J. Eur. Math. Soc. 28, 5107–5111 (2026) DOI 10.4171/JEMS/1801

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Corrigendum to "Families of singular Kähler–Einstein metrics"

Received February 5, 2026

Abstract. We correct an error in [J. Eur. Math. Soc. 25, 2697–2762 (2023)].

Keywords: singular Kähler–Einstein metrics, families of varieties.

The purpose of this corrigendum is to point out a mistake in [1] and explain how to correct it. We thank Rui Tang for pointing out the error.

In Proposition D and Theorem E of the introduction of *loc. cit.*, the lower bound for the potential of the Kähler–Einstein metric is incorrect. The corresponding error in the body of the text lies in Claim 4.5 and Lemma 4.6, which affects Proposition 5.9 and Theorem 5.11 (the last two statements being simply a repetition of Proposition D and Theorem E). We will give below a correct version of Claim 4.5 and Lemma 4.6 along with their proofs, and the corrected statements for Proposition D and Theorem E (the proof being identical to that in [1] once Claim 4.5 and Lemma 4.6 have been fixed).

- Proposition D and Proposition 5.9 should be replaced by the following text.

Given a stable variety X and a log resolution of singularities $f : Y \to X$ write $K_Y = f^*K_X + \sum_{i\in I} a_i E_i$, where $E = \sum_{i\in I} E_i$ is a divisor with simple normal crossings whose components are either exceptional or sit above the divisorial components of $\mathrm{Sing}(X)$. We set $J := \{i \in I;\ a_i = -1\}$, which we assume to be non-empty, and define the non-klt locus of X by $\mathrm{Nklt}(X) = \bigcup_{j\in J} f(E_j)$; it is independent of f. Given $K \subset I$,

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Mathematics Subject Classification 2020: 32Q20 (primary); 32U05, 32W20 (secondary).

we set $E_K = \bigcap_{i \in K} E_i$ and introduce the invariants

$$\nu(f) := \max\{|K|;\ K \subset J,\ E_K \neq \emptyset\} \quad \text{and} \quad \nu := \inf_f \nu(f), \tag{1}$$

where the infimum is taken over all the log resolutions of X. We have $\nu \in \{1, \dots, n\}$.

Proposition 1. *For any $\varepsilon > 0$, there is a constant C_ε such that*

$$C_1 \geqslant \varphi \geqslant -(n + \nu + \varepsilon) \log(-\log|s|) - C_\varepsilon, \tag{2}$$

where $(s = 0)$ is any reduced divisor containing the non-klt locus of X and ν is as in (1).

This estimate is almost optimal as the following two examples show.

Example 2. Let $X = \overline{\mathbb{B}^n/\Gamma}^{\mathrm{BBS}}$ be the Baily–Borel–Satake compactification of a ball quotient, where $\Gamma \subset \mathrm{PU}(n, 1)$ is a discrete group acting freely with finite covolume. Then X is a normal stable variety and it admits a resolution $f : \overline{X} \to X$ which is a toroidal compactification of $\mathbb{B}^n/\Gamma$ obtained by adding disjoint abelian varieties D_i. That is, $\nu = \nu(f) = 1$. If one sets $D = \sum D_i$, the potential φ of the Kähler–Einstein metric on $(\overline{X}, D)$ with respect to a smooth form in $c_1(K_{\overline{X}} + D)$ satisfies

$$\varphi = -(n + 1) \log(-\log|s_D|) + O(1)$$

if $(s_D = 0) = D$.

Example 3. Let $X = \overline{\Delta^2/\Gamma}^{\mathrm{BBS}}$ be the Baily–Borel–Satake compactification of a Hilbert modular surface, where $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ is a discrete group whose diagonal action on the bi-disk Δ^2 is free with finite covolume. It is a normal stable surface and the exceptional divisor D of its minimal resolution $f : \overline{X} \to X$ consists of cycles of $\mathbb{P}^1$'s. In particular, we have $\nu = \nu(f) = 2$. Moreover, it is known that the potential φ of the Kähler–Einstein metric on $(\overline{X}, D)$ with respect to a smooth form in $c_1(K_{\overline{X}} + D)$ satisfies

$$\varphi = -4 \log(-\log|s_D|) + O(1)$$

if $(s_D = 0) = D$ (see [2, pp. 57–58]).

- Theorem E and Theorem 5.11 should be replaced by the following text.

Given a semistable model $f : \mathcal{Y} \to \mathcal{X}$ write $K_{\mathcal{Y}} + Y_0 = f^*(K_{\mathcal{X}} + X_0) + \sum_{i \in I} a_i E_i$, where Y_0 is the strict transform of X_0. We set $J := \{i \in I;\ a_i = -1\}$, which we assume to be non-empty and define the non-klt locus of $(\mathcal{X}, X_0)$ by $\mathrm{Nklt}(\mathcal{X}, X_0) = \bigcup_{j \in J} f(E_j)$; it is independent of f. Given $K \subset I$, we set $E_K = \bigcap_{i \in K} E_i$ and introduce the invariants

$$\nu(f) := \max\{|K|;\ K \subset J,\ E_K \neq \emptyset\} \quad \text{and} \quad \nu := \inf_f \nu(f), \tag{3}$$

where the infimum is taken over all the semistable models $f : \mathcal{Y} \to \mathcal{X}$. We have $\nu \in \{1, \dots, n\}$.

Theorem 4. *Let $\mathcal{X}$ be a normal Kähler space and let $\pi : \mathcal{X} \to \mathbb{D}$ be a proper, surjective, holomorphic map such that*

- *each schematic fiber X_t has semi-log-canonical singularities,*
- *$K_{\mathcal{X}/\mathbb{D}}$ is an ample $\mathbb{Q}$-line bundle.*

In particular, X_t is a stable variety for any $t \in \mathbb{D}$. Assume additionally that the central fiber X_0 is irreducible.

Let $\omega_{X_t} + dd^c\varphi_t$ be the Kähler–Einstein metric of X_t and let $D = (s = 0) \subset \mathcal{X}$ be a divisor which contains $\mathrm{Nklt}(\mathcal{X}, X_0)$. *Fix some smooth hermitian metric $|\cdot|$ on $\mathcal{O}_{\mathcal{X}}(D)$. Up to shrinking $\mathbb{D}$, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$C_1 \geqslant \varphi_t \geqslant -(n + \nu + \varepsilon)\log(-\log|s|) - C_\varepsilon$$

on X_t for any $t \in \mathbb{D}$, where ν is defined in (3).

- Claim 4.5 should be replaced by the following.

Claim 5. *There exists $\delta > 0$ small enough such that for any $\varepsilon > 0$, there exists a constant C_ε such that for any $t \in \mathbb{D}$,*

$$\int_{f^{-1}(U_t)} e^{(\nu(f)+\varepsilon)\psi_F - \delta\psi_{\mathrm{klt}}} f^*(\Omega_t \wedge \bar{\Omega}_t)^{1/m} \leqslant C_\varepsilon,$$

where $\nu(f)$ is defined in (3).

Proof. The statement is local on $\mathcal{Y}$, so it is enough to control the integrals over V_t. Up to relabeling, one can assume that $Y_0^{(k_0)} \cap V = (z_0 = 0)$, $F \cap V = (z_1 \cdots z_s = 0)$ so that for $s + 1 \leqslant i \leqslant p$, $f^*\Omega_t$ has a pole of order at most $m - 1$ along $(z_i = 0)$. Given the definition of $\nu(f)$, we have

$$\nu(f) \geqslant s. \tag{4}$$

We implicitly assumed that V meets $Y_0^{(k_0)}$; this actually does not matter much for the computation which is insensitive to whether that condition is fulfilled or not. Using [1, (4.5)], our integral is bounded by the following one:

$$\int_{V_t} \frac{1}{\prod_{i=1}^s |z_i|^2} \cdot \frac{1}{(-\log\prod_{i=1}^s |z_i|)^{\nu(f)+\varepsilon}} \cdot \prod_{i=s+1}^p \frac{1}{|z_i|^{2(\delta-a_i)}}\, d\lambda_{\mathbb{C}^n},$$

where $-1 < a_i < 0$ and $V = \prod_{i=0}^n\{|z_i| < 1\} \subset \mathbb{C}^{n+1}$ and $V_t = V \cap \{z_0 \cdots z_p = t\}$. By Fubini's theorem, one can reduce the integral to $V_t^p := V_t \cap \mathbb{C}^{p+1}$ (i.e. fixing $z_{p+1}, \ldots, z_n$). There is no harm in assuming that $\delta < \min_i \frac{1+a_i}{2}$ so that the integral is bounded by

$$\int_{V_t^p} \frac{1}{\prod_{i=1}^s |z_i|^2} \cdot \frac{1}{(-\log\prod_{i=1}^s |z_i|)^{\nu(f)+\varepsilon}} \cdot \prod_{i=s+1}^p \frac{1}{|z_i|^{2(1-\delta/2)}}\, d\lambda_{\mathbb{C}^p}$$

Using polar coordinates, one can assume that t is real (in $(0, 1)$) and the integral becomes, over $W_t := \{(r_i)_{1\leqslant i\leqslant p} \in [0,1/2]^p;\ r_1\dots r_p \geqslant t\}$,

$$\int_{W_t} \frac{1}{\prod_{i=1}^s r_i \cdot (-\log \prod_{i=1}^s r_i)^{\nu(f)+\varepsilon}} \cdot \prod_{i=s+1}^{p} \frac{1}{r_i^{1-\delta}}\, d\lambda_{\mathbb{R}^p}.$$

As $W_t \subset \prod_{i=1}^p \{t \leqslant r_i \leqslant 1/2\}$ and the function $r \mapsto 1/r^{1-\delta}$ is integrable on $[0,1/2]$, one can appeal to Fubini's theorem to reduce the problem to showing convergence of the integral

$$\int_{[0,1]^s} \frac{dr_1\cdots dr_s}{\prod_{i=1}^s r_i \cdot (-\log \prod_{i=1}^s r_i)^{\nu(f)+\varepsilon}}. \tag{5}$$

Now, we inductively use the formula for $\alpha > 1$:

$$\int_{[0,1]^2} \frac{dx\,dy}{xy(-\log xy)^\alpha} = \frac{1}{\alpha-1}\int_0^1 \frac{dx}{x(-\log x)^{\alpha-1}}$$

to see that the integral $\int_{[0,1]^s} \frac{dx_1\dots dx_s}{x_1\dots x_s(-\log x_1\dots x_s)^\alpha}$ converges if and only if $\alpha > s$. The conclusion now follows from the observation above and the inequality (4). ■

The function ψ_F is well defined on $\mathcal{Y}$ but it does not necessarily come from $\mathcal{X}$. Given that $\mathrm{Nklt}(\mathcal{X}, X_0)$ is an analytic set in $\mathcal{X}$ and up to shrinking $\mathbb{D}$ a little, one can construct a function ρ such that

- $\rho \leqslant -1$ on $\mathcal{X}$.
- ρ is quasi-psh and has analytic singularities along $\mathrm{Nklt}(\mathcal{X}, X_0)$; in particular, it is identically $-\infty$ on that set.

We set

$$\psi := -\log(-\rho) \quad \text{on } \mathcal{X}.$$

Up to scaling ρ, one can assume that

$$f^*\psi \leqslant \psi_F. \tag{6}$$

Next, we introduce for $\varepsilon > 0$ the function $\gamma_\varepsilon := \gamma - (n+\nu(f)+2\varepsilon)\psi$ defined on U. In other words, one has

$$e^{(n+\nu(f)+2\varepsilon)\psi}(\Omega\wedge\bar{\Omega})^{1/m} = e^{-\gamma_\varepsilon}\omega^n. \tag{7}$$

Lemma 6. *With the notation above, there exists a constant $\widetilde{C}_\varepsilon$ such that*

$$\int_{U_t} |\gamma_\varepsilon|^{n+\varepsilon} e^{-\gamma_\varepsilon}\omega_t^n \leqslant \widetilde{C}_\varepsilon$$

for any $t \in \mathbb{D}$.

Proof. In order to compute the integral, we pull it back by f and work on V_t. We have successively

$$\begin{aligned}|f^*\gamma_\varepsilon| &\lesssim -\log|s_E| + \log(-\log|s_F|)\\ &\lesssim -\log|s_F| - \log|s_{F_{\mathrm{klt}}}|.\end{aligned}$$

The first inequality is a combination of [1, eq. (4.8)] and (6). To obtain the second inequality, we use the fact that $E = F \cup F_{\mathrm{klt}}$ to split the term $\log|s_E|$, while $\log(-\log|s_F|)$ can be absorbed by the more singular $-\log|s_F|$. The integral to bound becomes

$$\int_{V_t} [(-\log|s_F|)^{n+\varepsilon} + (-\log|s_{F_{\mathrm{klt}}}|)^{n+\varepsilon}] e^{(n+\nu(f)+2\varepsilon)\psi_F} f^*(\Omega_t \wedge \overline{\Omega}_t)^{1/m},$$

which is itself controlled by

$$\int_{V_t} e^{(\nu(f)+\varepsilon)\psi_F} f^*(\Omega_t \wedge \overline{\Omega}_t)^{1/m} + \int_{V_t} e^{(n+1)\psi_F - \delta\psi_{\mathrm{klt}}} f^*(\Omega_t \wedge \overline{\Omega}_t)^{1/m}$$

for any given $\delta > 0$. The lemma now follows from Claim 5. ■

References

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