J. Eur. Math. Soc. 28, 4639–4674 (2026) DOI 10.4171/JEMS/1803

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Minimal resolutions of monomial ideals

Dedicated to the memory of John ("Jack") Eagon

Received October 14, 2022; revised January 19, 2026

Abstract. An explicit, closed-form combinatorial minimal free resolution of an arbitrary monomial ideal I in a polynomial ring in n variables over a field of characteristic 0 (and almost all positive characteristics) is defined canonically, without any choices, using higher-dimensional generalizations of combined spanning trees for cycles and cocycles (*hedges*) in the upper Koszul simplicial complexes of I at lattice points in $\mathbb{Z}^n$. The differentials in these *sylvan resolutions* are expressed as matrices whose entries are sums over lattice paths of weights determined combinatorially by sequences of hedges (*hedgerows*) along each lattice path. This combinatorics enters via an explicit matroidal expression for Moore–Penrose pseudoinverses as weighted averages of splittings defined by hedges. The translation from Moore–Penrose combinatorics to free resolutions relies on Wall complexes, which construct minimal free resolutions of graded ideals from vertical splittings of Koszul bicomplexes. The algebra of Wall complexes applied to individual hedgerows yields explicit but noncanonical combinatorial minimal free resolutions of arbitrary monomial ideals in any characteristic.

Keywords: monomial ideal, minimal free resolution, simplicial complex, sylvan matrix, Moore–Penrose pseudoinverse, simplicial homology, lattice path, spanning tree, Koszul complex.

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Mathematics Subject Classification 2020: 05E40 (primary); 05E45, 13D02, 13F55, 57Q05, 05C05, 13C13, 13F20 (secondary).

1. Introduction

Overview

Irving Kaplansky had a habit of circulating to his students precise problem lists for potential dissertation topics. One such list contained a problem on ideals generated by subdeterminants of a matrix, which resulted in [14]. A later list, we speculate, concerned monomial ideals, resulting in Taylor's thesis [31], the first general construction of free resolutions for arbitrary monomial ideals. Since then the problem of finding minimal free resolutions of monomial ideals in polynomial rings has been central to the combinatorial side of commutative algebra, stimulating an enormous amount of research on the algebraic, combinatorial, and homological structure of monomial ideals, including hundreds of research papers and several influential books. The ultimate goal is a free resolution that is universal, canonical, and minimal, with closed-form combinatorial formulas for the differentials. This means that the construction should work for any monomial ideal, involve no choices, have no redundancy in algebraic or numerical senses, and be explicit in terms of the discrete input that determines a monomial ideal.

This goal has long been deemed impossible. Indeed, by the time Bayer and Sturmfels introduced the hull resolution [4], a nonminimal construction that works for any monomial ideal and more general monomial modules, they noted that "unlike minimal resolutions, it respects symmetry and is free from arbitrary choices" [4, p. 123]. Current research on minimal free resolutions of monomial ideals, which remains as active as ever, routinely and explicitly laments the lack of a general handle on the differentials in minimal free resolutions of monomial ideals; see [1, 9, 34] for a sample recently predating this paper.[1]

The sylvan resolutions introduced here encompass two decisive advances. First, we introduce *sylvan matrices* to represent homomorphisms between simplicial homology groups in a manner that circumvents selection of bases for homology and therefore allows for symmetry in minimal resolutions (Definition 3.8). Second, this potential for symmetry is realized on arbitrary monomial ideals (Theorem 3.10) in canonical, closed-form combinatorial expressions for sylvan matrix entries, over fields of characteristic 0 and most positive characteristics, using Moore–Penrose pseudoinverses of simplicial differentials, characterized in terms of matroidal data—higher-dimensional analogues of spanning trees for cycles and cocycles—following Berg [6]. In any characteristic, the spanning-tree framework produces noncanonical but nonetheless universal combinatorial constructions of minimal free resolutions of monomial ideals (Corollary 9.5).

In the time since the first version of this paper was publicly posted in 2019, it has found applications to fundamental problems extending far beyond monomial ideals. Indeed, our constructive method for free resolutions is the key point in bounding the projective dimension of an arbitrary (not necessarily finitely generated or graded) module over the

[1] The MathSciNet review of [34], the year before a preprint version of this article was publicly posted, begins, "The present paper is related with the important and wide open problem of explicitly describing minimal free resolutions of monomial ideals."

ring of polynomials with real exponents [18], which has the consequence of identifying smooth objects in the category of quantum noncommutative toric varieties as introduced by Katzarkov, Lupercio, Meersseman, and Verjovsky in 2020 [21]. And our main result lifts to monomial modules [22], yielding universal, canonical, closed-form minimal resolutions of lattice ideals, where explicit resolutions had been unknown outside of the toric case [32]. Looking farther ahead, a version of our construction taking into account representation-theoretic information in the presence of a group action could produce explicit minimal resolutions in richer contexts, such as determinantal ideals or, with the usual 1-parameter scaling action, any standard-graded ideal. (The monomial case here arises from the usual n-parameter torus action.) Sylvan homomorphisms in such contexts would be nontrivial linear maps among sections of bundles canonically associated to projective algebraic varieties.

1.1. Prior work

Formulas for the Betti numbers—the ranks of free modules in a minimal free resolution—based on the combinatorial topology of simplicial complexes have been known since the 1970s through work of Hochster [19] and others, but the differentials of the resolutions have remained elusive. All prior studies that produce differentials in resolutions of monomial ideals have dispensed with one or more of the desired properties. For example, Taylor's resolution [31] is not minimal; Lyubeznik's improvement on it [24] is not minimal or canonical; Eliahou–Kervaire resolutions of stable ideals [17] are not universal; Eagon's resolutions by Wall complex [13] are not a priori combinatorial or canonical, although the point of our work here is that combinatorics of Moore–Penrose pseudoinverses remedies both of these; hull resolutions [4] are not minimal; Scarf resolutions of generic monomial ideals [3, 28] are not universal, although they can be made universal by generic deformation, sacrificing minimality and canonicality; Yuzvinsky's resolutions using splittings in the manner of Eagon, applied to LCM lattices instead of Koszul simplicial complexes [35], are not a priori combinatorial, and Yuzvinsky claims they are not canonical, although Moore–Penrose pseudoinverses remedy the latter and can in principle remedy the former (a topic for future work); resolutions for shellable monomial ideals [2] are not universal; planar graph resolutions [26] are not universal, being defined only in three variables and not even canonical there; resolutions supported on order complexes of Betti posets [33] are not minimal; and Buchberger resolutions [29] are not canonical or minimal.

The strongest combinatorial structural result for minimal free resolutions of monomial ideals available before the current work is that they all admit hcw-poset structures [9]—essentially poset generalizations of the cellular structures in [4]. However, this poset framework does not construct resolutions but rather imposes structures a posteriori on a given resolution. That said, in a development subsequent to the current work, Tchernev produced canonical, minimal, universal resolutions that are not closed-form but can be seen as combinatorial in the sense of being algorithmic [32].

1.2. Koszul simplicial complexes and Hochster's formula

The combinatorics of minimal resolutions of monomial ideals is grounded in the local combinatorics near lattice points in the partially ordered set of exponent vectors.

Definition 1.1. For a monomial ideal I and a nonnegative integer vector $\mathbf{b} \in \mathbb{N}^n$ with n entries, the (*upper*) *Koszul simplicial complex* of I in degree $\mathbf{b}$ is

$$K^{\mathbf{b}} I = \{\tau \in \{0,1\}^n \mid \mathbf{x}^{\mathbf{b}-\tau} \in I\}.$$

That is, standing at the lattice point $\mathbf{b}$, thought of as an exponent vector on a monomial in the ideal I, one looks backward to see which (combinations of distinct) coordinate directions one can move along to remain in I.

Theorem 1.2 (Hochster's formula). *Fix a monomial ideal $I \subseteq \Bbbk[\mathbf{x}]$ and a degree vector $\mathbf{b} \in \mathbb{N}^n$. There is a natural isomorphism of vector spaces*

$$\mathrm{Tor}_i(\Bbbk, I)_{\mathbf{b}} = \widetilde{H}_{i-1}(K^{\mathbf{b}} I; \Bbbk).$$

Consequently, the Betti numbers of I in degree $\mathbf{b}$ can be expressed as

$$\beta_{i,\mathbf{b}}(I) = \dim_{\Bbbk} \widetilde{H}_{i-1}(K^{\mathbf{b}} I; \Bbbk).$$

For an exposition and proof, see [27, Theorem 1.34] and surrounding material.

Theorem 1.2 is the sense in which simplicial homology categorifies monomial Betti numbers. At issue in Kaplansky's problem is how to categorify the differentials in a minimal free resolution of I. More precisely, any attempt to produce general minimal free resolutions of arbitrary monomial ideals must reduce—explicitly or implicitly—to solving the following concrete problem.

Problem 1.3. *For a monomial ideal $I \subseteq \Bbbk[\mathbf{x}]$, produce vector space homomorphisms*

$$\bigoplus_{\mathbf{a} \prec \mathbf{b}} \widetilde{H}_{i-1}(K^{\mathbf{a}} I; \Bbbk) \leftarrow \widetilde{H}_i(K^{\mathbf{b}} I; \Bbbk)$$

for all $i \in \mathbb{N}$ and multigraded degrees $\mathbf{b} \in \mathbb{N}^n$ whose induced $\Bbbk[\mathbf{x}]$-module homomorphisms

$$\bigoplus_{\mathbf{a} \in \mathbb{N}^n} \widetilde{H}_{i-1}(K^{\mathbf{a}} I; \Bbbk) \otimes_{\Bbbk} \Bbbk[\mathbf{x}](-\mathbf{a}) \leftarrow \bigoplus_{\mathbf{b} \in \mathbb{N}^n} \widetilde{H}_i(K^{\mathbf{b}} I; \Bbbk) \otimes_{\Bbbk} \Bbbk[\mathbf{x}](-\mathbf{b})$$

constitute a free resolution of I.

Such a free resolution would automatically be minimal, by the Betti number computation in Theorem 1.2. To connect the vector space homomorphisms to the induced module homomorphisms in more detail, first see the right-hand side of the vector space homomorphism as $\mathrm{Tor}_{i+1}(\Bbbk, I)_{\mathbf{b}}$. Thinking of it as (the $\Bbbk$-linear span of) a basis for the $(i+1)$st syzygies in degree $\mathbf{b}$, the differential in a minimal free resolution preserves the degree $\mathbf{b}$ while taking this basis to homological stage i. The summands in homological stage i that contribute nonzero components to degree $\mathbf{b}$ have the natural form $\operatorname{Tor}_i(\Bbbk, I)_{\mathbf{a}} \otimes_{\Bbbk} \Bbbk[\mathbf{x}]$ for some $\mathbf{a} \prec \mathbf{b}$. The degree $\mathbf{b}$ component of this free summand is

$$(\operatorname{Tor}_i(\Bbbk, I)_{\mathbf{a}} \otimes_{\Bbbk} \Bbbk[\mathbf{x}])_{\mathbf{b}} = \mathbf{x}^{\mathbf{b}-\mathbf{a}} \operatorname{Tor}_i(\Bbbk, I)_{\mathbf{a}} = \widetilde{H}_{i-1}(K^{\mathbf{a}} I; \Bbbk)$$

as an ungraded vector space. Taking the direct sum over $\mathbf{a} \prec \mathbf{b}$ yields the left-hand side of the vector space homomorphism in Problem 1.3.

1.3. Sylvan combinatorics of the canonical differential

Given a free resolution whose syzygy modules have specified bases, the differentials can be expressed by matrices of scalars using monomial matrices [25, Section 3]. However, one of the fundamental obstacles to overcome in expressing an explicit, closed-form description of a minimal free resolution of an arbitrary monomial ideal is how to present a homomorphism canonically between homology vector spaces, which do not possess natural bases. Our approach is to specify a linear map $\widetilde{C}_{i-1} K^{\mathbf{a}} I \leftarrow \widetilde{C}_i K^{\mathbf{b}} I$ from i-chains to $(i-1)$-chains using their natural bases but then ensure that this linear map induces a well defined homology homomorphism $\widetilde{H}_{i-1} K^{\mathbf{a}} I \leftarrow \widetilde{H}_i K^{\mathbf{b}} I$. (The field $\Bbbk$ is fixed throughout and suppressed from the notation.) Thus, given any cycle of dimension i, expressed in the basis of i-simplices in $K^{\mathbf{b}} I$, the closed-form description acts on each term in the cycle to produce a cycle expressed in the basis of $(i-1)$-simplices in $K^{\mathbf{a}} I$. Different input cycles can yield different output cycles, a priori, even when they represent the same homology class, as long as homologous input cycles yield homologous output cycles. That said, our formulation takes homologous cycles to the same cycle, inducing a homomorphism $\widetilde{Z}_{i-1} K^{\mathbf{a}} I \leftarrow \widetilde{H}_i K^{\mathbf{b}} I$. This is part of the central result, Definition 3.8 and Theorem 3.10: the closed-form specification of the differential in our *canonical sylvan resolution* in characteristic 0 and almost all positive characteristics.

In more detail, Definition 3.8 and Theorem 3.10 formulate an explicit linear map

$$\widetilde{C}_{i-1} K^{\mathbf{a}} I \xleftarrow{D} \widetilde{C}_i K^{\mathbf{b}} I.$$

Specifying D is the same as specifying its entries $D_{\sigma\tau}$ for $\tau \in \widetilde{C}_i K^{\mathbf{b}} I$ and $\sigma \in \widetilde{C}_{i-1} K^{\mathbf{a}} I$. (See Convention 4.2 for the resulting matrix notation.) The combinatorics is matroidal, generalizing that of spanning trees in graphs, applied to the upper Koszul simplicial complexes of I at lattice points in $\mathbb{Z}^n$. The entries $D_{\sigma\tau}$ are expressed as weighted sums over all saturated decreasing lattice paths from $\mathbf{b}$ to $\mathbf{a}$, where the weights come from

- coefficients of faces in unique circuits or boundaries obtained by throwing one additional facet into a higher-dimensional analogue of a spanning tree, and
- determinants of submatrices indexed by the appropriate rows and columns.

The determinants are unavoidable in high dimension. They reflect the fact that integer boundaries of individual faces can contribute to bases for sublattices of varying index.

1.4. Methods

The apparent obstruction to constructing closed-form minimal resolutions has been how to appropriately relate the various Koszul simplicial homology groups that categorify the multigraded Betti numbers. Canonical homomorphisms among subquotients of the relevant homology groups constitute the spectral sequence (Corollary 6.8) of an appropriately constructed Koszul bicomplex (Definition 6.2), but alas, there is no categorically natural way to lift these homomorphisms on subquotients to homomorphisms on the intact homology. The method of Wall complexes here, following Eagon [13] (see Section 7 for an exposition), observes that any choice of splitting for the vertical differential forces the Koszul simplicial homology to split in such a way that the spectral sequence differentials collate into a compendium differential that solves Problem 1.3.

The splittings, and subsequently the Wall differentials, are made canonical by using Moore–Penrose pseudoinverses, which explains the exclusion of finitely many positive characteristics. The splittings are then made combinatorially explicit by applying summation formulas for the pseudoinverse [6, Theorem 1], [5, Theorem 2.1], which are so rarely cited that they must be largely unknown to algebraists. We rephrase these formulas as weighted averages of splittings (Corollary 5.9 and Proposition 5.10) in the context of the theory of higher-dimensional analogues of spanning trees initiated by Kalai [20], as developed by Duval, Klivans, and Martin [10, 11], Petersson [30], and Lyons [23]. Versions of Corollaries 5.9 and 5.11 in which the determinants are interpreted as orders of certain torsion subgroups in homology were proved by Catanzaro, Chernyak, and Klein [7, 8].

1.5. Noncanonical sylvan resolutions

Other combinatorial splittings of the vertical Koszul differential, arising from individual hedgerows (Definition 3.1 with Definitions 2.1 and 2.4; see Remark 9.9) contributing summands to the Moore–Penrose pseudoinverse formula, still produce combinatorial minimal resolutions. These could be suited to algorithmic computation (Remark 9.7). Moreover, these splittings—and hence the corresponding sylvan minimal resolutions—require no division and hence work over any field (Corollary 9.5). Certain existing families of minimal resolutions appear to be sylvan, with apt choices of hedges, that is, they can be constructed as Wall complexes for suitable splittings (Remark 9.9).

1.6. Logical structure

To make the prerequisites clear, the paper begins with a short, direct path to a rigorous statement of the main result—the combinatorial description of canonical minimal free resolutions of monomial ideals in Theorem 3.10. Thus Sections 2 and 3 are self-contained introductions to the relevant simplicial notions and the combinatorial assembly of these along descending lattice paths. The proof of Theorem 3.10 must wait until Section 8, as it relies on the Hedge Formula (Corollary 5.9), the Wall construction of minimal free resolutions via Koszul splittings (Corollary 7.11), and the Koszul simplicial formula for those (Theorem 8.1). No intervening result relies on the statement of Theorem 3.10.

1.7. Conventions

Convention 1.4 (Cellular notions). A CW complex K has its set K_i of i-faces and integer reduced chain groups $\widetilde{C}_i^{\mathbb{Z}} K = \mathbb{Z}\{K_i\}$ with differential $\partial_i : \widetilde{C}_i^{\mathbb{Z}} K \to \widetilde{C}_{i-1}^{\mathbb{Z}} K$. Tensoring with any field $\Bbbk$, such as the fields $\mathbb{Q}$ or $\mathbb{C}$ of rational or complex numbers, yields the reduced chain complex $\widetilde{C}_\bullet^{\Bbbk} K = \Bbbk \otimes_{\mathbb{Z}} \widetilde{C}_\bullet K = \Bbbk\{K_i\}$ over $\Bbbk$, with differential also denoted by ∂. The same conventions hold for cycles $\widetilde{Z}_i K = \ker \partial_i \subseteq \widetilde{C}_i K$, boundaries $\widetilde{B}_i K = \operatorname{im} \partial_{i+1} \subseteq \widetilde{C}_i K$, and reduced homology $\widetilde{H}_i K = \widetilde{Z}_i K / \widetilde{B}_i K$. To unclutter the notation, it helps to omit the superscript $\mathbb{Z}$ or $\Bbbk$ when the context is clear. The sign on a facet σ of a cell τ in the boundary $\partial\tau$ is written $(-1)^{\sigma \subset \tau}$.

Convention 1.5 (Polynomial and monomial notions). Fix, once and for all, an ideal I in the polynomial ring $\Bbbk[\mathbf{x}]$ in n variables $\mathbf{x} = x_1, \ldots, x_n$ over a field $\Bbbk$ that is assumed throughout to be arbitrary unless otherwise stated. Assume that I is a monomial ideal unless otherwise explicitly stated. Monomials in $\Bbbk[\mathbf{x}]$ are denoted by $\mathbf{x}^{\mathbf{a}}$ for lattice points $\mathbf{a} \in \mathbb{N}^n$. Unadorned tensor products $\otimes$ are understood as $\otimes_{\Bbbk}$.

2. Shrubs, stakes, and hedges

Analogues in higher dimension of graph-theoretic spanning trees or spanning forests play key roles. A shrubbery is a lift to $\Bbbk^n$ of a basis for the boundaries B under a map from $\Bbbk^n$; in a simplicial or cell complex setting, this lift consists of a set of faces. For the matroidally dual notion, a stake set spans a coordinate subspace whose complement is transverse to the boundary subspace B; in cellular settings, typically stake set faces have dimension one less than those in a shrubbery because they lie in the target instead of the source of the differential. See Remark 2.5 for more comments on the terminology.

Definition 2.1. Fix nonnegative integers $m, n \in \mathbb{N}$.

(1) A *shrubbery* for a surjection $B \twoheadleftarrow \Bbbk^n$ is a subset $T \subseteq \{e_1, \ldots, e_n\}$ of the standard basis such that the composite $B \twoheadleftarrow \Bbbk^n \hookleftarrow \Bbbk\{T\}$ is an isomorphism ∂_T.

(2) A *stake set* for an injection $\Bbbk^m \hookleftarrow B$ is a subset $S \subseteq \{e_1, \ldots, e_m\}$ of the standard basis such that the composite $\Bbbk\{S\} \twoheadleftarrow \Bbbk^m \hookleftarrow B$ is an isomorphism ∂_S. Basis vectors in a stake set are called *stakes*.

(3) If $\Bbbk^m \xleftarrow{\partial} \Bbbk^n$ is a linear map with image $B = \partial(\Bbbk^n)$, then a *hedge* for ∂ is a choice ST of a *shrubbery* T *for* ∂ as in item (1) and a *stake set* S *for* ∂ as in item (2).

Remark 2.2. Given a subspace $A \subseteq \Bbbk^\ell$, a shrubbery T for the surjection $\Bbbk^\ell / A \twoheadleftarrow \Bbbk^\ell$ is defined by the condition $\Bbbk^\ell = A \oplus \Bbbk^T$, while a stake set S for the injection $\Bbbk^\ell \hookleftarrow A$ is defined by the condition $\Bbbk^\ell = \Bbbk^{\overline{S}} \oplus A$.

Remark 2.3. A shrubbery and a stake set have simple interpretations in terms of a matrix for a linear map $\Bbbk^m \xleftarrow{\partial} \Bbbk^n$. For concreteness, assume that vectors in $\Bbbk^n$ are represented as columns of size n and vectors in $\Bbbk^m$ are represented as columns of size m, so ∂ is represented by an $m \times n$ matrix $M \in \Bbbk^{m\times n}$ with entries in $\Bbbk$. If rank $M = r$ then

- a shrubbery is a choice of r independent columns of M, and
- a stake set is a choice of r independent rows of M:

$$\begin{array}{ccc} & \text{shrubbery} = r\text{-subset here} & \\ & \begin{array}{ccc} e_1 & \cdots & e_n \end{array} & \\ \text{stake set} = r\text{-subset here} \quad \begin{array}{c} e_1 \\ \vdots \\ e_m \end{array} & \left[\begin{array}{ccc} & & \\ & M & \\ & & \end{array}\right] & \\ \Bbbk^m & \xleftarrow{\qquad\qquad\qquad} & \Bbbk^n \end{array}$$

Choosing a shrubbery and a stake set means choosing a maximal invertible submatrix.

Specializing Definition 2.1 to the setting of cell complexes yields cases that are used so often that they merit their own definition environment.

Definition 2.4. Fix a CW complex K and a field $\Bbbk$.

(1) A *shrubbery* in dimension i is a shrubbery $T_i \subseteq K_i$ for $\widetilde{B}^{\Bbbk}_{i-1} K \twoheadleftarrow \widetilde{C}^{\Bbbk}_i K$.

(2) A *stake set* in dimension $i-1$ is a stake set for $\widetilde{C}^{\Bbbk}_{i-1} K \hookleftarrow \widetilde{B}^{\Bbbk}_{i-1} K$.

(3) A *hedge* in K of dimension i is a choice of shrubbery in K_i and stake set in K_{i-1}.

A hedge of dimension i may be expressed as $ST_i = (S_{i-1}, T_i)$.

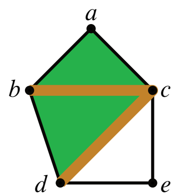
Remark 2.5. In the literature shrubberies are often known as "spanning trees", or "spanning forests", or some variant; see [7, 12], for example. We avoid these terms because they are inapt in certain ways—subcomplexes whose facets form shrubberies need not be connected in any appropriate sense (so should not be called trees) if the ambient CW complex is disconnected, and there could be forests that span in some appropriate sense but are nonetheless not spanning forests—and their precise definitions vary from paper to paper. But our botanical terminology grew from language of forests and trees, which also explains our shrubbery symbol "T" that classically stands for "tree". Regardless of terminology or notation, a hedge is matroidal information, given by subsets of fixed bases, and is hence combinatorial in nature.

Remark 2.6. Following Remark 2.3, a hedge of dimension i is equivalent to a maximal invertible submatrix in the differential $\widetilde{C}_{i-1} K \xleftarrow{\partial_i} \widetilde{C}_i K$:

$$\begin{array}{cccc} \begin{array}{r} (i-1)\text{-faces of } K \searrow \\ (\text{stake set here}) \end{array} & \begin{array}{ccc} \tau_1 & \cdots & \tau_n \end{array} & \longleftarrow & \begin{array}{l} i\text{-faces of } K \\ (\text{shrubbery here}) \end{array} \\ \begin{array}{c} \sigma_1 \\ \vdots \\ \sigma_m \end{array} & \left[\begin{array}{ccc} & & \\ & \partial_i & \\ & & \end{array}\right] & & \\ \widetilde{C}_{i-1} K & \xleftarrow{\qquad\qquad\qquad} & \widetilde{C}_i K & \end{array}$$

Remark 2.7. Definitions 2.1.(1) and (2) are plainly dual: $S \subseteq \Bbbk^m$ is a stake set for ∂ if and only if the corresponding dual basis vectors are a shrubbery for the transpose $\partial^\top$. (Hence a shrubbery and a stake set for the same differential ∂ are equinumerous.) For this reason, stake sets have been called "cotrees" in persistent homology [15].

Example 2.8. The simplicial complex



is missing the triangle cde, so $T_2 = \{abc, bcd\}$ is a shrubbery in dimension 2 (shaded triangles, in green). Indeed, the boundaries of these triangles span $\widetilde{B}_1$ because they are the only 2-simplices, and these boundaries are linearly independent because they involve distinct edge sets. The set $S_1 = \{bc, cd\}$ (thickened edges, in brown) remains linearly independent modulo $\widetilde{B}_1$ because a cycle (let alone a boundary) must have at least three edges. Since S_1 has the same size as T_2 (Remark 2.7), S_1 is a stake set in dimension 1. Hence $ST_2 = (S_1, T_2) = (\{bc, cd\}, \{abc, bcd\})$ is a hedge of dimension 2.

The horticultural picture that goes with the terminology extends: each stake is tied to the end of a unique shrub: the chain s in the next result (see Definition 2.15 (2)).

Lemma 2.9. *Fix a hedge ST for $\Bbbk^m \xleftarrow{\partial} \Bbbk^n$ and a stake $\sigma \in S$. There is a unique chain $s \in \Bbbk\{T\}$, the* shrub *of σ, whose boundary has coefficient* 1 *on σ and* 0 *on all other stakes in S.*

Proof. In the notation of Definition 2.1, $s = (\partial_S \circ \partial_T)^{-1}\sigma$, so that $\partial_S \circ \partial_T s = \sigma$. ∎

Example 2.10. Applying Lemma 2.9 to the stake cd in Example 2.8 yields the linear combination $s(cd) = -abc + bcd$ of shrubbery faces. The boundary of s involves three additional edges: ab, ac, and bd, but—as per Lemma 2.9—none of these are stakes.

The usual property of a spanning tree in a graph is that every edge outside of the spanning tree closes a unique circuit with the tree edges. The analogous well known combinatorics of shrubberies is described by way of a trivial lemma, in which the linear map α_U is simply projection modulo $\Bbbk\{U\}$, so ρ has components $r \in \Bbbk\{U\}$ and $\rho - r$.

Lemma 2.11. *Fix a field $\Bbbk$ and a vector subspace $A \subseteq \Bbbk^\ell$. For each subset U of the standard basis $\{e_1, \ldots, e_\ell\}$ such that $\Bbbk^\ell = \Bbbk\{U\} \oplus A$ there is a linear map $\alpha_U : \Bbbk^\ell \twoheadrightarrow A$ that takes $\rho \in \Bbbk^\ell$ to the unique vector $\rho - r \in A$ such that $r \in \Bbbk\{U\}$.* ∎

Definition 2.12. Lemma 2.11 is applied particularly when

(1) $U = T$ is a shrubbery for the surjection $\Bbbk^\ell/A \twoheadleftarrow \Bbbk^\ell$, in which case α_U is called the *circuit projection* and denoted ζ_T, or

(2) $U = \overline{S}$ is the complement of a stake set S for the injection $\Bbbk^\ell \hookleftarrow A$, in which case α_U is called the *boundary projection* and denoted β_S.

If T and S come from $\Bbbk^m \xleftarrow{\partial} \Bbbk^n$, then $\zeta_T : \Bbbk^n \twoheadrightarrow \ker\partial$ and $\beta_S : \Bbbk^m \twoheadrightarrow B = \partial(\Bbbk^n)$.

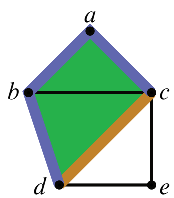
(1′) In cellular settings, when $T_i \subseteq K_i$ is a shrubbery of dimension i, the circuit projection $\zeta_{T_i} : \widetilde{C}_i^{\Bbbk} K \twoheadrightarrow \widetilde{Z}_i^{\Bbbk} K$ has a combinatorial interpretation: each i-face $\tau \notin T_i$ lies in a unique T_i*-circuit* $\zeta_{T_i}(\tau) = \tau - t \in \widetilde{Z}_i^{\Bbbk} K$ that is a cycle with coefficient 1 on τ in the CW complex with facets $\{\tau\} \cup T_i$. (If $\tau \in T_i$ then $t = \tau$, so $\tau - t = 0$.)

(2′) Similarly, when $S_i \subseteq K_i$ is a stake set, the boundary projection $\beta_{S_i} : \widetilde{C}_i^{\Bbbk} K \twoheadrightarrow \widetilde{B}_i^{\Bbbk} K$ takes each stake to the unique boundary with coefficient 1 on σ. The combinatorial interpretation in homological stage $i-1$ views the boundary $\beta_{S_{i-1}}(\sigma) = \sigma - r \in \sigma + \Bbbk\{\overline{S}_{i-1}\}$ in terms of the *hedge rim* r of σ.

Remark 2.13. The boundary projection $\beta_{S_{i-1}}(\sigma)$ of σ in Definition 2.12 (2′) is the boundary of the shrub s from Lemma 2.9 for any choice of shrubbery T_i; see Lemma 5.4.

The notion of T_i-circuit is the familiar one from graph theory, where the union of a spanning tree and an edge not in the spanning tree forms a unique cycle. The notion of hedge rim is dual but may be less familiar, so here is an illustration.

Example 2.14. Returning to the situation of Examples 2.8 and 2.10, again choose the hedge $ST_2 = (S_1, T_2) = (\{bc, cd\}, \{abc, bcd\})$ of dimension 2. The shrub and hedge rim of the stake cd are



$$s(cd) = -abc + bcd,$$

$$r(cd) = ab - ac + bd,$$

respectively, as discussed in Example 2.10.

Circuits, shrubs, and hedge rims are the core simplicial combinatorial players.

Definition 2.15. Fix a CW complex K and a field $\Bbbk$.

(1) Fix a shrubbery T_{i-1}. An $(i-1)$-face σ is *cycle-linked* to any $(i-1)$-face $\sigma' \in K_{i-1}$ with nonzero coefficient in the circuit $\zeta_{T_{i-1}}(\sigma)$ defined in Definition 2.12.

(2) Fix a hedge ST_i in K. A stake $\sigma \in S_{i-1}$ is *chain-linked* to an i-face $\tau \in K_i$ if τ has nonzero coefficient in the shrub of σ (Lemma 2.9).

(3) Fix a stake set S_i. An i-face $\rho \in K_i$ is *boundary-linked* to $\rho' \in K_i$ if ρ' has nonzero coefficient in the hedge rim of ρ: the chain $r(\rho) = \rho - \beta_{S_i}(\rho)$ (Definition 2.12 (2′)).

Write $c_\sigma(\sigma', T_{i-1})$ and $c_\sigma(\tau, ST_i)$ and $c_\rho(\rho', S_i)$ for the coefficients on σ' and τ and ρ' in the circuit, shrub, and hedge rim of an $(i-1)$-face σ, an $(i-1)$-stake σ, and an i-face ρ.

Remark 2.16. The three types of linkages behave in slightly different ways.

(1) A shrub τ is never cycle-linked to any face, because its circuit is $\tau - \tau = 0$. In contrast, every non-shrub (that is, every face that does not lie in the given shrubbery) is always cycle-linked to itself with coefficient 1 by Definition 2.12 (1′).

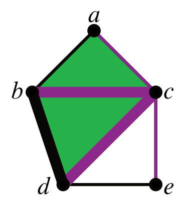
(2) A stake $\sigma \in S_{i-1}$ can only be chain-linked to a face $\tau \in T_i$. Indeed, although Definition 2.15 (2) allows $\tau \in K_i$, in fact the shrub s in Lemma 2.9 only has nonzero coefficients on faces in T_i.

(3) Similarly, a face ρ can only be boundary-linked to a non-stake $\rho' \in \overline{S}_i$, because although Definition 2.15 (3) allows $\rho' \in K_i$, in fact the hedge rim $r(\rho)$ only has nonzero coefficients on non-stakes by Definition 2.12 (2′).

Example 2.17. Return again to the simplicial complex in Examples 2.8, 2.10, and 2.14.

(1) The edges $T_1 = \{ac, bc, cd, ce\}$ form a shrubbery in dimension 1 (highlighted in purple), also known as a spanning tree in the edge graph. The 1-face $\sigma = bd$ (in **bold black**) is cycle-linked only to itself and bc and cd (in **bold purple**), the latter both with coefficient $c_\sigma(bc, T_1) = c_\sigma(cd, T_1) = -1$, because

$$\zeta_{T_1}(\sigma) = \sigma - bc - cd \qquad \text{in}$$



(2) Again choose the hedge $ST_2 = (S_1, T_2) = (\{bc, cd\}, \{abc, bcd\})$. By Example 2.14 the stake $\sigma = cd$ is chain-linked to the two 2-faces abc and bcd with coefficients

$$c_\sigma(abc, ST_2) = -1 \quad \text{and} \quad c_\sigma(bcd, ST_2) = 1.$$

(3) Let $i = 1$ and again choose $S_1 = \{bc, cd\}$. By Example 2.14 the 1-face $\rho = cd$ is boundary-linked to the three 1-faces ab, ac, and bd, with coefficients

$$c_\rho(ab, S_1) = 1, \quad c_\rho(ac, S_1) = -1, \quad \text{and} \quad c_\rho(bd, S_1) = 1.$$

Remark 2.18. The idiosyncratic choices of homological stages in Definition 2.15 are meant to smooth the introduction of hedgerows (Definition 3.1) as much as possible. Roughly speaking, a hedgerow starts from a sequence of simplicial complexes, from which it selects a shrubbery in dimension $i - 1$ at one end, a stake set in dimension i at the other end, and a hedge of dimension i for each of the intervening simplicial complexes. Only one homological stage i is treated at a time, and the stake sets need not relate to the shrubberies in any particular way. In fact, the goal in Definition 3.8 is to sum over all choices of hedgerow—with the sequence of simplicial complexes fixed—by way of chain-linked fences (Definition 3.5): sequences of faces, each linked to the next in specified ways using a hedgerow.

For CW complexes, hedges and shrubberies yield bases consisting of faces and boundaries of faces, but only over fields. Over the integers, the subgroups generated by these

$\mathbb{Z}$-linearly independent sets need not be saturated. The square determinants measuring the deviations act as weights in combinatorial formulas for projections and splittings. The remainder of this section provides a framework to work with these integers, even though a field $\Bbbk$ has been fixed from the start.

Definition 2.19. An *integral structure* on one of the maps in Definition 2.1 is a free abelian group $B^{\mathbb{Z}}$ and a homomorphism $B^{\mathbb{Z}} \to B$ inducing an isomorphism $B^{\mathbb{Z}} \otimes \Bbbk \xrightarrow{\sim} B$. The *square determinants* $\det(\partial_T)^2$ and $\det(\partial_S)^2$ in the presence of integral structures are the (images in $\Bbbk$ of the) squares of the determinants of $B^{\mathbb{Z}} \leftarrow \mathbb{Z}\{T\}$ and $\mathbb{Z}\{S\} \leftarrow B^{\mathbb{Z}}$.

Remark 2.20. The square of the determinant of a homomorphism $B^{\mathbb{Z}} \leftarrow \mathbb{Z}\{T\}$ or $\mathbb{Z}\{S\} \leftarrow B^{\mathbb{Z}}$ can be calculated using any basis of $B^{\mathbb{Z}}$: choosing another basis of $B^{\mathbb{Z}}$ yields the same determinant up to a sign that becomes irrelevant upon squaring.

Definition 2.21. Fix a CW complex K. Define the following integer sums over shrubberies, stake sets, and hedges using the natural integral structure $\widetilde{B}^{\mathbb{Z}}_{i-1}K$ on ∂_i:

$$\Delta_i^T K = \sum_{T_i} \det(\partial_{T_i})^2, \quad \Delta_{i-1}^S K = \sum_{S_{i-1}} \det(\partial_{S_{i-1}})^2, \quad \text{and} \quad \Delta_i^{ST} K = (\Delta_{i-1}^S K)(\Delta_i^T K).$$

The symbol "K" may be omitted if the context is clear. A field $\Bbbk$ is *torsionless* for K if for all i the characteristic of $\Bbbk$ does not divide these numbers or the order of the torsion subgroup of $\widetilde{C}_i^{\mathbb{Z}} K/\widetilde{B}_i^{\mathbb{Z}} K$.

Remark 2.22. Concretely, the absolute value of the determinant ∂_{T_i} is the index of the subgroup generated by the shrubbery T_i columns of the matrix for ∂_i (see Remark 2.6, for example) inside of the integer lattice generated by all of the columns. Similarly for $\partial_{S_{i-1}}$, but using rows—in particular, the stake set S_{i-1} rows—instead of columns.

Remark 2.23. The determinants in Definition 2.21 admit combinatorial interpetations as orders of torsion subgroups of CW subcomplexes of K whose facets are determined by the shrubberies and stake sets; see [7], for example.

3. Canonical sylvan resolutions

The differential in the canonical sylvan resolution is based on averaging over all hedge choices along lattice paths. This section develops notation for Koszul hedges along lattice paths. Other than the statement of the main theorem at the end (Theorem 3.10), it contains no results or proofs—only definitions and notation.

Definition 3.1. Fix a monomial ideal I and a saturated decreasing lattice path λ from $\mathbf{b}$ to $\mathbf{a}$, meaning that adjacent nodes in the path differ by a standard basis vector of the integer lattice $\mathbb{Z}^n$. Write $\Lambda(\mathbf{a}, \mathbf{b})$ for the set of such paths. A *hedgerow* on λ is

- a stake set $S_i^{\mathbf{b}} \subseteq K_i^{\mathbf{b}} I$ at the upper endpoint $\mathbf{b}$,

- a hedge $ST_i^{\mathbf{c}} = (S_{i-1}^{\mathbf{c}}, T_i^{\mathbf{c}})$ on the Koszul simplicial complex $K^{\mathbf{c}}I$ for each interior (i.e., non-endpoint) vertex $\mathbf{c}$ of λ, and
- a shrubbery $T_{i-1}^{\mathbf{a}} \subseteq K_{i-1}^{\mathbf{a}}I$ at the lower endpoint $\mathbf{a}$.

It is convenient to write

$$ST_i^{\lambda} = (S_{i-1}^{\lambda}, T_i^{\lambda})$$

for this hedgerow on λ, where—to emphasize—this notation carries an implicit stake set $S_i^{\mathbf{b}}$ and shrubbery $T_{i-1}^{\mathbf{a}}$ at the upper and lower endpoints. It is also convenient to write

$$\mathbf{a} = \mathbf{b}_\ell, \mathbf{b}_{\ell-1}, \dots, \mathbf{b}_1, \mathbf{b}_0 = \mathbf{b}$$

to denote the lattice points on the path λ, even if sometimes $\mathbf{c}$ is written for an otherwise unnamed vertex $\mathbf{b}_j$. The path $\lambda = (\lambda_1, \dots, \lambda_\ell)$ can be identified with the sequence

$$\lambda_j = \mathbf{b}_{j-1} - \mathbf{b}_j \in \{0,1\}^n \quad \text{for } j = 1, \dots, \ell$$

of steps in the path, so $\mathbf{x}^{\lambda_j} \in \{x_1, \dots, x_n\}$ is one of the n variables.

Remark 3.2. It may be helpful to visualize a hedgerow on λ as

$$\begin{array}{ccccccccccc}
T_{i-1}^{\mathbf{a}} & & ST_i^{\mathbf{b}_{\ell-1}} & & & & ST_i^{\mathbf{b}_2} & & ST_i^{\mathbf{b}_1} & & S_i^{\mathbf{b}} \\
\mathbf{a} = \mathbf{b}_\ell & \longleftarrow & \mathbf{b}_{\ell-1} & \longleftarrow & \cdots & \longleftarrow & \mathbf{b}_2 & \longleftarrow & \mathbf{b}_1 & \longleftarrow & \mathbf{b}_0 = \mathbf{b} \\
 & \lambda_\ell & & \lambda_{\ell-1} & & \lambda_3 & & \lambda_2 & & \lambda_1 &
\end{array}$$

Note that the shrubbery and stake set at lattice point $\mathbf{b}_j$ involve faces of the same simplicial complex but of differing dimensions i and $i-1$, and the faces involved at differing lattice points lie in different simplicial complexes. The visualization emphasizes that the hedgerow amounts to a choice of maximal invertible submatrix of the differential ∂_i in the Koszul simplicial complex of I at each interior lattice point in the path plus a row basis for ∂_{i+1} at the initial lattice point $\mathbf{b}$ and a column basis for ∂_{i-1} at the terminal endpoint $\mathbf{a}$.

Definition 3.3. Fix a monomial ideal I and a hedgerow ST_i^{λ} on a lattice path $\lambda \in \Lambda(\mathbf{a}, \mathbf{b})$. Applying Definition 2.21 at each lattice point in λ yields *vertex hedge weights*

$$\delta_{i,\mathbf{a}} = \det(\partial_{T_{i-1}^{\mathbf{a}}}), \quad \delta_{i,\mathbf{b}_j} = \det(\partial_{S_{i-1}^{\mathbf{b}_j}}) \det(\partial_{T_i^{\mathbf{b}_j}}), \quad \delta_{i,\mathbf{b}} = \det(\partial_{S_i^{\mathbf{b}}}),$$

which are viewed as lying in the field $\Bbbk$. The corresponding *vertex path weights*

$$\Delta_{i-1}^{\mathbf{a}} = \Delta_{i-1}^{T} K^{\mathbf{a}} I, \quad \Delta_i^{\mathbf{b}_j} = \Delta_i^{ST} K^{\mathbf{b}_j} I, \quad \Delta_i^{\mathbf{b}} = \Delta_i^{S} K^{\mathbf{b}} I$$

take into account all choices of hedges in the sequence of Koszul simplicial complexes of I along the lattice path λ. Multiplying the vertex hedge weights and vertex path weights along λ yields the *hedgerow vertex weight* and *path vertex weight*

$$\delta_{i,\lambda} ST_i^{\lambda} = \prod_{j=0}^{\ell} \delta_{i,\mathbf{b}_j} \quad \text{and} \quad \Delta_{i,\lambda} I = \prod_{j=0}^{\ell} \Delta_i^{\mathbf{b}_j}$$

again viewed as elements of the field $\Bbbk$.

Remark 3.4. Visualizing the determinants in Definition 3.3 as in Remark 3.2 yields

$$\begin{array}{ccccccccccc} \delta_{i,\mathbf{a}} & & \delta_{i,\mathbf{b}_{\ell-1}} & & & & \delta_{i,\mathbf{b}_2} & & \delta_{i,\mathbf{b}_1} & & \delta_{i,\mathbf{b}} \\ \mathbf{a} = \mathbf{b}_\ell & \longleftarrow & \mathbf{b}_{\ell-1} & \longleftarrow & \cdots & \longleftarrow & \mathbf{b}_2 & \longleftarrow & \mathbf{b}_1 & \longleftarrow & \mathbf{b}_0 = \mathbf{b} \end{array}$$

for the vertex hedge weights, which enter into the weights on chain-link fences (see Definition 3.7). Using Δ in place of δ visualizes the vertex path weights, which occur only via the path vertex weight $\Delta_{i,\lambda} I$—rather than individually as vertex weights—and that occurrence is in the combinatorial formula for sylvan homomorphisms (Definition 3.8). The path vertex weight $\Delta_{i,\lambda} I$ is analogous to a free energy or partition function.

Definition 3.5. Fix a lattice path $\lambda \in \Lambda(\mathbf{a}, \mathbf{b})$. A *chain-link fence* φ from an i-simplex τ to an $(i-1)$-simplex σ along λ is a choice of hedgerow ST_i^λ and a sequence

$$\begin{array}{ccccccccc} & & \tau_{\ell-1} & & \cdots & & \tau_1 & & \tau_0 \,—\, \tau \\ & / & & \backslash \quad / & & \backslash \quad / & & \backslash \quad / & \\ \sigma \,—\, \sigma_\ell & & & \sigma_{\ell-1} & & \sigma_2 & & \sigma_1 & \\ \mathbf{a} = \mathbf{b}_\ell & & & \mathbf{b}_{\ell-1} & \cdots \mathbf{b}_2 & & \mathbf{b}_1 & & \mathbf{b}_0 = \mathbf{b} \end{array}$$

in which $\tau_j \in K_i^{\mathbf{b}_j} I$ and $\sigma_j \in K_{i-1}^{\mathbf{b}_j} I$ (the degrees are depicted beneath the fence) and

$\tau_0 \,—\, \tau$ the simplex τ is boundary-linked to τ_0 via the stake set $S_i^{\mathbf{b}}$;

$\backslash$ the simplex $\sigma_j \in S_{i-1}^{\mathbf{b}_j}$ for $j = 1, \ldots, \ell - 1$ is a stake chain-linked to τ_j;

$/$ the simplex σ_j for $j = 1, \ldots, \ell$ equals the facet $\tau_{j-1} - \lambda_j$ of the simplex τ_{j-1};

$\sigma \,—\, \sigma_\ell$ the simplex $\sigma_\ell \in K_{i-1}^{\mathbf{a}}$ is cycle-linked to σ.

The i-simplex τ and $(i-1)$-simplex σ are the (*initial* and *terminal*) *posts* of the fence φ. The set $\Phi(\lambda)$ of chain-link fences along λ has the subset $\Phi_{\sigma\tau}(\lambda)$ with posts τ and σ. The fence φ is said to be *subordinate* to the chosen hedgerow ST_i^λ, written $\varphi \vdash ST_i^\lambda$.

Definition 3.6. Let $(-1)^{\sigma \subset \tau}$ be the coefficient of σ in $\partial\tau$. In any chain-link fence φ, each fence edge has an *edge weight* resulting from this sign or from Definition 2.15:

$\tau_0 \,—\, \tau$ has boundary-link weight $c_\tau(\tau_0, S_i^{\mathbf{b}})$;

$\tau_j \diagdown \sigma_j$ has chain-link weight $c_{\sigma_j}(\tau_j, ST_i^{\mathbf{b}_j})$;

$\sigma_j \diagup \tau_{j-1}$ has containment weight $(-1)^{\sigma_j \subset \tau_{j-1}}$; and

$\sigma \,—\, \sigma_\ell$ has cycle-link weight $c_{\sigma_\ell}(\sigma, T_{i-1}^{\mathbf{a}})$.

The *fence edge weight* of the chain-link fence φ is the product e_φ of its edge weights.

Definition 3.7. The *weight* of a chain-link fence $\varphi \vdash ST_i^\lambda$ is the product

$$w_\varphi = e_\varphi \delta_\varphi$$

of its fence edge weight e_φ and its *fence vertex weight* $\delta_\varphi = \delta_{i,\lambda} ST_i^\lambda$.

Definition 3.8. Fix a monomial ideal I. The *canonical sylvan homomorphism*

$$\widetilde{C}_{i-1}K^{\mathbf{a}}I \xleftarrow{D=D^{\mathbf{ab}}} \widetilde{C}_i K^{\mathbf{b}}I$$

is given by its *sylvan matrix*, whose entry $D_{\sigma\tau}$ for $\tau \in K_i^{\mathbf{b}}I$ and $\sigma \in K_{i-1}^{\mathbf{a}}I$ is the sum of the weights of all chain-link fences from τ to σ along all lattice paths from $\mathbf{b}$ to $\mathbf{a}$:

$$D_{\sigma\tau} = \sum_{\lambda\in\Lambda(\mathbf{a},\mathbf{b})} \frac{1}{\Delta_{i,\lambda}I} \sum_{\varphi\in\Phi_{\sigma\tau}(\lambda)} w_\varphi.$$

Remark 3.9. In more pictorial detail, the canonical sylvan matrix looks like

$$\begin{array}{ccc}
(i-1)\text{-faces of } K^{\mathbf{a}} \searrow & \begin{array}{ccc} \tau_1 & \cdots & \tau_n \end{array} & \longleftarrow\ i\text{-faces of } K^{\mathbf{b}} \\
\begin{array}{c} \sigma_1 \\ \vdots \\ \sigma_m \end{array} & \begin{bmatrix} & & \\ & D^{\mathbf{ab}} & \\ & & \end{bmatrix} & \\
\widetilde{H}_{i-1}K^{\mathbf{a}} \otimes \langle \mathbf{x}^{\mathbf{a}} \rangle & \longleftarrow\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!- & \widetilde{H}_i K^{\mathbf{b}} \otimes \langle \mathbf{x}^{\mathbf{b}} \rangle
\end{array}$$

in which the source and target vector spaces are indicated according to Theorem 3.10 rather than as the chain groups in Definition 3.8. Specific examples of sylvan matrices, and the chain-link fence combinatorics that computes their entries, occupy Section 4.

Theorem 3.10. *Fix a monomial ideal I and a field $\Bbbk$ that is torsionless for all Koszul simplicial complexes of I. The canonical sylvan homomorphism for each comparable pair $\mathbf{a} \prec \mathbf{b}$ of lattice points induces a homomorphism $\widetilde{Z}_{i-1}K^{\mathbf{a}}I \leftarrow \widetilde{Z}_i K^{\mathbf{b}}I$ that vanishes on $\widetilde{B}_i K^{\mathbf{b}}I$, and hence it induces a well defined* canonical sylvan homology morphism *$\widetilde{H}_{i-1}K^{\mathbf{a}}I \leftarrow \widetilde{H}_i K^{\mathbf{b}}I$. The induced homomorphisms*

$$\widetilde{H}_{i-1}K^{\mathbf{a}}I \otimes \Bbbk[\mathbf{x}](-\mathbf{a}) \leftarrow \widetilde{H}_i K^{\mathbf{b}}I \otimes \Bbbk[\mathbf{x}](-\mathbf{b})$$

of $\mathbb{N}^n$-graded free $\Bbbk[\mathbf{x}]$-modules constitute a minimal free resolution of I.

The proof is at the end of Section 8.

4. Examples: Sylvan block matrix notation

In the figures accompanying the following examples, hollow red spots represent degrees with nontrivial $\widetilde{H}_1$, hollow blue spots represent nontrivial $\widetilde{H}_0$, and hollow black spots represent nontrivial $\widetilde{H}_{-1}$. In each degree, the corresponding Koszul simplicial complex is displayed around the circle—slightly below in the partial order on $\mathbb{R}^3$, and at constant coordinate sum—in the appropriate color using smaller solid dots.

Example 4.1. One of the sticking points for any construction of canonical minimal free resolutions of monomial ideals is $I = \langle xy, yz, xz \rangle$, which has Betti number $\beta_{1,111}(I) = 2$ arising from the Koszul simplicial complex $K^{111}I$ whose facets are the three vertices depicted in Figure 1 as little blue dots. The two linearly independent syzygies in

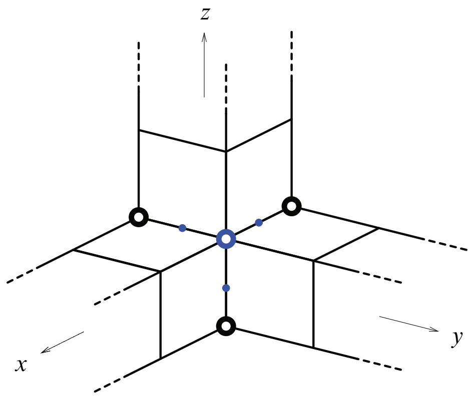


Fig. 1. The staircase diagram for $I = \langle xy, yz, xz\rangle$.

degree 111 split equally among the three generators, which is problematic for any construction that produces matrices for the differential, because any choice of basis breaks the symmetry. The sylvan method avoids this problem by defining the homomorphism on chains instead of on homology classes. Here is how it works for $I = \langle xy, yz, xz\rangle$.

Start by computing, say, the sylvan matrix $D^{110,111}$. There is only one lattice path $\lambda \in \Lambda(110, 111)$, since the length is 1. For this path, the stake set at the initial post 111 is forced to be $S_0^{111} = \{\}$ because $K^{111}I$ has dimension 0 and hence no faces of dimension 1 to take the boundary of. The shrubbery at the terminal post 110 is forced to be the empty set of faces, which is written $T_{-1}^{110} = \{\}$ so as not to confuse it with the set $\{\varnothing\}$ consisting of the empty face. As there is only one hedgerow on λ, and all of the square determinants δ^2 equal 1 because the number of vertices is too small for torsion in homology, $\Delta_{0,\lambda} = 1$. To construct a chain-link fence along λ, only one terminal post $\sigma = \varnothing$ is available. Trying x as initial post yields no fences: the boundary link $x \overset{}{—} x$ is forced, because x is the unique non-stake τ' in $K^{111}I$ such that $x - \tau'$ is a boundary, but then $z = \lambda_1$ is not a face of x, so the fence has no continuation.

$$\begin{array}{ccc} & x \overset{1}{—} x \\ /_0 & \end{array} \qquad\qquad \begin{array}{ccc} & y \overset{1}{—} y \\ /_0 & \end{array} \qquad\qquad \begin{array}{cc} & z \overset{1}{—} z \\ & /_1 \\ \varnothing \overset{1}{—} \varnothing & \end{array}$$

Similarly, no fence has initial post y. But the initial post z yields one fence, as depicted. The conclusion is that $\Phi_{\varnothing,x} = \varnothing = \Phi_{\varnothing,y}$ and $|\Phi_{\varnothing,z}| = 1$, with all of the edge coefficients and torsion numbers equal to 1. This determines the top row of the matrix

$$\begin{array}{c} \widetilde{H}_{-1}K^{110} \otimes \langle xy\rangle \\ \oplus \\ \widetilde{H}_{-1}K^{101} \otimes \langle xz\rangle \\ \oplus \\ \widetilde{H}_{-1}K^{011} \otimes \langle yz\rangle \end{array} \xleftarrow{\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} \varnothing \\ \varnothing \\ \varnothing \end{array} & \begin{bmatrix} [0 & 0 & 1] \\ [0 & 1 & 0] \\ [1 & 0 & 0] \end{bmatrix} \end{array}} \widetilde{H}_0 K^{111} \otimes \langle xyz\rangle$$

in which the other two rows are determined by symmetry. The blocks in this matrix are the sylvan matrices $D^{110,111}$, $D^{101,111}$, and $D^{011,111}$. Its rows and columns are labeled by the faces of the corresponding Koszul simplicial complexes. The cycles $x-y$, $z-x$, and $y-z$ correspond to the column vectors

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

The image of, say, $x-y$ is the $\mathbb{N}^3$-degree 111 element $x \cdot \varnothing - y \cdot \varnothing$ that is x times the free generator of $\widetilde{H}_{-1} K^{011} \otimes \langle yz \rangle$ minus y times the free generator of $\widetilde{H}_{-1} K^{101} \otimes \langle xz \rangle$. Looking back at the staircase drawn at the beginning of this example, the face x of the cycle $x-y$ has moved back parallel to the x-axis to become $x \cdot \varnothing$, and $-y$ has moved back parallel to the y-axis to become $-y \cdot \varnothing$. This behavior is fundamental to the sylvan construction: faces move back in directions parallel to axes, picking up the corresponding monomial coefficients as they go.

Convention 4.2. The general block matrix notational device illustrated by Example 4.1 works as follows. Choose an order in which to list the $\mathbb{N}^n$-degrees $\mathbf{a}$ and $\mathbf{b}$ of nonzero Betti numbers $\beta_{i,\mathbf{a}}(I)$ and $\beta_{i+1,\mathbf{b}}(I)$ in homological stages i and $i+1$, say $\mathbf{a}_1, \mathbf{a}_2, \ldots$ and $\mathbf{b}_1, \mathbf{b}_2, \ldots$. The pq block is the matrix $D^{\mathbf{a}_p\mathbf{b}_q}$, which takes chains in $\widetilde{C}_i K^{\mathbf{b}_q} I$, thought of as column vectors with entries indexed by the i-simplices in $K_i^{\mathbf{b}_q} I$, to chains in $\widetilde{C}_{i-1} K^{\mathbf{a}_p} I$, thought of as column vectors with entries indexed by the $(i-1)$-simplices in $K_{i-1}^{\mathbf{a}_p} I$. The orderings on the $\mathbb{N}^n$-degrees $\mathbf{a}_p$ and $\mathbf{b}_q$ are depicted by writing ordered direct sums vertically. The orderings on the simplices are depicted by labeling the rows and columns of each block.

Example 4.3. It might be helpful to see a more complicated canonical sylvan resolution, with multiple lattices paths from $\mathbf{b}$ to $\mathbf{a}$ and chain-link fences along a fixed lattice path. Alas, the square determinants are all 1 in this example, since nontrivial such invariants can only occur with enough vertices that writing down the full canonical sylvan resolution would be an ineffective use of space.

Let $I = \langle xy, y^3, z \rangle$, whose staircase is depicted in Figure 2. The three large black dots are the generators, the three large blue dots are the first syzygies, and the large red dot is the second syzygy. The little dots and edges behind and below each of the large dots form its Koszul simplicial complex: a triangle (a nontrivial loop in $\widetilde{H}_1$) for the second syzygy; a disconnected union of two vertices or an edge and a vertex (nontrivial $\widetilde{H}_0$) for the first syzygies; and just the empty face (nontrivial $\widetilde{H}_{-1}$; not drawn) for the generators.

The canonical sylvan resolution of I, notated as per Convention 4.2, is as follows:

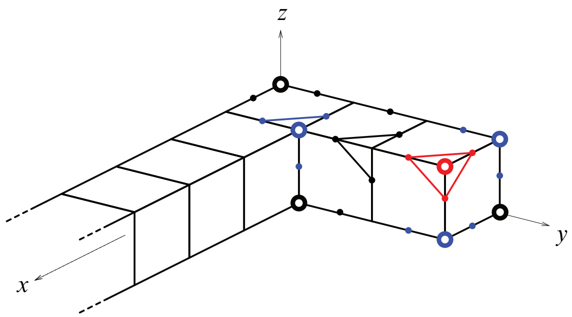


Fig. 2. The staircase diagram for $I = \langle xy, y^2, z\rangle$.

$$
\begin{array}{c}
\tilde H_{-1}K^{110}\otimes\langle xy\rangle\\ \oplus\\ \tilde H_{-1}K^{030}\otimes\langle y^3\rangle\\ \oplus\\ \tilde H_{-1}K^{001}\otimes\langle z\rangle
\end{array}
\xleftarrow{\;\begin{array}{c|ccc} & xyz & xy & yz\\ \hline \varnothing & [0\,0\,1] & [0\,1] & [0\,0]\\ \varnothing & [0\,0\,0] & [1\,0] & [0\,1]\\ \varnothing & [1\,1\,0] & [0\,0] & [1\,0]\end{array}\;}
\begin{array}{c}
\tilde H_{0}K^{111}\otimes\langle xyz\rangle\\ \oplus\\ \tilde H_{0}K^{130}\otimes\langle xy^3\rangle\\ \oplus\\ \tilde H_{0}K^{031}\otimes\langle y^3z\rangle
\end{array}
\xleftarrow{\;\begin{array}{c|ccc} & zy & yx & xz\\ \hline x & 4/9 & 5/9 & 0\\ y & 1/9 & -1/9 & 0\\ z & -5/9 & -4/9 & 0\\ \hline x & -1/2 & 0 & -1/2\\ y & 1/2 & 0 & 1/2\\ \hline y & 0 & -1/2 & -1/2\\ z & 0 & 1/2 & 1/2\end{array}\;}
\tilde H_{1}K^{131}\otimes\langle xy^3z\rangle
$$

Before getting to hedges and fences that compute the sylvan matrix entries, several features are to be noted. The column vector $(1,1,1)^\top$ generates $\tilde H_1 K^{131}$ and maps to the sum $(1,0,-1)^\top \oplus (-1,1)^\top \oplus (-1,1)^\top$ in the $\mathbb{N}^3$-degree 131 component of the middle free module. (The relevant fractions magically either cancel or sum to 1.) This statement should be viewed geometrically on the staircase: the cycle that is the red triangle $zy + yx + xz$ maps to the difference of the two vertices closest to it in each of the blue simplicial complexes. At 111 this is $x - z$; at 130 this is $y - x$; at 031 this is $z - y$. The direct sum of these three blue cycles yields a vector of length 9 that lies in the kernel of the 3×9 matrix—that is, the (3 block) $\times$ (3 block) matrix.

For a sample sylvan matrix computation for $F_0 \leftarrow F_1$, consider the 1×3 sylvan matrix $D^{001,111}$ in the lower-left corner. It involves two lattice paths in $\Lambda(001, 111)$:

$$
\begin{array}{rccc}
\lambda: & 001 \text{ ——— } & 011 \text{ ——— } & 111\\
ST_0^\lambda: & T_{-1} = \{\} & T_0 = \{y\} & S_0 = \{x\}\\
 & & S_{-1} = \{\varnothing\} & \text{or } \{y\}
\end{array}
$$

and

$$
\begin{array}{rccc}
\lambda': & 001 \text{ ——— } & 101 \text{ ——— } & 111\\
ST_0^\lambda: & T_{-1} = \{\} & T_0 = \{x\} & S_0 = \{x\}\\
 & & S_{-1} = \{\varnothing\} & \text{or } \{y\}
\end{array}
$$

both of which have two choices for the initial S_0 and hence have

$$\Delta_{0,\lambda'} = \Delta_{0,\lambda} = 1 \cdot 1 \cdot 2 = 2.$$

For λ, regardless of whether $S_0 = \{x\}$ or $S_0 = \{y\}$, the initial post z is boundary-linked only to $z \in \overline{S}_0$, which does not contain y and hence yields no fences along λ. The initial post y is boundary-linked to y if $S_0 = \{x\}$ and to x if $S_0 = \{y\}$. However, the case of $\tau_0 — \tau = y — y$ has no continuation because x is not a face of y. In contrast, the case of $\tau_0 — \tau = x — y$ yields a valid fence. The same logic shows that $\tau_0 — \tau = y — x$ when $S_0 = \{x\}$ has no continuation but $\tau_0 — \tau = x — x$ when $S_0 = \{y\}$ yields a valid fence. These possibilities for $\lambda = 001 — 011 — 111$ are summarized as follows.

$$
\begin{array}{ccccc}
\begin{array}{l} z \overset{1}{—} z \\ /_0 \\ \\ S_0 = \{x\} \\ \quad \text{or } \{y\} \end{array}
&
\begin{array}{l} y \overset{1}{—} y \\ /_0 \\ \\ S_0 = \{x\} \\ \end{array}
&
\begin{array}{rcl} & y & \quad x \overset{1}{—} y \\ & /_1\ \ \backslash^1\ \ /_1 & \\ \varnothing \overset{1}{—} \varnothing & & \varnothing \\ & & S_0 = \{y\} \\ \end{array}
&
\begin{array}{l} y \overset{1}{—} x \\ /_0 \\ \\ S_0 = \{x\} \\ \end{array}
&
\begin{array}{rcl} & y & \quad x \overset{1}{—} x \\ & /_1\ \ \backslash^1\ \ /_1 & \\ \varnothing \overset{1}{—} \varnothing & & \varnothing \\ & & S_0 = \{y\} \\ \end{array}
\end{array}
$$

The lattice path λ therefore contributes $\frac{1}{2}[1\ 1\ 0]$ to $D^{001,111}$.

For a sample sylvan matrix computation for $F_1 \leftarrow F_2$, compute $D^{111,131}$ using the sole lattice path $\lambda \in \Lambda(111, 131)$, namely

$$
\begin{array}{rccc}
\lambda: & 111 & ——— \ 121 \ ——— & 131 \\
ST_0^\lambda: & T_0 = \{x\} & T_1 = \{zy, yx\} & S_1 = \{\} \\
& \text{or } \{y\} & S_0 = \{y, z\} & \\
& \text{or } \{z\} & \text{or } \{x, z\} & \\
& & \text{or } \{x, y\} &
\end{array}
$$

The vector space of homological stage 0 boundaries in K^{121} has dimension 2, so any pair of vertices is a stake set because no single vertex is a boundary. Thus

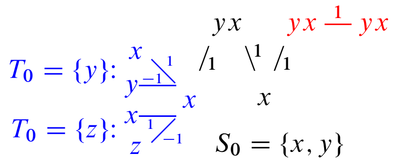
$$\Delta_{1,111-131} = 3 \cdot 3 \cdot 1 = 9.$$

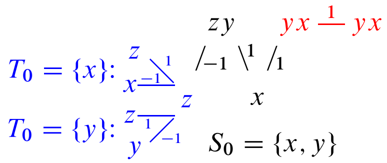
The initial boundary-link $xz — xz$ has no continuation because y is not a face of xz; this explains the zero column in the top block of $F_1 \leftarrow F_2$.

The initial boundary-link $yx — yx$ yields a much more interesting computation. When the stake set $S_0 = \{y, z\}$ is selected at the interior lattice point 121, the facet x of yx is forced by Definition 3.5 (/), but x is not a stake, so no continuation is possible. In contrast, when $S_0 = \{x, z\}$ is selected, four chain-link fences result: two from the choice of $T_0 = \{y\}$, because the circuit of x with respect to the shrubbery $\{y\}$ is $x - y$, so each of the terms x and $-y$ contributes a cycle-link; and two from $T_0 = \{z\}$ for a similar reason with z in place of y. To save space, these four fences are drawn as a single fork.

$$
\begin{array}{lcr}
\begin{array}{l} \quad yx \overset{1}{—} yx \\ /_1 \\ x \notin S_0 \\ S_0 = \{y, z\} \end{array}
& \qquad &
\begin{array}{lccr}
& & yx & \quad yx \overset{1}{—} yx \\
T_0 = \{y\}: & \begin{array}{l} x \searrow^{1} \\ y \overset{-1}{—} \end{array} & /_1\ \ \backslash^1\ \ /_1 & \\
& x & x & \\
T_0 = \{z\}: & \begin{array}{l} x \overset{1}{—} \\ z \nearrow_{-1} \end{array} & & S_0 = \{x, z\}
\end{array}
\end{array}
$$

When $S_0 = \{x, y\}$ is selected, eight chain-link fences result, by reasoning as in the four fences just constructed. (In fact, the first four fences, depicted in the left-hand diagram here, have exactly the same sequences of simplices and coefficients as the four fences just produced. They count separately because they are subordinate to a different hedgerow.) The reason there are twice as many is that the shrub of x for the hedge $ST_1 = (\{x, y\}, \{zy, yx\})$ is $zy + yx$, the unique path in K^{121} that joins the non-stake z to the stake x. Thus the fence with $S_0 = \{x, y\}$ bifurcates at x into yx and zy.





Now count weighted fences as a sum of three terms, one from each fork:

$$
\begin{aligned}
9D_{x,yx} &= 2 + 2 + 1 = 5,\\
9D_{y,yx} &= -1 - 1 + 1 = -1,\\
9D_{z,yx} &= -1 - 1 - 2 = -4,
\end{aligned}
$$

where $9 = \Delta_{1,111-131}$, as calculated before. This explains the middle column in the top block of $F_1 \leftarrow F_2$.

5. Hedge splittings and Moore–Penrose pseudoinverses

Definition 5.1. Fix a linear map $\Bbbk^m \xleftarrow{\partial} \Bbbk^n$. For any hedge ST, the *hedge splitting*

$$\partial^+_{ST} : \Bbbk^m \to \Bbbk^n$$

is defined by its action on the basis $\bar{S} \cup \partial T$:

(1) $\partial^+_{ST}(\sigma) = 0$ for any non-stake $\sigma \in \bar{S}$ and

(2) $\partial^+_{ST}(\partial\tau) = \tau$ for any face $\tau \in T$.

Hedge splittings are related to circuits, shrubs, and hedge rims as follows.

Proposition 5.2. *In the setting of Definition 5.1, with a fixed linear map $\Bbbk^m \xleftarrow{\partial} \Bbbk^n$ and hedge ST, if τ is a standard basis vector then*

$$\partial^+_{ST}\partial(\tau) = (\mathbb{1} - \zeta_T)(\tau),$$

where $\mathbb{1}$ is the identity on $\Bbbk^n$ and ζ_T is the circuit projection from Definition 2.12.

Proof. Since $\tau - \zeta_T(\tau)$ involves only basis vectors in T, it is fixed by $\partial^+_{ST}\partial$. Therefore

$$\tau - \zeta_T(\tau) = \partial^+_{ST}\partial(\tau - \zeta_T(\tau)) = \partial^+_{ST}\partial(\tau)$$

for all standard basis vectors $\tau \in \Bbbk^n$ because $\zeta_T(\tau) \in \ker \partial$. ∎

Proposition 5.3. *In the setting of Definition 5.1, with a fixed linear map $\Bbbk^m \xleftarrow{\partial} \Bbbk^n$ and hedge ST, the shrub of any stake $\sigma \in S$ (the chain s in Lemma 2.9) equals $\partial^+_{ST}(\sigma)$.*

Proof. Definition 5.1 implies that $s = \partial^+_{ST}\partial(s)$. But $\sigma - \partial s$ is a linear combination of non-stakes by Lemma 2.9, so $\partial^+_{ST}\partial(s) = \partial^+_{ST}(\sigma)$. ■

Lemma 5.4. *Fix a linear map $\Bbbk^m \xleftarrow{\partial} \Bbbk^n$, a stake set S for ∂, and a stake $\sigma \in S$. The boundary of the shrub $\partial^+_{ST}(\sigma)$ of σ in the hedge ST depends only on S and σ, not on the shrubbery T. More precisely, $\partial\partial^+_{ST}(\sigma) = \beta_S(\sigma)$ is the boundary projection of σ.*

Proof. $\partial^+_{ST}(\sigma) = \partial^+_{ST}(\beta_S(\sigma))$ by Definition 5.1 (1), since $\beta_S(\sigma) \in \sigma + \Bbbk\{\overline{S}\}$. But $\beta_S(\sigma)$ is fixed by $\partial\partial^+_{ST}$ because it lies in the image of ∂. ■

Definition 5.5. The element $\beta_S(\sigma) \in \partial(\Bbbk^n)$ in Lemma 5.4 is the *shrub boundary* of σ.

Definition 5.6. The *Moore–Penrose pseudoinverse* of a linear map $\Bbbk^m \xleftarrow{\partial} \Bbbk^n$ over a subfield $\Bbbk \subseteq \mathbb{C}$ of the complex numbers is the unique homomorphism $\partial^+ : \Bbbk^m \to \Bbbk^n$ with

(1) $\partial\partial^+\partial = \partial$,

(2) $\partial^+\partial\partial^+ = \partial^+$,

(3) $(\partial\partial^+)^* = \partial\partial^+$,

(4) $(\partial^+\partial)^* = \partial^+\partial$,

where * is conjugate transpose. When ∂ is the differential of a CW complex K, the indices are such that ∂ and ∂^+ pass between two fixed homological stages, so ∂ would mean $C_{i-1} \xleftarrow{\partial_i} C_i$ and then ∂^+ would mean $\partial_i^+ : C_{i-1} \to C_i$.

Theorem 5.7 ([6, Theorem 1]). *Fix a linear map $\mathbb{C}^m \xleftarrow{\partial} \mathbb{C}^n$ of complex vector spaces. The component x_j^+ of the Moore–Penrose solution $\mathbf{x}^+$ of $\partial\mathbf{x} = \mathbf{z}$ is expressed as a sum over all size $r = \operatorname{rank}(\partial)$ subsets $S \subseteq \{1, \ldots, m\}$ and $T \subseteq \{1, \ldots, n\}$ with $j \in T$:*

$$x_j^+ = (\partial^+\mathbf{z})_j = \frac{1}{\sum_{S,T} |\det \partial_{S\times T}|^2} \sum_{\substack{|S|=r\\|T|=r\\j\in T}} |\det \overline{\partial}_{S\times T} \det((\partial_{S\times T})_j[\mathbf{z}_S])|,$$

where the bar over ∂ denotes complex conjugation and

- *$\partial_{S\times T}$ restricts ∂ to its submatrix with rows and columns indexed by S and T,*
- *$\mathbf{z}_S$ restricts the column vector $\mathbf{z}$ to its entries indexed by S, and*
- *$(\partial_{S\times T})_j[\mathbf{z}_S]$ replaces column j of $\partial_{S\times T}$ with $\mathbf{z}_S$.*

Remark 5.8. To be faithful to [6], the sums in Theorem 5.7 are taken over arbitrary size r subsets. This contrasts with most of the summations in this paper, which restrict to summands where S is a stake set and T is a shrubbery. However, Berg's sums might as well be over hedges ST, as the determinant in every remaining term vanishes; this is a simple but key point in the proof of the following consequence.

Corollary 5.9 (Hedge Formula). *Fix a subfield $\Bbbk \subseteq \mathbb{C}$ of the complex numbers. The Moore–Penrose pseudoinverse of a linear map $\Bbbk^m \xleftarrow{\partial} \Bbbk^n$ is a sum over hedges ST for ∂:*

$$\partial^+ = \frac{1}{\sum_{ST} |\det \partial_{S\times T}|^2} \sum_{ST} |\det \partial_{S\times T}|^2 \partial^+_{ST} .$$

Proof. First assume $\Bbbk = \mathbb{C}$. By Remark 5.8, the summations in Berg's formula (Theorem 5.7) can be taken over all hedges ST for ∂.

Fix a hedge ST and $\mathbf{z} \in \Bbbk^m$. In Berg's formula, the hedge ST contributes

$$\sum_{j\in T} |\det \overline{\partial}_{S\times T} \det((\partial_{S\times T})_j[\mathbf{z}_S])|\, e_j \tag{$*$}$$

to $\partial^+\mathbf{z}$. The goal is to check that $(*)$ equals $|\det \partial_{S\times T}|^2 \partial^+_{ST}\mathbf{z}$. It suffices to check this equality when $\mathbf{z} \in \overline{S} \cup \{\partial\tau \mid \tau \in T\}$, which is a basis of $\Bbbk^m$.

If $\mathbf{z} \in \overline{S}$, then $\mathbf{z}_S = 0$, so $(*)$ vanishes because $(\partial_{S\times T})_j[\mathbf{z}_S]$ has a zero column. This is as desired, by Definition 5.1 (1). On the other hand, let $\mathbf{z} = \partial\tau$ for some $\tau \in T$. In the summand indexed by j in $(*)$, either $\tau = e_j$, in which case $(\partial_{S\times T})_j[\mathbf{z}_S] = \partial_{S\times T}$, or else $\tau \neq e_j$, in which case $(\partial_{S\times T})_j[\mathbf{z}_S]$ has a repeated column and thus has vanishing determinant. Therefore $(*)$ reduces to one summand

$$|\det \overline{\partial}_{S\times T} \det \partial_{S\times T}|\tau = |\det \partial_{S\times T}|^2\tau = |\det \partial_{S\times T}|^2 \partial^+_{ST}\partial\tau = |\det \partial_{S\times T}|^2 \partial^+_{ST}\mathbf{z}$$

as desired, where the middle equality is by Definition 5.1 (2).

For $\Bbbk \subseteq \mathbb{C}$, the formula for ∂^+ is defined and has the claimed properties after tensoring with $\mathbb{C}$. By faithfulness, it must have had those properties before tensoring. ∎

Proposition 5.10. *Fix a linear map $\mathbb{Z}^m \xleftarrow{\partial} \mathbb{Z}^n$. Over any field $\Bbbk$ in which the denominator $\sum_{ST} \det(\partial_{S\times T})^2$ and the order of the torsion subgroup of $\mathbb{Z}^m/\partial(\mathbb{Z}^n)$ are invertible, the summation formula for ∂^+ defines a splitting $\partial^+_\Bbbk$ of the map $\Bbbk^m \xleftarrow{\partial_\Bbbk} \Bbbk^n$ induced by $\otimes \Bbbk$.*

Proof. Over the rationals $\mathbb{Q}$, the Moore–Penrose pseudoinverse of ∂ is an orthogonal projection $\pi_B : \mathbb{Q}^m \twoheadrightarrow B$ followed by an isomorphism $B \xrightarrow{\sim} L^\perp$ of the image $B = \partial_\mathbb{Q}(\mathbb{Q}^m)$ with the orthogonal complement in $\mathbb{Q}^n$ of the kernel $L = \ker(B \twoheadleftarrow \mathbb{Q}^n)$.

View the inclusion $B^\mathbb{Z} = \partial(\mathbb{Z}^n) \subseteq \mathbb{Z}^m$ as taking place inside of $\mathbb{Q}^m$, which contains B as well as $B^\mathbb{Z}$ and $\mathbb{Z}^m$. Let R be the localization of $\mathbb{Z}$ by inverting the denominator and the torsion order. Then $\pi_B(R^m)$ lies in $B \cap R^m$ because the denominator is inverted. But because the torsion order is inverted, $B \cap R^m = B^\mathbb{Z} \otimes R$, a direct summand B^R of R^m. Since π_B fixes B and hence B^R, it follows that $\pi_B(R^m) = B^R$. This surjectivity of $\pi^R_B : R^m \twoheadrightarrow B^R$ persists modulo any prime of R, by right-exactness of tensor products, and subsequently under any extension of scalars.

The isomorphism $B \xrightarrow{\sim} L^\perp \subseteq \mathbb{Q}^n$ over $\mathbb{Q}$ restricts to an isomorphism of B^R with a subgroup of R^n because it is split by $\partial_\mathbb{Q}$. This splitting is preserved by arbitrary tensor products, including quotients modulo primes and subsequent extension of scalars. ∎

Corollary 5.11 (Projection Hedge Formula). *Fix a subfield $\Bbbk \subseteq \mathbb{C}$ and a subspace $A \subseteq \Bbbk^\ell$. Using $\alpha_U : \Bbbk^\ell \twoheadrightarrow A$ from Lemma 2.11, the orthogonal projection $\pi_A : \Bbbk^\ell \twoheadrightarrow A$ is*

$$\pi_A = \frac{1}{\sum_U |\det \eta_U|^2} \sum_U |\det \eta_U|^2 \alpha_U,$$

where the sums are equivalently over all

(1) *shrubberies $T = U$ for $\Bbbk^\ell/A \twoheadleftarrow \Bbbk^\ell$, in which case $\eta_U = \partial_T$ (Definition 2.1 (1)), or*

(2) *stake sets $S = \bar{U}$ for $\Bbbk^\ell \hookleftarrow A$, in which case $\eta_U = \partial_S$ (Definition 2.1 (2)),*

and the determinants are calculated using U along with any basis of $\Bbbk^\ell/A$ or A.

Proof. The formula does not depend on the basis of A because the change-of-basis determinant multiplies the numerator and the denominator equally.

The equivalence of the shrubbery and stake set formulations is plausible by Remark 2.2. Thus each shrubbery T uniquely corresponds to a stake set $\overline{T}$. While it is not necessarily the case that $\det \partial_T = \det \partial_S$ for $U = T = \overline{S}$, the ratios between these determinants are constant as U varies, because

$$\bigwedge^d \Bbbk^S \otimes \bigwedge^{\ell-d} \Bbbk^{\overline{S}} = \bigwedge^\ell \Bbbk^\ell = \bigwedge^d A \otimes \bigwedge^{\ell-d} \Bbbk^\ell/A,$$

where $d = |S|$, while the maps $\bigwedge^d \Bbbk^S \leftarrow \bigwedge^d A$ and $\bigwedge^{\ell-d} \Bbbk^{\overline{S}} \to \bigwedge^{\ell-d} \Bbbk^\ell/A$ go in opposite directions.

To prove the stake set formulation, set $\ell = m$ and choose any linear transformation $\Bbbk^m \xleftarrow{\partial} \Bbbk^n$ with image $A = B$. Multiply the Hedge Formula (Corollary 5.9) on the left by ∂. This yields left-hand side $\partial\partial^+$ and, by Lemma 5.4, right-hand side

$$\frac{1}{\sum_{ST} |\det \partial_{S\times T}|^2} \sum_{ST} |\det \partial_{S\times T}|^2 \partial\partial^+_{ST} = \frac{1}{\sum_{ST} |\det \partial_{S\times T}|^2} \sum_{ST} |\det \partial_{S\times T}|^2 \beta_S.$$

Using any basis of A to compute the determinants $\partial\partial_T$ and $\det \partial_S$, the fact that $\partial_{S\times T} = \partial_S \circ \partial_T$ implies that $\det(\partial_{S\times T})^2 = \det(\partial_S)^2 \det(\partial_T)^2$. Therefore a factor of $\sum_T |\det \partial_T|^2$ pulls out of the numerator as well as the denominator, leaving the desired sum. ∎

Proposition 5.12. *Fix an integral structure $B^{\mathbb{Z}} \to B$ (Definition 2.19) on a surjection, injection, or based linear map ∂ as in Definition 2.1. If the order of the torsion subgroup of $\mathbb{Z}^\ell/B^{\mathbb{Z}}$ and denominator $\sum_U \det(\partial_U)^2$ are invertible in $\Bbbk$, then the formula for π_A in Corollary 5.11 defines a surjection $\pi_A : \Bbbk^\ell \twoheadrightarrow A$ that splits the inclusion $\Bbbk^\ell \hookleftarrow A$ when*

- *$\ell = n$ and $A = \ker(B \overset{\partial}{\twoheadleftarrow} \Bbbk^n)$, the sum is interpreted as being over shrubberies $T = U$ (so ∂_U is the isomorphism ∂_T from Definition 2.1), and $\alpha_U = \zeta_T$; or*
- *$\ell = m$ and $A = B$, the sum is interpreted as being over stake sets $S = \bar{U}$ (so ∂_U is the isomorphism ∂_S from Definition 2.1), and $\alpha_U = \beta_S$.*

Proof. Argue as in the second paragraph of the proof of Proposition 5.10, mutatis mutandis. All the integral structure does is to make the determinants well defined. ∎

6. Koszul bicomplexes

Constructions involving Koszul complexes, particularly with two sets of variables, are needed in Section 8. These are contained in Convention 6.1, Definition 6.2, and Lemma 6.3. Beyond that, the remainder of this brief section presents some resolution and spectral sequence results that motivate the Wall complex construction in Section 7; see Remark 6.9.

Convention 6.1 (Koszul complex notation). Let V be a vector space over $\Bbbk$ of dimension n that is $\mathbb{N}^n$-graded to have one basis vector $z_1, \ldots, z_n$ in each of the degrees $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of the variables $x_1, \ldots, x_n$ of the polynomial ring $\Bbbk[\mathbf{x}]$. The Koszul complexes on the variables $\mathbf{x}$ and on variables $\mathbf{y} = y_1, \ldots, y_n$ are denoted by

$$\mathbb{K}^{\mathbf{x}}_{\bullet} = \textstyle\bigwedge^{\bullet} V \otimes \Bbbk[\mathbf{x}] \quad \text{and} \quad \mathbb{K}^{\mathbf{y}}_{\bullet} = \textstyle\bigwedge^{\bullet} V \otimes \Bbbk[\mathbf{y}]$$

with their usual $\mathbb{N}^n$-graded differentials. Thus, for example, the degree $\mathbf{b}$ differential

$$(\mathbb{K}^{\mathbf{y}}_{i-1})_{\mathbf{b}} = \bigoplus_{|\sigma|=i-1} \mathbf{z}^{\sigma} \otimes \mathbf{y}^{\mathbf{b}-\sigma} \leftarrow \bigoplus_{|\sigma|=i} \mathbf{z}^{\sigma} \otimes \mathbf{y}^{\mathbf{b}-\sigma} = (\mathbb{K}^{\mathbf{y}}_{i})_{\mathbf{b}}$$

is induced by the $\mathbb{N}^n$-graded $\Bbbk[\mathbf{y}]$-linear map $\Bbbk[\mathbf{y}] \leftarrow V \otimes \Bbbk[\mathbf{y}]$ that sends $y_j \leftarrow\!\shortmid z_j \otimes 1$:

$$\sum_{j\in\sigma} \pm\, \mathbf{z}^{\sigma-\mathbf{e}_j} \otimes \mathbf{y}^{\mathbf{b}+\mathbf{e}_j-\sigma} \leftarrow\!\shortmid \mathbf{z}^{\sigma} \otimes \mathbf{y}^{\mathbf{b}-\sigma}.$$

The symbol $\mathbf{z}^{\sigma}$ denotes the exterior product of the basis vectors of V indexed by the simplex $\sigma \subseteq \{1, \ldots, n\}$, which is also identified with its characteristic vector in $\{0,1\}^n$. The monomial $\mathbf{z}^{\sigma}$ is a $\Bbbk[\mathbf{x}]$-basis for a rank 1 free $\mathbb{N}^n$-graded summand $\mathbb{K}^{\mathbf{x}}_{\sigma}$ of $\mathbb{K}^{\mathbf{x}}_{|\sigma|}$, and similarly for $\mathbb{K}^{\mathbf{y}}_{\sigma}$. For any $\Bbbk[\mathbf{x}]$-module M, write $M^{\mathbf{y}}$ for the corresponding $\Bbbk[\mathbf{y}]$-module, and let $M^{-\mathbf{y}}$ be the same module but where each variable y_j acts by $-y_j$. These conventions work for any list of linear forms in place of $\mathbf{x}$ or $\mathbf{y}$; for example, we use $\mathbf{x} + \mathbf{y} = x_1 + y_1, \ldots, x_n + y_n$.

Definition 6.2. Using equality signs for natural isomorphisms of modules over the ring $\Bbbk[\mathbf{x}, \mathbf{y}] = \Bbbk[\mathbf{x}] \otimes \Bbbk[\mathbf{y}]$, the *Koszul bicomplex* $\mathbb{K}_{\bullet\bullet}$ is equivalently

$$\begin{array}{c} \Bbbk[\mathbf{x}] \otimes \mathbb{K}^{\mathbf{y}}_{\bullet} = \Bbbk[\mathbf{x}] \otimes \textstyle\bigwedge^{\bullet} V \otimes \Bbbk[\mathbf{y}] = \mathbb{K}^{\mathbf{x}}_{\bullet} \otimes \Bbbk[\mathbf{y}] \\ \| \\ \mathbb{K}^{\mathbf{x}+\mathbf{y}}_{\bullet} \end{array}$$

with

- *horizontal differential* induced by $\mathbb{K}^{\mathbf{x}}_{\bullet}$
- *vertical differential* induced by $\mathbb{K}^{\mathbf{y}}_{\bullet}$ and
- *total differential* induced by $\mathbb{K}^{\mathbf{x}+\mathbf{y}}_{\bullet}$

where $\mathbf{x}+\mathbf{y} = x_1 + y_1, \ldots, x_n + y_n$ lies in $\Bbbk[\mathbf{x},\mathbf{y}]$. For any $\Bbbk[\mathbf{x}]$-module M, write

$$\mathbb{K}_{\bullet\bullet}(M) = \mathbb{K}_{\bullet\bullet} \otimes_{\Bbbk[\mathbf{y}]} M^{-\mathbf{y}}$$

for the *Koszul bicomplex of* M.

The following lemma is nothing more than Definition 6.2 expressed explicitly in coordinates for a monomial ideal I, which is required for concrete computation of differentials in the proof of Theorem 8.1.

Lemma 6.3. *$\mathbb{K}_{\bullet\bullet}(I) \cong \bigwedge^{\bullet} V \otimes I^{\mathbf{y}} \otimes \Bbbk[\mathbf{x}]$ has a $\Bbbk$-linear basis $\mathbf{z}^\tau \otimes \mathbf{y}^{\mathbf{b}} \otimes \mathbf{x}^{\mathbf{a}}$ for*

- $\mathbf{z}^\tau \in \bigwedge^{|\tau|} V$,
- $\mathbf{y}^{\mathbf{b}} \in I^{\mathbf{y}}$, *and*
- $\mathbf{x}^{\mathbf{a}} \in \Bbbk[\mathbf{x}]$.

The $\mathbb{N}^n$-degree of $\mathbf{z}^\tau \otimes \mathbf{y}^{\mathbf{b}} \otimes \mathbf{x}^{\mathbf{a}}$ is $\tau + \mathbf{b} + \mathbf{a}$. The differentials of $\mathbb{K}_{\bullet\bullet}$ in this basis are

$$\begin{array}{ccc} \sum_{k\in\tau}(-1)^{\tau\smallsetminus k\subset\tau}\mathbf{z}^{\tau-\mathbf{e}_k} \otimes \mathbf{y}^{\mathbf{b}} \otimes \mathbf{x}^{\mathbf{a}+\mathbf{e}_k} & \xleftarrow{d_1} & \mathbf{z}^\tau \otimes \mathbf{y}^{\mathbf{b}} \otimes \mathbf{x}^{\mathbf{a}} \\ & & \downarrow d \\ & & \sum_{k\in\tau}(-1)^{\tau\smallsetminus k\subset\tau}\mathbf{z}^{\tau-\mathbf{e}_k} \otimes \mathbf{y}^{\mathbf{b}+\mathbf{e}_k} \otimes \mathbf{x}^{\mathbf{a}}. \end{array}$$

Proposition 6.4. *For any $\Bbbk[\mathbf{x}]$-module M, the total complex $\mathbb{K}_{\bullet}^{\mathbf{x}+\mathbf{y}}(M)$ is a $\Bbbk[\mathbf{x}]$-free resolution of M as a module over $\Bbbk[\mathbf{x},\mathbf{y}]$ on which y_j acts as $-x_j$ for $j = 1, \ldots, n$.*

Proof. The total complex is free over $\Bbbk[\mathbf{x}]$ because $\mathbb{K}_{\bullet}^{\mathbf{x}+\mathbf{y}}(M) = \mathbb{K}_{\bullet}^{\mathbf{x}} \otimes \Bbbk[\mathbf{y}] \otimes_{\Bbbk[\mathbf{y}]} M^{-\mathbf{y}}$. That this complex can also be expressed as $\mathbb{K}_{\bullet\bullet} \otimes_{\Bbbk[\mathbf{x},\mathbf{y}]} \Bbbk[\mathbf{x},\mathbf{y}] \otimes_{\Bbbk[\mathbf{y}]} M^{-\mathbf{y}}$ means that it is the Koszul complex for the sequence $\mathbf{x}+\mathbf{y}$ acting on $\Bbbk[\mathbf{x},\mathbf{y}] \otimes_{\Bbbk[\mathbf{y}]} M^{-\mathbf{y}}$. This sequence is regular, so $\mathbb{K}_{\bullet}^{\mathbf{x}+\mathbf{y}}(M)$ is acyclic. Its nonzero homology is naturally the $\Bbbk[\mathbf{y}]$-module $M^{-\mathbf{y}}$ with an action of $\Bbbk[\mathbf{x}]$ in which the variable x_j acts on $M^{-\mathbf{y}}$ the way $-y_j$ acts. As a $\Bbbk[\mathbf{x}]$-module, this is just M. ∎

Remark 6.5. If $I \subseteq \Bbbk[\mathbf{x}]$ is any ideal and K is any $\Bbbk[\mathbf{x},\mathbf{y}]$-module, such as $\mathbb{K}_{\bullet}^{\mathbf{x}+\mathbf{y}}$, then

$$K \otimes_{\Bbbk[\mathbf{y}]} I^{-\mathbf{y}} \cong K \otimes_{\Bbbk[\mathbf{x},\mathbf{y}]} \Bbbk[\mathbf{x},\mathbf{y}] \otimes_{\Bbbk[\mathbf{y}]} I^{-\mathbf{y}} \cong K \otimes_{\Bbbk[\mathbf{x},\mathbf{y}]} I^{-\mathbf{y}}\Bbbk[\mathbf{x},\mathbf{y}],$$

where the second isomorphism is by flatness of $\Bbbk[\mathbf{x},\mathbf{y}]$ over $\Bbbk[\mathbf{y}]$. Thus $\mathbb{K}_{\bullet\bullet}(I)$ is the ordinary Koszul complex of the ideal $I^{-\mathbf{y}}\Bbbk[\mathbf{x},\mathbf{y}]$. Moreover, note that when I is graded, $I^{-\mathbf{y}} = I^{\mathbf{y}}$; indeed, $M^{-\mathbf{y}} = M^{\mathbf{y}}$ as $\Bbbk$-vector spaces for any $\mathbb{Z}$-graded $\Bbbk[\mathbf{x}]$-module M.

Lemma 6.6. *When M is a $\mathbb{Z}$-graded $\Bbbk[\mathbf{x}]$-module with $\mathbf{x} = x_1, \ldots, x_n$, double indexing*

$$\mathbb{K}_{pq}(M) = \Bbbk[\mathbf{x}] \otimes (\mathbb{K}_{p+q}^{\mathbf{y}} \otimes_{\Bbbk[\mathbf{y}]} M^{-\mathbf{y}})_p = \Bbbk[\mathbf{x}] \otimes \bigwedge^{p+q} V \otimes M_{-q}$$

makes the Koszul bicomplex of M into a fourth-quadrant bicomplex $\mathbb{K}_{\bullet\bullet}(M)$ of modules over $\Bbbk[\mathbf{x}]$ concentrated in a strip between the diagonals $p+q=0$ and $p+q=n$.

Proof. The main content of the claim is

$$(\mathbb{K}^{\mathbf{y}}_{p+q} \otimes_{\Bbbk[\mathbf{y}]} M^{-\mathbf{y}})_p = \bigwedge\nolimits^{p+q} V \otimes M^{-\mathbf{y}}_{-q} = \bigwedge\nolimits^{p+q} V \otimes M_{-q},$$

after which the lemma is proved by tensoring with $\Bbbk[\mathbf{x}]$. ∎

Theorem 6.7. *Given any $\mathbb{Z}$-graded $\Bbbk[\mathbf{x}]$-module M, its Koszul bicomplex $\mathbb{K}_{\bullet\bullet}(M)$ has vertical-then-horizontal spectral sequence*

$$\operatorname{Tor}_{p+q}(\Bbbk, M)_p \otimes \Bbbk[\mathbf{x}] \Rightarrow H_{p+q}\mathbb{K}^{\mathbf{x}+\mathbf{y}}_{\bullet}(M).$$

Proof. The vertical homology of $\mathbb{K}_{\bullet\bullet}(M)$ at the location pq is $\Bbbk[\mathbf{x}] \otimes \operatorname{Tor}_{p+q}(\Bbbk, M)_p$ by Lemma 6.6 (particularly its indexing). The rest is by Definition 6.2. ∎

Corollary 6.8. *For a monomial ideal I, the vertical-then-horizontal spectral sequence of its Koszul bicomplex $\mathbb{K}_{\bullet\bullet}(I)$ is*

$$\bigoplus_{|\mathbf{a}|=p} \widetilde{H}_{p+q-1} K^{\mathbf{a}} I \otimes \Bbbk[\mathbf{x}] \Rightarrow H_{p+q}\mathbb{K}^{\mathbf{x}+\mathbf{y}}_{\bullet}(I).$$

Proof. Apply Hochster's formula (Theorem 1.2) to Theorem 6.7. ∎

Remark 6.9. The spectral sequence in Corollary 6.8 has $\widetilde{H}_{i-1} K^{\mathbf{a}} I$ at $p = |\mathbf{a}|$ and $q = i - p$ in E^1_{pq}. It yields arrows

$$\begin{tikzcd}[column sep=small]
\ddots & & & & \\
\displaystyle\bigoplus_{|\mathbf{a}|=|\mathbf{b}|-4} \widetilde{H}_{i-1} K^{\mathbf{a}} I & & & & \\
& \displaystyle\bigoplus_{|\mathbf{a}|=|\mathbf{b}|-3} \widetilde{H}_{i-1} K^{\mathbf{a}} I & & & \\
& & \displaystyle\bigoplus_{|\mathbf{a}|=|\mathbf{b}|-2} \widetilde{H}_{i-1} K^{\mathbf{a}} I & & \\
& & & \displaystyle\bigoplus_{|\mathbf{a}|=|\mathbf{b}|-1} \widetilde{H}_{i-1} K^{\mathbf{a}} I & \widetilde{H}_i K^{\mathbf{b}} I \arrow[l] \arrow[ull] \arrow[uulll] \arrow[uuullll]
\end{tikzcd}$$

as in Problem 1.3, but by the nature of spectral sequences, these arrows only give rise to homomorphisms between subquotients of the relevant free modules and therefore cannot directly be differentials in a free resolution of I. The next section is the remedy.

7. Minimal free resolutions from Wall complexes

The default coefficient ring in this section is an arbitrary ring R.

This section is mainly a summary of constructions and main results on Wall complexes from [13], without repeating proofs. The goal is to deduce that derived Wall complexes of Koszul bicomplexes are minimal free resolutions (Corollary 7.11).

The key point is that the Wall complex should resolve the given ideal I, and not some associated graded module $\operatorname{gr} I$.

Definition 7.1. Fix a ring R and a doubly indexed array $W_{\bullet\bullet}$ of R-modules with maps $\omega_j : W_{pq} \to W_{p-j,q+j-1}$ for $j \in \mathbb{N}$ (the index pq on ω_j is suppressed). Assume that for each element $w \in W_{pq}$, only finitely many images $\omega_j(w)$ are nonzero. Set

$$W_i = \bigoplus_{p+q=i} W_{pq} \quad \text{and} \quad D_i = \sum_{j=0}^{\infty} \omega_j : W_i \to W_{i-1}.$$

These data constitute a *Wall complex* if $D^2 = 0$, and $W_\bullet$ with the differential D is the *total complex* of $W_{\bullet\bullet}$.

Definition 7.2. Fix a bicomplex $C_{\bullet\bullet}$ of R-modules with vertical differential $d = d_0$ and horizontal differential d_1. A *vertical splitting* of $C_{\bullet\bullet}$ consists of a differential

$$d^+ = d^+_{pq} : C_{pq} \to C_{p,q+1}$$

with $dd^+d = d$ and $d^+dd^+ = d^+$. Being a differential means $d^+d^+ = 0$.

Thus d^+ is a vertical cohomological differential, going up columns opposite to the homological vertical differential d. The following is elementary [13, Proposition 1.1].

Proposition 7.3. *A vertical splitting of $C_{\bullet\bullet}$ is equivalent to a direct sum decomposition*

$$C_{pq} = B'_{p,q-1} \oplus \underbrace{H_{pq} \oplus B_{pq}}_{Z_{pq}}$$

in which, for all indices p and q,

- $H_{pq} \oplus B_{pq} = Z_{pq} = \ker d_{pq}$ *and*
- $\bar{d}_{p,q-1} : B'_{p,q-1} \xrightarrow{\sim} B_{p,q-1} = \operatorname{im} d_{pq}$, *where $\bar{d}_{p,q-1}$ is the restriction of d_{pq} to $B'_{p,q-1}$.*

More precisely, a vertical splitting is constructed from this direct sum decomposition by

$$d^+_{pq} = \iota_{pq} \circ \bar{d}^{-1}_{pq} \circ \pi_{pq},$$

where $\pi_{pq} : C_{pq} \twoheadrightarrow B_{pq}$ projects to the summand B_{pq} and $\iota_{pq} : B'_{pq} \hookrightarrow C_{p,q+1}$ is inclusion.

The homomorphisms whose composites define Wall complexes from bicomplexes are elementary to isolate.

Lemma 7.4. *Fix a vertical splitting of $C_{\bullet\bullet}$, with notation as in Proposition 7.3.*

(1) *The homology projection $C_{\bullet\bullet} \twoheadrightarrow H_{\bullet\bullet}$ is $P = \mathbb{1} - dd^+ - d^+d$.*

(2) *The composite of the upward and leftward differentials induces homomorphisms*

$$\begin{array}{cc} C_{p-1,q+1} & \\ & \nwarrow^{d^+d_1} \\ & C_{pq} \end{array}$$

Together, these homomorphisms induce morphisms $\omega_j : H_{pq} \to H_{p-j,q+j-1}$ for $j \geq 1$ via

$$\omega_j = P(d_1 d^+)^{j-1} d_1. \qquad \blacksquare$$

Remark 7.5. Pictorially, the homomorphisms $\omega_1, \omega_2, \omega_3, \omega_4, \ldots$ can be depicted as lying along the arrows in Remark 6.9.

Definition 7.6. The *derived Wall complex* of a bicomplex $C_{\bullet\bullet}$ with vertical differential d split by d^+ is $H_{\bullet\bullet}$ with the differentials $\omega_0 = 0$ and ω_i from Lemma 7.4 for $i \geq 1$.

Remark 7.7. Let $C_{\bullet\bullet}$ be a vertically split bicomplex. The derived Wall complex selects a split submodule $H_{pq} \subseteq Z_{pq}$ inside the vertical cycles of C_{pq} that maps isomorphically to the vertical homology—naturally defined as the quotient of these same cycles modulo the boundary submodule—which we denote by $\widetilde{H}_{pq} = Z_{pq}/B_{pq}$ so as not to confuse it with the submodule H_{pq}. The Wall differential $\omega_j = P(d_1 d^+)^{j-1} d_1$ assumes that its input is an element of the split homology submodule H_{pq}. In applications such as to Problem 1.3, where one wishes to specify homomorphisms

$$\widetilde{\omega}_j : \widetilde{H}_{pq} \to \widetilde{H}_{p-j,q+j-1}$$

on natural homology, the input should be a homology class—specified as a cycle that is well defined only up to adding a boundary element, rather than specified as an element of the split submodule H_{pq}—but at a cost: $\widetilde{\omega}_j$ must first project Z_{pq} to H_{pq} to ensure that the Wall differential acts indistinguishably on different cycles representing the same homology class. This projection is

$$\mathbb{1} - dd^+ : Z_{pq} \to H_{pq}.$$

On the other hand, $\widetilde{\omega}_j$ need not be forced to produce output that lies in the split submodule $H_{p-j,q+j-1}$; it need only produce a cycle in $Z_{p-j,q+j-1}$, since the output of $\widetilde{\omega}_j$ is to be understood modulo B_{pq}. That means $\widetilde{\omega}_j$ can use the simpler projection

$$\mathbb{1} - d^+ d : C_{pq} \to Z_{pq}$$

from chains to cycles instead of the split homology projection $P = \mathbb{1} - dd^+ - d^+ d$ from Lemma 7.4 (1). In total, then, the projection dd^+ moves from the left end of the expression defining $\omega_j = (\mathbb{1} - dd^+ - d^+ d)(d_1 d^+)^{j-1} d_1$ to the right end of the expression

$$\widetilde{\omega}_j = (\mathbb{1} - d^+ d)(d_1 d^+)^{j-1} d_1 (\mathbb{1} - dd^+) : Z_{pq} \to Z_{p-j,q+j-1},$$

which defines a differential—the same differential as ω_j defines—because

$$\begin{aligned}(\mathbb{1} - dd^+)(\mathbb{1} - d^+ d) &= \mathbb{1} - dd^+ - d^+ d + dd^+ d^+ d \\ &= \mathbb{1} - dd^+ - d^+ d\end{aligned}$$

occurs between the factors of $(d_1 d^+)^{j-1} d_1$ and $(d_1 d^+)^{j'-1} d_1$ in the square of the Wall differential either way.

It is these differentials $\widetilde{\omega}_j$, rather than ω_j from Lemma 7.4, that solve Problem 1.3 and give rise to the combinatorics in Section 3. We therefore record this shift from split homology $H_{\bullet\bullet}$ to natural homology $\widetilde{H}_{\bullet\bullet}$ formally, the proof being in Remark 7.7.

Definition 7.8. The *natural Wall complex* of a bicomplex $C_{\bullet\bullet}$ with vertical differential d split by d^+ is $\widetilde{H}_{\bullet\bullet}$ with the differentials $\widetilde{\omega}_0 = 0$ and $\widetilde{\omega}_i$ from Remark 7.7 for $i \geq 1$.

Proposition 7.9 ([13, Theorem 1.2]). *The derived Wall complex $H_{\bullet\bullet}$ of a vertically split bicomplex $C_{\bullet\bullet}$ is a Wall complex as long as the local finiteness of $\omega_\bullet$ is satisfied. The total complex of $H_{\bullet\bullet}$ has a filtration by taking successively more columns, starting from the left. The spectral sequence $HE^\bullet$ for this filtration of $H_{\bullet\bullet}$ is the same as the vertical-then-horizontal spectral sequence $E^\bullet$ of $C_{\bullet\bullet}$, in the sense that $HE^r_{pq} \cong E^r_{pq}$ for $r \geq 1$.*

Proposition 7.10. *Using the natural Wall complex $\widetilde{H}_{\bullet\bullet}$ in place of $H_{\bullet\bullet}$ and $\widetilde{\omega}_\bullet$ in place of $\omega_\bullet$ in Proposition 7.9, its conclusions hold verbatim.* ∎

Corollary 7.11. *Fix an arbitrary standard $\mathbb{Z}$-graded module M over $\Bbbk[\mathbf{x}]$. The total complex $W_\bullet(M)$ of the derived or natural Wall complex $W_{\bullet\bullet}(M)$ for any vertical splitting of the Koszul bicomplex $\mathbb{K}_{\bullet\bullet}(M)$ is a minimal free resolution of M.*

Proof. Remark 7.7 explains why the derived and natural Wall complexes have the same homology, so it is only necessary to prove the derived Wall case, for which [16, Lemma 3.5] provides a brief proof in the generality of abelian categories. ∎

8. Monomial resolutions from splittings

The first result in this section accomplishes the noncombinatorial part of the proof of Theorem 3.10. It is separated from the rest of the proof because it applies in much more generality than the canonical sylvan setting, which results from the choice to use Moore–Penrose pseudoinverses as the Koszul simplicial splittings. Notation for saturated decreasing lattice paths is as in Definition 3.1: $\lambda \in \Lambda(\mathbf{a}, \mathbf{b})$ is described as $\mathbf{b} = \mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_{\ell-1}, \mathbf{b}_\ell = \mathbf{a}$ or its successive differences $(\lambda_1, \ldots, \lambda_\ell)$ with $\lambda_j = \mathbf{b}_{j-1} - \mathbf{b}_j$.

Theorem 8.1. *Fix a monomial ideal I. Any splittings $\partial^{\mathbf{b}+}$ of the differentials $\partial^{\mathbf{b}}$ of the Koszul simplicial complexes $K^{\mathbf{b}} I$ for $\mathbf{b} \in \mathbb{N}^n$ that are themselves differentials satisfying*

(1) $\partial^{\mathbf{b}}\partial^{\mathbf{b}+}\partial^{\mathbf{b}} = \partial^{\mathbf{b}}$ *and*

(2) $\partial^{\mathbf{b}+}\partial^{\mathbf{b}}\partial^{\mathbf{b}+} = \partial^{\mathbf{b}+}$

yield a minimal free resolution of I whose differential from homological stage $i+1$ to stage i has its component $\widetilde{H}_{i-1} K^{\mathbf{a}} I \otimes \Bbbk[\mathbf{x}](-\mathbf{a}) \leftarrow \widetilde{H}_i K^{\mathbf{b}} I \otimes \Bbbk[\mathbf{x}](-\mathbf{b})$ induced by the map

$$\widetilde{H}_{i-1} K^{\mathbf{a}} I \xleftarrow{D} \widetilde{H}_i K^{\mathbf{b}} I$$

in $\mathbb{N}^n$-degree $\mathbf{b}$ that acts on any i-cycle in $\widetilde{Z}_i K^{\mathbf{b}} I$ via

$$D = \sum_{\lambda \in \Lambda(\mathbf{a},\mathbf{b})} (\mathbb{1}^{\mathbf{a}} - \partial_i^{\mathbf{a}+}\partial_i^{\mathbf{a}}) d_1^{\lambda_\ell} \Big(\prod_{j=1}^{\ell-1} \partial_i^{\mathbf{b}_j+} d_1^{\lambda_j} \Big) (\mathbb{1}^{\mathbf{b}} - \partial_{i+1}^{\mathbf{b}} \partial_{i+1}^{\mathbf{b}+}),$$

where $d_1^{\lambda_j}$ takes $\tau \subseteq \{1, \ldots, n\}$ to 0 if $\lambda_j \notin \tau$ and to $(-1)^{\tau \smallsetminus \lambda_j \subset \tau} \tau \smallsetminus \lambda_j$ if $\lambda_j \in \tau$, and where $\mathbb{1}^{\mathbf{c}}$ is the identity map on the relevant degree $\mathbf{c}$ Koszul simplicial chain group.

Remark 8.2. To visualize the formula for D, read the "∂" and "d" maps in the following diagram from right to left, ignoring height, but note that the products of the rightmost up-down and leftmost down-up maps must be subtracted from the identity.

$$
\begin{array}{cccccccccc}
\widetilde{C}_{i-1}K^{\mathbf{a}}I & \xleftarrow{d_1^{\lambda_\ell}} & & & & & & & & \\
\partial_i^{\mathbf{a}+}\uparrow\downarrow\partial_i^{\mathbf{a}} & & \uparrow\partial_i^{\mathbf{b}_{\ell-1}+} & & & & & & & \\
 & & \widetilde{C}_{i-1}K^{\mathbf{b}_{\ell-1}}I & \xleftarrow{d_1^{\lambda_{\ell-1}}} & & & & & & \\
 & & & & \ddots & \uparrow\partial_i^{\mathbf{b}_2+} & & & & \\
 & & & & & \widetilde{C}_{i-1}K^{\mathbf{b}_2}I & \xleftarrow{d_1^{\lambda_2}} & & & \\
 & & & & & & & \uparrow\partial_i^{\mathbf{b}_1+} & & \partial_{i+1}^{\mathbf{b}}\downarrow\uparrow\partial_{i+1}^{\mathbf{b}+} \\
 & & & & & & & \widetilde{C}_{i-1}K^{\mathbf{b}_1}I & \xleftarrow{d_1^{\lambda_1}} & \widetilde{C}_iK^{\mathbf{b}}I
\end{array}
$$

This diagram, when rotated counterclockwise by $\pi/4$, is a zoomed-in, labeled version of the chain-link fence in Definition 3.5. Compare also Remarks 7.5 and 6.9.

The proof uses the notation of Lemma 6.3 for concrete computations of differentials.

Proof of Theorem 8.1. The vertical differentials of the Koszul bicomplex $\mathbb{K}_{\bullet\bullet}(I)$ are obtained from the chain complexes of Koszul simplicial complexes of I by tensoring with $\Bbbk[\mathbf{x}]$ over $\Bbbk$ (use Lemma 6.3 if this is not clear from Convention 6.1 and Definition 6.2). The given splittings $\partial^{\mathbf{b}+}$ thus induce a vertical splitting d^+ of $\mathbb{K}_{\bullet\bullet}(I)$.

The natural Wall complex (Definition 7.8) of this vertically split Koszul bicomplex minimally resolves I by Corollary 7.11. The differentials in this resolution are, by Definition 7.1, $D = \sum_j \widetilde{\omega}_j$ for the homomorphisms $\widetilde{\omega}_j = (\mathbb{1} - d^+d)(d_1d^+)^{j-1}d_1(\mathbb{1} - dd^+)$ from Remark 7.7. The goal is to determine the action of d^+ on $\mathbb{N}^n$-degree $\mathbf{b}$ Koszul cycles in $\widetilde{Z}_iK^{\mathbf{b}}I \otimes \Bbbk[\mathbf{x}] \subseteq (\mathbb{K}^{\mathbf{y}}_{\bullet})_{\mathbf{b}} \otimes \Bbbk[\mathbf{x}]$, and then the action of d on the output of this d^+, and the action of d_1 on the output of this d, and so on.

The reason why this requires care is that the action of d^+ on an $\mathbb{N}^n$-degree $\mathbf{b}$ element of $\mathbb{K}_{\bullet\bullet}(I)$ depends on how the element decomposes in the basis from Lemma 6.3: the vertical splitting is $\partial^{\mathbf{b}+}$ only on basis vectors of the form $\mathbf{z}^\tau \otimes \mathbf{y}^{\mathbf{b}-\tau} \otimes \mathbf{x}^{\mathbf{a}}$. It is therefore crucial that the $\mathbb{N}^n$-degree $\mathbf{b}$ elements in $(\mathbb{K}^{\mathbf{y}}_{\bullet})_{\mathbf{b}} \otimes \Bbbk[\mathbf{x}]$ all have the form $\mathbf{z}^\tau \otimes \mathbf{y}^{\mathbf{b}-\tau} \otimes 1$ and are not (say) mixtures in which the $\mathbf{x}$-factors have nonzero $\mathbb{N}^n$-degree.

In contrast to d^+, the actions of d_1 and d do not depend on the tensor decomposition. Let us start with d. The isomorphism $(\mathbb{K}^{\mathbf{y}}_{\bullet}(I^{\mathbf{y}}))_{\mathbf{b}} \cong \widetilde{C}_{\bullet}K^{\mathbf{b}}I$ of the $\mathbb{N}^n$-graded components of the columns of $\mathbb{K}_{\bullet\bullet}(I)$ with chain complexes of Koszul simplicial complexes identifies $\mathbf{z}^\tau \otimes \mathbf{y}^{\mathbf{b}-\tau}$ with the face τ. Hence, Lemma 6.3 identifies d with the simplicial boundary operator $\partial^{\mathbf{b}}$ of $\widetilde{C}_{\bullet}K^{\mathbf{b}}I$. This is true regardless of the $\mathbf{x}$-factor and, indeed, regardless of the $\mathbb{N}^n$-degree, although of course for different $\mathbb{N}^n$-degrees the differential occurs in a different Koszul simplicial complex. Importantly, the **zy**-degree, for purposes of the splitting d^+, does not change under d, as is visible from Lemma 6.3.

Similarly, d_1 acts on simplices $\tau = \mathbf{z}^\tau \otimes \mathbf{y}^{\mathbf{c}-\tau} \otimes \mathbf{x}^{\mathbf{b}-\mathbf{c}}$ of $K^{\mathbf{c}} I$ (thought of as residing in $\mathbb{N}^n$-degree $\mathbf{b}$ of $\widetilde{C}_{\bullet} K^{\mathbf{c}} I \otimes \Bbbk[\mathbf{x}]$) as the boundary operator, but in this case the **zy**-degree of the boundary face $\tau - \mathbf{e}_k$ has **zy**-degree $\mathbf{c} - \mathbf{e}_k$. Therefore $d_1 = d_1^{\mathbf{e}_1} + \cdots + d_1^{\mathbf{e}_n}$ decomposes into the components that alter the **zy**-degree by $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Substituting this decomposition of d_1 back into the formula for $\widetilde{\omega}_j$ and introducing the $\mathbb{N}^n$-degree indices $\mathbf{b}, \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{\ell-1}, \mathbf{b}_\ell = \mathbf{a}$ on the upward and downward differentials yields the sum over saturated decreasing lattice paths $\lambda \in \Lambda(\mathbf{a}, \mathbf{b})$, as desired. ∎

Remark 8.3. The proof of Theorem 8.1 shows that the only summand $\widetilde{\omega}_j$ contributing to the component $\widetilde{H}_{i-1} K^{\mathbf{a}} I \otimes \Bbbk[\mathbf{x}](-\mathbf{a}) \leftarrow \widetilde{H}_i K^{\mathbf{b}} I \otimes \Bbbk[\mathbf{x}](-\mathbf{b})$ induced by the homomorphism $\widetilde{H}_{i-1} K^{\mathbf{a}} I \xleftarrow{D} \widetilde{H}_i K^{\mathbf{b}} I$ is $\widetilde{\omega}_\ell$, where $\ell = |\mathbf{b}| - |\mathbf{a}|$.

Proof of Theorem 3.10. Given Theorem 8.1, it remains only to prove that the formula for D in Theorem 8.1 specializes to the canonical sylvan homomorphism with entries

$$D_{\sigma\tau} = \sum_{\lambda \in \Lambda(\mathbf{a},\mathbf{b})} \frac{1}{\Delta_{i,\lambda} I} \sum_{\varphi \in \Phi_{\sigma\tau}(\lambda)} w_\varphi$$

from Definition 3.8 when all of the splittings $\partial^{\mathbf{b}_j +}$ are the Moore–Penrose pseudoinverses of the differentials $\partial^{\mathbf{b}_j}$. The proof is lattice path by lattice path, so fix henceforth a saturated decreasing lattice path λ from $\mathbf{b}$ to $\mathbf{a}$ of length $\ell = |\mathbf{b}| - |\mathbf{a}|$. Fix as well the simplices $\tau \in K_i^{\mathbf{b}} I$ and $\sigma \in K_{i-1}^{\mathbf{a}} I$.

The Projection Hedge Formula (Proposition 5.12) for $\pi_{B_i} = \partial_{i+1}\partial_{i+1}^+$ shows that

$$\begin{aligned} \mathbb{1} - \partial_{i+1}\partial_{i+1}^+ &= \frac{1}{\Delta_i^S}\Big(\Delta_i^S \mathbb{1} - \sum_{S_i} \delta_{S_i}^2 \beta_{S_i}\Big) \\ &= \frac{1}{\Delta_i^S} \sum_{S_i} \delta_{S_i}^2 (\mathbb{1} - \beta_{S_i}). \end{aligned}$$

In the image of τ under this homomorphism at $\mathbf{b} \in \mathbb{N}^n$, the coefficient on τ_0 in the summand for the stake set $S_i^{\mathbf{b}}$ is the weight of the boundary-link $\tau_0 — \tau$ by Definition 2.15 (3). Summing over stake sets and dividing by $\Delta_i^S K^{\mathbf{b}} I$ yields the $\tau_0\tau$ matrix entry in $\partial_{i+1}\partial_{i+1}^+$.

Let $j \geq 1$. In the image of any i-simplex τ_{j-1} under $d_1 = d_1^{\mathbf{e}_1} + \cdots + d_1^{\mathbf{e}_n}$, where $d_1^{\mathbf{e}_k}$ alters the **zy**-degree by $\mathbf{e}_k$ (see the end of the proof of Theorem 8.1), the coefficient on σ_j is 0 unless $\sigma_j = \tau_{j-1} - \lambda_j$ as in Definition 3.5, in which case the coefficient output by Theorem 8.1 is a sign—the correct one for the containment $\sigma_j \diagup \tau_{j-1}$ by Definition 3.6.

In the image of any $(i-1)$-simplex σ_j under the Moore–Penrose pseudoinverse $\partial_i^{\mathbf{b}_j +}$ of the boundary $\partial_i^{\mathbf{b}_j}$, the coefficient on τ_j in the summand indexed by the hedge $ST_i^{\mathbf{b}_j}$ in the Hedge Formula (Corollary 5.9) is the weight of the chain-link $\tau_j \diagdown \sigma_j$ by Proposition 5.3. Summing over hedges and dividing by $\Delta_i^{ST} K^{\mathbf{b}_j} I$ yields the $\sigma_j\tau_j$ matrix entry in $\partial_i^{\mathbf{b}_j +}$.

In the image of any $(i-1)$-simplex σ_ℓ under the orthogonal projection $\mathbb{1} - \partial_i^{\mathbf{a}+}\partial_i^{\mathbf{a}}$ to the cycles $\widetilde{Z}_{i-1}K^{\mathbf{a}}I$, the coefficient on σ in the summand indexed by the shrubbery $T_{i-1}^{\mathbf{a}}$ in the Projection Hedge Formula (Corollary 5.11 and Proposition 5.12) is the weight of the cycle-link σ — σ_ℓ Definition 2.15 (1). Summing over shrubberies and dividing by $\Delta_{i-1}^{T}K^{\mathbf{a}}I$ yields the $\sigma_\ell\sigma$ matrix entry in $\mathbb{1} - \partial_i^{\mathbf{a}+}\partial_i^{\mathbf{a}}$.

Summing the products of these three kinds of sums and the sign over all chain-link fences from τ to σ yields the matrix entry $D_{\sigma\tau}$ by matrix multiplication from elementary linear algebra. Definition 3.8 expresses this product of sums as a sum of products. ∎

9. Noncanonical sylvan resolutions

The master formula for Wall resolutions from Koszul simplicial splittings (Theorem 8.1) has the consequence that once the canonicality requirement is dropped, our constructions work universally, combinatorially, and minimally. The format is basically the same as Theorem 3.10, but there is no division and only edge weights e_φ enter—not the vertex weights δ_φ; recall Definitions 3.6 and 3.7 (via Definition 3.3) for these weights.

Remark 9.1. Since the main result in this section, Corollary 9.5, occurs at the end of the paper, it is worth taking precise account of the relatively meager prerequisites—beyond standard constructions like Koszul simplicial complexes $K^{\mathbf{b}}I$—on which its statement (but not its proof) relies. It requires the notions of

- shrubbery, stake, hedge (Definition 2.4) and their coefficients (Definition 2.15);
- hedgerow (Definition 3.1) to assemble this combinatorics along lattice paths;
- chain-link fence (Definition 3.5) with edge weights (Definition 3.6); and
- community (Definition 9.2) and hedge splitting (Definition 5.1) for the differential.

Definition 9.2. Fix a CW complex K. A *community* in K is a sequence

$$ST_\bullet = (ST_0, ST_1, ST_2, \dots) \quad \text{of hedges with} \quad T_i \cap S_i = \{\} \text{ for all } i.$$

Proposition 9.3. *Any community $ST_\bullet$ induces a differential $\partial^+_{ST_\bullet}$ over $\Bbbk$ such that*

(1) $\partial_i \partial^+_{ST_i} \partial_i = \partial_i$ *and*

(2) $\partial^+_{ST_i} \partial_i \partial^+_{ST_i} = \partial^+_{ST_i}$.

Proof. The disjointness of T_{i-1} and S_{i-1} means that $T_{i-1} \subseteq \overline{S}_{i-1}$, so $\partial^+_{ST_{i+1}}\partial^+_{ST_i} = 0$ by Definition 5.1. Property (1) follows from Proposition 5.2 because $\zeta_{T_i}(\tau)$ is a cycle—that is, $\partial_i \zeta_{T_i}(\tau) = 0$. Property (2) is immediate from Definition 5.1, with both sides of the equation being 0 for non-stakes and τ when applied to any boundary $\partial_i \tau$. ∎

Definition 9.4. Fix a monomial ideal I and a community (Definition 9.2 and Proposition 9.3) for each Koszul simplicial complex $K^{\mathbf{b}}I$. These data endow each lattice path $\lambda \in \Lambda(\mathbf{a}, \mathbf{b})$ with a fixed hedgerow ST_i^λ for $i = 0, \dots, n$. The *sylvan homomorphism*

$$\widetilde{C}_{i-1}K^{\mathbf{a}}I \xleftarrow{D = D^{\mathbf{ab}}} \widetilde{C}_i K^{\mathbf{b}}I$$

for these data is given by its *sylvan matrix*, whose entry $D_{\sigma\tau}$ for $\tau \in K_i^{\mathbf{b}} I$ and $\sigma \in K_{i-1}^{\mathbf{a}} I$ is the sum, over all lattice paths from $\mathbf{b}$ to $\mathbf{a}$, of the edge weights of all chain-link fences from τ to σ that are subordinate (Definition 3.5) to the relevant hedgerow ST_i^{λ}:

$$D_{\sigma\tau} = \sum_{\lambda \in \Lambda(\mathbf{a},\mathbf{b})} \sum_{\substack{\varphi \in \Phi_{\sigma\tau}(\lambda) \\ \varphi \vdash ST_i^{\lambda}}} e_{\varphi}.$$

Corollary 9.5. *Fix a monomial ideal I and a community for each Koszul simplicial complex $K^{\mathbf{b}} I$. The sylvan homomorphism for these data on each comparable pair $\mathbf{a} \prec \mathbf{b}$ of lattice points induces a homomorphism $\widetilde{Z}_{i-1} K^{\mathbf{a}} I \leftarrow \widetilde{Z}_i K^{\mathbf{b}} I$ that vanishes on $\widetilde{B}_i K^{\mathbf{b}} I$, and hence it induces a well defined* sylvan homology morphism $\widetilde{H}_{i-1} K^{\mathbf{a}} I \leftarrow \widetilde{H}_i K^{\mathbf{b}} I$. *The induced homomorphisms*

$$\widetilde{H}_{i-1} K^{\mathbf{a}} I \otimes \Bbbk[\mathbf{x}](-\mathbf{a}) \leftarrow \widetilde{H}_i K^{\mathbf{b}} I \otimes \Bbbk[\mathbf{x}](-\mathbf{b})$$

of $\mathbb{N}^n$-graded free $\Bbbk[\mathbf{x}]$-modules constitute a minimal free resolution of I.

Proof. The proof of Theorem 3.10 in Section 8, which is already done lattice path by lattice path, works mutatis mutandis in this setting but simplifies because the fixed hedgerows eliminate the summations over stake sets, hedges, and shrubberies. ∎

Remark 9.6. In general, a minimal free resolution ought to be called *sylvan* if its differentials are expressed as linear combinations of those for individual choices of hedges. Thus the canonical resolutions in Theorem 3.10 are sylvan because its differentials are weighted averages of differentials from hedges, and the resolutions in Corollary 9.5 are sylvan because each fixes single choices of hedges. Of course, all minimal free resolutions of a given graded ideal are isomorphic; the question is how the resolution is expressed. Usually in commutative algebra the differentials are expressed by selecting bases for the syzygies. In contrast, the sylvan method avoids choosing such bases, even in Corollary 9.5, because the syzygies are naturally homology vector spaces. Instead, the sylvan method selects bases for chains in a manner that descends to homology.

Remark 9.7. Canonical sylvan resolutions in Theorem 3.10 are not suited to efficient algorithms, as they require storage, manipulation, and sums over bases for chains in simplicial complexes. In contrast, noncanonical sylvan resolutions could potentially lead to efficient algorithmic computation of free resolutions, since they select bases not for chains but for homology and cohomology (see Remark 2.7) in each $\mathbb{N}^n$-degree while never actually computing homology. It helps that communities have been independently and simultaneously invented in the context of persistent homology, where they are called "tripartitions" [15]. The computational algebra of these could be particularly helpful, as relevant algorithms have been implemented.

Example 9.8. Noncanonical sylvan resolutions provide combinatorial minimal free resolutions of monomial ideals whose Betti numbers vary with the characteristic of the field, such as the Stanley–Reisner ideal of the six-vertex triangulation of the real projective plane. In any characteristic other than 2, this ideal has a minimal cellular free resolution of length 2; see [27, Section 4.3.5], for instance. But in characteristic 2, the top Betti number is at homological stage 3, namely $\beta_{3,\mathbb{1}}(I) = 1$, where $\mathbb{1} = (1,1,1,1,1,1)$. A sylvan resolution compensates by selecting hedges that respect the dependencies in characteristic 2. In particular, although the stake set $S_2^{\mathbb{1}} = \{\}$ at $\mathbb{1}$ is forced, because $K^{\mathbb{1}}I = \mathbb{RP}^2$ has dimension 2, the stake sets at degrees $\mathbb{1} - \mathbf{e}_i$ that differ from $\mathbb{1}$ by a standard basis vector $\mathbf{e}_i$ have cardinality 1 in characteristic 2, each consisting of any single edge in the relevant Koszul simplicial complex, thereby allowing the construction of chain-link fences to get started.

Remark 9.9. As noted in Section 1.3, a primary impediment to constructing canonical minimal free resolutions of arbitrary monomial ideals has been that writing down matrices for the differentials requires choices of bases in which to express those matrices. Sylvan matrices (Definition 3.8) get around the problem by specifying homomorphisms at the level of chains rather than homology classes. The canonical construction comes, however, at a cost of taking weighted averages, which have denominators, thus eliminating finitely many primes as characteristics for the field $\Bbbk$. Corollary 9.5 says that if one is willing to forgo canonicality, then selecting bases for homology and cohomology (see Remark 2.7) yields universal, closed-form, combinatorial minimal free resolutions without restriction on the characteristic, as should be expected. Judicious such selections can recover known special classes of resolutions of monomial ideals, such as the Eliahou–Kervaire resolution of any Borel-fixed or stable ideal [17] (see also [27, Chapter 2]) or any planar graph resolution of a trivariate ideal [26] (see also [27, Chapter 3]). These assertions require proof; they are topics for subsequent papers.

Remark 9.10. Sylvan methods put an upper bound on bad characteristics, but they do not furnish an obvious way to detect them precisely. Indeed, denominators in sylvan matrices are often not genuinely bad. See Example 4.3, whose denominators are filled with 2's and 3's even though there are no bad characteristics in three variables.

Acknowledgements. A great debt goes to Joel Roberts, with whom we discussed preliminary ideas along these lines decades ago. EM wishes to thank Art Duval for an enlightening conversation about simplicial spanning trees. Alexandre Tchernev provided extremely valuable feedback on a preprint version of this paper. Erin Delargy and Benjamin Greene helped evaluate the manuscript revisions that were suggested by two unusually thoughtful referees.

Funding. EM and EO had support from NSF DMS-1702395. EO was supported for a semester by NSF DMS-1406371.

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