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Columnar order in random packings of 2×2 squares on the square lattice

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Abstract. We study random packings of 2×2 squares with centers on the square lattice $\mathbb{Z}^2$, in which the probability of a packing is proportional to λ to the power of the number of squares. We prove that for large λ , typical packings exhibit columnar order, in which either the centers of most tiles agree in the parity of their x -coordinate or the centers of most tiles agree in the parity of their y -coordinate. This manifests itself in the existence of four extremal and periodic Gibbs measures in which the rotational symmetry of the lattice is broken while the translational symmetry is only broken along a single axis. We further quantify the decay of correlations in these measures, obtaining a slow rate of exponential decay in the direction of preserved translational symmetry and a fast rate in the direction of broken translational symmetry. Lastly, we prove that every periodic Gibbs measure is a mixture of these four measures. Additionally, our proof introduces an apparently novel extension of the chessboard estimate, from finite-volume torus measures to all infinite-volume periodic Gibbs measures.

Keywords: hard-core model, random packings, chessboard estimate, disagreement percolation, columnar order, liquid crystal.

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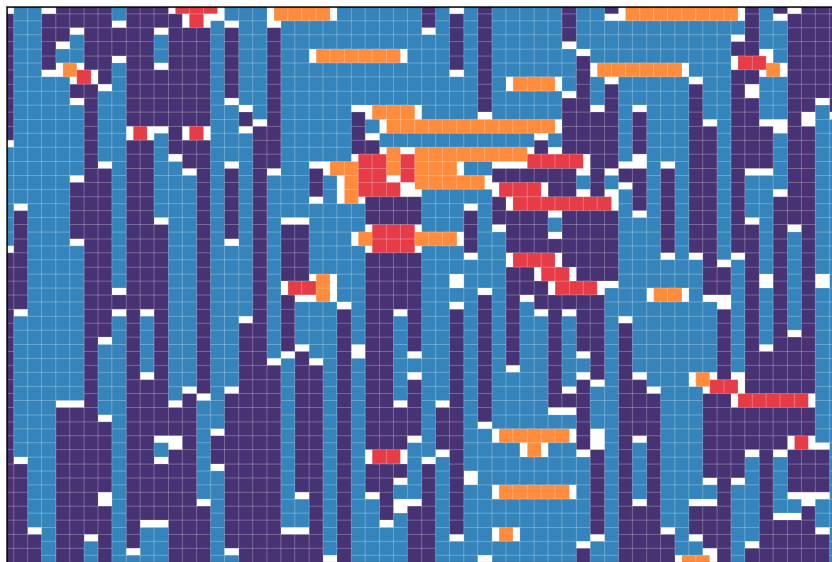
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1. Introduction

In this paper we study the 2×2 hard-square model on the square lattice $\mathbb{Z}^2$. We prove that the model admits *columnar order* at high fugacity, resulting in multiple Gibbs measures. Moreover, the set of periodic Gibbs measures is characterized.

1.1. The model

The square lattice $\mathbb{Z}^2$ is embedded in $\mathbb{R}^2$ in the natural way. With each vertex $(x, y) \in \mathbb{Z}^2$ is associated a *tile* $T_{(x,y)}$ that is a closed 2×2 square, with sides parallel to the axes, which is centered at that vertex. Hard-square configurations are sets of tiles whose interiors are pairwise disjoint. More precisely, a configuration is represented by a function



Columnar order in (a portion of) a high-density packing ($\lambda = 130$). The four colors correspond to the parities of the x - and y -coordinates of each tile.

$\sigma : \mathbb{Z}^2 \rightarrow \{0, 1\}$, with the value $\sigma(v) = 1$ indicating the presence of a tile centered at v , so that the *space of configurations* is

$$\Omega := \{\sigma \in \{0, 1\}^{\mathbb{Z}^2} : \text{if } \text{int}(T_u) \cap \text{int}(T_v) \neq \emptyset \text{ for } u \neq v, \text{ then } \sigma(u)\sigma(v) = 0\}$$

where $\text{int}(\cdot)$ stands for the interior of a set. Given a bounded set $\Lambda \subset \mathbb{R}^2$ and a configuration $\rho \in \Omega$, the space of *configurations with ρ -boundary conditions outside Λ* is

$$\Omega_{\Lambda}^{\rho} := \{\sigma \in \Omega : \sigma(v) = \rho(v) \text{ on } \mathbb{Z}^2 \setminus \text{int}(\Lambda)\}.$$

Given additionally a *fugacity* parameter $\lambda > 0$, the corresponding *finite-volume 2×2 hard-square model* is the probability measure $\mu_{\Lambda, \lambda}^{\rho}$ on Ω_{Λ}^{ρ} defined by

$$\mu_{\Lambda, \lambda}^{\rho}(\sigma) \propto \lambda^{\sum_{v \in \text{int}(\Lambda) \cap \mathbb{Z}^2} \sigma(v)} \quad (1.1)$$

where we use $\propto$ to denote that the left-hand side is proportional to the right-hand side. In words, the probability of a configuration is proportional to λ raised to the power of the number of tiles in Λ .

We describe our results in the language of infinite-volume Gibbs measures, defined via the standard Dobrushin–Lanford–Ruelle prescription. More precisely, a probability measure μ on Ω is an (infinite-volume) *Gibbs measure* for the 2×2 hard-square model at fugacity λ if for every bounded $\Lambda \subset \mathbb{R}^2$ the following holds: Let σ be sampled from μ . Conditionally on σ restricted to $\mathbb{Z}^2 \setminus \text{int}(\Lambda)$, the distribution of σ is given by $\mu_{\Lambda, \lambda}^{\rho}$ with ρ being any configuration which coincides with σ on $\mathbb{Z}^2 \setminus \text{int}(\Lambda)$.

Given a sublattice $\mathcal{L} \subset \mathbb{Z}^2$ and a probability measure μ on Ω , say that μ is $\mathcal{L}$ -invariant if it is invariant under all translations by vectors from $\mathcal{L}$ and say that μ is $\mathcal{L}$ -ergodic if it is $\mathcal{L}$ -invariant and assigns probability 0 or 1 to $\mathcal{L}$ -invariant events. A probability measure on Ω is called *periodic* if it is $\mathcal{L}$ -invariant under some *full-rank* sublattice $\mathcal{L} \subset \mathbb{Z}^2$. A probability measure on Ω is called *extremal* if it assigns probability 0 or 1 to all tail events (the tail sigma-algebra is the intersection over all finite $D \subset \mathbb{Z}^2$ of the sigma-algebra generated by σ restricted to D^c).

1.2. Discussion and results

Classical methods may be used to show that the 2×2 hard-square model is disordered at *low fugacity*, in the sense that it has a *unique* Gibbs measure. This follows from either the Dobrushin uniqueness theorem [20] or the disagreement percolation method of van den Berg [78], with the latter method proving uniqueness for $\lambda < p_c/(1 - p_c)$ (see also [80, Theorem 2.3]), where p_c is the site percolation threshold of the square lattice with nearest-neighbor and next-nearest-neighbor interactions (i.e., $u, v \in \mathbb{Z}^2$ are adjacent if $\|u - v\|_\infty = 1$, this is dual to the standard nearest-neighbor site percolation on $\mathbb{Z}^2$; numerical estimates give $p_c \approx 0.407$ [48, 88]).

In this work we study the *high-fugacity* regime of the 2×2 hard-square model. To gain intuition, it is instructive to consider the set of fully packed configurations; configurations in which the union of the tiles covers the whole of $\mathbb{R}^2$. In related models, such as the nearest-neighbor hard-core model on $\mathbb{Z}^d$ (see Section 1.3.2), one finds that there are a *finite number* of fully packed configurations, which are moreover periodic. In some such cases, a Peierls-type argument, or Pirogov–Sinai theory [65, 66] (see also [23, Chapter 7]), allow one to deduce that typical configurations sampled in the high-fugacity regime behave like “small perturbations” of one of the fully packed configurations in the sense that they coincide with this configuration at most places. By contrast, one easily checks that the 2×2 hard-square model admits a *continuum* (i.e., $2^{\aleph_0}$) of fully packed configurations, obtained in the following way (see Figure 1.1): start with the “square lattice configuration” $\sigma_0 \in \Omega$, which is defined by $\sigma_0(x, y) = 1$ if and only if both x and y are odd. From σ_0 , one can create a continuum of other fully packed configurations by “sliding” columns of tiles down by one lattice site; more precisely, for each $t : 2\mathbb{Z} + 1 \rightarrow \{0, 1\}$ one obtains a fully packed configuration σ^t by setting $\sigma^t(x, y) = 1$ if and only if x is odd

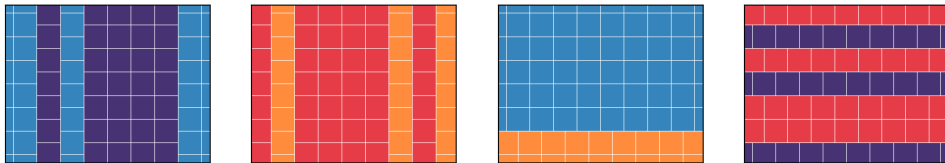


Fig. 1.1. Representatives of the four kinds of fully packed configurations of the 2×2 hard-square model. The colors of the tiles correspond to the parities of their x - and y -coordinates (see Figure 2.1 for the precise correspondence).

and $y \equiv t(x) \bmod 2$. In a similar manner, one can create a continuum of fully packed configurations by starting from σ_0 and sliding rows of tiles to the right by one lattice site. Additional fully packed configurations may be generated from the ones described so far by translating a fully packed configuration by one lattice site up, or by one lattice site to the right.

The existence of this continuum of fully packed configurations, sometimes termed the “sliding phenomenon”, forms an obstacle to Pirogov–Sinai theory, and is the reason that no rigorous results on the high-fugacity regime have appeared so far. Fröhlich–Israel–Lieb–Simon [25, Model 4.4] discussed the 2×2 hard-square model and wrote that the “conventional wisdom” is that it has a *unique* Gibbs state in the high-fugacity regime. Recently, Mazel–Stuhl–Suhov [51] conjectured that the 2×2 hard-square model has a unique Gibbs measure in the high-fugacity regime, as part of a more general conjecture on models with the sliding phenomenon. In the physics literature, while early results [5, 57] were inconclusive regarding the existence of a phase transition, modern studies (see [22, 54, 69] and references therein) indicate a *columnar phase* appearing in the high-fugacity regime of the 2×2 hard-square model. Our work clarifies the situation by proving that the model admits *multiple* (periodic) Gibbs measures at high fugacity, in agreement with the modern physics literature and in contrast to the expectations expressed in [25, 51].

Configurations sampled from these Gibbs measures display a columnar (or row) ordering of tiles similar to the fully packed configurations described above; however, the sampled configurations are not simply perturbations of some fully packed configuration. Instead, they should be thought of as a perturbation of a random tiling formed from a “union of one-dimensional systems” in the sense that we now describe (see Figure 1.2).

Denote by $\mu_{(\text{ver},0)}^{\text{UID}}$ the 2×2 hard-square model at fugacity λ conditioned so that the x -coordinates of the corners of all tiles are even. By this we mean that in the columns with odd x -coordinate we see a sample from μ^{1D} , the unique Gibbs measure of the nearest-neighbor hard-core model on $\mathbb{Z}$ at fugacity λ , and these samples are independent between the different columns (see Figure 1.2, middle). At high fugacity, samples from μ^{1D} give a very dense tiling, with long intervals of fully packed tiles separated by a single skip

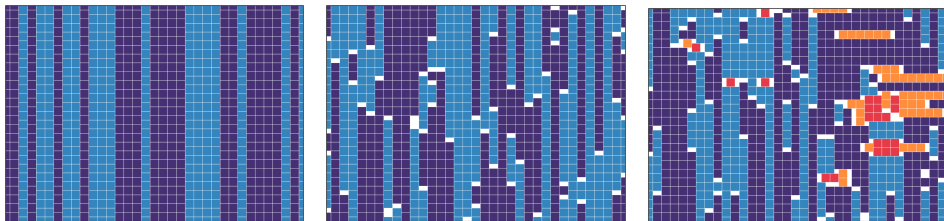


Fig. 1.2. Left: A fully packed configuration, arranged in columns. Middle: A sample from a union of independent “one-dimensional columnar systems” at high fugacity (denoted by $\mu_{(\text{ver},0)}^{\text{UID}}$ in the text). Right: A sample from the high-fugacity regime of the 2×2 hard-square model. We prove the existence of a phase for the high-fugacity 2×2 hard-square model with properties resembling a “small perturbation” of $\mu_{(\text{ver},0)}^{\text{UID}}$.

(or, more rarely, multiple skips). The typical length of the fully packed intervals is of order $\sqrt{\lambda}$, which is also the natural length scale at which μ^{1D} decorrelates. Our first theorem states the existence of a Gibbs measure $\mu_{(\text{ver},0)}$ for the high-fugacity 2×2 hard-square model, which may heuristically be regarded as a “small perturbation” of $\mu_{(\text{ver},0)}^{\text{1D}}$. As discussed after the theorem, this implies the existence of multiple Gibbs measures at high fugacity.

Theorem 1.1. *There exists $0 < \lambda_0 < \infty$ such that the 2×2 hard-square model at each fugacity $\lambda > \lambda_0$ admits a Gibbs measure $\mu_{(\text{ver},0)}$ satisfying:*

- (1) (Invariance and extremality) $\mu_{(\text{ver},0)}$ is $(2\mathbb{Z} \times \mathbb{Z})$ -invariant and extremal. In particular, $\mu_{(\text{ver},0)}$ is $(2\mathbb{Z} \times \mathbb{Z})$ -ergodic.
- (2) (Columnar order) For all $(x, y) \in \mathbb{Z}^2$,

$$\mu_{(\text{ver},0)}(\sigma(x, y) = 1) = \begin{cases} \Theta(\lambda^{-1}), & x \equiv 0 \pmod{2}, \\ \frac{1}{2} - \Theta(\lambda^{-1/2}), & x \equiv 1 \pmod{2}, \end{cases} \quad (1.2)$$

where $a = \Theta(b)$ indicates that $ca \leq b \leq Ca$ for some universal $C, c > 0$.

- (3) (Decay of correlations) Let $f, g : \Omega \rightarrow [-1, 1]$. Suppose that $f(\sigma)$ depends only on the restriction of σ to a set $A \subset \mathbb{Z}^2$ and similarly $g(\sigma)$ depends only on the restriction of σ to $B \subset \mathbb{Z}^2$. Then

$$|\text{Cov}_{\mu_{(\text{ver},0)}}(f, g)| \leq \sum_{u \in A} \sup_{v \in B} \alpha(u, v) \quad (1.3)$$

where $\text{Cov}_{\mu}(f, g)$ is the covariance of $f(\sigma)$ and $g(\sigma)$ when σ is sampled from μ and

$$\alpha((x_1, y_1), (x_2, y_2)) := \min \left\{ C e^{-c|x_2 - x_1| - c \frac{|y_2 - y_1|}{\sqrt{\lambda}}}, \left(\frac{C \log \lambda}{\sqrt{\lambda}} \right)^{1_{x_1 \neq x_2}} \right\} \quad (1.4)$$

for some universal $C, c > 0$.

The theorem thus establishes that at high fugacity, the model admits a Gibbs measure which is invariant under translations in the vertical direction and satisfies that tiles preferentially occupy vertices with odd x -coordinate (columnar order). This implies the existence of at least three other Gibbs measures $\mu_{(\text{ver},1)}$, $\mu_{(\text{hor},0)}$ and $\mu_{(\text{hor},1)}$: The measure $\mu_{(\text{ver},1)}$ is created by translating the measure $\mu_{(\text{ver},0)}$ by one lattice site to the right. The measures $\mu_{(\text{hor},0)}$ and $\mu_{(\text{hor},1)}$ are formed from $\mu_{(\text{ver},0)}$ and $\mu_{(\text{ver},1)}$, respectively, by exchanging the x - and y - axes. The four measures are distinct (by (1.2)), with the $\mu_{(\text{ver},i)}$ being $(2\mathbb{Z} \times \mathbb{Z})$ -invariant and extremal, while the $\mu_{(\text{hor},i)}$ are $(\mathbb{Z} \times 2\mathbb{Z})$ -invariant and extremal.

Due to the columnar order, the measure $\mu_{(\text{ver},0)}$ breaks the lattice's 90° rotational symmetry and also its translational symmetry in the x -coordinate, while preserving the translational symmetry in the y -coordinate. The asymmetric role of the two lattice directions is further manifested in the decay of correlations property. It is shown that $\mu_{(\text{ver},0)}$ exhibits exponential decay of correlations in the x -direction with correlation length of

order 1 (at most), but exponential decay in the y -direction is relatively slow, with a correlation length of order $\sqrt{\lambda}$ (at most). The second term in the minimum in (1.4) additionally shows that already the correlations of events in adjacent (or nearby) columns are rather small when λ is large. See also Remark 9.4 and Section 10.1 for further discussion of correlation decay.

Once the existence of multiple extremal Gibbs measures has been established, one may wonder whether other extremal Gibbs measures exist, or in other words, whether the four measures exhaust all the possible ways in which the model may be ordered. Our second theorem establishes that there are no other *periodic* and extremal measures.

Theorem 1.2. *There exists $0 < \lambda_0 < \infty$ such that the following holds for the hard-square model at each fugacity $\lambda > \lambda_0$. There is a unique Gibbs measure $\mu_{(\text{ver},0)}$ satisfying the properties listed in Theorem 1.1. Moreover, every periodic Gibbs measure is a convex combination of $\mu_{(\text{ver},0)}$, $\mu_{(\text{ver},1)}$, $\mu_{(\text{hor},0)}$ and $\mu_{(\text{hor},1)}$.*

An overview of our proof appears in Section 1.4 below.

Our proof uses the *chessboard estimate*, a consequence of the reflection positivity enjoyed by the 2×2 hard-square model. Our work introduces an apparently novel extension of the chessboard estimate which may be of interest also for other models. In its standard form, the chessboard estimate applies to *finite-volume* Gibbs measures on a discrete torus. In Proposition 3.8 we show that the estimate may be used directly in *infinite volume*, applying to all (infinite-volume) periodic Gibbs measures of the model. This extension is especially useful in the proofs in Part II of the paper. We formulate the extension solely for the 2×2 hard-square model but the proof provided is applicable to other models.

1.3. Background and related work

This subsection discusses related literature on hard-core models and liquid crystals.

1.3.1. General hard-core models. A *hard-core configuration* on a graph $G = (V(G), E(G))$ (called an independent set in combinatorics) is a function $\sigma : V(G) \rightarrow \{0, 1\}$ such that if $\{u, v\} \in E(G)$ then $\sigma(u)\sigma(v) = 0$. The *hard-core model* on a finite G , at fugacity λ , is the probability measure μ on hard-core configurations defined by $\mu(\sigma) \propto \lambda^{\sum_{v \in V(G)} \sigma(v)}$. The hard-core model further arises as a zero-temperature limit of the anti-ferromagnetic Ising model with a carefully chosen external field (see, e.g., [62, around (62)] for the construction on $\mathbb{Z}^d$). The definition extends, via the usual prescriptions, to hard-core measures with given boundary conditions and Gibbs measures on infinite graphs. A basic challenge in statistical physics is to characterize the Gibbs measures of the hard-core model on an infinite graph at different values of fugacity. As mentioned, general results [20, 78] lead to the unicity of Gibbs measures at sufficiently low fugacities (on bounded-degree graphs). Thus, the main interest is in other, intermediate and high, regimes of fugacity.

1.3.2. The nearest-neighbor hard-core model on $\mathbb{Z}^d$. A prototypical example is the hard-core model on the lattice $\mathbb{Z}^d$ (with nearest-neighbor edges). The model admits exactly two fully packed configurations σ^{even} and σ^{odd} (configurations where $\max\{\sigma(u), \sigma(v)\} = 1$ whenever u is adjacent to v), with $\sigma^{\text{even}}(v) = 1$ on the even sublattice (even sum of coordinates) and $\sigma^{\text{odd}}(v) = 1$ on the odd sublattice. A seminal result of Dobrushin [21] states that at high fugacity the model has two extremal Gibbs measures $\mu^{\text{even}}, \mu^{\text{odd}}$, invariant under parity-preserving shifts, such that samples of μ^{even} (resp. μ^{odd}) coincide with σ^{even} (resp. σ^{odd}) at most vertices (the density of coinciding vertices goes to 1 as $\lambda \rightarrow \infty$). In fact, one can define the measures $\mu^{\text{even}}, \mu^{\text{odd}}$ at all fugacities, as suitable infinite-volume limits, and establish that the model admits multiple Gibbs measures if and only if $\mu^{\text{even}} \neq \mu^{\text{odd}}$. It is believed that there is a unique transition point $\lambda_c(d)$ from a unique to multiple Gibbs measures on $\mathbb{Z}^d$, and that for $\lambda > \lambda_c(d)$ all periodic Gibbs measures are convex combinations of $\mu^{\text{even}}, \mu^{\text{odd}}$; however, these assertions have not been confirmed yet. There are examples of graphs on which there are multiple transitions between uniqueness and multiplicity of Gibbs measures as λ increases [13]. The fact that $\lambda_c(d) \rightarrow 0$ as $d \rightarrow \infty$ (for a suitable definition of $\lambda_c(d)$) was proved by Galvin–Kahn [27], with the rate of decay improved in [60]. It is believed that $\lambda_c(d)$ behaves like $d^{-1+o(1)}$ as $d \rightarrow \infty$ but this also remains unproved.

1.3.3. The monomer-dimer model. A phenomenon of a different nature was discovered by Heilmann–Lieb [33, 34], in their study of the *monomer-dimer model*. Monomer-dimer configurations on a graph H are subsets of edges with the property that no two edges share a common endpoint (called matchings in combinatorics). The probability assigned to a configuration by the monomer-dimer model on a finite graph is proportional to λ raised to the number of edges in the configuration. The monomer-dimer model is thus equivalent to the hard-core model on the line graph of H (the graph with vertex set $E(H)$ and with distinct $e, e' \in E(H)$ adjacent if $e \cap e' \neq \emptyset$). Heilmann and Lieb made the surprising discovery that the monomer-dimer model has a unique Gibbs measure on all bounded-degree graphs H at all finite fugacities! An alternative proof was found by van den Berg [79]. Thus, this model never exhibits a phase transition from unique to multiple Gibbs measures (except in the sense that the limiting model as $\lambda \rightarrow \infty$, termed the *dimer model*, may have multiple Gibbs measures).

1.3.4. Euclidean balls with centers on $\mathbb{Z}^d$ and other lattice packings. One may also study the hard-core model on a modified version of $\mathbb{Z}^d$ having a different adjacency structure. The version where each $u \in \mathbb{Z}^d$ is adjacent to all $v \in \mathbb{Z}^d$ with $0 < \|u - v\|_2 < D$ (without loss of generality we assume that D can be realized as the distance between some pair of points in $\mathbb{Z}^d$) has received special attention, motivated by a continuum version of the hard-core model. Various behaviors have been predicted for different regimes of D and the fugacity. The above-mentioned nearest-neighbor hard-core model arises by setting $D = \sqrt{2}$, while the 2×2 hard-square model studied here is obtained in two dimensions ($d = 2$) when $D = 2$. Recently, a comprehensive study of the high-fugacity behavior for the two- and three-dimensional models with various values of D was undertaken

by Mazel–Stuhl–Suhov [51, 53] (also on the triangular and hexagonal lattice [50]; see also [52]).

In two dimensions, the work [51] characterizes the periodic hard-core configurations of maximal density for each D . Moreover, Pirogov–Sinai theory is used to show that, with few exceptions, the extremal Gibbs measures arise as perturbations of (a subset of) these configurations in a suitable sense. The exceptions are a finite number of values of D (the list of which was confirmed independently in [51] and in [42]) for which there are *infinitely many* periodic hard-core configurations of maximal density; these always come about as a result of a “sliding instability” in the configurations (similarly to the sliding phenomenon described above for the 2×2 hard-square model). Pirogov–Sinai theory does not apply in these exceptional cases and their high-fugacity behavior remained unclear. It was conjectured in [51] that in these cases there is a unique Gibbs measure at high fugacities. As mentioned above, our study clarifies the case of the 2×2 hard-square model, refuting the conjecture in this case. Further discussion is in Section 10.4.

In three dimensions, the work [53] studies an infinite family of values of D , discovering a rich set of possibilities for the corresponding periodic configurations of maximal density and drawing conclusions on the periodic and extremal Gibbs measures of the model. The case $D = 2$ corresponds to the packing of $2 \times 2 \times 2$ cubes with centers on $\mathbb{Z}^3$ and exhibits the sliding phenomenon. It is conjectured in [53] that sliding leads to the unicity of periodic and extremal Gibbs measures at high fugacities. In Section 10.2 we discuss our predictions for the hard-cube model on $\mathbb{Z}^d$.

Jauslin–Lebowitz [38, 39] studied random packings of a (general) tile in $\mathbb{R}^d$ and its lattice translates (i.e., the hard-core model on a discrete periodic graph in $\mathbb{R}^d$). Their work also excludes sliding cases (including the 2×2 hard-square model), by requiring that the specified tile fully tiles $\mathbb{R}^d$ in a finite number of periodic and isometric ways, and further limiting the ways in which defects may occur in these periodic tilings. Using Pirogov–Sinai theory, they prove the existence of high-fugacity extremal and periodic Gibbs states corresponding to crystalline order according to the possible periodic tilings. They further establish that the pressure and correlation functions have expansions in powers of the inverse fugacity with a positive radius of convergence.

1.3.5. Liquid crystals. The term “columnar order” that we use originates in the study of liquid crystals [15]. There, one studies a material composed of molecules in three-dimensional space and classifies its state according to the symmetries of its structure. In a *gas* or *liquid* state, the molecules are disordered in the sense that their distribution retains both the (continuous) rotational and translational symmetries of $\mathbb{R}^3$. On the other end of the spectrum are *crystal* states, in which the symmetry group is *discrete*. Liquid crystals are “intermediate” states of matter, in which the symmetries of $\mathbb{R}^3$ are partially broken. Three of the main categories of such states are the *nematic*, in which rotational symmetry is broken while the full $\mathbb{R}^3$ translational invariance is preserved, *smectic*, in which rotational symmetry is broken and also translational symmetry is broken along *one axis* (an $\mathbb{R}^2$ translational symmetry is retained), and *columnar*, in which the rotational symmetry is broken and also translational symmetry is broken along *two axes* (an $\mathbb{R}$ trans-

lational symmetry is retained). A seminal work in the physics of liquid crystals is that of Onsager [59], who considered long, thin, rod-like molecules in $\mathbb{R}^3$ with a pure hard-core interaction (i.e., molecules are only constrained not to overlap) and predicted a transition from a disordered to a nematic phase as the density of the molecules increases.

In two dimensions, we use the term *nematic* to refer to a model in which rotational symmetry is broken while translational symmetry is retained, and the term *columnar* to refer to a model in which rotational symmetry is broken while the translational symmetry is broken only along a single axis. While the terminology of liquid crystal phases was originally introduced in the continuum, it is also used for lattice models with a similar meaning, classifying models in terms of which of the lattice symmetries are broken.

We are not aware of previous mathematically rigorous proofs of columnar order in a hard-core model. Nematic order has been given rigorous proof in several models, including the following:

- Heilmann–Lieb [35] established rotational symmetry breaking for several lattice models using reflection positivity methods and conjectured the absence of translational order.

Their models include a dimer model with attractive forces on the square lattice [35, Model I] for which the conjecture was recently established by Jauslin–Lieb [40] using Pirogov–Sinai methods, completing the proof of nematic order (Jauslin–Lieb add that there is little doubt that similar proofs could be devised for the other models in [35]). Alberici [3] studied the same model in the case of non-equal horizontal and vertical dimer activities and proved the absence of translational order using a cluster expansion.

- Ioffe–Velenik–Zahradník [37] established a nematic phase for a system of horizontal and vertical rods on a square lattice having unit width and varying lengths, with hard-core interactions and specific length-dependent activities; in the case where all lengths are allowed, they proved their result via an exact mapping to an Ising model.
- Disertori–Giuliani [17] considered rods of unit width and fixed length k on the square lattice with pure hard-core interaction and proved that for large k , the system has a nematic phase in an *intermediate* density regime via coarse-graining to an effective contour model and Pirogov–Sinai methods (see also Section 10.2).
- Disertori–Giuliani–Jauslin [18] considered anisotropic plates with a finite number of allowed orientations in the *continuum* $\mathbb{R}^3$ with pure hard-core interactions and established a nematic phase for an intermediate density regime.

We also mention the work of Bricmont–Kuroda–Lebowitz [12, Concluding Remarks] where, following Ruelle [70], rotational symmetry breaking is proved in a system of zero-width rods in $\mathbb{R}^2$ with finitely many orientations allowed. Nematic order has further been conjectured in several models, including the following: Abraham–Heilmann [1] introduced a three-dimensional model which extends the two-dimensional model of Heilmann–Lieb [35, Model I], proved rotational symmetry breaking and conjectured the absence of translational order. Angelescu–Zagrebnev [4] and Zagrebnev [87] studied

molecules on a lattice with an internal (“spin”) degree of freedom with *continuous* rotational symmetry. They showed, using a combination of the infra-red bound and chess-board estimates, that rotational symmetry is broken at low temperature, and conjectured that a nematic phase appears. We also mention the works [7, 8, 58] in which models with a continuum of ground states are analyzed, with the conclusion being that the ground-state degeneracy is (partially) lifted at low positive temperatures due to the different excitations available to each degenerate configuration.

1.4. Proof overview

1.4.1. Existence of multiple Gibbs measures. In Part I, the model is shown to admit several Gibbs measures via an involved Peierls-type argument. The keys to the argument are a “coarse-grained identification” of vertically ordered and horizontally ordered regions, and a proof that the resulting interfaces are rare. We proceed to describe these points, starting with some key concepts that we introduce.

Sticks: Tiles are classified into four types according to the parities of the coordinates of their bottom left corner. A stick edge of a configuration is an edge of $\mathbb{Z}^2$ which lies on the boundary of two tiles of different type. A stick is a maximal path of stick edges. It is easily seen that sticks are either horizontal or vertical segments and that sticks of different orientation cannot meet. The idea is that long vertical (horizontal) sticks should be abundant in regions of vertical (horizontal) columnar order, while the interfaces between differently ordered regions are not crossed by long sticks (see Figure 1.3).

Properly-divided rectangles: To classify regions into vertically and horizontally ordered we consider the crossing of rectangles by sticks. We call a rectangle R *divided* if there is a stick crossing R in the horizontal or vertical direction; importantly, a rectangle cannot be divided *both* horizontally and vertically. For technical reasons, we further define R to be properly divided if it is divided by a stick which also divides R^- , a rectangle concentric to R having slightly smaller dimensions; specifically, we fix a large integer N (independent of λ), suppose the dimensions of R are integers divisible by N and let the dimensions of R^- be $\frac{N-2}{N} \text{Width}(R) \times \frac{N-2}{N} \text{Height}(R)$ (see Figure 4.1). Rectangles are thus classified into properly divided vertically, properly divided horizontally or not properly divided. The concept of properly divided rectangles is only used with squares when proving the existence of multiple Gibbs measures. Rectangular R are used in the proofs of our other results.

Identification of interfaces: Let $b(\lambda)$ be an (integer) length scale, later chosen to satisfy (1.6) below. We consider the square grid $b(\lambda)\mathbb{Z} \times b(\lambda)\mathbb{Z}$, and associate with each of its points a square of side length $Nb(\lambda)$ with its bottom-left corner at that point. These grid squares are thus partially overlapping, with the amount of overlap chosen to ensure the following property: two squares associated to neighboring positions on the grid cannot be properly divided in distinct directions. Thus, if grid squares are properly divided vertically in one region and horizontally in another region then necessarily these regions

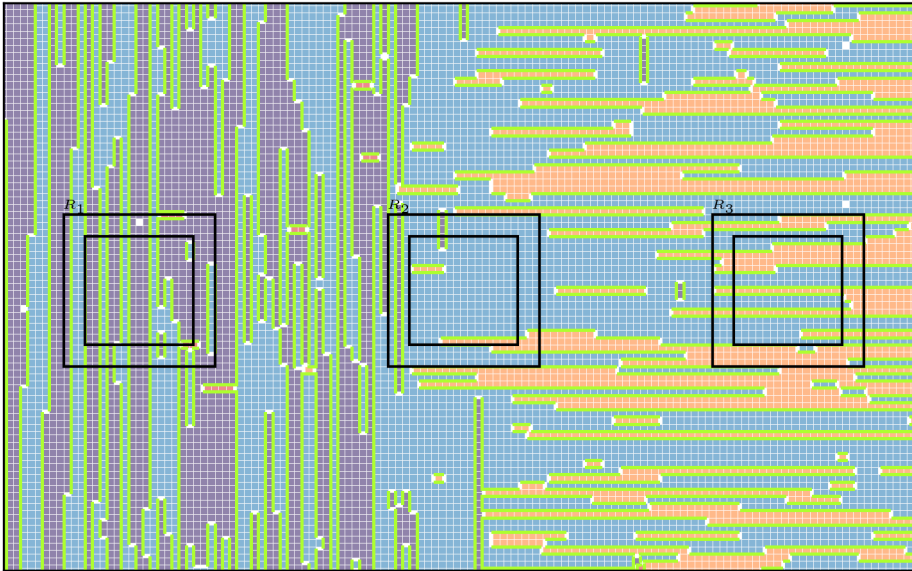


Fig. 1.3. Sticks (green lines) in a configuration. On the left there is an abundance of vertical sticks while on the right there is an abundance of horizontal sticks. The interface region is not crossed by long sticks (of either orientation), a feature which we rely upon in order to prove that interface regions are rare. The rectangles R_1 , R_2 , R_3 are drawn with their concentric R^- rectangles (with $N = 7$). The rectangles R_1 and R_3 are properly divided by vertical and horizontal sticks, respectively, while R_2 is not properly divided.

are separated by a “contour” of (points associated to) grid squares that are *not* properly divided.

Multiple Gibbs measures from a Peierls-type argument: Let μ be a periodic Gibbs measure (at least one such measure exists by compactness arguments). Our main technical lemma, Lemma 4.1, implies that for suitably chosen $b(\lambda)$ and N , in samples from μ , long contours of grid squares that are not properly divided are highly unlikely to occur at any given position (see next paragraph). A union bound over contours then shows that, μ -almost surely, there is either an infinite connected component of grid squares that are properly divided horizontally or an infinite connected component of grid squares that are properly divided vertically, but not both. Due to the lattice’s 90° rotational symmetry, this implies the existence of at least two periodic Gibbs measures.

The basic estimate: Lemma 4.1 implies, for a large constant N , that rectangles R whose width and height are at most $c_0\lambda^{1/2}$ satisfy

$$\mu(R \text{ is not properly divided}) \leq e^{-c_0 \text{Area}(R)\lambda^{-1/2}} \quad (1.5)$$

where μ stands for any periodic Gibbs measure and $c_0 > 0$ is a small universal constant. The lemma moreover shows that this estimate is multiplicative in the sense that the probability that n disjoint rectangles of the same dimensions (technically required to have their

corners on a shift of the grid $\text{Width}(R)\mathbb{Z} \times \text{Height}(R)\mathbb{Z}$) are all non-properly-divided is at most the right-hand side of (1.5) raised to the power n . To use the estimate (1.5) in our Peierls-type argument we choose $b(\lambda)$ to be a *mesoscopic* length scale, satisfying

$$C\lambda^{1/4} < Nb(\lambda) < c_0\lambda^{1/2} \quad (1.6)$$

for a large universal constant $C > 0$. The fugacity λ is required to be large in order for this interval to be non-empty.

Remark 1.3. We offer some motivation for the form of the bound (1.5). Recall that typical ordered vertical regions heuristically behave like small perturbations of a “union of 1D vertical systems”, i.e., samples from the measure $\mu_{(\text{ver},0)}^{\text{UID}}$ (Figure 1.2, middle) or its shift to the right by one lattice site. Suppose we wanted to prove (1.5) and its multiplicative version for $\mu_{(\text{ver},0)}^{\text{UID}}$. Let us show that there is a lower bound on the probability which matches (the multiplicative version of) (1.5) up to the constant in the exponent. Indeed, let $R_1, \dots, R_n$ be rectangles of equal dimensions positioned exactly one below the other (the top edge of R_i is the bottom edge of R_{i-1}). One way in which none of the R_i will be properly divided is if the configuration restricted to $\bigcup R_i$ is the “square lattice configuration” (defined by $\sigma(x, y) = \mathbf{1}_{x,y \equiv 1 \pmod{2}}$). What is the probability of this event under the measure $\mu_{(\text{ver},0)}^{\text{UID}}$? Say that a face of $\mathbb{Z}^2$ is a *vacancy* if it is uncovered by the tiles of the configuration. The event occurs when there are no vacancies in $\bigcup R_i$ and all columns in the 1D systems “enter $\bigcup R_i$ with the same phase”. Since vacancy pairs (as vacancies necessarily come in pairs) are distributed roughly as a Poisson process with intensity proportional to $\lambda^{-1/2}$ we conclude that the $\mu_{(\text{ver},0)}^{\text{UID}}$ probability of the event is roughly $2^{-\frac{1}{2} \text{Width}(R_1)} e^{-c'n \text{Area}(R_1)\lambda^{-1/2}}$ for some $c' > 0$ (the first factor accounts for the phases). In comparison, the upper bound resulting from (1.5) is $e^{-c_0 n \text{Area}(R_1)\lambda^{-1/2}}$, which has the same form as $n \rightarrow \infty$.

Applying the chessboard estimate: The first step in the proof of the bound (1.5) is to apply the chessboard estimate. This step is described with the following notation: Fix a rectangle $\Lambda \subset \mathbb{Z}^2$ which we view as the domain. For a configuration $\sigma \in \Omega$, define its weight in Λ to be

$$w_{\Lambda,\lambda}(\sigma) := \lambda^{-\frac{1}{4} \#\{\text{vacancies of } \sigma \text{ in } \Lambda\}}. \quad (1.7)$$

For a rectangle S denote $\mathcal{L}^S := \text{Width}(S)\mathbb{Z} \times \text{Height}(S)\mathbb{Z}$. Given an event E we write $Z_{\Lambda,\lambda}^{\text{per}}(E)$ for the sum of $w_{\Lambda,\lambda}(\sigma)$ over all $\sigma \in E$ with σ periodic with respect to Λ (i.e., invariant under translations by elements of $\mathcal{L}^\Lambda$). We also set $Z_{\Lambda,\lambda}^{\text{per}} := Z_{\Lambda,\lambda}^{\text{per}}(\Omega)$. Informally, the weight (1.7) is proportional to the analog of (1.1) in periodic boundary conditions (see also around (2.3)).

The proof of (1.5) and its multiplicative extension is reduced, via the chessboard estimate (or rather, its infinite-volume extension in Section 3.2), to showing that

$$\frac{Z_{\Lambda,\lambda}^{\text{per}}(E_{R,R^-})}{Z_{\Lambda,\lambda}^{\text{per}}} \leq \left(e^{-c_0 \text{Area}(R)\lambda^{-1/2}} \right)^{\frac{\text{Area}(\Lambda)}{\text{Area}(R)}} \quad (1.8)$$

where E_{R,R^-} is the “disseminated version” of the event that R is not properly divided, i.e., the event that no translates of R by elements of $\mathcal{L}^R$ are properly divided. The chessboard estimate requires that the dimensions of Λ are even integer multiples of the corresponding dimensions of R .

The bound (1.8) is proved via an upper bound on the numerator and a lower bound on the denominator. We proceed to describe these two bounds.

Lower bound via 1D systems: The lower bound on $Z_{\Lambda,\lambda}^{\text{per}}$ is easy. It is obtained by restricting to configurations corresponding to the “union of 1D systems” (i.e., all tiles have the same horizontal parity), for which an explicit solution of a linear recurrence relation (Proposition 4.2) yields, for some $c_1 > 0$,

$$Z_{\Lambda,\lambda}^{\text{per}} \geq e^{c_1 \lambda^{-1/2} \text{Area}(\Lambda)}. \quad (1.9)$$

Upper bound for the disseminated event: The difficult part lies in obtaining an upper bound on $Z_{\Lambda,\lambda}^{\text{per}}(E_{R,R^-})$. By several steps of simplification (see next paragraph), we pass to bounding instead the value of $Z_{\Lambda,\lambda}^1(E_M)$ which is defined as follows. The event E_M is that all sticks are of length at most M , and for an event E , the value of $Z_{\Lambda,\lambda}^1(E)$ is the sum of weights of configurations in $E \cap \Omega_\Lambda^\rho$ (as defined in Section 1.1) for the fully packed boundary condition $\rho(x, y) = \mathbf{1}_{x,y \equiv 1 \pmod{2}}$. The value of M is chosen to be $c_2 \lambda^{1/2}$.

Simplification steps: First, it is shown that the effect of boundary conditions is insignificant for the purposes of (1.8) (asymptotically as $\Lambda \uparrow \mathbb{Z}^2$), thus one may estimate $Z_{\Lambda,\lambda}^1(E_{R,R^-})$ instead of $Z_{\Lambda,\lambda}^{\text{per}}(E_{R,R^-})$. Second, we introduce an event $E_{M,A}$ that requires the sticks whose extension to a line crosses a translate of R^- by an element of $\mathcal{L}^R$ to have length at most M (the notation $E_{M,A}$ is used for consistency with the main text; there A is a set parameterizing the event while here we define the event as a special case resulting from a specific choice of A). One checks that

$$E_{R,R^-} \subset E_{M,A} \quad \text{when} \quad M \geq \max \{2 \text{Width}(R), 2 \text{Height}(R)\} \quad (1.10)$$

(this holds for the R of (1.5) by choosing $c_0 < c_2/2$). Third and lastly, it remains to bound $Z_{\Lambda,\lambda}^1(E_{M,A})$ from above in terms of $Z_{\Lambda,\lambda}^1(E_M)$. The proof of this bound is quite technical, but the essential idea is simply that the additional constraints imposed by E_M on top of those of $E_{M,A}$ concern only a small fraction (approximately $4/N$) of the area of Λ : namely, they concern the complement of the union of the translates of R^- by elements of $\mathcal{L}^R$.

Eventually, for the chosen value of M , the simplification steps result in the following bound for large Λ :

$$Z_{\Lambda,\lambda}^{\text{per}}(E_{R,R^-}) \leq e^{c_3 \lambda^{-1/2} \text{Area}(\Lambda)} Z_{\Lambda,\lambda}^1(E_M) \quad (1.11)$$

where for a fixed c_2 , we have $c_3 \rightarrow 0$ as $N \rightarrow \infty$.

Configuration without long sticks: We are left with the more essential task of bounding $Z_{\Lambda, \lambda}^1(E_M)$. This is achieved by our key Lemma 4.6, which shows that for some large $C_4 > 0$, if $M < \lambda^{1/2}/C_4$ then

$$Z_{\Lambda, \lambda}^1(E_M) \leq \left(1 + \frac{C_4 M}{\lambda}\right)^{\text{Area}(\Lambda)}. \quad (1.12)$$

The proof is via direct combinatorial counting arguments. A first observation is that sticks must end at vacant faces. Therefore, configurations contain “connected components” composed of sticks and vacancies together. We consider all possibilities for such connected components, up to translations. Then, the proof of (1.12) reduces to suitably bounding the sum $\sum_H \lambda^{-\frac{1}{4}v_H}$ where H ranges over all those possibilities in which all sticks have length at most M and v_H denotes the number of vacancies in H . To this end, we define k_H , a quantity such that $k_H - 2$ is the number of “degrees of freedom” one has for extending and contracting the sticks of H (this is generally less than the number of sticks in H since following the sticks in a cycle of H must lead back to the starting point). Geometric considerations lead to the bound $v_H \geq \max\{2(k_H - 1), 4\}$. This supplies necessary control for the requisite bound on the above sum.

Conclusion of the basic estimate: As mentioned above, the basic estimate (1.5) follows from (1.8). The latter bound, on the probability of $E_{R, R-}$, then follows by combining (1.9), (1.11) and (1.12), under the assumption that

$$c_1 - c_3 - C_4 c_2 > c_0 > 0.$$

For (1.11), we also require that $c_0 < c_2/2$ so that the condition of (1.10) holds.

To satisfy this, the constants are chosen in the following order: First, c_1 and C_4 are fixed. Then we take $c_2 = c_1/(2C_4)$, and subsequently select N so that $c_3 < c_1/2$. This allows one to choose c_0 satisfying the two inequalities of the previous paragraph.

1.4.2. Fine properties of Gibbs measures and the characterization of periodic Gibbs measures. In Part II, we prove the fine properties and characterization results stated in Theorems 1.1 and 1.2. We briefly describe our proofs here.

Let $\mathcal{L}$ be a sufficiently sparse full-rank lattice $\mathcal{L}$. The previous arguments imply that for every $\mathcal{L}$ -ergodic Gibbs measure, either most long sticks are vertical (“ver” measure) or most long sticks are horizontal (“hor” measure). The next step is to refine this classification, proving that $\mathcal{L}$ -ergodic Gibbs measures come in exactly one of four “phases” (ver, 0), (ver, 1), (hor, 0), (hor, 1) according to the orientation of most long sticks and their parity (the parity of a vertical stick is the parity of its x -coordinate and the parity of a horizontal stick is the parity of its y -coordinate).

First, we use an inductive procedure on length scales, employing a Peierls-type argument driven by the basic estimate (1.5) in each step, showing that for “ver” measures, even very thin rectangles, of dimensions $C \times c\sqrt{\lambda}$ for a large universal $C > 0$ and small universal $c > 0$, are typically properly divided vertically. Via an additional Peierls-type

argument, this allows the further classification into the phases $(\text{ver}, 0)$ and $(\text{ver}, 1)$ by noting that if two long vertical sticks of opposite parities are near each other, then there is a long rectangle bounded between them which must contain an atypically large number of vacancies (tail bounds for the number of vacancies are readily obtained by the infinite-volume version of the chessboard estimate). Classification of “hor” measures is analogous.

The fact that there exist exactly four $\mathcal{L}$ -ergodic measures, which are furthermore extremal, along with quantitative decay of correlation estimates and precise invariance properties, is achieved by the method of disagreement percolation [78, 80] (see Theorem 8.2): Let μ, μ' be Gibbs measures and let σ, σ' be two *independent* samples from μ and μ' , respectively. If, almost surely, there is no infinite path where σ, σ' disagree then $\mu = \mu'$ and this common measure is extremal. Moreover, decay of correlation estimates are obtained by bounding the probability that disagreement paths connect distant vertices. This reduces our task to that of controlling long disagreement paths between independent samples from $\mathcal{L}$ -ergodic measures of the same phase. A key fact is that in a one-dimensional system, disagreement paths must terminate at the first vacancy (in any of the two configurations). This allows one to control the length of disagreement paths in regions where both configurations have their tiles arranged in columns (or rows) of the same parity. Other regions are rare, with suitable quantitative control, by the assumption that the measures have the same phase and by the thin rectangle crossing results of the previous paragraph.

Lastly, the proof of the estimate $\mu_{(\text{ver}, 0)}(\sigma(x, y) = 1) = O(\lambda^{-1})$ for even x (part of the columnar order property (1.2)) relies on the fact that between two vertical sticks of even parity, if there is some tile with an even x -coordinate of its center then there are at least four vacancies in its row between the two sticks. The other parts of (1.2) are relatively simple consequences of the previously obtained information (Section 9.3).

1.5. Reader's guide

The fundamental task of proving that the 2×2 hard-square model admits multiple Gibbs measures at high fugacity is achieved in Part I. Section 2 contains the basic definitions used throughout and some simple estimates on the effect of boundary conditions. Section 3 establishes reflection positivity of the 2×2 hard-square model, presents the chessboard estimate in finite volume and extends its applicability to infinite volume. Section 4 introduces the notion of sticks and proves that mesoscopic rectangles are typically divided by sticks. This fact is then used in Section 5 to derive the existence of multiple Gibbs measures via a Peierls-type argument.

Part II is devoted to proving the existence of a Gibbs measure with the properties stated in Theorem 1.1, and proving the characterization of periodic Gibbs measures stated in Theorem 1.2. Section 6 sets up a convenient framework for Peierls-type arguments. Classification of periodic-ergodic Gibbs measures into four phases is established in Section 7. The disagreement percolation method is introduced in Section 8, where it is used to prove extremality of the periodic-ergodic Gibbs measures and to bound their rate of cor-

relation decay, as well as to characterize the periodic Gibbs measures. Lastly, columnar order, as well as additional correlation decay estimates, are established in Section 9.

Part III is devoted to further discussion and open questions.

Part I

Existence of multiple Gibbs measures

In this part we prove the existence of multiple Gibbs measures for the 2×2 hard-square model on the square lattice $\mathbb{Z}^2$ at high fugacity (Corollary 5.3). The results of this part will further be instrumental in the second part, where we prove the more refined results stated in the introduction.

2. Preliminaries

2.1. Basic definitions

In this section we provide some of the basic definitions used throughout the paper. The definitions from the introduction are repeated, sometimes in a different (but equivalent) formulation.

Elementary objects: We use the convention $\mathbb{N} := \{1, 2, \dots\}$. We use the standard coordinate system in $\mathbb{R}^2$ where the x -axis points to the right and the y -axis points upwards.

In this paper, the term rectangle refers to an axis parallel rectangle with corners in integer coordinates, formally viewed as *closed* set. For $x, y \in \mathbb{Z}$ and $K, L \in \mathbb{N}$, define

$$R_{K \times L, (x, y)} := [x, x + K] \times [y, y + L] \subset \mathbb{R}^2,$$

that is, the rectangle with its bottom left corner at (x, y) and with side lengths K and L . For a rectangle $R = R_{K \times L, (x, y)}$ denote $\text{Width}(R) = K$, $\text{Height}(R) = L$, $\text{Perimeter}(R) = 2K + 2L$ and $\text{Area}(R) = KL$ (but note that R contains $(K + 1)(L + 1)$ points of $\mathbb{Z}^2$). When K, L, x, y are even we say that R is an *even rectangle*. We also use the shorthand $R_{K \times L} := R_{K \times L, (0, 0)}$. The boundary of R as a set in $\mathbb{R}^2$ is denoted ∂R .

For sets A, B, C and a function $f : A \rightarrow B$ we denote by $f|_C$ the restriction of f to $A \cap C$. We denote the indicator function of a set A by $\mathbf{1}_A$.

We consider two graphs having the vertex set $\mathbb{V} := \mathbb{Z}^2$ (we will use these two notations interchangeably). The *nearest-neighbors graph* is $(\mathbb{V}, \mathbb{E}_{\square})$ where

$$\mathbb{E}_{\square} = \{uv : u, v \in \mathbb{V}, \|v - u\|_1 = 1\}$$

is the set of edges connecting nearest neighbors in $\mathbb{Z}^2$. This is a planar graph, and we consider its edges to be embedded in $\mathbb{R}^2$ as line segments. Therefore its *faces* may be thought of as 1×1 squares. We define $\mathbb{F} := \{R_{1 \times 1, v} : v \in \mathbb{V}\}$. We also consider the *nearest-and-next-nearest-neighbors graph* $(\mathbb{V}, \mathbb{E}_{\boxtimes})$ where

$$\mathbb{E}_{\boxtimes} = \{uv : u, v \in \mathbb{V}, \|v - u\|_{\infty} = 1\}.$$

When discussing vertices in $\mathbb{V}$ we use terms such as “ $\square$ -adjacent” and “ $\boxtimes$ -connected component” with the obvious meaning.

With each vertex (x, y) is associated a tile that is a 2×2 square centered at that vertex: $T_{(x,y)} := R_{2 \times 2, (x-1, y-1)}$. We define the *parity* of a tile centered at (x, y) to be $(x - 1 \bmod 2, y - 1 \bmod 2)$, so there are four possible parities for a tile. The first component of the parity of a tile is termed its *horizontal parity* while the second component is termed its *vertical parity*.

Configuration spaces: We think of hard square configurations as sets of tiles whose interiors are pairwise disjoint. Formally, a configuration is represented by a function $\sigma : \mathbb{V} \rightarrow \{0, 1\}$, with the value 1 corresponding to centers of tiles, so that the space of configurations is

$$\Omega := \{\sigma \in \{0, 1\}^{\mathbb{V}} : \sigma(u) = \sigma(v) = 1 \Rightarrow \text{int}(T_v) \cap \text{int}(T_u) = \emptyset\}$$

where $\text{int}(\cdot)$ stands for the interior of a set. The space Ω is equipped with the standard Borel measurable structure (induced by the product topology).

Let Λ be a rectangle. We will work with several restricted sets of configurations, corresponding to different choices of boundary conditions for Λ :

- Define the set of Λ -periodic configurations as

$$\Omega_{\Lambda}^{\text{per}} := \{\sigma \in \Omega : \forall v \in \mathbb{V}, \sigma(v) = \sigma(v + (\text{Width}(\Lambda), 0)) = \sigma(v + (0, \text{Height}(\Lambda)))\}.$$

We emphasize that only the dimensions of Λ enter into the definition of $\Omega_{\Lambda}^{\text{per}}$. To stress this point, we will often use the notation $\Lambda = R_{K \times L}$ (omitting the corner position) when working with periodic boundary conditions.

- Let $\rho \in \Omega$, and define the set of configurations with ρ -boundary conditions outside Λ to be

$$\Omega_{\Lambda}^{\rho} := \{\sigma \in \Omega : \forall v \in \mathbb{V} \setminus \text{int}(\Lambda), \rho(v) = \sigma(v)\}. \quad (2.1)$$

- We give names to two special cases of this definition. We write Ω_{Λ}^0 for the case ρ is identically 0 and call Ω_{Λ}^0 the set of configurations with *free boundary conditions* outside Λ . When Λ is an *even* rectangle, we also write Ω_{Λ}^1 for the case where $\rho(x, y) = \mathbf{1}_{x, y=1 \bmod 2}$ and call Ω_{Λ}^1 the set of configurations with *fully packed boundary conditions* outside Λ . We point out that for an even rectangle Λ , free and fully packed boundary conditions, realized by Ω_{Λ}^0 and Ω_{Λ}^1 , induce the same set of feasible configurations in Λ (and thus the same partition function and measure, according to the definitions below), but the distinction between them will be convenient in Section 4.

For a set Λ and a measurable function $f : \Omega \rightarrow \mathbb{R}$, say that f is Λ -local if

$$\sigma|_{\Lambda} = \sigma'|_{\Lambda} \Rightarrow f(\sigma) = f(\sigma'), \quad \forall \sigma, \sigma' \in \Omega. \quad (2.2)$$

An event E is called Λ -local if $\mathbf{1}_E$ is Λ -local.

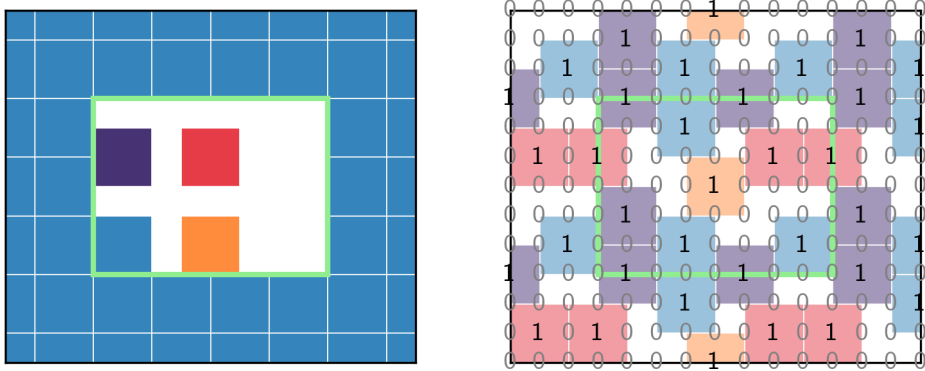


Fig. 2.1. Two configurations in Ω . The boundary of the even rectangle $\Lambda = \mathbb{R}_{8 \times 6, (0,0)}$ is shown in green. Left: A configuration in $\sigma \in \Omega_{\Lambda}^1$. The tiles inside Λ are $T_{(1,1)}, T_{(1,4)}, T_{(4,1)}, T_{(4,4)}$, of parities $(0,0), (0,1), (1,0), (1,1)$ respectively, and colored blue, orange, deep blue, red, respectively. Throughout the paper, we color tiles according to their parities in this way. Right: A configuration $\sigma \in \Omega_{\Lambda}^{\text{per}}$. The values of σ appear centered on points of $\mathbb{Z}^2$, and the corresponding tiles are in the background.

Measures: Given a configuration $\sigma \in \Omega$ and a face $f \in \mathbb{F}$, we say that f is a *vacancy* (or that f is vacant) if it is not contained in a tile of σ . Otherwise we say that it is *occupied*.

Let $\lambda > 0$ denote *fugacity*. Informally, in the hard-square model, the probability of a configuration is proportional to λ^n where n is the number of tiles. Equivalently, it is proportional to $\lambda^{-v/4}$ where v is the number of vacancies. Formally, for a rectangle Λ , we define the weight of a configuration according to the number of vacant faces as follows:

$$w_{\Lambda, \lambda}(\sigma) := \lambda^{-\frac{1}{4} \#\{f \in \mathbb{F} : f \subset \Lambda \text{ and } f \text{ is vacant in } \sigma\}}. \quad (2.3)$$

For $*$ denoting either $^{\text{per}}$ or $^{\rho}$, define the *hard-square Gibbs measure* $\mu_{\Lambda, \lambda}^*$, at fugacity λ in the finite volume Λ with boundary conditions $*$ as the measure on Ω_{Λ}^* assigning probability

$$\mu_{\Lambda, \lambda}^*(\sigma) = \frac{w_{\Lambda, \lambda}(\sigma)}{Z_{\Lambda, \lambda}^*}$$

to each configuration σ , where $Z_{\Lambda, \lambda}^* := \sum_{\sigma \in \Omega_{\Lambda}^*} w_{\Lambda, \lambda}(\sigma)$ is the partition function. The assumption that Λ is a rectangle ensures that this coincides with the definitions (1.1) and (1.7) given in the introduction. It is convenient to further define the weight of an event $E \subset \Omega$ under the boundary conditions $*$ to be

$$Z_{\Lambda, \lambda}^*(E) := \sum_{\sigma \in E \cap \Omega_{\Lambda}^*} w_{\Lambda, \lambda}(\sigma), \quad (2.4)$$

so that $Z_{\Lambda, \lambda}^* = Z_{\Lambda, \lambda}^*(\Omega)$ and $\mu_{\Lambda, \lambda}^*(E) = Z_{\Lambda, \lambda}^*(E)/Z_{\Lambda, \lambda}^*$. Throughout the text, λ will denote the fugacity and we will often omit it from the notation.

A measure μ on Ω with the natural sigma-algebra is said to be an *infinite-volume Gibbs measure* if for every rectangle Λ and measurable function f , almost surely for σ

sampled from μ we have

$$\mu(f \mid \sigma \mid_{\mathbb{V} \setminus \text{int}(\Lambda)}) = \mu_{\lambda, \Lambda}^{\sigma}(f). \quad (2.5)$$

Note that while we only use rectangular domains in this definition, the resulting infinite-volume Gibbs measures are the same as the ones defined after (1.1) in the introduction.

Transformations: Let $\eta : \mathbb{V} \rightarrow \mathbb{V}$ (usually η will be the restriction of an isometry of $\mathbb{R}^2$). For $\sigma \in \Omega$, define $\eta\sigma := \sigma \circ \eta$. For a function $f : \Omega \rightarrow \mathbb{R}$, define ηf by

$$\eta f(\sigma) = f(\eta\sigma) = f(\sigma \circ \eta). \quad (2.6)$$

Analogously, for an event $E \subset \Omega$ define $\eta E := \{\sigma \in \Omega : \eta\sigma \in E\}$. For a measure μ on Ω , define $\eta\mu$ by $\eta\mu(E) = \mu(\eta E)$.

For $v \in \mathbb{V}$, define $\eta_v : \mathbb{V} \rightarrow \mathbb{V}$ by $\eta_v(u) = u + v$.

Constant notation: Many of our claims introduce constants in phrasings such as “There is $c > 0$ such that ...”. For clarity, when referring to such constants in later parts of the argument we add the number of the claim as a subscript (e.g. $c_{4.4}$).

2.2. Comparison of boundary conditions

Fix a fugacity parameter $\lambda > 0$. The goal of this section is to bound the effects of boundary conditions on the expectation value of observables. The approach is standard, though some care is needed due to the hard constraints inherent in the model. We present two bounds of this type, applicable in slightly different settings.

Proposition 2.1. *Let Λ be a rectangle and let $\rho \in \Omega$. Define $m^{\rho, \Lambda} : \Omega \rightarrow \Omega_{\Lambda}^{\rho}$ as follows: let $m^{\rho, \Lambda}(\sigma)$ be obtained from σ by first setting $\sigma(v)$ to $\rho(v)$ for every $v \in \mathbb{V} \setminus \text{int}(\Lambda)$, and then removing any tile that has its center in $\text{int}(\Lambda)$ and overlaps another tile.*

Let $E \subset \Omega$. Then

$$Z_{\Lambda}^{*}(E) \leq C(\lambda)^{\text{Perimeter}(\Lambda)} Z_{\Lambda}^{\rho}(m^{\rho, \Lambda}(E))$$

where $C(\lambda)$ depends only on λ and $*$ stands for either a configuration in Ω or the symbol per.

Proof. For $\sigma' \in m^{\rho, \Lambda}(E)$ we have

$$\#\{\sigma \in E \cap \Omega_{\Lambda}^{*} : m^{\rho, \Lambda}(\sigma) = \sigma'\} \leq 2^{\frac{3}{2}\text{Perimeter}(\Lambda)}. \quad (2.7)$$

Indeed, if $*$ is a configuration in Ω then all σ in this set are identical except on the points of $\text{int}(\Lambda) \cap \mathbb{V}$ which are adjacent to $\partial\Lambda$. If $*$ = per, configurations in this set coincide on all points of $\text{int}(\Lambda) \cap \mathbb{V}$ which are not adjacent to $\partial\Lambda$, which implies (2.7) when taking into account the periodicity constraint.

Additionally, for any $\sigma \in \Omega$,

$$w_{\Lambda, \lambda}(\sigma) \leq (\max\{\lambda, \lambda^{-1}\})^{\frac{1}{4}\text{Perimeter}(\Lambda)} w_{\Lambda, \lambda}(m^{\rho, \Lambda}(\sigma)).$$

Thus

$$\begin{aligned}
 Z_{\Lambda}^*(E) &= \sum_{\sigma \in E \cap \Omega_{\Lambda}^*} w_{\Lambda, \lambda}(\sigma) = \sum_{\sigma' \in m^{\rho, \Lambda}(E)} \sum_{\substack{\sigma \in E \cap \Omega_{\Lambda}^* \\ m^{\rho, \Lambda}(\sigma) = \sigma'}} w_{\Lambda, \lambda}(\sigma) \\
 &\leq \sum_{\sigma' \in m^{\rho, \Lambda}(E)} 2^{\frac{3}{2} \text{Perimeter}(\Lambda)} (\max\{\lambda, \lambda^{-1}\})^{\frac{1}{4} \text{Perimeter}(\Lambda)} w_{\Lambda, \lambda}(\sigma') \\
 &= C(\lambda)^{\text{Perimeter}(\Lambda)} Z_{\Lambda}^{\rho}(m^{\rho, \Lambda}(E))
 \end{aligned}$$

where the last equality is since $m^{\rho, \Lambda}(E) \subset \Omega_{\Lambda}^{\rho}$. $\blacksquare$

Proposition 2.2. *Let $\Lambda' \subset \Lambda$ be rectangles and further assume that the Euclidean distance from Λ' to $\mathbb{R}^2 \setminus \Lambda$ is at least 2. Let $f : \Omega \rightarrow [0, \infty)$ be a Λ' -local function. Then there is $C(\lambda)$ (depending only on λ) such that*

$$\mu_{\Lambda'}(f) \leq C(\lambda)^{\text{Perimeter}(\Lambda)} \mu_{\Lambda}(f)$$

where μ_{Λ} may stand for $\mu_{\Lambda}^{\text{per}}$, μ_{Λ}^{ρ} for some $\rho \in \Omega$, or an infinite-volume Gibbs measure μ , and, independently, $\mu_{\Lambda'}$ may stand for $\mu_{\Lambda'}^{\text{per}}$, $\mu_{\Lambda'}^{\rho'}$ for some $\rho' \in \Omega$, or an infinite-volume Gibbs measure μ' .

Proof. By the DLR condition (2.5), we have $\mu(f) = \mu(\mu_{\Lambda}^{\sigma}(f))$. Similarly, $\mu_{\Lambda}^{\text{per}} = \mu_{\Lambda}^{\text{per}}(\mu_{\Lambda}^{\sigma}(f))$. The analogous equalities hold with Λ' instead of Λ . Therefore it suffices to prove for all $\rho, \rho' \in \Omega$ that

$$\mu_{\Lambda'}^{\rho'}(f) \leq C(\lambda)^{\text{Perimeter}(\Lambda)} \mu_{\Lambda}^{\rho}(f). \quad (2.8)$$

Let E be the event $\{\sigma : \sigma|_{\partial\Lambda'} = \rho'|_{\partial\Lambda'}\}$. The fact that f is Λ' -local implies that

$$\mu_{\Lambda'}^{\rho'}(f) = \mu_{\Lambda}^{\rho}(f | E).$$

We conclude that

$$\mu_{\Lambda'}^{\rho'}(f) = \frac{\mu_{\Lambda}^{\rho}(f \cdot \mathbf{1}_E)}{\mu_{\Lambda}^{\rho}(E)} \leq \frac{\mu_{\Lambda}^{\rho}(f)}{\mu_{\Lambda}^{\rho}(E)}$$

and thus (2.8) will follow once we show that $\mu_{\Lambda}^{\rho}(E) \geq C(\lambda)^{-\text{Perimeter}(\Lambda)}$. The last inequality is a simple consequence of the fact that the Euclidean distance from Λ' to $\mathbb{R}^2 \setminus \Lambda$ is at least 2 (proved similarly to Proposition 2.1). $\blacksquare$

3. Chessboard estimates

In this section we state the chessboard estimate for the 2×2 hard-square model, after setting up the necessary definitions. We do not give a proof, but refer to [6, 24, 74], [23, Chapter 10], [61, Section 2.7.1] and [32] for pedagogical references.

Additionally we prove a version of the chessboard estimate applicable to periodic infinite-volume Gibbs measures.

We use the following definitions for a rectangle $R = R_{K \times L, (x_0, y_0)}$:

- Define the grid of R and its origin-shifted version:

$$G^R := (x_0 + K\mathbb{Z}) \times (y_0 + L\mathbb{Z}) \quad \text{and} \quad \mathcal{L}^R := K\mathbb{Z} \times L\mathbb{Z}. \quad (3.1)$$

- Let T^R denote the group generated by reflections of $\mathbb{R}^2$ through the horizontal and vertical lines that intersect G^R . More precisely, T^R is the set of $\tau : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfying, for some $m, n \in \mathbb{Z}$,

$$\text{either } \tau(x, y)_1 = 2(x_0 + mK) - x \quad \text{or} \quad \tau(x, y)_1 = x + 2mK,$$

and

$$\text{either } \tau(x, y)_2 = 2(y_0 + nL) - y \quad \text{or} \quad \tau(x, y)_2 = y + 2nL.$$

- Importantly, for each $v \in G^R$, there is a unique isometry $\tau_{R,v}$ of T^R which maps R to $R_{K \times L, v}$.
- For $f : \Omega \rightarrow \mathbb{R}$ and $\tau \in T^R$ recall that τf is defined by (2.6) where τ is implicitly restricted to $\mathbb{Z}^2$.
- Recall that an R -local function (or event) is defined by (2.2). It will be essential that R is *closed*.

3.1. Finite volume

Throughout this subsection we fix a rectangle Λ and derive properties of the measure μ_Λ^{per} . We remind the reader that μ_Λ^{per} is supported on Λ -periodic configurations (i.e., $\sigma \in \Omega_\Lambda^{\text{per}}$) and though we stick to our convention of regarding configurations as defined on the infinite lattice $\mathbb{Z}^2$, the reader should keep in mind that configurations $\sigma \in \Omega_\Lambda^{\text{per}}$ are naturally defined on the torus $\mathbb{Z}^2 / \mathcal{L}^\Lambda$.

We start in Section 3.1.1 by showing that μ_Λ^{per} has reflection positivity with respect to reflection lines passing through vertices in $\mathbb{V}$. This is a standard consequence of the fact that the model has only nearest-neighbor and next-nearest-neighbor interactions (i.e., interactions involving only the eight nearest vertices). We continue in Section 3.1.2 to derive the chessboard estimate for μ_Λ^{per} – a standard consequence of reflection positivity. We give the name chessboard seminorm (see (3.2)) to the function $\|\cdot\|_{R|\Lambda}$ which appears on the right-hand side of the chessboard estimate (see also [6, (5.47)]) and derive some of its basic properties.

3.1.1. Reflection positivity. We now establish the basic reflection positivity property of the 2×2 hard-square model.

Lemma 3.1 (Reflection positivity). *Let $R = R_{K \times L, (x_0, y_0)}$ be a rectangle and let f be an R -local function. Then*

$$\mu_\Lambda^{\text{per}}(f \cdot \tau f) \geq 0$$

when either $\Lambda = R_{2K \times L}$ and τ is the reflection $\tau_{R, (x_0 + K, y_0)}$, or $\Lambda = R_{K \times 2L}$ and τ is the reflection $\tau_{R, (x_0, y_0 + L)}$.

Proof. We prove only the first case because the second one is analogous. Denote $R_1 := R_{K \times L, (x_0 + K, y_0)}$ and observe that τf is an R_1 -local function. Let $\mathcal{F}$ be the sigma-algebra generated by the $\sigma(x, y)$ with $x \equiv x_0 \pmod{K}$. The fact that the model has only nearest-neighbor and next-nearest-neighbor interactions, which are also symmetric, implies that f and τf are independent and identically distributed under μ_Λ^{per} conditioned on $\mathcal{F}$. Thus,

$$\mu_\Lambda^{\text{per}}(f \cdot \tau f \mid \mathcal{F}) = (\mu_\Lambda^{\text{per}}(f \mid \mathcal{F}))^2 \geq 0$$

and taking expectations of both sides concludes the proof. $\blacksquare$

3.1.2. The chessboard seminorm and the chessboard estimate. We now discuss the chessboard estimate for the measure μ_Λ^{per} .

We say that R is a *block* of Λ when R and Λ are rectangles such that $2 \text{Width}(R)$ divides $\text{Width}(\Lambda)$ and $2 \text{Height}(R)$ divides $\text{Height}(\Lambda)$. In this case we make the following definitions, which depend on R and on the dimensions of Λ :

- Set $T_\Lambda^R := T^R / \mathcal{L}^\Lambda$, i.e., the quotient of the group T^R by the group of translations by vectors of $\mathcal{L}^\Lambda$. Our assumption that R is a block of Λ implies that the latter group is indeed a subgroup of the former. Note that $\#T_\Lambda^R = \frac{\text{Area}(\Lambda)}{\text{Area}(R)}$.
- We observe that while an element $\tau \in T_\Lambda^R$ is formally an equivalence class of isometries, it may also be thought of as a single isometry of the torus $\mathbb{Z}^2 / \mathcal{L}^\Lambda$. Thus, $\sigma \circ \tau$ is well defined for $\sigma \in \Omega_\Lambda^{\text{per}}$, which allows us, given $f : \Omega \rightarrow \mathbb{R}$, to further define $\tau f : \Omega_\Lambda^{\text{per}} \rightarrow \mathbb{R}$ (via (2.6)).
- For an R -local function $f : \Omega \rightarrow \mathbb{R}$ define

$$\|f\|_{R|\Lambda} := \left[\mu_\Lambda^{\text{per}} \left(\prod_{\tau \in T_\Lambda^R} \tau f \right) \right]^{1/\#T_\Lambda^R}. \quad (3.2)$$

Note that the expectation in this definition is necessarily non-negative by reflection positivity (Lemma 3.1), so that $\|f\|_{R|\Lambda}$ is well defined and satisfies

$$\|f\|_{R|\Lambda} \geq 0. \quad (3.3)$$

Note also that $\|\cdot\|_{R|\Lambda}$ still depends on λ but, for brevity, we omit this dependence from the notation.

For an R -local event E we write $\|E\|_{R|\Lambda} := \|\mathbf{1}_E\|_{R|\Lambda}$.

We call $\|\cdot\|_{R|\Lambda}$ the (R, Λ) -*chessboard seminorm*. The name “seminorm” is justified by Proposition 3.3 below. The notation $\|\cdot\|_{R|\Lambda}$, as an alternative to the $\mathfrak{z}$ notation used in [6, (5.47)], is chosen to better remind the reader of the seminorm properties.

Proposition 3.2 (Chessboard estimate). *Let R be a block of Λ , let $A \subset T_\Lambda^R$, and let $(f_\tau)_{\tau \in A}$ be R -local functions. Then*

$$\mu_\Lambda^{\text{per}} \left(\prod_{\tau \in A} \tau f_\tau \right) \leq \prod_{\tau \in A} \|f_\tau\|_{R|\Lambda}.$$

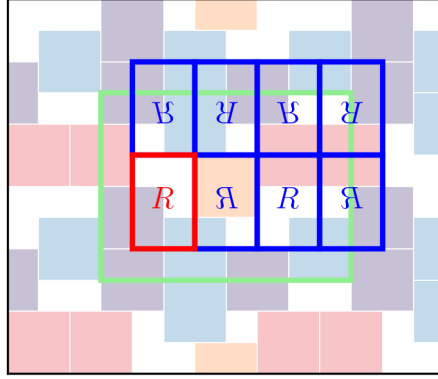


Fig. 3.1. In the background the rectangle $\Lambda = R_{8 \times 6, (0,0)}$ is shown in green, with a configuration in $\Omega_{\Lambda}^{\text{per}}$. The red rectangle $R = R_{2 \times 3, (1,1)}$ is a block of Λ . The red and blue rectangles are images of R under elements of T^R that form a representative set of T_{Λ}^R .

Proof. This is a standard consequence of reflection positivity (Lemma 3.1); see [23, Theorem 10.11, Remark 10.15] or [6, Theorem 5.8]. In both references, the proof is given for the case of reflection positivity “through edges/bonds” and it is remarked that an analogous result holds for reflection positivity “through vertices/sites” (as in our case). ■

We proceed to record several basic properties of the chessboard seminorm $\|\cdot\|_{R|\Lambda}$. The first two properties justify the name seminorm while the last two imply that $\|\cdot\|_{R|\Lambda}$ restricted to R -local events is an outer measure (as in [6, Lemma 5.9]; countable subadditivity follows from additivity as there are only finitely many R -local events).

Proposition 3.3 (Positive homogeneity, triangle inequality and monotonicity). *The mapping $f \mapsto \|f\|_{R|\Lambda}$, where f ranges over R -local functions, satisfies the following properties:*

- (1) (Homogeneity) $\|\alpha f\|_{R|\Lambda} = |\alpha| \|f\|_{R|\Lambda}$ for $\alpha \in \mathbb{R}$. In particular, $\|f\|_{R|\Lambda} \geq 0$.
- (2) (Triangle inequality) $\|f_0 + f_1\|_{R|\Lambda} \leq \|f_0\|_{R|\Lambda} + \|f_1\|_{R|\Lambda}$.
- (3) (Monotonicity:) $\|g\|_{R|\Lambda} \geq \|f\|_{R|\Lambda}$ whenever $g \geq f \geq 0$.

Proof. Since $\#T_{\Lambda}^R$ is even, we have $\prod_{\tau \in T_{\Lambda}^R} \tau(\alpha f) = |\alpha|^{\#T_{\Lambda}^R} \prod_{\tau \in T_{\Lambda}^R} \tau f$. It follows from the definition of $\|\cdot\|_{R|\Lambda}$ that $\|\alpha f\|_{R|\Lambda} = |\alpha| \|f\|_{R|\Lambda}$.

The triangle inequality follows from

$$\begin{aligned}
 (\|f_0 + f_1\|_{R|\Lambda})^{\#T_{\Lambda}^R} &= \\
 & \text{(by definition)} = \mu_{\Lambda}^{\text{per}} \left(\prod_{\tau \in T_{\Lambda}^R} \tau(f_0 + f_1) \right) \\
 & \text{(expanding brackets)} = \sum_{r: T_{\Lambda}^R \rightarrow \{0,1\}} \mu_{\Lambda}^{\text{per}} \left(\prod_{\tau \in T_{\Lambda}^R} \tau f_{r(\tau)} \right)
 \end{aligned}$$

$$\begin{aligned}
(\text{by the chessboard estimate}) &\leq \sum_{r: T_\Lambda^R \rightarrow \{0,1\}} \prod_{\tau \in T_\Lambda^R} \|f_{r(\tau)}\|_{R|\Lambda} \\
(\text{factorizing}) &= (\|f_0\|_{R|\Lambda} + \|f_1\|_{R|\Lambda})^{\#T_\Lambda^R}.
\end{aligned}$$

Monotonicity follows from the definition of $\|f\|_{R|\Lambda}$ by the monotonicity of μ_Λ^{per} . ■

Lastly, we note a simple relation between the chessboard seminorms of rectangles with different dimensions.

Lemma 3.4 (“Recursive chessboard estimate”). *Let R and S be blocks of Λ and assume that the corners of S are in G^R . Let $A \subset T^R$ be such that $\bigcup_{\tau \in A} \tau R \subset S$. For each $\tau \in A$, let f_τ be an R -local function. Then*

$$\left\| \prod_{\tau \in A} \tau f_\tau(\sigma) \right\|_{S|\Lambda} \leq \prod_{\tau \in A} \|f_\tau\|_{R|\Lambda}.$$

Proof. Denote $g := \prod_{\tau \in A} \tau f_\tau(\sigma)$, so that g is an S -local function. Our assumptions imply that $T_\Lambda^S \subset T_\Lambda^R$. By the definition of g ,

$$\prod_{\iota \in T_\Lambda^S} \iota g = \prod_{\iota \in T_\Lambda^S} \prod_{\tau \in A} \iota \tau f_\tau \quad \text{on } \Omega_\Lambda^{\text{per}}. \quad (3.4)$$

Our assumption that $\bigcup_{\tau \in A} \tau R \subset S$ shows that each choice of $\iota \in T_\Lambda^S$ and $\tau \in A$ gives a distinct element $\iota \tau \in T_\Lambda^R$. Therefore, by the chessboard estimate,

$$\mu_\Lambda^{\text{per}} \left(\prod_{\iota \in T_\Lambda^S} \prod_{\tau \in A} \iota \tau f_\tau \right) \leq \prod_{\iota \in T_\Lambda^S} \prod_{\tau \in A} \|f_\tau\|_{R|\Lambda} = \left(\prod_{\tau \in A} \|f_\tau\|_{R|\Lambda} \right)^{\#T_\Lambda^S}. \quad (3.5)$$

Substituting (3.4) in the left-hand side of (3.5), we find that, by the definition (3.2),

$$\|g\|_{S|\Lambda} \leq \prod_{\tau \in A} \|f_\tau\|_{R|\Lambda}. \quad \blacksquare$$

3.2. Infinite volume

Recall that we call an (infinite-volume) Gibbs measure periodic if it is invariant under translations by some full-rank sublattice of $\mathbb{Z}^2$. Our goal in this section is to provide a version of the chessboard estimate applicable to periodic Gibbs measures (Proposition 3.8 below). This will be used in later sections to apply a Peierls-type argument directly in infinite volume.

We have not seen the chessboard estimate formulated directly in infinite volume before, though we mention that a different approach was used by Biskup–Kotecký [9] in order to apply a Peierls argument driven by a chessboard estimate to periodic (infinite-volume) Gibbs measures.

3.2.1. The chessboard seminorm and its basic properties. We begin by defining a “limit of $\|\cdot\|_{R|\Lambda}$ as $\Lambda \rightarrow \infty$ ”. Let R be a rectangle and let f be an R -local function. Define

$$\|f\|_R := \limsup_{n \rightarrow \infty} \|f\|_{R|R_n! \times n!}, \quad (3.6)$$

noting that R is a block of $R_n! \times n!$ for almost all n .

Remark 3.5. In fact, $\lim_{(m,n) \rightarrow (\infty, \infty)} \|f\|_{R|2m \text{ Width}(R) \times 2n \text{ Height}(R)}$ exists, but the proof of this fact is complicated by the “boundary overlaps between blocks” in the definition of $\|f\|_{R|\Lambda}$. To avoid proving this fact, we have chosen the somewhat arbitrary definition above.

The basic properties of $\|\cdot\|_{R|\Lambda}$ transfer directly to the limiting definition (3.6).

Proposition 3.6 (Positive homogeneity, triangle inequality and monotonicity for infinite volume). *The mapping $f \mapsto \|f\|_R$, where f ranges over R -local functions, satisfies the properties stated in Proposition 3.3.*

Proof. The properties follow from Proposition 3.3 and the subadditivity of the upper limit. ■

Lemma 3.4 also admits an immediate extension.

Lemma 3.7 (“Recursive chessboard estimate” for infinite volume). *Let R and S be rectangles and assume that the corners of S are in G^R . Let $A \subset T^R$ be such that $\bigcup_{\tau \in A} \tau R \subset S$. For each $\tau \in A$, let f_τ be an R -local function. Then*

$$\left\| \prod_{\tau \in A} \tau f_\tau(\sigma) \right\|_S \leq \prod_{\tau \in A} \|f_\tau\|_R.$$

Proof. The inequality follows from Lemma 3.4 using the definition (3.6) (making use of the fact that the definition (3.6) involves a limsup rather than a liminf). ■

3.2.2. The chessboard estimate. We now prove the following statement.

Proposition 3.8 (Chessboard estimate for infinite volume). *Let R be a rectangle. Let $A \subset T^R$ be finite and let $(f_\tau)_{\tau \in A}$ be R -local functions. Then*

$$\mu\left(\prod_{\tau \in A} \tau f_\tau\right) \leq \prod_{\tau \in A} \|f_\tau\|_R$$

for all periodic Gibbs measures μ .

The following lemma is the main tool in the proof.

Lemma 3.9. *Let R be a rectangle and let f be an R -local function. Then*

$$\mu(f) \leq \|f\|_R \quad (3.7)$$

for all periodic Gibbs measures μ .

The proof of the lemma relies on the following two auxiliary claims. The first claim is a weak form of Proposition 3.8 which follows from the finite-volume chessboard estimate and a comparison of boundary conditions.

Claim 3.10. *Let R be a rectangle and let g be a non-negative R -local function. Let $(A_n)_{n \geq 1}$ be finite subsets of T^R satisfying*

$$\frac{\text{diam}(\bigcup_{\tau \in A_n} \tau R)}{\#A_n} \xrightarrow{n \rightarrow \infty} 0 \quad (3.8)$$

where we denote by $\text{diam}(\cdot)$ the diameter of subsets of $\mathbb{R}^2$. Then

$$\limsup_{n \rightarrow \infty} \left[\mu \left(\prod_{\tau \in A_n} \tau g \right) \right]^{1/\#A_n} \leq \|g\|_R$$

for all Gibbs measures μ .

Proof. Fix $n \geq 1$. Denote $\Lambda_m := R_{m! \times m!, (-m!/2, -m!/2)}$ for $m \geq 2$.

We use Proposition 2.2 to show that for sufficiently large m ,

$$\mu \left(\prod_{\tau \in A_n} \tau g \right) \leq C_{2.2}(\lambda)^{2\text{diam}(\bigcup_{\tau \in A_n} \tau R)} \mu_{\Lambda_m}^{\text{per}} \left(\prod_{\tau \in A_n} \tau g \right),$$

as the distance from the “bounding box” of $\bigcup_{\tau \in A_n} \tau R$ to Λ_m^c is at least 2 and where we bounded the perimeter of the bounding box by twice its diameter.

For sufficiently large m , the rectangle R is a block of Λ_m and $\bigcup_{\tau \in A_n} \tau R \subset \Lambda_m$, so that the elements of A_n belong to distinct equivalence classes in $T_{\Lambda_m}^R$. Therefore, by the finite-volume chessboard estimate of Proposition 3.2,

$$\mu_{\Lambda_m}^{\text{per}} \left(\prod_{\tau \in A_n} \tau g \right) \leq \|g\|_{R|\Lambda_m}^{\#A_n}.$$

Combining the last two displays shows that

$$\left[\mu \left(\prod_{\tau \in A_n} \tau g \right) \right]^{1/\#A_n} \leq C(\lambda)^{\frac{\text{diam}(\bigcup_{\tau \in A_n} \tau R)}{\#A_n}} \|g\|_{R|\Lambda_m}.$$

The claim follows by taking upper limits, first as $m \rightarrow \infty$ and then as $n \rightarrow \infty$, and using our assumption (3.8) and the definition (3.6) of $\|g\|_R$. ■

The second claim is a simple application of Taylor’s theorem.

Claim 3.11. *Let $M \in \mathbb{N}$. Let $S_n \subset M\mathbb{Z}^2$ be finite and let $A_n := \{\tau_s : s \in S_n\}$, where $\tau_s : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the shift $\tau_s(u) := u + s$. Let $0 < \varepsilon < 1/2$ and let $g : \Omega \rightarrow [1 - \varepsilon, 1 + \varepsilon]$ be a measurable function. Then*

$$\mu \left(\left[\prod_{\tau \in A_n} \tau g \right]^{1/\#A_n} \right) = \mu(g) + O(\varepsilon^2) \quad (3.9)$$

for every $M\mathbb{Z}^2$ -invariant μ . Here $O(\varepsilon^2)$ denotes an expression whose absolute value is at most $C\varepsilon^2$ for a universal constant $C > 0$.

Proof. Set $g = 1 + \varepsilon f$, so that $|f| \leq 1$. Then

$$\begin{aligned}
 \mu\left(\left[\prod_{\tau \in A_n} \tau g\right]^{1/\#A_n}\right) &= \mu\left(\exp\left(\frac{1}{\#A_n} \sum_{\tau \in A_n} \log(1 + \varepsilon \tau f)\right)\right) \\
 (\text{by Taylor's theorem}) &= \mu\left(\exp\left(\frac{1}{\#A_n} \sum_{\tau \in A_n} (\varepsilon \tau f + O(\varepsilon^2))\right)\right) \\
 &= \mu\left(\exp\left(\varepsilon \left(\frac{1}{\#A_n} \sum_{\tau \in A_n} \tau f\right) + O(\varepsilon^2)\right)\right) \\
 (\text{by Taylor's theorem}) &= \mu\left(1 + \varepsilon \left(\frac{1}{\#A_n} \sum_{\tau \in A_n} \tau f\right) + O(\varepsilon^2)\right) \\
 &= 1 + \frac{\varepsilon}{\#A_n} \sum_{\tau \in A_n} \mu(\tau f) + O(\varepsilon^2) \\
 (\text{since } \mu \text{ is } M\mathbb{Z}^2 \text{ invariant}) &= 1 + \varepsilon \mu(f) + O(\varepsilon^2) \\
 &= \mu(g) + O(\varepsilon^2). \quad \blacksquare
 \end{aligned}$$

We now deduce Lemma 3.9.

Proof of Lemma 3.9. Let $\mathcal{L} \subset \mathbb{Z}^2$ be a full-rank sublattice and let μ be an $\mathcal{L}$ -invariant Gibbs measure. Recall the definition of $\mathcal{L}^R$ from (3.1) and note that $\mathcal{L} \cap 2\mathcal{L}^R$ is also a full-rank sublattice. Let $M \in \mathbb{N}$ be such that $M\mathbb{Z}^2 \subset \mathcal{L} \cap 2\mathcal{L}^R$. For $n \in \mathbb{N}$, set $S_n := \{M, 2M, \dots, Mn\}^2 \subset M\mathbb{Z}^2$ and let A_n be as in Claim 3.11. Observe that $A_n \subset T^R$ as $S_n \subset 2\mathcal{L}^R$. For $0 < \varepsilon < \frac{1}{2 \max |f|}$, set $g = 1 + \varepsilon f$ and observe that

$$\begin{aligned}
 1 + \varepsilon \|f\|_R &\stackrel{(i)}{\geq} \|g\|_R \stackrel{(ii)}{\geq} \limsup_{n \rightarrow \infty} \left[\mu\left(\prod_{\tau \in A_n} \tau g\right) \right]^{1/n^2} \\
 &\stackrel{(iii)}{\geq} \limsup_{n \rightarrow \infty} \mu\left(\left[\prod_{\tau \in A_n} \tau g\right]^{1/n^2}\right) \\
 &\stackrel{(iv)}{=} \mu(g) + O((\varepsilon \max |f|)^2) \\
 &= 1 + \varepsilon \mu(f) + O(\varepsilon^2 (\max |f|)^2)
 \end{aligned}$$

where (i) follows from subadditivity of $\|\cdot\|_R$ (Proposition 3.6), (ii) follows from Claim 3.10, noting that A_n satisfies (3.8), (iii) follows from Jensen's inequality, and (iv) follows from Claim 3.11, noting that μ is $M\mathbb{Z}^2$ -invariant. The lemma follows by letting $\varepsilon \rightarrow 0$. $\blacksquare$

Finally, we turn to the proof of the infinite-volume chessboard estimate.

Proof of Proposition 3.8. Recall the definition of G^R from (3.1). Let S be a rectangle whose corners lie in G^R and which contains $\bigcup_{\tau \in A} \tau R$. Observe that $\prod_{\tau \in A} \tau f_\tau(\sigma)$ is an S -local function. Applying Lemmas 3.9 and 3.7, we get

$$\mu\left(\prod_{\tau \in A} \tau f_\tau(\sigma)\right) \leq \left\| \prod_{\tau \in A} \tau f_\tau(\sigma) \right\|_S \leq \prod_{\tau \in A} \|f_\tau\|_R. \quad \blacksquare$$

4. Mesoscopic rectangles are divided by sticks

The goal of this work is to establish a form of columnar, or row, order for the 2×2 hard-square model at high fugacity. Recall from the introduction (see Figure 1.2) that in a vertically ordered state of this type, the tiles organize in columns of width 2 which are non-interacting at most places. That is, the system may be thought of as a perturbation of a product system in which each column follows a one-dimensional hard-square model. It is instructive to note that one-dimensional systems at high fugacity λ consist of segments of fully packed tiles whose lengths are typically of order $\sqrt{\lambda}$ – a mesoscopic length scale which will be important in our arguments.

Motivated by this description, we introduce the notion of sticks. Informally, a vertical (resp. horizontal) stick is the line separating two finite columns (rows) in which the tiles are fully packed but have a different vertical (resp. horizontal) offset. Columnar order leads to an abundance of vertical sticks of mesoscopic length (of order $\sqrt{\lambda}$) while row order similarly leads to an abundance of horizontal sticks of mesoscopic length. Importantly, vertical and horizontal sticks cannot meet. Using this fact, it will be shown that, in a suitable sense, the interface between regions of columnar and row order is characterized by the presence of mesoscopic rectangles which are not divided by a stick.

The goal of this section is to prove Lemma 4.1 below, which roughly states that mesoscopic rectangles are divided by a stick with high probability. By the loose term “mesoscopic rectangle” we mean a rectangle whose side lengths are small compared to $\sqrt{\lambda}$ but whose area is large compared to $\sqrt{\lambda}$. Moreover, this probabilistic estimate applies multiplicatively to collections of disjoint mesoscopic rectangles using the chess-board estimate of Proposition 3.8. This will lead, in Section 5, to the existence of multiple Gibbs measures (one with a predominance of vertical sticks and another with a predominance of horizontal sticks) through a Peierls-type argument.

We proceed with the formal definitions and results.

Sticks: Let $\sigma \in \Omega$, and let the following definitions depend on σ . Recall from Section 2.1 that the parity of a tile centered at (x, y) is $(x - 1 \bmod 2, y - 1 \bmod 2)$. An edge of $\mathbb{E}_{\square}$ that bounds two faces in $\mathbb{F}$ that are respectively contained in two tiles of σ having *distinct* parities, is called a *stick edge* (for σ); each stick edge is naturally vertical or horizontal. A *stick* is a maximal path of stick edges (possibly infinite in one or both directions).

A case analysis shows that a vertical stick edge can never meet a horizontal stick edge at a vertex. Thus a stick may be viewed as a vertical or horizontal segment in $\mathbb{R}^2$. Note also that sticks are pairwise disjoint.

In later sections we will classify sticks according to their orientation and parity. We say that a stick is of type $(\text{ver}, 0)$, and call it “a $(\text{ver}, 0)$ stick”, if it is vertical and passes through points with even x -coordinate. Equivalently, a stick is of type $(\text{ver}, 0)$ if it bounds only on tiles with horizontal parity 0. We define analogously the types $(\text{ver}, 1)$, $(\text{hor}, 0)$ and $(\text{hor}, 1)$.

Let $R = R_{K \times L, (x, y)}$ be a rectangle and consider a vertical segment whose endpoints are (x_1, y_1) and (x_1, y_2) with $y_1 < y_2$. We say that the segment *vertically divides* R if

$y_1 \leq y \leq y + L \leq y_2$ and $x < x_1 < x + K$. We make an analogous definition for horizontal segments. A segment is said to *divide* R if it is either a vertical segment dividing R vertically or a horizontal segment dividing R horizontally.

Note that with these definitions, the event that a stick divides R is R -local. This may be seen using the fact that if an edge $e \in \mathbb{E}_\square$ is such that its image in $\mathbb{R}^2$ is contained in R but not contained in ∂R , then the event that e is a stick edge is R -local.

Our goal in this section is to prove the following probabilistic bound on the prevalence of dividing sticks in mesoscopic rectangles. Recall the infinite-volume chessboard seminorm $\|\cdot\|_R$ defined in (3.6).

Lemma 4.1. *There is $c > 0$ such that if rectangles $S \subset R$ satisfy*

$$1/c \leq \text{Width}(R), \text{Height}(R) \leq c\lambda^{1/2}, \quad (4.1)$$

$$\text{Width}(S) \geq (1-c)\text{Width}(R) \quad \text{and} \quad \text{Height}(S) \geq (1-c)\text{Height}(R), \quad (4.2)$$

then

$$\|\text{no stick divides both } R \text{ and } S\|_R \leq e^{-c \text{Area}(R)\lambda^{-1/2}}. \quad (4.3)$$

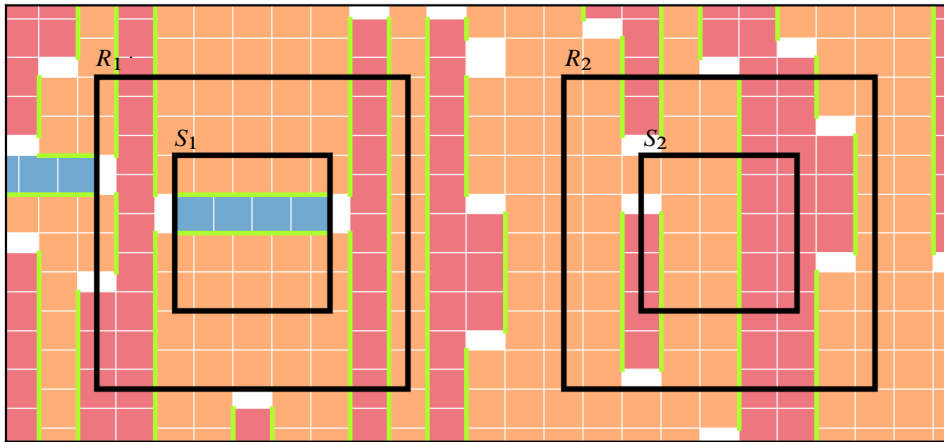


Fig. 4.1. The sticks of the configuration are highlighted in green. No stick divides both R_1 and S_1 although each of them is divided by a stick. A stick divides both R_2 and S_2 . In the terminology of Section 5, if $R_1 = R_{16 \times 16, (0,0)}$ and $N = 4$ then $S_1 = R_1^-$, $S_2 = R_2^-$ and R_2 is properly divided by a (ver, 1) stick while R_1 is not properly divided. In symbols, $(0, 0) \notin \Psi^{4 \times 4}$ and $(6, 0) \in \Psi_{(\text{ver}, 1)}^{4 \times 4}$.

4.1. One-dimensional systems

The proof of the probability estimate (4.3) involves giving an upper bound on the total weight of configurations (mostly) without long sticks and comparing it with a lower bound on the total weight of all configurations (with suitable boundary conditions). The first task will be performed in the subsequent sections whereas here we focus on the simpler

second task. Having in mind that high-fugacity systems are expected to order in a columnar fashion (as we aim to prove in this paper), it is natural to obtain a lower bound for the two-dimensional system via lower bounds for one-dimensional systems (which should be thought of as single columns of tiles in the two-dimensional system). We proceed to develop such bounds.

It is simplest to define the one-dimensional model as the restriction of the two-dimensional model to a rectangle of width 2 (the width of a single tile). Thus we define the partition function of a one-dimensional system of size L with free boundary conditions by

$$Z_{L,1D}^0 := Z_{R_{2 \times L}}^0$$

and the partition function of a one-dimensional system of size L with periodic boundary conditions by

$$Z_{L,1D}^{\text{per}} := Z_{R_{2 \times L}}^{\text{per}}(E),$$

where E is the event that all tiles have even horizontal parity, and using the notation (2.4) for the weight of an event. In these definitions we again follow our convention of omitting the fugacity parameter λ from the notation.

The next proposition provides a lower bound for the partition function of periodic one-dimensional systems.

Proposition 4.2. $Z_{L,1D}^{\text{per}} \geq (1 + \frac{1}{2}\lambda^{-1/2})^L$ for all $\lambda > 0$ and all even $L \geq 0$.

Proof. Let A be the set of configurations in $\Omega_{R_{2 \times L}}^{\text{per}}$ where all tiles have even horizontal parity. There is a one-to-one correspondence between configurations in A and the set B of sequences $r \in \{0, 1\}^{\{0, 1, \dots, L\}}$ such that $r_0 = r_L$ and $r_i r_{i+1} = 0$ for $0 \leq i < L$; the correspondence is defined by $(r(\sigma))_i = \sigma(1, i)$.

Recalling the formula (2.3) for the weight of a configuration, we note the identity

$$w_{R_{2 \times L}, \lambda}(\sigma) = \lambda^{-\frac{1}{2} \sum_{i=0}^{L-1} (1-r(\sigma)_i)(1-r(\sigma)_{i+1})}.$$

This follows by observing that the vacancies in σ necessarily come in horizontally adjacent pairs, and that such pairs correspond in $r(\sigma)$ to pairs of consecutive 0 values. Thus

$$Z_{L,1D}^{\text{per}} = \sum_{\sigma \in A} w_{R_{2 \times L}, \lambda}(\sigma) = \sum_{r \in B} \lambda^{-\frac{1}{2} \sum_{i=0}^{L-1} (1-r_i)(1-r_{i+1})} = \text{Tr} \left(\begin{pmatrix} \lambda^{-1/2} & 1 \\ 1 & 0 \end{pmatrix}^L \right)$$

where the 0 in the matrix corresponds to the restriction of not having consecutive 1 values in r . The eigenvalues of $\begin{pmatrix} \lambda^{-1/2} & 1 \\ 1 & 0 \end{pmatrix}$ are

$$\gamma_{\pm} = \frac{\lambda^{-1/2} \pm \sqrt{\lambda^{-1} + 4}}{2},$$

whence, for even L ,

$$Z_{L,1D}^{\text{per}} = \gamma_+^L + \gamma_-^L \geq \gamma_+^L \geq \left(1 + \frac{1}{2}\lambda^{-1/2}\right)^L. \quad \blacksquare$$

The above one-dimensional bound implies the following lower bound for the partition function of two-dimensional systems.

Corollary 4.3. *For each $c < \frac{1}{4}$, there is λ_0 such that for all $\lambda > \lambda_0$ and all even rectangles Λ ,*

$$Z_{\Lambda, \lambda}^{\text{per}} \geq e^{c\lambda^{-1/2} \text{Area}(\Lambda)}.$$

Proof. The total weight of configurations in $\Omega_{\Lambda, \lambda}^{\text{per}}$ with all tiles having even horizontal parity (as in the center panel of Figure 1.2) is $(Z_{\text{Height}(\Lambda), 1\text{D}}^{\text{per}})^{\text{Width}(\Lambda)/2}$. Thus, by Proposition 4.2,

$$Z_{\Lambda, \lambda}^{\text{per}} \geq (Z_{\text{Height}(\Lambda), 1\text{D}}^{\text{per}})^{\text{Width}(\Lambda)/2} \geq \left(1 + \frac{1}{2}\lambda^{-1/2}\right)^{\text{Area}(\Lambda)/2}$$

from which the corollary follows. $\blacksquare$

We also prove a lower bound on $Z_{L, 1\text{D}}^0$ that will be used in Proposition 4.15. This bound is useful in particular when L has the same order of magnitude as $\lambda^{1/2}$.

Proposition 4.4. $Z_{L, 1\text{D}}^0 \geq 1 + \frac{L^2}{8\lambda}$ for all $\lambda > 0$ and all even $L \geq 0$.

Proof. Since L is even, there is a configuration in $\Omega_{R_{2 \times L}}^0$ in which $R_{2 \times L}$ is fully packed with tiles, and this configuration has weight 1. We also consider configurations having one tile less than the fully packed configuration. Each such configuration has weight λ^{-1} , and one checks that the number of such configurations is exactly $\frac{L/2 \cdot (L/2 + 1)}{2}$. This shows that $Z_{L, 1\text{D}}^0 \geq 1 + \frac{L^2 + 2L}{8\lambda}$, from which the proposition follows. $\blacksquare$

We will make extensive use of the following simple corollary in Part II (it will not be used in Part I).

Corollary 4.5. *Let $f \in \mathbb{F}$ be a face. Then*

- (1) $\|f \text{ is vacant}\|_f \leq \lambda^{-1/4}$,
- (2) *for each $c < 1/4$ and sufficiently large λ , $\|f \text{ is occupied}\|_f \leq 1 - c\lambda^{-1/2}$.*

Proof. Consider an even rectangle Λ , and a face f . Then f is a block of Λ . The empty configuration $0 \in \Omega_{\Lambda}^{\text{per}}$ has weight $w_{\Lambda, \lambda}(0) = \lambda^{-\frac{1}{4} \text{Area}(\Lambda)}$. A fully packed configuration has weight 1, whence $Z_{\Lambda}^{\text{per}} \geq 1$. This gives

$$\begin{aligned} \|f \text{ is vacant}\|_{f|\Lambda} &= (\mu_{\Lambda}^{\text{per}}(\text{all faces of } \Lambda \text{ are vacant}))^{1/\text{Area}(\Lambda)} \\ &= \left(\frac{w_{\Lambda, \lambda}(0)}{Z_{\Lambda}^{\text{per}}} \right)^{1/\text{Area}(\Lambda)} \leq \lambda^{-1/4}. \end{aligned}$$

For the second item, note that a fully packed configuration in $\Omega_{\Lambda}^{\text{per}}$ is either composed of fully packed columns or of fully packed rows (see Figure 1.1). In the case of columns, say, there is a global choice of parity for the horizontal offset of the columns and, for each column, two possibilities to choose its vertical offset. Thus

$$Z_{\Lambda}^{\text{per}}(\Lambda \text{ is fully packed}) \leq 2 \cdot 2^{\text{Width}(\Lambda)} + 2 \cdot 2^{\text{Height}(\Lambda)}. \quad (4.4)$$

Let $c < 1/4$, and choose some $c < c_{4.3} < 1/4$. Then

$$\begin{aligned}
 \|f \text{ occupied}\|_{f|\Lambda} &= (\mu_{\Lambda}^{\text{per}}(\text{all faces of } \Lambda \text{ are occupied}))^{1/\text{Area}(\Lambda)} \\
 &\stackrel{(\text{by (4.4)})}{\leq} \left(\frac{2 \cdot 2^{\text{Width}(\Lambda)} + 2 \cdot 2^{\text{Height}(\Lambda)}}{Z_{\Lambda}^{\text{per}}} \right)^{1/\text{Area}(\Lambda)} \\
 &\stackrel{(\text{Corollary 4.3})}{\leq} \left(\frac{2 \cdot 2^{\text{Width}(\Lambda)} + 2 \cdot 2^{\text{Height}(\Lambda)}}{e^{c_{4.3} \lambda^{-1/2} \text{Area}(\Lambda)}} \right)^{1/\text{Area}(\Lambda)} \\
 &\stackrel{\frac{\text{Perimeter}(\Lambda)}{\text{Area}(\Lambda)} \rightarrow 0}{\longrightarrow} e^{-c_{4.3} \lambda^{-1/2}} \\
 &\left(\begin{array}{l} \text{for sufficiently large } \lambda, \\ \text{since } c < c_{4.3} \end{array} \right) \leq 1 - c \lambda^{-1/2}.
 \end{aligned}$$

The bounds on $\|\cdot\|_f$ now follow using definition (3.6). ■

4.2. Configurations without long sticks

For $M \geq 1$, denote by $E_M \subset \Omega$ the set of configurations in which all sticks are of length at most M . For an even rectangle Λ , consider $E_M \cap \Omega_{\Lambda}^1$, the set of configurations *with fully packed boundary conditions* in E_M . This subsection is devoted to proving the following “weighted counting lemma”, bounding the total weight of all such configurations.

Lemma 4.6. *There exists $C > 0$ such that for every $\lambda > 0$, $M \geq 1$ and even rectangle Λ , if $M < \lambda^{1/2}/C$ then*

$$Z_{\Lambda}^1(E_M) \leq \left(1 + \frac{CM}{\lambda} \right)^{\text{Area}(\Lambda)}.$$

Remark 4.7. The bound of the lemma is sharp up to the value of the constant C , at least when $M \leq \max\{\text{Width}(\Lambda), \text{Height}(\Lambda)\}$. Let us sketch how a matching lower bound may be obtained. Observe that E_M contains the set of configurations

$$\tilde{E}_M := \{\sigma \in \Omega : \forall (x, y) \in \mathbb{Z}^2, (2 \mid x \text{ or } M \mid y) \Rightarrow \sigma(x, y) = 0\}.$$

Indeed, for $\sigma \in \tilde{E}_M$ all stick edges are vertical, and no stick intersects a line of the form $y = y_0$ with M dividing y_0 , limiting the length of vertical sticks to be at most M . Assume for simplicity that $\Lambda = R_{K \times L, (0,0)}$ where M divides L . Then configurations in $\Omega_{\Lambda}^0 \cap \tilde{E}_M$ are sums of the form $\sum_{i=0}^{K/2-1} \sum_{j=0}^{L/M-1} \sigma_{i,j}$ where $\sigma_{i,j} \in \Omega_{R_{2 \times M, (2i, Mj)}}^0$. This, together with Proposition 4.4, shows that $Z_{\Lambda}^1(\tilde{E}_M) = Z_{\Lambda}^0(\tilde{E}_M) = (Z_{M, \text{1D}}^0)^{\frac{KL}{2M}} \geq (1 + cM^2/\lambda)^{\text{Area}(\Lambda)/M}$, which matches the upper bound of the lemma, up to the value of C , since $M < \lambda^{1/2}/C$.

4.2.1. Components. Recall the definition of a stick edge and further define a *vacancy edge* as an edge in $\mathbb{E}_{\square}$ that bounds a vacant face. A *regular edge* is defined as one that is neither a stick edge nor a vacancy edge.

We define a *marked graph* as a directed graph where each edge is marked as either horizontal or vertical, and also marked as either a vacancy edge, a stick edge or a regular edge. Formally, it is a triplet (V, E, f) where (V, E) is a directed graph and $f : E \rightarrow \{\text{“horizontal”}, \text{“vertical”}\} \times \{\text{“vacancy”}, \text{“stick”}, \text{“regular”}\}$.

For a configuration σ its *configuration graph* G_σ is defined to be a marked graph obtained as follows. We direct each edge of $(\mathbb{V}, \mathbb{E}_\square)$ either upwards or to the right, and mark it as horizontal or vertical, in accordance with our standard embedding of $(\mathbb{V}, \mathbb{E}_\square)$ in the plane. Then we mark each edge with the information of whether it is a stick, vacancy or regular edge in σ . Finally, we remove the regular edges (while keeping all vertices). We note for later use that every vertex in a configuration graph is either isolated, an internal vertex of a stick (in which case it has degree exactly 2) or incident to a vacancy in σ , and these cases are mutually exclusive. In particular, there are no vertices of degree exactly 1.

Define $\mathcal{H}$ to be the family of abstract marked graphs that may appear as finite connected components of a configuration graph. We emphasize that $\mathcal{H}$ includes the *trivial* graph, having a single vertex and no edges. The word “abstract” is used to signify that two elements of $\mathcal{H}$ are considered equal if they are isomorphic as marked directed graphs, which means that an isomorphism must preserve the directions and markings of the edges. Formally we write

$$\mathcal{H} := \{H : \text{there exist } \sigma \in \Omega \text{ such that } H \text{ is a finite connected component of } G_\sigma\}.$$

Each $H \in \mathcal{H}$ can be realized as a connected component of some G_σ , by definition, and each such realization yields an embedding of H in $\mathbb{R}^2$. By the *image* of such an embedding we mean the set in $\mathbb{R}^2$ formed by the union of all of its vertices and edges. For a non-trivial H , this is the same as the union of all edges.

Proposition 4.8. *Let $H \in \mathcal{H}$ be non-trivial. Suppose H appears as a connected component of both G_{σ_1} and G_{σ_2} , for some $\sigma_1, \sigma_2 \in \Omega$. Then the two resulting embeddings of H , as well as the vacancies and tiles of σ_1 and σ_2 bounding on the images of these embeddings, are the same up to a global translation.*

Proof. The embeddings are the same up to a global translation since H is connected and the vector in $\mathbb{R}^2$ pointing from the head of an edge in G_σ to its tail is uniquely determined by its marking as “horizontal” or “vertical”. To show that the vacancies and tiles bounding on the images of the embeddings are the same up to a global translation, let us fix a G_σ for which H is a connected component and a face f bounding on an edge e of the resulting embedding, and explain how the information on the embedding uniquely determines whether the face is a vacancy or part of a tile in σ , and in the latter case, the parity of the tile.

It is easy to see that f is a vacancy in σ if and only if it is surrounded by vacancy edges (all of which are necessarily in H). Thus suppose, without loss of generality, that e is horizontal, that f is the face directly above it, and that f is in a tile. Denote the other three edges in $\mathbb{E}_\square$ incident to the right end of e by e_1, e_2, e_3 , in clockwise order. Consider the first of these edges that appears in G_σ . If it is e_1 or e_3 , a case analysis shows that the

tile covering f has its center above the left end of e . If it is e_2 , then there is also a tile directly above e_2 , and the parity of this tile is the same as that of the tile directly above e (they may or may not be the same tile). By an inductive argument (as H is finite), we may assume that the tile directly above e_2 is already known. ■

4.2.2. *The partition function of $\mathcal{H}_M$.* For $M \geq 1$, define $\mathcal{H}_M \subset \mathcal{H}$ by

$$\begin{aligned} \mathcal{H}_M &:= \{H \in \mathcal{H} : \text{paths of stick edges in } H \text{ have length at most } M\} \\ &= \{H : \text{there exist } \sigma \in E_M \text{ such that } H \text{ is a finite connected component of } G_\sigma\}. \end{aligned}$$

Consider G_σ for some $\sigma \in \Omega$. Note that the four bounding edges of a vacancy are necessarily in the same component of G_σ ; we then say that the vacancy *belongs* to the component of its bounding edges. For a finite component H of G_σ , denote by v_H the number of vacancies that belong to it (see Figure 4.3). By Proposition 4.8, we may define v_H for an abstract $H \in \mathcal{H}$, without mention of σ . We define the *weight* of H to be $\lambda^{-v_H/4}$. We will deduce Lemma 4.6 from the following bound on the total weight of $\mathcal{H}_M$.

Lemma 4.9. *There exists $C > 0$ such that for every $\lambda > 0$ and $M \geq 1$, if $M < \lambda^{1/2}/C$ then*

$$\sum_{H \in \mathcal{H}_M} \lambda^{-v_H/4} \leq 1 + \frac{CM}{\lambda}.$$

We again remark that the bound is sharp, up to the value of C , since $\mathcal{H}_M$ contains the trivial graph, and also all marked graphs with two vertical sticks of equal length $1 \leq k \leq M$ bounded by a pair of vacancies at both ends.

Before proving the lemma, let us explain how it implies the main result of this subsection.

Proof of Lemma 4.6. Fix an even rectangle Λ , $\lambda > 0$ and $M \geq 1$. For each $H \in \mathcal{H}_M$ we designate one vertex as the root, with the designation being arbitrary except for the requirement that if H is non-trivial then the root is not the head of a directed edge. It is possible to satisfy this requirement since H cannot have a directed cycle (as edges are directed right/up).

For every $\sigma \in E_M \cap \Omega_\Lambda^1$, define a function $f_\sigma : \mathbb{V} \rightarrow \mathcal{H}_M$ as follows. For $v \in \mathbb{V}$, if v happens to be the root of a non-trivial component H of G_σ then $f_\sigma(v) = H$. Otherwise $f_\sigma(v)$ is the trivial graph. We will rely on the fact that f_σ determines σ , which is implied by Proposition 4.8.

Since outside of Λ the configurations in Ω_Λ^1 are fully packed with tiles of the same parity, f_σ must assign the trivial graph to any v outside of $\mathbb{V} \cap \Lambda$. Taking into account the requirement on the root we see that f_σ is in fact constant outside of the set V , defined to be the set of lower left corners of faces inside Λ (vertices at the right or top boundaries of Λ which are in non-trivial components of G_σ are necessarily heads of directed edges since their degree in G_σ must be at least 2, while their out-degree is at most 1).

Now

$$\begin{aligned}
 Z_{\Lambda}^1(E_M) &= \sum_{\sigma \in E_M \cap \Omega_{\Lambda}^1} w_{\Lambda, \lambda}(\sigma) \\
 &\stackrel{(1)}{\leq} \sum_{f: V \rightarrow \mathcal{H}_M} \prod_{v \in V} \lambda^{-v_{f(v)}/4} \\
 &= \left(\sum_{H \in \mathcal{H}_M} \lambda^{-v_H/4} \right)^{\#V} \stackrel{(2)}{\leq} \left(1 + \frac{CM}{\lambda} \right)^{\text{Area}(\Lambda)},
 \end{aligned}$$

where (1) holds because the mapping $\sigma \mapsto f_{\sigma|V}$ is injective and $w_{\Lambda, \lambda}(\sigma) = \prod_{v \in V} \lambda^{-v_{f_{\sigma}(v)}/4}$, and (2) holds by Lemma 4.9 and $\#V = \text{Area}(\Lambda)$. ■

The remainder of the subsection is devoted to the proof of Lemma 4.9.

4.2.3. Lower bounds on v_H . We proceed to obtain lower bounds for the number of vacancies belonging to a non-trivial $H \in \mathcal{H}$. We first prove a simple lower bound, showing that $v_H \geq 4$, and then prove a more involved lower bound in terms of the number of vertical and horizontal subcomponents of H (as defined below).

Proposition 4.10. *If $H \in \mathcal{H}$ is non-trivial then $v_H \geq 4$.*

Proof. Consider H as a component of G_{σ} for some σ . Since the end of a stick is necessarily a corner of a vacancy, at least one vacancy must belong to H . Among the leftmost vacancies of H , the topmost one must differ from the bottommost one, since otherwise there is a unique leftmost vacancy f , and this is not possible. Indeed, assuming this by contradiction, a case analysis shows that from the edge bounding on the left of f must extend a vertical stick, either from the bottom or from the top vertex of the edge, and this stick must have a vacancy at its other end, contradicting that f is the unique leftmost vacancy.

Likewise there is no unique rightmost vacancy, no unique topmost vacancy, and no unique bottommost vacancy. Therefore the topmost rightmost vacancy, the topmost leftmost vacancy, the bottommost leftmost vacancy and the bottommost rightmost vacancy are all distinct from each other. ■

Let H be a marked graph. Define H_{ver} as the graph obtained from H by removing its horizontal stick edges (while keeping all vertices and all vacancy edges). We define the *vertical subcomponents* of H to be the connected components of H_{ver} that are not trivial graphs (equivalently, the ones that have at least one edge), and denote their cardinality by $k_{H, \text{ver}}$. We note for later reference that for non-trivial H , if a vertex v is isolated in H_{ver} (i.e., its connected component is trivial) then it necessarily was an internal vertex of a horizontal stick in H (using the fact that H has no vertex of degree 1). We define H_{hor} , *horizontal subcomponents* of H and $k_{H, \text{hor}}$ analogously. Then define $k_H = k_{H, \text{ver}} + k_{H, \text{hor}}$ (see Figure 4.3).

Consider some $H \in \mathcal{H}$ as a component of G_σ for some σ . The bounding edges of a vacancy all belong to a single vertical subcomponent of H , and the same is true for a horizontal one. Therefore we may say that the vacancy *belongs* to a single vertical subcomponent and a single horizontal subcomponent.

Lemma 4.11. *Let H be a finite component of G_σ for some $\sigma \in \Omega$. Let A be a vertical subcomponent of H and let B be a horizontal subcomponent H . Suppose that A and B share a vertex. Then there are at least two vacancies of σ that belong to both A and B .*

Proof. The shared vertex must be incident to a vacancy edge for σ , otherwise the vertex is incident to both a horizontal and a vertical stick edge, and this is not possible. Therefore A shares at least one vacancy with B ; denote it by f . If f shares an edge with another vacant face, we are done since the two vacancies must belong to both A and B . Therefore we assume that f is an *isolated vacancy*.

Adjacent to f must be four tiles. We may assume without loss of generality (by applying reflections and translations) that they are arranged as in Figure 4.2. Additionally, since H is finite, we may assume (by modifying σ away from H) that $\sigma \in \Omega_\Lambda^1$ for some even rectangle Λ .

Consider the union U of all the tiles whose parity has an odd vertical component; this includes the deep blue and red tiles in the figure (recall the color convention introduced in Figure 2.1). The boundary of U is the image of a subgraph I of $(\mathbb{V}, \mathbb{E}_\square)$ (i.e., it is a union of points in $\mathbb{Z}^2$ and segments of length 1 connecting some of them). We observe that U is bounded, by the definition of Ω_Λ^1 , and necessarily $I \subset (G_\sigma)_{\text{ver}}$.

As the image of I is a boundary of a region in the plane, all the degrees of I are necessarily even. Therefore, any path in I whose internal vertices have degree 2 in I may be extended to a (simple) cycle in I . Let C^{ver} be the extension of the pink path in Figure 4.2 to a cycle in I . Then C^{ver} is a subgraph of A : this is because $C^{\text{ver}} \subset I \subset (G_\sigma)_{\text{ver}}$ and C^{ver} is connected and intersects A , which is a connected component of $(G_\sigma)_{\text{ver}}$.

Repeating the analogous steps with the yellow path in Figure 4.2 (whose image lies in the boundary of the union of tiles whose parity has an odd *horizontal* component) gives rise to a cycle C^{hor} which is a subgraph of B .

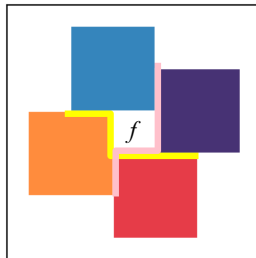


Fig. 4.2. The pink and yellow paths intersect near the vacant face f . They extend to cycles C^{ver} and C^{hor} , which must intersect near an additional vacancy. For better visibility, the paths are slightly offset.

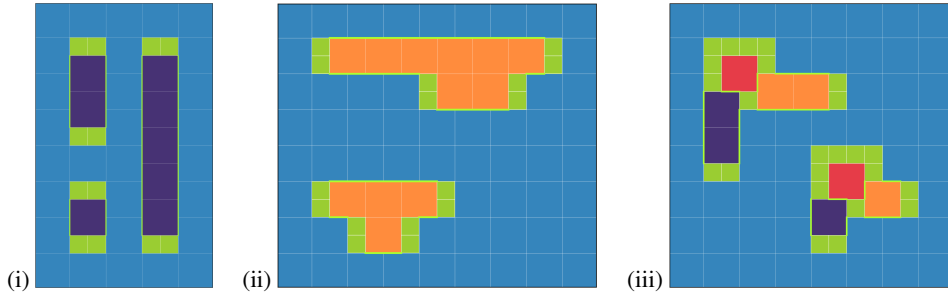


Fig. 4.3. Each image depicts several components with the same compressed version, with sticks and vacancies colored green. Below are the number of vacancies and number of vertical and horizontal subcomponents for a single component in each image:

- (i) $v_H = 4$, $k_H = 3$, $k_{H,\text{ver}} = 1$, $k_{H,\text{hor}} = 2$,
- (ii) $v_H = 8$, $k_H = 5$, $k_{H,\text{ver}} = 4$, $k_{H,\text{hor}} = 1$,
- (iii) $v_H = 12$, $k_H = 4$, $k_{H,\text{ver}} = 2$, $k_{H,\text{hor}} = 2$.

The cycles C^{ver} and C^{hor} must intersect in a shared vertex of A and B that is not a corner of f . This is because two cycles in the plane that intersect transversally at a point, must intersect at an additional point. The new shared vertex is necessarily a corner of a vacancy (as explained at the beginning of the proof), which is necessarily distinct from f . Thus there are at least two vacancies belonging to both A and B . ■

We proceed to deduce a lower bound for v_H which improves upon that of Proposition 4.10 when $k_H > 3$.

Corollary 4.12. *If $H \in \mathcal{H}$ is non-trivial then $v_H \geq 2(k_H - 1)$.*

Proof. Consider H as a component of G_σ for some σ .

We construct an auxiliary bipartite graph, whose vertices are the horizontal and the vertical subcomponents of H . A horizontal subcomponent is adjacent to a vertical one if they share at least one vertex.

Lemma 4.11 implies that two components are adjacent in the auxiliary graph iff there are at least *two* vacancies that belong to both components. This shows that v_H is at least twice the number of edges in the auxiliary graph (since each vacancy belongs to a unique vertical and a unique horizontal subcomponent).

By definition, k_H is the number of vertices in the auxiliary graph. Note that the auxiliary graph is connected since H is connected. Therefore the number of edges in the auxiliary graph is at least $k_H - 1$. Together with the previous paragraph, this gives the lemma. ■

4.2.4. Compressed graphs. For a marked graph H , which is either in $\mathcal{H}$ or is a vertical or horizontal subcomponent of a graph in $\mathcal{H}$, define its *compressed version* $\text{comp}(H)$ as follows: every stick (that is, a maximal path of stick edges) is replaced with a single directed edge pointing from the beginning of the path to its end, removing all the original

internal vertices (noting that such internal vertices necessarily have degree 2 in H). The new edge is marked “stick” and also “vertical” or “horizontal” in accordance with the stick that it replaced.

The idea here is that $\text{comp}(I) = \text{comp}(H)$ if I and H are “the same up to extending and contracting sticks” (see Figure 4.3). Lemma 4.14 roughly follows from the fact that $k_H - 2$ is the number of “degrees of freedom” in choosing an I with $\text{comp}(I) = \text{comp}(H)$.

Proposition 4.13. *Let $H \in \mathcal{H}$ be non-trivial. There are exactly $k_{H,\text{ver}}$ connected components in $\text{comp}(H)_{\text{ver}}$, all of which are non-trivial. The analogous claim holds for horizontal subcomponents, so that, in particular, $k_{\text{comp}(H)} = k_H$.*

Proof. Recall that, since H is non-trivial, the only trivial connected components in H_{ver} arise from internal vertices of horizontal sticks in H . This implies that all connected components of $\text{comp}(H)_{\text{ver}}$ are non-trivial. It remains to check that for each vertex $v \in \text{comp}(H)$, the connected component of v in $\text{comp}(H)_{\text{ver}}$ necessarily equals the compressed version of the connected component of v in H_{ver} . The analogous claims hold for horizontal subcomponents. ■

Lemma 4.14. *Let $M \geq 1$ and let $H \in \mathcal{H}$ be non-trivial. Then*

$$\#\{I \in \mathcal{H}_M : \text{comp}(I) = \text{comp}(H)\} \leq M^{k_H-2}.$$

Proof. Given $I \in \mathcal{H}$ we may assign lengths to the stick edges of $\text{comp}(I)$ such a way that each is assigned the length of the path that it replaced. Then $\text{comp}(I)$ together with these lengths contains sufficient information to reconstruct I .

Fix $M \geq 1$ and $H \in \mathcal{H}$. An assignment of lengths to the stick edges of $\text{comp}(H)$ is termed *valid* if it arises from some $I \in \mathcal{H}$ with $\text{comp}(I) = \text{comp}(H)$. If, additionally, all the assigned lengths are at most M , then we say the assignment is *M-valid*. Thus, the lemma will follow once we prove that the number of *M-valid* length assignments to $\text{comp}(H)$ is at most M^{k_H-2} . We will show that

$$\text{there are at most } M^{k_{H,\text{ver}}-1} \text{ possibilities for the restriction of an } M\text{-valid length assignment to the horizontal stick edges of } \text{comp}(H). \quad (4.5)$$

This, together with the analogous statement for the restriction to the vertical stick edges will imply the lemma (recalling that $k_H = k_{H,\text{ver}} + k_{H,\text{hor}}$).

We first make the following observation. Suppose $\text{comp}(H)$ is endowed with a valid length assignment, and assign length 1 to all the vacancy edges of $\text{comp}(H)$. Consider a closed walk on $\text{comp}(H)$. Then the sum of signed lengths of horizontal edges in the walk (where edges that are walked in the opposite direction are counted with a minus sign) necessarily equals zero, since it represents the total horizontal movement for a closed walk on a component of G_σ for some σ .

Now consider a maximal spanning forest of $\text{comp}(H)_{\text{ver}}$, having $k_{H,\text{ver}}$ components by Proposition 4.13. As a spanning forest of $\text{comp}(H)$, it may be extended to a spanning

tree of $\text{comp}(H)$, by adding $k_{H,\text{ver}} - 1$ edges of $\text{comp}(H)$. Denote the set of added edges by E . There are at most $M^{k_{H,\text{ver}}-1}$ possibilities for the restriction to E of an M -valid length assignment, as the assigned length of each edge is in $\{1, \dots, \lfloor M \rfloor\}$.

We now prove that the restriction to the horizontal edges of $\text{comp}(H)$ of a valid length assignment is determined by its restriction to E , which proves (4.5) and thus finishes the proof of the lemma. By construction, the only horizontal edges in the spanning tree are vacancy edges and the edges in E . Thus, a length assignment to the edges in E determines the length of all horizontal edges in the spanning tree. Lastly, any horizontal edge in $\text{comp}(H)$ which is not in the tree, necessarily closes a cycle with the edges in the tree and thus its length is determined by the lengths of the horizontal edges in the tree by the observation above. ■

4.2.5. *Proof of Lemma 4.9.* For $v \geq 4$, denote

$$\mathcal{H}'_v := \{\text{comp}(H) : H \in \mathcal{H} \text{ and } v_H = v\}.$$

We claim that $\#\mathcal{H}'_v$ grows at most exponentially in v , say

$$\#\mathcal{H}'_v \leq C_1^v. \quad (4.6)$$

This follows from the following two facts: The number of (unlabeled, simple) planar graphs on v vertices grows at most exponentially with v [77], which implies the same for marked planar graphs (as the number of edges of a planar graph is at most a constant times its number of vertices). The number of vertices in $\text{comp}(H)$, for a non-trivial $H \in \mathcal{H}$, is at most $4v_H$ (since, in the realization of H as a component of some G_σ , every vertex which is not an internal vertex of a stick is incident to a vacancy).

Let $C := \max\{4C_1^2, 2C_1^4\}$. Fix some $\lambda > 0$ and $M \geq 1$ satisfying $M < \lambda^{1/2}/C$, so that in particular $C_1\lambda^{-1/4}M^{1/2} < 1/2$. Then

$$\begin{aligned} \sum_{H \in \mathcal{H}_M} \lambda^{-v_H/4} &= \\ \left(\begin{array}{c} \text{by Proposition 4.10,} \\ \text{and summing over possibilities} \\ \text{for the compressed version of } H \end{array} \right) &= 1 + \sum_{v \geq 4} \sum_{H' \in \mathcal{H}'_v} \sum_{\substack{H \in \mathcal{H}_M \\ \text{comp}(H) = H'}} \lambda^{-v/4} \\ \left(\begin{array}{c} \text{by Lemma 4.14} \\ \text{and Proposition 4.13} \end{array} \right) &\leq 1 + \sum_{v \geq 4} \sum_{H' \in \mathcal{H}'_v} \lambda^{-v/4} M^{k_{H'}-2} \\ \left(\begin{array}{c} \text{by Proposition 4.13,} \\ \text{Corollary 4.12 and } M \geq 1 \end{array} \right) &\leq 1 + \sum_{v \geq 4} \sum_{H' \in \mathcal{H}'_v} \lambda^{-v/4} M^{v/2-1} \\ \text{(by inequality (4.6))} &\leq 1 + \sum_{v \geq 4} C_1^v \lambda^{-v/4} M^{v/2-1} \\ \text{(rearrangement)} &\leq 1 + M^{-1} \sum_{v \geq 4} (C_1 \lambda^{-1/4} M^{1/2})^v \\ \left(\begin{array}{c} \text{using } C_1 \lambda^{-1/4} M^{1/2} < 1/2 \\ \text{and summing a geometric series} \end{array} \right) &\leq 1 + 2C_1^4 \lambda^{-1} M \leq 1 + \frac{CM}{\lambda}. \quad \blacksquare \end{aligned}$$

4.3. Configurations mostly without long sticks

Recall that E_M is the event that all sticks have length at most M , and that the weight of E_M under fully packed boundary conditions was estimated in Lemma 4.6. The proof of our main lemma, Lemma 4.1, requires an extension of Lemma 4.6 in which the weight of a larger event is estimated. The larger event is parameterized by a collection of horizontal and vertical line segments and consists of configurations in which all sticks are of length at most M , *except* maybe the sticks contained in one of the segments of the collection. We proceed to describe this extension.

Let $M \geq 1$ and let A be a collection of vertical and horizontal line segments of the form $\{x = x_0, y_0 \leq y \leq y_1\}$ or $\{y = y_0, x_0 \leq x \leq x_1\}$ with $x_0, y_0, x_1, y_1 \in \mathbb{Z}$. Define $E_{M,A}$ to be the event that every stick longer than M is fully contained in one of the segments in A . Denote by $\text{len}(A)$ the total length of the segments in A .

Proposition 4.15. *There exists $C \geq 1$ such that the following holds for all fugacities $\lambda > 0$. Let Λ be an even rectangle. Let $M > C$ and let A be a collection of line segments as above. Then*

$$Z_{\Lambda}^1(E_{M,A}) \leq \exp\left(\frac{C\lambda}{M^3} \text{len}(A)\right) Z_{\Lambda}^1(E_M). \quad (4.7)$$

Proof. Fix M , A and Λ as above. For the span of this proof, we say that a stick or segment is *long* if its length is more than M .

We assume without loss of generality that all segments in A are contained in Λ , since replacing each segment in A by its intersection with Λ leaves the set $\Omega_{\Lambda}^1 \cap E_{M,A}$ unaltered. Similarly, assume that no segment in A lies at distance exactly 1 from an edge of Λ parallel to it (as a stick contained in such a segment implies that a tile is centered on a point in $\partial\Lambda$, violating the boundary conditions). We also assume that all line segments in A are long, because the event $E_{M,A}$ is invariant under the removal of segments from A whose length is at most M .

Choose a collection of vertical and horizontal line segments $I_1, \dots, I_N$ of length $\lceil \frac{1}{2}M \rceil$ whose endpoints are on $\mathbb{Z}^2$ and whose union equals the union of the segments in A , in such a way that

$$N \leq \frac{3\text{len}(A)}{M}.$$

One may check that if a long stick is contained in a segment of A then necessarily the stick contains one of $I_1, \dots, I_N$.

For each $1 \leq i \leq N$, let D_i be the event that I_i is not contained in a long stick.

For $0 \leq k \leq N$, set $F_k := E_{M,A} \cap \bigcap_{i=1}^k D_i$, so that $F_0 = E_{M,A}$ and $F_N = E_M$. Observe that

$$\frac{Z_{\Lambda}^1(E_{M,A})}{Z_{\Lambda}^1(E_M)} = \prod_{i=1}^N \frac{Z_{\Lambda}^1(F_{i-1})}{Z_{\Lambda}^1(F_i)}. \quad (4.8)$$

We will show that for a sufficiently large universal constant C , for each $1 \leq i \leq N$ and $M > C$, we have

$$\frac{Z_{\Lambda}^1(F_{i-1})}{Z_{\Lambda}^1(F_i)} \leq 1 + \frac{C\lambda}{M^2}. \quad (4.9)$$

This suffices for the proposition, as substituting this bound into (4.8) implies (4.7) (with a larger C) by using the bound on N .

Fix $1 \leq i \leq N$. We will define a mapping $m : (F_{i-1} \setminus F_i) \rightarrow 2^{F_{i-1}}$ such that

$$m(\sigma_1) \cap m(\sigma_2) = \emptyset \quad \text{for distinct } \sigma_1, \sigma_2 \in F_{i-1} \setminus F_i, \quad (4.10)$$

and, for $M > C$,

$$Z_\Lambda^1(m(\sigma)) \geq \left(1 + \frac{M^2}{C\lambda}\right) w_\Lambda(\sigma) \quad \text{for } \sigma \in (F_{i-1} \setminus F_i) \cap \Omega_\Lambda^1. \quad (4.11)$$

The existence of m with these properties implies that

$$Z_\Lambda^1(F_{i-1}) \geq \left(1 + \frac{M^2}{C\lambda}\right) Z_\Lambda^1(F_{i-1} \setminus F_i)$$

by summing over (4.11). The last display and the fact that $F_i \subset F_{i-1}$ imply (4.9).

We proceed to define the mapping m . Assume without loss of generality that I_i is vertical. Choose J to be a rectangle contained in Λ with $\text{Width}(J) = 2$ and with one of its vertical sides coinciding with I_i (this is possible by the first two assumptions made at the beginning of the proof). Define $m(\sigma) := \Omega_J^\sigma$, where we recall from (2.1) that Ω_J^σ is the set of configurations which agree with σ on all tiles which are not fully contained in J . To show that m is correctly defined we must prove that $\Omega_J^\sigma \subset F_{i-1}$, which we shall do in the last two paragraphs of the proof.

To show (4.10), let $\sigma_1, \sigma_2 \in F_{i-1} \setminus F_i$, and assume that $\Omega_J^{\sigma_1} \cap \Omega_J^{\sigma_2} \neq \emptyset$. Then $\sigma_1|_{\text{int}(J)^c} = \sigma_2|_{\text{int}(J)^c}$. In particular, σ_1 agrees with σ_2 on the tiles bounding on I_i on the side opposite to J . Thus, using the fact that I_i is contained in a stick of both σ_1 and σ_2 (since $\sigma_1, \sigma_2 \notin D_i$) we deduce that $\sigma_1|_{\text{int}(J)} = \sigma_2|_{\text{int}(J)}$, whence $\sigma_1 = \sigma_2$.

We now show (4.11). Fix $\sigma \in (F_{i-1} \setminus F_i) \cap \Omega_\Lambda^1$. Let σ_0 be the configuration obtained from σ by removing all tiles fully contained in J and denote the union of these tiles by J_σ . Since $\sigma \notin D_i$, the rectangle J is fully covered by tiles of σ , whence J_σ is a rectangle of width 2 and its height is even and satisfies $\lceil \frac{1}{2}M \rceil - 2 \leq \text{Height}(J_\sigma) \leq \lceil \frac{1}{2}M \rceil$. We then observe that

$$m(\sigma) = \Omega_J^\sigma = \Omega_{J_\sigma}^\sigma = \sigma_0 + \Omega_{J_\sigma}^0 := \{\sigma_0 + \tilde{\sigma} : \tilde{\sigma} \in \Omega_{J_\sigma}^0\}. \quad (4.12)$$

We note that $w_\Lambda(\sigma_0 + \tilde{\sigma}) = w_\Lambda(\sigma)w_{J_\sigma}(\tilde{\sigma})$ for all $\tilde{\sigma} \in \Omega_{J_\sigma}^0$; also, $m(\sigma) \subset \Omega_\Lambda^1$ (since $\sigma \in \Omega_\Lambda^1$ and $J \subset \Lambda$) and thus

$$Z_\Lambda^1(m(\sigma)) = Z_\Lambda^0(J_\sigma)w_\Lambda(\sigma).$$

Recalling that $Z_\Lambda^0(J_\sigma) = Z_{\text{Height}(J_\sigma), 1D}^0$ and that $\text{Height}(J_\sigma)$ is even and at least $\frac{1}{2}M - 2$, we obtain (4.11) from the one-dimensional estimate of Proposition 4.4 by choosing C large enough and using the hypothesis $M > C$.

It remains to prove that $\Omega_J^\sigma \subset F_{i-1}$ for each $\sigma \in F_{i-1} \setminus F_i$. Fix $\sigma \in F_{i-1} \setminus F_i$ and some $\sigma' \in m(\sigma) = \Omega_{J_\sigma}^\sigma$. We have $\sigma \in F_{i-1}$, which means that each long stick of σ is contained in a segment of A , and does not contain any of $I_1, \dots, I_{i-1}$. We will show that

each long stick of σ' is contained in a stick of σ . Thus each long stick of σ' is contained in a segment of A , and does not contain any of $I_1, \dots, I_{i-1}$, whence $\sigma' \in F_{i-1}$.

We now show that each long stick of σ' is contained in a stick of σ . Indeed, every horizontal stick edge in σ' is a stick edge in σ . Every vertical stick edge in σ' that is not a stick edge in σ bounds on a tile contained in J_σ which does not appear in σ . As $\text{Height}(J_\sigma)$ is even, J_σ contains pairs of horizontally adjacent vacancies of σ' above and below that tile. This implies that such vertical stick edges are part of sticks whose length is less than $\text{Height}(J_\sigma) \leq \lceil \frac{1}{2}M \rceil \leq M$. ■

4.4. Proof of the main lemma

Proof of Lemma 4.1. Let $c > 0$ be a constant sufficiently small to satisfy some assumptions to follow. Let $S \subset R$ be rectangles satisfying the hypotheses (4.1) and (4.2) of the lemma. Note in particular that we may assume the fugacity λ large by taking c sufficiently small in (4.1).

Denote by f the indicator of the (R -local) event that no stick divides both R and S . Fix an arbitrary even rectangle Λ for which R is a block. Recalling (3.6), the definition of $\|f\|_R$, we aim to bound

$$\|f\|_{R|\Lambda} = \mu_\Lambda^{\text{per}} \left(\prod_{\tau \in T_\Lambda^R} \tau f \right)^{\frac{\text{Area}(R)}{\text{Area}(\Lambda)}}$$

and take the limit as $\text{Width}(\Lambda), \text{Height}(\Lambda) \rightarrow \infty$. We assume for convenience that R and Λ have their bottom left corner at the origin. Denote by $E_{R,S}$ the event that for all $\tau \in T^R$, there is no stick dividing both τR and τS , and observe that on $\Omega_\Lambda^{\text{per}}$, the function $\prod_{\tau \in T_\Lambda^R} \tau f$ is the indicator of $E_{R,S}$.

Recall Proposition 2.1 and the function $m^{\rho, \Lambda}$ defined there. We use this proposition to bound $\mu_\Lambda^{\text{per}}(E_{R,S})$. Fix $\rho \in \Omega$ to be the fully packed configuration $\rho(x, y) = \mathbf{1}_{x, y \equiv 1 \pmod{2}}$. By the proposition,

$$Z_\Lambda^{\text{per}}(E_{R,S}) \leq C_{2.1}(\lambda)^{\text{Perimeter}(\Lambda)} Z_\Lambda^1(m^{\rho, \Lambda}(E)).$$

Aiming to apply the bound of Proposition 4.15, we will choose M and A so that

$$m^{\rho, \Lambda}(E_{R,S}) \subset E_{M,A}. \quad (4.13)$$

We postpone the choice of M and assume for now

$$M \geq 2 \max \{ \text{Width}(R), \text{Height}(R) \}, \quad (4.14)$$

and let A be the set of integer translates of the sides of Λ which are both contained in Λ and disjoint from $\bigcup_{\tau \in T^R} \text{int}(\tau S)$.

Let us check that (4.13) holds. Indeed, let $\sigma \in E_{R,S}$ and denote $\sigma' = m^{\rho, \Lambda}(\sigma)$. To show that $\sigma' \in E_{M,A}$, we consider a stick s' of σ' with length more than M , and show that it is contained in a segment of A . Note that $s' \subset \Lambda$ by the choice of ρ . Assume without

loss of generality that s' is vertical and consider its extension to a horizontal translation of a vertical side of Λ . If this extension is a side of Λ then we are done, as it is an element of A . Otherwise, by the choice of ρ and using the fact that Λ is an even rectangle, every edge of s' is also a stick edge in σ , and thus s' is contained in a stick s of σ . The stick s is of length at least $2 \text{Height}(R)$ and thus must divide some rectangle τR or lie on its boundary. But, as $\sigma \in E_{R,S}$, it cannot divide τS . This implies that the extension of s' is disjoint from $\bigcup_{\tau \in T^R} \text{int}(\tau S)$ and thus an element of A . We conclude that (4.13) holds.

So far, we have

$$\|f\|_{R|\Lambda}^{\frac{\text{Area}(\Lambda)}{\text{Area}(R)}} = \mu_{\Lambda}^{\text{per}}(E_{R,S}) = \frac{Z_{\Lambda}^{\text{per}}(E_{R,S})}{Z_{\Lambda}^{\text{per}}} \leq \frac{C_{2.1}(\lambda)^{\text{Perimeter}(\Lambda)} Z_{\Lambda}^1(E_{M,A})}{Z_{\Lambda}^{\text{per}}}.$$

Putting $\Lambda = R_{n! \times n!}$ and taking the limit $n \rightarrow \infty$ we learn that

$$\|f\|_R = \limsup_{n \rightarrow \infty} \|f\|_{R|\Lambda} \leq \limsup_{\text{Width}(\Lambda), \text{Height}(\Lambda) \rightarrow \infty} \left(\frac{Z_{\Lambda}^1(E_{M,A})}{Z_{\Lambda}^{\text{per}}} \right)^{\frac{\text{Area}(R)}{\text{Area}(\Lambda)}} \quad (4.15)$$

and we are left with bounding the right-hand side.

Combining the main results of the previous three subsections: Corollary 4.3, Lemma 4.6 and Proposition 4.15, we have

$$\frac{Z_{\Lambda}^1(E_{M,A})}{Z_{\Lambda}^{\text{per}}} \leq \frac{\exp\left(\frac{C_{4.15}\lambda}{M^3} \text{len}(A)\right) \left(1 + \frac{C_{4.6}M}{\lambda}\right)^{\text{Area}(\Lambda)}}{\exp(c_{4.3}\lambda^{-1/2} \text{Area}(\Lambda))} \quad (4.16)$$

given that we fix some $0 < c_{4.3} < 1/4$, say $c_{4.3} = 1/8$, and make the assumptions that λ is sufficiently large and that

$$C_{4.15} < M < \lambda^{1/2}/C_{4.6}. \quad (4.17)$$

Let us bound $\text{len}(A)$. The number of vertical segments in A is

$$\text{Width}(\Lambda) + 1 - [\text{Width}(S) - 1] \frac{\text{Width}(\Lambda)}{\text{Width}(R)},$$

thus their total length is

$$\begin{aligned} & \text{Area}(\Lambda) \left(1 - \frac{\text{Width}(S)}{\text{Width}(R)} + \frac{1}{\text{Width}(R)} + \frac{1}{\text{Width}(\Lambda)} \right) \\ & \leq \text{Area}(\Lambda) \left(1 - (1 - c) + \frac{1}{1/c} + \frac{1}{2/c} \right) \leq 3c \text{Area}(\Lambda), \end{aligned}$$

using the hypotheses (4.2) and (4.1) and the fact that R is a block of Λ . A similar bound holds for the horizontal segments, and thus

$$\text{len}(A) \leq 6c \text{Area}(\Lambda).$$

We set

$$M = \frac{c_{4.3}}{2C_{4.6}} \lambda^{1/2}$$

and require c to be sufficiently small for (4.1) to imply (4.14), and λ to be sufficiently large to imply (4.17). Substitute the last two displays into (4.16) to obtain

$$\begin{aligned} \left(\frac{Z_{\Lambda}^1(E_{M,A})}{Z_{\Lambda}^{\text{per}}} \right)^{\frac{\text{Area}(R)}{\text{Area}(\Lambda)}} &\leq \exp \left(\left[6c C_{4.15} \frac{\lambda^{3/2}}{M^3} + C_{4.6} M \lambda^{-1/2} - c_{4.3} \right] \text{Area}(R) \lambda^{-1/2} \right) \\ &\leq \exp \left(\left[6c C_{4.15} \frac{\lambda^{3/2}}{M^3} - c_{4.3}/2 \right] \text{Area}(R) \lambda^{-1/2} \right) \\ &\leq \exp(-c \text{Area}(R) \lambda^{-1/2}), \end{aligned}$$

where c is chosen sufficiently small for the last inequality. Combining this with (4.15) we get the lemma. $\blacksquare$

5. Existence of multiple Gibbs measures

In this section we prove Theorem 5.2 which shows, for periodic Gibbs measures with sufficiently large fugacity, that almost surely exactly one of two symmetric invariant events holds. Corollary 5.3 follows from this the non-uniqueness of Gibbs measures.

For the rest of the paper, fix some integer $N > 2$ such that $2/N \leq c_{4.1}$ (where $c_{4.1}$ is the constant from the statement of Lemma 4.1). For a rectangle R with dimensions divisible by N , denote by R^- the rectangle sharing its center with R and having

$$(\text{Width}(R^-), \text{Height}(R^-)) = \frac{N-2}{N} (\text{Width}(R), \text{Height}(R)).$$

Observe that our choice of N ensures that the assumption (4.2) of Lemma 4.1 is satisfied when $S = R^-$. We say that a stick *divides R properly* if it divides both R and R^- (see Figure 4.1).

For $K, L \in \mathbb{N}$ and a configuration $\sigma \in \Omega$, define a set $\Psi_{(\text{ver},0)}^{K \times L}(\sigma) \subset \mathbb{V}$ as follows: for $(x, y) \in \mathbb{V}$, set $R = R_{KN \times LN, (xK, yL)}$ and say that (x, y) is in $\Psi_{(\text{ver},0)}^{K \times L}(\sigma)$ if R is divided properly by some $(\text{ver}, 0)$ stick of σ . We make three analogous definitions by putting $(\text{ver}, 1)$, $(\text{hor}, 0)$ or $(\text{hor}, 1)$ instead of $(\text{ver}, 0)$ in the definition above. Also define

$$\begin{aligned} \Psi_{\text{ver}}^{K \times L}(\sigma) &:= \Psi_{(\text{ver},0)}^{K \times L}(\sigma) \cup \Psi_{(\text{ver},1)}^{K \times L}(\sigma), \\ \Psi_{\text{hor}}^{K \times L}(\sigma) &:= \Psi_{(\text{hor},0)}^{K \times L}(\sigma) \cup \Psi_{(\text{hor},1)}^{K \times L}(\sigma), \\ \Psi^{K \times L}(\sigma) &:= \Psi_{\text{ver}}^{K \times L}(\sigma) \cup \Psi_{\text{hor}}^{K \times L}(\sigma). \end{aligned}$$

The following lemma is key to our use of the Peierls argument. It shows that regions with long vertical sticks must be separated from regions with long horizontal sticks.

Lemma 5.1. *Let $K, L \in \mathbb{N}$ and $\sigma \in \Omega$. If $u \in \Psi_{\text{ver}}^{K \times L}(\sigma)$ and $v \in \Psi_{\text{hor}}^{K \times L}(\sigma)$ then u, v are not neighbors in $(\mathbb{V}, \mathbb{E}_{\square})$.*

Proof. Assume $u \in \Psi_{\text{ver}}^{K \times L}(\sigma)$ and $v \in \Psi_{\text{hor}}^{K \times L}(\sigma)$ and assume by contradiction that u, v are neighbors in $(\mathbb{V}, \mathbb{E}_{\square})$. The situation has enough symmetry that we may assume that

$u = (0, 0)$ and $v = (0, 1)$. Then $(0, 0) \in \Psi_{\text{ver}}^{K \times L}(\sigma)$ implies that $R = R_{KN \times LN, (0, 0)}$ is divided by a vertical stick. Also, $(0, 1) \in \Psi_{\text{hor}}^{K \times L}(\sigma)$ implies that a horizontal stick divides $R_{KN \times LN, (0, L)}$ and $R_{K(N-2) \times L(N-2), (K, 2L)}$. This horizontal stick must also divide R , and thus intersects the vertical stick. As sticks cannot intersect, this is a contradiction. ■

In the next theorem, as well as in many of our later uses, the discussion focuses on a Gibbs measure μ , and the notation $\Psi_{\text{ver}}^{K \times L}$ (and its relatives) is used without explicit mention of σ . In these cases it is understood that σ is randomly sampled from μ .

Theorem 5.2. *There are $C, c > 0$ such that the following holds. Let $b \in \mathbb{N}$ satisfy $C\lambda^{1/4} < b < c\lambda^{1/2}$. Let μ be a periodic Gibbs measure. Then*

$$\mu(\text{exactly one of } \Psi_{\text{ver}}^{b \times b} \text{ and } \Psi_{\text{hor}}^{b \times b} \text{ has an infinite } \square\text{-component}) = 1. \quad (5.1)$$

Proof. We will prove that for each $C_0 > 0$, under the hypotheses of the theorem, for each finite $A \subset \mathbb{V}$ we have

$$\mu(A \cap \Psi^{b \times b} = \emptyset) \leq e^{-C_0 \#A}. \quad (5.2)$$

Taking C_0 large enough, by the Peierls argument, (5.2) implies that $\Psi^{b \times b}$ almost surely contains a unique infinite $\square$ -component I (we assume here familiarity with the Peierls argument; however, Lemma 6.2 below provides a proof). The infinite component I must be contained in either $\Psi_{\text{ver}}^{b \times b}$ or $\Psi_{\text{hor}}^{b \times b}$ since their union is $\Psi^{b \times b}$, and for $u \in \Psi_{\text{ver}}^{b \times b}$, $v \in \Psi_{\text{hor}}^{b \times b}$ it cannot be that $(u, v) \in \mathbb{E}_{\square}$ by Lemma 5.1. Thus it remains to prove (5.2).

Choose $c = c_{4.1}/N$ and let $C > 0$ be a constant, sufficiently large to satisfy some assumptions to follow. Let $b \in \mathbb{N}$, $\lambda > 0$ satisfy $C\lambda^{1/4} < b < c\lambda^{1/2}$, and let μ be a periodic Gibbs measure for the fugacity λ . Fix a finite $A \subset \mathbb{V}$.

Choose $v \in \{0, \dots, N-1\}^2$ such that $A' := (v + N\mathbb{Z}^2) \cap A$ satisfies $\#A' \geq \#A/N^2$. Set $R = R_{bN \times bN, bv}$. Lemma 4.1 is applicable to R and $S = R^-$. Indeed, (4.2) holds by the choice of N , the right part of (4.1) holds by the choice of c , and the left part of (4.1) then holds for sufficiently large C (as $b > C\lambda^{1/4} > C^2/c$). Thus the lemma yields

$$\|f\|_R \leq e^{-c_{4.1} \text{Area}(R)\lambda^{-1/2}} = e^{-c_{4.1}(bN)^2\lambda^{-1/2}}$$

where f is the indicator of the event that R is not divided properly. The definition of A' implies that for each $u \in A'$, the indicator of the event that $u \notin \Phi^{b \times b}$ is of the form τf , where $\tau \in T^R$ is distinct for each u . By the infinite-volume chessboard estimate (Proposition 3.8), this says that

$$\begin{aligned} \mu(A \cap \Psi^{b \times b} = \emptyset) &\leq \mu(A' \cap \Psi^{b \times b} = \emptyset) \leq (\|f\|_R)^{\#A'} \\ &\leq e^{-c_{4.1}(bN)^2\lambda^{-1/2}\#A'} \leq e^{-c_{4.1}b^2\lambda^{-1/2}\#A}. \end{aligned}$$

Inequality (5.2) now follows from the assumption that $b > C\lambda^{1/4}$ when C is sufficiently large. ■

Corollary 5.3. *There exists λ_0 such that for all $\lambda > \lambda_0$ there are at least two periodic Gibbs measures.*

Proof. By compactness, there is a subsequence of $(\mu_{R_L \times L, (-L/2, -L/2)}^{\text{per}})_{L \in 2\mathbb{N}}$ which converges in distribution to a Gibbs measure μ . The periodic boundary conditions ensure that μ is $\mathbb{Z}^2$ -invariant.

Let λ be sufficiently large so that we may choose $b \in \mathbb{N}$ satisfying $C_{5.2}\lambda^{1/4} < b < c_{5.2}\lambda^{1/2}$. Let E_{ver} be the event that $\Psi_{\text{ver}}^{b \times b}$ has an infinite $\square$ -component and define E_{hor} analogously. Theorem 5.2 shows that

$$\mu(\text{exactly one of } E_{\text{ver}} \text{ or } E_{\text{hor}} \text{ occurs}) = 1. \quad (5.3)$$

Assume without loss of generality that $\mu(E_{\text{ver}}) > 0$. Write μ_{ver} for the measure μ conditioned on E_{ver} . The fact that E_{ver} is a $b\mathbb{Z}^2$ -invariant event implies that μ_{ver} is a Gibbs measure, and is $b\mathbb{Z}^2$ -invariant.

Define $\mu_{\text{hor}} := \tau\mu_{\text{ver}}$ where τ is the reflection defined by $\tau(x, y) = (y, x)$. Then $\mu_{\text{ver}}, \mu_{\text{hor}}$ are $b\mathbb{Z}^2$ -invariant Gibbs measures and they are distinct since $\mu_{\text{hor}}(E_{\text{hor}}) = 1$ while $\mu_{\text{ver}}(E_{\text{hor}}) = 0$ by (5.3). ■

Part II

Characterization of the periodic Gibbs measures

In this part we show that at high fugacity the set of periodic Gibbs measures is the convex hull of exactly four periodic and extremal Gibbs measures, proving Theorem 1.2. We also investigate some properties of the extreme measures leading to a proof of Theorem 1.1.

6. Peierls-type arguments and strongly percolating sets

In this section we introduce the non-standard terminology of ε -strongly percolating sets. We use it to state and prove the Peierls argument and some related propositions. The reason for doing so is that this terminology will allow us in later sections to easily apply the Peierls argument repeatedly, and on grids with different spacing.

6.1. Definitions

Let B be a random set (with respect to a measure $\mathbb{P}$), and let $\varepsilon \geq 0$. Say that B is ε -rare if for every finite set A ,

$$\mathbb{P}(A \subset B) \leq \varepsilon^{\#A}.$$

For the rest of the paper, fix

$$\varepsilon_0 = 1/21. \quad (6.1)$$

For a random set $\Psi \subset \mathbb{V}$ say that Ψ is ε -strongly percolating if either $\varepsilon \geq \varepsilon_0$ or there is an ε -rare set B (on the same probability space) such that Ψ almost surely contains an infinite $\square$ -component of $\mathbb{V} \setminus B$. If Ψ is ε -strongly percolating for some $0 < \varepsilon < \varepsilon_0$, we

say that Ψ is strongly percolating (if $\varepsilon \geq \varepsilon_0$, the statement that Ψ is ε -strongly percolating is vacuous). We denote

$$\mathfrak{p}_{\mathbb{P}}(\Psi) = \inf \{ \varepsilon \geq 0 : \Psi \text{ is } \varepsilon\text{-strongly percolating} \}$$

and usually omit $\mathbb{P}$ from the notation.

To get a feeling for the above definitions, note the following two points: An ε -strongly percolating set is a random set, which necessarily contains an infinite $\square$ -component if $\varepsilon < \varepsilon_0$ (while nothing is guaranteed if $\varepsilon \geq \varepsilon_0$). In addition, the smaller ε is, the “larger” the ε -strongly percolating set is, in the sense that the condition of being ε -strongly percolating becomes stricter as ε decreases. With this in mind, $\mathfrak{p}_{\mathbb{P}}(\Psi)$ is a “measure of the size of Ψ ”, with smaller values corresponding to larger size.

For sets $U, V, B \subset \mathbb{V}$, we say for a set B that it $\square$ -separates U from V if there is no $\square$ -path starting in U , ending in V , and contained in $\mathbb{V} \setminus B$.

The following additional definitions come into play near the end of the section. Recall from Section 2.1 that translation by a vector v is denoted by η_v . For an event $E \subset \Omega$, define the random set $\bar{E} : \Omega \rightarrow 2^{\mathbb{V}}$ by

$$\bar{E}(\sigma) := \{v \in \mathbb{V} : \sigma \in \eta_v E\}.$$

Intuitively, E should be thought of as a local property of the configuration and $\bar{E}$ is the random set of positions where this local property holds.

For a set (or a random set) Ψ , define

$$X_{K \times L} \Psi := \{(x, y) \in \mathbb{V} : (Kx, Ly) \in \Psi\}.$$

In other words, $X_{K \times L} \Psi$ is formed by restricting Ψ to the grid $K\mathbb{Z} \times L\mathbb{Z}$ and then rescaling so that this grid becomes $\mathbb{V}$.

6.2. Reformulation of the Peierls argument

We now formulate and prove several Peierls-type results using the terminology of strongly percolating sets.

The following lemma is a standard fact on the connectivity of separating sets in $\mathbb{V}$. We provide a proof for completeness, following the ideas of Timár [76], as we do not have a reference for the precise result that we need. The lemma is similar to a special case of [76, Theorem 3].

Proposition 6.1 (Connectivity of minimal separators). *Let $U, V \subset \mathbb{V}$ be two $\boxtimes$ -connected sets and let $B \subset \mathbb{V}$ be a minimal (with respect to inclusion) set that $\square$ -separates U from V . Then B is $\boxtimes$ -connected.*

Proof. Introduce two auxiliary vertices u, v and consider the graph

$$G_{U,V} = (\mathbb{V} \cup \{u, v\}, \mathbb{E}_{\square} \cup \{uw : w \in U\} \cup \{vw : w \in V\} \cup \mathbb{E}_{\boxtimes}[U] \cup \mathbb{E}_{\boxtimes}[V])$$

where $E[A]$ stands for the set of edges in E with both endpoints in A . Note that B is a minimal set separating u from v in $G_{U,V}$.

Define $\mathcal{C}$ to be the set of cycles consisting of the 4-cycles in $(\mathbb{V}, \mathbb{E}_{\square})$ and all the triangles in $G_{U,V}$. Let us show that $\mathcal{C}$ generates the cycle space of $G_{U,V}$ (the set of spanning subgraphs with even degrees, viewed as a vector space over the two-element finite field). Let C be an element of the cycle space of $G_{U,V}$. We show how to add to it cycles from $\mathcal{C}$ to obtain the empty graph. Whenever $\deg_C(u) \neq 0$, pick two neighbors of u in C , $u_1, u_2 \in U$. Since U is $\boxtimes$ -connected, there is a path in $\mathbb{E}_{\boxtimes}[U]$ from u_1 to u_2 . By adding, for each edge e in the path, the triangle incident to e and u , we decrease $\deg_C(u)$ by 2 without altering $\deg_C(v)$. Repeating this process, and its analog for v , we can make sure that C has no edges incident to u or v . Then, we may add to C triangles from $(\mathbb{V}, \mathbb{E}_{\boxtimes})$ until $C \subset \mathbb{E}_{\square}$. It is known that the cycle space of $(\mathbb{V}, \mathbb{E}_{\square})$ is generated by its 4-cycles. Thus we have shown that $\mathcal{C}$ generates the cycle space of $G_{U,V}$.

Our goal is to show that B is $\boxtimes$ -connected. Thus it suffices to take an arbitrary partition, $B = B_1 \uplus B_2$, and find an edge $w_1 w_2 \in \mathbb{E}_{\boxtimes}$ with $w_1 \in B_1$ and $w_2 \in B_2$.

Consider the set T_0 of edges incident to B in $G_{U,V}$. The set T_0 separates u from v , so let us choose a subset $T \subset T_0$ which is a minimal set of edges separating u from v . By the minimality of B , every vertex $w \in B$ is an endpoint of an edge in T .

If an edge in T is incident to both B_1 and B_2 , then it is in $\mathbb{E}_{\boxtimes}$ and we are done. Otherwise, let T_1, T_2 be the sets of edges in T incident to B_1, B_2 respectively; they are both non-empty and form a partition $T = T_1 \uplus T_2$. Thus by [76, Lemma 1], there is a cycle $C \in \mathcal{C}$, that contains edges e_1, e_2 from T_1 and T_2 respectively. The edges e_1, e_2 are respectively incident to vertices $w_1 \in B_1, w_2 \in B_2$. In particular, $w_1, w_2 \in \mathbb{V} \cap C$, and considering the case where C is a triangle and the case where C is a 4-cycle in $\mathbb{E}_{\square}$, it is obvious that $w_1 w_2 \in \mathbb{E}_{\boxtimes}$. ■

Lemma 6.2 (Peierls argument). *If $\varepsilon < \varepsilon_0$ and B is an ε -rare set, then $\mathbb{V} \setminus B$ almost surely has a unique infinite $\square$ -component I . Moreover, each $\boxtimes$ -component of $\mathbb{V} \setminus I$ is finite.*

Proof. Let B be an ε -rare set and assume $\varepsilon < \varepsilon_0$. We first show that $\mathbb{V} \setminus B$ almost surely has an infinite $\square$ -component. Denote $C_n = \{-n, -n+1, \dots, n\}^2$. Consider the random set D_n of all points from which a $\square$ -path to C_n exists in $\mathbb{V} \setminus B$. Whenever this set is finite, there exists $m > n$ and a finite set $B' \subset B$ minimal among the sets separating C_n from $\mathbb{V} \setminus C_m$. Such B' is $\boxtimes$ -connected by Proposition 6.1, and must contain points $(-s, 0)$, $(t, 0)$ for some s, t between n and $m+1$. Thus B must contain a $\boxtimes$ -path of length at least $n+s$ starting at $(-s, 0)$ for some $s \geq n$. Using the fact that B is ε -rare and summing over such paths gives

$$\mathbb{P}(D_n \text{ is finite}) \leq \sum_{s=n}^{\infty} 8 \cdot 7^{n+s-1} \varepsilon^{n+s+1} \xrightarrow{n \rightarrow \infty} 0,$$

using $\varepsilon < \varepsilon_0 \leq 1/7$. Thus almost surely D_n is infinite for some n . Whenever D_n is infinite, there is some point in C_n that is contained in an infinite $\square$ -connected component of $\mathbb{V} \setminus B$. Thus $\mathbb{V} \setminus B$ almost surely has an infinite $\square$ -component.

Let I be such an infinite $\square$ -component of $\mathbb{V} \setminus B$. Whenever $\mathbb{V} \setminus I$ has an infinite $\boxtimes$ -component J , by Proposition 6.1 there is a $\boxtimes$ -connected set $B' \subset B$ that $\square$ -separates I from J , and B' is infinite (it is easy to see that a set separating two infinite sets of vertices in $(\mathbb{V}, \mathbb{E}_{\square})$ must be infinite). But the probability that B contains an infinite $\boxtimes$ -connected set is 0 (since B is ε -rare for $\varepsilon < 1/7$). Thus $\mathbb{V} \setminus I$ has no infinite $\boxtimes$ -component. ■

The following is an immediate corollary of the definition of p and Lemma 6.2.

Corollary 6.3. *Let $\varepsilon \geq 0$. Let B be an ε -rare set. Then $\mathbb{V} \setminus B$ is ε -strongly percolating, and in particular $p(\mathbb{V} \setminus B) \leq \varepsilon$.*

The next lemma strengthens the Peierls-type result of Lemma 6.2 by bounding the probability that in the complement of a strongly percolating set, the connected component of a given point has a large diameter.

Lemma 6.4 (Quantitative Peierls argument). *Let $u \in \mathbb{V}$, let $d \geq 0$ and let $\Psi \subset \mathbb{V}$ be a random set. Let E be the event that there exists a $\boxtimes$ -path in $\mathbb{V} \setminus \Psi$ starting at u and ending at some point in $\{v \in \mathbb{V} : \|v - u\|_{\infty} \geq d\}$. Then*

$$\mathbb{P}(E) \leq (p(\Psi)/\varepsilon_0)^{1+d}.$$

Proof. It suffices to show $\mathbb{P}(E) \leq (\varepsilon/\varepsilon_0)^{1+d}$ for each $\varepsilon > p(\Psi)$. If $\varepsilon \geq \varepsilon_0$ there is nothing to prove. Otherwise, let B be an ε -rare set such that Ψ contains I , the infinite $\square$ -component of $\mathbb{V} \setminus B$ (it exists by Lemma 6.2).

When $u \in \mathbb{V} \setminus \Psi$, consider C , the $\boxtimes$ -connected component of $\mathbb{V} \setminus \Psi$ containing u . Since $p(\Psi) < \varepsilon_0$, C is almost surely finite, and it is $\square$ -separated from I by B (since B $\square$ -separates I from its complement). Let B' be some minimal subset of B that $\square$ -separates C from I . Then by Proposition 6.1, B' is $\boxtimes$ -connected.

Denote $u = (x_u, y_u)$. Denote by E_1 the event that $u \in \mathbb{V} \setminus \Psi$ and there exists a point $v = (x_v, y_v) \in C$ with $x_v - x_u \geq d$. Fix an outcome $\sigma \in E_1$. Consider the straight infinite $\square$ -paths extending from u to the left and from v to the right respectively. These paths intersect I by Lemma 6.2, thus since B' separates C from I , there must be $s, t \geq 0$ such that $(x_u - s, y_u), (x_v + t, y_v) \in B'$. Since B' is $\boxtimes$ -connected it contains a $\boxtimes$ -path connecting these two points, which is of length at least $s + d$. Summing over the possibilities for s and over $\boxtimes$ -paths of length $s + d$ starting at $(x_u - s, y_u)$, and using the fact that $B' \subset B$ and B is ε -rare, it follows that

$$\begin{aligned} 4\mathbb{P}(E_1) &\leq 4 \sum_{s=0}^{\infty} 8 \cdot 7^{d+s-1} \varepsilon^{d+s+1} \leq \frac{32}{49} \sum_{s=0}^{\infty} (7\varepsilon)^{d+s+1} \\ &= \frac{32}{49(1-7\varepsilon)} (7\varepsilon)^{d+1} \leq \left(\frac{\varepsilon}{\varepsilon_0} \right)^{d+1} \end{aligned}$$

where we have used $\varepsilon < \varepsilon_0$. Define E_2, E_3, E_4 in the same way as E_1 except that the condition $x_v - x_u \geq d$ is replaced by $x_v - x_u \leq -d$, $y_v - y_u \geq d$, or $y_v - y_u \leq -d$ respectively. The bound on $\mathbb{P}(E_1)$ applies analogously for $\mathbb{P}(E_2)$, $\mathbb{P}(E_3)$, and $\mathbb{P}(E_4)$. Since $E = E_1 \cup E_2 \cup E_3 \cup E_4$, the lemma follows from the bound. ■

6.3. Relations between random sets

In this section we show that the properties of being rare and strongly percolating are maintained under various set operations, provided the ε parameter in these properties is sufficiently small.

Lemma 6.5 (Union of rare sets). *Suppose B_i is ε_i -rare for $i \in \{1, \dots, k\}$. Then $\bigcup_{i=1}^k B_i$ is $(k \max_{i=1, \dots, k} \varepsilon_i^{1/k})$ -rare.*

Proof. Denote $B = \bigcup_{i=1}^k B_i$. Let $A \subset \mathbb{V}$. For a function $f : A \rightarrow \{1, \dots, k\}$, there is some i such that $\#f^{-1}(i) \geq \#A/k$, thus the probability that $v \in B_{f(v)}$ for each $v \in A$ is at most $\varepsilon_i^{\#A/k}$. By a union bound over all $f : A \rightarrow \{1, \dots, k\}$, the probability that $A \subset B$ is at most

$$\varepsilon := \left(k \max_{i \in \{1, \dots, k\}} \varepsilon_i^{1/k} \right)^{\#A},$$

thus B is ε -rare. ■

Lemma 6.6 (Intersection of strongly percolating sets).

$$p\left(\bigcap_{i=1}^k \Psi_i\right) \leq k \left[\max_{i=1, \dots, k} p(\Psi_i) \right]^{1/k} \quad \text{for every random } \Psi_1, \dots, \Psi_k \subset \mathbb{V}.$$

Proof. We may assume $k \geq 1$ since $p(\mathbb{V}) = 0$. Let $\varepsilon_i > p(\Psi_i)$ for $i \in \{1, \dots, k\}$ and set $\varepsilon := k \max_{i \in \{1, \dots, k\}} \varepsilon_i^{1/k}$. It suffices to prove that $\bigcap_{i=1}^k \Psi_i$ is ε -strongly percolating. We may assume that $\varepsilon < \varepsilon_0$, otherwise there is nothing to prove. In particular, since $\varepsilon_0 < 1$, for each $i \in \{1, \dots, k\}$ $\varepsilon_i < \varepsilon_0$. So by the definition of strong percolation we may fix random sets B_i and I_i such that B_i is ε_i -rare, I_i is an infinite $\square$ -component of $\mathbb{V} \setminus B_i$, and $I_i \subset \Psi_i$. Denote $B = \bigcup_{i=1}^k B_i$. By Lemma 6.5, B is ε -rare. Thus by Lemma 6.2 there is a set I which is an infinite $\square$ -component of $\mathbb{V} \setminus B$, such that all the $\boxtimes$ -components of $\mathbb{V} \setminus I$ are finite. For each $i \in \{1, \dots, k\}$ we have $I \subset \mathbb{V} \setminus B_i$, and since it is infinite and $\boxtimes$ -connected, it must be that $I \subset I_i$. Thus $I \subset \bigcap_{i=1}^k I_i \subset \bigcap_{i=1}^k \Psi_i$. ■

The next lemma considers “strong percolation of events on subgrids of $\mathbb{V}$ ”. The following is an intuitive description of items (1) and (2) therein: We associate to an event E two random sets, thought of as percolation processes: Given $k, K, l, L \in \mathbb{N}$ satisfying $k \mid K$ and $l \mid L$, we have the percolation $\Psi := X_{k \times l} \bar{E}$ corresponding to the $k\mathbb{Z} \times l\mathbb{Z}$ grid. In addition, we define a block percolation Ψ' corresponding to the $K\mathbb{Z} \times L\mathbb{Z}$ grid in which a point is open if all the points of Ψ in the corresponding block of the $k\mathbb{Z} \times l\mathbb{Z}$ grid are open. We show that sufficiently strong percolation of Ψ implies strong percolation of Ψ' , and vice versa.

Lemma 6.7. *Let $E \subset \Omega$ be an event. Let $k, K, l, L \in \mathbb{N}$ where $k \mid K$ and $l \mid L$. Denote*

$$H := \{\eta_{(x,y)} : (x, y) \in (k\mathbb{Z} \times l\mathbb{Z}) \cap ([0, K) \times [0, L))\}$$

and $r := \frac{KL}{kl} = \#H$. Denote

$$\Psi := X_{k \times l} \bar{E}, \quad \Psi'_\eta := X_{K \times L} \overline{\eta E} \text{ for each } \eta \in H, \quad \Psi' := \bigcap_{\eta \in H} \Psi'_\eta.$$

Then:

- (1) $\mathfrak{p}(\Psi') \leq r \mathfrak{p}(\Psi)$.
- (2) $\mathfrak{p}(\Psi) \leq \sqrt[r]{\mathfrak{p}(\Psi')}$.
- (3) $\mathfrak{p}(\Psi) \leq [r[\max_{\eta \in H} \mathfrak{p}(\Psi'_\eta)]^{1/r}]^{1/r}$.
- (4) If $E \subset \Omega$ is R -local for $R = \mathbf{R}_{K \times L, (0,0)}$, and E is invariant under reflections through the vertical and horizontal lines passing through the center of R , then for each periodic Gibbs measure μ ,

$$\mathfrak{p}_\mu(\Psi) \leq \sqrt[r]{\|E^c\|_R}. \quad (6.2)$$

Proof. Define $f : \mathbb{V} \rightarrow \mathbb{V}$ by $f(x, y) = (\lfloor xk/K \rfloor, \lfloor yl/L \rfloor)$ and note that

$$\Psi' = \{v \in \mathbb{V} : f^{-1}(v) \subset \Psi\}. \quad (6.3)$$

(1) Assume without loss of generality that $r \mathfrak{p}(\Psi) < \varepsilon_0$. Let ε satisfy $\mathfrak{p}(\Psi) < \varepsilon$ and $r\varepsilon < \varepsilon_0$. Choose random sets B, I such that B is ε -rare, $I \subset \Psi$, and I is a unique (by Lemma 6.2) infinite $\square$ -component of $\mathbb{V} \setminus B$. Then, by a union bound, the set $B' := f(B)$ is $r\varepsilon$ -rare. Thus by Lemma 6.2, $\mathbb{V} \setminus B'$ has an infinite $\square$ -component I' . It is easily seen that $f^{-1}(I')$ is infinite, $\square$ -connected and disjoint from $f^{-1}(B')$. Since $B \subset f^{-1}(B')$, it follows that $f^{-1}(I')$ is disjoint from B and thus $f^{-1}(I') \subset I$, as I is the unique infinite $\square$ -component of $\mathbb{V} \setminus B$. For each $v \in I'$, we have $f^{-1}(v) \subset I \subset \Psi$. Thus by (6.3), we have $I' \subset \Psi'$. The existence of B', I' as above shows that $\mathfrak{p}(\Psi') \leq r\varepsilon$.

(2) Assume without loss of generality that $\sqrt[r]{\mathfrak{p}(\Psi')} < \varepsilon_0$. Let ε satisfy $\mathfrak{p}(\Psi') < \varepsilon$ and $\sqrt[r]{\varepsilon} < \varepsilon_0$. Choose random sets B', I' such that B' is ε -rare, $I' \subset \Psi'$, and I' is an infinite $\square$ -component of $\mathbb{V} \setminus B'$. This easily implies that $f^{-1}(I')$ is an infinite $\square$ -connected component of $\mathbb{V} \setminus f^{-1}(B')$. Since B' is ε -rare, $f^{-1}(B')$ is $\sqrt[r]{\varepsilon}$ -rare. It remains to note that $f^{-1}(I') \subset f^{-1}(\Psi') \subset \Psi$ by (6.3). The existence of $f^{-1}(B'), f^{-1}(I')$ as above shows that $\mathfrak{p}(\Psi) \leq \sqrt[r]{\varepsilon}$.

(3) By Lemma 6.6, $\mathfrak{p}(\Psi') \leq r \sqrt[r]{\max \{\mathfrak{p}(\Psi'_\eta) : \eta \in H\}}$. The result follows by (2).

(4) Define $B := X_{k \times l} \bar{E}^c$. Denote $\varepsilon = \|E^c\|_R$. We will show that B is $\sqrt[r]{\varepsilon}$ -rare. This suffices by Corollary 6.3 since $\Psi = \mathbb{V} \setminus B$.

Let $A \subset \mathbb{V}$ be finite. Choose $\eta = \eta_{(x_0, y_0)} \in H$ such that for $G := (\frac{K\mathbb{Z} + x_0}{k} \times \frac{L\mathbb{Z} + y_0}{l})$ and $A' := A \cap G$ we have $\#A' \geq \#A/r$.

By the hypotheses of the current item, for each $v \in K\mathbb{Z} \times L\mathbb{Z} = G_R$, we have $\tau_{R,v} E^c = \eta_v E^c$ (recall G_R and $\tau_{R,v}$ from Section 3). Let σ be sampled from $\eta\mu$. Then the chessboard estimate for infinite volume (Lemma 3.8) implies that

$$X_{K \times L} \bar{E}^c(\sigma) = \{(x, y) \in \mathbb{V} : \eta\sigma \in \tau_{R, (Kx, Ly)} E^c\} \text{ is } \varepsilon\text{-rare}. \quad (6.4)$$

By the one-to-one mapping $m : \mathbb{V} \rightarrow G$, $m(x, y) = (\frac{Kx+x_0}{k}, \frac{Ly+y_0}{l})$ it follows that $G \cap B$ is ε -rare. Thus

$$\mu(A \subset B) \leq \mu(A' \subset G \cap B) \leq \varepsilon^{\#A'} \leq (\sqrt[\ell]{\varepsilon})^{\#A}. \quad \blacksquare$$

6.4. Splitting strongly percolating sets

The next proposition shows that, in an ergodic setting, when a strongly percolating set is split into two *separated* random sets, then one of the two resulting sets is itself strongly percolating (in particular, this set contains an infinite $\square$ -component with probability 1).

Proposition 6.8. *Let μ be an $\mathcal{L}$ -ergodic measure on Ω for some lattice $\mathcal{L} \subset K\mathbb{Z} \times L\mathbb{Z}$. Let $E, F \subset \Omega$ be events. Assume that*

$$E \cap \eta_{(K,0)}F = E \cap \eta_{(0,L)}F = \eta_{(K,0)}E \cap F = \eta_{(0,L)}E \cap F = \emptyset.$$

Then

$$\min \{p_\mu(X_{K \times L} \bar{E}), p_\mu(X_{K \times L} \bar{F})\} \leq p_\mu(X_{K \times L} \overline{E \cup F}).$$

Proof. Denote $\Psi_E := X_{K \times L} \bar{E}$, $\Psi_F := X_{K \times L} \bar{F}$, $\Psi := X_{K \times L} \overline{E \cup F} = \Psi_E \cup \Psi_F$, and $\varepsilon := p_\mu(X_{K \times L} \overline{E \cup F})$. If $\varepsilon \geq \varepsilon_0$ there is nothing to prove. Otherwise, there is a random ε -rare set B such that Ψ almost surely contains an infinite $\square$ -component I of $\mathbb{V} \setminus B$. By Lemma 6.2, the set Ψ almost surely has a unique infinite component; denote it by I' . The random set I' is defined from Ψ up to measure 0 (without dependence on B or I) and satisfies $I \subset I'$.

Thus the events $\{I' \subset \Psi_E\}$ and $\{I' \subset \Psi_F\}$ are properly defined and $\mathcal{L}$ -invariant up to measure 0. By the $\mathcal{L}$ -ergodicity they each have probability 0 or 1. The assumption of the proposition ensures that no element of Ψ_E is $\square$ -adjacent to an element of Ψ_F , and thus each component of Ψ is contained in either Ψ_E or Ψ_F . This holds in particular for the component I' , thus the union of the two events above holds almost surely, and as they are 0-1 events, one of them holds almost surely. Thus one of the events $\{I \subset \Psi_E\}$ and $\{I \subset \Psi_F\}$ holds almost surely, and this implies by definition that either Ψ_E or Ψ_F is ε -strongly percolating. $\blacksquare$

7. Four phases

In this section, we improve upon the result of Section 5. On the intuitive level, there it was shown that mesoscopic sticks in a configuration are either mostly vertical or mostly horizontal. Here we extend this, by showing that the offset of mesoscopic sticks (horizontal offset for vertical sticks, and vice versa) is either mostly even or mostly odd. We also get better quantitative control over the “density” of the sticks. We choose a length scale b comparable to $\lambda^{1/2}$, and take α to be a sufficiently large universal constant. We show that when sticks are mostly vertical and with even offset, most rectangles of dimensions $N\alpha \times Nb$ will be divided by a vertical stick of even offset (recall that N was defined as a universal constant).

The above is stated formally in Theorem 7.1, which is the main result of the section. This theorem gives quantitative results that will be used in later sections to prove the main theorems stated in the introduction. Additionally, it already implies an extension of Corollary 5.3: that for all sufficiently large λ , there is a set of four affinely independent periodic Gibbs measures.

The theorems are stated for ergodic Gibbs measures, sometimes requiring ergodicity with respect to a very sparse lattice such as $\mathfrak{b}!\mathbb{Z}^2$. All these theorems may be seen to have implications for any periodic Gibbs measure using the ergodic decomposition theorem (as will be done later, in the proof of item 4 of Lemma 8.1). Also note that we could have replaced $\mathfrak{b}!\mathbb{Z}^2$ by any lattice $\mathcal{L} \subset \mathfrak{b}!\mathbb{Z}^2$.

Recall from Section 5 the definitions of N and $\Psi^{K \times L}, \Psi_{\text{ver}}^{K \times L}, \Psi_{\text{hor}}^{K \times L}, \Psi_{(\text{ver}, 0)}^{K \times L}, \Psi_{(\text{ver}, 1)}^{K \times L}, \Psi_{(\text{hor}, 0)}^{K \times L}, \Psi_{(\text{hor}, 1)}^{K \times L}$. Recall also ε_0 from (6.1). Define

$$D^{K \times L} := \{\sigma \in \Omega : (0, 0) \in \Psi^{K \times L}(\sigma)\}.$$

Equivalently, $D^{K \times L}$ is the event that the rectangles

$$R = R_{NK \times NL, (0, 0)} \quad \text{and} \quad R^- = R_{(N-2)K \times (N-2)L, (K, L)}$$

are both divided by a stick. Then for each $(x, y) \in \mathbb{V}$, the event $\eta_{(Kx, Ly)} D^{K \times L}$ holds iff $(x, y) \in \Psi^{K \times L}$, thus $\Psi^{K \times L} = X_{K \times L} \overline{D^{K \times L}}$. Similarly define

$$D_{\pi}^{K \times L} = \{\sigma \in \Omega : (0, 0) \in \Psi_{\pi}^{K \times L}(\sigma)\}$$

for $\pi \in \{\text{ver}, \text{hor}, (\text{ver}, 0), (\text{ver}, 1), (\text{hor}, 0), (\text{hor}, 1)\}$ and note that statements analogous to those made for $D^{K \times L}$ hold for $D_{\pi}^{K \times L}$, assuming that K and L are even.

For the rest of the paper, fix

$$\mathfrak{b} = \mathfrak{b}(\lambda) := 2 \left\lfloor \frac{c_{4.1} \lambda^{1/2}}{2N} \right\rfloor. \quad (7.1)$$

Theorem 7.1. *There is $c > 0$ such that for each sufficiently large $\alpha \in 2\mathbb{N}$ the following holds:*

For all sufficiently large λ and every $\mathfrak{b}!\mathbb{Z}^2$ -ergodic Gibbs measure, exactly one of $\Psi_{(\text{ver}, 0)}^{\alpha \times \mathfrak{b}}, \Psi_{(\text{ver}, 1)}^{\alpha \times \mathfrak{b}}, \Psi_{(\text{hor}, 0)}^{\mathfrak{b} \times \alpha}$, and $\Psi_{(\text{hor}, 1)}^{\mathfrak{b} \times \alpha}$ is $e^{-c\alpha}$ -strongly percolating, while each of the others almost surely has only finite $\boxtimes$ -components.

Let

$$\mathcal{P} := \{(\text{ver}, 0), (\text{ver}, 1), (\text{hor}, 0), (\text{hor}, 1)\}.$$

Following Theorem 7.1, for a $\mathfrak{b}!\mathbb{Z}^2$ -ergodic Gibbs measure μ , we write

$$\text{Phase}(\mu) = \pi$$

for the element $\pi \in \mathcal{P}$ corresponding to the set which percolates. Formally, the notation Phase also depends on a choice of α , and only makes sense when λ is chosen large as a function of α , but we omit explicit mention of this in the notation as we will use Phase

in situations where α and λ will be suitably fixed in advance. The following statement explains how Phase transforms under isometries τ of $\mathbb{Z}^2$. Recall from Section 2.1 that we write $\tau\mu$ for the push-forward of μ under τ and that $\eta_{(x,y)}$ denotes translation by the vector (x, y) .

Proposition 7.2. *Let $\alpha \in 2\mathbb{N}$ be sufficiently large. Let λ be sufficiently large (as a function of α). Let μ be a $\mathfrak{b}!\mathbb{Z}^2$ -ergodic Gibbs measure. Then:*

- (1) *Phase(μ) = (ver, j) if and only if Phase($\eta_{(1,0)}\mu$) = (ver, $1 - j$) for $j \in \{0, 1\}$.*
- (2) *If Phase(μ) $\in \{(\text{ver}, 0), (\text{ver}, 1)\}$ then Phase($\eta_{(0,1)}\mu$) = Phase(μ).*
- (3) *Let $\tau : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ be the reflection $\tau(x, y) = (y, x)$. Then Phase(μ) = (ver, j) if and only if Phase($\tau\mu$) = (hor, j), for $j \in \{0, 1\}$.*
- (4) *Let $\tau : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ be the reflection $\tau(x, y) = (-x, y)$. Then Phase(μ) = Phase($\tau\mu$).*

The rest of the section is devoted to the proof of the two results above.

7.1. Proof of Theorem 7.1

We divide the proof of Theorem 7.1 into parts, corresponding to the three items of the lemma below.

The first item is similar to Theorem 5.2. Unlike Theorem 5.2, the item concerns an ergodic measure rather than just a periodic one it gives a quantitative result, and it concerns rectangles rather than just squares (i.e., α does not necessarily equal $\mathfrak{b}$). The proof is similar, except for the use of the terminology that was introduced in Section 6, and an additional step where the ergodicity is used to draw a stronger conclusion.

The second item concerns the “density” of the sticks. It shows that thin rectangles aligned with the preferred direction of sticks are usually divided *in this direction*.

The last item implies Theorem 7.1 directly.

We point out that while the first two items do not refer to the fugacity explicitly, they contain an implicit requirement that λ be large in the assumption that $\alpha_0 \leq \mathfrak{b}$.

Lemma 7.3. *There exist $c, \alpha_0 > 0$ such that for every $\lambda > 0$, each $\alpha \in 2\mathbb{N}$ satisfying $\alpha_0 \leq \alpha \leq \mathfrak{b}$, and every $\mathfrak{b}!\mathbb{Z}^2$ -ergodic Gibbs measure, the following hold:*

- (1) *One of $\Psi_{\text{ver}}^{\alpha \times \mathfrak{b}}$ and $\Psi_{\text{hor}}^{\alpha \times \mathfrak{b}}$ is $e^{-c\alpha}$ -strongly percolating.*
- (2) *One of $\Psi_{\text{ver}}^{\alpha \times \mathfrak{b}}$ and $\Psi_{\text{hor}}^{\mathfrak{b} \times \alpha}$ is $e^{-c\alpha}$ -strongly percolating.*
- (3) *There is a universal $\lambda_0(\alpha)$ such that for $\lambda > \lambda_0(\alpha)$, exactly one of $\Psi_{(\text{ver},0)}^{\alpha \times \mathfrak{b}}$, $\Psi_{(\text{ver},1)}^{\alpha \times \mathfrak{b}}$, $\Psi_{(\text{hor},0)}^{\mathfrak{b} \times \alpha}$, and $\Psi_{(\text{hor},1)}^{\mathfrak{b} \times \alpha}$ is $e^{-c\alpha}$ -strongly percolating while each of the others almost surely has only finite $\boxtimes$ -components.*

Proof. Let c, α_0 be universal constants with c sufficiently small and α_0 sufficiently large to satisfy some assumptions to follow. Let λ, α and a Gibbs measure be as above. By slightly decreasing c , for each item, instead of showing that one of the sets Ψ is $e^{-c\alpha}$ -strongly percolating, it suffices to prove that one of the random sets Ψ satisfies $\mathfrak{p}_\mu(\Psi) \leq e^{-c\alpha}$.

Proof of (1). Note that $D^{\alpha \times \mathfrak{b}}$ is R -local for $R = R_{\alpha N \times \mathfrak{b} N, (0,0)}$, and thus by Lemma 6.7 (4),

$$\mathfrak{p}(\Psi^{\alpha \times \mathfrak{b}}) = \mathfrak{p}(X_{\alpha \times \mathfrak{b}} \overline{D^{\alpha \times \mathfrak{b}}}) \leq \sqrt[N^2]{\|\Omega \setminus D^{\alpha \times \mathfrak{b}}\|_R}.$$

Lemma 4.1 is applicable to R and $S = R^-$. Indeed, (4.2) holds by the choice of N (at the beginning of Section 5), the right part of (4.1) holds by the definition of $\mathfrak{b}$, and the left part then holds by assuming $\alpha_0 \geq \frac{1}{N c_{4.1}}$. Thus the lemma yields

$$\|\Omega \setminus D^{\alpha \times \mathfrak{b}}\|_R \leq e^{-c_{4.1} \text{Area}(R) \lambda^{-1/2}} = e^{-c_{4.1} \alpha \mathfrak{b} N^2 \lambda^{-1/2}} \leq e^{-2c_{4.1} N \alpha} \quad (7.2)$$

where the last inequality is obtained by requiring $\alpha_0 \geq 2$, and noting that by $\alpha_0 \leq \mathfrak{b}$ and (7.1) we have $\mathfrak{b} \leq 2c_{4.1} \lambda^{1/2} N$. Combining the last two displays, and taking sufficiently small c , it follows that

$$\mathfrak{p}(\Psi^{\alpha \times \mathfrak{b}}) \leq e^{-c\alpha}. \quad (7.3)$$

Note that by Lemma 5.1 and the assumption $\alpha \in 2\mathbb{N}$, the conditions of Proposition 6.8 are satisfied for

$$K = \alpha, \quad L = \mathfrak{b}, \quad E = D_{\text{ver}}^{\alpha \times \mathfrak{b}}, \quad F = D_{\text{hor}}^{\alpha \times \mathfrak{b}}.$$

Thus

$$\min \{\mathfrak{p}(\Psi_{\text{ver}}^{\alpha \times \mathfrak{b}}), \mathfrak{p}(\Psi_{\text{hor}}^{\alpha \times \mathfrak{b}})\} \leq \mathfrak{p}(\Psi^{\alpha \times \mathfrak{b}}) \leq e^{-c\alpha}. \quad \blacksquare$$

Proof of (2). Denote $\varepsilon = e^{-c\alpha_0}/\varepsilon_0$. We keep c as in (1) and possibly increase α_0 so that $3\varepsilon < 1$.

(1) implies that one of $\Psi_{\text{ver}}^{\mathfrak{b} \times \mathfrak{b}}$ and $\Psi_{\text{hor}}^{\mathfrak{b} \times \mathfrak{b}}$ is $e^{-\mathfrak{b}c}$ -strongly percolating for μ . Assume without loss of generality that $\Psi_{\text{ver}}^{\mathfrak{b} \times \mathfrak{b}}$ is. We prove by decreasing induction that $\mathfrak{p}(\Psi_{\text{ver}}^{\alpha \times \mathfrak{b}}) \leq e^{-\alpha c}$ for each $\alpha \in 2\mathbb{N}$ satisfying $\alpha_0 \leq \alpha \leq \mathfrak{b}$.

Let $\alpha \in \mathbb{N}$ satisfy $\alpha_0 \leq \alpha < \mathfrak{b}$ and assume the induction hypothesis that $\mathfrak{p}(\Psi_{\text{ver}}^{(\alpha+2) \times \mathfrak{b}}) \leq e^{-c(\alpha+2)}$. By (1), one of $\Psi_{\text{ver}}^{\alpha \times \mathfrak{b}}$ and $\Psi_{\text{hor}}^{\alpha \times \mathfrak{b}}$ is $e^{-c\alpha}$ -strongly percolating. Assume by contradiction that $\Psi_{\text{hor}}^{\alpha \times \mathfrak{b}}$ is. Thus by Lemma 6.4 (with $d = 0$) and the choice of ε ,

$$\max \{\mu(\Omega \setminus D_{\text{ver}}^{(\alpha+2) \times \mathfrak{b}}), \mu(\Omega \setminus D_{\text{hor}}^{\alpha \times \mathfrak{b}}), \mu(\Omega \setminus \eta_{(2\alpha,0)} D_{\text{hor}}^{\alpha \times \mathfrak{b}})\} \leq \varepsilon.$$

By the assumption $3\varepsilon < 1$, the event $D_{\text{ver}}^{(\alpha+2) \times \mathfrak{b}} \cup D_{\text{hor}}^{\alpha \times \mathfrak{b}} \cup \eta_{(2\alpha,0)} D_{\text{hor}}^{\alpha \times \mathfrak{b}}$ holds with positive probability. That is, there is an outcome σ for which $R_{\alpha \times N \mathfrak{b}, (0,0)}$ and $R_{\alpha \times N \mathfrak{b}, (2\alpha,0)}$ are divided horizontally, and $R_{(N\alpha+2N) \times N \mathfrak{b}, (0,0)}$ is divided vertically. By assuming $\alpha_0 \geq N$, the union of the former two rectangles contains the latter one, while all their vertical dimensions are the same. This implies an intersection of sticks, which is a contradiction. Thus $\mathfrak{p}(\Psi_{\text{ver}}^{\alpha \times \mathfrak{b}}) \leq e^{-c\alpha}$, completing the induction step. $\blacksquare$

As preparation for the proof of (3), we make a definition, and a claim, that will also be used later on. Define the event

$$G^{\alpha \times \mathfrak{b}} := \{\sigma \in \Omega : R_{(N+1)\alpha \times (N-1)\mathfrak{b}, (0,0)} \text{ is not divided by vertical sticks of both parities}\}. \quad (7.4)$$

Claim 7.4. *There is a universal $\lambda_0(\alpha)$ such that for $\lambda > \lambda_0(\alpha)$,*

$$p(X_{\alpha \times b} \overline{G^{\alpha \times b}}) < e^{-c\alpha}.$$

Proof. Denote $R = R_{K \times L, (0,0)}$ with $K = (N+1)\alpha$, $L = (N-1)b$. Let E be the event that each row of faces in R has a vacant face. Note that $\Omega \setminus G^{\alpha \times b} \subset E$.

For a face f in R , Corollary 4.5 says that $\|f$ is vacant $\|_f \leq \lambda^{-1/4}$. Lemma 3.4 and Proposition 3.3 imply together that for a row of faces $S = R_{K \times 1, (0,\ell)}$ we have

$$\|S \text{ has a vacant face}\|_S \leq K\lambda^{-1/4}.$$

Again by Lemma 3.4 we get $\|E\|_R \leq (K\lambda^{-1/4})^L$.

By Lemma 6.7 (4),

$$\begin{aligned} p(X_{\alpha \times b} \overline{G^{\alpha \times b}}) &\leq \frac{(N+1)(N-1)}{\sqrt{\|\Omega \setminus G^{\alpha \times b}\|_R}} \leq \frac{(N+1)(N-1)}{\sqrt{\|E\|_R}} \\ &\leq ((N+1)\alpha\lambda^{-1/4})^{\frac{(N-1)b}{(N+1)(N-1)}} \leq e^{-c\alpha} \end{aligned}$$

where the last inequality holds by choosing $\lambda_0(\alpha)$ such that for $\lambda > \lambda_0$ we have $(N+1)\alpha\lambda^{-1/4} < e^{-1}$ and $\frac{b}{N+1} \geq \frac{c_4\lambda^{1/2}}{N(N+1)} \geq c\alpha$. ■

We continue with the proof of Lemma 7.3.

Proof of (3). By (2), we may assume that one of $\Psi_{\text{ver}}^{\alpha \times b}$ and $\Psi_{\text{hor}}^{b \times \alpha}$ is $e^{-c\alpha}$ -strongly percolating. We handle the case where $\Psi_{\text{ver}}^{\alpha \times b}$ percolates; the other case is similar. Thus

$$p(\Psi_{\text{ver}}^{\alpha \times b}) = p(X_{\alpha \times b} \overline{D_{\text{ver}}^{\alpha \times b}}) \leq e^{-c\alpha}. \quad (7.5)$$

By Lemma 6.6, and by (7.5) and Claim 7.4,

$$p(X_{\alpha \times b} \overline{D_{\text{ver}}^{\alpha \times b} \cap G^{\alpha \times b}}) \leq 2 \sqrt{\max\{p(X_{\alpha \times b} \overline{G^{\alpha \times b}}), p(\Psi_{\text{ver}}^{\alpha \times b})\}} \leq 2e^{-c\alpha/2}$$

when $\lambda > \lambda_0$ and $\alpha_0 \leq \alpha \leq b$, for c and α_0 of (2) and λ_0 of Claim 7.4.

At this point we fix c and α_0 to their final values. We may decrease c and choose α_0 sufficiently large depending on c such that under the assumptions of the current item,

$$p(X_{\alpha \times b} \overline{D_{\text{ver}}^{\alpha \times b} \cap G^{\alpha \times b}}) \leq e^{-c\alpha} < \varepsilon_0.$$

Note that the conditions of Proposition 6.8 are satisfied for

$$K = \alpha, \quad L = b, \quad E = D_{(\text{ver},0)}^{\alpha \times b} \cap G^{\alpha \times b}, \quad F = D_{(\text{ver},1)}^{\alpha \times b} \cap G^{\alpha \times b}. \quad (7.6)$$

Thus

$$\min\{p(X_{\alpha \times b} \overline{D_{(\text{ver},0)}^{\alpha \times b} \cap G^{\alpha \times b}}), p(X_{\alpha \times b} \overline{D_{(\text{ver},1)}^{\alpha \times b} \cap G^{\alpha \times b}})\} \leq e^{c\alpha}.$$

Assume that $p(X_{K \times L} \overline{D_{(\text{ver},0)}^{\alpha \times b} \cap G^{\alpha \times b}}) \leq e^{-c\alpha}$ (the other case is similar). Then in particular $\Psi_{(\text{ver},0)}^{\alpha \times b}$ is $e^{-c\alpha}$ -strongly percolating. In addition, as $e^{-c\alpha} < \varepsilon_0$ and

$$(D_{(\text{ver},0)}^{\alpha \times b} \cap G^{\alpha \times b}) \cap (D_{(\text{ver},1)}^{\alpha \times b} \cup D_{(\text{hor},0)}^{\alpha \times b} \cup D_{(\text{hor},1)}^{\alpha \times b}) = \emptyset,$$

the other three sets almost surely have only finite $\boxtimes$ -components. ■

7.2. Proof of Proposition 7.2

By the assumptions of the proposition, we may require that α is sufficiently large, and λ is large as a function of α . Thus Theorem 7.1, the definition of Phase and Claim 7.4 apply.

Proof of (3) and (4). Immediate from the definitions of $\Psi_{(\text{ver},0)}^{\alpha \times b}, \Psi_{(\text{ver},1)}^{\alpha \times b}, \Psi_{(\text{hor},0)}^{b \times \alpha}$, and $\Psi_{(\text{hor},1)}^{b \times \alpha}$. ■

Proof of (1). It suffices to rule out the following possibilities:

$$\text{Phase}(\mu) = \text{Phase}(\eta_{(1,0)}\mu) = (\text{ver}, i), \quad (7.7)$$

$$\text{Phase}(\mu) = (\text{hor}, i), \text{Phase}(\eta_{(1,0)}\mu) = (\text{ver}, j), \quad (7.8)$$

$$\text{Phase}(\mu) = (\text{ver}, i), \text{Phase}(\eta_{(1,0)}\mu) = (\text{hor}, j), \quad (7.9)$$

for any $i, j \in \{0, 1\}$.

Assume by contradiction that (7.7) holds. Applying Theorem 7.1 to μ and $\eta_{(1,0)}\mu$, and Claim 7.4 to μ , shows that

$$\max \{p_\mu(X_{\alpha \times b} \overline{D_{(\text{ver},0)}^{\alpha \times b}}), p_\mu(X_{\alpha \times b} \overline{\eta_{(1,0)} D_{(\text{ver},0)}^{\alpha \times b}}), p_\mu(X_{\alpha \times b} \overline{G^{\alpha \times b}})\} < e^{-c\alpha}$$

for some universal $c > 0$. Applying Lemma 6.4 (with $d = 0$) thus gives

$$\max \{\mu(\Omega \setminus D_{(\text{ver},0)}^{\alpha \times b}), \mu(\Omega \setminus \eta_{(1,0)} D_{(\text{ver},0)}^{\alpha \times b}), \mu(\Omega \setminus G^{\alpha \times b})\} < e^{-c\alpha}/\varepsilon_0.$$

Taking α large enough that $e^{-c\alpha}/\varepsilon_0 < 1/3$ we conclude that

$$\mu(D_{(\text{ver},0)}^{\alpha \times b} \cap \eta_{(1,0)} D_{(\text{ver},0)}^{\alpha \times b} \cap G^{\alpha \times b}) > 0.$$

However, this is a contradiction since the event on the left-hand side is empty.

Now assume that (7.8) holds. Then by Theorem 7.1,

$$\max \{p_\mu(X_{\alpha \times b} \overline{D_{\text{hor}}^{b \times \alpha}}), p_\mu(X_{b \times \alpha} \overline{\eta_{(1,0)} D_{\text{ver}}^{\alpha \times b}})\} < e^{-c\alpha}$$

which as before leads to

$$\mu(D_{\text{hor}}^{b \times \alpha} \cap \eta_{(1,0)} D_{\text{ver}}^{\alpha \times b}) > 0.$$

The event on the left-hand side is empty when $\alpha \leq b$ and $0 \leq 1 < N\alpha + 1 \leq Nb$, since then each horizontal stick that divides $R_{Nb \times N\alpha, (0,0)}$ crosses each vertical stick that divides $R_{N\alpha \times Nb, (1,0)}$. This holds when λ is sufficiently large as a function of α , and thus we have a contradiction.

Possibility (7.9) leads similarly to a contradiction, by considering the two events $\eta_{(\alpha,0)} D_{\text{ver}}^{\alpha \times b}, \eta_{(1,0)} D_{\text{hor}}^{b \times \alpha}$ instead of $D_{\text{hor}}^{b \times \alpha}, \eta_{(1,0)} D_{\text{ver}}^{\alpha \times b}$. ■

Proof of (2). It suffices to rule out the following possibilities:

$$\text{Phase}(\mu) = (\text{ver}, i), \quad \text{Phase}(\eta_{(0,1)}\mu) = (\text{ver}, 1 - i), \quad (7.10)$$

$$\text{Phase}(\mu) = (\text{ver}, i), \quad \text{Phase}(\eta_{(0,1)}\mu) = (\text{hor}, j), \quad (7.11)$$

$$\text{Phase}(\mu) = (\text{hor}, i), \quad \text{Phase}(\eta_{(0,1)}\mu) = (\text{ver}, j), \quad (7.12)$$

for any $i, j \in \{0, 1\}$. Possibility (7.10) leads to a contradiction in a manner similar to the previous ones, by noting that

$$D_{(\text{ver}, i)}^{\alpha \times \mathfrak{b}} \cap \eta_{(0,1)} D_{(\text{ver}, 1-i)}^{\alpha \times \mathfrak{b}} \cap \eta_{(0,1)} G^{\alpha \times \mathfrak{b}} = \emptyset,$$

and each of the events in this intersection has high probability. Possibilities (7.11) and (7.12) are impossible, as switching the x - and y -axis leads respectively to possibilities (7.8) and (7.9). ■

8. Characterization of the invariant Gibbs measures and decay of correlations

Throughout this section and Section 9, fix $\alpha \in 2\mathbb{N}$ large enough (its value is a large universal constant). Also fix λ_0 to be a threshold depending on α and chosen sufficiently large for the following arguments, and assume $\lambda > \lambda_0$. Lastly, we continue to use the length scale $\mathfrak{b}$ defined in (7.1) and introduce a third (and final) length scale

$$c = c(\lambda) := \lfloor \sqrt{\alpha} \rfloor \mathfrak{b}.$$

We will use the Phase notation introduced after Theorem 7.1.

In this section we establish significant parts of our main results. We prove Theorems 1.1 (1) and 1.2. We explicitly state these results (together with some byproducts) in the following lemma.

Lemma 8.1. *Let $\lambda > \lambda_0$. For each $\pi \in \mathcal{P}$, there is a unique $\mathfrak{b}\mathbb{Z}^2$ -ergodic Gibbs measure, denoted μ_π , with $\text{Phase}(\mu_\pi) = \pi$. In addition:*

- (1) μ_π is extremal for each $\pi \in \mathcal{P}$.
- (2) μ_π is $2\mathbb{Z} \times \mathbb{Z}$ -invariant when $\pi \in \{(\text{ver}, 0), (\text{ver}, 1)\}$ and $\mathbb{Z} \times 2\mathbb{Z}$ -invariant when $\pi \in \{(\text{hor}, 0), (\text{hor}, 1)\}$.
- (3) $\mu_{(\text{ver}, 1)}$ is created by translating $\mu_{(\text{ver}, 0)}$ by one lattice space in the horizontal direction. The measures $\mu_{(\text{hor}, 0)}$ and $\mu_{(\text{hor}, 1)}$ are formed from $\mu_{(\text{ver}, 0)}$ and $\mu_{(\text{ver}, 1)}$, respectively, by switching the x - and y -axes.
- (4) Every periodic Gibbs measure is a convex combination of $(\mu_\pi)_{\pi \in \mathcal{P}}$.

In addition, we establish the quantitative decay of correlations estimate corresponding to the first term in the minimum in (3) of Theorem 1.1.

Following these facts, the tasks remaining to complete the proofs of our main results are to prove that $\mu_{(\text{ver}, 0)}$ satisfies (2) (columnar order) of Theorem 1.1 and to refine the quantitative decay of correlations estimate to include the second term in the minimum in (3) of Theorem 1.1. These tasks will be taken up in Section 9.

8.1. Disagreement percolation

The proofs of our main results are based on the concept of disagreement percolation, as introduced by van den Berg [78] and further studied in the context of the hard-core model

by van den Berg and Steif [80]. The following theorem states the results that will be used. For two configurations $\sigma, \sigma' \in \Omega$, denote their *disagreement set* by

$$\Delta_{\sigma, \sigma'} := \{v \in \mathbb{Z}^2 : \sigma(v) \neq \sigma'(v)\}.$$

Define a *path of disagreement* as a $\boxtimes$ -path in $\Delta_{\sigma, \sigma'}$. The motivation for considering the $\boxtimes$ -connectivity in particular is that our model is a random Markov field with respect to the graph $(\mathbb{V}, \mathbb{E}_{\boxtimes})$.

Theorem 8.2. *Let μ, μ' be Gibbs measures. Let σ, σ' be independent samples from μ, μ' , respectively. Suppose that*

$$\mathbb{P}(\Delta_{\sigma, \sigma'} \text{ has an infinite } \boxtimes\text{-connected component}) = 0. \quad (8.1)$$

Then:

- (1) $\mu = \mu'$ and μ is extremal.
- (2) Let $f, g : \Omega \rightarrow [-1, 1]$. Suppose that f is A -local and g is B -local for $A, B \subset \mathbb{V}$ (locality is defined in (2.2)). Then

$$\text{Cov}(f(\sigma), g(\sigma)) \leq 2\mathbb{P}(a \text{ } \boxtimes\text{-path in } \Delta_{\sigma, \sigma'} \text{ intersects } A \text{ and } B) \quad (8.2)$$

with $\text{Cov}(\cdot, \cdot)$ denoting the covariance between two random variables.

The equality of the measures under the assumption (8.1) is proved in [78, Theorem 1]. Extremality also follows, as one may apply the equality clause to the measures in the extremal decomposition of μ (for extremal decomposition, see [28, Theorem (7.26)]). The covariance bound (8.2) is an extension of [80, Theorem 2.4]. For completeness, a self-contained proof is provided in Section 8.4. While we state the theorem for the specific hard-core model studied here, we remark that the disagreement percolation method applies to general Markov random fields, defined on general graphs.

The following lemma gives a quantitative bound on the size of disagreement components, and in particular shows that disagreement components do not percolate. The lemma is proved in Section 8.3. After its statement we proceed to derive the main results of the current section.

Lemma 8.3. *Let $\lambda > \lambda_0$. There exist universal $C, c > 0$ such that the following holds. Let μ, μ' be $\mathfrak{b}!\mathbb{Z}^2$ -ergodic Gibbs measures with $\text{Phase}(\mu) = \text{Phase}(\mu') = (\text{ver}, 0)$. Let σ, σ' be independent samples from μ, μ' , respectively. Then for all $A \subset \mathbb{Z}^2$ and $B \subset \mathbb{Z}^2$,*

$$\mathbb{P}(a \text{ } \boxtimes\text{-path in } \Delta_{\sigma, \sigma'} \text{ intersects } A \text{ and } B) \leq \sum_{u \in A} \sup_{v \in B} \alpha_1(u, v)$$

where for $u = (x_1, y_1)$ and $v = (x_2, y_2) \in \mathbb{Z}^2$,

$$\alpha_1(u, v) := C \exp\left(-c|x_2 - x_1| - c \frac{|y_2 - y_1|}{\sqrt{\lambda}}\right).$$

In particular, there are no infinite disagreement components in the sense that (8.1) holds.

Applying Theorem 8.2 to the conclusions of the lemma above yields the following proposition, which is the key to the proof of Lemma 8.1.

Proposition 8.4. *Let $\lambda > \lambda_0$. Then there is a unique $\mathfrak{b}!\mathbb{Z}^2$ -ergodic Gibbs measure μ with $\text{Phase}(\mu) = (\text{ver}, 0)$, and μ is extremal.*

Proof. Let $\lambda > \lambda_0$. Let μ, μ' be $\mathfrak{b}!\mathbb{Z}^2$ -ergodic Gibbs measures with $(\text{ver}, 0) = \text{Phase}(\mu) = \text{Phase}(\mu')$, and let σ, σ' be independent samples from μ, μ' respectively. By Lemma 8.3, condition (8.1) holds, thus by Theorem 8.2 (1), $\mu = \mu'$, and μ is extremal. ■

Finally, Lemma 8.3, put together with Theorem 8.2 (2), also yields the quantitative decay of correlations estimate corresponding to the first term in the minimum in Theorem 1.1 (3). This is used later in Section 9.2, where a complete proof of item (3) is given.

8.2. Proof of Lemma 8.1

Here we prove the main result of the current section. The proof relies only on Propositions 8.4 and 7.2.

Proof of Lemma 8.1. Let $\lambda > \lambda_0$. We require that α and λ_0 are sufficiently large so that Phase is defined for every $\mathfrak{b}!\mathbb{Z}^2$ -ergodic Gibbs measure, as explained immediately after Theorem 7.1. Proposition 8.4 justifies the notation μ_π and proves that μ_π is extremal in the case of $\pi = (\text{ver}, 0)$.

Let μ be a $\mathfrak{b}!\mathbb{Z}^2$ -ergodic Gibbs measure with $\text{Phase}(\mu) = (\text{ver}, 1)$. By Proposition 7.2 (1), $\text{Phase}(\eta_{(-1,0)}\mu) = (\text{ver}, 0)$. Thus $\mu = \eta_{(1,0)}\mu_{(\text{ver},0)}$, showing the uniqueness of μ .

Let $i \in \{0, 1\}$. Let μ be a $\mathfrak{b}!\mathbb{Z}^2$ -ergodic Gibbs measure with $\text{Phase}(\mu) = (\text{hor}, i)$. By Proposition 7.2 (3), for τ defined by $\tau(x, y) = (y, x)$, we have $\text{Phase}(\tau^{-1}\mu) = (\text{ver}, i)$. Thus $\mu = \tau\mu_{(\text{ver},i)}$, showing the uniqueness of μ .

The above arguments show the uniqueness for each $\pi \in \mathcal{P}$, justify the notation μ_π and prove items (1) and (3). Item (2) then follows from Proposition 7.2 (1, 2) together with the uniqueness just shown.

The uniqueness results above show that $(\mu_\pi)_{\pi \in \mathcal{P}}$ are the only $\mathfrak{b}!\mathbb{Z}^2$ -ergodic Gibbs measures. We now show how this implies item (4).

We will use the fact that a Gibbs measure invariant with respect to a full-rank lattice $\mathcal{L}$ has a unique decomposition as a mixture of $\mathcal{L}$ -ergodic Gibbs measures. This is stated and proved in [28, Theorem (14.17)]. More formally, for each $\mathcal{L}$ -invariant Gibbs measure μ , there is a unique measure w_μ on the space of all $\mathcal{L}$ -ergodic measures on Ω such that $\mu = \int v dw_\mu(v)$, and the unique measure w_μ is supported on the set of $\mathcal{L}$ -ergodic Gibbs measures.

Let $\mathcal{L}$ be a full-rank lattice such that $\mathcal{L} \subset \mathfrak{b}!\mathbb{Z}^2$. We claim that every $\mathcal{L}$ -ergodic Gibbs measure is one of $(\mu_\pi)_{\pi \in \mathcal{P}}$. Indeed, let μ be an $\mathcal{L}$ -ergodic Gibbs measure. Define ν to be the average of all the shifts of μ by elements of $\mathfrak{b}!\mathbb{Z}^2/\mathcal{L}$. Then ν is $\mathfrak{b}!\mathbb{Z}^2$ -invariant and thus by the decomposition theorem, ν is a linear combination of $(\mu_\pi)_{\pi \in \mathcal{P}}$. This decomposition is also the unique decomposition of ν as a mixture of $\mathcal{L}$ -ergodic measures, since $(\mu_\pi)_{\pi \in \mathcal{P}}$

are extremal, and in particular $\mathcal{L}$ -ergodic. But as ν is defined as an average of $\mathcal{L}$ -ergodic measures, namely the shifts of μ , it follows that each shift of μ and in particular μ itself must be one of $(\mu_\pi)_{\pi \in \mathcal{P}}$.

Let μ be a periodic measure, invariant with respect to a full-rank lattice $\mathcal{L}$. Assume without loss of generality that $\mathcal{L} \subset \mathfrak{b}\mathbb{Z}^2$. Then by the claim of the previous paragraph that $(\mu_\pi)_{\pi \in \mathcal{P}}$ are the only $\mathcal{L}$ -ergodic Gibbs measures, the decomposition theorem implies item (4). ■

8.3. Tail bounds for the connectivity of disagreement components

We now prove Lemma 8.3. Recall the variables and assumptions introduced at the beginning of the section. Let $\lambda > \lambda_0$ and let μ, μ' be $\mathfrak{b}\mathbb{Z}^2$ -ergodic Gibbs measures satisfying $\text{Phase}(\mu) = \text{Phase}(\mu') = (\text{ver}, 0)$. Let σ, σ' be independent samples from μ, μ' , respectively.

Heuristically, to prove Lemma 8.3 one needs to show that long disagreement paths are rare. To this end we will define “sealed rectangles” and “semisealed rectangles”. A rectangle R will be defined to be semisealed in σ if σ satisfies, in the vicinity of R , a set of conditions which are typical of a configuration drawn from a $(\text{ver}, 0)$ Gibbs measure. The rectangle R is said to be sealed in (σ, σ') if it satisfies these conditions for both σ and σ' .

The conditions are designed in such a way that the assumption that R is sealed ensures that a disagreement path starting in R can only reach points in the vicinity of R . Therefore a long path of disagreement will imply a long sequence of neighboring non-sealed rectangles, which will be shown to be unlikely by a Peierls argument.

8.3.1. Semisealed rectangles. Here we consider only σ , the same considerations apply also to σ' . We say that $R_{N\alpha \times Nc, (0,0)}$ is *semisealed* if the event $\Sigma \subset \Omega$ holds, where $\Sigma = \Sigma_1 \cap \Sigma_2$ and Σ_1, Σ_2 are defined below. More generally, for $(x, y) \in \mathbb{Z}^2$, we say that $R_{N\alpha \times Nc, (N\alpha x, Ncy)}$ is semisealed if $\eta_{(N\alpha x, Ncy)} \Sigma$ holds. Our goal now is to prove “strong percolation of semisealed rectangles” in the sense of Lemma 8.5 below. We first prove the lemma assuming Proposition 8.6, and then prove the proposition.

Consider events $\Sigma_0, \Sigma_1, \Sigma_2$ defined as follows:

- $\sigma \in \Sigma_0$ iff every $N\alpha \times 1$ rectangle contained in

$$R_{N\alpha \times 3Nc, (-N\alpha, -Nc)} \cup R_{N\alpha \times 3Nc, (+N\alpha, -Nc)}$$

intersects the interior of a tile with even horizontal parity.

- $\sigma \in \Sigma_1$ iff all tiles of σ with center in $R_{N\alpha \times 3Nc, (0, -Nc)}$ have even horizontal parity.
- $\sigma \in \Sigma_2$ iff every $1 \times Nc$ rectangle contained in

$$R_{N\alpha \times Nc, (0, -Nc)} \cup R_{N\alpha \times Nc, (0, +Nc)}$$

contains a vacant face of σ .

Recall that α and λ_0 were introduced at the beginning of the section.

Lemma 8.5. *For every $\varepsilon > 0$ we may choose α sufficiently large, and λ_0 sufficiently large as a function of α , such that $X_{N\alpha \times Nc} \overline{\Sigma}(\sigma)$ is ε -strongly percolating.*

Proof. Let $\delta > 0$. Recalling the definitions of b and c , we check that for every sufficiently large α there is a sufficiently large λ_0 such that all the bounds of Proposition 8.6 are less than δ .

For each $\eta \in \{\eta_v : v \in \alpha\mathbb{Z} \times b\mathbb{Z}\}$, and every $b!\mathbb{Z}^2$ -ergodic Gibbs measure μ , $\text{Phase}(\mu) = \text{Phase}(\eta\mu)$. Thus the bounds of Proposition 8.6 hold also for translations of the relevant events by elements of $\alpha\mathbb{Z} \times b\mathbb{Z}$. Thus we may apply Lemma 6.7 (3) three times, with $k = N\alpha$, $l = Nb$ and respectively with

$$\begin{aligned} K &= 3N\alpha, & L &= 3Nb, & E &= \Sigma_0, \\ K &= 3N\alpha, & L &= 5Nb, & E &= \Sigma_1 \cup \Sigma_0^c, \\ K &= N\alpha, & L &= 3Nb, & E &= \Sigma_2, \end{aligned}$$

to obtain respectively

$$\begin{aligned} p_\mu(X_{N\alpha \times Nc} \overline{\Sigma_0}) &\leq \sqrt[9]{9 \sqrt[9]{\delta}}, \\ p_\mu(X_{N\alpha \times Nc} \overline{\Sigma_1 \cup \Sigma_0^c}) &\leq \sqrt[15]{15 \sqrt[15]{\delta}}, \\ p_\mu(X_{N\alpha \times Nc} \overline{\Sigma_2}) &\leq \sqrt[3]{3 \sqrt[3]{\delta}}. \end{aligned}$$

Note that $\Sigma = \Sigma_1 \cap \Sigma_2 \supset \Sigma_0 \cap (\Sigma_1 \cup \Sigma_0^c) \cap \Sigma_2$, thus by Lemma 6.6,

$$\begin{aligned} p_\mu(X_{N\alpha \times Nc} \overline{\Sigma}) &\leq 3 \left(\max \{ p_\mu(X_{N\alpha \times Nc} \overline{\Sigma_0}), \right. \\ &\quad p_\mu(X_{N\alpha \times Nc} \overline{\Sigma_1 \cup \Sigma_0^c}), \\ &\quad \left. p_\mu(X_{N\alpha \times Nc} \overline{\Sigma_2}) \} \right)^{1/3}. \end{aligned}$$

Given ε , we are done by taking δ sufficiently small. ■

Proposition 8.6. *There is a universal $c > 0$ such that*

- (1) $p_\mu(X_{3N\alpha \times 3Nc} \overline{\Sigma_0}) \leq \frac{9N^2c}{b} e^{-c\alpha}$;
- (2) $p_\mu(X_{3N\alpha \times 5Nc} \overline{\Sigma_1 \cup \Sigma_0^c}) \leq (3Nc + 1)(6N\alpha)^4 \lambda^{-1}$;
- (3) $p_\mu(X_{N\alpha \times 3Nc} \overline{\Sigma_2}) \leq 2Na e^{-c\lambda^{-1/2}Nc}$.

Proof of (1). We shall apply Lemma 6.7 (1) for

$$\begin{aligned} K &= 3N\alpha, & k &= \alpha, & L &= 3Nc, & l &= b, \\ E &= D_{(\text{ver}, 0)}^{\alpha \times b}. \end{aligned}$$

We let H be as defined in the lemma. The lemma yields

$$\mathfrak{p}_\mu \left(X_{3N\alpha \times 3Nc} \bigcap_{\eta \in H} \overline{\eta E} \right) \leq \frac{9N^2c}{b} \mathfrak{p}_\mu(X_{\alpha \times b} \bar{E}). \quad (8.3)$$

As $\Psi_{(\text{ver},0)}^{\alpha \times b}(\sigma) = X_{\alpha \times b} \bar{E}$, Theorem 7.1 yields

$$\mathfrak{p}_\mu(X_{\alpha \times b} \bar{E}) \leq e^{-c_{7.1}\alpha}. \quad (8.4)$$

Note that the application of the theorem above is the only place in the proof of the current proposition where we used the assumptions on α and λ being sufficiently large and the assumption that $\text{Phase}(\mu) = (\text{ver}, 0)$. It is also the only place in the current subsection where we directly use the assumption that $\text{Phase}(\mu) = (\text{ver}, 0)$.

If $\sigma \in \bigcap_{\eta \in H} \eta E$ then in particular for each $0 \leq i < \bigcap_{i=0}^{3c/b-1}$ we have $\sigma \in \eta_{(-N\alpha, -Nc+iNb)} E$, so $R_i := R_{N\alpha \times Nb, (-N\alpha, -Nc+iNb)}$ is divided by a $(\text{ver}, 0)$ stick. Each $N\alpha \times 1$ rectangle in $R_{N\alpha \times 3Nc, (-N\alpha, -Nc)}$ is contained in some R_i , and thus intersects a $(\text{ver}, 0)$ stick of σ , and thus also intersects the interior of a tile in σ with even horizontal parity. A similar argument holds for $N\alpha \times 1$ rectangles contained in $R_{N\alpha \times 3Nc, (+N\alpha, -Nc)}$. Thus

$$\bigcap_{\eta \in H} \eta E \subset \Sigma_0, \quad (8.5)$$

and the claim follows by (8.3)–(8.5). $\blacksquare$

Proof of (2). Denote $R = R_{3N\alpha \times 5Nc, (-N\alpha, -2Nc)}$. By Corollary 6.3,

$$\mathfrak{p}_\mu(X_{3N\alpha \times 5Nc} \overline{\Sigma_1 \cup \Sigma_0^c}) \leq \|\Sigma_0 \setminus \Sigma_1\|_R,$$

and thus it suffices to bound the right-hand side.

Assume $\sigma \in \Sigma_0 \setminus \Sigma_1$. Then since $\sigma \notin \Sigma_1$, the rectangle $R_{N\alpha \times 3Nc, (0, -Nc)}$ contains the center of a tile with odd horizontal parity. Equivalently, there is a point $(x_0, y_0) \in [0, N\alpha] \times [-Nc, 2Nc]$ with x_0 even and $\sigma(x_0, y_0) = 1$. Consider the four rectangles

$$\begin{aligned} &[-N\alpha, x_0] \times [y_0 - 1, y_0], \quad [x_0, 2N\alpha] \times [y_0 - 1, y_0], \\ &[-N\alpha, x_0] \times [y_0, y_0 + 1], \quad [x_0, 2N\alpha] \times [y_0, y_0 + 1]. \end{aligned}$$

As $\sigma \in \Sigma_0$, each of them intersects the interior of a tile of σ that has even horizontal parity. Each also intersects $T_{(x_0, y_0)}$, which has odd horizontal parity. Thus each contains a vacant face of σ . Hence whenever $\sigma \in \Sigma_0 \setminus \Sigma_1$, the rectangle R contains a $3N\alpha \times 2$ rectangle that contains four vacancies of σ . This corresponds to at most $(3Nc + 1)(6N\alpha)^4$ sets of four faces in R such that for each $\sigma \in \Sigma_0 \setminus \Sigma_1$, one of those sets has all of its faces vacant.

For four given faces in R , the event that they are all vacant has chessboard norm at most λ^{-1} by Corollary 4.5 and Proposition 3.4. The bound on $\|\Sigma_0 \setminus \Sigma_1\|_R$ follows from the subadditivity and positivity of the chessboard seminorm. $\blacksquare$

Proof of (3). For $i \in \mathbb{Z}$, $0 \leq i < N\alpha$, $j \in \{-Nc, Nc\}$ denote $R_{ij} = R_{1 \times Nc, (i, j)}$ and let E_{ij} be the event that R_{ij} contains no vacant face. By Corollary 4.5 and Proposition 3.4,

$$\|E_{ij}\|_{R_{ij}} \leq (1 - c_{4.5}\lambda^{-1/2})^{Nc} \leq e^{-c_{4.5}\lambda^{-1/2}Nc}.$$

Denote $R = R_{N\alpha \times 3Nc, (0, -Nc)}$. By Corollary 6.3, Proposition 3.4, and subadditivity and positivity of the chessboard seminorm,

$$p_\mu(X_{N\alpha \times 3Nc} \overline{\Sigma_2}) \leq \|\Sigma_2^c\|_R = \left\| \bigcup_{i,j} E_{ij} \right\|_R \leq \sum_{i,j} \|E_{ij}\|_{R_i} \leq 2N\alpha e^{-c_{4.5}\lambda^{-1/2}Nc}. \quad \blacksquare$$

8.3.2. Bounding the disagreement components. A rectangle $R_{N\alpha \times Nc, (N\alpha x_0, Ncy_0)}$ is called *sealed* in (σ, σ') if it is *semisealed* in both σ and σ' . The following deterministic statement shows that large $\boxtimes$ -components of $\Delta_{\sigma, \sigma'}$ are disjoint from the union of the rectangles that are sealed in (σ, σ') .

Proposition 8.7. *Let $\sigma, \sigma' \in \Omega$. Suppose that $S = R_{N\alpha \times Nc, (N\alpha x_0, Ncy_0)}$ is sealed in (σ, σ') . Let $v = (x_1, y_1) \in S$. If $v \in \Delta_{\sigma, \sigma'}$ then the $\boxtimes$ -component of v in $\Delta_{\sigma, \sigma'}$ is contained in $\{(x_1, y) \in \mathbb{V} : Nc(y_0 - 1) < y < Nc(y_0 + 2)\}$.*

Proof. It suffices to handle the case of $x_0 = y_0 = 0$. The assumption that S is sealed means that $\sigma, \sigma' \in \Sigma$. Since $\sigma, \sigma' \in \Sigma_1$, for each (x, y) with $0 \leq x \leq N\alpha$, $-Nc \leq y \leq 2Nc$ and even x , we have $\sigma(x, y) = \sigma'(x, y) = 0$. In particular, each point in $S \cap \Delta_{\sigma, \sigma'}$ must have an odd first coordinate. Thus x_1 is odd.

As $\sigma \in \Sigma_2$, the rectangle $R_{1 \times Nc, (x_1, Nc)}$ contains a vacant face, $R_{1 \times 1, (x_1, y)}$, of σ . Thus $\sigma(x_1, y) = \sigma(x_1, y + 1) = 0$. Since either $\sigma'(x_1, y) = 0$ or $\sigma'(x_1, y + 1) = 0$, and $Nc \leq y < 2Nc$, there is a point $(x_1, y_3) \notin \Delta_{\sigma, \sigma'}$ with $Nc \leq y_3 \leq 2Nc$ (with either $y_3 = y$ or $y_3 = y + 1$).

By a similar argument, there is a point $(x_1, y_4) \notin \Delta_{\sigma, \sigma'}$ with $-Nc \leq y_4 \leq 0$. By the first paragraph, $\Delta_{\sigma, \sigma'} \cap \{(x, y) : x \in \{x_1 - 1, x_1 + 1\}, y_4 \leq y \leq y_3\} = \emptyset$.

Thus the $\boxtimes$ -component of v in $\Delta_{\sigma, \sigma'}$ is contained in $B := \{(x_1, y) \in \mathbb{V} : y_4 < y < y_3\}$ since we have shown that every point outside of B and $\boxtimes$ -adjacent to a point in B is not in $\Delta_{\sigma, \sigma'}$. As $B \subset \{(x_1, y) \in \mathbb{V} : -Nc < y < 2Nc\}$, the proof is complete. $\blacksquare$

Denote

$$\Pi = \Pi(\sigma, \sigma') := X_{N\alpha \times Nc} \overline{\Sigma}(\sigma) \cap X_{N\alpha \times Nc} \overline{\Sigma}(\sigma').$$

This random set represents the set of sealed rectangles, as $R_{N\alpha \times Nc, (N\alpha x, Ncy)}$ is sealed iff $(x, y) \in \Pi$. Fix some ε with $0 < \varepsilon < \varepsilon_0$. By Lemma 8.5, we may choose λ_0 sufficiently large so that $p(X_{N\alpha \times Nc} \overline{\Sigma}(\sigma)) < (\varepsilon/2)^2$. Thus by Lemma 6.6, we have

$$p(\Pi) \leq 2 \sqrt{\max \{p(X_{N\alpha \times Nc} \overline{\Sigma}(\sigma)), p(X_{N\alpha \times Nc} \overline{\Sigma}(\sigma'))\}} < \varepsilon. \quad (8.6)$$

We proceed to prove Lemma 8.3.

Proof of Lemma 8.3. Define $f : \mathbb{V} \rightarrow \mathbb{V}$ by $f(x, y) = (\lfloor \frac{x}{N\alpha} \rfloor, \lfloor \frac{y}{Nc} \rfloor)$. Let $u = (x_u, y_u)$, $v = (x_v, y_v)$ be in $\mathbb{V}$. Suppose that $u - v \notin \{0\} \times [-4Nc, 4Nc]$ and that a $\boxtimes$ -path P in $\Delta_{\sigma, \sigma'}$ connects u to v .

We claim that this implies that $f(u)$ and $f(v)$ are connected by a $\boxtimes$ -path of points (x, y) satisfying $(x, y) \notin \Pi$. Each point w on P is connected by a $\boxtimes$ -path in $\Delta_{\sigma, \sigma'}$ to a point w' such that $w - w' \notin \{0\} \times [-2Nc, 2Nc]$. By Proposition 8.7, this means that each point in P is contained in a non-sealed rectangle, i.e., a rectangle of the form $R_{N\alpha \times Nc, (N\alpha x, Ncy)}$ where $(x, y) \notin \Pi$. The claim follows, as $\{f(w) : w \in P\}$ contains the required path.

Fix a point $u \in A$. Denote $\alpha = \sup_{v \in B} \alpha_1(u, v)$ and $d' = \inf_{v \in B} \|f(u) - f(v)\|_\infty$. Then $d' \geq \frac{1}{2} \inf_{v \in B} (\frac{|x_u - x_v|}{N\alpha} + \frac{|y_u - y_v|}{Nc}) - 1$. Choosing c, C appropriately, we get

$$(\varepsilon/\varepsilon_0)^{d'+1} \leq \sup_{v \in B} \alpha_1(u, v) \quad (8.7)$$

and $\alpha_1(u, v) \geq 1$ whenever $u - v \in \{0\} \times [-4Nc, 4Nc]$. To prove the lemma, by a union bound it suffices to show that

$$\mathbb{P}(\text{a } \boxtimes\text{-path in } \Delta_{\sigma, \sigma'} \text{ intersects } \{u\} \text{ and } B) \leq \sup_{v \in B} \alpha_1(u, v).$$

When there is $v \in B$ with $u - v \in \{0\} \times [-4Nc, 4Nc]$, there is nothing to prove as the right-hand side is at least 1. Otherwise, by the claim, if a $\boxtimes$ -path in $\Delta_{\sigma, \sigma'}$ connects u to some $v \in B$, then $f(u)$ is connected to $f(v)$ by a $\boxtimes$ -path disjoint from Π . By (8.6) and Lemma 6.4, the probability that this holds for some $v \in B$ is at most $(\varepsilon/\varepsilon_0)^{d'+1}$, and by (8.7) the proof is complete. ■

8.4. Disagreement percolation – proofs

In this section we provide a proof of Theorem 8.2. The proof is based on the following lemma.

Lemma 8.8. *Let $A \subset \mathbb{V}$ be a finite set. For $\sigma, \sigma' \in \Omega$, define $C_A(\sigma, \sigma')$ to be the set of points which are connected to a point in A by a path of disagreement (“the cluster of disagreement of A ”). Let $m : \Omega^2 \rightarrow \Omega^2$ be defined by $m(\sigma, \sigma') = (\omega, \omega')$ where*

$$\omega(v), \omega'(v) = \begin{cases} \sigma'(v), \sigma(v), & C_A(\sigma, \sigma') \text{ is finite and } v \in C_A(\sigma, \sigma'), \\ \sigma(v), \sigma'(v), & \text{otherwise.} \end{cases} \quad (8.8)$$

Let (σ, σ') be sampled from Ω^2 with the measure $\mu \times \mu'$. Then (σ, σ') has the same distribution as $m(\sigma, \sigma')$.

Proof. Denote by $C'_A(\sigma, \sigma')$ the “exterior $\boxtimes$ -boundary of $C_A(\sigma, \sigma')$ ”. More precisely, it is the set of vertices that are in A or $\boxtimes$ -adjacent to a vertex in $C_A(\sigma, \sigma')$, but not in $C_A(\sigma, \sigma')$.

Consider the family $\mathcal{F}$ of events $E \subset \Omega^2$ consisting of

- (1) events contained in the event that $C_A(\sigma, \sigma')$ is infinite,

- (2) events of the form $E = \{(\sigma, \sigma') : \sigma|_D = \rho|_D, \sigma'|_D = \rho'|_D\}$ where $\rho, \rho' \in \Omega$, and D is finite and $C_A(\rho, \rho') \cup C'_A(\rho, \rho') \subset D$.

The family $\mathcal{F}$ is a π -system. To see that it generates the sigma-algebra of Ω^2 , consider for an event E its partition according to the possibilities for $C_A(\sigma, \sigma')$ being finite, and the possibility that it is infinite. This gives a *countable* partition of E , and each part is easily seen to be in the sigma-algebra generated by $\mathcal{F}$. Thus it remains to show for an event $E \in \mathcal{F}$ that

$$\mathbb{P}((\sigma, \sigma') \in E) = \mathbb{P}((\sigma, \sigma') \in m^{-1}(E)). \quad (8.9)$$

For the case of (1), this is since $m^{-1}(E) = E$. For (2), fix ρ, ρ', D and the corresponding event E , as in (2). Denote $C = C_A(\rho, \rho')$ and $C' = C'_A(\rho, \rho')$. Then $C' \subset D \setminus C$ and $\rho|_{C'} = \rho'|_{C'}$. Thus by the domain Markov property,

$$\begin{aligned} \mathbb{P}(\sigma|_C = \rho|_C \mid \sigma|_{D \setminus C} = \rho|_{D \setminus C}) &= \mathbb{P}(\sigma'|_C = \rho|_C \mid \sigma'|_{D \setminus C} = \rho'|_{D \setminus C}), \\ \mathbb{P}(\sigma'|_C = \rho'|_C \mid \sigma'|_{D \setminus C} = \rho'|_{D \setminus C}) &= \mathbb{P}(\sigma|_C = \rho'|_C \mid \sigma|_{D \setminus C} = \rho|_{D \setminus C}). \end{aligned} \quad (8.10)$$

Since $C' \subset D$, we have $C_A(\sigma, \sigma') = C$ for all $(\sigma, \sigma') \in E$. Thus

$$m^{-1}(E) = \{(\sigma, \sigma') : \sigma|_{D \setminus C} = \rho|_{D \setminus C}, \sigma'|_{D \setminus C} = \rho'|_{D \setminus C}, \sigma|_C = \rho'|_C, \sigma'|_C = \rho|_C\}.$$

Writing E in an analogous way, we show (8.9) by expressing each side as a product of four terms and then using (8.10). ■

Proof of Theorem 8.2. Let μ, μ' and σ, σ' be as in the theorem and assume (8.1). Define $(\omega, \omega') = m(\sigma, \sigma')$ as in Lemma 8.8.

To show that $\mu = \mu'$, it suffices to prove $\mu(f) = \mu'(f)$ for every function $f : \Omega \rightarrow \mathbb{R}$ which is A -local for some *finite* set $A \subset \mathbb{V}$. Define $C_A = C_A(\sigma, \sigma')$ as in Lemma 8.8. By the assumption (8.1), C_A is almost surely finite, thus (8.8) gives that almost surely $\omega|_A = \sigma'|_A$ and thus almost surely $f(\omega) = f(\sigma')$. In addition, by Lemma 8.8, σ and ω have the same distribution, and thus

$$\mathbb{E}(f(\sigma)) = \mathbb{E}(f(\omega)) = \mathbb{E}(f(\sigma')).$$

Thus we have shown $\mu = \mu'$. We first prove (2) and then conclude from it the extremality of μ . Let f, g, A, B be as in (2). As f, g may be approximated (in L^2) by functions depending on restrictions to finite sets, and as replacing A, B with finite subsets only decreases the right-hand side of (8.2), we may assume without loss of generality that A and B are finite. Again define $C_A = C_A(\sigma, \sigma')$ as in Lemma 8.8. Let E be the event that a $\boxtimes$ -path in $\Delta_{\sigma, \sigma'}$ intersects both A and B . Then E^c is the event that C_A is disjoint from B . Since C_A is almost surely finite, (8.8) implies that $\omega|_B = \sigma|_B$ and $g(\omega) = g(\sigma)$ hold almost surely on E^c . As before, $f(\omega) = f(\sigma')$ almost surely. Thus $\mathbb{P}(f(\omega) \cdot g(\omega) \neq f(\sigma') \cdot g(\sigma)) \leq \mathbb{P}(E)$. By Lemma 8.8, since f, g are bounded between -1 and 1 , and by the independence of σ, σ' , we have

$$\begin{aligned} \mathbb{E}(f(\sigma)g(\sigma)) &= \mathbb{E}(f(\omega)g(\omega)) \\ &\leq \mathbb{E}(f(\sigma')g(\sigma)) + 2\mathbb{P}(E) = \mathbb{E}(f(\sigma))\mathbb{E}(g(\sigma)) + 2\mathbb{P}(E). \end{aligned}$$

Finally, to show that μ is extremal, it suffices to show that the covariance is 0 between every bounded $f, \tilde{g}$ where f is A -local for a finite set A , and $\tilde{g}$ is measurable with respect to the tail sigma-algebra. This may be seen by approximating $\tilde{g}$ by a function g which is B -local where B is the complement of a large box around the origin. The assumption (8.1) shows that $\mathbb{P}(E) \rightarrow 0$ as the box grows to infinity. ■

9. Columnar order

The measure $\mu_{(\text{ver},0)}$ referred to in Theorem 1.1 has been defined in the previous section (in Lemma 8.1) for $\lambda > \lambda_0$. In this section we prove some additional properties of $\mu_{(\text{ver},0)}$, completing the proof of the theorem. We increase λ_0 to be sufficiently large for the arguments in this section.

9.1. Offset tiles are rare

The measure $\mu_{(\text{ver},0)}$ is characterized by columns of tiles with even horizontal parity, i.e., tiles whose center has an odd first coordinate. Here we bound the probability that a tile with odd horizontal parity (“an offset tile”) appears in a given position. In Section 9.3 we show that the bound is sharp up to a multiplicative constant.

Theorem 9.1. *There is $C > 0$ such that for $\lambda > \lambda_0$, every $(x, y) \in \mathbb{V}$ with even x satisfies*

$$\mu_{(\text{ver},0)}(\sigma(x, y) = 1) \leq C\lambda^{-1}.$$

Proof. Fix λ sufficiently large for the following computations and fix $(x_T, y_T) \in \mathbb{V}$ with even x_T . We denote $\mu = \mu_{(\text{ver},0)}$ and aim to show for some universal $C > 0$ that $\mu(\sigma(x_T, y_T) = 1) \leq C\lambda^{-1}$. For $\sigma \in \Omega$ define

$$X_{-\downarrow}(\sigma) = \max \{x \in \mathbb{Z} : x < x_T \text{ and } R_{1 \times 1, (x-1, y_T-1)} \text{ is a vacant face in } \sigma\},$$

$$X_{-\uparrow}(\sigma) = \max \{x \in \mathbb{Z} : x < x_T \text{ and } R_{1 \times 1, (x-1, y_T)} \text{ is a vacant face in } \sigma\},$$

$$X_{+\downarrow}(\sigma) = \min \{x \in \mathbb{Z} : x > x_T \text{ and } R_{1 \times 1, (x, y_T-1)} \text{ is a vacant face in } \sigma\},$$

$$X_{+\uparrow}(\sigma) = \min \{x \in \mathbb{Z} : x > x_T \text{ and } R_{1 \times 1, (x, y_T)} \text{ is a vacant face in } \sigma\}.$$

These variables are almost surely finite.

Fix some integers $x_{-\downarrow}, x_{-\uparrow} < x_T$ and $x_{+\downarrow}, x_{+\uparrow} > x_T$. Define an event

$$J := \{\sigma \in \Omega : \sigma(x_T, y_T) = 1 \text{ and}$$

$$(X_{-\downarrow}(\sigma), X_{-\uparrow}(\sigma), X_{+\downarrow}(\sigma), X_{+\uparrow}(\sigma)) = (x_{-\downarrow}, x_{-\uparrow}, x_{+\downarrow}, x_{+\uparrow})\},$$

and denote

$$x_- := \min \{x_{-\downarrow}, x_{-\uparrow}\}, \quad x_+ := \max \{x_{+\downarrow}, x_{+\uparrow}\}.$$

We claim that for some universal constants $c, C > 0$,

$$\mu(J) \leq C\lambda^{-1}e^{-c(x_+-x_-)}. \quad (9.1)$$

The theorem follows by summing over the possible values of $x_{-\downarrow}, x_{-\uparrow}, x_{+\downarrow}, x_{+\uparrow}$. We now prove (9.1).

Consider the segment $s = [x_-, x_+] \times \{y_T\}$. For $\sigma \in J$, no $(\text{ver}, 0)$ stick intersects s . This implies that whenever s divides $R_{\alpha N \times bN, (\alpha x, by)}$, we have $(x, y) \notin \Psi_{(\text{ver}, 0)}^{\alpha \times b}$. Thus the set

$$A_0 = \{(x, \lfloor y_T/b \rfloor) : x \in \mathbb{Z}, x_0 < \alpha x < \alpha x + N\alpha < x_1\}$$

is disjoint from $\Psi_{(\text{ver}, 0)}^{\alpha \times b}(\sigma)$ for each $\sigma \in J$. Note that $\#A_0 \geq (x_1 - x_0)/\alpha - N - 1$.

By Corollary 4.5, $\mu(J) \leq \lambda^{-1}$. Fix some $\varepsilon_1 > 0$ satisfying $\varepsilon_1 < \varepsilon_0$ and define $\tilde{\Omega}$ to be an arbitrary event satisfying $J \subset \tilde{\Omega}$ and $\mu(\tilde{\Omega}) = \lambda^{-1}\varepsilon_1^{-4}$. Define a measure $\tilde{\mu}$ to be μ conditioned on $\tilde{\Omega}$.

We define a random set

$$\Theta(\sigma) := \begin{cases} \emptyset, & \sigma \in J, \\ \mathbb{V}, & \text{otherwise.} \end{cases}$$

We claim that for sufficiently large λ ,

$$\mathfrak{p}_{\tilde{\mu}}(\Theta \cup \Psi_{(\text{ver}, 0)}^{\alpha \times b}) \leq \varepsilon_1. \quad (9.2)$$

Given this claim, Lemma 6.4 implies that

$$\tilde{\mu}(J) \leq \tilde{\mu}(A_0 \cap (\Theta \cup \Psi_{(\text{ver}, 0)}^{\alpha \times b})) = \emptyset \leq (\varepsilon_1/\varepsilon_0)^{\#A_0},$$

and (9.1) follows by taking into account the measure of $\tilde{\Omega}$ and the size of A_0 . Thus it remains to prove (9.2).

We claim that in the current situation, Lemma 6.7 (4) holds (for all k, l, K, L, R, E, r, H satisfying its conditions) when we replace the conclusion (6.2) with

$$\mathfrak{p}_{\tilde{\mu}}(\Theta \cup X_{k \times l} \bar{E}) \leq \sqrt[r]{\max\{\lambda^{-1/4}, \|E^c\|_R\}}. \quad (9.3)$$

The proof is similar, except that $\varepsilon := \max\{\lambda^{-1/4}, \|E^c\|_R\}$ and that (6.4) is proved differently: we let $\eta \in H$ and show that $X_{K \times L} \bar{E}^c$ is ε -rare for the measure $\eta\tilde{\mu}$. Indeed, let $A \subset \mathbb{V}$ be a non-empty finite set. Then one may check using the chessboard estimate that $\eta\mu(J \cap \{A \subset X_{K \times L} \bar{E}^c\}) \leq \lambda^{-1}(\|f\|_R)^{\#A-4}$. Thus $\tilde{\mu}(A \subset X_{K \times L} \bar{E}^c) \leq \varepsilon^{\#A}$.

Recall $G^{\alpha \times b}$ from (7.4).

We show that for sufficiently large λ ,

$$\mathfrak{p}_{\tilde{\mu}}(\Theta \cup (\Psi^{\alpha \times b} \cap X_{\alpha \times b} \overline{G^{\alpha \times b}})) < \varepsilon_1. \quad (9.4)$$

Assume that λ_0 is such that $\lambda^{-1/4}$ is smaller than the bound of (7.2). Then we obtain $\mathfrak{p}_{\tilde{\mu}}(\Theta \cup \Psi^{\alpha \times b}) \leq e^{-c_{7.3}\alpha}$, in the same way as (7.3) is obtained in the proof of Lemma 7.3, except that instead of using Lemma 6.7, we use (9.3).

Similarly, we obtain $p_{\tilde{\mu}}(\Theta \cup X_{\alpha \times b} \overline{G^{\alpha \times b}}) \leq e^{-c_{7.4}\alpha}$ as in Claim 7.4 except that again in the proof, instead of Lemma 6.7, we use (9.3).

Combining the bounds of the last two paragraphs, Lemma 6.6 gives (9.4) when α is taken large enough. By (9.4), there is an ε_1 -rare set B (with respect to $\tilde{\mu}$) for which I , the unique infinite $\square$ -component of $\mathbb{V} \setminus B$, is contained in $\Theta \cup (\Psi^{\alpha \times b} \cap X_{\alpha \times b} \overline{G^{\alpha \times b}})$. Thus I is ε_1 -strongly percolating and almost surely connected with respect to $\tilde{\mu}$.

On the event J (up to measure 0) the following holds: I is connected and $I \subset \Psi^{\alpha \times b} \cap X_{\alpha \times b} \overline{G^{\alpha \times b}}$. Both I and $\Psi_{(\text{ver},0)}^{\alpha \times b}$ have the $\boxtimes$ -components of their complements almost surely finite. Thus $\Psi_{(\text{ver},0)}^{\alpha \times b} \cap I \neq \emptyset$, and since $I \subset \Psi^{\alpha \times b}$, Lemma 5.1 implies that $I \subset \Psi_{\text{ver}}^{\alpha \times b}$. The fact that the conditions of Proposition 6.8 are satisfied for (7.6) implies that there is no edge $uv \in \mathbb{E}_{\square}$ with $u \in \Psi_{(\text{ver},0)}^{\alpha \times b} \cap X_{\alpha \times b} \overline{G^{\alpha \times b}}$ and $v \in \Psi_{(\text{ver},1)}^{\alpha \times b} \cap X_{\alpha \times b} \overline{G^{\alpha \times b}}$. Thus $I \subset \Psi_{(\text{ver},0)}^{\alpha \times b}$.

Therefore with respect to $\tilde{\mu}$ the set I is ε_1 -strongly percolating and almost surely contained in $\Theta \cup \Psi_{(\text{ver},0)}^{\alpha \times b}$. This completes the proof of (9.2). ■

9.2. Correlations

We now prove Theorem 1.1 (3).

Lemma 9.2. *There exists a universal $C > 0$ such that the following holds for $\lambda > \lambda_0$. Let σ, σ' be independently sampled from $\mu_{(\text{ver},0)}$. Then for all $A \subset \mathbb{Z}^2$ and $B \subset \mathbb{Z}^2$,*

$$\mathbb{P}(a \boxtimes\text{-path in } \Delta_{\sigma,\sigma'} \text{ intersects } A \text{ and } B) \leq \sum_{u \in A} \sup_{v \in B} \alpha_2(u, v)$$

where for $u = (x_1, y_1)$ and $v = (x_2, y_2) \in \mathbb{Z}^2$,

$$\alpha_2(u, v) := \left(\frac{C \log \lambda}{\sqrt{\lambda}} \right)^{\mathbf{1}_{x_1 \neq x_2}}.$$

Proof. Let $\lambda > \lambda_0$. Let σ, σ' be independently sampled from $\mu_{(\text{ver},0)}$. Fix $u = (x_1, y_1)$. Let $d \in \mathbb{N}$. Let E be the event that a $\boxtimes$ -path in $\Delta_{\sigma,\sigma'}$ connects u to a point outside of $\{(x_1, y) \in \mathbb{V} : |y - y_1| < d\}$. It suffices to show that

$$\mathbb{P}(E) \leq \frac{C \log \lambda}{\sqrt{\lambda}}. \quad (9.5)$$

If x_1 is even, then by Theorem 9.1 there is probability of at most $2C_{9.1}\lambda^{-1}$ that $u \in \Delta_{\sigma,\sigma'}$, and thus (9.5) follows immediately. Assume that x_1 is odd. Let E_1 be the event that $\Delta_{\sigma,\sigma'}$ intersects $\{(x, y) \in \mathbb{V} : |y - y_1| \leq d, |x - x_1| = 1\}$. Then by Theorem 9.1, $\mathbb{P}(E_1) \leq 4(2d + 1)C_{9.1}\lambda^{-1}$. Let E_2 be the event that u is connected by a $\boxtimes$ -path to a point in $B = \{(x_1, y) \in \mathbb{V} : |y - y_1| = d\}$. By Lemma 8.3 (taking $A = \{u\}$ and B as above), $\mathbb{P}(E_2) \leq C_{8.3} \exp(-c_{8.3}d/\sqrt{\lambda})$. As $E \subset E_1 \cup E_2$, for $d = \lfloor \frac{\sqrt{\lambda} \log \lambda}{c_{8.3}} \rfloor$ and sufficiently large λ_0 , (9.5) is satisfied. ■

Corollary 9.3. *Let $\lambda > \lambda_0$. Then $\mu_{(\text{ver},0)}$ satisfies item (3) of Theorem 1.1.*

Proof. Let $\lambda > \lambda_0$. Let σ, σ' be independently sampled from $\mu_{(\text{ver},0)}$. Let $u \in \mathbb{V}$ and B be a finite subset of $\mathbb{V}$. Define $B_1 = \{v \in B : \alpha_1(u, v) < \alpha_2(u, v)\}$ and $B_2 = B \setminus B_1$. We apply Lemma 8.3 to $\{u\}$ and B_1 , and Lemma 9.2 to $\{u\}$ and B_2 . By a union bound, this shows that

$$\begin{aligned} \mathbb{P}(\text{a } \boxtimes\text{-path in } \Delta_{\sigma, \sigma'} \text{ intersects } \{u\} \text{ and } B) &\leq \sup_{v \in B_1} \alpha_1(u, v) + \sup_{v \in B_2} \alpha_2(u, v) \\ &\leq 2 \sup_{v \in B} \min \{\alpha_1(u, v), \alpha_2(u, v)\} \\ &\leq \frac{1}{2} \sup_{v \in B} \alpha(u, v) \end{aligned}$$

where for the last inequality we require $C_{1.1} \geq 4 \max \{C_{8.3}, C_{9.2}\}$ and $c_{1.1} \leq c_{8.3}$. We then use a union bound over $u \in A$ to show that the right-hand side of (8.2) is at most $\sum_{v \in A} \sup_{v \in B} \alpha(u, v)$.

We finish the proof using Theorem 8.2, recalling that (8.1) holds by Lemma 8.3. ■

Remark 9.4. The correlation decay estimate (1.3) of Theorem (1.1) shows, in particular, that $\text{Cov}_{\mu_{(\text{ver},0)}}(\sigma(1, 0), \sigma(3, 0)) \leq \frac{C \log \lambda}{\sqrt{\lambda}}$. It is natural to ask how sharp this bound is. We believe that $\text{Cov}_{\mu_{(\text{ver},0)}}(\sigma(1, 0), \sigma(3, 0))$ is of order $\lambda^{-1/2}$ so that our bound has the correct power of λ but adds an unnecessary logarithmic term. Indeed, van den Berg–Steif [80, Theorem 2.4] give a precise formula in terms of disagreement paths for such covariances and we believe that in our setup the terms in this formula are dominated by the disagreement paths that start at $(1, 0)$, go vertically to distance of order $\sqrt{\lambda}$, move horizontally to the column of $(3, 0)$ and then move vertically to $(3, 0)$. Such disagreement paths should occur with probability of order $\lambda^{-1/2}$.

9.3. Probability for a tile at a given position

We now give for each point in $\mathbb{Z}^2$ an estimate for the probability (with respect to $\mu_{(\text{ver},0)}$) that a tile is centered at it.

Theorem 9.5. *Let $\lambda > \lambda_0$. Then $\mu_{(\text{ver},0)}$ satisfies item (2) of Theorem 1.1.*

Proof. Fix (x, y) where x is odd. Denote

$$\begin{aligned} E_1 &= \{\sigma(x, y) + \sigma(x, y + 1) = 1\}, \\ E_2 &= \{\sigma(x - 1, y) + \sigma(x - 1, y + 1) + \sigma(x + 1, y) + \sigma(x + 1, y + 1) \geq 1\}, \\ E_3 &= \{\text{the faces } R_{1 \times 1, (x-1, y)} \text{ and } R_{1 \times 1, (x, y)} \text{ are vacant}\}, \end{aligned}$$

and note that $\{E_1, E_2, E_3\}$ is a partition of Ω . By Theorem 9.1, $\mu_{\text{ver},0}(E_2) \leq 4C_{9.1}\lambda^{-1}$. By Corollary 4.5, $\mu_{\text{ver},0}(E_3) \leq \lambda^{-1/2}$ and $\mu_{\text{ver},0}(E_1) \leq (1 - c\lambda^{-1/2})^2$. Thus $\mu_{\text{ver},0}(E_1) = 1 - \Theta(\lambda^{-1/2})$. Since $\mu_{\text{ver},0}$ is $2\mathbb{Z} \times \mathbb{Z}$ -translation invariant, $\mu_{\text{ver},0}(E_1) = 2\mu_{\text{ver},0}(\sigma(x, y) = 1)$ and the theorem follows for the case of odd x .

Now fix (x, y) where x is even. By Theorem 9.1, it remains to show that $\mu_{\text{ver},0}(\sigma(x, y) = 1) = \Omega(\lambda^{-1})$ (note that here the symbol Ω represents the asymptotic notation rather than the set of configurations). Now denote

$$\begin{aligned} E &= \{\sigma(x, y) = 1\}, \\ E_1 &= \{\sigma(x-1, y) = 1\}, \\ E_2 &= \{\sigma(x+1, y) = 1\}. \end{aligned}$$

By the previous case, $\mu_{\text{ver},0}(E_1) = \mu_{\text{ver},0}(E_2) = \Omega(1)$. By Corollary 9.3, $\text{Cov}(E_1, E_2) = o(1)$. Thus $\mu_{\text{ver},0}(E_1 \cap E_2) = \Omega(1)$. By a local surgery (remove one of the two tiles and slide the other), it follows that $\mu_{\text{ver},0}(E) \geq \lambda^{-1} \mu_{\text{ver},0}(E_1 \cap E_2) = \Omega(\lambda^{-1})$. ■

Part III

Concluding remarks

10. Discussion and open questions

In this section we discuss some of the predictions, open questions and research directions related to this work.

10.1. The 2×2 hard-square model

Intermediate fugacity and critical behavior: This work establishes that the 2×2 hard-square model exhibits columnar order in the high-fugacity regime. As discussed in the introduction, classical results imply that the model is disordered with a unique Gibbs measure in the low-fugacity regime. What happens at intermediate fugacities? The physics literature predicts a single transition point from the disordered to the columnar phase, with the transition being continuous and belonging to the Ashkin–Teller universality class [69] (at a point close to the Ising universality class [68, Figure 5]). These predictions have not been mathematically justified.

Boundary conditions: Our work characterizes the periodic Gibbs measures of the 2×2 hard-square model. However, we do not prove that any specific sequence of finite volumes and boundary conditions converges in the infinite-volume limit.

A related question is whether non-periodic Gibbs measures exist for the model. We expect the answer is negative, as in other two-dimensional models [2, 14, 19, 29, 31, 36, 71].

Decay of correlations: Theorem 1.1 gives an upper bound on the exponential rate of correlation decay in $\mu_{(\text{ver},0)}$ which is anisotropic. Specifically, the correlation length in the horizontal direction is at most a universal constant, while in the vertical direction it is at most $C\sqrt{\lambda}$. On a mesoscopic scale (for distances $1 \ll d \leq \sqrt{\lambda}$), it is clear from our results that correlations are indeed anisotropic. However, we do not establish lower bounds for the correlations as the distances grow without bound (i.e., when λ is fixed and

$d \rightarrow \infty$). A natural question is whether in the limit of large distances, the exponential rate of correlation decay is indeed highly anisotropic as our bound suggests.

The same question may be asked for models where a proof of nematic order was given, such as those listed in Section 1.3.5. We mention in particular the result of Jauslin–Lieb, where the proven correlation bounds [40, (19)] are very similar in form to those of the present work.

10.2. Cubes and rods on $\mathbb{Z}^d$

We briefly discuss related models in which the 2×2 hard squares are replaced by cubes and rods on $\mathbb{Z}^d$.

Cubes: For $k \times \cdots \times k$ cubes on $\mathbb{Z}^d$ we expect the high-fugacity regime to behave similarly to our results for the 2×2 hard-square model. In particular, we conjecture that there are exactly dk^{d-1} extremal and periodic Gibbs measures (where d accounts for the possible orientations of columns and k^{d-1} accounts for translations perpendicular to the columns). Elements of our approach may well be relevant to proving such a result, at least when $d = 2$. However, even in two dimensions our analysis does not apply, due to the absence of reflection positivity when $k \geq 3$. In higher dimensions, the case $k = 2$ may be more accessible as reflection positivity is again available.

Interestingly, a recent physics study [85] (see also [84]) predicts that the $2 \times 2 \times 2$ hard-cube model undergoes *three* phase transitions as the fugacity increases. One of the predicted phases is a sublattice phase, at intermediate fugacity, where cubes preferentially occupy one of the eight sublattices.

As a possible complication, we point out that tilings of cubes in high dimensions present new phenomena. For instance, refuting a conjecture of Keller [41], it has been shown [45, 47] that for $d \geq 8$ one may tile $\mathbb{R}^d$ with unit cubes in a way that no two cubes share a complete $(d - 1)$ -dimensional face (while this is not possible for $d \leq 7$ [11, 63, 64]). Moreover, these tilings may be chosen so that all cube centers lie in $\frac{1}{2}\mathbb{Z}^d$ (equivalently, a tiling with this feature is possible using $2 \times \cdots \times 2$ cubes with centers in $\mathbb{Z}^d$) and the tiling is $2\mathbb{Z}^d$ -translation invariant.

We also note a connection between cube packings and a famous problem in information theory. The Shannon capacity of a graph G is defined as $\lim_{d \rightarrow \infty} (\alpha(G^{\boxtimes d}))^{1/d}$ where $\alpha(H)$ is the size of the largest independent set in the graph H and $G^{\boxtimes d}$ stands for the strong product of G with itself d times [73]. The Shannon capacity remains unknown even for fairly simple graphs. In particular, the Shannon capacity of the cycle C_k with k odd has not been determined for $k \geq 7$ (see, e.g., [10]). The connection is that $\alpha(C_k^d)$ equals the maximal number of $2 \times \cdots \times 2$ cubes with centers having integer coordinates that may be packed in the torus $(\mathbb{R}/k\mathbb{Z})^d$.

Rods in two dimensions: The random packing of $1 \times k$ and $k \times 1$ tiles on $\mathbb{Z}^2$ has been studied extensively in the physics literature [30, 44, 49, 72, 86] (see also the literature reviews in the theses [43, 54]). The following behavior is predicted: For $k \leq 6$, the model is disordered for all fugacities. For $k \geq 7$, the model exhibits two phase transitions. At low

fugacity the model is disordered (low-density disordered, LDD), at intermediate fugacities the model has a nematic phase, and at high fugacities the model is again disordered (high-density disordered, HDD).

The case $k = 2$ is the monomer–dimer model. As discussed in Section 1.3.3, it was shown to have a unique Gibbs measure for all fugacities. As mentioned in Section 1.3.5, Disertori–Giuliani [17] rigorously established the nematic phase at an *intermediate* range of fugacities for large values of k .

The properties of the HDD phase are unclear, though simulations clearly demonstrate that horizontal and vertical rods appear with equal density (unlike in the nematic phase). It would be interesting to improve our understanding of this phase and a starting point may be the study of the fully packed regime (the limit $\lambda = \infty$). Is there a unique maximal-entropy Gibbs measure in this case? Estimates of the entropy-per-site are provided by [16, 26].

Rods in three dimensions: The predicted phase diagram for $1 \times 1 \times k$ rods (and their lattice rotations) on $\mathbb{Z}^3$ is less complete; see [16, 72, 83] for recent results.

10.3. A simplified lattice model with nematic order

In developing the technique of this paper the following simplified spin model proved handy for pointing out the essential features. We describe this model for its intrinsic interest and with the hope that it may lend similar help to the study of some of the models described above.

Oriented monomer model: Configurations are functions $\sigma : \mathbb{Z}^d \rightarrow \{0, e_1, \dots, e_d\}$ with e_i being the i th vector of the standard basis of $\mathbb{R}^d$. The state 0 represents a vacancy while each state e_i may be thought of as a “monomer oriented in the i th coordinate direction”. Oriented monomers of equal orientations which are adjacent in the direction of their orientation are thought to join together to form rods (an oriented monomer which is not joined in this way is thought of as a rod of length 1). A configuration may thus be imagined as a packing of rods. We wish to study the model in which the probability of a configuration is proportional to $t^{2N(\sigma)}$ with $N(\sigma)$ representing the number of rods (i.e., the fugacity $\lambda = t^2$). Equivalently, the weight of a configuration assigns weight t to each end of a rod. An essential simplification available for this model is that the probability measure may be represented by nearest-neighbor interactions, via the Hamiltonian

$$H(\sigma) = \sum_{uv \in \mathbb{E}_\square} |\langle u - v, \sigma(u) - \sigma(v) \rangle|$$

with $\langle \cdot, \cdot \rangle$ denoting the standard inner product and $\mathbb{E}_\square$ denoting the edge set of $\mathbb{Z}^d$. It is straightforward that $H(\sigma) = 2N(\sigma)$. We then define the probability of a configuration σ to be proportional to $e^{-\beta H(\sigma)} = t^{H(\sigma)}$ with $t = e^{-\beta}$. This way of writing the model shows that it is reflection positive (for reflections through planes of vertices) in all dimensions.

We are interested in the low-fugacity / low-temperature regime of the model. There, since rod ends are disfavored, the rods that appear tend to be long and orientational sym-

metry breaking may occur. Indeed, we expect this regime to exhibit a nematic phase, with exactly d extremal and periodic Gibbs measures, with each measure characterized by an orientation in which monomers appear abundantly while monomers of other orientations are rare. Moreover, in the i th such measure (where e_j for $j \neq i$ are rare) we expect a typical configuration to resemble a perturbation of a union of one-dimensional systems in the i th direction in which only the states 0 and e_i are allowed. In particular, the density of e_i should be approximately $1/2$ since such one-dimensional systems are invariant to swapping the two allowed states (0 and e_i). We are able to prove these properties when $d = 2$ using the techniques of the current work (thinking of the sticks of the 2×2 hard-square model as the rods of the oriented monomer model).

We point out connections between the oriented monomer model and the existing literature. First, if one removes the possibility of vacancies then, in two dimensions, one recovers the “exactly solvable” version of the model of Ioffe–Velenik–Zahradník [37] which has an exact mapping to the Ising model. Second, there is also similarity with the two-dimensional interacting dimer model studied by Heilmann–Lieb [35, Model I] and Jauslin–Lieb [40]: The models become identical when a specific relation between the dimer activity and dimer interaction energy is imposed and, further, the interacting dimers are replaced by interacting oriented monomers. Lastly, the oriented monomer model with a fixed number of vacancies is equivalent to a model studied in [46].

As mentioned, we hope that the oriented monomer model may be handy in understanding orientational order in higher dimensions because we believe that it represents some of the essential difficulties in those problems, while cutting down on some technicalities. Specifically, it resembles the lattice rod models of Section 10.2 but has the advantage of having reflection positivity and the further advantage that the nematic phase is expected at a perturbative regime (low fugacity) rather than at intermediate fugacity. In addition, it may be of help in analyzing the lattice hard cubes packing model. As mentioned, this was indeed the case for us when studying the hard-square model.

10.4. Packing Euclidean disks on the lattice with the sliding phenomenon

Continuing a discussion from Section 1.3.4, we consider hard-core models of Euclidean disks of fixed diameter D with centers restricted to lie on a planar lattice. There is a finite list of diameters for which the maximal-density packings exhibit a “sliding instability”. For these diameters, it is not known whether there are multiple Gibbs measures at high fugacity (except for the one case resolved by the current work) and our goal in this section is to speculate on this question.

10.4.1. Basic definitions. The base lattice $\mathbb{W}$ is either the square ($\mathbb{Z}^2$), the triangular ($\mathbb{A}_2$) or the hexagonal/honeycomb ($\mathbb{H}_2$) lattice, normalized so that nearest-neighbor points are at (Euclidean) distance 1. Configurations are packings of Euclidean disks of diameter D with disjoint interiors and centers on the base lattice. We restrict to values of D which are attainable as the Euclidean distance between points in the base lattice (for $\mathbb{Z}^2$, all values of the form $D^2 = a^2 + b^2$ and for $\mathbb{A}_2$ and $\mathbb{H}_2$ all values of the form

$\mathbb{Z}^2$:

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	...	21	...	31	...	34	...	39
D^2	2	4	5	8	9	10	13	16	17	18	20	25	26	29	...	45	...	72	...	80	...	90

Tab. 1. The correspondence between the k -NN and D^2 notation for $\mathbb{Z}^2$. The first few cases of sliding are highlighted.

$\mathbb{H}_2$:

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	...	46
D^2	3	4	7	9	12	13	16	19	21	25	27	28	31	36	...	133

Tab. 2. The correspondence between the k -NN and D^2 notation for $\mathbb{A}_2$ and $\mathbb{H}_2$. All the cases of sliding in $\mathbb{H}_2$ are highlighted.

$D^2 = a^2 + b^2 + ab$). The model is known in the physics literature as the k -nearest-neighbor (k -NN) hard-core lattice gas, with k being the number of distinct positive lattice distances smaller than D (see Tables 1 and 2).

A maximal-density *periodic* packing is called a *periodic ground state* (PGS). Following [50–52], the *sliding phenomenon* is defined to occur for the pair $(\mathbb{W}, D)$ if there are infinitely many PGSs (this definition suffices for our planar setting; see [53] for a study of $\mathbb{Z}^3$).

10.4.2. Geometric characterization of periodic ground states. We now introduce key notions from the approach used by Mazel–Stuhl–Suhov (MSS) [50–52] to give a geometric description of the set of PGSs. This approach allows one to determine the cases where sliding occurs, and provides further information regarding the ground states of these cases, which should be important in understanding the high-fugacity behavior (cf. the discussion of fully packed configurations in Section 1.2). The notions described here are used below to comment on the sliding cases. For simplicity, we restrict to the case $\mathbb{W} = \mathbb{Z}^2$; the treatment of the other cases follows similar ideas, and we indicate some of the differences in Section 10.4.4.

MSS make the following definitions: A $\mathbb{Z}^2$ -*triangle* is a triangle with vertices on $\mathbb{Z}^2$. A $\mathbb{Z}^2$ -triangle with side lengths $\geq D$ and angles $\leq 90^\circ$ is called an *M-triangle* if it has minimal area among such $\mathbb{Z}^2$ -triangles. Given a configuration, the triangles forming the Delauney triangulation of the set of disk centers are called *C-triangles* (of the configuration). A configuration is said to be *perfect* if all of its C-triangles are M-triangles.

Theorem 10.1 ([51]). *A periodic configuration is a PGS iff it is perfect.*

The theorem follows from the following claims: (i) Every triangulation by M-triangles is the Delauney triangulation of a configuration with density $1/S(D)$, with $S(D)$ denoting twice the area of an M-triangle. (ii) A perfect configuration exists, since an M-triangle may be extended to a triangulation consisting of translations and 180° rotations of it. (iii) All configurations have density at most $1/S(D)$ and a non-perfect periodic configuration has density that is strictly less than $1/S(D)$. The last claim is non-trivial and is established in [51, Lemmas 3.5, 3.6].

The theorem provides a handy tool to study PGSs for a given exclusion diameter D . As a first step, one should understand the M-triangles for that D (a task which may be carried out by a computer search). Then, one may study the ways in which these triangles may be assembled together to form periodic triangulations.

A necessary condition for sliding to occur for a given exclusion diameter D is the existence of two distinct M-triangles with a common edge, termed the *sliding base*, both having their third vertex on the same side of the common edge. See Figure 10.1 for the case $D^2 = 29$.

10.4.3. Sliding on $\mathbb{Z}^2$. The list of sliding cases on $\mathbb{Z}^2$ was confirmed by [42,51] (a partial list is also in [56]). The first nine cases are highlighted in Table 1. We recall that the first among these cases ($D^2 = 4$) is the subject of the current work. At this point we refer the reader to [51, Section 2.2] for the list of sliding cases on $\mathbb{Z}^2$, a visualization of sliding bases in some specific cases, and some discussion of the resulting ground states.

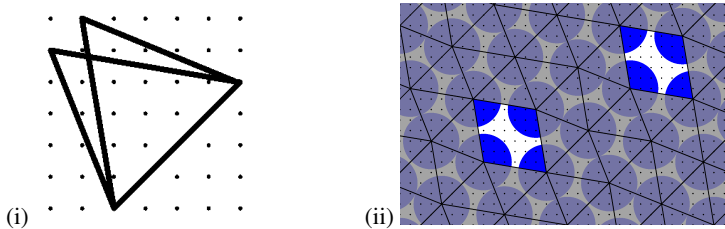


Fig. 10.1. (i) Two M-triangles for $D^2 = 29 = 5^2 + 2^2$, sharing their sliding base. (ii) A configuration for $D^2 = 29$. The C-triangles which are M-triangles are displayed in transparent grey color.

Monte Carlo simulations [55] carried out for $D^2 \leq 20$ indicate that the high-fugacity phase is columnar in all of the sliding cases (interestingly, two phase transitions are predicted for $D^2 = 8, 18, 20$). Here, we point to an extra feature present only in the sliding cases with $D^2 > 20$, which leads us to believe that the lattice's 90° rotational symmetry is broken in the high-fugacity phase (leading to multiple Gibbs measures).

In all sliding cases except for the first five cases (i.e., when $D^2 > 20$), the following property holds: each M-triangle has at most one sliding base. Consequently, two internally disjoint M-triangles that share an edge either do not have sliding bases, or have their unique sliding bases parallel to each other (see Figure 10.1). This leads to the following heuristic argument supporting the multiplicity of Gibbs measures in high fugacity: Consider a configuration sampled from a high-fugacity Gibbs measure. It is natural to believe that most C-triangles are M-triangles, leading to the existence of a unique infinite connected component of M-triangles. If this is the case, by the property above, either all the M-triangles in the unique infinite component have no sliding bases, or all of them have their sliding bases oriented parallel to the same line. We believe that the second possibility is entropically favored (see Figure 10.2). The orientation of the line to which the sliding bases are parallel is thus a (tail measurable and translation invariant) observable which

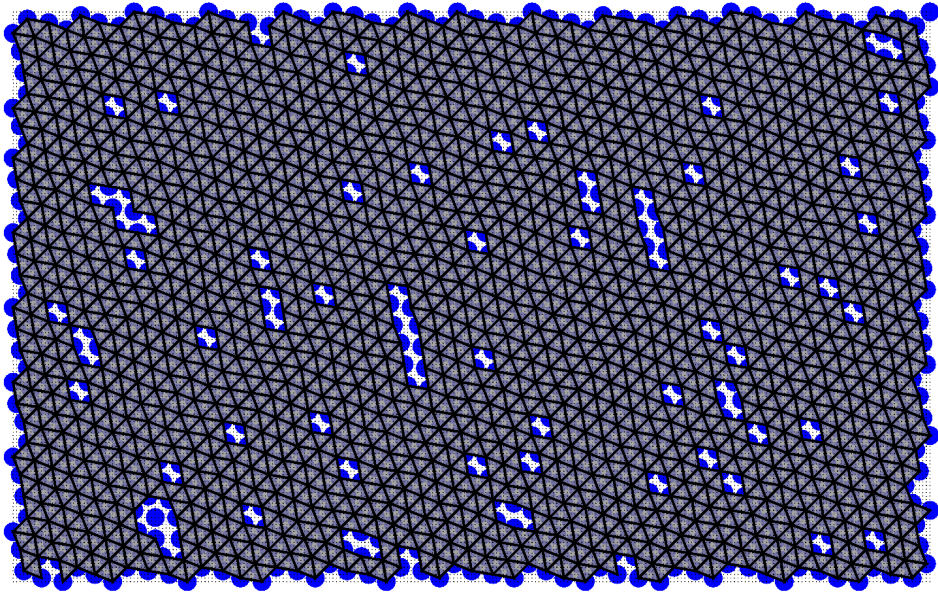


Fig. 10.2. Simulation for $D^2 = 29$ with fugacity $\lambda = 2000$ on a torus of dimensions 259×161 . In line with the proposed heuristic, all the M-triangles have their sliding base parallel to the line $x = y$, and the disks are arranged in columns parallel to this line.

may be used to distinguish different Gibbs measures (e.g., a high-fugacity ergodic Gibbs measure will be singular with respect to its 90° rotation).

In the cases where the above heuristic applies, we conjecture that typical configurations display the following order. All C-triangles but a rare set are M-triangles with a sliding base parallel to a shared direction. Thus centers are arranged in columns parallel to the shared direction. The columns are occasionally interrupted by gaps (analogous to the double vacancies in the 1D systems of the 2×2 hard-square model). In contrast to the case studied in this work (where correlations between neighboring columns are small), the columns interact more strongly since shifting a column parallel to its direction does not in general result in a valid configuration.

10.4.4. Sliding on $\mathbb{H}_2$. On the triangular lattice $\mathbb{A}_2$, due to its symmetry, there is always an equilateral lattice triangle with side length D . By arguments similar to those described in Section 10.4.2, it follows that for the disk models on $\mathbb{A}_2$ the sliding phenomenon never occurs.

For the case of $\mathbb{H}_2$, MSS analyze the PGSs using an approach similar to the one described above for $\mathbb{Z}^2$, with some modification. The notion of M-triangles is replaced by that of *MRA-triangles*, defined in [50, Section 4.2] using a notion of “redistributed area”. In the equivalent of Theorem 10.1 for the case of $\mathbb{H}_2$, the direct implication that every PGS is perfect still holds; however, the converse does not, since MRA-triangles do not

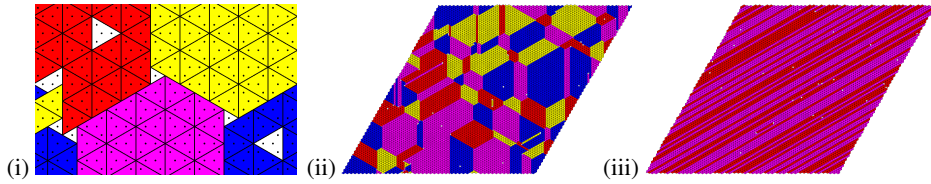


Fig. 10.3. Images of the $D^2 = 4$ model on the hexagonal lattice. The model is represented using triangular tiles, consisting of four lattice points. (i) A closeup image of a configuration. The black dots are elements of $\mathbb{H}_2$. (ii, iii) MCMC simulations with $\lambda = 700$ (see description in text).

necessarily all have the same area. It also happens that there are finitely many values of D where there are PGSSs that are not lattices in the sense of an additive group, but still no sliding occurs.

On $\mathbb{H}_2$, there are exactly four cases of sliding [50]: $D^2 = 4, 7, 31, 133$. We refer the reader to [50, Section 8] for a discussion and visualization of the resulting perfect configurations.

For the cases $D^2 = 31, 133$ we note that all MRA-triangles are of the same area, implying that every perfect configuration has maximal density. While the heuristic presented in the previous subsection does not apply to these cases, as there exist (equilateral) MRA-triangles for which all sides are sliding bases, the specific geometry of these cases still leads us to conjecture that columnar order arises at high fugacity, with three possible orientations.

For the case $D^2 = 4$, configurations may be equivalently represented as a packing of equilateral triangles of side length 2, with vertices restricted to the $\mathbb{A}_2$ lattice (in this equivalence, one rescales $\mathbb{H}_2$ to be dual to $\mathbb{A}_2$). This is illustrated in Figure 10.3(i), where tiles are painted in four colors corresponding to the parities of each tile's center when expressed in the basis $1, e^{2\pi i/3}$. MCMC simulations with local moves at fugacity $\lambda = 700$ did not converge and led to different results depending on the starting position. Figure 10.3(ii) depicts the result starting from an empty configuration, in which the domains of uniform color vaguely resemble the faces of a randomly deformed hexagonal lattice. Figure 10.3(iii) depicts the result starting from a fully packed configuration with tiles of a single color, in which columnar order is exhibited. Thewes–Fernandes [75, Section B] consider this model in the physical literature, predict a columnar high-fugacity phase and further discuss the intermediate fugacity regime.

For the case $D^2 = 7$, configurations may equivalently be represented as packings of “trimers”, where each trimer is a union of three pairwise neighboring faces of $\mathbb{H}_2$; see Figure 10.4. For the fully packed version of this model an exact solution was found by Verberkmoes–Nienhuis [81, 82] (see also Propp [67] for related enumeration problems). The case $D^2 = 7$ is discussed by Thewes–Fernandes [75, Section C] where, interestingly, it is predicted that the model is disordered at all finite fugacities.

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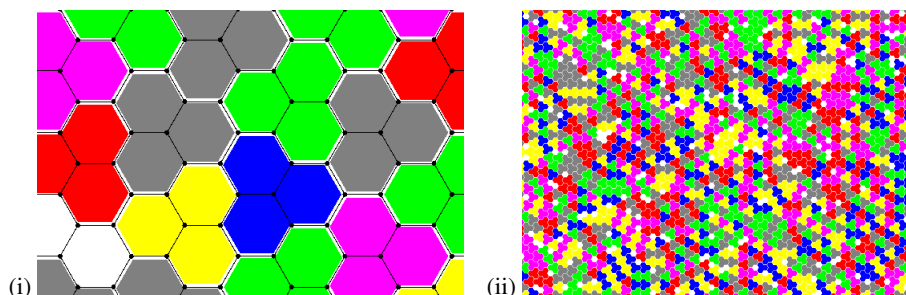


Fig. 10.4. (i) A configuration for $D^2 = 7$, viewed as a packing of “trimers” with centers on $\mathbb{H}_2$. (ii) A portion of a result of an MCMC simulation with $\lambda = 600$.

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