\,dv\bigg).$$

Applying Lemma 3.1, we get

$$\begin{aligned}|L(1/2+it,\pi)|^2 &\ll (\log t)\bigg(1+\int_{\mathbb{R}} U\Big(\frac{v}{M}\Big)|L(1/2+it+iv,\pi)|^2\,dv\bigg)\\ &\ll (\log t)\bigg(1+\sup_{1\le N\ll t^{3/2+\varepsilon}}\frac{|S(N)|}{N}\bigg),\end{aligned}$$

where

$$S(N) := \int_{\mathbb{R}} U\Big(\frac{v}{M}\Big)\bigg|\sum_{n=1}^{\infty}\lambda(1,n)\,n^{-i(t+v)}V\Big(\frac{n}{N}\Big)\bigg|^2\,dv$$

with $V\in C_c^\infty([1,2])$ depending on $t$ and satisfying $V^{(j)}(x)\ll_j 1$ for any $j\ge 0$ (see e.g. [19, Proposition 5.4]). Opening the square yields

$$S(N) = \sum_n \lambda(1,n)V\Big(\frac{n}{N}\Big)\sum_m \overline{\lambda(1,m)V\Big(\frac{m}{N}\Big)}\Big(\frac{m}{n}\Big)^{it}\int_{\mathbb{R}} U\Big(\frac{v}{M}\Big)\Big(\frac{m}{n}\Big)^{iv}\,dv.$$

By repeated integration by parts in the $v$-integral, we obtain arbitrary savings unless

$$|m-n| \ll Nt^{\varepsilon}/M.$$

Hence

$$\begin{aligned}S(N) = M\sum_{|h|\ll Nt^\varepsilon/M}\sum_n \lambda(1,n)\overline{\lambda(1,n+h)}V\Big(\frac{n}{N}\Big)\overline{V\Big(\frac{n+h}{N}\Big)}\Big(\frac{n+h}{n}\Big)^{it}\\ \times\int_{\mathbb{R}} U(v)\Big(\frac{n+h}{n}\Big)^{iMv}\,dv + O(t^{-999}).\end{aligned}$$

For $h=0$, (3.6) applied to the $n$-sum gives us

$$M\sum_n |\lambda(1,n)|^2\bigg|V\Big(\frac{n}{N}\Big)\bigg|^2\int_{\mathbb{R}} U(v)\,dv \ll MNt^{\varepsilon}.$$

Now performing a smooth dyadic subdivision of the $h$-sum for $h \neq 0$ yields

$$S(N) \ll \sup_{H \ll Nt^{\varepsilon}/M} \Bigg| \sum_{\pm} M \sum_{h} \sum_{n} \lambda(1,n)\overline{\lambda(1,n+h)} \\ \times V\left(\frac{n}{N}\right)\overline{V\left(\frac{n+h}{N}\right)}\varphi_{\pm}\left(\frac{h}{H}\right)\left(\frac{n+h}{n}\right)^{it} \int_{\mathbb{R}} U(v)\left(\frac{n+h}{n}\right)^{iMv} dv \Bigg| + MNt^{\varepsilon},$$

where $\varphi_{\pm}(y) = \varphi(\pm y)$ for some fixed function $\varphi \in C_c^{\infty}([1/2,2])$. Finally, we rewrite

$$\left(\frac{n+h}{n}\right)^{it} = e\left(\frac{t}{2\pi}\log\left(1+\frac{h}{n}\right)\right) = e\left(\frac{th}{2\pi n}\right)e\left(\frac{t}{2\pi}\left(\log\left(1+\frac{h}{n}\right)-\frac{h}{n}\right)\right).$$

Notice that when multiplied by the weight function $V(n/N)\varphi_{\pm}(h/H)$, the integral $\int_{\mathbb{R}} U(v)(\frac{n+h}{n})^{iMv}dv$ is $t^{\varepsilon}$-inert as a function of $n$ and $h$, while $e(\frac{t}{2\pi}(\log(1+\frac{h}{n})-\frac{h}{n}))$ is $t^{\varepsilon}$-inert as a function of $n$ and $h$ when $M \gg t^{1/2+\varepsilon}$. For brevity of notation, we define $W_{\pm}(x,y)$ as

$$V(x)\overline{V\left(x+\frac{Hy}{N}\right)}\varphi_{\pm}(y)e\left(\frac{t}{2\pi}\left(\log\left(1+\frac{Hy}{Nx}\right)-\frac{Hy}{Nx}\right)\right)\int_{\mathbb{R}} U(v)\left(\frac{Nx+Hy}{Nx}\right)^{iMv} dv.$$

As a result, we obtain the following lemma.

**Lemma 6.1.** *Let $\varepsilon, A > 0$ and let $t^{1/2+\varepsilon} \ll M \ll t^{1-\varepsilon}$. Then there exist $t^{\varepsilon}$-inert functions $W_{\pm} \in C_c^{\infty}([1,2]\times[\pm 1,\pm 2])$ such that*

$$|L(1/2+it,\pi)|^2 \ll (\log t)\left(1+\int_{\mathbb{R}} U\left(\frac{v}{M}\right)|L(1/2+it+iv,\pi)|^2\, dv\right) \\ \ll (\log t)\left(Mt^{\varepsilon} + \sup_{H \ll Nt^{\varepsilon}/M}\ \sup_{1 \leq N \ll t^{3/2+\varepsilon}} \sum_{\pm} \frac{|S^{\pm}_{M,H}(N)|}{N}\right),$$

*where*

$$S^{\pm}_{M,H}(N) := M\sum_{h}\sum_{n} \lambda(1,n)\overline{\lambda(1,n+h)}W_{\pm}\left(\frac{n}{N},\frac{h}{H}\right)e\left(\frac{th}{2\pi n}\right).$$

### 6.2. *Trivial bound*

Applying the Cauchy–Schwarz inequality together with Lemma 3.6, we get the trivial bound

$$S^{\pm}_{M,H}(N) \ll M \sum_{|h| \ll H}\left(\sum_{n \ll N} |\lambda(1,n)|^2\right)^{1/2}\left(\sum_{n \ll N} |\lambda(1,n+h)|^2\right)^{1/2} \ll MHNt^{\varepsilon}. \tag{6.2}$$

We will apply this bound for the cases with $H \ll N/t^{1-\varepsilon}$. Hence we will restrict our attention to $H \gg N/t^{1-\varepsilon}$ in the rest of the paper.

*6.3. Application of the delta method*

Now we focus on the case $H \gg N/t^{1-\varepsilon}$. To bound $S^{\pm}_{M,H}(N)$, we start by applying the DFI delta method (Lemma 4.1) to separate the oscillations. Let $U \in C_c^\infty([1/2, 5/2])$ be fixed such that $U(x) = 1$ for $3/4 \le x \le 9/4$. Then

$$\begin{aligned}
&S^{\pm}_{M,H}(N)\\
&\quad= M\sum_h\sum_n \lambda(1,n)W_{\pm}\Big(\frac{n}{N},\frac{h}{H}\Big)e\Big(\frac{th}{2\pi n}\Big)\sum_m \overline{\lambda(1,m)}U\Big(\frac{m}{N}\Big)\delta(m=n+h)\\
&\quad= \frac{M}{Q}\sum_{1\le q\le Q}\frac{1}{q}\sum_h\sum_n \lambda(1,n)W_{\pm}\Big(\frac{n}{N},\frac{h}{H}\Big)e\Big(\frac{th}{2\pi n}\Big)\sum_m \overline{\lambda(1,m)}U\Big(\frac{m}{N}\Big)\\
&\qquad\times \sideset{}{^\star}\sum_{\alpha \bmod q} e\Big(\frac{\alpha(n+h-m)}{q}\Big)\int_{\mathbb{R}} g(q,x)e\Big(\frac{(n+h-m)x}{qQ}\Big)\,dx.
\end{aligned}$$

We divide the $q$-sum into segments $C \le q < (1+10^{-10})C$, denoted $q \sim_0 C$, to obtain

$$S^{\pm}_{M,H}(N) \ll t^{\varepsilon}\sup_{1\le C\ll Q} S^{\pm}_{M,H,C}(N), \tag{6.3}$$

where

$$\begin{aligned}
S^{\pm}_{M,H,C}(N) = \frac{M}{Q}\sum_{q\sim_0 C}\frac{1}{q}\sum_h\sum_n \lambda(1,n)W_{\pm}\Big(\frac{n}{N},\frac{h}{H}\Big)e\Big(\frac{th}{2\pi n}\Big)\sum_m \overline{\lambda(1,m)}\\
\times U\Big(\frac{m}{N}\Big)\sideset{}{^\star}\sum_{\alpha \bmod q} e\Big(\frac{\alpha(n+h-m)}{q}\Big)\int_{\mathbb{R}} g(q,x)e\Big(\frac{(n+h-m)x}{qQ}\Big)\,dx.
\end{aligned}$$

*6.4. Dual summation formulas*

We shall now apply dual summation formulas to the $h$-, $m$- and $n$-sums.

*6.4.1. Analysis of the $h$-sum.* We start with an application of the Poisson summation formula to the $h$-sum mod $q$, giving us

$$\begin{aligned}
&\sum_h W_{\pm}\Big(\frac{n}{N},\frac{h}{H}\Big)e\Big(\frac{\alpha h}{q}+\frac{th}{2\pi n}+\frac{hx}{qQ}\Big)\\
&\quad= \sum_h \frac{1}{q}\sum_{\gamma \bmod q} e\Big(\frac{(\alpha+h)\gamma}{q}\Big)\int_{\mathbb{R}} W_{\pm}\Big(\frac{n}{N},\frac{y}{H}\Big)e\Big(\frac{ty}{2\pi n}+\frac{xy}{qQ}-\frac{hy}{q}\Big)\,dy\\
&\quad= H\sum_h \delta(\alpha \equiv -h \bmod q)\int_{\mathbb{R}} W_{\pm}\Big(\frac{n}{N},y\Big)e\Big(\frac{tHy}{2\pi n}+\frac{Hxy}{qQ}-\frac{hHy}{q}\Big)\,dy.
\end{aligned}$$

The above congruence condition and $(\alpha, q) = 1$ imply $(h, q) = 1$, so that when $h = 0$, the terms with $q > 1$ vanish. Since $H \gg N/t^{1-\varepsilon}$, choosing $Q \gg N/t^{1-\varepsilon}$ and repeated integration by parts gives us arbitrary savings unless

$$h \neq 0 \quad \text{and} \quad h \sim qt/N \sim Ct/N.$$

This gives arbitrary savings unless $C \gg N/t^{1+\varepsilon}$ and shows that $S^{\pm}_{M,H,C}(N)$ equals

$$\frac{MH}{Q} \sum_{\substack{H' \sim Ct/N \\ \text{dyadic}}} \sum_{q \sim_0 C} \frac{1}{q} \sum_{\substack{h \sim_0 H' \\ (h,q)=1}} \sum_n \lambda(1,n) \sum_m \overline{\lambda(1,m)} U\left(\frac{m}{N}\right) e\left(\frac{h(m-n)}{q}\right)$$
$$\times \int_{\mathbb{R}} \int_{\mathbb{R}} g(q,x) W_{\pm}\left(\frac{n}{N}, y\right) e\left(\frac{(n+Hy-m)x}{qQ} + \frac{tHy}{2\pi n} - \frac{hHy}{q}\right) dx\, dy + O(t^{-999}).$$

*6.4.2. Analysis of the m-sum.* Next we apply the Voronoi summation formula (Lemma 3.3) to the $m$-sum,

$$\sum_m \overline{\lambda(1,m)} U\left(\frac{m}{N}\right) e\left(\frac{hm}{q} - \frac{mx}{qQ}\right)$$
$$= q \sum_{\eta_1 = \pm} \sum_{m_0 | q} \sum_m \frac{\overline{\lambda(m,m_0)}}{m_0 m} S\left(\overline{h}, \eta_1 m; \frac{q}{m_0}\right) \tilde{U}_{\eta_1}\left(\frac{m_0^2 m}{q^3}, q, x\right) + O(t^{-999}),$$

where

$$\tilde{U}_{\eta_1}(a,b,c) = a^{2/3} \int_0^\infty U\left(\frac{w}{N}\right) \overline{U}_0(a,w) w^{-1/3} e\left(-\frac{cw}{bQ} + \eta_1 3(aw)^{1/3}\right) dw,$$

and

$$U_0(u,w) = \sum_{\ell=1}^{L} \frac{\gamma_\ell}{(uw)^{\frac{\ell-1}{3}}} \tag{6.4}$$

for a fixed large $L$, and $\gamma_\ell$ as given in Lemma 3.4. Note that $U_0(u,w)$ is a fixed flat function. By a change of variables, we obtain

$$\tilde{U}_{\eta_1}\left(\frac{m_0^2 m}{q^3}, q, x\right) = \left(\frac{m_0^2 m N}{q^3}\right)^{2/3} \int_0^\infty U(w) \overline{U}_0\left(\frac{m_0^2 m}{q^3}, Nw\right) w^{-1/3} e(f_1(w,x))\, dw,$$

where the phase function is

$$f_1(w,x) = \eta_1 3 \left(\frac{m_0^2 m N w}{q^3}\right)^{1/3} - \frac{Nwx}{qQ}.$$

Repeated integration by parts gives us arbitrary savings unless

$$m_0^2 m \ll N^2 t^{\varepsilon} / Q^3.$$

Writing

$$U_{\eta_1}\left(\frac{m_0^2 m}{q^3}, q, x\right) = \int_0^\infty U(w) \overline{U}_0\left(\frac{m_0^2 m}{q^3}, Nw\right) w^{-1/3} e(f_1(w,x))\, dw \tag{6.5}$$

we obtain

$$S^{\pm}_{M,H,C}(N)=\frac{MHN^{2/3}t^{\varepsilon}}{Q}\sum_{\eta_1=\pm}\sup_{H'\sim Ct/N}\sum_{q\sim_0 C}\frac{1}{q^2}\sum_{\substack{h\sim_0 H'\\(h,q)=1}}\sum_{n}\lambda(1,n)$$
$$\times\sum_{m_0|q}\sum_{m_0^2m\ll N^2t^{\varepsilon}/Q^3}\overline{\lambda(m,m_0)}\left(\frac{m_0}{m}\right)^{1/3}S\left(\bar{h},\eta_1 m;\frac{q}{m_0}\right)e\left(\frac{-hn}{q}\right)$$
$$\times\int_{\mathbb{R}}\int_{\mathbb{R}}g(q,x)W_{\pm}\left(\frac{n}{N},y\right)U_{\eta_1}\left(\frac{m_0^2m}{q^3},q,x\right)$$
$$\times e\left(\frac{(n+Hy)x}{qQ}+\frac{tHy}{2\pi n}-\frac{hHy}{q}\right)dx\,dy+O(t^{-99}).$$

*6.4.3. Analysis of the $n$-sum.* Finally, we apply Voronoi formula (Lemma 3.3) to the $n$-sum. Similar analysis to the one above yields

$$\sum_{n}\lambda(1,n)W_{\pm}\left(\frac{n}{N},y\right)e\left(\frac{-hn}{q}+\frac{nx}{qQ}+\frac{tHy}{2\pi n}\right)$$
$$=\frac{N^{2/3}}{q}\sum_{\eta_2=\pm}\sum_{n_0|q}\sum_{n}\lambda(n,n_0)\left(\frac{n_0}{n}\right)^{1/3}S\left(-\bar{h},\eta_2 n;\frac{q}{n_0}\right)W^{\pm}_{0,-\eta_2}(n_0^2n,q,x,y)$$
$$+O(t^{-999}),$$

where

$$W^{\pm}_{0,\eta_2}(n_0^2n,q,x,y)=\int_0^{\infty}W_{\pm}(w,y)U_0\left(\frac{n_0^2n}{q^3},Nw\right)w^{-1/3}$$
$$\times e\left(\frac{Nwx}{qQ}+\frac{tHy}{2\pi Nw}-\eta_2 3\left(\frac{n_0^2nNw}{q^3}\right)^{1/3}\right)dw,\qquad(6.6)$$

where $U_0$ is given in (6.4). Since $H\gg N/t^{1-\varepsilon}$, repeated integration by parts gives us arbitrary savings unless

$$n_0^2n\ll\left(\frac{N^2}{Q^3}+\frac{(CHt)^3}{N^4}\right)t^{\varepsilon}.$$

By choosing $Q^2\ll NM/t^{1-\varepsilon}$, the first term on the right side dominates, so that $n_0^2n\ll N^2t^{\varepsilon}/Q^3$. Changing $\eta_2$ to $-\eta_2$ for ease of notation shows that $S^{\pm}_{M,H,C}(N)$ equals

$$\frac{MHN^{4/3}t^{\varepsilon}}{Q}\sum_{\eta_1,\eta_2=\pm}\sup_{H'\sim Ct/N}\sum_{q\sim_0 C}\frac{1}{q^3}\sum_{\substack{h\sim_0 H'\\(h,q)=1}}\sum_{n_0|q}\sum_{n_0^2n\ll N^2t^{\varepsilon}/Q^3}\lambda(n,n_0)\left(\frac{n_0}{n}\right)^{1/3}$$
$$\times\sum_{m_0|q}\sum_{m_0^2m\ll N^2t^{\varepsilon}/Q^3}\overline{\lambda(m,m_0)}\left(\frac{m_0}{m}\right)^{1/3}S\left(\bar{h},\eta_1 m;\frac{q}{m_0}\right)S\left(\bar{h},\eta_2 n;\frac{q}{n_0}\right)$$
$$\times\int_{\mathbb{R}}\int_{\mathbb{R}}g(q,x)W^{\pm}_{0,\eta_2}(n_0^2n,q,x,y)U_{\eta_1}\left(\frac{m_0^2m}{q^3},q,x\right)e\left(\frac{Hxy}{qQ}-\frac{hHy}{q}\right)dy\,dx$$
$$+O(t^{-99}).\qquad(6.7)$$

*6.4.4. Analysis of integral transforms obtained by dual summations.* We now consider the $y, w$-integral. Then (6.6) and a change of variable give us

$$\begin{aligned}
&\int_{\mathbb{R}} W_{0,\eta_2}^{\pm}(n_0^2 n, q, x, y) e\left(\frac{Hxy}{qQ} - \frac{hHy}{q}\right) dy \\
&\quad = \int_0^\infty U_0\left(\frac{n_0^2 n}{q^3}, Nw\right) w^{-1/3} e\left(\frac{Nwx}{qQ} - \eta_2 3\left(\frac{n_0^2 nNw}{q^3}\right)^{1/3}\right) \\
&\qquad \times \int_{\mathbb{R}} W_{\pm}(w, y) e\left(\frac{tHy}{2\pi Nw} + \frac{Hxy}{qQ} - \frac{hHy}{q}\right) dy\, dw \\
&\quad = e\left(\frac{tx}{2\pi hQ}\right) \int_{\mathbb{R}} U_0\left(\frac{n_0^2 n}{q^3}, \frac{qt}{2\pi h} + Nu\right)\left(\frac{qt}{2\pi hN} + u\right)^{-1/3} \\
&\qquad \times e\left(\frac{Nxu}{qQ} - \eta_2 3\left(\frac{n_0^2 nN}{q^3}\left(\frac{qt}{2\pi hN} + u\right)\right)^{1/3}\right) \\
&\qquad \times \int_{\mathbb{R}} W_{\pm}\left(\frac{qt}{2\pi hN} + u, y\right) e\left(\frac{tHy}{2\pi N}\left(\frac{qt}{2\pi hN} + u\right)^{-1} + \frac{Hxy}{qQ} - \frac{hHy}{q}\right) dy\, du.
\end{aligned}$$

Since $C \gg N/t^{1+\varepsilon}$ and $H \ll Nt^{\varepsilon}/M$, choosing $MQ > t^{1+\varepsilon}$ gives us $\frac{Hxy}{qQ} \ll \frac{t^{1+\varepsilon}}{MQ} \ll 1$. Hence repeated integration by parts in the $y$-integral gives us arbitrary savings unless

$$\begin{aligned}
\left|\frac{tH}{2\pi N}\left(\frac{qt}{2\pi hN} + u\right)^{-1} - \frac{hH}{q}\right| \ll t^{\varepsilon/2} &\iff \left|\frac{1}{1 + \frac{2\pi hNu}{qt}} - 1\right| \ll \frac{qt^{\varepsilon}}{hH} \ll \frac{N}{Ht^{1-\varepsilon/2}} \\
&\iff |u| \ll \frac{N}{Ht^{1-\varepsilon/2}}.
\end{aligned}$$

Take a fixed function $U \in C_c^\infty([-2, -2])$ such that $U(x) = 1$ for $-1 \le x \le 1$, and let

$$\begin{aligned}
\phi_1(a, b, c, d) = \left(\frac{bt}{2\pi cN} + \frac{Na}{Ht^{1-\varepsilon}}\right)^{-1/3} &\int_{\mathbb{R}} W_{\pm}\left(\frac{bt}{2\pi cN} + \frac{Na}{Ht^{1-\varepsilon}}, y\right) \\
&\times e\left(\frac{tHy}{2\pi N}\left(\frac{bt}{2\pi cN} + \frac{Na}{Ht^{1-\varepsilon}}\right)^{-1} + \frac{dHy}{bQ} - \frac{cHy}{b}\right) dy.
\end{aligned}$$

Then $\phi_1$ is $t^{\varepsilon}$-inert and the $y, w$-integral becomes

$$\begin{aligned}
&e\left(\frac{tx}{2\pi hQ}\right) \int_{\mathbb{R}} U\left(\frac{uHt^{1-\varepsilon}}{N}\right) \phi_1\left(\frac{uHt^{1-\varepsilon}}{N}, q, h, x\right) U_0\left(\frac{n_0^2 n}{q^3}, \frac{qt}{2\pi h} + Nu\right) \\
&\qquad \times e\left(\frac{Nxu}{qQ} - \eta_2 3\left(\frac{n_0^2 nN}{q^3}\left(\frac{qt}{2\pi hN} + u\right)\right)^{1/3}\right) du + O(t^{-999}) \\
&\quad = \frac{N}{Ht^{1-\varepsilon}} e\left(\frac{tx}{2\pi hQ}\right) \int_{\mathbb{R}} U(u)\phi_1(u, q, h, x) U_0\left(\frac{n_0^2 n}{q^3}, \frac{qt}{2\pi h} + \frac{N^2 u}{Ht^{1-\varepsilon}}\right) \\
&\qquad \times e\left(\frac{N^2 xu}{qQHt^{1-\varepsilon}} - \eta_2 3\left(\frac{n_0^2 nN}{q^3}\left(\frac{qt}{2\pi hN} + \frac{Nu}{Ht^{1-\varepsilon}}\right)\right)^{1/3}\right) du + O(t^{-999}).
\end{aligned}$$

Defining

$$W_{\eta_2}^{\pm}(n_0^2 n, q, h, u) := U_0\left(\frac{n_0^2 n}{q^3}, \frac{qt}{2\pi h} + \frac{N^2 u}{Ht^{1-\varepsilon}}\right) \times e\left(-\eta_2 3\left(\frac{n_0^2 nN}{q^3}\left(\frac{qt}{2\pi hN} + \frac{Nu}{Ht^{1-\varepsilon}}\right)\right)^{1/3}\right), \tag{6.8}$$

we find that $S^{\pm}_{M,H,C}(N)$ equals

$$\frac{MN^{7/3}t^{\varepsilon}}{Qt^{1-\varepsilon}} \times \sum_{\eta_1,\eta_2=\pm} \sup_{H'\sim Ct/N} \sum_{q\sim_0 C} \frac{1}{q^3} \sum_{\substack{h\sim_0 H'\\(h,q)=1}} \sum_{n_0|q} \sum_{n_0^2 n \ll (N^2/Q^3+(CHt)^3/N^4)t^{\varepsilon}} \lambda(n,n_0)\left(\frac{n_0}{n}\right)^{1/3}$$
$$\times \sum_{m_0|q} \sum_{m_0^2 m\ll N^2t^{\varepsilon}/Q^3} \overline{\lambda(m,m_0)}\left(\frac{m_0}{m}\right)^{1/3} S\left(\overline{h}, \eta_1 m; \frac{q}{m_0}\right) S\left(\overline{h}, \eta_2 n; \frac{q}{n_0}\right)$$
$$\times \int_{\mathbb{R}} U(u) W_{\eta_2}^{\pm}(n_0^2 n, q, h, u) \int_{\mathbb{R}} g(q,x)\phi_1(u,q,h,x)$$
$$\times U_{\eta_1}\left(\frac{m_0^2 m}{q^3}, q, x\right) e\left(\frac{N^2 xu}{qQHt^{1-\varepsilon}} + \frac{tx}{2\pi hQ}\right) dx\, du + O(t^{-99}).$$

Applying smooth dyadic subdivisions to the $m_0^2 m, n_0^2 n$-sums, we get

$$S^{\pm}_{M,H,C}(N) \ll t^{\varepsilon} \sup_{\eta_1,\eta_2=\pm} \sup_{N_1\ll N^2t^{\varepsilon}/Q^3} \sup_{N_2\ll N^2t^{\varepsilon}/Q^3} \sup_{H'\sim Ct/N} \frac{MN^{7/3}}{Qt}$$
$$\times \sum_{q\sim_0 C} \frac{1}{q^3} \sum_{\substack{h\sim_0 H'\\(h,q)=1}} \sum_{m_0|q} \sum_{m} \overline{\lambda(m,m_0)}\left(\frac{m_0}{m}\right)^{1/3} \varphi\left(\frac{m_0^2 m}{N_1}\right) \sum_{n_0|q} \sum_{n} \lambda(n,n_0)$$
$$\times \left(\frac{n_0}{n}\right)^{1/3} \varphi\left(\frac{n_0^2 n}{N_2}\right) S\left(\overline{h}, \eta_1 m; \frac{q}{m_0}\right) S\left(\overline{h}, \eta_2 n; \frac{q}{n_0}\right) \int_{\mathbb{R}} U(u) W_{\eta_2}^{\pm}(n_0^2 n, q, h, u)$$
$$\times \int_{\mathbb{R}} g(q,x)\phi_1(u,q,h,x) U_{\eta_1}\left(\frac{m_0^2 m}{q^3}, q, x\right) e\left(\frac{N^2 xu}{qQHt^{1-\varepsilon}} + \frac{tx}{2\pi hQ}\right) dx\, du + t^{-99}$$

for some fixed $\varphi \in C_c^{\infty}([1/2, 2])$.

*6.4.5. $x$-integral.* We get arbitrary savings for $|x| \gg t^{\varepsilon}$ due to bounds on $g(q,x)$ as given in (4.1). So we can restrict to $|x| \ll t^{\varepsilon}$. Moreover, we expect to obtain cancellations in the $x$-integral when $|x| \gg CQt^{\varepsilon}/N$. Therefore we use a smooth partition of unity to split the $x$-integral into dyadic segments and a small 'remainder' part of size $CQt^{\varepsilon}/N$.

Let $\varphi : (0,\infty) \to [0,1]$ be a smooth compactly supported function satisfying

$$\operatorname{supp}(\varphi) \subset [1/2, 2] \quad \text{and} \quad \sum_{j\in\mathbb{Z}} \varphi\left(\frac{y}{2^j}\right) = 1 \text{ for } y \in (0,\infty). \tag{6.9}$$

Define $\varphi_0(0) = 1$, and for $y \in \mathbb{R} \setminus \{0\}$,

$$\varphi_0(y) := \sum_{i \geq 1} \varphi\left(\frac{2^i y}{CQt^\varepsilon/N}\right) + \varphi\left(\frac{-2^i y}{CQt^\varepsilon/N}\right),$$

$$\varphi_X(y) := \varphi\left(\frac{y}{X}\right) \quad \text{for } |X| \geq \frac{2^{j-1}CQt^\varepsilon}{N} \quad \text{and} \quad j \geq 1.$$

Then $\varphi_0$ is smooth and supported in $(\frac{-CQt^\varepsilon}{N}, \frac{CQt^\varepsilon}{N})$ with $\varphi_0(y)=1$ for $y \in [\frac{-CQt^\varepsilon}{2N}, \frac{CQt^\varepsilon}{2N}]$.

Therefore

$$\begin{aligned} S^{\pm}_{M,H,C}(N) \ll t^\varepsilon & \sup_{\substack{CQt^\varepsilon/N \leq |X| \leq t^\varepsilon \\ \text{or } X=0}} \sup_{N_1 \ll N^2 t^\varepsilon/Q^3} \sup_{N_2 \ll N^2 t^\varepsilon/Q^3} \sup_{H' \sim Ct/N} \sup_{\eta_1,\eta_2=\pm} MN^{7/3}/Qt \\ & \times \Bigg| \sum_{q \sim_0 C} \frac{1}{q^3} \sum_{\substack{h \sim_0 H' \\ (h,q)=1}} \sum_{m_0 | q} \sum_{m} \overline{\lambda(m,m_0)} \left(\frac{m_0}{m}\right)^{1/3} \varphi\left(\frac{m_0^2 m}{N_1}\right) \sum_{n_0|q} \sum_{n} \lambda(n,n_0) \left(\frac{n_0}{n}\right)^{1/3} \\ & \times \varphi\left(\frac{n_0^2 n}{N_2}\right) S\left(\overline{h}, \eta_1 m; \frac{q}{m_0}\right) S\left(\overline{h}, \eta_2 n; \frac{q}{n_0}\right) \int_{\mathbb{R}} U(u) W^{\pm}_{\eta_2}(n_0^2 n, q, h, u) \\ & \times \int_{\mathbb{R}} \varphi_X(x) g(q,x) \phi_1(u,q,h,x) U_{\eta_1}\left(\frac{m_0^2 m}{q^3}, q, x\right) e\left(\frac{N^2 x u}{qQHt^{1-\varepsilon}} + \frac{tx}{2\pi h Q}\right) dx\, du \Bigg| \\ & + t^{-99}. \end{aligned}$$

### 6.5. *Cauchy–Schwarz inequality and the duality principle*

We now apply the Cauchy–Schwarz inequality to take out everything in (6.7) except the $m, m_0, n, n_0$-sums and the $x$-integral, showing that $S^{\pm}_{M,H,C}(N)$ is

$$O\left(t^\varepsilon \sup_{\eta_1,\eta_2=\pm} \sup_{\substack{\frac{CQt^\varepsilon}{N} \leq |X| \leq t^\varepsilon \\ \text{or } X=0}} \sup_{N_1 \ll \frac{N^2 t^\varepsilon}{Q^3}} \sup_{N_2 \ll \frac{N^2 t^\varepsilon}{Q^3}} \sup_{H' \sim \frac{Ct}{N}} (\tilde{S}_{1,X}(N) \tilde{S}_2(N))^{1/2} + t^{-99}\right),$$

where

$$\begin{aligned} \tilde{S}_{1,X}(N) := \tilde{S}^{\pm,\eta_1}_{1,M,H,H',C,X}(N,N_1) = & \frac{MN^{7/3}}{Qt} \\ & \times \sum_{q \sim_0 C} \frac{1}{q^3} \sum_{\substack{h \sim_0 H' \\ (h,q)=1}} \int_{\mathbb{R}} U(u) \Bigg| \sum_{m_0|q} \sum_{m} \overline{\lambda(m,m_0)} \left(\frac{m_0}{m}\right)^{1/3} \varphi\left(\frac{m_0^2 m}{N_1}\right) S\left(\overline{h}, \eta_1 m; \frac{q}{m_0}\right) \\ & \times \int_{\mathbb{R}} \varphi_X(x) g(q,x) \phi_1(u,q,h,x) U_{\eta_1}\left(\frac{m_0^2 m}{q^3}, q, x\right) e\left(\frac{N^2 x u}{qQHt^{1-\varepsilon}} + \frac{tx}{2\pi h Q}\right) dx \Bigg|^2 du \end{aligned}$$

and

$$\tilde{S}_2(N) := \tilde{S}^{\pm,\eta_2}_{2,M,H,H',C}(N,N_2) = \frac{MN^{7/3}}{Qt} \sum_{q\sim_0 C} \frac{1}{q^3} \sum_{\substack{h\sim_0 H' \\ (h,q)=1}} \int_{\mathbb{R}} U(u) \times \left| \sum_{n_0|q} \sum_{n} \lambda(n,n_0) \left(\frac{n_0}{n}\right)^{1/3} \varphi\left(\frac{n_0^2 n}{N_2}\right) S\left(\overline{h}, \eta_2 n; \frac{q}{n_0}\right) W^{\pm}_{\eta_2}(n_0^2 n, q, h, u)\right|^2 du.$$

By the duality principle (Lemma 4.2) and Lemma 3.6, we get the following theorem.

**Theorem 6.2.** *For any positive numbers $\sqrt{t} \ll M \ll N$, $N/t^{1-\varepsilon} \ll H \ll Nt^{\varepsilon}/M$, $Q \gg N/t^{1-\varepsilon}$, $C \ll Q$, we have*

$$S^{\pm}_{M,H,C}(N) \ll t^{-99}$$

*unless $C \gg N/t^{1+\varepsilon}$. When $C \gg N/t^{1+\varepsilon}$, $S^{\pm}_{M,H,C}(N)$ is*

$$O\left(t^{\varepsilon} \sup_{|u|\ll 1} \sup_{\eta_1,\eta_2=\pm} \sup_{\substack{\frac{CQt^{\varepsilon}}{N} \le |X| \le t^{\varepsilon} \\ \text{or } X=0}} \sup_{N_1 \ll \frac{N^2 t^{\varepsilon}}{Q^3}} \sup_{N_2 \ll \frac{N^2 t^{\varepsilon}}{Q^3}} \sup_{H' \sim \frac{Ct}{N}} (S_{1,X}(N) S_2(N))^{1/2} + t^{-99}\right),$$

*where*

$$S_{1,X}(N) := S^{\pm,\eta_1}_{1,M,H,H',C,X}(N,N_1)$$
$$= \frac{MN_1^{1/3}N^{7/3}}{Qt^{1-\varepsilon}} \sup_{\|\beta\|_2=1} \sum_{m_0} \frac{1}{m_0} \sum_{m} \varphi\left(\frac{m_0^2 m}{N_1}\right) \Bigg| \sum_{q\sim_0 C/m_0} \frac{1}{q^{3/2}} \times \sum_{\substack{h\sim_0 H' \\ (h,qm_0)=1}} S(\overline{h}, \eta_1 m; q) \int_{\mathbb{R}} U(u)\beta(m_0 q, h, u) \int_{\mathbb{R}} \varphi_X(x) g(m_0 q, x) \phi_1(u, m_0 q, h, x) \times U_{\eta_1}\left(\frac{m}{q^3 m_0}, m_0 q, x\right) e\left(\frac{N^2 xu}{m_0 qQHt^{1-\varepsilon}} + \frac{tx}{2\pi hQ}\right) dx\, du\Bigg|^2, \tag{6.10}$$

*and*

$$S_2(N) := S^{\pm,\eta_2}_{2,M,H,H',C,u}(N,N_2)$$
$$= \frac{MN_2^{1/3}N^{7/3}}{Qt^{1-\varepsilon}} \sup_{\|\beta\|_2=1} \sum_{n_0} \frac{1}{n_0} \sum_{n} \varphi\left(\frac{n_0^2 n}{N_2}\right) \Bigg| \sum_{q\sim_0 C/n_0} \frac{1}{q^{3/2}} \sum_{\substack{h\sim_0 H' \\ (h,qn_0)=1}} S(\overline{h}, \eta_2 n; q) \times \int_{\mathbb{R}} U(u)\beta(n_0 q, h, u) W^{\pm}_{\eta_2}(n_0^2 n, n_0 q, h, u)\, du\Bigg|^2, \tag{6.11}$$

*with $U_{\eta_1}$ and $W^{\pm}_{\eta_2}$ as defined in (6.5) and (6.8) respectively.*

## 7. Opening the square and Poisson summation for $S_{1,X}(N)$

In this section we focus on the analysis of $S_{1,X}(N)$. We will then apply similar treatment to $S_2(N)$ in the next section.

### *7.1. Opening the square and Poisson summation*

Opening the square in (6.10), we get

$$\begin{aligned} S_{1,X}(N) = \frac{MN_1^{1/3}N^{7/3}}{Qt^{1-\varepsilon}} & \sup_{\|\beta\|_2=1} \sum_{m_0} \frac{1}{m_0} \sum_m \varphi\left(\frac{m_0^2 m}{N_1}\right) \sum\sum_{q_1,q_2\sim_0 C/m_0} \frac{1}{(q_1q_2)^{3/2}} \\ & \times \sum\sum_{\substack{h_1,h_2\sim_0 H' \\ (h_1,q_1m_0)=1 \\ (h_2,q_2m_0)=1}} S(\overline{h_1},\eta_1 m;q_1)S(\overline{h_2},\eta_1 m;q_2) \\ & \times \int_{\mathbb{R}} U(u_1)\beta(m_0q_1,h_1,u_1)\int_{\mathbb{R}} \varphi_X(x_1)g(m_0q_1,x_1)\phi_1(u_1,m_0q_1,h_1,x_1) \\ & \times U_{\eta_1}\left(\frac{m}{q_1^3 m_0},m_0q_1,x_1\right)e\left(\frac{N^2x_1u_1}{m_0q_1QHt^{1-\varepsilon}}+\frac{tx_1}{2\pi h_1 Q}\right)dx_1du_1 \\ & \times \int_{\mathbb{R}} U(u_2)\overline{\beta(m_0q_2,h_2,u_2)}\int_{\mathbb{R}} \varphi_X(x_2)\,\overline{g(m_0q_2,x_2)\phi_1(u_2,m_0q_2,h_2,x_2)} \\ & \times \overline{U_{\eta_1}\left(\frac{m}{q_2^3 m_0},m_0q_2,x_2\right)}e\left(-\frac{N^2x_2u_2}{m_0q_2QHt^{1-\varepsilon}}-\frac{tx_2}{2\pi h_2 Q}\right)dx_2\,du_2. \end{aligned}$$

Applying Poisson summation to the $m$-sum, we have

$$\begin{aligned} \sum_m \varphi\left(\frac{m_0^2m}{N_1}\right) & S(\overline{h_1},\eta_1 m;q_1)S(\overline{h_2},\eta_1 m;q_2) \\ & \times U_{\eta_1}\left(\frac{m}{q_1^3m_0},m_0q_1,x_1\right)\overline{U_{\eta_1}\left(\frac{m}{q_2^3m_0},m_0q_2,x_2\right)} \\ & = \frac{N_1}{m_0^2q_1q_2}\sum_m \mathcal{C}_{\eta_1}(m,h_1,h_2,q_1q_2)\mathcal{J}_0(m,m_0,q_1,q_2,x_1,x_2), \end{aligned}$$

where

$$\mathcal{C}_{\eta_1}(m,h_1,h_2,q_1q_2) = \sum_{\gamma \bmod q_1q_2} S(\overline{h_1},\eta_1\gamma;q_1)S(\overline{h_2},\eta_1\gamma;q_2)e\left(\frac{m\gamma}{q_1q_2}\right) \tag{7.1}$$

and

$$\begin{aligned} & \mathcal{J}_0(m,m_0,q_1,q_2,x_1,x_2) \\ & = \int_0^\infty \varphi(z)U_{\eta_1}\left(\frac{N_1z}{(m_0q_1)^3},m_0q_1,x_1\right)\overline{U_{\eta_1}\left(\frac{N_1z}{(m_0q_2)^3},m_0q_2,x_2\right)}e\left(-\frac{mN_1z}{m_0^2q_1q_2}\right)dz. \end{aligned} \tag{7.2}$$

Now applying stationary phase analysis with the calculations of Appendix A.1 gives us

$$S_{1,0}(N) \ll \frac{QMN_1^{4/3}N^{1/3}}{Ct^{1-\varepsilon}}\delta(N_1 \ll C^3t^\varepsilon/N) \sup_{\|\beta\|_2=1} \sum_{m_0} \sum_{q_1,q_2 \sim_0 C/m_0}\sum \frac{1}{q_1q_2}$$
$$\times \sum_{\substack{h_1,h_2\sim_0 H' \\ (h_1,q_1m_0)=1 \\ (h_2,q_2m_0)=1}}\sum \int_{|u_1|\ll 1}\int_{|u_2|\ll 1} |\beta(m_0q_1,h_1,u_1)\beta(m_0q_2,h_2,u_2)|\, du_2\, du_1$$
$$\times \sum_{|m|\ll C^2t^\varepsilon/N_1} |\mathscr{C}_{\eta_1}(m,h_1,h_2,q_1q_2)| + t^{-99}, \tag{7.3}$$

and for $X \neq 0$,

$$S_{1,X}(N) \ll \frac{CMN_1N^2}{Qt^{1-\varepsilon}}\delta(\eta_1 X > 0) \sup_{\|\beta\|_2=1} \sum_{m_0} m_0^{-3} \sum_{q_1,q_2\sim_0 C/m_0}\sum (q_1q_2)^{-5/2}$$
$$\times \sum_{\substack{h_1,h_2\sim_0 H' \\ (h_1,q_1m_0)=1 \\ (h_2,q_2m_0)=1}}\sum \sum_{|m|\ll \frac{CQ^2}{NX^2}t^\varepsilon} \mathscr{C}_{\eta_1}(m,h_1,h_2,q_1q_2)$$
$$\times \int_{\mathbb{R}}\int_{\mathbb{R}} U(u_1)U(u_2)\beta(m_0q_1,h_1,u_1)\overline{\beta(m_0q_2,h_2,u_2)}\mathscr{J}(q_1,q_2,\vec{v}\,)\, du_2\, du_1 + t^{-99}, \tag{7.4}$$

where for brevity of notation, we write $\vec{v} = (n, n_0, h_1, h_2, u_1, u_2)$,

$$\mathscr{J}(q_1,q_2,\vec{v}\,) = \int_{\mathbb{R}} \varphi_X(x_1)g(m_0q_1,x_1)\phi_1(u_1,m_0q_1,h_1,x_1)$$
$$\times \int_{\mathbb{R}} \varphi_X(x_2)\overline{g(m_0q_2,x_2)\phi_1(u_2,m_0q_2,h_2,x_2)}$$
$$\times \int_0^\infty \varphi(z)U_3\left(\frac{N_1Q^3z}{N^2|x_1|^3}, m_0q_1, x_1\right)\overline{U_3\left(\frac{N_1Q^3z}{N^2|x_2|^3}, m_0q_2, x_2\right)}$$
$$\times e\left(2\sqrt{\frac{N_1Qz}{m_0^2q_1^2|x_1|}} - 2\sqrt{\frac{N_1Qz}{m_0^2q_2^2|x_2|}} - \frac{mN_1z}{m_0^2q_1q_2}\right) dz$$
$$\times e\left(\frac{N^2x_1u_1}{m_0q_1QHt^{1-\varepsilon}} - \frac{N^2x_2u_2}{m_0q_2QHt^{1-\varepsilon}} + \frac{tx_1}{2\pi h_1Q} - \frac{tx_2}{2\pi h_2Q}\right) dx_2\, dx_1, \tag{7.5}$$

and $U_3$ is some $t^\varepsilon$-inert function supported on $[1/4, 25/4] \times \mathbb{R} \times \mathbb{R}$.

For $X \neq 0$, applying stationary phase analysis to the $x_1, x_2$-integrals and calculations of Appendix A.2 shows that there exists a $t^\varepsilon$-inert function $\Phi$ supported on $[1/4, 25/4] \times \mathbb{R}^4$ such that

$$S_{1,X}(N)=\frac{C^2M\sqrt{N_1}N^2X^{5/2}}{Q^{3/2}t^{1-\varepsilon}}\delta(\eta_1X>0)\sup_{\|\beta\|_2=1}\sum_{m_0}m_0^{-3}\sum\sum_{q_1,q_2\sim_0 C/m_0}(q_1q_2)^{-5/2}$$
$$\times\sum\sum_{\substack{h_1,h_2\sim_0 H'\\(h_1,q_1m_0)=1\\(h_2,q_2m_0)=1}}\sum_{|m|\ll\frac{CQ^2}{NX^2}t^{\varepsilon}}\mathcal{C}_{\eta_1}(m,h_1,h_2,q_1q_2)\int_{\mathbb{R}}\int_{\mathbb{R}}U(u_1)U(u_2)$$
$$\times\beta(m_0q_1,h_1,u_1)\overline{\beta(m_0q_2,h_2,u_2)}\int_0^\infty\varphi(z)\Phi\left(\frac{N_1Q^3z}{N^2X^3},m_0q_1,m_0q_2,u_1,u_2\right)$$
$$\times g(m_0q_1,Xx_{1,0})g(m_0q_2,Xx_{2,0})e\left(3\left(\frac{N_1tz}{2\pi h_1(m_0q_1)^2}\left(1+\frac{2\pi h_1N^2u_1}{m_0q_1Ht^{2-\varepsilon}}\right)\right)^{1/3}\right)$$
$$\times e\left(-3\left(\frac{N_1tz}{2\pi h_2(m_0q_2)^2}\left(1+\frac{2\pi h_2N^2u_2}{m_0q_2Ht^{2-\varepsilon}}\right)\right)^{1/3}-\frac{mN_1z}{m_0^2q_1q_2}\right)dz\,du_2\,du_1+O(t^{-99}),\tag{7.6}$$

and for $j=1,2$,

$$x_{j,0}=\left(\frac{2\pi h_j}{m_0q_jt}\right)^{2/3}\frac{Q(N_1z)^{1/3}}{X}\left(1+\frac{2\pi h_jN^2u_j}{m_0q_jHt^{2-\varepsilon}}\right)^{-2/3}.\tag{7.7}$$

*7.2. Bounding $S_{1,0}(N)$*

To bound $S_{1,0}(N)$, we are left to bound the character sum $\mathcal{C}_{\eta_1}(m,h_1,h_2,q_1q_2)$ in (7.3). Recalling the definition in (7.1) we get, for $m=0$,

$$\begin{aligned}\mathcal{C}_{\eta_1}(0,h_1,h_2,q_1q_2)&=\sum_{\gamma\bmod q_1q_2}S(\overline{h_1},\eta_1\gamma;q_1)S(\overline{h_2},\eta_1\gamma;q_2)\\&=\sideset{}{^\star}\sum_{\alpha_1\bmod q_1}e\left(\frac{\alpha_1\overline{h_1}}{q_1}\right)\sideset{}{^\star}\sum_{\alpha_2\bmod q_2}e\left(\frac{\alpha_2\overline{h_2}}{q_2}\right)\sum_{\gamma\bmod q_1q_2}e\left(\eta_1\frac{(\overline{\alpha_1}q_2+\overline{\alpha_2}q_1)\gamma}{q_1q_2}\right)\\&=q_1q_2\sideset{}{^\star}\sum_{\alpha_1\bmod q_1}e\left(\frac{\alpha_1\overline{h_1}}{q_1}\right)\sideset{}{^\star}\sum_{\alpha_2\bmod q_2}e\left(\frac{\alpha_2\overline{h_2}}{q_2}\right)\delta(\overline{\alpha_1}q_2\equiv-\overline{\alpha_2}q_1\bmod q_1q_2)\\&=q_1^2\sideset{}{^\star}\sum_{\alpha\bmod q_1}e\left(\frac{\alpha(\overline{h_1}-\overline{h_2})}{q_1}\right)\delta(q_1=q_2)\\&=q_1^2\sum_{q_1'|q_1}q_1'\mu\left(\frac{q}{q_1'}\right)\delta(h_1\equiv h_2\bmod q_1',\,q_1=q_2).\end{aligned}\tag{7.8}$$

Hence the contribution of $m=0$ to $S_{1,0}(N)$ is

$$\ll\frac{QMN_1^{4/3}N^{1/3}}{Ct^{1-\varepsilon}}\delta\left(N_1\ll\frac{C^3t^{\varepsilon}}{N}\right)\sup_{\|\beta\|_2=1}\sum_{m_0}\sum_{q\sim_0 C/m_0}\sum_{q'|q}q'$$
$$\times\sum\sum_{\substack{h_1,h_2\sim_0 H'\\(h_1,q_1m_0)=1\\(h_2,q_2m_0)=1\\h_1\equiv h_2\bmod q'}}\int_{|u_1|\ll1}\int_{|u_2|\ll1}|\beta(m_0q,h_1,u_1)\beta(m_0q,h_2,u_2)|\,du_2\,du_1.$$

Applying the AM-GM inequality

$$|\beta(m_0q,h_1,u_1)\beta(m_0q,h_2,u_2)| \ll |\beta(m_0q,h_1,u_1)|^2 + |\beta(m_0q,h_2,u_2)|^2,$$

we find that the above is

$$\ll \frac{QMN_1^{4/3}N^{1/3}}{Ct^{1-\varepsilon}}\delta\bigg(N_1 \ll \frac{C^3t^\varepsilon}{N}\bigg) \sup_{\|\beta\|_2=1} (|\mathfrak{s}_{0,1}| + |\mathfrak{s}_{0,2}|),$$

where for brevity of notation, we have temporarily defined

$$\mathfrak{s}_{0,\nu} := \sum_{m_0} \sum_{q\sim_0 C/m_0} \sum_{q'|q} q' \mathop{\sum\sum}_{\substack{h_1,h_2\sim_0 H' \\ h_1\equiv h_2 \bmod q'}} \int_{|u_1|\ll 1}\int_{|u_2|\ll 1} |\beta(m_0q,h_\nu,u_\nu)|^2\,du_2\,du_1$$

for $\nu = 1, 2$. Hence the contribution of $m = 0$ to $S_{1,0}(N)$ is

$$\ll \frac{QMN_1^{4/3}N^{1/3}}{t^{1-\varepsilon}}\bigg(1+\frac{t}{N}\bigg)\delta\bigg(N_1 \ll \frac{C^3t^\varepsilon}{N}\bigg). \tag{7.9}$$

For the case $m \neq 0$, the character sum is

$$\begin{aligned}
\mathcal{C}_{\eta_1}(m,h_1,h_2,q_1q_2) &= \sum_{\gamma \bmod q_1q_2} S(\overline{h_1},\eta_1\gamma;q_1)S(\overline{h_2},\eta_1\gamma;q_2)e\bigg(\frac{m\gamma}{q_1q_2}\bigg) \\
&= \sideset{}{^\star}\sum_{\alpha_1 \bmod q_1} e\bigg(\frac{\overline{\alpha_1 h_1}}{q_1}\bigg) \sideset{}{^\star}\sum_{\alpha_2 \bmod q_2} e\bigg(\frac{\overline{\alpha_2 h_2}}{q_2}\bigg) \sum_{\gamma \bmod q_1q_2} e\bigg(\frac{(\eta_1(\alpha_1q_2+\alpha_2q_1)+m)\gamma}{q_1q_2}\bigg) \\
&\ll q_1q_2 \sideset{}{^\star}\sum_{\alpha_1 \bmod q_1} \sideset{}{^\star}\sum_{\alpha_2 \bmod q_2} \delta(-\eta_1 m \equiv \alpha_1q_2+\alpha_2q_1 \bmod q_1q_2).
\end{aligned} \tag{7.10}$$

Hence (7.3) shows that the contribution of $m \neq 0$ to $S_{1,0}(N)$ is

$$\begin{aligned}
&\ll \frac{QMN_1^{4/3}N^{1/3}}{Ct^{1-\varepsilon}}\delta\bigg(N_1 \ll \frac{C^3t^\varepsilon}{N}\bigg) \sup_{\|\beta\|_2=1} \sum_{m_0} \mathop{\sum\sum}_{q_1,q_2\sim_0 C/m_0} \mathop{\sum\sum}_{\substack{h_1,h_2\sim_0 H' \\ (h_1,q_1m_0)=1 \\ (h_2,q_2m_0)=1}} \\
&\quad \int_{|u_1|\ll 1}\int_{|u_2|\ll 1} |\beta(m_0q_1,h_1,u_1)\beta(m_0q_2,h_2,u_2)|\,du_2\,du_1 \\
&\quad \times \sum_{|m|\ll C^2t^\varepsilon/N_1} \sideset{}{^\star}\sum_{\alpha_1 \bmod q_1} \sideset{}{^\star}\sum_{\alpha_2 \bmod q_1} \delta(-\eta_1 m \equiv \alpha_1q_2+\alpha_2q_1 \bmod q_1q_2) + t^{-99}.
\end{aligned}$$

Applying the AM-GM inequality as in the $m = 0$ case, we find that the above is

$$\ll \frac{QMN_1^{4/3}N^{1/3}}{Ct^{1-\varepsilon}}\delta\bigg(N_1 \ll \frac{C^3t^\varepsilon}{N}\bigg) \sup_{\|\beta\|_2=1} (|\mathfrak{s}_{1,1}| + |\mathfrak{s}_{1,2}|),$$

where for brevity, we have set

$$\begin{aligned} \mathfrak{s}_{1,\nu} := \sum_{m_0} \sum\sum_{q_1,q_2\sim_0 C/m_0} \sum\sum_{h_1,h_2\sim_0 H'} \sum_{|m|\ll C^2t^\varepsilon/N_1} \\ \sideset{}{^\star}\sum_{\alpha_1 \bmod q_1} \sideset{}{^\star}\sum_{\alpha_2 \bmod q_2} \delta(-\eta_1 m \equiv \alpha_1 q_2 + \alpha_2 q_1 \bmod q_1q_2) \\ \times \int_{|u_\nu|\ll 1} |\beta(m_0 q_\nu, h_\nu, u_\nu)|^2\, du_\nu. \end{aligned}$$

To analyse the congruence condition, we let $q_0 = (q_1, q_2)$, $q_1 = q_0 q_{1,0} q_1'$, $q_2 = q_0 q_{2,0} q_2'$ be such that $q_{1,0}, q_{2,0} \mid q_0^\infty$ and $(q_0, q_1') = (q_0, q_2') = 1$. Then the congruence condition decomposes into $q_0 \mid m$ and

$$\begin{aligned} -\eta_1 \frac{m}{q_0} &\equiv \alpha_1 q_{2,0} q_2' + \alpha_2 q_{1,0} q_1' \bmod q_0 q_{1,0} q_{2,0}, \\ \alpha_1 &\equiv -\eta_1 \frac{m\overline{q_{2,0}q_2'}}{q_0} \bmod q_1', \\ \alpha_2 &\equiv -\eta_1 \frac{m\overline{q_{1,0}q_1'}}{q_0} \bmod q_2'. \end{aligned} \tag{7.11}$$

This leads to

$$\begin{aligned} \mathfrak{s}_{1,\nu} \ll &\sum_{m_0} \sum_{q_0} \sum_{\substack{q_{1,0},q_{2,0}\mid q_0^\infty \\ m_0q_0q_{1,0}q_1', m_0q_0q_{2,0}q_2' \sim C}} \sum_{\substack{(q_1',q_2')=1 \\ (q_1'q_2',q_0)=1}} \sum\sum_{h_1,h_2\sim H'} \sum_{|m'|\ll \frac{C^2t^\varepsilon}{q_0N_1}} \\ &\sideset{}{^\star}\sum_{\substack{\alpha_1 \bmod q_0q_{1,0} \\ -\eta_1 m' \equiv \alpha_1 q_{2,0}q_2' + \alpha_2 q_{1,0}q_1' \bmod q_0q_{1,0}q_{2,0}}} \sideset{}{^\star}\sum_{\alpha_2 \bmod q_0q_{2,0}} \int_{|u_\nu|\ll 1} |\beta(m_0q_0q_{\nu,0}q_\nu', h_\nu, u_\nu)|^2\, du_\nu \\ \ll &\frac{C^2T^\varepsilon}{N_1} H' \sum_{m_0} \sum_{q_0} \sum_{\substack{q_{1,0},q_{2,0}\mid q_0^\infty \\ m_0q_0q_{1,0}q_1', m_0q_0q_{2,0}q_2' \sim C}} \sum_{\substack{(q_1',q_2')=1 \\ (q_1'q_2',q_0)=1}} \sum_{h_\nu \sim H'} \int_{|u_\nu|\ll 1} |\beta(m_0q_0q_{\nu,0}q_\nu', h_\nu, u_\nu)|^2\, du_\nu. \end{aligned} \tag{7.12}$$

Here we have used the fact that, for any $m_1', q_0, q_{1,0}, q_{2,0}, q_1', q_2'$,

$$\sideset{}{^\star}\sum_{\substack{\alpha_1 \bmod q_0q_{1,0} \\ -\eta_1 m' \equiv \alpha_1 q_{2,0}q_2' + \alpha_2 q_{1,0}q_1' \bmod q_0q_{1,0}q_{2,0}}} \sideset{}{^\star}\sum_{\alpha_2 \bmod q_0q_{2,0}} \dots \ll q_0,$$

because the congruence condition determines $\alpha_1 \bmod q_{1,0}$, and $\alpha_2$ is completely determined once $\alpha_1$ is chosen. Since $\|\beta\|_2 = 1$ and $H' \sim Ct/N$, we have

$$\mathfrak{s}_{1,\nu} \ll \frac{C^4 t^{1+\varepsilon}}{N_1 N}$$

for $\nu = 1, 2$ and hence the contribution of $m \neq 0$ to $S_{1,0}(N)$ is

$$\ll \frac{QMN_1^{4/3}N^{1/3}}{Ct^{1-\varepsilon}}\delta\left(N_1 \ll \frac{C^3t^\varepsilon}{N}\right)\frac{C^4t}{N_1N} \ll \frac{C^3QMN_1^{1/3}t^\varepsilon}{N^{2/3}}\delta\left(N_1 \ll \frac{C^3t^\varepsilon}{N}\right). \tag{7.13}$$

We remark that the point counting computation in (7.11) and (7.12) will be repeated several times in a similar fashion throughout the paper. Since such computations are uninspiring and tedious, we will simply refer to these lines whenever we apply a similar point counting bound.

Combining (7.9) and (7.13), we get

$$S_{1,0}(N) \ll \frac{C^3QMN_1^{1/3}t^\varepsilon}{N^{2/3}}\delta\left(N_1 \ll \frac{C^3t^\varepsilon}{N}\right). \tag{7.14}$$

## 8. Opening the square and Poisson summation for $S_2(N)$

In this section, we apply similar treatment to $S_2(N)$. Recall that in (6.11) we have

$$S_2(N) = \frac{MN_2^{1/3}N^{7/3}}{Qt^{1-\varepsilon}}\sup_{\|\beta\|_2=1}\sum_{n_0}\frac{1}{n_0}\sum_n\varphi\left(\frac{n_0^2n}{N_2}\right)\Bigg|\sum_{q\sim_0 C/n_0}\frac{1}{q^{3/2}}\sum_{\substack{h\sim_0 H'\\(h,qn_0)=1}}S(\overline{h},\eta_2n;q)$$
$$\times\int_{\mathbb{R}}U(u)\beta(n_0q,h,u)W_{\eta_2}^{\pm}(n_0^2n,n_0q,h,u)\,du\Bigg|^2.$$

Opening the square, we get

$$S_2(N) = \frac{MN_2^{1/3}N^{7/3}}{Qt^{1-\varepsilon}}\sup_{\|\beta\|_2=1}\sum_{n_0}\frac{1}{n_0}\sum_n\varphi\left(\frac{n_0^2n}{N_2}\right)\sum\sum_{q_1,q_2\sim_0 C/n_0}\frac{1}{(q_1q_2)^{3/2}}$$
$$\times\sum\sum_{\substack{h_1,h_2\sim_0 H'\\(h_1,q_1n_0)=1\\(h_2,q_2n_0)=1}}S(\overline{h_1},\eta_2n;q_1)S(\overline{h_2},\eta_2n;q_2)\int_{\mathbb{R}}\int_{\mathbb{R}}U(u_1)U(u_2)\beta(n_0q_1,h_1,u_1)$$
$$\times\overline{\beta(n_0q_2,h_2,u_2)}W_{\eta_2}^{\pm}(n_0^2n,n_0q_1,h_1,u_1)\overline{W_{\eta_2}^{\pm}(n_0^2n,n_0q_2,h_2,u_2)}\,du_2\,du_1. \tag{8.1}$$

Applying Poisson summation to the $n$-sum, we have

$$\sum_n\varphi\left(\frac{n_0^2n}{N_2}\right)S(\overline{h_1},\eta_2n;q_1)S(\overline{h_2},\eta_2n;q_2)$$
$$\times W_{\eta_2}^{\pm}(n_0^2n,n_0q_1,h_1,u_1)\overline{W_{\eta_2}^{\pm}(n_0^2n,n_0q_2,h_2,u_2)}$$
$$= \frac{N_2}{n_0^2q_1q_2}\sum_n\mathcal{C}_{\eta_2}(n,h_1,h_2,q_1q_2)\mathcal{J}_2(q_1,q_2,\vec{v}),$$

where

$$\mathcal{C}_{\eta_2}(n,h_1,h_2,q_1q_2) = \sum_{\gamma \bmod q_1q_2}S(\overline{h_1},\eta_2\gamma;q_1)S(\overline{h_2},\eta_2\gamma;q_2)e\left(\frac{n\gamma}{q_1q_2}\right)$$

is the same character sum defined in (7.1) and

$$\mathcal{J}_2(q_1,q_2,\vec{v})=\int_0^\infty \varphi(z)W_{\eta_2}^{\pm}(N_2z,n_0q_1,h_1,u_1)\overline{W_{\eta_2}^{\pm}(N_2z,n_0q_2,h_2,u_2)}e\left(-\frac{nN_2z}{n_0^2q_1q_2}\right)dz$$

with $\vec{v}=(n,n_0,h_1,h_2,u_1,u_2)$ as before. Repeated integration by parts in the $z$-integral gives us arbitrary savings unless

$$|n|\ll CN^{1/3}t^{\varepsilon}/N_2^{2/3}.$$

Inserting this back into (8.1), we get

$$\begin{aligned} S_2(N) &= \frac{MN_2^{4/3}N^{7/3}}{Qt^{1-\varepsilon}}\sup_{\|\beta\|_2=1}\sum_{n_0}n_0^{-3}\sum_{|n|\ll CN^{1/3}t^{\varepsilon}/N_2^{2/3}}\sum\sum_{q_1,q_2\sim_0 C/n_0}(q_1q_2)^{-5/2}\\ &\times\sum\sum_{\substack{h_1,h_2\sim_0 H'\\(h_1,q_1n_0)=1\\(h_2,q_2n_0)=1}}\mathcal{C}_{\eta_2}(n,h_1,h_2,q_1q_2)\int_{\mathbb{R}}\int_{\mathbb{R}}U(u_1)U(u_2)\beta(n_0q_1,h_1,u_1)\overline{\beta(n_0q_2,h_2,u_2)}\\ &\times\mathcal{J}_2(q_1,q_2,\vec{v}\,)\,du_2\,du_1+O(t^{-99}). \end{aligned}\tag{8.2}$$

## 9. Analysis of $S_{1,X}(N)$ and $S_2(N)$

Recall that $N_2\ll N^2t^{\varepsilon}/Q^3$. Using $N_1\sim N^2X^3/Q^3$, splitting the $n$-sum into $n\geq 0$ and $n<0$, and collecting the above analysis (6.8), (7.6), (7.14) and (8.2), we obtain the following lemma.

**Lemma 9.1.** *For $0\leq|X|\ll t^{\varepsilon}$, we have*

$$\begin{aligned} S_{1,X}(N) \ll{}& \frac{C^2QMN_1^{4/3}N^{1/3}}{t^{1-\varepsilon}}\delta\left(\eta_1X\geq 0,\ N_1\sim\frac{N^2X^3}{Q^3}\right)\\ &\times\left(\frac{Ct}{N_1N}\delta\left(N_1\ll\frac{C^3t^{\varepsilon}}{N}\right)+\sup_{\pm}|\mathcal{R}_1^{\pm}(N)|\right)+O(t^{-99}) \end{aligned}$$

*and*

$$S_2(N)\ll\frac{MN_2^{4/3}N^{7/3}}{Qt^{1-\varepsilon}}\sup_{\pm}|\mathcal{R}_2^{\pm}(N)|+O(t^{-99}),$$

*where for $j=1,2$,*

$$\begin{aligned} \mathcal{R}_j^{\pm}(N) = {}& \sup_{\|\beta\|_2=1}\sum_{n_0}n_0^{-3}\sum_{n\ll CN^{1/3}t^{\varepsilon}/N_j^{2/3}}\sum\sum_{q_1,q_2\sim_0 C/n_0}(q_1q_2)^{-5/2}\\ &\times\sum\sum_{\substack{h_1,h_2\sim_0 H'\\(h_1,q_1n_0)=1\\(h_2,q_2n_0)=1}}\mathcal{C}_{\eta_j}(\pm n,h_1,h_2,q_1q_2)\int_{\mathbb{R}}\int_{\mathbb{R}}U(u_1)U(u_2)\beta(n_0q_1,h_1,u_1)\\ &\times\overline{\beta(n_0q_2,h_2,u_2)}\mathcal{J}_j(q_1,q_2,\vec{v}\,)\,du_2\,du_1 \end{aligned}$$

*and*

$$
\begin{aligned}
&\mathcal{J}_j(q_1,q_2,\vec{v}\,)\\
&\quad=\int_0^\infty \varphi(z)\Phi_{j,\eta_j}(n_0q_1,n_0q_2,u_1,u_2,z)\big(g(n_0q_1,Xx_{1,0})g(n_0q_2,Xx_{2,0})\big)^{2-j}\\
&\qquad\times e\bigg(-3\eta_j'\bigg(\frac{N_jNz}{(n_0q_1)^3}\bigg(\frac{n_0q_1t}{2\pi h_1N}+\frac{Nu_1}{Ht^{1-\varepsilon}}\bigg)\bigg)^{1/3}\\
&\qquad+3\eta_j'\bigg(\frac{N_jNz}{(n_0q_2)^3}\bigg(\frac{n_0q_2t}{2\pi h_2N}+\frac{Nu_2}{Ht^{1-\varepsilon}}\bigg)\bigg)^{1/3}\mp\frac{nN_jz}{n_0^2q_1q_2}\bigg)\,dz,
\end{aligned}
$$

*and where $\eta_1'=1,\eta_2'=\eta_2$, $x_{1,0},x_{2,0}$ are defined in (7.7) and $\Phi_{j,\eta_j}$ is a $t^\varepsilon$-inert function. Moreover, $x_{1,0},x_{2,0}$ are flat with respect to $q_1,q_2,z$.*

With the above lemma, we are left to bound $\mathcal{R}_j^{\pm}(N)$. Since the treatment for $\pm=+$ and $\pm=-$ is the same, we will assume $\pm=+$ and denote $\mathcal{R}_j(N)=\mathcal{R}_j^{+}(N)$ henceforth. We separate the analysis into four cases depending on the size of $n$.

### 9.1. *Diagonal contribution*

We first deal with the case $n=0$. Its contribution to $\mathcal{R}_j(N)$ is equal to

$$
\begin{aligned}
&\sup_{\|\beta\|_2=1}\sum_{n_0}n_0^{-3}\sum_{q_1,q_2\sim_0 C/n_0}\sum(q_1q_2)^{-5/2}\sum_{\substack{h_1,h_2\sim_0 H'\\(h_1,q_1n_0)=1\\(h_2,q_2n_0)=1}}\sum\mathcal{C}_{\eta_j}(0,h_1,h_2,q_1q_2)\\
&\qquad\times\int_{\mathbb{R}}\int_{\mathbb{R}}U(u_1)U(u_2)\beta(n_0q_1,h_1,u_1)\overline{\beta(n_0q_2,h_2,u_2)}\mathcal{J}_j(q_1,q_2,\vec{v}_0)\,du_2\,du_1,
\end{aligned}
$$

where $\vec{v}_0=(0,n_0,h_1,h_2,u_1,u_2)$. Recall that (7.8) gives us

$$
\mathcal{C}_{\eta_j}(0,h_1,h_2,q_1q_2)=q_1^2\sum_{q_1'|q_1}q_1'\mu\bigg(\frac{q}{q_1'}\bigg)\delta(h_1\equiv h_2 \bmod q_1',\ q_1=q_2).
$$

Using (4.3) and since $x_{j,0}$ is flat with respect to $z$, repeated integration by parts in the $z$-integral in $\mathcal{J}_j$ with $q_1=q_2=q$ gives us arbitrary savings unless

$$
\begin{aligned}
&\bigg|\bigg(\frac{N_jN}{(n_0q)^3}\bigg(\frac{n_0qt}{2\pi h_1N}+\frac{Nu_1}{Ht^{1-\varepsilon}}\bigg)\bigg)^{1/3}-\bigg(\frac{N_jN}{(n_0q)^3}\bigg(\frac{n_0qt}{2\pi h_2N}+\frac{Nu_2}{Ht^{1-\varepsilon}}\bigg)\bigg)^{1/3}\bigg|\ll t^\varepsilon\\
&\iff\bigg|\frac{n_0qt}{2\pi N}(h_1^{-1}-h_2^{-1})+\frac{N}{Ht^{1-\varepsilon}}(u_1-u_2)\bigg|\ll\frac{Ct^\varepsilon}{(N_jN)^{1/3}}\\
&\iff 0\neq|h_2-h_1|\ll\frac{C^2t^{1+\varepsilon}}{N_j^{1/3}N^{4/3}}+\frac{Ct^\varepsilon}{H}\ \text{or}\ \bigg(h_1=h_2\ \text{and}\ |u_2-u_1|\ll\frac{CHt^{1+\varepsilon}}{N_j^{1/3}N^{4/3}}\bigg).
\end{aligned}
$$

Note that (4.2) gives us $\mathcal{J}_j(q_1,q_2,\vec{v}_0)\ll t^\varepsilon$ for both $j=1,2$.

Applying the AM-GM inequality

$$|\beta(n_0q_1,h_1,u_1)\beta(n_0q_2,h_2,u_2)| \ll |\beta(n_0q_1,h_1,u_1)|^2 + |\beta(n_0q_2,h_2,u_2)|^2$$

shows that the contribution of $n = 0$ to $\mathcal{R}_j(N)$ is

$$\ll \frac{t^\varepsilon}{C^3} \sup_{\|\beta\|_2=1} (|\mathfrak{s}_{4,1}| + |\mathfrak{s}_{4,2}|),$$

where for brevity, we have set

$$\mathfrak{s}_{4,\nu} := \sum_{n_0} \sum_{q\sim_0 C/n_0} \sum_{q'|q} q' \sum_{\substack{h_1,h_2\sim_0 H' \\ h_1\equiv h_2 \bmod q' \\ |h_2-h_1| \ll \frac{C^2t^{1+\varepsilon}}{N_j^{1/3}N^{4/3}} + \frac{Ct^\varepsilon}{H}}} \int_{|u_1|\ll 1} \int_{|u_2|\ll 1} |\beta(n_0q,h_\nu,u_\nu)|^2$$
$$\times \left(\delta(h_1 \neq h_2) + \delta\left(h_1 = h_2,\ |u_2-u_1| \ll \frac{CHt^{1+\varepsilon}}{N_j^{1/3}N^{4/3}}\right)\right) du_2\, du_1$$

for $\nu = 1, 2$. Applying a point counting argument similar to (7.12), the contribution of $n = 0$ to $\mathcal{R}_j(N)$ is

$$\ll \frac{t^\varepsilon}{C^2H} + \frac{Ht^{1+\varepsilon}}{CN_j^{1/3}N^{4/3}}. \tag{9.1}$$

### 9.2. *Contribution when n is small*

Now we deal with the case $0 \neq nN_j \ll C^2t^\varepsilon$. In that case, $e(-\frac{nN_jz}{n_0^2q_1q_2})$ is $t^\varepsilon$-inert. Using (4.3) and since $x_{j,0}$ is flat with respect to $z$, repeated integration by parts in the $z$-integral in $\mathcal{J}_j$ gives us arbitrary savings unless

$$\left|\left(\frac{N_jN}{(n_0q_1)^3}\left(\frac{n_0q_1t}{2\pi h_1N} + \frac{Nu_1}{Ht^{1-\varepsilon}}\right)\right)^{1/3} - \left(\frac{N_jN}{(n_0q_2)^3}\left(\frac{n_0q_2t}{2\pi h_2N} + \frac{Nu_2}{Ht^{1-\varepsilon}}\right)\right)^{1/3}\right| \ll t^\varepsilon$$
$$\iff \left|\frac{t}{2\pi n_0^2N}\left(\frac{1}{q_1^2h_1} - \frac{1}{q_2^2h_2}\right) + \frac{N}{n_0^3Ht^{1-\varepsilon}}\left(\frac{u_1}{q_1^3} - \frac{u_2}{q_2^3}\right)\right| \ll \frac{t^\varepsilon}{C^2(N_jN)^{1/3}}$$
$$\iff \left|q_2 - \sqrt{\frac{h_1}{h_2}}q_1\right| \ll \frac{C^2t^\varepsilon}{n_0(N_jN)^{1/3}} + \frac{CN}{n_0Ht^{1-\varepsilon}}.$$

Together with (7.10) giving us

$$\mathcal{C}_{\eta_j}(n,h_1,h_2,q_1,q_2) \ll q_1q_2 \sideset{}{^\star}\sum_{\alpha_1 \bmod q_1} \sideset{}{^\star}\sum_{\alpha_2 \bmod q_2} \delta(n \equiv \alpha_1q_2 + \alpha_2q_1 \bmod q_1q_2),$$

the contribution of $0 \neq nN_j \ll C^2t^\varepsilon$ to $\mathcal{R}_j(N)$ is

$$\ll t^{\varepsilon} \sup_{\|\beta\|_2=1} \sum_{n_0} n_0^{-3} \sum_{0\neq n\ll \frac{C^2t^\varepsilon}{N_j}} \sum_{h_1,h_2\sim_0 H'} \sum\sum_{\substack{q_1,q_2\sim_0 C/n_0\\ |q_2-\sqrt{h_1/h_2}q_1|\ll \frac{C^2t^\varepsilon}{n_0(N_jN)^{1/3}}+\frac{CN}{n_0Ht^{1-\varepsilon}}}} (q_1q_2)^{-3/2}$$
$$\times \sideset{}{^\star}\sum_{\alpha_1 \bmod q_1} \sideset{}{^\star}\sum_{\alpha_2 \bmod q_2} \int_{|u_1|\ll 1}\int_{|u_2|\ll 1} |\beta(n_0q_1,h_1,u_1)|\,|\beta(n_0q_2,h_2,u_2)|\,du_2\,du_1$$
$$\times \big(g(n_0q_1,Xx_{1,0})g(n_0q_2,Xx_{2,0})\big)^{2-j}\delta(n\equiv \alpha_1q_2+\alpha_2q_1 \bmod q_1q_2).$$

Applying the AM-GM inequality

$$|\beta(n_0q_1,h_1,u_1)\beta(n_0q_2,h_2,u_2)| \ll |\beta(n_0q_1,h_1,u_1)|^2+|\beta(n_0q_2,h_2,u_2)|^2$$

with (4.2) giving us $g(m_0q_j,Xx_{j,0})\ll t^{\varepsilon}$, the same point counting analysis as in $S_{1,0}(N)$ using (7.11) and (7.12) implies that the contribution of $0\neq nN_j\ll C^2t^{\varepsilon}$ to $\mathcal{R}_j(N)$ is

$$\ll \frac{t^{\varepsilon}}{C^3}\frac{C^2}{N_j}\frac{Ct}{N}\left(1+\frac{C^2}{(N_jN)^{1/3}}+\frac{CN}{Ht}\right) \ll \frac{t^{1+\varepsilon}}{N_jN}\left(1+\frac{C^2}{(N_jN)^{1/3}}+\frac{CN}{Ht}\right). \tag{9.2}$$

Notice we have this contribution only when $N_j\ll C^2t^{\varepsilon}$.

### 9.3. *Integral analysis*

Now we continue with the analysis when $nN_j\gg C^2t^{\varepsilon}$. Recalling the definition of $\mathcal{J}_j(q_1,q_2,\vec{v}\,)$ in Lemma 9.1, applying stationary phase analysis with the calculations of Appendix A.3 yields

$$\mathcal{J}_j(q_1,q_2,\vec{v}\,)=\frac{C}{\sqrt{nN_j}}\big(g(n_0q_1,Xx_1)g(n_0q_2,Xx_2)\big)^{2-j}\Phi_2\big(z_0(n_0q_1,n_0q_2),q_1,q_2;\nu\big)$$
$$\times e(z_0(n_0q_1,n_0q_2))+O(t^{-A}) \tag{9.3}$$

for any $A>0$, with some flat function $\Phi_2$ supported on $[1/2,5/2]\times\mathbb{R}^7$, and

$$\begin{aligned} x_j &:= x_j(z_0(n_0q_1,n_0q_2))\\ &=\left(\frac{2\pi h_j}{n_0q_jt}\right)^{2/3}\frac{Q(N_1z_0(n_0q_1,n_0q_2))^{1/3}}{X}\left(1+\frac{2\pi h_jN^2u}{n_0q_jHt^{2-\varepsilon}}\right)^{-2/3}\end{aligned}$$

for $j=1,2$. Inserting this back into (8.2) shows that the contribution of $nN_j\gg C^2t^{\varepsilon}$ to $\mathcal{R}_j(N)$ is equal to

$$\frac{1}{C^2\sqrt{N_j}}\sup_{\|\beta\|_2=1}\sum_{n_0}\sum_{C^2t^\varepsilon/N_j\ll n\ll CN^{1/3}t^\varepsilon/N_j^{2/3}}\frac{1}{\sqrt{n}}\sum\sum_{q_1,q_2\sim_0 C/n_0}\frac{1}{q_1q_2}$$
$$\times \sum\sum_{\substack{h_1,h_2\sim_0 H'\\ (h_1,q_1n_0)=1\\ (h_2,q_2n_0)=1}} \mathcal{C}_{\eta_j}(n,h_1,h_2,q_1q_2)\int_{\mathbb{R}}\int_{\mathbb{R}} U(u_1)U(u_2)\beta(n_0q_1,h_1,u_1)$$
$$\times\overline{\beta(n_0q_2,h_2,u_2)}\Phi\big(z_0(n_0q_1,n_0q_2),q_1,q_2;\nu\big)e\big(F_3(z_0(n_0q_1,n_0q_2))\big)\,du_2\,du_1$$
$$+O(t^{-999}),$$

where we define $\Phi(z_0(n_0q_1,n_0q_2),q_1,q_2;\nu)$ as

$$\Phi(\cdots):=\frac{C^3}{(n_0^2q_1q_2)^{3/2}}\Phi_2\big(z_0(n_0q_1,n_0q_2)q_1,q_2;\nu\big)\big(g(n_0q_1,Xx_1)g(n_0q_2,Xx_2)\big)^{2-j}.$$

By (4.2) and (4.3), $\Phi$ is a $t^{\varepsilon}$-inert function supported on $[1/2,5/2]\times\mathbb{R}^7$ up to an arbitrarily small error.

### 9.4. *Contribution when n is of intermediate size*

For a technical reason, we deal with the case

$$\frac{C^2t^{\varepsilon}}{N_j}\ll n\ll\frac{CN^{4/3}}{N_j^{2/3}Ht^{1-\varepsilon}}$$

before proceeding further. Note that the restriction of $z_0(n_0q_1,n_0q_2)\sim 1$ implies

$$\left|n_0q_2\left(\frac{n_0q_1t}{2\pi h_1N}+\frac{Nu_1}{Ht^{1-\varepsilon}}\right)^{1/3}-n_0q_1\left(\frac{n_0q_2t}{2\pi h_2N}+\frac{Nu_2}{Ht^{1-\varepsilon}}\right)^{1/3}\right|\sim\frac{nN_j^{2/3}}{N^{1/3}}$$
$$\iff\left|n_0q_2\left(\frac{n_0q_1t}{h_1N}\right)^{1/3}-n_0q_1\left(\frac{n_0q_2t}{h_2N}\right)^{1/3}\right|\ll\frac{nN_j^{2/3}}{N^{1/3}}+\frac{CN}{Ht^{1-\varepsilon}}\ll\frac{CN}{Ht^{1-\varepsilon}}$$
$$\iff\left|q_2-\sqrt{\frac{h_1}{h_2}}q_1\right|\ll\frac{CN}{n_0Ht^{1-\varepsilon}}.$$

By a similar procedure to the case $0\neq nN_j\ll C^2t^{\varepsilon}$, the AM-GM inequality

$$|\beta(n_0q_1,h_1,u_1)\beta(n_0q_2,h_2,u_2)|\ll|\beta(n_0q_1,h_1,u_1)|^2+|\beta(n_0q_2,h_2,u_2)|^2$$

and (7.10) imply that the contribution of $\frac{C^2t^{\varepsilon}}{N_j}\ll n\ll\frac{CN^{4/3}}{N_j^{2/3}Ht^{1-\varepsilon}}$ to $\mathcal{R}_j(N)$ is

$$\ll\frac{t^{\varepsilon}}{C^2\sqrt{N_j}}\sup_{\|\beta\|_2=1}(|\mathfrak{s}_{3,1}|+|\mathfrak{s}_{3,2}|),$$

where for brevity, we have set

$$\mathfrak{s}_{3,\nu}:=\sum_{0\neq n\ll\frac{CN^{4/3}}{N_j^{2/3}Ht^{1-\varepsilon}}}\frac{1}{\sqrt{n}}\sum_{n_0}\sum_{h_1,h_2\sim_0H'}\sum_{\substack{q_1,q_2\sim_0C/n_0\\|q_2-\sqrt{h_1/h_2}q_1|\ll\frac{CN}{n_0Ht^{1-\varepsilon}}}}$$
$$\sum_{\alpha_1\bmod q_1}^{\star}\sum_{\alpha_2\bmod q_2}^{\star}\int_{|u_1|\ll 1}|\beta(n_0q_\nu,h_\nu,u_\nu)|^2\,du_1\delta(n\equiv\alpha_1q_2+\alpha_2q_1\bmod q_1q_2)$$

for $\nu=1,2$. By the same analysis as for $S_{1,0}(N)$ using (7.11), the contribution of $\frac{C^2t^{\varepsilon}}{N_j}\ll n\ll\frac{CN^{4/3}}{N_j^{2/3}Ht^{1-\varepsilon}}$ to $\mathcal{R}_j(N)$ is

$$\ll\frac{t^{\varepsilon}}{C^2\sqrt{N_j}}\frac{\sqrt{C}N^{2/3}}{N_j^{1/3}\sqrt{Ht}}\frac{Ct}{N}\left(1+\frac{CN}{Ht}\right)=\frac{t^{1/2+\varepsilon}}{\sqrt{CH}N_j^{5/6}N^{1/3}}\left(1+\frac{CN}{Ht}\right).\tag{9.4}$$

Notice we have this contribution only when $N_j\ll\frac{C^{3/2}N^2}{(Ht)^{3/2}}t^{\varepsilon}$.

### 9.5. *Contribution when n is large*

Denote

$$\begin{aligned} N_S &:= \Big(\frac{C^2}{N_j} + \frac{CN^{4/3}}{N_j^{2/3}Ht}\Big)t^{\varepsilon}, \\ N_B &:= \frac{CN^{1/3}t^{\varepsilon}}{N_j^{2/3}}., \end{aligned} \tag{9.5}$$

Denote by $\mathcal{R}_j^*(N)$ the contribution of $n \gg N_S$ to $\mathcal{R}_j(N)$ in Lemma 9.1, i.e.

$$\begin{aligned} \mathcal{R}_j^*(N) = \sup_{\|\beta\|_2=1} \sum_{n_0} n_0^{-3} \sum_{N_S \ll n \ll N_B} \sum\sum_{q_1,q_2 \sim_0 C/n_0} (q_1q_2)^{-5/2} \\ \times \sum\sum_{\substack{h_1,h_2 \sim_0 H' \\ (h_1,q_1n_0)=1 \\ (h_2,q_2n_0)=1}} \mathcal{C}_{\eta_j}(n,h_1,h_2,q_1q_2) \\ \times \int_{\mathbb{R}}\int_{\mathbb{R}} U(u_1)U(u_2)\beta(n_0q_1,h_1,u_1)\overline{\beta(n_0q_2,h_2,u_2)}\mathcal{J}_j(q_1,q_2,\vec{v}\,)\,du_2\,du_1. \end{aligned}$$

*9.5.1. Character sum analysis.* Recall that by (7.10), $\mathcal{C}_{\eta_j}(n,h_1,h_2,q_1q_2)$ equals

$$q_1q_2 \sideset{}{^\star}\sum_{\alpha_1 \bmod q_1} e\Big(\frac{\overline{\alpha_1 h_1}}{q_1}\Big) \sideset{}{^\star}\sum_{\alpha_2 \bmod q_2} e\Big(\frac{\overline{\alpha_2 h_2}}{q_2}\Big)\delta(-\eta_j n \equiv \alpha_1q_2 + \alpha_2q_1 \bmod q_1q_2).$$

We apply the same analysis as in (7.11): Write $(q_1,q_2) = q_0$, let $q_{1,0}, q_1', q_{2,0}, q_2'$ be unique positive integers such that $q_{1,0}, q_{2,0} \mid q_0^{\infty}$, $(q_1'q_2', q_0) = 1$ with $q_1 = q_0q_{1,0}q_1'$ and $q_2 = q_0q_{2,0}q_2'$. Then the congruence condition above breaks down into $q_0 \mid n$ and

$$\begin{aligned} -\eta_j\frac{n}{q_0} &\equiv \alpha_1q_{2,0}q_2' + \alpha_2q_{1,0}q_1' \bmod q_0q_{1,0}q_{2,0}, \\ \alpha_1 &\equiv -\eta_j\frac{n}{q_0}\overline{q_{2,0}q_2'} \bmod q_1', \\ \alpha_2 &\equiv -\eta_j\frac{n}{q_0}\overline{q_{1,0}q_1'} \bmod q_2'. \end{aligned}$$

Then the Chinese Remainder Theorem yields

$$\begin{aligned} \mathcal{C}_{\eta_j}(n,h_1,h_2,q_1q_2) = q_1q_2e\Big(-\eta_j\frac{\overline{nh_1q_{1,0}q_{2,0}q_2'}}{q_1'} - \eta_j\frac{\overline{nh_2q_{2,0}q_{1,0}q_1'}}{q_2'}\Big) \\ \times \sideset{}{^\star}\sum_{\beta_1 \bmod q_0q_{1,0}} \sideset{}{^\star}\sum_{\beta_2 \bmod q_0q_{2,0}} e\Big(\frac{\beta_1\overline{h_1q_1'}}{q_0q_{1,0}} + \frac{\beta_2\overline{h_2q_2'}}{q_0q_{2,0}}\Big) \\ \times \delta\Big(q_0 \mid n,\ -\eta_j\frac{n}{q_0} \equiv \overline{\beta_1}q_{2,0}q_2' + \overline{\beta_2}q_{1,0}q_1' \bmod q_0q_{1,0}q_{2,0}\Big). \end{aligned}$$

By reciprocity together with a change of variable, $\mathcal{C}_{\eta_j}(n, h_1, h_2, q_1q_2)$ equals

$$q_1q_2 e\left(\eta_j \frac{\overline{q_1'}q_{2,0}q_2'}{nh_1q_{1,0}} + \eta_j \frac{\overline{q_2'}q_{1,0}q_1'}{nh_2q_{2,0}} - \eta_j \frac{q_{2,0}q_2'}{nh_1q_{1,0}q_1'} - \eta_j \frac{q_{1,0}q_1'}{nh_2q_{2,0}q_2'}\right)$$
$$\times \sideset{}{^\star}\sum_{\beta_1 \bmod q_0q_{1,0}} \sideset{}{^\star}\sum_{\beta_2 \bmod q_0q_{2,0}} e\left(\frac{\beta_1}{q_0q_{1,0}} + \frac{\beta_2}{q_0q_{2,0}}\right)$$
$$\times \delta\left(q_0 \mid n,\ -\eta_j\frac{n}{q_0} \equiv \overline{\beta_1h_1q_1'}q_{2,0}q_2' + \overline{\beta_2h_2q_2'}q_{1,0}q_1' \bmod q_0q_{1,0}q_{2,0}\right).$$

Before we proceed further, we first remove the coprimality condition $(q_1', q_2') = 1$ by Möbius inversion, i.e. for any function $F$,

$$\sum_{(q_1',q_2')=1} F(q_1', q_2') = \sum_r \mu(r) \sum_{q_1',q_2'} F(q_1'r, q_2'r);$$

this yields

$$\mathcal{R}_j^*(N) = \sup_{\|\beta\|_2=1} \sum_{n_0} n_0^{-3} \sum_{N_S \ll n \ll N_B} \sum\sum\sum\sum\sum\sum_{\substack{q_1=q_0q_{1,0}q_1'r, q_2=q_0q_{2,0}q_2'r \\ q_{1,0}q_{2,0}\mid q_0^\infty, (q_1'q_2'r, q_0)=1 \\ q_1,q_2\sim_0 C/n_0}} \mu(r)(q_1q_2)^{-5/2}$$
$$\times \sum\sum_{\substack{h_1,h_2\sim_0 H' \\ (h_1,q_1n_0)=1 \\ (h_2,q_2n_0)=1}} \mathcal{C}_{\eta_j}(n, h_1, h_2, q_1q_2)$$
$$\times \int_{\mathbb{R}}\int_{\mathbb{R}} U(u_1)U(u_2)\beta(n_0q_1, h_1, u_1)\overline{\beta(n_0q_2, h_2, u_2)}\mathcal{J}_j(q_1, q_2, \vec{v}\,)\, du_2\, du_1.$$

To prepare ourselves for a character sum analysis in later steps, we pull out $(n, q_0^\infty)$, $(h_2, (nh_1)^\infty)$ and the square-full part of $nh_1$. Write $n = n_1n_2n'$, $h_1 = h_{1,0}h_1'$ and $h_2 = h_{2,0}h_2'$ where $n_1 \mid q_0^\infty$, $(n_2h_{1,0}h_{2,0}, n'h_1'h_2'q_0) = (n'h_1', h_2'q_0) = 1$, $h_{2,0} \mid (n_2h_{1,0})^\infty$, $n'h_1'$ is square-free and $n_2h_{1,0}h_{2,0}$ is square-full, i.e. for any prime $p \mid n_2h_{1,0}h_{2,0}$, we have $p^2 \mid n_2h_{1,0}h_{2,0}$. With this notation, we arrive at

$$\mathcal{R}_j^*(N) = \sup_{\|\beta\|_2=1} \sum_{n_0} n_0^{-3} \sum_{q_0} \sum_{q_0\mid n_1\mid q_0^\infty} \sum\sum\sum_{\substack{n_2h_{1,0}h_{2,0}\ \square\text{-full} \\ h_{2,0}\mid(n_2h_{1,0})^\infty \\ (n_2,q_0)=1}} \sum\sum\sum_{\substack{N_S\ll n=n_1n_2n'\ll N_B \\ h_1=h_{1,0}h_1', h_2=h_{2,0}h_2'\sim_0 H' \\ n'h_1'\ \square\text{-free}, (n_1'h_1',h_2')=1 \\ (n'h_1'h_2', n_1h_{1,0}h_{2,0}q_0)=1}}$$
$$\sum\sum\sum\sum\sum_{\substack{q_1=q_0q_{1,0}q_1'r, q_2=q_0q_{2,0}q_2'r \\ q_{1,0}q_{2,0}\mid q_0^\infty, (q_1'q_2'r, q_0)=1 \\ (q_1,h_1)=(q_2,h_2)=1 \\ q_1,q_2\sim_0 C/n_0}} \mu(r)(q_1q_2)^{-5/2}\mathcal{C}_{\eta_j}(n, h_1, h_2, q_1q_2)$$
$$\times \int_{\mathbb{R}}\int_{\mathbb{R}} U(u_1)U(u_2)\beta(n_0q_1, h_1, u_1)\overline{\beta(n_0q_2, h_2, u_2)}\mathcal{J}_j(q_1, q_2, \vec{v}\,)\, du_2\, du_1.$$

*9.5.2. Second Cauchy–Schwarz inequality and Poisson summation.* Notice that by the same point counting argument as in (7.12), we have

$$\sup_{\|\beta\|_2=1} \sum\sum\sum_{n_2 h_{1,0} h_{2,0}\ \square\text{-full}} \frac{1}{\sqrt{n_2 h_{2,0}}} \sum_{q_1 \ll C} \sum\sum\sum\sum_{\substack{n_0 q_0 q_{1,0} r | q_1 \\ q_{1,0} | q_0^\infty}} \sum_{\substack{q_{2,0} \ll C \\ q_{2,0} | q_0^\infty}} \sum\sum_{\substack{0 \neq n_1 n' \ll N_B \\ q_0 | n_1 | q_0^\infty}} \frac{1}{n'}$$
$$\times \sum\sum_{h_1', h_2' \ll H'} \frac{1}{h_2'} \int_{|u_1| \ll 1} |\beta(q_1, h_{1,0} h_1', u_1)|^2 \, du_1 \int_{|u_2| \ll 1} du_2 \ll t^\varepsilon.$$

Let $\Upsilon \in C_c^\infty([1 - 10^{-9}, 1 + 10^{-9}])$ be a fixed function such that $\Upsilon(x) = 1$ for $1 \leq x < 1 + 10^{-10}$. Now we apply the Cauchy–Schwarz inequality to take out $n_0, n_1, n', q_0, q_{1,0}, q_{2,0}, q_1', r, h_{1,0}, h_1', h_{2,0}, h_2'$-sums and the $u_1, u_2$-integrals. Together with (9.3) and the character sum analysis above, we get

$$\begin{aligned}
\mathcal{R}_j^*(N) \ll \frac{t^\varepsilon}{C^2 \sqrt{N_j}} & \sup_{\|\beta\|_2=1} \sup_{|u_1| \ll 1} \Bigg\{ \int_{|u_2| \ll 1} \sum_{n_0} \sum_{q_0} \sum_{q_0 | n_1 | q_0^\infty} \sum_{r} \sum\sum\sum_{\substack{n_2 h_{1,0} h_{2,0}\ \square\text{-full} \\ h_{2,0} | (n_2 h_{1,0})^\infty \\ (n_2, q_0) = 1}} \\
& \times \sum\sum\sum_{\substack{N_S \ll n = n_1 n_2 n' \ll N_B \\ h_1 = h_{1,0} h_1', h_2 = h_{2,0} h_2' \sim_0 H' \\ n' h_1'\ \square\text{-free}, (n_1' h_1', h_2') = 1 \\ (n' h_1' h_2', n_1 h_{1,0} h_{2,0} q_0) = 1}} \sum\sum_{q_{1,0}, q_{2,0} | q_0^\infty} \frac{\sqrt{h_{2,0}} h_2'}{n_1 \sqrt{n_2}} \sum_{q_1'} \Upsilon\left( \frac{n_0 q_0 q_{1,0} q_1' r}{C} \right) \\
& \times \sum\sum_{\substack{q_2 = n_0 q_0 q_{2,0} q_2' r \sim_0 C \\ q_3 = n_0 q_0 q_{2,0} q_3' r \sim_0 C}} \mathcal{D}_1(q_1') \overline{\beta(q_2, h_2, u_2)} \beta(q_3, h_2, u_2) \Psi(n_0 q_0 q_{1,0} q_1' r, q_2, q_3) \\
& \times e\big(F_3(z_0(n_0 q_0 q_{1,0} q_1' r, q_2)) - F_3(z_0(n_0 q_0 q_{1,0} q_1' r, q_3))\big) \, du_2 \Bigg\}^{1/2} + t^{-999},
\end{aligned} \tag{9.6}$$

where

$$\begin{aligned}
\mathcal{D}_1(q_1') = e & \left( \eta_j \frac{\overline{q_1'} q_{2,0} (q_2' - q_3')}{n h_1 q_{1,0}} + \eta_j \frac{(\overline{q_2'} - \overline{q_3'}) q_{1,0} q_1'}{n h_2 q_{2,0}} \right) \\
& \times \sideset{}{^\star}\sum_{\beta_1 \bmod q_0 q_{1,0}} \sideset{}{^\star}\sum_{\beta_1' \bmod q_0 q_{1,0}} \sideset{}{^\star}\sum_{\beta_2 \bmod q_0 q_{2,0}} \sideset{}{^\star}\sum_{\beta_2' \bmod q_0 q_{2,0}} e\left( \frac{\beta_1 - \beta_1'}{q_0 q_{1,0}} + \frac{\beta_2 - \beta_2'}{q_0 q_{2,0}} \right) \\
& \times \delta \Bigg( -\eta_j \frac{n}{q_0} \equiv \overline{\beta_1 h_1 q_1'} q_{2,0} q_2' + \overline{\beta_2 h_2 q_2'} q_{1,0} q_1' \bmod q_0 q_{1,0} q_{2,0}, \\
& - \eta_j \frac{n}{q_0} \equiv \overline{\beta_1' h_1 q_1'} q_{2,0} q_3' + \overline{\beta_2' h_2 q_3'} q_{1,0} q_1' \bmod q_0 q_{1,0} q_{2,0} \Bigg),
\end{aligned} \tag{9.7}$$

$$\begin{aligned}
\Psi(n_0 q_0 q_{1,0} q_1' r, q_2, q_3) = & \frac{C^3}{(q_2 q_3)^{3/2}} \Phi(z_0(n_0 q_0 q_{1,0} q_1' r, q_2), n_0 q_0 q_{1,0} q_1' r, q_2; \nu) \\
& \times \overline{\Phi(z_0(n_0 q_0 q_{1,0} q_1' r, q_3), n_0 q_0 q_{1,0} q_1' r, q_3; \nu)} \\
& \times e\left( -\eta_j \frac{q_{2,0} (q_2' - q_3')}{n h_1 q_{1,0} q_1'} - \eta_j \frac{q_{1,0} q_1'}{n h_2 q_{2,0} q_2'} + \eta_j \frac{q_{1,0} q_1'}{n h_2 q_{2,0} q_3'} \right)
\end{aligned} \tag{9.8}$$

with

$$z_0(x,q) = \frac{\sqrt{N}}{n^{3/2}N_j}\left(-\eta_j' q\left(\frac{xt}{2\pi h_1 N} + \frac{Nu_1}{Ht^{1-\varepsilon}}\right)^{1/3} + \eta_j' x\left(\frac{qt}{2\pi h_2 N} + \frac{Nu_2}{Ht^{1-\varepsilon}}\right)^{1/3}\right)^{3/2},$$

and

$$F_3(z_0(x,q)) = \frac{2nN_j}{xq} z_0(x,q)$$

as before. Here coprimalities between variables are assumed such that $D_1(q_1')$ is well-defined. In particular, the condition $(q_1', nh_1q_0) = (q_2'q_3', nh_2q_0) = 1$ is assumed.

Consider the $q_1'$-sum. Notice that $\mathcal{D}_1(q_1')$ is determined by $q_1' \bmod nh_1h_2q_0q_{1,0}q_{2,0}$, hence applying Poisson summation gives us

$$\begin{aligned}\sum_{q_1'} &\Upsilon\left(\frac{n_0q_0q_{1,0}q_1'r}{C}\right)\mathcal{D}_1(q_1')\varphi\left(\frac{n_0q_0q_{1,0}q_1'r}{C}\right)\Psi(n_0q_0q_{1,0}q_1'r, q_2, q_3)\\ &\qquad \times e(F_3(z_0(n_0q_0q_{1,0}q_1'r, q_2)) - F_3(z_0(n_0q_0q_{1,0}q_1'r, q_3)))\\ &= \frac{C}{n_0nh_1h_2q_0^2q_{1,0}^2q_{2,0}r}\sum_{q_1'}\sum_{\gamma \bmod nh_1h_2q_0q_{1,0}q_{2,0}} \mathcal{D}_1(\gamma)e\left(\frac{q_1'\gamma}{nh_1h_2q_0q_{1,0}q_{2,0}}\right)\\ &\qquad \times I_2(q_1', q_2, q_3),\end{aligned}$$

where $I_2(q_1', q_2, q_3)$ is the integral

$$\int_0^\infty \Upsilon(y)\Psi(Cy, q_2, q_3)e\Bigg(F_3(z_0(Cy,q_2)) - F_3(z_0(Cy,q_3)) - \frac{q_1'Cy}{n_0nh_1h_2q_0^2q_{1,0}^2q_{2,0}r}\Bigg)\,dy. \tag{9.9}$$

Differentiating gives us

$$\frac{d}{dy}z_0(Cy,q) = z_0(Cy,q)^{1/3}\left[\frac{-\eta_j' qCt}{4\pi nh_1N^{2/3}N_j^{2/3}}\left(\frac{Cyt}{2\pi h_1N} + \frac{Nu_1}{Ht^{1-\varepsilon}}\right)^{-2/3} + \frac{3\eta_j' N^{1/3}C}{2nN_j^{2/3}}\left(\frac{qt}{2\pi h_2 N} + \frac{Nu_2}{Ht^{1-\varepsilon}}\right)^{1/3}\right]. \tag{9.10}$$

In order to bound the higher derivatives, we temporarily set

$$\begin{aligned} G &:= \frac{N}{n^3N_j^2}q^3\frac{Ct}{2\pi h_1 N} \sim \frac{N}{n^3N_j^2}q^3,\\ K &:= \frac{N}{n^3N_j^2}q^3\frac{Nu_1}{Ht^{1-\varepsilon}} \ll Gt^{-\varepsilon/2},\\ J &:= \frac{CN^{1/3}}{nN_j^{2/3}}\left(\frac{qt}{2\pi h_2N} + \frac{Nu_2}{Ht^{1-\varepsilon}}\right)^{1/3} \sim G^{1/3},\end{aligned}$$

where we have used the conditions $H \gg N/t^{1-\varepsilon}$ and $q \sim C$. Then we may write

$$z_0(Cy,q) = J^{3/2}\left(-\eta_j'\left(\frac{G}{J^3}y + \frac{K}{J^3}\right)^{1/3} + \eta_j' y\right)^{3/2},$$

so that the coefficients of $y$ are of size $O(1)$ and one can see that $\frac{d^k}{dy^k}z^{2/3} \ll J$ for any $k \geq 0$. By Faà di Bruno's formula, we have

$$\left|\frac{d^k}{dy^k}z_0(Cy,q)\right|, \left|\frac{d^k}{dy^k}\Psi(Cy,q_2,q_3)\right| \ll_k \left(\frac{CN^{1/3}}{nN_j^{2/3}}\right)^k,$$
$$\left|\frac{d^k}{dy^k}F_3(z_0(Cy,q))\right| \ll_k \left(\frac{(N_jN)^{1/3}}{C}\right)^k \tag{9.11}$$

for any $k \geq 0$ and $q \sim C$. Hence repeated integration by parts gives us arbitrary savings unless

$$|q_1'| \ll \frac{n_0 n h_1 h_2 q_0^2 q_{1,0}^2 q_{2,0} r}{C}\left(1 + \frac{CN^{1/3}}{nN_j^{2/3}} + \frac{(N_jN)^{1/3}}{C}\right)t^{\varepsilon} \ll Q_1 Q_B,$$

where

$$Q_1 := n_1 n_2 h_{1,0} h_{2,0} q_0 q_{1,0}^2 \quad \text{and} \quad Q_B := \frac{n_0 n' q_0 q_{2,0} r N_j^{1/3} t^{2+\varepsilon}}{h_{1,0} h_{2,0} N^{5/3}}. \tag{9.12}$$

Here we have used $C^2 t^{\varepsilon}/N_j = N_S \ll n$ and $h_1, h_2 \sim H' \sim Ct/N$. Inserting the above back into (9.6) yields

$$\begin{aligned}\mathcal{R}_j^*(N) \ll \frac{t^{\varepsilon}}{C^{3/2}\sqrt{N_j}} \sup_{\|\beta\|_2=1} \sup_{|u_1|\ll 1} \Bigg\{\int_{|u_2|\ll 1} \sum_{n_0} \sum_{q_0} \sum_{q_0|n_1|q_0^{\infty}} \sum_{r} \sum\sum\sum_{\substack{n_2 h_{1,0} h_{2,0}\ \square\text{-full} \\ h_{2,0}|(n_2 h_{1,0})^{\infty} \\ (n_2,q_0)=1}} \\ \sum\sum\sum_{\substack{N_S \ll n=n_1 n_2 n' \ll N_B \\ h_1=h_{1,0}h_1', h_2=h_{2,0}h_2' \sim_0 H' \\ n'h_1'\ \square\text{-free}, (n_1'h_1', h_2')=1 \\ (n'h_1'h_2', n_1 h_{1,0} h_{2,0} q_0)=1}} \sum\sum_{q_{1,0}, q_{2,0}|q_0^{\infty}} \sum\sum_{\substack{q_2=n_0 q_0 q_{2,0} q_2' r \sim_0 C \\ q_3=n_0 q_0 q_{2,0} q_3' r \sim_0 C}} \frac{\overline{\beta(q_2,h_2,u_2)}\beta(q_3,h_2,u_2)}{n_0 n_1^2 n_2^{3/2} n' h_1 \sqrt{h_{2,0}} q_0^2 q_{1,0}^2 q_{2,0} r} \\ \times \sum_{|q_1'|\ll Q_1 Q_B} \sum_{\gamma \bmod n h_1 h_2 q_0 q_{1,0} q_{2,0}} \mathcal{D}_1(\gamma) e\left(\frac{q_1'\gamma}{n h_1 h_2 q_0 q_{1,0} q_{2,0}}\right) I_2(q_1', q_2, q_3)\, du_2\Bigg\}^{1/2} \\ + t^{-999}. \end{aligned} \tag{9.13}$$

We first deal with the character sum

$$\sum_{\gamma \bmod n h_1 h_2 q_0 q_{1,0} q_{2,0}} \mathcal{D}_1(\gamma) e\left(\frac{q_1'\gamma}{n h_1 h_2 q_0 q_{1,0} q_{2,0}}\right).$$

Recalling the definition of $\mathcal{D}_1$ in (9.7), the Chinese Remainder Theorem implies that the above equals

$$h_2' S\big(q_1'\overline{n_1 n_2 h_{1,0} h_2 q_0 q_{1,0} q_{2,0}} + \eta_j(\overline{q_2'} - \overline{q_3'}) q_{1,0}\overline{n_1 n_2 h_2 q_{2,0}}, \\ \eta_j(q_2' - q_3') q_{2,0}\overline{n_1 n_2 h_{1,0} q_{1,0}}; n' h_1'\big) \\ \times \delta\big(q_1' \equiv -\eta_j q_0 q_{1,0}^2 h_1(\overline{q_2'} - \overline{q_3'}) \bmod h_2'\big)\mathcal{D}_0(q_2', q_3'),$$

where $M_0 = n_1 n_2 h_{1,0} h_{2,0} q_0 q_{1,0} q_{2,0}$ and $\mathcal{D}_0(q_2', q_3')$ equals

$$\sum_{\gamma \bmod M_0} e\left(\frac{q_1'\gamma\overline{n' h_1' h_2'}}{M_0} + \eta_j \frac{q_{2,0}(q_2' - q_3')\overline{\gamma n' h_1'}}{n_1 n_2 h_{1,0} q_{1,0}} + \eta_j \frac{(\overline{q_2'} - \overline{q_3'}) q_{1,0}\overline{\gamma n' h_2'}}{n_1 n_2 h_{2,0} q_{2,0}}\right) \\ \times \sideset{}{^\star}\sum_{\beta_1 \bmod q_0 q_{1,0}} \sideset{}{^\star}\sum_{\beta_1' \bmod q_0 q_{1,0}} \sideset{}{^\star}\sum_{\beta_2 \bmod q_0 q_{2,0}} \sideset{}{^\star}\sum_{\beta_2' \bmod q_0 q_{2,0}} e\left(\frac{\beta_1 - \beta_1'}{q_0 q_{1,0}} + \frac{\beta_2 - \beta_2'}{q_0 q_{2,0}}\right) \\ \times \delta\Bigg(-\eta_j \frac{n}{q_0} \equiv \overline{\beta_1 h_1 \gamma} q_{2,0} q_2' + \overline{\beta_2 h_2 q_2'} q_{1,0}\gamma \bmod q_0 q_{1,0} q_{2,0}, \\ -\eta_j \frac{n}{q_0} \equiv \overline{\beta_1' h_1 \gamma} q_{2,0} q_3' + \overline{\beta_2' h_2 q_3'} q_{1,0}\gamma \bmod q_0 q_{1,0} q_{2,0}\Bigg) \\ \ll n_1 n_2 h_{1,0} h_{2,0} q_0^3 q_{1,0} q_{2,0}. \tag{9.14}$$

Hence the Weil bound for Kloosterman sums gives us

$$\sum_{\gamma \bmod n h_1 h_2 q_0 q_{1,0} q_{2,0}} \mathcal{D}_1(\gamma) e\left(\frac{q_1'\gamma}{n h_1 h_2 q_0 q_{1,0} q_{2,0}}\right) \\ \ll n_1 n_2 h_{1,0} h_2 q_0^3 q_{1,0} q_{2,0} t^\varepsilon \sqrt{n' h_1' (q_1', q_2' - q_3', n' h_1')} \\ \times \delta\big(q_1' \equiv -\eta_j q_0 q_{1,0}^2 h_1(\overline{q_2'} - \overline{q_3'}) \bmod h_2'\big). \tag{9.15}$$

Next we analyse the $y$-integral with the phase function being

$$F_3(z_0(Cy, q_2)) - F_3(z_0(Cy, q_3)) - \frac{q_1' C y}{n_0 n h_1 h_2 q_0^2 q_{1,0}^2 q_{2,0} r} \\ = \frac{2n N_j}{Cy}\left(\frac{z_0(Cy, q_2)}{q_2} - \frac{z_0(Cy, q_3)}{q_3}\right) - \frac{q_1' C y}{n_0 n h_1 h_2 q_0^2 q_{1,0}^2 q_{2,0} r}.$$

Denote

$$G(y, z) = G_{h_1, h_2}(y, z) = F_3(z_0(Cy, Cz)) \\ = \sqrt{\frac{2t}{\pi n}}\left(-\eta_j'\left(\frac{z}{h_1 y} + \frac{2\pi N^2 u_1 z}{C H t^{2-\varepsilon} y^2}\right)^{1/3} + \eta_j'\left(\frac{y}{h_2 z} + \frac{2\pi N^2 u_2 y}{C H t^{2-\varepsilon} z^2}\right)^{1/3}\right)^{3/2}. \tag{9.16}$$

We start by analysing the case when $|q_1'| \ll Q_1 Q_S$ (including $q_1' = 0$), where $Q_1$ is defined in (9.12) and

$$Q_S := \frac{n_0 q_0 q_{2,0} r C^3 t^{2+\varepsilon}}{n_1 n_2 n h_{1,0} h_{2,0} (N_j N)^{4/3}}. \tag{9.17}$$

**Remark 9.2.** The threshold $Q_1 Q_S$ is chosen such that when $|q_1'| \gg Q_1 Q_S$, the phase satisfies

$$\frac{q_1' C y}{n_0 n h_1 h_2 q_0^2 q_{1,0}^2 q_{2,0} r} \gg \left(\frac{C N^{1/3}}{n N_j^{2/3}}\right)^2 t^{\varepsilon}.$$

Since the weight function $\Psi(Cy, q_2, q_3)$ consists of an inert function with $z_0$ as an argument, we can apply Lemma 5.2 (2, 3) in the range $|q_1'| \gg Q_1 Q_S$. This condition is ultimately used in Lemma 9.5.

Applying stationary phase analysis with the calculations of Appendix A.4 shows that the $y$-integral is bounded by

$$\ll \frac{n N_j^{2/3} t^{\varepsilon}}{C N^{1/3}} \delta\left(|q_2 - q_3| \ll \frac{C^3 t^{\varepsilon}}{n N_j}\right) + t^{-999}.$$

Together with (9.15), this shows that the contribution of $|q_1'| \ll Q_1 Q_S$ to $\mathcal{R}_2^*(N)$ in (9.13) is

$$\begin{aligned} &\ll \frac{t^{\varepsilon}}{C^{3/2}\sqrt{N_j}} \\ &\quad \times \sup_{\|\beta\|_2=1} \sup_{|u_1|\ll 1} \Bigg\{ \int_{|u_2|\ll 1} \sum_{n_0} \sum_{q_0} \sum_{q_0 | n_1 | q_0^{\infty}} \sum_{r} \sum\sum\sum_{\substack{n_2 h_{1,0} h_{2,0}\ \square\text{-full} \\ h_{2,0} | (n_2 h_{1,0})^{\infty} \\ (n_2, q_0)=1}} \sum\sum\sum_{\substack{N_S \ll n = n_1 n_2 n' \ll N_B \\ h_1 = h_{1,0} h_1', h_2 = h_{2,0} h_2' \sim_0 H' \\ n' h_1'\ \square\text{-free}, (n_1' h_1', h_2')=1 \\ (n' h_1' h_2', n_1 h_{1,0} h_{2,0} q_0)=1}} \\ &\quad \sum\sum_{q_{1,0}, q_{2,0} | q_0^{\infty}} \sum\sum_{\substack{q_2 = n_0 q_0 q_{2,0} q_2' r \sim_0 C \\ q_3 = n_0 q_0 q_{2,0} q_3' r \sim_0 C \\ |q_2 - q_3| \ll \frac{C^3 t^{\varepsilon}}{n N_j}}} \sum_{\substack{|q_1'| \ll Q_1 Q_S \\ q_1' \equiv -\eta_j q_0 h_1 (\overline{q_2'} - \overline{q_3'}) \bmod h_2'}} |\beta(q_2, h_2, u_2)\beta(q_3, h_2, u_2)| du_2 \\ &\quad \times \frac{N_j^{2/3}}{C N^{1/3}} \frac{\sqrt{n_2 n' h_{2,0}} h_2' q_0}{n_0 \sqrt{h_1'} q_{1,0} r} \sqrt{(q_1', q_2' - q_3', n' h_1')} \Bigg\}^{1/2}. \end{aligned}$$

Recall the definition of $N_S$ and $N_B$ in (9.5). Applying the AM-GM inequality

$$|\beta(q_2, h_2, u_2)\beta(q_3, h_2, u_2)| \ll |\beta(q_2, h_2, u_2)|^2 + |\beta(q_3, h_2, u_2)|^2$$

and the same point counting argument as in (7.12) implies that the contribution of $|q_1'| \ll Q_1 Q_S$ to $\mathcal{R}_2^*(N)$ in (9.13) is

$$\begin{aligned} &\ll \frac{t^{\varepsilon}}{C^{3/2}\sqrt{N_j}} \Bigg\{ \underbrace{\frac{H'^2 N_B^2 N_j^{2/3}}{C N^{1/3}}}_{q_1'=0, q_2=q_3} + \underbrace{\frac{C^2 \sqrt{H' N_B}}{(N_j N)^{1/3}}}_{q_1'=0, q_2 \neq q_3} + \underbrace{\frac{C^2 \sqrt{H' N_B} t^2}{N_j^{2/3} N^{5/3}}}_{q_1' \neq 0, q_2 = q_3} + \underbrace{\frac{C^5 \sqrt{H'} t^2}{\sqrt{N_S} (N_j N)^{5/3}}}_{q_1' \neq 0, q_2 \neq q_3} \Bigg\}^{1/2} \\ &\ll t^{\varepsilon} \left( \frac{t}{(N_j N)^{5/6}} + \frac{t^{1/4}}{N_j^{5/6} N^{1/3}} + \frac{t^{5/4}}{N_j N} + \frac{C^{3/4} t^{5/4}}{(N_j N)^{13/12}} \right) =: \mathcal{B}_1. \end{aligned} \tag{9.18}$$

Inserting the above analysis back into (9.13) yields

$$\mathcal{R}_j^*(N) \ll \frac{t^\varepsilon}{C^{3/2}\sqrt{N_j}} \sup_{\|\beta\|_2=1} \sup_{|u_1|\ll 1} \Bigg\{ \int_{|u_2|\ll 1} \sum_{n_0} \sum_{q_0} \sum_{q_0|n_1|q_0^\infty} \sum_r \mathop{\sum\sum\sum}_{\substack{n_2 h_{1,0} h_{2,0}\ \square\text{-full} \\ h_{2,0}|(n_2 h_{1,0})^\infty \\ (n_2,q_0)=1}}$$

$$\mathop{\sum\sum\sum}_{\substack{N_S \ll n = n_1 n_2 n' \ll N_B \\ h_1 = h_{1,0} h_1', h_2 = h_{2,0} h_2' \sim_0 H' \\ n' h_1'\ \square\text{-free}, (n_1' h_1', h_2')=1 \\ (n' h_1' h_2', n_1 h_{1,0} h_{2,0} q_0)=1}} \mathop{\sum\sum}_{q_{1,0}, q_{2,0}|q_0^\infty} \mathop{\sum\sum}_{\substack{q_2 = n_0 q_0 q_{2,0} q_2' r \sim_0 C \\ q_3 = n_0 q_0 q_{2,0} q_3' r \sim_0 C}} \frac{h_2' \overline{\beta(q_2, h_2, u_2)} \beta(q_3, h_2, u_2)}{n_0 n_1^2 n_2^{3/2} n' h_1 \sqrt{h_{2,0} q_0^2 q_{1,0}^2} q_{2,0} r}$$

$$\times \sum_{Q_1 Q_S \ll |q_1'| \ll Q_1 Q_B} \mathcal{D}_2(q_2', q_3') I_2(q_1', q_2, q_3)\, du_2 \Bigg\}^{1/2} + \mathcal{B}_1,$$

where $\mathcal{D}_2(q_2', q_3')$ equals

$$\begin{aligned} S\big(q_1' \overline{n_1 n_2 h_{1,0} h_2 q_0 q_{1,0} q_{2,0}} + \eta_j (\overline{q_2'} - \overline{q_3'}) q_{1,0} \overline{n_1 n_2 h_2 q_{2,0}}, \\ \eta_j (q_2' - q_3') q_{2,0} \overline{n_1 n_2 h_{1,0} q_{1,0}}; n' h_1'\big) \\ \times \delta(q_1' \equiv -\eta_j q_0 q_{1,0}^2 h_1 (\overline{q_2'} - \overline{q_3'}) \bmod h_2') \mathcal{D}_0(q_2', q_3') \end{aligned} \tag{9.19}$$

and $\mathcal{D}_0(q_2', q_3')$ is as in (9.14).

*9.5.3. Third Cauchy–Schwarz inequality and Poisson summation.* Now we consider the remaining case $|q_1'| \gg Q_1 Q_S$ by iterating the above process.

Notice that by the same counting argument as in (7.12),

$$\sup_{\|\beta\|_2=1} \sum_{q_2 \ll C} \mathop{\sum\sum\sum\sum}_{\substack{n_0 q_0 q_{2,0} q_2' r = q_2 \\ q_{2,0}|q_0^\infty}} \sum_{\substack{q_{1,0} \ll C \\ q_{1,0}|q_0^\infty}} \sum_{q_0|n_1|q_0^\infty} \mathop{\sum\sum\sum}_{\substack{n_2 h_{1,0} h_{2,0}\ \square\text{-full} \\ h_{2,0}|(n_2 h_{1,0})^\infty}} \sum_{n' \ll N_B} \mathop{\sum\sum}_{h_1', h_2' \ll H'} \sum_{0 < |q_1'| \ll Q_1 Q_B}$$

$$\frac{1}{n_1 n_2^{3/2} n' h_{1,0}^{3/2} h_{2,0} h_1' q_0 q_{1,0} q_{2,0} |q_1'|} \int_{|u_2| \ll 1} |\beta(q_2, h_{2,0} h_2', u_2)|^2\, du_2 \sum_{\gamma \bmod n_1 n_2 h_{1,0} h_{2,0} q_0 q_{1,0} q_{2,0}}$$

$$\Bigg| \sum_{\beta_1 \bmod q_0 q_{1,0}}^\star \sum_{\beta_2 \bmod q_0 q_{2,0}}^\star$$

$$\delta\left( -\eta_j \frac{n_1 n_2 n'}{q_0} \equiv \overline{\beta_1 h_{1,0} h_1' \gamma q_{2,0} q_2'} + \overline{\beta_2 h_{2,0} h_2' q_2'} q_{1,0} \gamma \bmod q_0 q_{1,0} q_{2,0} \right) \Bigg|^2 \ll t^\varepsilon.$$

Recalling the definition of $\mathcal{D}_2$ in (9.19) and applying the Cauchy–Schwarz inequality to take out $n_0, n_1, n_2, n', q_0, q_{1,0}, q_{2,0}, q_1', q_2', h_{1,0}, h_1', h_{2,0}, h_2', \gamma$-sums we get, for the same fixed $\Upsilon \in C_c^\infty([1 - 10^{-9}, 1 + 10^{-9}])$ as before,

$$\mathcal{R}_j^*(N) \ll \frac{t^{\varepsilon}}{C^{3/2}\sqrt{N_j}} \sup_{\|\beta\|_2=1} \sup_{|u_1|\ll 1} \Bigg\{ \int_{|u_2|\ll 1} \sum_{n_0} \sum_{q_0} \sum\sum_{q_{1,0},q_{2,0}|q_0^\infty} \sum_{q_0|n_1|q_0^\infty} \sum_{r}$$

$$\sum\sum\sum_{\substack{n_2 h_{1,0} h_{2,0} \,\square\text{-full} \\ h_{2,0}|(n_2h_{1,0})^\infty \\ (n_2,q_0)=1}} \sum\sum\sum_{\substack{N_S \ll n=n_1n_2n' \ll N_B \\ h_1=h_{1,0}h_1', h_2=h_{2,0}h_2' \sim_0 H' \\ n'h_1' \,\square\text{-free}, (n_1'h_1',h_2')=1 \\ (n'h_1'h_2', n_1h_{1,0}h_{2,0}q_0)=1}} \sum\sum_{\substack{q_3=n_0q_0q_{2,0}q_3'r\sim_0 C \\ q_4=n_0q_0q_{2,0}q_4'r\sim_0 C}}$$

$$\sum_{Q_1Q_S \ll |q_1'| \ll Q_1Q_B} \sum_{q_2=n_0q_0q_{2,0}q_2'r} \Upsilon\left(\frac{q_2}{C}\right) \frac{h_2'^2 |q_1'| \beta(q_3,h_2,u_2)\overline{\beta(q_4,h_2,u_2)}}{n_0^2n_1^3n_2^{3/2}n'\sqrt{h_{1,0}}h_1'q_0^3q_{1,0}^3q_{2,0}r^2}$$

$$\times \mathcal{D}_2'(q_2',q_3')\mathcal{D}_2'(q_2',q_4')\overline{\mathcal{D}_0(q_3',q_4')} I_2(q_1',q_2,q_3)\overline{I_2(q_1',q_2,q_4)} du_2 \Bigg\}^{1/4}$$

$$+ \mathcal{B}_1, \tag{9.20}$$

where $\mathcal{D}_2'(q_2',q')$ equals

$$S(q_1'\overline{n_1n_2h_{1,0}h_2q_0q_{1,0}q_{2,0}} + \eta_j(\overline{q_2'}-\overline{q'})q_{1,0}\overline{n_1n_2h_2q_{2,0}},$$
$$\eta_j(q_2'-q')q_{2,0}\overline{n_1n_2h_{1,0}q_{1,0}}; n'h_1')$$
$$\times \delta(q_1' \equiv -\eta_j q_0 q_{1,0}^2 h_1(\overline{q_2'}-\overline{q'}) \bmod h_2')$$

with $\mathcal{D}_0(q_3',q_4')$ as in (9.14). Notice that $\mathcal{D}_2'$ is defined mod $n'h_1'h_2'$. Now we apply Poisson summation to the $q_2'$-sum to get

$$\sum_{q_2'} \Upsilon\left(\frac{n_0q_0q_{2,0}q_2'r}{C}\right)\mathcal{D}_2'(q_2',q_3')\overline{\mathcal{D}_2'(q_2',q_4')}I_2(q_1',q_2,q_3)\overline{I_2(q_1',q_2,q_4)}$$
$$= \frac{C}{n_0n'h_1'h_2'q_0q_{2,0}r} \sum_{q_2'} \sum_{\gamma \bmod n'h_1'h_2'} \mathcal{D}_2'(\gamma,q_3')\overline{\mathcal{D}_2'(\gamma,q_4')} e\left(\frac{\gamma q_2'}{n'h_1'h_2'}\right)$$
$$\times \int_0^\infty \Upsilon(y_3) I_2(q_1',Cy_3,q_3)\overline{I_2(q_1',Cy_3,q_4)} e\left(-\frac{q_2'Cy_3}{n_0n'h_1'h_2'q_0q_{2,0}r}\right) dy_3.$$

By a similar computation to that for $\mathcal{D}_1$ in the previous section with the Chinese Remainder Theorem, the character sum simplifies to

$$\sum_{\gamma \bmod n'h_1'h_2'} \mathcal{D}_2'(\gamma,q_3')\overline{\mathcal{D}_2'(\gamma,q_4')} e\left(\frac{\gamma q_2'}{n'h_1'h_2'}\right)$$
$$= \delta(q_3' \equiv q_4' \bmod h_2') e\left(\frac{h_{1,0}q_0q_{1,0}^2q_2'\overline{n'(h_1q_0q_{1,0}^2\overline{q_3'} - \eta_j q_1')}}{h_2'}\right) \mathcal{D}_3(q_3',q_4'),$$

where

$$\begin{aligned}\mathcal{D}_3(q_3',q_4') = \sideset{}{^\star}\sum_{\gamma \bmod n'h_1'} e\Big(\frac{\gamma q_2'\overline{h_2'}}{n'h_1'}\Big) \\ \times S\big(q_1'\overline{n_1n_2h_{1,0}h_2q_0q_{1,0}q_{2,0}} + \eta_j(\overline{\gamma}-\overline{q_3'})q_{1,0}\overline{n_1n_2h_2q_{2,0}}, \\ \eta_j(\gamma-q_3')q_{2,0}\overline{n_1n_2h_{1,0}q_{1,0}}; n'h_1'\big) \\ \times S\big(q_1'\overline{n_1n_2h_{1,0}h_2q_0q_{1,0}q_{2,0}} + \eta_j(\overline{\gamma}-\overline{q_4'})q_{1,0}\overline{n_1n_2h_2q_{2,0}}, \\ \eta_j\eta_j(\gamma-q_4')q_{2,0}\overline{n_1n_2h_{1,0}q_{1,0}}; n'h_1'\big). \qquad (9.21)\end{aligned}$$

Applying Lemma B.1, we obtain

$$\mathcal{D}_3(q_3',q_4') \ll (n'h_1')^2 t^\varepsilon \delta(*_0) + (n'h_1')^{3/2}t^\varepsilon, \qquad (9.22)$$

where $*_0$ is the condition

$$\begin{aligned}*_0 = \{&q_3' \equiv q_4' \bmod n'h_1' \\ &\text{or } q_1'\overline{h_{1,0}q_0q_{1,0}} \pm q_{1,0}(\overline{q_3'}-\overline{q_4'}) \equiv 0 \bmod n'h_1' \\ &\text{or } q_1'\overline{h_{1,0}q_0q_{1,0}} - \eta_j q_{1,0}\overline{q_3'} \equiv 0 \bmod n'h_1' \\ &\text{or } q_1'\overline{h_{1,0}q_0q_{1,0}} - \eta_j q_{1,0}\overline{q_4'} \equiv 0 \bmod n'h_1'\}.\end{aligned}$$

Using (9.11) as before, repeated integration by parts now gives us arbitrary savings unless

$$|q_2'| \ll \frac{n_0n'h_1'h_2'q_0q_{2,0}r}{C}\Big(1+\frac{CN^{1/3}}{nN_j^{2/3}}+\frac{(N_jN)^{1/3}}{C}\Big)t^\varepsilon \ll Q_B,$$

with $Q_B$ defined in (9.12). Inserting this back into (9.20), we get

$$\begin{aligned}\mathcal{R}_j^*(N) \ll \frac{t^\varepsilon}{C^{3/2}\sqrt{N_j}} \sup_{\|\beta\|_2=1}\sup_{|u_1|\ll 1}\Big\{C\int_{|u_2|\ll 1}\sum_{n_0}\sum_{q_0}\sum\sum_{q_{1,0},q_{2,0}|q_0^\infty}\sum_{q_0|n_1|q_0^\infty}\sum_r \\ \sum\sum\sum_{\substack{n_2h_{1,0}h_{2,0}\ \square\text{-full} \\ h_{2,0}|(n_2h_{1,0})^\infty \\ (n_2,q_0)=1}} \sum\sum\sum_{\substack{N_S\ll n=n_1n_2n'\ll N_B \\ h_1=h_{1,0}h_1',h_2=h_{2,0}h_2'\sim_0 H' \\ n'h_1'\ \square\text{-free},(n_1'h_1',h_2')=1 \\ (n'h_1'h_2',n_1h_{1,0}h_{2,0}q_0)=1}} \sum\sum_{\substack{q_3=n_0q_0q_{2,0}q_3'r\sim_0 C \\ q_4=n_0q_0q_{2,0}q_4'r\sim_0 C \\ q_3'\equiv q_4' \bmod h_2'}} \\ \sum_{Q_1Q_S\ll|q_1'|\ll Q_1Q_B}\sum_{|q_2'|\ll Q_B}\frac{h_2'|q_1'|\beta(q_3,h_2,u_2)\overline{\beta(q_4,h_2,u_2)}}{n_0^3n_1^3n_2^{3/2}n'^2\sqrt{h_{1,0}}h_1'^2q_0^4q_{1,0}^3q_{2,0}^2r^3} \\ \times \mathcal{D}_3(q_3',q_4')\mathcal{D}_{3,0}(q_3',q_4')I_3(q_1',q_2',q_3,q_4)\,du_2\Big\}^{1/4} + \mathcal{B}_1, \qquad (9.23)\end{aligned}$$

where $\mathcal{D}_{3,0}(q_3',q_4')$ equals

$$e\Big(\frac{h_{1,0}q_0q_{1,0}^2q_2'\overline{n'(h_1q_0q_{1,0}^2\overline{q_3'}-\eta_jq_1')}}{h_2'}\Big)\overline{\mathcal{D}_0(q_3',q_4')} \ll n_1n_2h_{1,0}h_{2,0}q_0^3q_{1,0}q_{2,0} \qquad (9.24)$$

by (9.14), and $I_3(q_1', q_2', q_3, q_4)$ equals

$$\int_0^\infty \Upsilon(y_3) I_2(q_1', Cy_3, q_3)\overline{I_2(q_1', Cy_3, q_4)} e\left(-\frac{q_2' C y_3}{n_0 n' h_1' h_2' q_0 q_{2,0} r}\right) dy_3. \tag{9.25}$$

Now we want to again separate the contribution when $|q_2'|$ is small, arguing similarly to the case $|q_1'| \ll Q_1 Q_S$. To be precise, we first deal with the case

$$|q_2'| \ll Q_S = \frac{n_0 q_0 q_{2,0} r C^3 t^{2+\varepsilon}}{n_1 n_2 n h_{1,0} h_{2,0} (N_j N)^{4/3}}$$

including the case $q_2' = 0$. Applying Lemma 5.2 (1) to the $y_3$-integral, we get arbitrary savings unless there exists $y_3' \in [1/2, 5/2]$ such that

$$\left| \frac{d}{dy_3}\bigg|_{y=y_3'} (G(y_1, y_3) - G(y_2, y_3)) \right| \ll \frac{C^2 N^{2/3} t^\varepsilon}{n^2 N_j^{4/3}}.$$

Applying the same analysis as in the previous subsection (to get (A.2)) with Lemma A.1, (A.1) on $z_0(Cy_1, y_3), z_0(Cy_2, y_3) \sim 1$ in $\Psi$, the above bound for the derivative yields

$$|y_1 - y_2| \ll \frac{C^2 t^\varepsilon}{n N_j}. \tag{9.26}$$

Inserting these conditions into $I_3(q_1', q_2', q_3, q_4)$, writing $y_2 = y_1 + v$ and bounding the $y_3$-integral trivially yields, for $|q_2'| \ll Q_S$,

$$\begin{aligned} I_3(q_1', q_2', q_3, q_4) \ll t^\varepsilon \sup_{y_3 \sim 1} \bigg| \int_{|v| \ll \frac{C^2 t^\varepsilon}{n N_j}} \int_0^\infty & \Upsilon(y_1)\Upsilon(y_1+v)\Psi(Cy_1, Cy_3, q_3) \\ \times \overline{\Psi(C(y_1+v), Cy_3, q_4)} e\bigg(G(y_1, y_3) - G\left(y_1, \frac{q_3}{C}\right) & - G(y_1+v, y_3) + G\left(y_1+v, \frac{q_4}{C}\right) \\ & + \frac{q_1' C v}{n_0 n h_1 h_2 q_0^2 q_{1,0}^2 q_{2,0} r}\bigg) dy_1\, dv \bigg| + t^{-A} \end{aligned}$$

for any $A > 0$. Applying Taylor expansion to the functions involving $1 + \frac{v}{y_1}$ and $1 + \frac{v}{y_1}$, using $H \gg N/t^{1-\varepsilon}$, and bounding the $v_1$-integral trivially, there exists some $\frac{CN}{n^{3/2} N_j \sqrt{t}} t^\varepsilon$-inert function $\varphi$ such that $I_3(q_1', q_2', q_3, q_4)$ is

$$O\left(\frac{C^2 t^\varepsilon}{n N_j} \sup_{|v_1| \ll \frac{C^2 t^\varepsilon}{n N_j}} \left| \int_0^\infty \Upsilon(y)\Psi_2(Cy, q_3, q_4) e\left(-G\left(y, \frac{q_3}{C}\right) + G\left(y, \frac{q_4}{C}\right)\right) dy \right| + t^{-A}\right)$$

with

$$\Psi_2(Cy, q_3, q_4) = \Psi(Cy, Cy_3, q_3)\overline{\Psi(C(y+v), Cy_3, q_4)}\varphi(y, y_3, v, q_3, q_4),$$

which is a $(\frac{CN^{1/3}}{nN_j^{2/3}} + \frac{CN}{n^{3/2}N_j\sqrt{t}})t^\varepsilon$-inert function. Applying the same repeated integration by parts process to the $y$-integral using (9.11), we get

$$I_3(q_1', q_2', q_3, q_4) \ll \frac{C^2 t^\varepsilon}{nN_j}\delta\left(|q_3 - q_4| \ll \left(\frac{C^2}{(N_j N)^{1/3}} + \frac{C^2 N^{1/3}}{\sqrt{nt}N_j^{2/3}}\right)t^\varepsilon\right) + t^{-A}$$

for any $A > 0$.

Inserting (9.24) and the above bound for $I_3$ into (9.23), we deduce that the contribution of $|q_2'| \ll Q_S$ is

$$\ll \frac{t^\varepsilon}{C^{3/2}\sqrt{N_j}} \sup_{\|\beta\|_2=1} \sup_{|u_1|\ll 1} \Bigg\{ C \int_{|u_2|\ll 1} \sum_{n_0}\sum_{q_0} \sum\sum_{q_{1,0},q_{2,0}|q_0^\infty} \sum_{q_0|n_1|q_0^\infty} \sum_r$$

$$\sum\sum\sum_{\substack{n_2 h_{1,0} h_{2,0}\ \square\text{-full} \\ h_{2,0}|(n_2 h_{1,0})^\infty \\ (n_2,q_0)=1}} \sum\sum\sum_{\substack{N_S \ll n = n_1 n_2 n' \ll N_B \\ h_1 = h_{1,0}h_1', h_2 = h_{2,0}h_2' \sim_0 H' \\ n'h_1'\ \square\text{-free}, (n'h_1', h_2')=1 \\ (n'h_1'h_2', n_1 h_{1,0} h_{2,0} q_0)=1}} \sum\sum_{\substack{q_3 = n_0 q_0 q_{2,0} q_3' r \sim_0 C \\ q_4 = n_0 q_0 q_{2,0} q_4' r \sim_0 C \\ |q_3 - q_4| \ll (\frac{C^2}{(N_j N)^{1/3}} + \frac{C^2 N^{1/3}}{\sqrt{nt}N_j^{2/3}})t^\varepsilon \\ q_3' \equiv q_4' \bmod h_2'}}$$

$$\sum_{Q_1 Q_S \ll |q_1'| \ll Q_1 Q_B} \sum_{|q_2'| \ll Q_S} \frac{\sqrt{h_{1,0}}h_2 |q_1'|\, |\beta(q_3, h_2, u_2)\beta(q_4, h_2, u_2)|}{n_0^3 n_1^2 \sqrt{n_2} n'^2 h_1'^2 q_0 q_{1,0}^2 q_{2,0} r^3}$$

$$\times \frac{C^2}{nN_j} |\mathcal{D}_3(q_3', q_4')| \mathcal{D}_{3,0}(q_3', q_4')\, du_2 \Bigg\}^{1/4}.$$

Recall that $Q_S, Q_B$ are defined in (9.17) and (9.12) respectively. Applying the AM-GM inequality

$$|\beta(q_2, h_2, u_2)\beta(q_3, h_2, u_2)| \ll |\beta(q_2, h_2, u_2)|^2 + |\beta(q_3, h_2, u_2)|^2$$

and using (9.22) and (9.24) with the same counting argument as in (7.12) shows that the above is

$$\ll \frac{t^\varepsilon}{C^{3/2}\sqrt{N_j}} \Bigg\{ \sum_{n \ll CN^{1/3}t^\varepsilon/N_j^{2/3}} \frac{C^3}{nN_j}\left(\frac{nN_j^{1/3}t^2}{N^{5/3}}\right)^2 \left(1 + \frac{C^3 t^2}{n(N_j N)^{4/3}}\right)\frac{1}{n^2}\left(\frac{Ct}{N}\right)^{-1}$$

$$\times \Bigg(\underbrace{\frac{C^2}{(N_j N)^{1/3}} + \frac{C^2 N^{1/3}}{\sqrt{nt}N_j^{2/3}}}_{q_3' \neq q_4'} + \underbrace{\frac{Ct}{N}}_{q_3' = q_4'}\Bigg)\left(\frac{nCt}{N}\right)^{3/2}\Bigg\}^{1/4}$$

$$\ll \frac{t^{9/8+\varepsilon}}{C^{1/4}(N_j N)^{5/6}}\left(1 + \frac{\sqrt{Ct}}{N_j^{1/6}N^{5/12}}\right)\left(\frac{C^2}{(N_j N)^{1/3}} + \frac{Ct}{N}\right)^{1/4}.$$

Indeed, with the same process we get the following technical lemma that we will apply repeatedly later.

**Lemma 9.3.** *Let $q_3, q_4 \sim C$. Let $B \ll \frac{C^2 N^{2/3} t^\varepsilon}{n^2 N_j^{4/3}}$ and let $\eta = 0$ or $1$. Let $W$ be a $(\frac{CN^{1/3}}{nN_j^{2/3}} + \eta \frac{CN}{n^{3/2} N_j \sqrt{t}}) t^\varepsilon$-inert function and let $J$ be equal to*

$$\int_0^\infty \Upsilon(y) \int_0^\infty \Upsilon(y_1) \int_0^\infty \Upsilon(y_2) \Psi(Cy_1, Cy, q_3) \overline{\Psi(Cy_2, Cy, q_4)} W(y, y_1, y_2, q_3, q_4) \times e\left( G(y_1, y) - G\left(y_1, \frac{q_3}{C}\right) - G(y_2, y) + G\left(y_2, \frac{q_4}{C}\right) + (1-\eta) B y \right) dy_2\, dy_1\, dy.$$

*Then there exists some $(\frac{CN^{1/3}}{nN_j^{2/3}} + \frac{CN}{n^{3/2} N_j \sqrt{t}}) t^\varepsilon$-inert function $\tilde{W}$ and some function $F$ such that $J$ equals*

$$\int_{1/2}^{5/2} \int_{|v| \ll ((1-\eta)\frac{C^2}{nN_j} + \eta(\frac{C}{(N_j N)^{1/3}} + \frac{CN^{1/3}}{\sqrt{nt} N_j^{2/3}})) t^\varepsilon} F(u, v) \times \int_0^\infty \Upsilon(y) \Upsilon(y+v) \Psi(Cy, Cu, q_3) \times \overline{\Psi(C(y+v), Cu, q_4)} \tilde{W}(u, y, y+v, q_3, q_4) e\left( -G\left(y, \frac{q_3}{C}\right) + G\left(y, \frac{q_4}{C}\right) \right) dy\, dv\, du.$$

*Moreover,*

$$J \ll \left( (1-\eta) \frac{C^2}{nN_j} + \eta \left( \frac{C}{(N_j N)^{1/3}} + \frac{CN^{1/3}}{\sqrt{nt} N_j^{2/3}} \right) \right) t^\varepsilon \times \delta\left( |q_3 - q_4| \ll \left( \frac{C^2}{(N_j N)^{1/3}} + \frac{C^2 N^{1/3}}{\sqrt{nt} N_j^{2/3}} \right) t^\varepsilon \right) + t^{-A}$$

*for any $A > 0$.*

Altogether we have

$$\mathcal{R}_j^*(N) \ll \frac{t^\varepsilon}{C^{3/2}\sqrt{N_j}} \sup_{\|\beta\|_2 = 1} \sup_{|u_1| \ll 1} \Bigg\{ C \int_{|u_2| \ll 1} \sum_{n_0} \sum_{q_0} \sum_{q_{1,0}, q_{2,0} | q_0^\infty} \sum_{q_0 | n_1 | q_0^\infty} \sum_r \sum_{\substack{n_2 h_{1,0} h_{2,0}\ \square\text{-full} \\ h_{2,0} | (n_2 h_{1,0})^\infty \\ (n_2, q_0) = 1}} \sum_{\substack{N_S \ll n = n_1 n_2 n' \ll N_B \\ h_1 = h_{1,0} h_1', h_2 = h_{2,0} h_2' \sim_0 H' \\ n' h_1'\ \square\text{-free}, (n_1' h_1', h_2') = 1 \\ (n' h_1' h_2', n_1 h_{1,0} h_{2,0} q_0) = 1}} \sum_{\substack{q_3 = n_0 q_0 q_{2,0} q_3' r \sim_0 C \\ q_4 = n_0 q_0 q_{2,0} q_4' r \sim_0 C \\ q_3' \equiv q_4' \bmod h_2'}} \sum_{Q_1 Q_S \ll |q_1'| \ll Q_1 Q_B} \sum_{Q_S \ll |q_2'| \ll Q_B} \frac{h_2' |q_1'| \beta(q_3, h_2, u_2) \overline{\beta(q_4, h_2, u_2)}}{n_0^3 n_1^3 n_2^{3/2} n'^2 \sqrt{h_{1,0}} h_1'^2 q_0^4 q_{1,0}^3 q_{2,0}^2 r^3} \times \mathcal{D}_3(q_3', q_4') \mathcal{D}_{3,0}(q_3', q_4') I_3(q_1', q_2', q_3, q_4)\, du_2 \Bigg\}^{1/4} + \mathcal{B}_1 + \mathcal{B}_2,$$

where $\mathcal{B}_1$ is defined in (9.18) and

$$\mathcal{B}_2 := \frac{t^{9/8+\varepsilon}}{C^{1/4}(N_j N)^{5/6}} \left( 1 + \frac{\sqrt{Ct}}{N_j^{1/6} N^{5/12}} \right) \left( \frac{C^2}{(N_j N)^{1/3}} + \frac{Ct}{N} \right)^{1/4}. \tag{9.27}$$

*9.5.4. Iteration of the above process.* Iterating the above process, we obtain the following lemma.

**Lemma 9.4.** *For any positive integer $\nu \geq 2$, we have*

$$\mathcal{R}_j^*(N) \ll |\mathcal{A}_\nu| + \sum_{j=1}^{\nu} \mathcal{B}_j,$$

*where $\mathcal{A}_\nu$ equals*

$$\frac{t^\varepsilon}{C^{3/2}\sqrt{N_j}} \sup_{\|\beta\|_2=1} \sup_{|u_1|\ll 1} \Bigg\{ \int_{|u_2|\ll 1} \sum_{n_0}\sum_{q_0} \sum\sum_{q_{1,0},q_{2,0}|q_0^\infty} \sum_{q_0|n_1|q_0^\infty} \sum_r \sum_{Q_1Q_S\ll|q_1'|\ll Q_1Q_B}$$
$$\sum_{Q_S\ll|q_2'|,\ldots,|q_\nu'|\ll Q_B}\!\!\cdots\!\!\sum \sum\sum\sum_{\substack{n_2h_{1,0}h_{2,0}\ \square\text{-full}\\ h_{2,0}|(n_2h_{1,0})^\infty\\ (n_2,q_0)=1}} \sum\sum\sum_{\substack{N_S\ll n=n_1n_2n'\ll N_B\\ h_1=h_{1,0}h_1',h_2=h_{2,0}h_2'\sim_0 H'\\ n'h_1'\ \square\text{-free},(n_1'h_1',h_2')=1\\ (n'h_1'h_2',n_1h_{1,0}h_{2,0}q_0)=1}} \sum\sum_{\substack{q_{\nu+1}=n_0q_0q_{2,0}q_{\nu+1}'r\sim_0 C\\ q_{\nu+2}=n_0q_0q_{2,0}q_{\nu+2}'r\sim_0 C\\ q_{\nu+1}'\equiv q_{\nu+2}' \bmod h_2'}}$$
$$\left(\frac{\sqrt{h_{1,0}h_{2,0}}C}{n_0^2n_1n'h_1'q_0^2q_{1,0}q_{2,0}r^2}\right)^{2^{\nu-1}} \frac{n_0h_2'r}{n_1n_2^{3/2}h_{1,0}^{3/2}h_{2,0}q_{1,0}C} \prod_{j=1}^{\nu-1}|q_j|^{2^{\nu-j}-1}D(\nu)$$
$$\times \beta'(q_{\nu+1},h_2,u_2)\overline{\beta'(q_{\nu+2},h_2,u_2)}\mathcal{D}_{\nu+1}(q_{\nu+1}',q_{\nu+2}')\mathcal{D}_0'(q_{\nu+1}',q_{\nu+2}')$$
$$\times I_{\nu+1}(q_1',\ldots,q_\nu',q_{\nu+1},q_{\nu+2})\,du_2\Bigg\}^{2^{-\nu}},$$

*with the notations $\beta' = \beta$ or $\overline{\beta}$, $\mathcal{D}_0' = \mathcal{D}_0$ or $\overline{\mathcal{D}_0}$ depending on $\nu$, with $\mathcal{B}_1$, $\mathcal{D}_0$, $\mathcal{D}_3$, $I_3$ as defined in (9.18), (9.14), (9.21) and (9.25) respectively, and*

$$D(\nu) = \begin{cases} e\left(\dfrac{h_{1,0}q_0q_{1,0}^2q_2'\overline{n'(h_1q_0q_{1,0}^2\overline{q_3'} - \eta_jq_1')}}{h_2'}\right), & \nu = 2,\\ e\left(\dfrac{q_\nu'q_{\nu+1}'\overline{n'h_1'}}{h_2'}\right), & \nu > 2.\end{cases}$$

*The character sum $\mathcal{D}_\nu(q_{\nu+1}',q_{\nu+2}')$ and the integral $I_\nu(q_1',\ldots,q_\nu',q_{\nu+1},q_{\nu+2})$ are defined recursively by*

$$\mathcal{D}_{\nu+1}(q_{\nu+1}',q_{\nu+2}') = \sum_{\gamma \bmod n'h_1'} \mathcal{D}_\nu(\gamma,q_{\nu+1}')\overline{\mathcal{D}_\nu(\gamma,q_{\nu+2}')}e\left(\frac{\gamma q_\nu'\overline{h_2'}}{n'h_1'}\right)$$

*and*

$$I_{\nu+1}(q_1',\ldots,q_\nu',q_{\nu+1},q_{\nu+2}) = \int_0^\infty \Upsilon(y_{2^\nu-1})I_\nu(q_1',\ldots,q_{\nu-1}',Cy_{2^\nu-1},q_{\nu+1})$$
$$\times\overline{I_\nu(q_1',\ldots,q_{\nu-1}',Cy_{2^\nu-1},q_{\nu+2})}e\left(-\frac{q_\nu'Cy_{2^\nu-1}}{n_0n'h_1h_2q_0q_{2,0}r}\right)dy_{2^\nu-1},$$

*and for* $\ell \geq 2$,

$$\mathcal{B}_\ell = \frac{\sqrt{C}t^{2+\varepsilon}}{N_j N^{3/2}} \left\{ \frac{N_j^{2/3} N^{8/3}}{C^3 t^{7/2}} \left(1 + \frac{C^2 t^2}{N_j^{2/3} N^{5/3}}\right) \left(\frac{Ct}{N} + \frac{C^2}{(N_j N)^{1/3}}\right) \right\}^{2-\ell}.$$

*Moreover,*

$$\mathcal{D}_{\nu+1}(q'_{\nu+1}, q'_{\nu+2}) \ll (n'h'_1)^{2^{\nu-1}-1/2} t^\varepsilon \left(\sqrt{n'h'_1}\,\delta(*_{\nu-2}) + 1\right),$$

*where* $*_{\nu-2}$ *is the condition*

$$\begin{aligned} *_{\nu-2} = \{ & q'_{\nu+1} \equiv q'_{\nu+2} \bmod n'h'_1 \\ & \textit{or } q'_1 \overline{q_0 q_{1,0}} \pm h_1 q_{1,0} (\overline{q'_{\nu+1}} - \overline{q'_{\nu+2}}) \equiv 0 \bmod n'h'_1 \\ & \textit{or } q'_1 \overline{q_0 q_{1,0}} - \eta_j h_1 q_{1,0} \overline{q'_{\nu+1}} \equiv 0 \bmod n'h'_1 \\ & \textit{or } q'_1 \overline{q_0 q_{1,0}} - \eta_j h_1 q_{1,0} \overline{q'_{\nu+2}} \equiv 0 \bmod n'h'_1 \}. \end{aligned}$$

*Proof.* The proof is presented in Appendix B.2. ∎

With the above lemma and the bound for $\mathcal{D}_0$ in (9.14), we are left to bound $I_{\nu+1}(q'_1, \ldots, q'_\nu, q_{\nu+1}, q_{\nu+2})$. We do so with the following lemma.

**Lemma 9.5.** *Let* $q'_1, \ldots, q'_\nu, q_{\nu+1}, q_{\nu+2}$ *be integers such that*

$$|q'_1| \gg Q_1 Q_S, \quad |q'_2|, \ldots, |q'_\nu| \gg Q_S \quad \textit{and} \quad q_{\nu+1}, q_{\nu+2} \sim C.$$

*With* $I_3$ *defined in* (9.25) *and* $I_\nu$ *defined in Lemma* 9.4, *we have*

$$\begin{aligned} I_{\nu+1}(q'_1, \ldots, q'_\nu, q_{\nu+1}, q_{\nu+2}) \ll_\nu & \prod_{j=2}^{\nu} \left(1 + \frac{(N_j N)^{1/3} n_0 n' h_1 h_2 q_0 q_{2,0} r}{\sqrt{\min\{q''_u : 1 \leq u \leq j-1\} q''_j C^2}}\right)^{2^{\nu-j}} \\ & \times \frac{(n_1 n_2 h_{1,0} h_{2,0} q_0 q_{1,0}^2)^{2^{\nu-2}} (n_0 n' h_1 h_2 q_0 q_{2,0} r)^{2^{\nu-1}-1/2}}{q'^{2^{\nu-2}}_1 q'^{2^{\nu-3}}_2 \cdots q'^{2^{-1}}_\nu C^{2^{\nu-1}-1/2}} t^\varepsilon \end{aligned}$$

*for* $\nu \geq 2$.

*Proof.* The proof is presented in Appendix B.3 via a careful iterative stationary phase analysis. Indeed, we have to use (3) instead of (2) of Lemma 5.2, and the parameter $U$ for each stationary phase step has to be chosen carefully to yield a recursive structure. ∎

Inserting the above bound into the bound of $\mathcal{R}_j^*(N)$ in Lemma 9.4, together with

$$\mathcal{D}(k+1) \ll 1, \quad \mathcal{D}_0 \ll n_1 n_2 h_{1,0} h_{2,0} q_0^3 q_{1,0} q_{2,0}$$

as in (9.14) and the last statement of Lemma 9.4 on $\mathcal{D}_{\nu+1}(q'_{\nu+1}, q'_{\nu+2})$, we get, for any $\nu \geq 2$,

$$\begin{aligned}
&\mathcal{R}_j^*(N)\\
&\ll \frac{t^\varepsilon}{C^{3/2}\sqrt{N_j}} \sup_{\|\beta\|_2=1} \sup_{|u_1|\ll 1} \Bigg\{ \int_{|u_2|\ll 1} \sum_{n_0,r} \sum_{q_0} \sum\sum_{q_{1,0},q_{2,0}|q_0^\infty} \sum_{q_0|n_1|q_0^\infty} \sum_{Q_1 Q_S \ll |q_1'| \ll Q_1 Q_B}\\
&\quad \sum_{Q_S \ll |q_2'|,\dots,|q_\nu'| \ll Q_B} \cdots \sum \sum\sum\sum_{\substack{n_2 h_{1,0} h_{2,0}\ \square\text{-full}\\ h_{2,0}|(n_2 h_{1,0})^\infty\\ (n_2,q_0)=1}} \sum\sum\sum_{\substack{N_S \ll n=n_1n_2n' \ll N_B\\ h_1=h_{1,0}h_1',h_2=h_{2,0}h_2' \sim_0 H'\\ n'h_1'\ \square\text{-free},(n_1'h_1',h_2')=1\\ (n'h_1'h_2',n_1h_{1,0}h_{2,0}q_0)=1}} \sum\sum_{\substack{q_{\nu+1}=n_0q_0q_{2,0}q_{\nu+1}'r\sim_0 C\\ q_{\nu+2}=n_0q_0q_{2,0}q_{\nu+2}'r\sim_0 C\\ q_{\nu+1}'\equiv q_{\nu+2}' \bmod h_2'}}\\
&\quad \left(\frac{\sqrt{h_{1,0}h_{2,0}}C}{n_0^2 n_1 n' h_1' q_0^2 q_{1,0} q_{2,0} r^2}\right)^{2^{\nu-1}} \frac{n_0 h_2' r}{n_1 n_2^{3/2} h_{1,0}^{3/2} h_{2,0} q_{1,0} C} \prod_{j=1}^{\nu-1} |q_j|^{2^{\nu-j}-1} |\beta(q_{\nu+1},h_2,u_2)|\\
&\quad \times |\beta(q_{\nu+2},h_2,u_2)| du_2 (n'h_1')^{2^{\nu-1}-1/2} \big(\sqrt{n'h_1'}\delta(*_{\nu-2})+1\big) n_1 n_2 h_{1,0} h_{2,0} q_0^3 q_{1,0} q_{2,0}\\
&\quad \times \prod_{j=2}^{\nu} \left(1+\frac{(N_jN)^{1/3} n_0 n' h_1 h_2 q_0 q_{2,0} r}{\sqrt{\min\{q_u'' : 1\le u\le j-1\} q_j'' C^2}}\right)^{2^{\nu-j}}\\
&\quad \times \frac{(n_1 n_2 h_{1,0} h_{2,0} q_0 q_{1,0}^2)^{2^{\nu-2}} (n_0 n' h_1 h_2 q_0 q_{2,0} r)^{2^{\nu-1}-1/2}}{q_1'^{2^{\nu-2}} q_2'^{2^{\nu-3}} \cdots q_\nu'^{2^{-1}} C^{2^{\nu-1}-1/2}} \Bigg\}^{2^{-\nu}} + \sum_{\ell=1}^{\nu} \mathcal{B}_\ell
\end{aligned}$$

with $(*_{\nu-2})$ as defined in Lemma 9.4. Recall the definition of $Q_B$ in (9.12). Applying the AM-GM inequality

$$|\beta(q_{\nu+1},h_2,u_2)\beta(q_{\nu+2},h_2,u_2)| \ll |\beta(q_{\nu+1},h_2,u_2)|^2 + |\beta(q_{\nu+2},h_2,u_2)|^2$$

and the same counting argument as in (7.12) implies that the above is

$$\begin{aligned}
&\mathcal{R}_j^*(N) \ll\\
&\frac{t^\varepsilon}{C^{3/2}\sqrt{N_j}} \Bigg\{ \sum_{n \ll CN^{1/3}t^\varepsilon/N_j^{2/3}} \frac{n^{2^{\nu-1}-1/2}}{\sqrt{C}} \left(\frac{Ct}{N}\right)^{2^\nu} \left(\frac{nN_j^{1/3}t^2}{N^{5/3}}\right)^{2^{\nu-1}-3/2} \left(\frac{Ct}{N}+C\right)\Bigg\}^{2^{-\nu}}\\
&\qquad + \nu \sup_{1\le \ell\le \nu} \mathcal{B}_\ell \ll \frac{\sqrt{C} t^{2+\varepsilon+99/2^\nu}}{N_j N^{3/2}} + \nu \sup_{1\le\ell\le\nu} \mathcal{B}_\ell \qquad (9.28)
\end{aligned}$$

for $N \gg t^{1-\varepsilon}$.

**Remark 9.6.** To get the above bound in the first line, first observe that the worst case scenario for $q_j'$ is when $q_2',\dots,q_\nu'$ are as large as possible, and $|q_1'|$ is either 1 or as large as possible.

### 9.6. *Bound for $\mathcal{R}_j(N)$*

Recall the definition of $\mathcal{R}_j(N)$ in Lemma 9.1, and the definition of $\mathcal{B}_\ell$ in (9.18) and in Lemma 9.4. Taking $\nu = t^\varepsilon$ in (9.28) and combining it with the bounds (9.1), (9.2) and

(9.4) we get, for $N \gg t^{1-\varepsilon}$,

$$\begin{aligned}\mathcal{R}_j(N) \ll t^{\varepsilon}\bigg\{&\frac{1}{C^2H}+\frac{Ht}{CN_j^{1/3}N^{4/3}}+\frac{t}{N_jN}\bigg(1+\frac{C^2}{(N_jN)^{1/3}}+\frac{CN}{Ht}\bigg)\delta(N_j \ll C^2t^{\varepsilon})\\ &+\frac{\sqrt{t}}{\sqrt{CH}N_j^{5/6}N^{1/3}}\bigg(1+\frac{CN}{Ht}\bigg)\delta\bigg(N_j \ll \frac{C^{3/2}N^2}{(Ht)^{3/2}}t^{\varepsilon}\bigg)+\frac{\sqrt{C}t^2}{N_jN^{3/2}}\\ &+\frac{t}{(N_jN)^{5/6}}+\frac{t^{1/4}}{N_j^{5/6}N^{1/3}}+\frac{t^{5/4}}{N_jN}+\frac{C^{3/4}t^{5/4}}{(N_jN)^{13/12}}\\ &+\sup_{2\le\ell\le t^{\varepsilon}}\frac{\sqrt{C}t^2}{N_jN^{3/2}}\bigg\{\frac{N_j^{2/3}N^{8/3}}{C^3t^{7/2}}\bigg(1+\frac{C^2t^2}{N_j^{2/3}N^{5/3}}\bigg)\bigg(\frac{Ct}{N}+\frac{C^2}{(N_jN)^{1/3}}\bigg)\bigg\}^{2-\ell}\bigg\}. \end{aligned} \tag{9.29}$$

## 10. Final bound

We choose $Q$ such that $\frac{N}{t^{1-\varepsilon}} < Q < \frac{N}{\sqrt{Ht}}t^{\varepsilon}$. Inserting (9.29) into Lemma 9.1, the conditions $N/t^{1+\varepsilon} \ll C \ll Qt^{\varepsilon}, t^{1/2+\varepsilon} \ll H \ll Nt^{\varepsilon}/M, t^{1/2+\varepsilon} \ll M \ll t^{1-\varepsilon}, N \ll t^{3/2+\varepsilon}$, $N_1 \sim N^2X^3/Q^3$ and $N_2 \ll (N^2/Q^3+(CHt)^3/N^4)t^{\varepsilon} \ll N^2t^{\varepsilon}/Q^3$ (by the choice of $Q$) imply

$$\begin{aligned}&(S_{1,X}(N)S_2(N))^{1/2}\\ &\quad\ll \frac{CM(N_1N_2)^{2/3}N^{4/3}}{t^{1-\varepsilon}}\bigg\{\bigg(\frac{Ct}{N_1N}\delta\bigg(N_1 \ll \frac{C^3t^{\varepsilon}}{N}\bigg)+\mathcal{R}_1(N)\bigg)\mathcal{R}_2(N)\bigg\}^{1/2}\\ &\quad\ll t^{\varepsilon}\bigg\{Q^3M+\frac{MHN^2}{Q^3}+Q^{5/3}MN^{1/3}\bigg(1+\frac{Q^{4/3}}{N^{1/3}}+\frac{QN}{Ht}\bigg)\\ &\qquad+\frac{Q^{5/4}MN^2}{(Ht)^{5/4}}\bigg(1+\frac{QN}{Ht}\bigg)+\frac{MN^{3/2}}{\sqrt{Q}}+QMN^{3/4}t^{1/4}\\ &\qquad+\sqrt{QN}Mt\bigg(1+\frac{N}{Q^{5/4}t^{7/8}}\bigg(1+\frac{Q\sqrt{t}}{N^{3/4}}\bigg)\bigg(\bigg(\frac{Qt}{N}\bigg)^{1/4}+\frac{Q^{3/4}}{N^{1/4}}\bigg)\bigg)\bigg\},\end{aligned}$$

including the case $X=0$.

We choose $Q = M^{5/6}H^{1/3}N^{1/3}/t^{3/4}$. Then $Q$ satisfies $\frac{N}{t^{1-\varepsilon}} < Q < \frac{N}{\sqrt{Ht}}t^{\varepsilon}$ when $H \gg \frac{N^2}{M^{5/2}t^{3/4-\varepsilon}}$. Inserting the bound into Theorem 6.2 and then using (6.3) we get, for $t^{1/2+\varepsilon} \ll M \ll t^{1-\varepsilon}$, $N \ll t^{3/2+\varepsilon}$, $\sqrt{t}+\frac{N^2}{M^{5/2}t^{3/4-\varepsilon}} \ll H \ll \frac{Nt^{\varepsilon}}{M}$,

$$\begin{aligned}\frac{S^{\pm}_{M,H}(N)}{N} \ll t^{\varepsilon}\bigg\{&\frac{M^{5/2}}{t^{3/4}}+\frac{t^{9/4}}{M^{3/2}}+\frac{M^{29/9}}{H^{1/9}t^{7/6}}+\frac{M^{23/8}t^{3/16}}{H^{3/2}}+\frac{M^{7/12}t^{7/8}}{H^{1/6}}\\ &+M^{3/2}t^{1/8}+\frac{M^{5/4}t^{5/8}}{N^{1/6}}\bigg\}.\end{aligned}$$

**Remark 10.1.** The optimal choice of $Q$ presented above is determined by minimizing $MHN^2/Q^3 + \sqrt{QN}Mt$ restricted to $Q < Nt^\varepsilon/\sqrt{Ht}$.

Inserting the above bound together with the trivial bound (6.2) into Lemma 6.1, we get

$$
\begin{aligned}
&|L(1/2+it,\pi)|^2 \ll (\log t)\Big(1+\int_{\mathbb{R}} U\Big(\frac{v}{M}\Big)|L(1/2+it+iv,\pi)|^2\,dv\Big) \\
&\ll t^\varepsilon\Big(\frac{M^{5/2}}{t^{3/4}}+\frac{t^{9/4}}{M^{3/2}}+\sup_H \min\Big\{\frac{M^{29/9}}{H^{1/9}t^{7/6}}+\frac{M^{23/8}t^{3/16}}{H^{3/2}}+\frac{M^{7/12}t^{7/8}}{H^{1/6}}, MH\Big\} \\
&\quad + M^{3/2}t^{1/8}+\sup_N \min\Big\{\frac{M^{5/4}t^{5/8}}{N^{1/6}}, N\Big\}\Big) \\
&\ll t^\varepsilon\Big(\frac{M^{5/2}}{t^{3/4}}+\frac{t^{9/4}}{M^{3/2}}+\frac{M^3}{t^{21/20}}+M^{7/4}t^{3/40}+M^{9/14}t^{3/4}+M^{3/2}t^{1/8}+M^{15/14}t^{15/28}\Big) \\
&\ll t^\varepsilon\Big(\frac{t^{9/4}}{M^{3/2}}+\frac{M^3}{t^{21/20}}+M^{7/4}t^{3/40}+M^{15/14}t^{15/28}\Big).
\end{aligned}
$$

The optimal choice of $M$ is $M = t^{2/3}$, which yields

$$
|L(1/2+it,\pi)|^2 \ll (\log t)\Big(1+\int_{\mathbb{R}} U\Big(\frac{v}{t^{2/3}}\Big)|L(1/2+it+iv,\pi)|^2\,dv\Big) \ll t^{5/4+\varepsilon}.
$$

This completes the proof of Theorems 1.1 and 1.2.

## Appendix A. Integral analysis

### *A.1. Stationary phase analysis of $\mathcal{J}_0(m, m_0, q_1, q_2, x_1, x_2)$*

Let $x_1, x_2$ be such that $\varphi_X(x_j) \neq 0$ for $j = 1, 2$, with $\varphi_X$ as defined in (6.9). Recall from (7.2) that

$$
\begin{aligned}
&\mathcal{J}_0(m, m_0, q_1, q_2, x_1, x_2) \\
&\quad = \int_0^\infty \varphi(z) U_{\eta_1}\Big(\frac{N_1 z}{(m_0 q_1)^3}, m_0 q_1, x_1\Big)\overline{U_{\eta_1}\Big(\frac{N_1 z}{(m_0 q_2)^3}, m_0 q_2, x_2\Big)}\, e\Big(-\frac{m N_1 z}{m_0^2 q_1 q_2}\Big)\,dz
\end{aligned}
$$

and

$$
U_{\eta_1}(a, b, c) = \int_0^\infty U(w)\overline{U}_0(a, Nw) w^{-1/3} e\Big(-\frac{cNw}{bQ} + \eta_1 3(aNw)^{1/3}\Big)\,dw
$$

for $j = 1, 2$ with $U_0$ being a fixed flat function.

The oscillatory integral in $U_{\eta_1}$ has the phase function

$$
f_1(w) = -\frac{Nwx_j}{m_0 q_j Q} + \eta_1 3\Big(\frac{N_1 N w z}{(m_0 q_j)^3}\Big)^{1/3}.
$$

Differentiating gives us

$$f_1'(w) = -\frac{Nx_j}{m_0 q_j Q} + \eta_1 \left(\frac{N_1 N z}{(m_0 q_j)^3 w^2}\right)^{1/3},$$
$$f_1''(w) = -\eta_1 \frac{2}{3}\left(\frac{N_1 N z}{(m_0 q_j)^3 w^5}\right)^{1/3},$$
$$f_1^{(j)}(w) \ll_j \frac{(N_1 N)^{1/3}}{C} \quad \text{for any } j \geq 2.$$

For $X = 0$, the support of $\varphi_0(x_j)$ yields $\frac{Nx_j}{CQ} \ll t^\varepsilon$, and hence Lemma 5.2 (1) gives us arbitrary savings unless

$$(N_1 N/C^3)^{1/3} \ll t^\varepsilon \iff N_1 \ll x\frac{C^3 t^\varepsilon}{N}.$$

In such a case, $U_{\eta_1}$ is $t^\varepsilon$-inert. Now by repeated integration by parts in the $z$-integral in $\mathcal{J}_0$, we get arbitrary savings unless

$$|m| \ll C^2 t^\varepsilon / N_1.$$

The function $\mathcal{J}_0(m, m_0, q_1, q_2, x_1, x_2)$ is $t^\varepsilon$-inert when $|m| \ll C^2 t^\varepsilon / N_1$.

For $X \neq 0$, the support of $\varphi_X(x_j)$ yields $\frac{N|x_j|}{CQ} \sim \frac{N|X|}{CQ} \gg t^\varepsilon$, and Lemma 5.2 (1) gives us arbitrary savings unless

$$\eta_1 X > 0 \quad \text{and} \quad \frac{(N_1 N)^{1/3}}{C} \sim \frac{N|X|}{CQ} \gg t^\varepsilon.$$

In such a case, we have $f_1''(w) \gg t^\varepsilon$. Computing the stationary phase point $f_1'(w_0) = 0$, we get

$$w_0 = \sqrt{\frac{N_1 Q^3 z}{N^2 |x_j|^3}} \quad \text{and} \quad f_1(w_0) = \frac{2Nx_j w_0}{m_0 q_j Q} = 2\sqrt{\frac{N_1 Q z}{(m_0 q_j)^2 |x_j|}}.$$

Hence for any $A > 0$, Lemma 5.2 (2) yields

$$U_{\eta_1}\left(\frac{N_1 z}{(m_0 q_j)^3}, m_0 q_j, x_j\right) = \frac{\delta(\eta_1 X > 0)}{\sqrt{f_1''(w_0)}} U_2\left(\frac{N_1 Q^3 z}{N^2 |x_j|^3}\right) e\left(2\sqrt{\frac{N_1 Q z}{(m_0 q_j)^2 |x_j|}}\right) + O(t^{-A})$$

for some $t^\varepsilon$-inert function $U_2 \in C_c^\infty([1/4, 25/4])$.

Write

$$U_3\left(\frac{N_1 Q^3 z}{N^2 |x_j|^3}, m_0 q_j, x_j\right) = \frac{(N_1 N)^{1/6}}{\sqrt{C}} \frac{1}{\sqrt{f_1''(w_0)}} U_2\left(\frac{N_1 Q^3 z}{N^2 |x_j|^3}\right).$$

Then $U_3$ is a $t^\varepsilon$-inert function supported on $[1/4, 25/4] \times \mathbb{R} \times \mathbb{R}$ such that

$$U_{\eta_1}\left(\frac{N_1 z}{(m_0 q_j)^3}, m_0 q_j, x_j\right) = \frac{\sqrt{C}}{(N_1 N)^{1/6}} \delta(\eta_1 X > 0) U_3\left(\frac{N_1 Q^3 z}{N^2 |x_j|^3}, m_0 q_j, x_j\right) \times e\left(2\sqrt{\frac{N_1 Q z}{(m_0 q_j)^2 |x_j|}}\right) + O(t^{-A}).$$

Inserting this back into (7.2) yields

$$\begin{aligned}
&\mathcal{J}_0(m, m_0, q_1, q_2, x_1, x_2) \\
&= \frac{C}{(N_1 N)^{1/3}} \delta(\eta_1 X > 0) \int_0^\infty \varphi(z) U_3\left(\frac{N_1 Q^3 z}{N^2 |x_1|^3}, m_0 q_1, x_1\right) \overline{U_3\left(\frac{N_1 Q^3 z}{N^2 |x_2|^3}, m_0 q_2, x_2\right)} \\
&\qquad \times e\left(2\sqrt{\frac{N_1 Q z}{(m_0 q_1)^2 |x_1|}} - 2\sqrt{\frac{N_1 Q z}{(m_0 q_2)^2 |x_2|}} - \frac{m N_1 z}{m_0^2 q_1 q_2}\right) dz.
\end{aligned}$$

Repeated integration by parts gives us arbitrary savings unless

$$\frac{|m| N_1}{m_0^2 q_1 q_2} \ll \sqrt{\frac{N_1 Q}{C^2 |X|}} \iff |m| \ll \frac{C\sqrt{Q}}{\sqrt{N_1 |X|}} t^\varepsilon \sim \frac{C Q^2}{N X^2} t^\varepsilon.$$

### *A.2. Stationary phase analysis of $\mathcal{J}(q_1, q_2, \vec{v})$*

Recall from (7.5) that

$$\begin{aligned}
\mathcal{J}(q_1, q_2, \vec{v}) &= \int_{\mathbb{R}} \varphi_X(x_1) g(m_0 q_1, x_1) \phi_1(u_1, m_0 q_1, h_1, x_1) \\
&\quad \times \int_{\mathbb{R}} \varphi_X(x_2) \overline{g(m_0 q_2, x_2) \phi_1(u_2, m_0 q_2, h_2, x_2)} \int_0^\infty \varphi(z) U_3\left(\frac{N_1 Q^3 z}{N^2 |x_1|^3}, m_0 q_1, x_1\right) \\
&\quad \times \overline{U_3\left(\frac{N_1 Q^3 z}{N^2 |x_2|^3}, m_0 q_2, x_2\right)} e\left(2\sqrt{\frac{N_1 Q z}{m_0^2 q_1^2 |x_1|}} - 2\sqrt{\frac{N_1 Q z}{m_0^2 q_2^2 |x_2|}} - \frac{m N_1 z}{m_0^2 q_1 q_2}\right) dz \\
&\quad \times e\left(\frac{N^2 x_1 u_1}{m_0 q_1 Q H t^{1-\varepsilon}} - \frac{N^2 x_2 u_2}{m_0 q_2 Q H t^{1-\varepsilon}} + \frac{t x_1}{2\pi h_1 Q} - \frac{t x_2}{2\pi h_2 Q}\right) dx_2\, dx_1,
\end{aligned}$$

and $U_3$ is some $t^\varepsilon$-inert function supported on $[1/4, 25/4] \times \mathbb{R} \times \mathbb{R}$. Consider

$$\begin{aligned}
&\int_{\mathbb{R}} \varphi_X(x_1) g(m_0 q_1, x_1) \phi_1(u_1, m_0 q_1, h_1, x_1) U_3\left(\frac{N_1 Q^3 z}{N^2 |x_1|^3}, m_0 q_1, x_1\right) \\
&\qquad \times e\left(2\sqrt{\frac{N_1 Q z}{m_0^2 q_1^2 |x_1|}} + \frac{N^2 x_1 u_1}{m_0 q_1 Q H t^{1-\varepsilon}} + \frac{t x_1}{2\pi h_1 Q}\right) dx_1 \\
&= X \int_{\mathbb{R}} \varphi(x_1) g(m_0 q_1, X x_1) \phi_1(u_1, m_0 q_1, h_1, X x_1) \\
&\qquad \times U_3\left(\frac{N_1 Q^3 z}{N^2 X^3 x_1^3}, m_0 q_1, X x_1\right) e(F_1(x_1))\, dx_1.
\end{aligned}$$

The phase function is

$$F_1(x_1) = 2\sqrt{\frac{N_1 Qz}{m_0^2 q_1^2 |X| x_1}} + \frac{N^2 X x_1 u_1}{m_0 q_1 Q H t^{1-\varepsilon}} + \frac{tXx_1}{2\pi h_1 Q}.$$

Recall that $N_1 \sim N^2X^3/Q^3$ by the support of $U_3$. Differentiating, we obtain

$$F_1'(x_1) = -\sqrt{\frac{N_1 Qz}{m_0^2 q_1^2 |X| x_1^3}} + \frac{N^2 X u_1}{m_0 q_1 Q H t^{1-\varepsilon}} + \frac{tX}{2\pi h_1 Q},$$

$$F_1''(x_1) = \frac{3}{2}\sqrt{\frac{N_1 Qz}{m_0^2 q_1^2 |X| x_1^5}} \sim \frac{\sqrt{N_1 Q}}{C\sqrt{X}},$$

$$F_1^{(j)}(x_1) \ll_j \frac{\sqrt{N_1 Q}}{C\sqrt{X}} \quad \text{for any } j \geq 2.$$

The stationary point $x_{1,0}$ satisfying $F_1'(x_{1,0}) = 0$ is

$$x_{1,0} = \left(\frac{2\pi h_1}{m_0 q_1 t}\right)^{2/3} \frac{Q(N_1 z)^{1/3}}{|X|} \left(1 + \frac{2\pi h_1 N^2 u_1}{m_0 q_1 H t^{2-\varepsilon}}\right)^{-2/3},$$

and

$$F_1(x_{1,0}) = 3\sqrt{\frac{N_1 Qz}{m_0^2 q_1^2 |X| x_{1,0}}} = 3\left(\frac{N_1 t z}{2\pi h_1 (m_0 q_1)^2}\left(1 + \frac{2\pi h_1 N^2 u_1}{m_0 q_1 H t^{2-\varepsilon}}\right)\right)^{1/3}.$$

Since $X \gg CQt^{\varepsilon}/N$, we have

$$F_1''(x_1) \sim \frac{\sqrt{N_1 Q}}{C\sqrt{|X|}} \sim \frac{N|X|}{CQ} \gg t^{\varepsilon}.$$

Together with (4.3), Lemma 5.2 (2) implies that there exists a $t^{\varepsilon}$-inert function $\tilde{\Phi}$ supported on $[1/4, 25/4] \times \mathbb{R}^2$ such that the $x_1$-integral is equal to

$$\begin{aligned} X \int_{\mathbb{R}} \varphi(x_1) g(m_0 q_1, X x_1) \phi_1(u_1, m_0 q_1, h_1, X x_1) U_3\left(\frac{N_1 Q^3 z}{N^2 X^3 x_1^3}, m_0 q_1, X x_1\right) \\ \times e(F_1(x_1))\, dx_1 \\ = X\sqrt{\frac{C\sqrt{X}}{\sqrt{N_1 Q}}} t^{\varepsilon} \tilde{\Phi}\left(\frac{N_1 Q^3 z}{N^2 X^3}, m_0 q_1, u_1\right) g(m_0 q_1, X x_{1,0}) \\ \times e\left(3\left(\frac{N_1 t z}{2\pi h_1 (m_0 q_1)^2}\left(1 + \frac{2\pi h_1 N^2 u_1}{m_0 q_1 H t^{2-\varepsilon}}\right)\right)^{1/3}\right) + O(t^{-A}) \end{aligned}$$

for any $A > 0$. Applying the same argument to the $x_2$-integral, we find that there exists a $t^{\varepsilon}$-inert function $\Phi$ supported on $[1/4, 25/4] \times \mathbb{R}^4$ such that $\mathcal{J}(q_1, q_2, \vec{v}\,)$ equals

$$\frac{CX^{5/2}t^{\varepsilon}}{\sqrt{N_1 Q}}\int_0^{\infty}\varphi(z)\Phi\left(\frac{N_1Q^3z}{N^2X^3}, m_0q_1, m_0q_2, u_1.u_2\right)g(m_0q_1, Xx_{1,0})$$
$$\times g(m_0q_2, Xx_{2,0})e\left(3\left(\frac{N_1tz}{2\pi h_1(m_0q_1)^2}\left(1+\frac{2\pi h_1N^2u_1}{m_0q_1Ht^{2-\varepsilon}}\right)\right)^{1/3}\right)$$
$$\times e\left(-3\left(\frac{N_1tz}{2\pi h_2(m_0q_2)^2}\left(1+\frac{2\pi h_2N^2u_2}{m_0q_2Ht^{2-\varepsilon}}\right)\right)^{1/3}-\frac{mN_1z}{m_0^2q_1q_2}\right)dz+O(t^{-A})$$

for any $A>0$, and

$$x_{j,0}=\left(\frac{2\pi h_j}{m_0q_jt}\right)^{2/3}\frac{Q(N_1z)^{1/3}}{X}\left(1+\frac{2\pi h_jN^2u}{m_0q_jHt^{2-\varepsilon}}\right)^{-2/3}$$

for $j=1,2$. Note that $x_{j,0}$ is flat with respect to $z$.

### *A.3. Stationary phase analysis of $\mathcal{J}_j(q_1,q_2,\vec{v})$*

Let $nN_j\gg C^2t^{\varepsilon}$. Recall the definition of $\mathcal{J}_j(q_1,q_2,\vec{v})$ in Lemma 9.1:

$$\mathcal{J}_j(q_1,q_2,\vec{v})=\int_0^{\infty}\varphi(z)\Phi_{j,\eta_j}(n_0q_1,n_0q_2,u_1,u_2)\big(g(n_0q_1,Xx_{1,0})g(n_0q_2,Xx_{2,0})\big)^{2-j}$$
$$\times e\left(-3\eta_j'\left(\frac{N_jNz}{(n_0q_1)^3}\left(\frac{n_0q_1t}{2\pi h_1N}+\frac{Nu_1}{Ht^{1-\varepsilon}}\right)\right)^{1/3}\right.$$
$$\left.+3\eta_j'\left(\frac{N_jNz}{(n_0q_2)^3}\left(\frac{n_0q_2t}{2\pi h_2N}+\frac{Nu_2}{Ht^{1-\varepsilon}}\right)\right)^{1/3}-\frac{nN_jz}{n_0^2q_1q_2}\right)dz.$$

The phase function, denoted $F_3(z)$, equals

$$-\eta_j'3\left(\frac{N_jNz}{(n_0q_1)^3}\left(\frac{n_0q_1t}{2\pi h_1N}+\frac{Nu_1}{Ht^{1-\varepsilon}}\right)\right)^{1/3}+\eta_j'3\left(\frac{N_jNz}{(n_0q_2)^3}\left(\frac{n_0q_2t}{2\pi h_2N}+\frac{Nu_2}{Ht^{1-\varepsilon}}\right)\right)^{1/3}$$
$$-\frac{nN_jz}{n_0^2q_1q_2}.$$

Differentiating, we obtain

$$F_3'(z)=-\eta_j'\left(\frac{N_jN}{(n_0q_1)^3z^2}\left(\frac{n_0q_1t}{2\pi h_1N}+\frac{Nu_1}{Ht^{1-\varepsilon}}\right)\right)^{1/3}$$
$$+\eta_j'\left(\frac{N_jN}{(n_0q_2)^3z^2}\left(\frac{n_0q_2t}{2\pi h_2N}+\frac{Nu_2}{Ht^{1-\varepsilon}}\right)\right)^{1/3}-\frac{nN_j}{n_0^2q_1q_2},$$
$$F_3''(z)=\eta_j'\frac{2}{3}\left(\frac{N_jN}{(n_0q_1)^3z^5}\left(\frac{n_0q_1t}{2\pi h_1N}+\frac{Nu_1}{Ht^{1-\varepsilon}}\right)\right)^{1/3}$$
$$-\eta_j'\frac{2}{3}\left(\frac{N_jN}{(n_0q_2)^3z^5}\left(\frac{n_0q_2t}{2\pi h_2N}+\frac{Nu_2}{Ht^{1-\varepsilon}}\right)\right)^{1/3},$$

$$F_3^{(j)}(z) \ll_j (N_j N)^{1/3}\left(\frac{1}{n_0 q_1}\left(\frac{n_0 q_1 t}{2\pi h_1 N}+\frac{N u_1}{H t^{1-\varepsilon}}\right)^{1/3} - \frac{1}{n_0 q_2}\left(\frac{n_0 q_2 t}{2\pi h_2 N}+\frac{N u_2}{H t^{1-\varepsilon}}\right)^{1/3}\right)$$

for any $j \geq 2$. The stationary point of $F_3(z)$, denoted $z_0(n_0q_1, n_0q_2)$, equals

$$\frac{\sqrt{N}}{n^{3/2}N_j}\left(-\eta_j' n_0 q_2\left(\frac{n_0 q_1 t}{2\pi h_1 N}+\frac{N u_1}{H t^{1-\varepsilon}}\right)^{1/3}+\eta_j' n_0 q_1\left(\frac{n_0 q_2 t}{2\pi h_2 N}+\frac{N u_2}{H t^{1-\varepsilon}}\right)^{1/3}\right)^{3/2},$$

so that

$$F_3(z_0(n_0q_1, n_0q_2)) = \frac{2nN_j z_0}{n_0^2 q_1 q_2} = 2\sqrt{\frac{N}{n}}\left(-\eta_j'\left(\frac{q_2 t}{2\pi h_1 q_1 N}\left(1+\frac{2\pi h_1 N^2 u_1}{n_0 q_1 H t^{2-\varepsilon}}\right)\right)^{1/3} + \eta_j'\left(\frac{q_1 t}{2\pi q_2 h_2 N}\left(1+\frac{2\pi h_2 N^2 u_2}{n_0 q_2 H t^{2-\varepsilon}}\right)\right)^{1/3}\right)^{3/2}.$$

By (4.3) and the fact that $x_{j,0}$ is flat with respect to $z$ and $nN_j \gg C^2 t^\varepsilon$, Lemma 5.2 (1) gives us arbitrary savings unless

$$F_3''(z) \sim nN_j/C^2 \gg t^\varepsilon.$$

Hence Lemma 5.2 (2) yields

$$\mathcal{J}_j(q_1, q_2, \vec{v}\,) = \frac{C}{\sqrt{nN_j}}\big(g(n_0q_1, Xx_1)g(n_0q_2, Xx_2)\big)^{2-j}\,\Phi_2\big(z_0(n_0q_1, n_0q_2), q_1, q_2; \nu\big) \times e(F_3(z_0(n_0q_1, n_0q_2))) + O(t^{-A})$$

for any $A > 0$, with some flat function $\Phi_2$ supported on $[1/2, 5/2] \times \mathbb{R}^7$, and

$$x_j := x_j(z_0(n_0q_1, n_0q_2)) = \left(\frac{2\pi h_j}{n_0 q_j t}\right)^{2/3}\frac{Q(N_1 z_0(n_0q_1, n_0q_2))^{1/3}}{X}\left(1+\frac{2\pi h_j N^2 u}{n_0 q_j H t^{2-\varepsilon}}\right)^{-2/3}$$

for $j = 1, 2$.

### A.4. *Stationary phase analysis of* (9.13)

Here we analyse the $y$-integral

$$\int_0^\infty \Upsilon(y)\Psi(Cy, q_2, q_3)e\left(F_3(z_0(Cy, q_2)) - F_3(z_0(Cy, q_3)) - \frac{q_1' C y}{n_0 n h_1 h_2 q_0^2 q_{1,0}^2 q_{2,0} r}\right)dy$$

in (9.13) when $|q_1'| \ll Q_1 Q_S$ with

$$Q_S = \frac{n_0 q_0 q_{2,0} r C^3 t^{2+\varepsilon}}{n_1 n (N_j N)^{4/3}}$$

as defined in (9.17). Recall that

$$G(y,z)=G_{h_1,h_2}(y,z)=F_3(z_0(Cy,Cz))$$
$$=\sqrt{\frac{2t}{\pi n}}\left(-\eta_j'\left(\frac{z}{h_1y}+\frac{2\pi N^2u_1z}{CHt^{2-\varepsilon}y^2}\right)^{1/3}+\eta_j'\left(\frac{y}{h_2z}+\frac{2\pi N^2u_2y}{CHt^{2-\varepsilon}z^2}\right)^{1/3}\right)^{3/2}.$$

Lemma A.1 gives us

$$\frac{\partial G}{\partial y}\left(y,\frac{q}{C}\right)$$
$$=\eta_j'\sqrt{\frac{t}{2\pi n}}\left(-\eta_j'\left(\frac{q}{h_1Cy}+\frac{2\pi qN^2u_1}{C^2Ht^{2-\varepsilon}y^2}\right)^{1/3}+\eta_j'\left(\frac{Cy}{h_2q}+\frac{2\pi CN^2u_2y}{q^2Ht^{2-\varepsilon}}\right)^{1/3}\right)^{1/2}$$
$$\times\left\{\left(\frac{q}{h_1Cy^4}+\frac{2\pi qN^2u_1}{C^2Ht^{2-\varepsilon}y^5}\right)^{1/3}+\left(\frac{C}{h_2qy^2}+\frac{2\pi CN^2u_2}{q^2Ht^{2-\varepsilon}y^2}\right)^{1/3}\right.$$
$$\left.+\frac{2\pi qN^2u_1}{C^2Ht^{2-\varepsilon}y^3}\left(\frac{q}{h_1Cy}+\frac{2\pi qN^2u_1}{C^2Ht^{2-\varepsilon}y^2}\right)^{-2/3}\right\}.$$

Using (9.11) to bound the derivatives of $\Psi$ (see (9.8) for the definition of $\Psi$) and applying Lemma 5.2 (1) to the $y$-integral gives us arbitrary savings unless there exists $y'\in[1-10^{-9},1+10^{-9}]$ such that $z_0(Cy',q_j)\sim 1$ for $j=1,2$ and

$$\frac{d}{dy}\bigg|_{y=y'}\left(G\left(y,\frac{q_2}{C}\right)-G\left(y,\frac{q_3}{C}\right)\right)\ll\frac{C^2N^{2/3}t^{\varepsilon}}{n^2N_j^{4/3}}.$$

Note that for any $y$ such that $z_0(Cy,q)\sim 1$, together with the condition $n\gg N_S\gg\frac{CN^{4/3}}{N_j^{2/3}Ht^{1-\varepsilon}}$, we have

$$\left|-\eta_j'q\left(\frac{Cty}{2\pi h_1N}+\frac{Nu_1}{Ht^{1-\varepsilon}}\right)^{1/3}+\eta_j'Cy\left(\frac{qt}{2\pi h_2N}+\frac{Nu_2}{Ht^{1-\varepsilon}}\right)^{1/3}\right|\sim\frac{nN_j^{2/3}}{N^{1/3}},$$
$$\implies\left|\left(\frac{q}{h_1Cy}+\frac{2\pi qN^2u_1}{C^2Ht^{2-\varepsilon}y^2}\right)^{1/3}-\left(\frac{Cy}{h_2q}+\frac{2\pi CN^2u_2y}{q^2Ht^{2-\varepsilon}}\right)^{1/3}\right|\sim\frac{nN_j^{2/3}}{C^{4/3}t^{1/3}}.\tag{A.1}$$

Hence when such a $y'$ exists, the bound for the derivative above implies

$$\left|\left(\left(\frac{q_2}{h_1Cy'}+\frac{2\pi q_2N^2u_1}{C^2Ht^{2-\varepsilon}y'^2}\right)^{1/3}-\left(\frac{Cy'}{h_2q_2}+\frac{2\pi CN^2u_2y'}{q_2^2Ht^{2-\varepsilon}}\right)^{1/3}\right)\right.$$
$$\times\left(\left(\frac{q_2}{h_1Cy'}+\frac{2\pi q_2N^2u_1}{C^2Ht^{2-\varepsilon}y'^2}\right)^{1/3}+\left(\frac{Cy'}{h_2q_2}+\frac{2\pi CN^2u_2y'}{q_2^2Ht^{2-\varepsilon}}\right)^{1/3}\right.$$
$$\left.+\frac{2\pi q_2N^2u_1}{C^2Ht^{2-\varepsilon}y'^2}\left(\frac{q_2}{h_1Cy'}+\frac{2\pi q_2N^2u_1}{C^2Ht^{2-\varepsilon}y'^2}\right)^{-2/3}\right)^2$$
$$-\left(\left(\frac{q_3}{h_1Cy'}+\frac{2\pi q_3N^2u_1}{C^2Ht^{2-\varepsilon}y'^2}\right)^{1/3}-\left(\frac{Cy'}{h_2q_3}+\frac{2\pi CN^2u_2y'}{q_3^2Ht^{2-\varepsilon}}\right)^{1/3}\right)$$

$$\times\Bigg(\bigg(\Big(\frac{q_3}{h_1Cy'}+\frac{2\pi q_3N^2u_1}{C^2Ht^{2-\varepsilon}y'^2}\Big)^{1/3}+\Big(\frac{Cy'}{h_2q_3}+\frac{2\pi CN^2u_2y'}{q_3^2Ht^{2-\varepsilon}}\Big)^{1/3}$$

$$+\frac{2\pi q_3N^2u_1}{C^2Ht^{2-\varepsilon}y'^2}\Big(\frac{q_3}{h_1Cy'}+\frac{2\pi q_3N^2u_1}{C^2Ht^{2-\varepsilon}y'^2}\Big)^{-2/3}\bigg)^2\Bigg|$$

$$\ll\frac{C^2N^{2/3}}{n^2N_j^{4/3}}\frac{n(N_jN)^{1/3}}{Ct^{1-\varepsilon}}=\frac{CN}{nN_jt^{1-\varepsilon}}.$$

To simplify this condition, observe that the expression inside the absolute value can be expressed as $\mathcal{T}_2-\mathcal{T}_3$, where for suitable $A,B,C,D$,

$$\mathcal{T}_j:=\Big(q_j^{1/3}A^{1/3}-q_j^{-1/3}\Big(B+\frac{C}{q_j}\Big)^{1/3}\Big)\Big(q_j^{1/3}A^{1/3}+q_j^{-1/3}\Big(B+\frac{C}{q_j}\Big)^{1/3}+q_j^{1/3}D\Big)^2$$

$$=q_j(A+2A^{2/3}D+A^{1/3}D^2)+q_j^{1/3}(A^{2/3}-D^2)\Big(B+\frac{C}{q_j}\Big)^{1/3}$$

$$-\frac{1}{q_j^{1/3}}(A^{1/3}+2D)\Big(B+\frac{C}{q_j}\Big)^{2/3}-\frac{1}{q_j}\Big(B+\frac{C}{q_j}\Big).$$

Applying the mean value theorem, we can factor out $q_2-q_3$ from the terms in $\mathcal{T}_2-\mathcal{T}_3$, simplifying which gives

$$|q_2-q_3|\ll\frac{C^3t^{\varepsilon}}{nN_j}. \tag{A.2}$$

Moreover, (A.1) yields

$$\Big|y^{2/3}-\Big(\frac{h_2q_2^2}{h_1C^2}\Big)^{1/3}\Big|\sim\frac{nN_j^{2/3}}{CN^{1/3}}+O\Big(\frac{N}{Ht^{1-\varepsilon}}\Big),\ \text{so}\ \Big|y-\frac{\sqrt{h_2}q_2}{\sqrt{h_1}C}\Big|\sim\frac{nN_j^{2/3}}{CN^{1/3}}.$$

Here we have used $|n|\gg\frac{CN^{4/3}}{N_j^{2/3}Ht^{1-\varepsilon}}$. Hence the $y$-integral is

$$\ll\frac{nN_j^{2/3}t^{\varepsilon}}{CN^{1/3}}\delta\Big(|q_2-q_3|\ll\frac{C^3t^{\varepsilon}}{nN_j}\Big)+t^{-A}$$

for any $A>0$.

### A.5. *Integral analysis involving* GL(3) *Bessel functions*

**Lemma A.1.** *Let $\eta=\pm1$ and let $h_1,h_2,A_1,A_2,x,y$ be such that $\eta(\frac{x}{h_2y})^{1/3}\geq\eta(\frac{y}{h_1x})^{1/3}$. Let*

$$g(x,y,h_1,h_2,A_1,A_2)=\Big(-\eta\Big(\frac{y}{h_1x}+\frac{A_1y}{x^2}\Big)^{1/3}+\eta\Big(\frac{x}{h_2y}+\frac{A_2x}{y^2}\Big)^{1/3}\Big)^{3/2}.$$

*Then*

$$g(x,y,h_1,h_2,A_1,A_2)=(-1)^{3/2}g(y,x,h_2,h_1,A_2,A_1).$$

*Differentiating yields*

$$\begin{aligned} g_x(x,y,h_1,h_2,A_1,A_2) &= (-1)^{3/2} g_y(y,x,h_2,h_1,A_2,A_1) \\ &= \frac{\eta}{2x}\left(-\eta\left(\frac{y}{h_1x}+\frac{A_1y}{x^2}\right)^{1/3}+\eta\left(\frac{x}{h_2y}+\frac{A_2x}{y^2}\right)^{1/3}\right)^{1/2} \\ &\quad\times\left(\left(\frac{y}{h_1x}+\frac{A_1y}{x^2}\right)^{1/3}+\left(\frac{x}{h_2y}+\frac{A_2x}{y^2}\right)^{1/3}+\frac{A_1y}{x^2}\left(\frac{y}{h_1x}+\frac{A_1y}{x^2}\right)^{-2/3}\right). \end{aligned}$$

**Lemma A.2.** *Let $\eta=\pm1$ and let $h_1,h_2,x,y$ be such that $\eta(\frac{x}{h_2y})^{1/3}\geq\eta(\frac{y}{h_1x})^{1/3}$. Let*

$$f(x,y,h_1,h_2)=\left(-\eta\left(\frac{y}{h_1x}\right)^{1/3}+\eta\left(\frac{x}{h_2y}\right)^{1/3}\right)^{3/2}.$$

*Then*

$$f(x,y,h_1,h_2)=(-1)^{3/2}f(y,x,h_2,h_1).$$

*Differentiating yields*

$$\begin{aligned} f_x(x,y,h_1,h_2) &= (-1)^{3/2} f_y(y,x,h_2,h_1) \\ &= \frac{\eta}{2}\left(-\eta\left(\frac{y}{h_1x}\right)^{1/3}+\eta\left(\frac{x}{h_2y}\right)^{1/3}\right)^{1/2}\left(\left(\frac{y}{h_1x^4}\right)^{1/3}+\frac{1}{(h_2x^2y)^{1/3}}\right), \end{aligned}$$

$$f_{xx}(x,y,h_1,h_2) = (-1)^{3/2} f_{yy}(y,x,h_2,h_1) =$$

$$\frac{1}{12}\left(-\eta\left(\frac{y}{h_1x}\right)^{1/3}+\eta\left(\frac{x}{h_2y}\right)^{1/3}\right)^{-1/2}\left(\frac{9y^{2/3}}{h_1^{2/3}x^{8/3}}-\frac{2}{(h_1h_2)^{1/3}x^2}-\frac{3}{h_2^{2/3}x^{4/3}y^{2/3}}\right),$$

$$f_{xy}(x,y,h_1,h_2) =$$

$$-\frac{1}{12}\left(-\eta\left(\frac{y}{h_1x}\right)^{1/3}+\eta\left(\frac{x}{h_2y}\right)^{1/3}\right)^{-1/2}\left(\frac{3}{(h_1^2x^5y)^{1/3}}+\frac{3}{(h_2^2xy^5)^{1/3}}-\frac{2}{(h_1h_2)^{1/3}xy}\right),$$

*and*

$$\begin{aligned} f_{xxy}(x,y,h_1,h_2) &= \frac{\eta}{72}\left(-\eta\left(\frac{y}{h_1x}\right)^{1/3}+\eta\left(\frac{x}{h_2y}\right)^{1/3}\right)^{-3/2} \\ &\quad\times\left(\frac{9}{h_2xy^2}+\frac{43}{h_1^{2/3}h_2^{1/3}x^{7/3}y^{2/3}}-\frac{17}{h_1^{1/3}h_2^{2/3}x^{5/3}y^{4/3}}-\frac{27}{h_1x^3}\right). \end{aligned}$$

Define

$$G(x,y)=G_{h_1,h_2}(x,y)=\sqrt{\frac{2t}{\pi n}}\,f(x,y,h_1,h_2)$$

as in the main sections. Then we have the following integral analysis lemmas.

**Lemma A.3.** *Let $1 \leq a, b, h_1/H', h_2/H' < 1 + 10^{-10}$. Fix $\Upsilon \in C_c^\infty([1-10^{-9}, 1+10^{-9}])$ and let $W$ be an $X$-inert function such that $W(y,a,b)$ contains the restriction*

$$\left|\left(\frac{a}{h_1 y}\right)^{1/3} - \left(\frac{y}{h_2 a}\right)^{1/3}\right| \sim \left|\left(\frac{b}{h_1 y}\right)^{1/3} - \left(\frac{y}{h_2 b}\right)^{1/3}\right| \sim \frac{nN_j^{2/3}}{C^{4/3}t^{1/3}}. \tag{A.3}$$

*Let $Z > 0$ and let $|q'| \gg \frac{X^2}{Z}t^\varepsilon$. Let*

$$J = \int_0^\infty \Upsilon(y)W(y,a,b)e\big(G(y,a) - G(y,b) - q'Zy\big)\,dy$$

*and let $y_0$ be such that*

$$G_x(y_0,a) - G_x(y_0,b) = q'Z.$$

*Let $U \gg \sqrt{\frac{nN_j^{2/3}}{CN^{1/3}q'Z}}t^\varepsilon$ and let $W_0 \in C_c^\infty([-2,2])$ with $W_0(x) = 1$ for $|x| \leq 1$. Then*

$$J = \int_0^\infty \Upsilon(y)W(y,a,b)W_0\left(\frac{y-y_0}{U}\right)e\big(G(y,a) - G(y,b) - q'Zy\big)\,dy + O(t^{-A})$$

*for any $A > 0$. Moreover,*

$$\frac{d^j y_0}{da^j}, \frac{d^j y_0}{db^j} \ll_j \left(\frac{nN_j^{2/3}}{CN^{1/3}} + \frac{(N_j N)^{1/3}}{Cq'Z}\right)^j \quad \text{for any } j \geq 1.$$

*Proof.* Since $W$ is $X$-inert and $|q'| \gg \frac{X^2}{Z}t^\varepsilon$, Lemma 5.2 (1) gives us arbitrary savings unless there exists $1 \leq y' < 1 + 10^{-10}$ such that

$$\frac{d}{dy}\bigg|_{y=y'}(G(y,a) - G(y,b)) = G_x(y',a) - G_x(y',b) \sim q'Z.$$

Applying the same analysis that concludes in (9.26) yields

$$|a-b| \sim \frac{nN_j^{1/3}}{N^{2/3}}q'Z. \tag{A.4}$$

Differentiating once more with Lemma A.2 yields

$$\frac{d^2}{dy^2}(G(y,a) - G(y,b)) = \sqrt{\frac{t}{72\pi n}}(F(a) - F(b)),$$

where $F(x) := G_{xx}(y,x)$ equals

$$\left(-\eta\left(\frac{x}{h_1 y}\right)^{1/3} + \eta\left(\frac{y}{h_2 x}\right)^{1/3}\right)^{-1/2}\left(\frac{9x^{2/3}}{h_1^{2/3}y^{8/3}} - \frac{2}{(h_1h_2)^{1/3}y^2} - \frac{3}{h_2^{2/3}x^{2/3}y^{4/3}}\right).$$

We want to compute the bounds for the derivatives of $G$ in order to apply the stationary phase analysis of Lemma 5.2. Notice that

$$F'(x) = \frac{\eta}{6}\left(-\eta\left(\frac{x}{h_1 y}\right)^{1/3} + \eta\left(\frac{y}{h_2 x}\right)^{1/3}\right)^{-3/2} \times\left(\frac{43}{(h_1^2 h_2 x^2 y^7)^{1/3}} + \frac{9}{h_2 x^2 y} - \frac{27}{h_1 y^3} - \frac{17}{(h_1 h_2^2 x^4 y^5)^{1/3}}\right).$$

Moreover, the fact that $(\frac{x}{h_1 y})^{1/3} - (\frac{y}{h_2 x})^{1/3}$ is increasing with respect to $x$ for $x, y > 0$ together with (A.3) gives us

$$\min_{x=a,b}\left|\left(\frac{x}{h_1 y}\right)^{1/3} - \left(\frac{y}{h_2 x}\right)^{1/3}\right| \le \left|\left(\frac{x}{h_1 y}\right)^{1/3} - \left(\frac{y}{h_2 x}\right)^{1/3}\right| \le \max_{x=a,b}\left|\left(\frac{x}{h_1 y}\right)^{1/3} - \left(\frac{y}{h_2 x}\right)^{1/3}\right|,$$

so

$$\left|\left(\frac{x}{h_1 y}\right)^{1/3} - \left(\frac{y}{h_2 x}\right)^{1/3}\right| \sim \frac{nN_j^{2/3}}{C^{4/3}t^{1/3}} \tag{A.5}$$

for any $x$ between $a$ and $b$. Together with $1 \le a, b, h_1/H', h_2/H' < 1 + 10^{-10}$, $1 - 10^{-9} \le y \le 1 + 10^{-9}$ by assumption and the support of $\Upsilon$, we get

$$F'(x) \gg \left(\frac{nN_j^{2/3}}{C^{4/3}t^{1/3}}\right)^{-3/2}\left(\frac{Ct}{N}\right)^{-1} = \frac{CN}{n^{3/2}N_j\sqrt{t}}$$

for all $x$ between $a$ and $b$. Hence the mean value theorem yields

$$\begin{aligned}\frac{d^2}{dy^2}(G(y,a) - G(y,b)) &= \sqrt{\frac{t}{72\pi n}}(F(a) - F(b)) \\ &\gg \min_{\min\{a,b\}\le x\le \max\{a,b\}} \sqrt{\frac{t}{72\pi n}}|F'(x)|\,|a-b| \gg \frac{CN}{n^2 N_j}|a-b| \sim \frac{CN^{1/3}}{nN_j^{2/3}}q'Z.\end{aligned}$$

With similar analysis we obtain

$$\frac{d^j}{dy^j}(G(y,a) - G(y,b)) \ll_j \left(\frac{CN^{1/3}}{nN_j^{2/3}}\right)^{j-1} q'Z \quad \text{for any } j \ge 2.$$

Let $y_0$ be as in the statement, i.e.

$$\left.\frac{d}{dy}\right|_{y=y_0}(G(y,a) - G(y,b)) = G_x(y_0,a) - G_x(y_0,b) = q'Z. \tag{A.6}$$

Then Lemma 5.2 (3) yields the first statement.

For the second statement about the derivatives of $y_0$, notice that differentiating (A.6) yields

$$\frac{dy_0}{da} = \frac{G_{xy}(y_0,a)}{G_{xx}(y_0,b) - G_{xx}(y_0,a)}.$$

The mean value theorem then implies that there exists some $x'$ between $a$ and $b$ such that

$$G_{xx}(y_0, b) - G_{xx}(y_0, a) = G_{xxy}(y_0, x')(b-a),$$

with $G_{xxy}(y_0, x')$ computed with Lemma A.2 like $F$ above. Then (A.4) and (A.5) together with $1 \le x', a, b, h_1/H', h_2/H' < 1 + 10^{-10}, 1 - 10^{-9} \le y_0 \le 1 + 10^{-9}$ give

$$\begin{aligned}\frac{dy_0}{da} = \frac{G_{xy}(y_0, a)}{(b-a)G_{xxy}(y_0, x')} &= \frac{6\eta}{a-b}\left(-\eta\left(\frac{a}{h_1 y_0}\right)^{1/3} + \eta\left(\frac{y_0}{h_2 a}\right)^{1/3}\right)^{-1/2} \\ &\quad \times \left(\frac{3}{(h_1^2 a y_0^5)^{1/3}} + \frac{3}{(h_2^2 a^5 y_0)^{1/3}} - \frac{2}{(h_1 h_2)^{1/3} a y_0}\right) \\ &\quad \times \left(-\eta\left(\frac{x'}{h_1 y_0}\right)^{1/3} + \eta\left(\frac{y_0}{h_2 x'}\right)^{1/3}\right)^{3/2} \\ &\quad \times \left(\frac{43}{(h_1^2 h_2 x'^2 y^7)^{1/3}} + \frac{9}{h_2 x'^2 y_0} - \frac{27}{h_1 y_0^3} - \frac{17}{(h_1 h_2^2 x'^4 y_0^5)^{1/3}}\right)^{-1} \\ &\sim \frac{N^{2/3}}{n N_j^{1/3} q' Z} \frac{n N_j^{2/3}}{C^{4/3} t^{1/3}}\left(\frac{Ct}{N}\right)^{1/3} = \frac{(N_j N)^{1/3}}{C q' Z}.\end{aligned}$$

Similar analysis yields

$$\frac{d^j y_0}{da^j}, \frac{d^j y_0}{db^j} \ll_j \left(\frac{n N_j^{2/3}}{C N^{1/3}}\right)^j \left(1 + \frac{N^{2/3}}{n N_j^{1/3} q' Z}\right)^j = \left(\frac{n N_j^{2/3}}{C N^{1/3}} + \frac{(N_j N)^{1/3}}{C q' Z}\right)^j. \quad \blacksquare$$

## Appendix B. Infinite Cauchy–Schwarz

### *B.1. Character sum analysis*

Let $a, b_1, b_2, b_3, c, q_1, q_2$ be integers such that $(q_1 q_2, c) = 1$ and $c \nmid b_2, b_3$ with $c$ being square-free. Let

$$\begin{aligned}\mathcal{C}(a, b_1, b_2, b_3, c, q_1, q_2) = \sum_{\gamma \bmod c}^{\star} e\left(\frac{a\gamma}{c}\right) S(b_1 - \overline{q_1} b_2 + \overline{\gamma} b_2, b_3(\gamma - q_1); c) \\ \times S(b_1 - \overline{q_2} b_2 + \overline{\gamma} b_2, b_3(\gamma - q_2); c).\end{aligned}$$

**Lemma B.1.** *If $q_1 \equiv q_2 \bmod c$ or $b_1 \pm b_2(\overline{q_1} - \overline{q_2}) \equiv 0 \bmod c$ or $b_1 - \overline{q_1} b_2 \equiv 0 \bmod c$ or $b_1 - \overline{q_2} b_2 \equiv 0 \bmod c$, then*

$$\mathcal{C}(a, b_1, b_2, b_3, c, q_1, q_2) \ll c^{2+\varepsilon}.$$

*Otherwise,*

$$\mathcal{C}(a, b_1, b_2, b_3, c, q_1, q_2) \ll c^{3/2+\varepsilon}.$$

*Proof.* Write $d_j(\gamma) = \gcd(b_1 - \overline{q_j}b_2 + \overline{\gamma}b_2, b_3(\gamma - q_j), c)$ for $j = 3, 4$. For the first statement, applying the Weil bound to the two Kloosterman sums yields

$$\mathcal{C}(a, b_1, b_2, b_3, c, q_1, q_2) \ll c^{1+\varepsilon} \sideset{}{^\star}\sum_{\gamma \bmod c} \sqrt{(d_3(\gamma)d_4(\gamma))} \ll c^{2+\varepsilon}.$$

For the second statement, we apply the method of [1]. Consider the Newton polyhedron $\Delta(f)$ of

$$\begin{aligned} f(x, y, z) = az + x(b_1 - \overline{q_1}b_2) + xz^{-1}b_2 + x^{-1}zb_3 - x^{-1}b_3q_1 + y(b_1 - \overline{q_2}b_2) \\ + yz^{-1}b_2 + y^{-1}zb_3 - y^{-1}b_3q_2. \end{aligned}$$

We have to consider two cases:

(1) $a \equiv 0 \bmod c$, while the other coefficients of $f$ are not 0 mod $c$.

(2) No coefficient of $f$ is 0 mod $c$.

The Newton polyhedron $\Delta(f)$ for case (2) is illustrated in Figure 1. One can check that the locus of $\partial f_\sigma/\partial x = \partial f_\sigma/\partial y = \partial f_\sigma/\partial z = 0$ is empty in $((\mathbb{Z}/c\mathbb{Z})^\times)^3$ for any face $\sigma$ of the Newton polyhedron $\Delta(f)$ that corresponds to any of the three cases unless $q_1 \equiv q_2 \bmod c$ or $b_1 \pm b_2(\overline{q_1} - \overline{q_2}) \equiv 0 \bmod c$. ∎

In general, we can extend the above lemma to the following setting. Let $\nu \geq 0$ be an integer and let $I_\nu \subset \mathcal{P}(\{1, 2, \ldots, 2^\nu\})$ be the set defined by

$$\begin{aligned} I_\nu = &\big\{\{2j + 1, 2j + 2\} : 0 \leq j \leq 2^{\nu-1} - 1\big\} \\ &\cup \big\{\{2^k(1 + 4j), 2^k(3 + 4j)\} : j, k \geq 0,\ 2^k(3 + 4j) \leq 2^\nu\} \setminus \{2^{\nu-1}, 2^\nu\}\big\}. \end{aligned}$$

Let $J_\nu = \{(j, S) : 1 \leq j \leq 2^\nu, S \in I_\nu \text{ and } j \in S\}$. Let $a_1, \ldots, a_{2^\nu}, a_S$ $(S \in I_\nu)$, $b_1, b_2, b_3, c, q_1, q_2$ be integers. Write $a_{I_\nu} = \{a_1, \ldots, a_{2^\nu}, a_S : S \in I_\nu\}$ and define

$$\begin{aligned} &\mathcal{C}_\nu(a_{I_\nu}, b_1, b_2, b_3, c, q_1, q_2) \\ &\quad = \sideset{}{^\star}\sum \cdots \sideset{}{^\star}\sum_{\{\alpha_{j,S} : (j,S)\in J_\nu\}, \beta_1, \beta_2, \gamma_1, \ldots, \gamma_{2^\nu}, \{\gamma_S : S \in I_\nu\} \bmod c} e\left(\frac{f_\nu(\alpha_{J_\nu}, \beta_1, \beta_2, \gamma_{I_\nu})}{c}\right) \end{aligned}$$

with

$$\begin{aligned} f_\nu(\alpha_{J_\nu}, \beta_1, \beta_2, \gamma_{I_\nu}) = &\sum_{j=1}^{2^\nu} a_j\gamma_j + \sum_{S \in I_\nu} a_S\gamma_S \\ &+ \sum_{(j,S)\in J_\nu} \big(\alpha_{j,S}(b_1 - \overline{\gamma_S}b_2 + \overline{\gamma_j}b_2) + \overline{\alpha_{j,S}}(\gamma_j - \gamma_S)b_3)\big)\beta_1(b_1 - \overline{q_1}b_2 + \overline{\gamma_{2^\nu-1}}b_2) \\ &+ \overline{\beta_1}(\gamma_{2^\nu-1} - q_1)b_3) + \beta_2(b_1 - \overline{q_2}b_2 + \overline{\gamma_{2^\nu}}b_2) + \overline{\beta_2}(\gamma_{2^\nu} - q_2)b_3), \end{aligned}$$

and the notation $\alpha_{J_\nu} = \{\alpha_{j,S} : (j, S) \in J_\nu\}$ and $\gamma_{I_\nu} = \{\gamma_1, \ldots, \gamma_{2^\nu}, \gamma_S : S \in I_\nu\}$.

**Lemma B.2.** *With notations as above, let $a_1, \ldots, a_{2^\nu}, a_S$ ($S \in I_\nu$), $b_1, b_2, b_3, c, q_1, q_2$ be integers such that $(q_1 q_2, c) = 1$ and $c \nmid b_2, b_3$. If $q_1 \equiv q_2 \bmod c$ or $b_1 \pm b_2(\overline{q_1} - \overline{q_2}) \equiv 0 \bmod c$ or $b_1 - \overline{q_1} b_2 \equiv 0 \bmod c$ or $b_1 - \overline{q_2} b_2 \equiv 0 \bmod c$, then*

$$\mathcal{C}_\nu(a_{I_\nu}, b_1, b_2, b_3, c, q_1, q_2) \ll c^{2^{\nu+1}+\varepsilon}.$$

*Otherwise,*

$$\mathcal{C}_\nu(a_{I_\nu}, b_1, b_2, b_3, c, q_1, q_2) \ll c^{2^{\nu+1}-1/2+\varepsilon}.$$

*Proof.* We apply the same method as in the proof of Lemma B.1.

For the second statement, we apply the method of [1]. Studying all the cases where each $a_j, a_S$ for $1 \le j \le 2^\nu$, $S \in I_\nu$ can be $0 \bmod c$ or not, one can check that the locus of $\partial f_{\nu\sigma}/\partial x_j = 0$ for all $1 \le j \le 2^{\nu+2} - 1$ is empty in $((\mathbb{Z}/c\mathbb{Z})^\times)^{2^{\nu+2}-1}$ for any face $\sigma$ of the Newton polyhedron $\Delta(f_\nu)$ that corresponds to any of the three cases unless $q_1 \equiv q_2 \bmod c$ or $b_1 \pm b_2(\overline{q_1} - \overline{q_2}) \equiv 0 \bmod c$.

For the first statement, we use induction. Note that $\nu = 0$ is precisely the statement of Lemma B.1. Notice that

$$\begin{aligned}\mathcal{C}_\nu(a_{I_\nu}, b_1, b_2, b_3, c, q_1, q_2) = \sideset{}{^\star}\sum_{\gamma_{2^{\nu-2},3\cdot 2^{\nu-2}}} & e\left(\frac{a_{\{2^{\nu-2},3\cdot 2^{\nu-2}\}}\gamma_{\{2^{\nu-2},3\cdot 2^{\nu-2}\}}}{c}\right) \\ & \times \mathcal{C}_{\nu-1}(a_{I_{\nu-1}}, b_1, b_2, b_3, c, \gamma_{2^{\nu-2},3\cdot 2^{\nu-2}}, q_1) \\ & \times \mathcal{C}_{\nu-1}(a'_{I_{\nu-1}}, b_1, b_2, b_3, c, \gamma_{2^{\nu-2},3\cdot 2^{\nu-2}}, q_2),\end{aligned}$$

with appropriate choices of $a_{I_{\nu-1}}$ and $a'_{I_{\nu-1}}$ given $a_{I_\nu}$. Hence we have

$$\begin{aligned}&\mathcal{C}_\nu(a_{I_\nu}, b_1, b_2, b_3, c, q_1, q_2) \\ &\qquad \ll \sideset{}{^\star}\sum_{\gamma_{2^{\nu-2},3\cdot 2^{\nu-2}} \bmod c} |\mathcal{C}_{\nu-1}(a_{I_{\nu-1}}, b_1, b_2, b_3, c, \gamma_{2^{\nu-2},3\cdot 2^{\nu-2}}, q_1)| \\ &\qquad\qquad \times |\mathcal{C}_{\nu-1}(a'_{I_{\nu-1}}, b_1, b_2, b_3, c, \gamma_{2^{\nu-2},3\cdot 2^{\nu-2}}, q_2)|.\end{aligned}$$

For $j = 1, 2$, write $(*_j)$ as the condition $\{\gamma_{2^{\nu-2},3\cdot 2^{\nu-2}} \equiv q_j \bmod c$ or $b_1 \pm b_2(\overline{\gamma_{2^{\nu-2},3\cdot 2^{\nu-2}}} - \overline{q_j}) \equiv 0 \bmod c$ or $b_1 - \overline{\gamma_{2^{\nu-2},3\cdot 2^{\nu-2}}} b_2 \equiv 0 \bmod c$ or $b_1 - \overline{q_j} b_2 \equiv 0 \bmod c\}$. By induction hypothesis, we have

$$|\mathcal{C}_{\nu-1}(a_{I_{\nu-1}}, b_1, b_2, b_3, c, \gamma_{2^{\nu-2},3\cdot 2^{\nu-2}}, q_j)| \ll c^{2^\nu - 1/2+\varepsilon}(\sqrt{c}\,\delta(*_j) + 1).$$

Hence

$$\mathcal{C}_\nu(a_{I_\nu}, b_1, b_2, b_3, c, q_1, q_2) \ll \sideset{}{^\star}\sum_{\gamma_{2^{\nu-2},3\cdot 2^{\nu-2}} \bmod c} c^{2^{\nu+1}-1+\varepsilon} \prod_{j=1,2} (\sqrt{c}\,\delta(*_j) + 1),$$

which yields the desired result. ∎

### *B.2. Proof of Lemma 9.4*

We argue by induction. Note that $\mathcal{B}_2$ coincides with the definition in (9.27) and the statement for $\nu = 2$ is proved in the previous subsection. Assume the statement is true for

$\nu = k$. Then

$$S_2(N)^* \ll |\mathcal{A}_k| + \sum_{j=1}^{k} \mathcal{B}_j,$$

with $\mathcal{A}_k$, $\mathcal{B}_1, \ldots, \mathcal{B}_k$ defined as in Lemma 9.4. Notice that by the same counting argument as in (7.12),

$$\sup_{\|\beta\|_2=1} \sum_{q_{k+1}\ll C} \sum_{\substack{n_0q_0q_{2,0}q'_{k+1}r=q_{k+1} \\ q_{2,0}|q_0^\infty}} \sum\sum\sum \sum_{\substack{q_{1,0}\ll C \\ q_{1,0}|q_0^\infty}} \sum_{q_0|n_1|q_0^\infty} \sum\sum\sum_{\substack{n_2h_{1,0}h_{2,0}\ \square\text{-full} \\ h_{2,0}|(n_2h_{1,0})^\infty}} \sum_{n'\ll N_B} \sum\sum_{h'_1,h'_2\ll H'}$$

$$\sum_{0<|q'_1|\ll Q_1Q_B} \sum\cdots\sum_{0<|q'_2|,\ldots,|q'_k|\ll Q_B} \frac{1}{n_1n_2^{3/2}n'h_{1,0}^{3/2}h_{2,0}h'_1q_0q_{1,0}q_{2,0}|q'_1\cdots q'_k|}$$

$$\times \int_{|u_2|\ll 1} |\beta(q_{k+1},h_2,u_2)|^2\,du_2 \sum_{\gamma \bmod n_1n_2h_{1,0}h_{2,0}q_0q_{1,0}q_{2,0}} \Bigg| D(k) \sideset{}{^\star}\sum_{\beta_1 \bmod q_0q_{1,0}}$$

$$\sideset{}{^\star}\sum_{\beta_2 \bmod q_0q_{2,0}} \delta\left(-\eta_j\frac{n}{q_0} \equiv \beta_1h_1q_{2,0}\gamma q'_{k+1} + \beta_2h_2q_{1,0}\gamma q'_{k+1} \bmod q_0q_{1,0}q_{2,0}\right)\Bigg|^2 \ll t^\varepsilon.$$

Recalling the definition of $\mathcal{D}_0$ in (9.14), and applying the Cauchy–Schwarz inequality to take out the $n, n_0, q_0, q_{1,0}, q_{2,0}, q'_1, q'_2, h_1, h_2$-sums, the $\gamma$-sum in $\mathcal{D}_0$ and the $u_2$-integral, we get, for the same fixed $\Upsilon \in C_c^\infty([1-10^{-9}, 1+10^{-9}])$ as before,

$$\mathcal{A}_k \ll \frac{t^\varepsilon}{C^{3/2}\sqrt{N_j}} \sup_{\|\beta\|_2=1} \sup_{|u_1|\ll 1} \Bigg\{ \int_{|u_2|\ll 1} \sum_{n_0}\sum_{q_0} \sum\sum_{q_{1,0},q_{2,0}|q_0^\infty} \sum_{q_0|n_1|q_0^\infty} \sum_r$$

$$\sum_{Q_1Q_S\ll|q'_1|\ll Q_1Q_B} \sum\cdots\sum_{Q_S\ll|q'_2|,\ldots,|q'_\nu|\ll Q_B} \sum\sum\sum_{\substack{n_2h_{1,0}h_{2,0}\ \square\text{-full} \\ h_{2,0}|(n_2h_{1,0})^\infty \\ (n_2,q_0)=1}} \sum\sum\sum_{\substack{N_S\ll n=n_1n_2n'\ll N_B \\ h_1=h_{1,0}h'_1,h_2=h_{2,0}h'_2\sim_0 H' \\ n'h'_1\ \square\text{-free},(n'_1h'_1,h'_2)=1 \\ (n'h'_1h'_2,n_1h_{1,0}h_{2,0}q_0)=1}}$$

$$\sum_{q_{k+1}=n_0q_0q_{2,0}q'_{k+1}r} \Psi\left(\frac{q_{k+1}}{C}\right) \sum\sum_{\substack{q_{k+2}=n_0q_0q_{2,0}q'_{k+2}r\sim_0 C \\ q_{k+3}=n_0q_0q_{2,0}q'_{k+3}r\sim_0 C \\ q'_{k+1}\equiv q'_{k+2}\equiv q'_{k+1} \bmod h'_2}} \left(\frac{\sqrt{h_{1,0}h_{2,0}}C}{n_0^2n_1n'h'_1q_0^2q_{1,0}q_{2,0}r^2}\right)^{2^k}$$

$$\times \left(\frac{n_0h'_2r\prod_{j=1}^{k-1}|q_j|^{2^{k-j}-1}}{n_1n_2^{3/2}h_{1,0}^{3/2}h_{2,0}q_{1,0}C}\right)^2 n_1n_2^{3/2}n'h_{1,0}^{3/2}h_{2,0}h'_1q_0q_{1,0}q_{2,0}|q'_1\cdots q'_k|$$

$$\times \beta'(q_{k+2},h_2,u_2)\overline{\beta'(q_{k+3},h_2,u_2)}\mathcal{D}_{k+1}(q'_{k+1},q'_{k+2})\overline{\mathcal{D}_{k+1}(q'_{k+1},q'_{k+3})}$$

$$\times \overline{\mathcal{D}'_0(q'_{k+2},q'_{k+3})} I_{k+1}(q'_1,\ldots,q'_k,q_{k+1},q_{k+2})$$

$$\times \overline{I_{k+1}(q'_1,\ldots,q'_k,q_{k+1},q_{k+3})}\,du_2 \Bigg\}^{2^{-k-1}}.$$

Note that $\mathcal{D}_k$ is defined $\bmod\, n'h_1'$. Applying Poisson summation to the $q'_{k+1}$-sum gives

$$\begin{aligned}
&\sum_{q'_{k+1}} \Upsilon\left(\frac{n_0 q_0 q_{2,0} q'_{k+1} r}{C}\right) \mathcal{D}_{k+1}(q'_{k+1}, q'_{k+2}) \overline{\mathcal{D}_{k+1}(q'_{k+1}, q'_{k+3})} \\
&\qquad \times \delta(q'_{k+1} \equiv q'_{k+2} \equiv q'_{k+3} \bmod h'_2) I_{k+1}(q'_1, \dots, q'_k, q_{k+1}, q_{k+2}) \\
&\qquad \times \overline{I_{k+1}(q'_1, \dots, q'_k, q_{k+1}, q_{k+3})} \\
&\quad = \frac{C}{n_0 n' h'_1 h'_2 q_0 q_{2,0} r} \sum_{q'_{k+1}} \sum_{\gamma \bmod n'h'_1 h'_2} \mathcal{D}_{k+1}(\gamma, q'_{k+2}) \overline{\mathcal{D}_{k+1}(\gamma, q'_{k+3})} \\
&\qquad \times \delta(\gamma \equiv q'_{k+2} \equiv q'_{k+3} \bmod h'_2) e\left(\frac{\gamma q'_{k+1}}{n'h'_1 h'_2}\right) \\
&\qquad \times \int_0^\infty \Upsilon(y_{2^{k+1}-1}) I_{k+1}(q'_1, \dots, q'_k, C y_{2^{k+1}-1}, q_{k+2}) \\
&\qquad \times \overline{I_{k+1}(q'_1, \dots, q'_k, C y_{2^{k+1}-1}, q_{k+3})} e\left(-\frac{q'_{k+1} C y_{2^{k+1}-1}}{n_0 n' h'_1 h'_2 q_0 q_{2,0} r}\right) dy_{2^{k+1}-1}.
\end{aligned}$$

For the character sum, the Chinese Remainder Theorem yields

$$\begin{aligned}
&\sum_{\gamma \bmod n'h'_1 h'_2} \mathcal{D}_{k+1}(\gamma, q'_{k+2}) \overline{\mathcal{D}_{k+1}(\gamma, q'_{k+3})} \delta(\gamma \equiv q'_{k+2} \equiv q'_{k+3} \bmod h'_2) e\left(\frac{\gamma q'_{k+1}}{n'h'_1 h'_2}\right) \\
&\qquad = e\left(\frac{q_{k+1'} q'_{k+2} \overline{n'h'_1}}{h'_2}\right) \delta(q'_{k+2} \equiv q'_{k+3} \bmod h'_2) \\
&\qquad\quad \times \sum_{\gamma \bmod n'h'_1} \mathcal{D}_{k+1}(\gamma, q'_{k+2}) \overline{\mathcal{D}_{k+1}(\gamma, q'_{k+3})} e\left(\frac{\gamma q'_{k+1} \overline{h'_2}}{n'h'_1}\right) \\
&\qquad = D(k+1) \mathcal{D}_{k+2}(q'_{k+2}, q'_{k+3}) \delta(q'_{k+2} \equiv q'_{k+3} \bmod h'_2).
\end{aligned}$$

For the integral, note that the phase function in the $y_{2^{k+1}-1}$-integral is

$$(-1)^{k+1}(G(y_{2^{k-1}-k+1}, y_{2^{k+1}-1}) - G(y_{2^k+2^{k-1}-k}, y_{2^{k+1}-1})) - \frac{q'_{k+1} C y_{2^{k+1}-1}}{n_0 n' h'_1 h'_2 q_0 q_{2,0} r}.$$

Hence repeated integration by parts gives us arbitrary savings unless $|q'_{k+1}| \ll Q_B$.

Let $\mathcal{B}_{k+1}$ be the contribution of $|q'_{k+1}| \ll Q_S$ including the case $q'_{k+1} = 0$. Then the above analysis yields

$$S_2(N)^* \ll |\mathcal{A}_{k+1}| + \sum_{j=1}^{k} \mathcal{B}_j + \tilde{\mathcal{B}}_{k+1},$$

with $\mathcal{A}_{k+1}, \mathcal{B}_1, \dots, \mathcal{B}_k$ as defined in the statement and

$$\tilde{\mathcal{B}}_{k+1} := \frac{t^{\varepsilon}}{C^{3/2}\sqrt{N_j}}$$
$$\times \sup_{\|\beta\|_2=1} \sup_{|u_1|\ll 1} \Bigg\{ \int_{|u_2|\ll 1} \sum_{n_0} \sum_{q_0} \sum\sum_{q_{1,0},q_{2,0}|q_0^\infty} \sum_{q_0|n_1|q_0^\infty} \sum_{r} \sum_{Q_1 Q_S \ll |q_1'| \ll Q_1 Q_B}$$
$$\sum\cdots\sum_{Q_S\ll|q_2'|,\ldots,|q_k'|\ll Q_B} \sum_{|q_{k+1}'|\ll Q_S} \sum\sum\sum_{\substack{n_2 h_{1,0} h_{2,0}\ \square\text{-full} \\ h_{2,0}|(n_2 h_{1,0})^\infty \\ (n_2,q_0)=1}} \sum\sum\sum_{\substack{N_S \ll n = n_1 n_2 n' \ll N_B \\ h_1 = h_{1,0}h_1', h_2 = h_{2,0}h_2' \sim_0 H' \\ n'h_1'\ \square\text{-free}, (n_1' h_1', h_2')=1 \\ (n'h_1'h_2', n_1 h_{1,0} h_{2,0} q_0)=1}}$$
$$\sum\sum_{\substack{q_{k+2}=n_0 q_0 q_{2,0} q_{k+2}' r \sim_0 C \\ q_{k+3}=n_0 q_0 q_{2,0} q_{k+3}' r \sim_0 C \\ q_{k+2}' \equiv q_{k+3}' \bmod h_2'}} \left( \frac{\sqrt{h_{1,0}h_{2,0}}C}{n_0^2 n_1 n' h_1' q_0^2 q_{1,0} q_{2,0} r^2} \right)^{2^k} \frac{n_0 h_2' r}{n_1 n_2^{3/2} h_{1,0}^{3/2} h_{2,0} q_{1,0} C}$$
$$\times \prod_{j=1}^{k} |q_j|^{2^{k+1-j}-1} D(k+1) \beta'(q_{k+2}, h_2, u_2) \overline{\beta'(q_{k+3}, h_2, u_2)}$$
$$\times \mathcal{D}_{k+2}(q_{k+2}', q_{k+3}') \mathcal{D}_0'(q_{k+2}', q_{k+3}') I_{k+2}(q_1', \ldots, q_{k+1}', q_{k+2}, q_{k+3})\, du_2 \Bigg\}^{2^{-k-1}}.$$

Applying Lemma 9.3 with $\eta = 0$ to the $y_{2^{k+1}-1}$-integral, then $\eta = 1$ ($2^k - 2$ times) to the remaining integrals, followed by the final statement on the single integral left after the process, we get, for $|q_{k+1}'| \ll Q_S$,

$$I_{k+2}(q_1', \ldots, q_{k+1}', q_{k+2}, q_{k+3}) \ll \frac{C^2 t^{\varepsilon}}{n N_j} \left( \frac{C}{(N_j N)^{1/3}} + \frac{C N^{1/3}}{\sqrt{nt} N_j^{2/3}} \right)^{2^k - 2}$$
$$\times \delta\left(|q_{k+2} - q_{k+3}| \ll \left( \frac{C^2}{(N_j N)^{1/3}} + \frac{C^2 N^{1/3}}{\sqrt{nt} N_j^{2/3}} ) t^{\varepsilon} \right) + t^{-A}\right.$$

for any $A > 0$. The fact that every use of Lemma 9.3 will truncate the difference between two variables of integration and finally truncate the difference between $q_{k+2}', q_{k+3}'$ can be seen by noticing that every integral $y_i$ appears in some $G(\cdot,\cdot)$ in the phase function of the $(2^{k+1} - 1)$-fold integral twice and no two of them appear in the same $G(\cdot,\cdot)$. Also, the way $q_{k+2}, q_{k+3}$ is constructed through iterations of the Cauchy–Schwarz inequality shows that the truncation between them occurs at last.

On the other hand, notice that the character sum $\mathcal{D}_{k+2}$ is exactly of the form defined before Lemma B.2. Indeed, we have

$$\mathcal{D}_{k+2}(q_{k+2}', q_{k+3}') = \mathcal{C}_{k-1}\big(a_{I_{k-1}}, q_1' \overline{n_1 n_2 h_{1,0} h_2 q_0 q_{1,0} q_{2,0}}, \eta_j q_{1,0} \overline{n_1 n_2 h_2 q_{2,0}},$$
$$\eta_j q_{2,0} \overline{n_1 n_2 h_{1,0} q_{1,0}}, n' h_1', q_{k+2}', q_{k+3}'\big),$$

with $a_{I_{k-1}}$ being an appropriate sequence of numbers consisting of $q_j'\overline{h_2'}$. Hence Lemma B.2 gives us

$$\mathcal{D}_{k+2}(q_{k+2}', q_{k+3}') \ll (n'h_1')^{2^k-1/2} t^{\varepsilon}\left(\sqrt{n'h_1'}\,\delta(*_{k-1}) + 1\right),$$

where $*_{k-1}$ is the condition

$$\begin{aligned} *_{k-1} = \{&q_{k+2}' \equiv q_{k+3}' \bmod n'h_1' \\ &\text{or } q_1'\overline{q_0 q_{1,0}} \pm h_1 q_{1,0}(\overline{q_{k+2}'} - \overline{q_{k+3}'}) \equiv 0 \bmod n'h_1' \\ &\text{or } q_1'\overline{q_0 q_{1,0}} - \eta_j h_1 q_{1,0}\overline{q_{k+2}'} \equiv 0 \bmod n'h_1' \\ &\text{or } q_1'\overline{q_0 q_{1,0}} - \eta_j h_1 q_{1,0}\overline{q_{k+3}'} \equiv 0 \bmod n'h_1'\}. \end{aligned}$$

This proves the last statement of the lemma.

Together with $\mathcal{D}(k+1) \ll 1$ and $\mathcal{D}_0 \ll n_1 n_2 h_{1,0} h_{2,0} q_0^3 q_{1,0} q_{2,0}$ as in (9.14), this yields

$$\begin{aligned} \tilde{\mathcal{B}}_{k+1} \ll{}& \frac{t^{\varepsilon}}{C^{3/2}\sqrt{N_j}} \sup_{\|\beta\|_2=1} \sup_{|u_1|\ll 1} \Bigg\{ \int_{|u_2|\ll 1} \sum_{n_0}\sum_{q_0} \sum_{q_{1,0},q_{2,0}|q_0^\infty}\sum_{q_0|n_1|q_0^\infty} \sum_r \\ & \sum_{Q_1 Q_S \ll |q_1'| \ll Q_1 Q_B} \sum_{Q_S \ll |q_2'|,\dots,|q_k'| \ll Q_B} \!\!\cdots\!\! \sum \ \sum_{|q_{k+1}'| \ll Q_S} \sum_{\substack{n_2 h_{1,0} h_{2,0}\ \square\text{-full} \\ h_{2,0}|(n_2 h_{1,0})^\infty \\ (n_2,q_0)=1}}\sum\sum \ \sum_{\substack{N_S \ll n = n_1 n_2 n' \ll N_B \\ h_1 = h_{1,0}h_1',\, h_2 = h_{2,0}h_2' \sim_0 H' \\ n'h_1'\ \square\text{-free},\,(n_1'h_1',h_2')=1 \\ (n'h_1'h_2',\, n_1 h_{1,0} h_{2,0} q_0)=1}}\sum\sum \\ & \sum_{\substack{q_{k+2} = n_0 q_0 q_{2,0} q_{k+2}' r \sim_0 C \\ q_{k+3} = n_0 q_0 q_{2,0} q_{k+3}' r \sim_0 C \\ q_{k+2}' \equiv q_{k+3}' \bmod h_2' \\ |q_{k+2} - q_{k+3}| \ll \left(\frac{C^2}{(N_j N)^{1/3}} + \frac{C^2 N^{1/3}}{\sqrt{nt} N_j^{2/3}}\right) t^{\varepsilon}}}\sum \left( \frac{\sqrt{h_{1,0}h_{2,0}}\,C}{n_0^2 n_1 n' h_1' q_0^2 q_{1,0} q_{2,0} r^2} \right)^{2^k} \frac{n_0 h_2' r}{n_1 n_2^{3/2} h_{1,0}^{3/2} h_{2,0} q_{1,0} C} \\ & \times \prod_{j=1}^{k} |q_j|^{2^{k+1-j}-1} |\beta(q_{k+2}, h_2, u_2)| \\ & \times |\beta'(q_{k+3}, h_2, u_2)| n_1 n_2 h_{1,0} h_{2,0} q_0^3 q_{1,0} q_{2,0} (n'h_1')^{2^k-1/2} t^{\varepsilon} \left(\sqrt{n'h_1'}\,\delta(*_{k-1}) + 1\right) \\ & \times \frac{C^2}{nN_j} \left( \frac{C}{(N_j N)^{1/3}} + \frac{C N^{1/3}}{\sqrt{nt} N_j^{2/3}} \right)^{2^k-2} du_2 \Bigg\}^{2^{-k-1}}. \end{aligned}$$

Recall the definition of $Q_S, Q_B$ in (9.17) and (9.12). Applying the AM-GM inequality

$$|\beta(q_{k+2}, h_2, u_2)\beta(q_{k+3}, h_2, u_2)| \ll |\beta(q_{k+2}, h_2, u_2)|^2 + |\beta(q_{k+3}, h_2, u_2)|^2$$

and the same counting argument as in (7.12) yields

$$\begin{aligned}\tilde{\mathcal{B}}_{k+1} \ll& \frac{t^{\varepsilon}}{C^{3/2}\sqrt{N_j}}\\ &\times\Bigg\{\frac{C^{2^k+1}}{N_j}\sum_{n\ll CN^{1/3}t^{\varepsilon}/N_j^{2/3}} n^{-3/2}\left(\frac{nN_j^{1/3}t^2}{N^{5/3}}\right)^{2^{k+1}-2}\left(1+\frac{C^3t^2}{n(N_jN)^{4/3}}\right)\sqrt{\frac{Ct}{N}}\\ &\times\Bigg(\underbrace{\frac{Ct}{N}}_{q_{k+2}=q_{k+3}}+\underbrace{\frac{C^2}{(N_jN)^{1/3}}+\frac{C^2N^{1/3}}{\sqrt{nt}N_j^{2/3}}}_{q_{k+2}\neq q_{k+3}}\Bigg)\left(\frac{C}{(N_jN)^{1/3}}+\frac{CN^{1/3}}{\sqrt{nt}N_j^{2/3}}\right)^{2^k-2}\Bigg\}^{2^{-k-1}}\\ \ll& \frac{\sqrt{C}t^{2+\varepsilon}}{N_jN^{3/2}}\left\{\frac{N_j^{2/3}N^{8/3}}{C^3t^{7/2}}\left(1+\frac{C^2t^2}{N_j^{2/3}N^{5/3}}\right)\left(\frac{Ct}{N}+\frac{C^2}{(N_jN)^{1/3}}\right)\right\}^{2^{-k-1}}.\end{aligned}$$

### *B.3. Proof of Lemma* 9.5

Recall that the relevant integral is defined recursively by

$$\begin{aligned}I_{\nu+1}(q_1',\ldots,q_\nu',q_{\nu+1},q_{\nu+2}) =& \int_0^\infty \Upsilon(y_{2^\nu-1})I_\nu(q_1',\ldots,q_{\nu-1}',Cy_{2^\nu-1},q_{\nu+1})\\ &\times\overline{I_\nu(q_1',\ldots,q_{\nu-1}',Cy_{2^\nu-1},q_{\nu+2})}e\left(-\frac{q_\nu'Cy_{2^\nu-1}}{n_0n'h_1h_2q_0q_{2,0}r}\right)dy_{2^\nu-1}\end{aligned}$$

with

$$\begin{aligned}I_3(q_1',q_2',q_3,q_4) =& \int_0^\infty \Upsilon(y_3)I_2(q_1',Cy_3,q_3,q_4)e\left(-\frac{q_2'Cy_3}{n_0n'h_1h_2q_0q_{2,0}r}\right)dy_3\\ =& \int_0^\infty \Upsilon(y_3)\int_0^\infty \Upsilon(y_1)\Psi(Cy_1,Cy_3,q_3)e\Bigg(G(y_1,y_3)-G\left(y_1,\frac{q_3}{C}\right)\\ &-\frac{q_1'Cy_1}{n_0nh_1h_2q_0^2q_{1,0}^2q_{2,0}r}\Bigg)dy_1\\ &\times\int_0^\infty \Upsilon(y_2)\overline{\Psi(Cy_2,Cy_3,q_4)}e\Bigg(-G(y_2,y_3)+G\left(y_2,\frac{q_4}{C}\right)\\ &+\frac{q_1'Cy_2}{n_0nh_1h_2q_0^2q_{1,0}^2q_{2,0}r}\Bigg)dy_2e\left(-\frac{q_2'Cy_3}{n_0n'h_1h_2q_0q_{2,0}r}\right)dy_3.\end{aligned}$$

Recall the definition of $G$ in (9.16). Applying Taylor expansion, we get

$$G(y,z)=\tilde{G}(y,z)+\theta(y,z),$$

where

$$\tilde{G}(y,z)=\sqrt{\frac{2t}{\pi n}}\left(-\eta_j'\left(\frac{z}{h_1y}\right)^{1/3}+\eta_j'\left(\frac{y}{h_2z}\right)^{1/3}\right)^{3/2}$$

and

$$\theta(y,z)=\sqrt{\frac{2t}{\pi n}}\sum_{j=1}^{\infty}a_j\left(-\eta_j'\left(\frac{z}{h_1y}\right)^{1/3}+\eta_j'\left(\frac{y}{h_2z}\right)^{1/3}\right)^{3/2-j}$$
$$\times\left(-\eta_j'\left(\frac{z}{h_1y}\right)^{1/3}\sum_{\ell=1}^{\infty}b_\ell\left(\frac{2\pi h_1N^2u_1}{CHt^{2-\varepsilon}y}\right)^{\ell}+\eta_j'\left(\frac{y}{h_2z}\right)^{1/3}\sum_{\ell=1}^{\infty}b_\ell\left(\frac{2\pi h_2N^2u_2}{CHt^{2+\varepsilon}z}\right)^{\ell}\right)^{j}$$

for some constants $a_j, b_\ell\in\mathbb{R}$. Then $e(\theta(y,z))$ is $\frac{N^{3/2}}{\sqrt{CnHt^{1-\varepsilon}}}$-inert and $I_3$ simplifies to

$$\begin{aligned}
&I_3(q_1',q_2',q_3,q_4)\\
&=\int_0^\infty\Upsilon(y_3)\int_0^\infty\Upsilon(y_1)\tilde{\Psi}(Cy_1,Cy_3,q_3)e\Bigg(\tilde{G}(y_1,y_3)-\tilde{G}\left(y_1,\frac{q_3}{C}\right)\\
&\qquad\qquad-\frac{q_1'Cy_1}{n_0nh_1h_2q_0^2q_{1,0}^2q_{2,0}r}\Bigg)dy_1\\
&\times\int_0^\infty\Upsilon(y_2)\overline{\tilde{\Psi}(Cy_2,Cy_3,q_4)}e\Bigg(-\tilde{G}(y_2,y_3)+\tilde{G}\left(y_2,\frac{q_4}{C}\right)\\
&\qquad\qquad+\frac{q_1'Cy_2}{n_0nh_1h_2q_0^2q_{1,0}^2q_{2,0}r}\Bigg)dy_2\,e\left(-\frac{q_2'Cy_3}{n_0n'h_1h_2q_0q_{2,0}r}\right)dy_3,
\end{aligned}$$

where

$$\tilde{\Psi}(x,y,z)=\Psi(x,y,z)e(\theta(x,y)-\theta(x,z))$$

is $(1+\frac{CN^{1/3}}{nN_j^{2/3}}+\frac{N^{3/2}}{\sqrt{CnHt}})t^{\varepsilon}$-inert by (9.11) and the above analysis of $\theta$.

Write $q_1''=q_1'/Q_1=q_1'/n_1n_2h_{1,0}h_{2,0}q_0q_{1,0}^2$ and $q_j''=q_j'$ for all $2\le j\le\nu$. Note that $n\gg N_S$ and $|q_j''|\gg Q_S$; this allows us to apply Lemma A.3 ($2^\nu-1$ times) to bound the integral. (The lower bounds for $q_j'$ and $n$ are needed so that the oscillation of $\tilde{\Psi}$ satisfies the condition in Lemma A.3.) For $1\le j\le\nu$, we choose

$$U_j=\left(1+\delta(j>1)\left(1+\frac{(N_jN)^{1/3}n_0n'h_1h_2q_0q_{2,0}r}{\sqrt{\min\{q_u'':1\le u\le j-1\}q_j''C^2}}\right)\right)\sqrt{\frac{n_0n'h_1h_2q_0q_{2,0}r}{q_j''C}}$$

as the parameter $U$ in Lemma A.3 for the $2^{\nu-j}$ integrals with $q_j'$. We first apply the lemma once to the integral with $q_\nu'$, then twice to the two integrals with $q_2'$ and go on until we apply it $2^{\nu-1}$ times to the $2^{\nu-1}$ integrals with $q_1'$. The choices of $U$ are made so that the conditions of Lemma A.3 are fulfilled in the next iteration.

**Remark B.3.** Explanation on the choice of $U_j$: Suppose we have applied Lemma A.3 to all the integrals involving $q_\nu'',\dots,q_k'$. Then a number of functions of the form $W_0(\frac{y_u-y_0}{U_j})$ is created, and they are $\frac{t^\varepsilon}{U_j}(1+\frac{(N_jN)^{1/3}}{Cq_j''Z})$-inert by Lemma A.3, where $Z=\frac{C}{n_0n'h_1h_2q_0q_{2,0}r}$. To apply Lemma A.3 to the integrals involving $q_{k-1}'$, we need to make sure $U_\nu,\dots,U_k$ are chosen such that

$$q_{k-1}''Z\gg\frac{t^\varepsilon}{U_j^2}\left(1+\frac{(N_jN)^{1/3}}{Cq_j''Z}\right)^2$$

for all $k\le j\le\nu$.

Applying the above procedure and bounding the resulting integral trivially yields

$$\begin{aligned}I_{\nu+1}(q_1',\dots,q_\nu',q_{\nu+1},q_{\nu+2}) &\ll_\nu U_1^{2^{\nu-1}}U_2^{2^{\nu-2}}\cdots U_\nu t^\varepsilon\\ &\ll_\nu \prod_{j=2}^{\nu}\biggl(1+\frac{(N_jN)^{1/3}n_0n'h_1h_2q_0q_{2,0}r}{\sqrt{\min\{q_u'':1\le u\le j-1\}q_j''C^2}}\biggr)^{2^{\nu-j}}\\ &\quad\times\frac{(n_1q_0q_{1,0}^2)^{2^{\nu-2}}(n_0n'h_1h_2q_0q_{2,0}r)^{2^{\nu-1}-1/2}}{q_1'^{2^{\nu-2}}q_2'^{2^{\nu-3}}\cdots q_\nu'^{2^{-1}}C^{2^{\nu-1}-1/2}}t^\varepsilon\end{aligned}$$

for any $\nu\ge 2$.

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## References

[1] Adolphson, A., Sperber, S.: Twisted exponential sums and Newton polyhedra. J. Reine Angew. Math. **443**, 151–177 (1993) Zbl 0853.11067 MR 1241131

[2] Aggarwal, K.: Weyl bound for GL(2) in $t$-aspect via a simple delta method. J. Number Theory **208**, 72–100 (2020) Zbl 1446.11092 MR 4032289

[3] Aggarwal, K.: A new subconvex bound for GL(3) $L$-functions in the $t$-aspect. Int. J. Number Theory **17**, 1111–1138 (2021) Zbl 1478.11072 MR 4270876

[4] Bernstein, J., Reznikov, A.: Subconvexity bounds for triple $L$-functions and representation theory. Ann. of Math. (2) **172**, 1679–1718 (2010) Zbl 1225.11068 MR 2726097

[5] Blomer, V.: Subconvexity for twisted $L$-functions on GL(3). Amer. J. Math. **134**, 1385–1421 (2012) Zbl 1297.11046 MR 2975240

[6] Blomer, V., Jana, S., Nelson, P. D.: The Weyl bound for triple product $L$-functions. Duke Math. J. **172**, 1173–1234 (2023) Zbl 1542.11049 MR 4576241

[7] Blomer, V., Khan, R., Young, M.: Distribution of mass of holomorphic cusp forms. Duke Math. J. **162**, 2609–2644 (2013) Zbl 1312.11028 MR 3127809

[8] Booker, A. R., Milinovich, M. B., Ng, N.: Subconvexity for modular form $L$-functions in the $t$ aspect. Adv. Math. **341**, 299–335 (2019) Zbl 1469.11280 MR 3872849

[9] Bourgain, J.: Decoupling, exponential sums and the Riemann zeta function. J. Amer. Math. Soc. **30**, 205–224 (2017) Zbl 1352.11065 MR 3556291

[10] Conrey, J. B., Iwaniec, H.: The cubic moment of central values of automorphic $L$-functions. Ann. of Math. (2) **151**, 1175–1216 (2000) Zbl 0973.11056 MR 1779567

[11] Goldfeld, D.: Automorphic forms and $L$-functions for the group $\mathrm{GL}(n,\mathbf{R})$. Cambridge Stud. Adv. Math. 99, Cambridge University Press, Cambridge (2006) Zbl 1108.11039 MR 2254662

[12] Good, A.: The square mean of Dirichlet series associated with cusp forms. Mathematika **29**, 278–295 (1982) Zbl 0497.10016 MR 0696884

[13] Graham, S. W., Kolesnik, G.: Van der Corput's method of exponential sums. London Math. Soc. Lecture Note Ser. 126, Cambridge University Press, Cambridge (1991) Zbl 0713.11001 MR 1145488

[14] Heath-Brown, D. R.: The twelfth power moment of the Riemann-function. Quart. J. Math. Oxford Ser. (2) **29**, 443–462 (1978) Zbl 0394.10020 MR 0517737

[15] Holowinsky, R., Munshi, R., Qi, Z.: Beyond the Weyl barrier for GL(2) exponential sums. Adv. Math. **426**, article no. 109099, 43 pp. (2023) Zbl 1527.11065 MR 4592267

[16] Huang, B.: Hybrid subconvexity bounds for twisted $L$-functions on GL(3). Sci. China Math. **64**, 443–478 (2021) Zbl 1469.11125 MR 4215995

[17] Ivić, A.: Hybrid moments of the Riemann zeta-function. J. Numbers **2015**, article no. 892324, 14 pp. Zbl 1423.11147 MR 3375901

[18] Iwaniec, H.: Fourier coefficients of cusp forms and the Riemann zeta-function. In: Séminaire de Théorie des Nombres, 1979–1980, Univ. Bordeaux I, Talence, exp. no. 18, 36 pp. (1980) Zbl 0438.10030 MR 0604215

[19] Iwaniec, H., Kowalski, E.: Analytic number theory. Amer. Math. Soc. Colloq. Publ. 53, American Mathematical Society, Providence, RI (2004) Zbl 1059.11001 MR 2061214

[20] Jutila, M.: Lectures on a method in the theory of exponential sums. Tata Institute of Fundamental Research Lectures on Mathematics and Physics 80, Springer, Berlin (1987) Zbl 0671.10031 MR 0910497

[21] Khan, R., Young, M. P.: Moments and hybrid subconvexity for symmetric-square $L$-functions. J. Inst. Math. Jussieu **22**, 2029–2073 (2023) Zbl 1551.11095 MR 4624955

[22] Kıral, E. M., Petrow, I., Young, M. P.: Oscillatory integrals with uniformity in parameters. J. Théor. Nombres Bordeaux **31**, 145–159 (2019) Zbl 1447.41015 MR 3994723

[23] Landau, E.: Über die Wurzeln der Zetafunktion. Math. Z. **20**, 98–104 (1924) JFM 50.0231.01 MR 1544664

[24] Li, X.: Bounds for GL(3) $\times$ GL(2) $L$-functions and GL(3) $L$-functions. Ann. of Math. (2) **173**, 301–336 (2011) Zbl 1320.11046 MR 2753605

[25] Lin, Y., Nunes, R., Qi, Z.: Strong subconvexity for self-dual GL(3) $L$-functions. Int. Math. Res. Notices **2023**, 11453–11470 Zbl 1541.11046 MR 4609788

[26] McKee, M., Sun, H., Ye, Y.: Improved subconvexity bounds for $GL(2) \times GL(3)$ and $GL(3)$ $L$-functions by weighted stationary phase. Trans. Amer. Math. Soc. **370**, 3745–3769 (2018) Zbl 1439.11128 MR 3766865

[27] Meurman, T.: On exponential sums involving the Fourier coefficients of Maass wave forms. J. Reine Angew. Math. **384**, 192–207 (1988) Zbl 0627.10017 MR 0929983

[28] Michel, P., Venkatesh, A.: The subconvexity problem for $\mathrm{GL}_2$. Publ. Math. Inst. Hautes Études Sci. **111**, 171–271 (2010) Zbl 1376.11040 MR 2653249

[29] Miller, S. D., Schmid, W.: Automorphic distributions, $L$-functions, and Voronoi summation for GL(3). Ann. of Math. (2) **164**, 423–488 (2006) Zbl 1162.11341 MR 2247965

[30] Munshi, R.: The circle method and bounds for $L$-functions—III: $t$-aspect subconvexity for $GL(3)$ $L$-functions. J. Amer. Math. Soc. **28**, 913–938 (2015) Zbl 1354.11036 MR 3369905

[31] Munshi, R.: Subconvexity for GL(3) $\times$ GL(2) $L$-functions in $t$-aspect. J. Eur. Math. Soc. **24**, 1543–1566 (2022) Zbl 1494.11048 MR 4404783

[32] Nelson, P. D.: Bounds for standard $L$-functions. arXiv:2109.15230v2 (2023)

[33] Nunes, R. M.: Subconvexity for GL(3) $L$-functions. arXiv:1703.04424v1 (2017)

[34] Qi, Z.: Subconvexity for $L$-functions on $\mathrm{GL}_3$ over number fields. J. Eur. Math. Soc. **26**, 1113–1192 (2024) Zbl 1545.11046 MR 4721030

[35] Young, M. P.: The second moment of GL(3) $\times$ GL(2) $L$-functions integrated. Adv. Math. **226**, 3550–3578 (2011) Zbl 1231.11112 MR 2764898