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The pong algebra

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Abstract. In an earlier paper, we described bordered algebras for knot Floer homology. In this paper, we introduce closely related differential graded algebra, the *pong algebra*, and compute the A_∞ structure on its homology.

1. Introduction

Knot Floer homology [15,23] is a knot invariant defined using methods from symplectic geometry. This invariant has many variants, the most general being a module over $\mathbb{Z}[u,v]$. Its $u = v = 0$ specialization, $\widehat{HFK}$, is a categorification of the Alexander polynomial.

In [16,17], we developed an algebraic invariant for knots, inspired by Floer homology. This invariant is the homology of a chain complex associated to a knot projection. The homology is a module over the ring $\mathbb{Z}[u,v]/uv = 0$. In [20], we identified these constructions with analogues in knot Floer homology with coefficients mod 2; specifically, the $u = v = 0$ specialization computes $\widehat{HFK}$ with $\mathbb{Z}/2\mathbb{Z}$ coefficients while the $v = 0$ specialization computes HFK^- over $\mathbb{Z}/2\mathbb{Z}[u]$.

The construction of the bordered knot invariants associates an algebra to a generic horizontal slice of the knot diagram and modules over this algebra to the upper and lower diagrams. Specifically, versions of the bordered algebras are indexed by pairs of integers m and k with $k < m$. These *bordered algebras* $\mathcal{C}(m,k)$ have idempotents $\mathbf{I}_\mathbf{x}$ corresponding to *idempotent states* $\mathbf{x}$, which are increasing sequences of integers $1 \leq x_1 < \cdots < x_k \leq m-1$. These idempotents generate a subalgebra $\mathcal{I}(m,k) \subset \mathcal{C}(m,k)$. The algebra also contains distinguished elements $U_1,\dots,U_m$, $L_2,\dots,L_{m-1}$, and $R_2,\dots,R_{m-1}$ satisfying various relations, including $R_i R_{i+1} = 0$ and $L_i L_{i-1} = 0$. (See Section 3.)

Our goal here is to enrich $\mathcal{C}(m,k)$, with an eye towards constructing a knot invariant over $\mathbb{Z}[u,v]$. To this end, we introduce a differential graded algebra, the *pong algebra*. Like the bordered algebras, the pong algebras are indexed by pairs of integers m and k; we denote them by $\mathcal{P}(m,k)$. The algebra $\mathcal{P}(m,k)$ is an algebra over

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the polynomial algebra over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ in m variables $v_1, \dots, v_m$. The definitions of these algebras are combinatorial and closely connected to the strands algebras which have appeared in bordered Floer homology [10]. These algebras are defined in Section 4. By construction, the pong algebra $\mathcal{P}(m,k)$ is a differential graded algebra, with a preferred $\mathbb{Z}$-grading (which drops by one under the boundary operator), and an additional $(\frac{1}{2}\mathbb{Z})^m$-grading, which is preserved by the differential.

The bordered algebra $\mathcal{C}(m,k)$ also has a *weight grading* with values in $(\frac{1}{2}\mathbb{Z})^m$, whose ith component $\mathfrak{w}_i$ is specified by

$$\mathfrak{w}_i(L_j) = \mathfrak{w}_i(R_j) = \begin{cases} \frac{1}{2} & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathfrak{w}_i(U_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

We extend this to a grading on $\mathcal{C}(m,k)[t]$ with the convention that $\mathfrak{w}_i(t) = 1$ for $i = 1, \dots, m$. We also introduce a further integer-valued grading gr on $\mathcal{C}(m,k)[t]$ so that

$$\operatorname{gr}(\mathcal{C}(m,k)) = 0,$$
$$\operatorname{gr}(t) = 2m - 2k - 2.$$

The main result here identifies the homology of the pong algebra with the bordered algebras and computes the induced A_∞ structure on the homology as follows.

Theorem 1.1. *When $m > 1$ and $0 < k < m-1$, the homology of $\mathcal{P}(m, m-k-1)$ is isomorphic, as an algebra, to $\mathcal{C}(m,k)[t]$, equipped with the above gradings. The homology of $\mathcal{P}(m, m-k-1)$, with its induced A_∞ structure, is uniquely characterized up to isomorphism by the following properties:*

- *the underlying algebra is $\mathcal{C}(m,k)[t]$,*
- *the actions respect the above gradings, in the sense that*

$$\mathfrak{w}_i(\mu_d(a_1, \dots, a_d)) = \sum_{j=1}^{d} \mathfrak{w}_i(a_j),$$

$$\operatorname{gr}(\mu_d(a_1, \dots, a_d)) = d - 2 + \sum_{j=1}^{d} \operatorname{gr}(a_j),$$

- *the μ_d are $\mathbb{F}[t]$-multilinear; i.e.,*

$$\mu_d(a_1, \dots, a_{j-1}, t \cdot a_j, a_{j+1}, \dots, a_d) = t \cdot \mu_d(a_1, \dots, a_d),$$

- *there is a non-zero operation μ_{2m-2k}.*

See Section 12.1 for a few remarks on the extreme cases when $k = 0$ or $m - 1$.

Our proof of Theorem 1.1 involves a blend of algebraic and holomorphic techniques. In particular, the proof relies on the construction of modules associated to Heegaard diagrams whose boundary is marked in a particular manner, the *pong-bordered boundary*.

In this article, we first describe the pong algebra. Next, we compute its homology as a $\mathcal{I}(m,k)$-bimodule in Section 6. The unique characterization of the $\mathcal{A}_\infty$ structure involves a computation in Hochschild cohomology, which we do in Section 11.

The Hochschild cohomology computation is facilitated by a certain Koszul duality result. (Compare also [12].)

Let $\mathcal{Q}(m,k)$ denote the quotient

$$\mathcal{Q}(m,k) = \frac{\mathcal{P}(m,k)}{v_1 = \cdots = v_m = 0}. \tag{1.1}$$

Theorem 1.2. *Given $0 < k < m$, there is a quasi-isomorphism*

$$\mathcal{Q}(m,k) \simeq \operatorname{Cobar}(\mathcal{C}(m,k)).$$

The above Koszul duality is proved as follows. First, we construct a type DD bimodule ${}^{\mathcal{C}(m,k)}\mathcal{X}^{\mathcal{Q}(m,k)}$. Next, we construct a bordered diagram whose boundary decomposes as a union of a pong-bordered piece, and another piece which is bordered as in [20]. Adapting methods of bordered Floer homology [10], we associate to this diagram a type AA bimodule, which we prove is the inverse to ${}^{\mathcal{C}(m,k)}\mathcal{X}^{\mathcal{Q}(m,k)}$, to prove Theorem 1.2. (The content of Theorem 1.2 is restated in Theorem 10.1, with more attention to gradings.)

Theorem 1.1 is proved using Hochschild cohomology. The computations in Hochschild cohomology are facilitated by the relationship between the cobar algebra and the much smaller $\mathcal{Q}(m,k)$ from Theorem 1.2, much in the spirit of [12].

This paper is organized as follows. In Section 2, we introduce much of the algebraic background and notation used throughout. In Section 3, we recall the bordered algebra $\mathcal{C}(m,k)$ from [20]. In Section 4, we define the pong algebra itself. In Section 5, we introduce some particular generators of the pong algebra which will be used throughout and verify some of their properties. In Section 6, we compute the homology of the pong algebra, obtaining a weak version of Theorem 1.1, on the level of vector spaces (rather than $\mathcal{A}_\infty$-algebras). Sections 7–10 are aimed at establishing Theorem 1.2. In Section 7, we define (algebraically) a type DD bimodule which connects $\mathcal{C}(m,k)$ and $\mathcal{Q}(m,k)$. In Section 8, we define its inverse bimodule, by adapting holomorphic techniques in the spirit of [10]; see also [20]. In Section 9, we verify these bimodules are quasi-inverses to one another. In Section 10, we review Koszul duality (in the spirit of [8]) to deduce Theorem 1.2 from the fact that the DD bimodule constructed before is quasi-invertible. In Section 11, we use deformation theory to prove that $\mathcal{C}(m,k)[t]$ can be given a unique non-trivial $\mathcal{A}_\infty$ structure. In Section 12, we show that the induced $\mathcal{A}_\infty$ structure on $H(\mathcal{P}(m, m-k-1)) \cong \mathcal{C}(m,k)[t]$ is non-trivial, completing the proof of Theorem 1.1.

Motivation for the constructions. We pause to explain some of the geometric content of these algebras. As explained in [20], the bordered algebras $\mathcal{C}(m,k)$ can be thought of as generated by k-tuples of Reeb chords, with products in the algebra recording the collisions that occur in certain ends of moduli spaces. In Section 8, we also interpret the pong algebra (and especially its $v_1 = \cdots = v_m = 0$ specialization) as an algebra of Reeb chords. (Compare the "strands algebras" of [10]; see also [3,4,22].)

Both algebras can be thought of as part of a more general construction which simultaneously generalizes both the pointed matched circles of [10] and the configuration of boundary circles from [20]. The Koszul duality result from Theorem 1.2 can be thought of as an analogue of the duality result from [8]. That paper sets up a Koszul duality between the algebra associated to a pointed matched circle (i.e., a handle decomposition of a surface with a single 0-handle and 2-handle) and its dual pointed matched circle. One could envision a more general picture, where one associates an algebra to a (more general) handle decomposition of a surface, which is Koszul dual to the algebra associated to its dual handle decomposition; compare also [6, 13, 14, 24].

In this interpretation, $\mathcal{C}(m,k)$ corresponds to the handle decomposition of the sphere with m zero-cells, $m-1$ one-cells, and one 2-cell. The zero-cells are arranged in a linear change, connected by the $m-1$ one-cells. The parameter k specifies the multiplicity of the symmetric product we are considering. The algebra $Q(m,\ell)$ corresponds to the dual handle decomposition, with one 0-cell and $m-1$ one-cells arranged in a bouquet, and m 2-cells. See Figure 1. The Koszul duality from Theorem 1.2 can be seen as a consequence of the duality between the two handle decompositions. This Koszul duality is further explored in [21].

Following Auroux [2], one can think of the generators of pong algebra as morphisms in a wrapped Fukaya category, the (specialized) $v_1 = \cdots = v_m = 0$ product as product on the partially wrapped Fukaya category of [1], and the unspecialized algebra actions as products in a relative Fukaya category. This latter identification is explored further in [18].

On an algebraic level, the algebras we consider here are closely related to the algebras associated to the extended affine symmetric groups of Manion and Rouquier [14]; see also Remark 4.4 below.

Constructing a signed lift of the pong algebra (to $\mathbb{Z}$ coefficients) is a straightforward task in the spirit of [19, Chapter 15]. Lifting the holomorphic aspects would require more work.

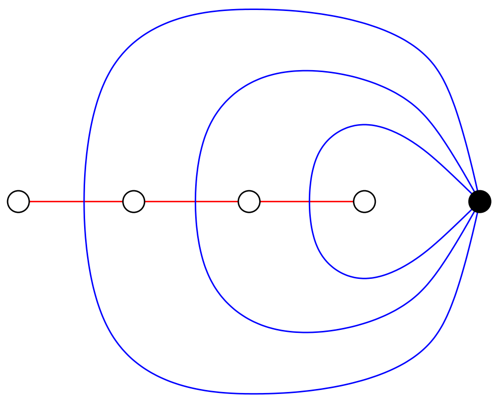


Figure 1. *Dual cell decomposition of* S^2. The linear arrangement corresponds to $\mathcal{C}$ and its dual wedge of circles corresponds to the pong algebra.

2. Algebraic background

We will use the algebraic underpinnings from [9].

2.1. Type D structures

If A is a DG algebra over a ring I, a *type D structure* is a left I-module X, equipped with a map

$$\delta^1\colon X \to A \otimes X,$$

satisfying the structural relation

$$(\mu_1 \otimes \mathrm{Id}_X) \circ \delta^1 + (\mu_2 \otimes \mathrm{Id}_X) \circ (\mathrm{Id}_A \otimes \delta^1) \circ \delta^1 = 0. \tag{2.1}$$

If A and B are two DG algebras over I, a *type DD bimodule* ${}^A X^B$ is an I-bimodule X, together with a structure map

$$\delta^1\colon X \to A \otimes_I X \otimes_I B,$$

which in turn can be seen as a map

$$\delta^1\colon X \to (A \otimes B^{\mathrm{op}})_{I \otimes I^{\mathrm{op}}} \otimes X,$$

where here op denotes the "opposite" module.

For A and B, we will also consider type DA bimodules ${}_A M^B$ and type AA bimodules ${}_A M_B$, as in [9, Section 2.24].

2.2. Grading conventions

Gradings for our present purposes are simpler than the gradings considered in [9, 10].

The DA algebras we consider here will be equipped with augmentations. Specifically, let $\mathfrak{k}$ be a ring (in our case, this is the ring of idempotents). A $\mathfrak{k}$-algebra is a graded, associated algebra, which is also a $\mathfrak{k}$-bimodule, so that the multiplication map $\mu_2\colon A \otimes A \to A$ is a map of $\mathfrak{k}$-bimodules.

Definition 2.1. An *augmented* $\mathfrak{k}$-algebra is an algebra, equipped with an algebra map $\varepsilon\colon A \to \mathfrak{k}$, called the *augmentation*, whose kernel, the *augmentation ideal*, is denoted A_+. Together, the unit (in $\mathfrak{k}$) and the augmentation provide a $\mathfrak{k}$-$\mathfrak{k}$-bimodule splitting

$$A \cong A_+ \oplus \mathfrak{k}.$$

The algebras in this paper will all have the following special structure.

Definition 2.2. A *Maslov–Alexander bigraded* algebra is an augmented DA algebra A, equipped with two gradings, one (homological) $\mathbb{Z}$-grading, denoted gr^A, and an Alexander grading, denoted $\vec{\mathfrak{w}}$, with values in

$$\frac{1}{2}(\mathbb{Z}^{\geq 0})^m \subset \frac{1}{2}\mathbb{Z}^m.$$

Both gradings are additive under multiplication, and differential drops the homological grading by one while preserving the algebra grading. The summand of A with vanishing Alexander grading is the ground ring, and the augmentation ideal A_+ is the portion in non-zero Alexander grading. Finally, A will have the following boundedness hypothesis: for each $\vec{w} \in \frac{1}{2}\mathbb{Z}^m$, the portion of A with Alexander grading $\vec{w}$ is a finitely generated $\mathfrak{k}$-bimodule.

A Maslov–Alexander bigraded type D structure X is one which is equipped with two gradings, one, denoted gr^X, with values in $\mathbb{Z}$ and another, denoted $\vec{v}$, with values by $\frac{1}{2}\mathbb{Z}^m$. These gradings are further required to be compatible with the structure map δ^1 in the sense that if $a \otimes y$ appears with non-zero coefficient in $\delta^1(x)$, then

$$\mathrm{gr}^X(x) - 1 = \mathrm{gr}^A(a) + \mathrm{gr}^X(y),$$
$$\vec{v}(x) = \vec{\mathfrak{w}}(a) + \vec{v}(y).$$

More generally, let ${}_AM^B$ be a type DA bimodule. Recall that DA bimodules are equipped with actions for all $p \geq 0$,

$$\delta^1_{p+1}\colon A^{\otimes(n-1)} \otimes M \to B \otimes M$$

satisfying a structure relation generalizing equation (2.1). (See [9, Definition 2.2.43].) We say that it is *Maslov–Alexander bigraded* if it is equipped with gr^M and $\vec{v}$ so that if $y \otimes b$ appears in $\delta^1_{1+p}(a_1, \ldots, a_p, x)$, then

$$\begin{aligned} \mathrm{gr}^X(x) + \sum_{i=1}^{p} \mathrm{gr}^A(a_i) &= \mathrm{gr}^X(y) + \mathrm{gr}^B(b) + 1 - p, \\ \vec{v}(x) + \sum_{i=1}^{p} \vec{\mathfrak{w}}^A(a_i) &= \vec{v}(y) + \vec{\mathfrak{w}}^B(b). \end{aligned}$$

Furthermore, we say ${}_A M^B$ is *(strictly) unital* if $\delta^1_2(\mathbf{1}, x) = x \otimes \mathbf{1}$ and for all $p > 1$, if $(a_1, \ldots, a_p)$ is a sequence of algebra elements such that some $a_i = \mathbf{1}$ then $\delta^1_{p+1}(a_1, \ldots, a_p, x) = 0$.

Similarly, if ${}_A M_B$ is a type AA bimodule, we say that it is *Maslov–Alexander bigraded* if for $y = m_{p|1|q}(a_1 \otimes \cdots \otimes a_p \otimes x \otimes b_1 \otimes \cdots \otimes b_q)$, then

$$\mathrm{gr}(y)^M = \mathrm{gr}^M(x) + p + q - 1 + \sum_{i=1}^{p} \mathrm{gr}^A(a_i) + \sum_{i=1}^{q} \mathrm{gr}^B(b_i), \tag{2.2}$$

$$\vec{v}(y) = \Big(\sum_{j=1}^{p} \vec{\mathfrak{w}}^A(a_j)\Big) + \vec{v}(x) + \Big(\sum_{j=1}^{q} \vec{\mathfrak{w}}^B(b_j)\Big). \tag{2.3}$$

The strict unitality condition in this case is that for all $x \in M$ and all algebra sequences $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ of algebra elements of length $p, q \geq 0$ with $p + q > 1$, with $\mathbf{1} \in \{a_1, \ldots, a_p, b_1, \ldots, b_q\}$, we have that

$$\begin{aligned} m_{1|1|0}(\mathbf{1}, x) &= x, \\ m_{0|1|1}(x, \mathbf{1}) &= x, \\ m_{p,1,q}(a_1, \ldots, a_p, x, b_1, \ldots, b_q) &= 0. \end{aligned}$$

2.3. Quasi-inverses

Definition 2.3. Fix dg algebras A and B, and let ${}^A X^B$ and ${}_B Y_A$ be a bimodules of type DD and AA, respectively, as discussed above. We say that X and Y are *quasi-inverses* if

$$\begin{aligned} {}^A X^B \otimes {}_B Y_A &\simeq {}^A \mathrm{Id}_A, \\ {}_B Y_A \otimes {}^A X^B &\simeq {}_B \mathrm{Id}^B. \end{aligned}$$

Let A and B be DG algebras over $\mathfrak{k}$. Let $\phi \colon A \to B$ be an $\mathcal{A}_\infty$ homomorphism. There is an associated type DA bimodule ${}^B[\phi]_A$, which is rank one as as a $\mathfrak{k}$ bimodule, with actions specified by ϕ.

The bimodules that arise in this way have a simple characterization.

Lemma 2.4 ([9, Lemma 2.2.50]). *Let A and B be a DG algebras over $\mathfrak{k}$. Suppose that ${}_A[M]^B$ is a type DA bimodule with the following properties:*

- $\delta^1_1 = 0$,
- *M has rank 1 as a $\mathfrak{k}$ bimodule.*

Then, there is an $\mathcal{A}_\infty$ homomorphism $\phi\colon A \to B$ so that ${}_A[M]^B = {}_A[\phi]^B$.

When A and B are Maslov–Alexander bigraded, an $\mathcal{A}_\infty$ homomorphism respects this structure if

$$\mathfrak{w}^B(\phi_p(a_1 \otimes \cdots \otimes a_p)) = \sum_{i=1}^{p} \mathfrak{w}^A(a_i),$$
$$\mathrm{gr}^B(\phi_p(a_1 \otimes \cdots \otimes a_p)) = \sum_{i=1}^{p} \mathrm{gr}^A(a_i) + p - 1.$$

Note that ϕ is a Maslov–Alexander graded $\mathcal{A}_\infty$ homomorphism if its associated bimodule ${}_A[\phi]^B$ is Maslov–Alexander bigraded, supported in vanishing bigrading.

3. The bordered algebras

We recall the bordered algebras of [16]; in fact, we use mostly notation from [20, Section 3.2]. The algebras we are interested in are denoted $\mathcal{C}(m,k)$, where (m,k) is a pair of integers with $0 \le k \le m-1$. These algebras are constructed by first constructing an $\mathbb{F}[U_1,\dots,U_m]$-algebra $\mathcal{B}_0(m,k)$, forming the quotient $\mathcal{B}(m,k)$ by an ideal, and then considering a subalgebra $\mathcal{C}(m,k)$. We recall the details presently, referring the reader to [16] for more details. (Note also that in [20], we considered the case where $m = 2k$, since these were the algebras that arose naturally in the case of "upper Heegaard diagrams"; but this hypothesis is not needed for the algebraic constructions.)

3.1. The algebras $\mathcal{B}_0$ and $\mathcal{B}$

First, we recall the construction of $\mathcal{B}_0(m,k)$, associated to a pair of integers (m,k) with $0 \le k \le m-1$. (The construction of $\mathcal{B}_0(m,k)$ also works in the cases where $k = m$ or $k = m+1$; but in those cases, the subalgebra $\mathcal{C}(m,k)$ we are interested in is zero.)

The base ring of $\mathcal{B}_0(m,k)$ is the polynomial algebra $\mathbb{F}[U_1,\dots,U_m]$. Idempotents correspond to k-element subsets $\mathbf{x}$ of $\{0,\dots,m\}$ called *idempotent states*.

Given idempotents states $\mathbf{x},\mathbf{y}$, the $\mathbb{F}[U_1,\dots,U_m]$-module $\mathbf{I}_{\mathbf{x}} \cdot \mathcal{B}_0(m,k) \cdot \mathbf{I}_{\mathbf{y}}$ is identified with $\mathbb{F}[U_1,\dots,U_m]$, given with a preferred generator $\gamma_{\mathbf{x},\mathbf{y}}$. To specify the product, we proceed as follows. Each idempotent state $\mathbf{x}$ has a *weight vector* $v^{\mathbf{x}} \in \mathbb{Z}^m$, with components $i = 1, \dots, m$ given by $v_i^{\mathbf{x}} = \#\{x \in \mathbf{x} \mid x \geq i\}$. The multiplication is specified by

$$\gamma_{\mathbf{x},\mathbf{y}} \cdot \gamma_{\mathbf{y},\mathbf{z}} = U_1^{n_1} \cdots U_m^{n_m} \cdot \gamma_{\mathbf{x},\mathbf{z}},$$

where

$$n_i = \frac{1}{2}(|v_i^{\mathbf{x}} - v_i^{\mathbf{y}}| + |v_i^{\mathbf{y}} - v_i^{\mathbf{z}}| - |v_i^{\mathbf{x}} - v_i^{\mathbf{z}}|).$$

Definition 3.1. The algebra $\mathcal{B}_0(m,k)$ is equipped with the distinguished elements $\{L_i\}_{i=1}^m$, $\{R_i\}_{i=1}^m$, and $\{U_i\}_{i=1}^m$, defined as follows. For $i = 1, \dots, m$, let L_i be the sum of $\gamma_{\mathbf{x},\mathbf{y}}$, taken over all pairs of idempotent states $\mathbf{x}, \mathbf{y}$ so that there is some integer s with $x_s = i$ and $y_s = i - 1$ and $x_t = y_t$ for all $t \neq s$. Similarly, let R_i denote the sum of all the $\gamma_{\mathbf{y},\mathbf{x}}$ taken over the same pairs of idempotent states as above. We will often think of U_i as an element of $\mathcal{B}_0(m,k)$; explicitly, it is the sum over all idempotents $\mathbf{x}$ of the element $\gamma_{\mathbf{x},\mathbf{x}} \cdot U_i$. With this definition, multiplication by $U_i \in \mathcal{B}_0(m,k)$ corresponds to the action of $U_i \in \mathbb{F}[U_1, \dots, U_m]$ on $\mathcal{B}_0(m,k)$.

Under the identification

$$\mathbf{I}_{\mathbf{x}} \cdot \mathcal{B}_0(m,k) \cdot \mathbf{I}_{\mathbf{y}} \cong \mathbb{F}[U_1, \dots, U_m],$$

the elements that correspond to monomials in the $U_1, \dots, U_m$ are called *pure algebra elements in* $\mathcal{B}_0(m,k)$. These elements are specified by their idempotents $\mathbf{x}$ and $\mathbf{y}$, and their *relative weight vector* $w(b) \in \mathbb{Q}^m$, which in turn is uniquely characterized by

$$\mathfrak{w}_i(\gamma_{\mathbf{x},\mathbf{y}}) = \frac{1}{2}|v_i^{\mathbf{x}} - v_i^{\mathbf{y}}|, \quad \mathfrak{w}_i(U_j \cdot b) = \mathfrak{w}_i(b) + \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

Let $\mathcal{J} \subset \mathcal{B}_0(m,k)$ be the two-sided ideal generated by $L_{i+1} \cdot L_i$, $R_i \cdot R_{i+1}$ and, for all choices of $\mathbf{x} = \{x_1, \dots, x_k\}$ with $\mathbf{x} \cap \{j-1, j\} = \emptyset$, the element $\mathbf{I}_{\mathbf{x}} \cdot U_j$. Then,

$$\mathcal{B}(m,k) = \mathcal{B}_0(m,k)/\mathcal{J}.$$

The pure algebra elements in $\mathcal{B}(m,k)$ are those elements that are images of pure algebra elements in $\mathcal{B}_0(m,k)$ under the above quotient map. We use the symbols L_i, R_i, and U_i to denote the elements of $\mathcal{B}(m,k)$ which are the images of the corresponding elements of $\mathcal{B}_0(m,k)$ as defined in Definition 3.1.

Definition 3.2. Idempotents $\mathbf{x} = x_1 < \cdots < x_k$ and $\mathbf{y} = y_1 < \cdots < y_k$ are said to be *too far* if for some $t \in \{1, \dots, k\}$, $|x_t - y_t| \geq 2$. If $\mathbf{x}$ and $\mathbf{y}$ are too far, then $\mathbf{I}_{\mathbf{x}} \cdot \mathcal{B}_0(m,k) \cdot \mathbf{I}_{\mathbf{y}} \subset \mathcal{J}$; i.e., $\mathbf{I}_{\mathbf{x}} \cdot \mathcal{B}(m,k) \cdot \mathbf{I}_{\mathbf{y}} = 0$.

We restate here the concrete description of the ideal $\mathcal{J}$ given in [16].

Proposition 3.3 ([16, Proposition 3.7]). *Suppose that $b = \mathbf{I_x} \cdot b \cdot \mathbf{I_y}$ is a pure algebra element. The element b is in $\mathcal{J}$ if and only if $\mathbf{x}$ and $\mathbf{y}$ are too far, or there is a pair of integers $i < j$ so that*

- $i, j \in \{0, \dots, m\} \setminus \mathbf{x} \cap \mathbf{y}$,
- *for all* $i < t < j$, $t \in \mathbf{x} \cap \mathbf{y}$,
- $\mathfrak{w}_t(b) \geq 1$ *for all* $t = i + 1, \dots, j$,
- $\#(x \in \mathbf{x} \mid x \leq i) = \#(y \in \mathbf{y} \mid y \leq i)$.

3.2. The algebra $\mathcal{C}$

Having defined $\mathcal{B}(m, k)$, we turn now to the construction of $\mathcal{C}(m, k)$, which is of primary interest here. To this end, let

$$\mathbf{J} = \sum_{\{\mathbf{x} \mid \mathbf{x} \cap \{0, m\} = \emptyset\}} \mathbf{I_x}.$$

The subalgebra $\mathcal{C}(m, k) \subset \mathcal{B}(m, k)$ is defined by

$$\mathcal{C}(m, k) = \mathbf{J} \cdot \mathcal{B}(m, k) \cdot \mathbf{J}.$$

There is a projection map $\pi \colon \mathcal{B}(m, k) \to \mathcal{C}(m, k)$ defined by

$$\pi(b) = \mathbf{J} \cdot b \cdot \mathbf{J}.$$

Equivalently, $\mathcal{C}(m, k)$ corresponds to the subalgebra whose idempotent subalgebra correspond to those idempotent states $\mathbf{x} \subset \{1, \dots, m-1\}$.

In Definition 3.1, we defined distinguished algebra elements $\{L_i\}_{i=1}^m$, $\{R_i\}_{i=1}^m$, and $\{U_i\}_{i=1}^m$ in $\mathcal{B}_0(m, k)$, which project to elements of $\mathcal{B}(m, k)$ (with the same notation). Note that $\pi(L_1) = \pi(R_1) = \pi(R_m) = \pi(L_m) = 0$. Let $\{L_i\}_{i=2}^{m-1}$, $\{R_i\}_{i=2}^{m-1}$, and $\{U_i\}_{i=1}^m$ be the elements of $\mathcal{C}(m, k)$ obtained by projecting (under π) the corresponding elements of $\mathcal{B}(m, k)$.

Given $i \in \{1, \dots, m-1\}$, let

$$[i] = \sum_{\{\mathbf{x} \mid i \in \mathbf{x}\}} \mathbf{I_x} \quad \text{and} \quad [\bar{i}] = \sum_{\{\mathbf{x} \mid i \notin \mathbf{x}\}} \mathbf{I_x}.$$

The following relations in $\mathcal{C}(m, k)$ hold for all $2 \leq i \leq m - 1$:

$$\begin{aligned} &([i] \cdot [\overline{i-1}]) \cdot L_i \cdot [i-1] \cdot [\bar{i}] = L_i, && ([i-1] \cdot [\bar{i}]) \cdot R_i \cdot [\overline{i-1}] \cdot [i] = R_i, \\ &L_i \cdot R_i = [\overline{i-1}] \cdot [i] \cdot U_i, && R_i \cdot L_i = [i-1] \cdot [\bar{i}] \cdot U_i. \end{aligned}$$

Moreover, for all $2 \leq i \leq m - 2$,

$$R_i \cdot R_{i+1} = 0, \quad L_{i+1} \cdot L_i = 0.$$

(The first four relations hold even in $\mathcal{B}_0(m, k)$; whereas the last two hold in $\mathcal{B}(m, k)$.)

3.3. Gradings

The weight vector $\mathfrak{w}$ induces a grading on $\mathcal{C}$ with values in $\frac{1}{2}\mathbb{Z}^m$. It will sometimes be helpful to view $\mathcal{C}$ as a Maslov–Alexander bigraded DG algebra, with trivial differential. When doing so, we will view the additional $\mathbb{Z}$-grading as also trivial; i.e., the algebra is supported entirely in $\mathrm{gr} = 0$.

4. Definition of the pong algebra

The aim of this section is to introduce the *pong algebra* with *m walls* and k strands, denoted $\mathcal{P}(m,k)$. Here, (m,k) is a pair of integers with $0 \leq k < m$ and $m > 1$. The base ring of the pong algebra is a polynomial ring $\mathbb{F}[v_1, \dots, v_m]$, and the algebra comes equipped with a preferred generating set, given by the so-called *lifted partial permutations*, which we define presently.

4.1. Lifted partial permutations

Let $r_t\colon \mathbb{R} \to \mathbb{R}$ be the reflection $r_t(x) = 2t - x$; and consider the subgroup G_m of the reflection group of the real line generated by $r_{\frac{1}{2}}$ and $r_{m-\frac{1}{2}}$. The quotient of the integral lattice by this group of rigid motions is naturally an $m-1$ point set; generated by $\{1, \dots, m-1\}$. Let

$$Q\colon \mathbb{Z} \to \{1, \dots, m-1\}$$

denote this quotient map. Explicitly, $Q(i)$ is the integer $1 \leq t \leq m-1$ so that $t \equiv i \mod 2m-2$ or $t \equiv 1-i \pmod{2m-2}$.

Note that G_m also acts on the set $\frac{1}{2} + \mathbb{Z}$. The quotient of $\frac{1}{2} + \mathbb{Z}$ by G_m is naturally the m-point set, $\{\frac{1}{2}, \dots, m - \frac{1}{2}\}$. We think of these points as being in one-to-one correspondence with the underlying variables in the pong algebra, where the point $j \in \{\frac{1}{2}, \dots, m - \frac{1}{2}\}$ corresponds to the variable $u_{\frac{1}{2}+j}$.

A G_m invariant subset $\widetilde{S}$ of $\mathbb{Z}$ has a natural quotient $\widetilde{S}/G_m$, which is a subset of $\{1, \dots, m-1\}$.

Definition 4.1. A *lifted partial permutation on k letters* is a pair $(\widetilde{S}, \tilde{f})$, where

- $\widetilde{S} \subset \mathbb{Z}$ is a G_m-invariant subset;
- $\tilde{f}\colon \widetilde{S} \to \mathbb{Z}$ is a G_m-equivariant map;

subject to the following two conditions:

- $\widetilde{S}/G_m$ consists of k elements,
- the induced map $\tilde{f}\colon \widetilde{S}/G_m \to \mathbb{Z}/G_m$ is injective.

Let $\mathfrak{P}$ denote the set of lifted partial permutations.

The following data specify a lifted partial permutation.

Definition 4.2. Given integers (k, m), we say that *pong data of type* (k, m) consists of a pair (S, f), where

$$S \subseteq \{1, \ldots, m-1\} = \mathbb{Z}/G_m$$

is a k element subset, and $f\colon S \to \mathbb{Z}$ is a map, so that the induced map $Q \circ f\colon S \to \mathbb{Z}/G_m$ is injective.

Lemma 4.3. *Given a lifted partial permutation $(\widetilde{S}, \tilde{f})$, we can associate pong data (S, f), where $S = \widetilde{S}/G_m$ and f is the restriction of $\tilde{f}$ to $\{1, \ldots, m-1\} \subset \mathbb{Z}$. This restriction map induces a one-to-one correspondence between lifted partial permutations (on k letters) and pong data (of type (k, m)).*

Proof. Restriction gives a map from lifted partial permutations to pong data. Conversely, given pong data, let $\widetilde{S} = Q^{-1}(S)$; so that any $\tilde{s} \in \widetilde{S}$ can be written as $\tilde{s} = \gamma \cdot s$, for a unique choice of $s \in \{1, \ldots, m-1\}$ and a unique $\gamma \in G_m$. Define the corresponding lifted partial permutation $\tilde{f}$ by

$$\tilde{f}(\tilde{s}) = \gamma \cdot f(s). \qquad \blacksquare$$

If a lifted partial permutation $(\widetilde{S}, \tilde{f})$ restricts to pong data (S, f), we call $(\widetilde{S}, \tilde{f})$ the *equivariant lift* of (S, f).

Write $S = s_1 < \cdots < s_k$. The partial permutation will be labeled by the k-tuple of ordered pairs $(s_1, f(s_1)), \ldots, (s_k, f(s_k))$.

Pong data (S, f) can be represented graphically as follows. Consider the infinite strip $\mathbb{R} \times [-1, 0]$; where $[-1, 0]$ is an interval which we think of as measuring the height. Draw a collection of straight segments connecting $(\tilde{s}, 0)$ for $\tilde{s} \in \widetilde{S}$ to $(f(\tilde{s}), -1)$. If we further draw all the translates of these segments by the action of G_m, we obtain the *lifted pong diagram*.

All of the information in the partial permutation is contained in the portion of this picture in the finite rectangle $[\frac{1}{2}, m - \frac{1}{2}] \times [-1, 0]$. For each $s \in S$, the segment from s to $f(s)/G_m \in \{1, \ldots, m-1\}$, called a *strand*, is transformed into a continuous, piecewise linear curve connecting $(s, 0)$ to $(Q(f(s)), -1)$ whose slope has fixed absolute value. The curves fail to be linear at the points where the horizontal coordinate is $\frac{1}{2}$ or $m - \frac{1}{2}$. Points of the first kind are called *left extrema* and those of the second are called *right extrema*. This picture is called the *pong diagram* for the lifted partial permutation.

Remark 4.4. The pong diagrams defined here can be viewed as a kind of $\mathbb{Z}/2\mathbb{Z}$ quotient of the cylindrical diagrams considered in [14].

Example 4.5. Let $m = 4$, and consider the lifted partial permutation f on 2 letters sending 1 to -2 and 2 to 1, which we write $((1, -2), (2, 1))$. The pong diagram is pictured in Figure 2, and its lift is shown in Figure 3.

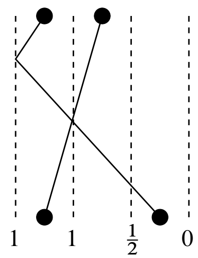


Figure 2. *Pong diagrams.* This is a pong diagram for the lifted partial permutation $((1, -2), (2, 1))$ (where the vertical coordinate has been scaled up). The dotted lines occur at horizontal positions $1/2$, $3/2$, $5/2$, and $7/2$, respectively. There is one left extremum in the picture, occurring on the strand out of 1; and there are no right extrema. The local multiplicities at these positions are recorded in the figure; i.e., in the notation of Definition 4.6, we have $\mathfrak{w}(f) = (1, 1, \frac{1}{2}, 0)$.

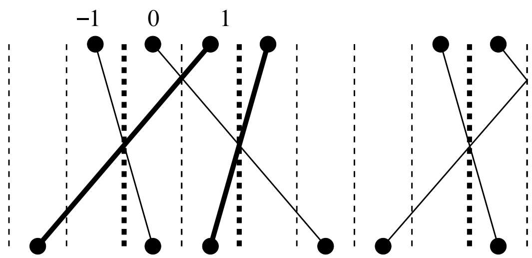


Figure 3. *Lifting the pong diagrams.* We have drawn here a lift of the pong diagram from Figure 2; drawing three copies of the fundamental domain. The darkened dashed lines represent the three translates of $3/2$, and the two darkened lines represent the two strands. The two intersection points of these lines with the darkened dashed lines demonstrate that $\mathfrak{w}_{3/2}(S, f) = 1$.

Definition 4.6. Let (S, f) be pong data, and let $\tilde{f}\colon \widetilde{S} \to \mathbb{Z}$ be its equivariant lift. For each $j \in \frac{1}{2} + \mathbb{Z}$, the *local multiplicity of the lift of f at j*, denoted $\mathfrak{w}_j(f)$, is given by

$$\mathfrak{w}_j(f) = \frac{1}{2}\#\bigl\{i \in \widetilde{S} \mid i < j < \tilde{f}(i) \text{ or } \tilde{f}(i) < j < i\bigr\}.$$

Note that for all $\gamma \in G_m$, $\mathfrak{w}_j(f) = \mathfrak{w}_{\gamma \cdot j}(f)$, so the local multiplicity descends to a function of $j \in (\frac{1}{2} + \mathbb{Z})/G_m = \{\frac{1}{2}, \ldots, m - \frac{1}{2}\}$; or equivalently, it gives vector $\mathfrak{w}(f) \in (\frac{1}{2}\mathbb{Z})^m$, which we call the *vector of local multiplicities*.

The local multiplicities of a lifted partial permutation can be read off from the pong diagram, as follows. The local multiplicity at $\frac{1}{2}$ is the number of left extrema in s; the local multiplicity at $m - \frac{1}{2}$ is the number of right extrema; and the local

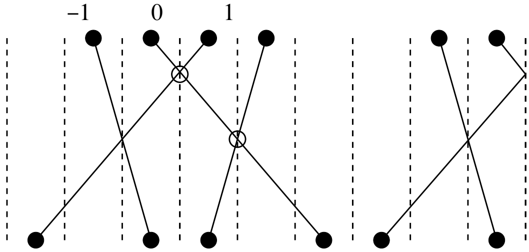


Figure 4. *Crossings.* We have illustrated here part of the lifted pong diagram corresponding to Figure 2, consisting of three fundamental domains. The two circled intersection points correspond to the two crossings in $\mathrm{Cross}(f)$.

multiplicity at j with $\frac{1}{2} < j < m - \frac{1}{2}$ is half the intersection number of the strands with the vertical segment at j.

Consider equivalence classes of pairs $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ under the equivalence relation generated by $(i, j) \sim (j, i)$ and $(i, j) \sim (\gamma \cdot i, \gamma \cdot j)$ for $\gamma \in G_m$. Let $\langle i, j \rangle$ denote the equivalence class of $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ under that equivalence relation.

Definition 4.7. A *lifted crossing* of (S, f) is an element (i, j) of $\widetilde{S} \times \widetilde{S}$ with the property that the sign of $i - j$ is opposite to the sign of $\tilde{f}(i) - \tilde{f}(j)$; i.e., $\mathrm{sgn}(i - j) \neq \mathrm{sgn}(\tilde{f}(i) - \tilde{f}(j))$. Note that any G_m orbit of a lifted crossing is a lifted crossing; also if (i, j) is a lifted crossing, then so is (j, i). A *crossing* is the equivalence class $\langle i, j \rangle$ of the crossing (i, j) under the above equivalence relation. The set of crossings of (S, f) is denoted $\mathrm{Cross}(S, f)$, or simply $\mathrm{Cross}(f)$ when S is clear from the context. See Figure 4 for an illustration.

To find all the crossings on a given strand connecting $(s, 0)$ and $(f(s), -1)$ in a lifted pong diagram, find all the intersection points of the corresponding segment with all of the segments connecting $(\tilde{t}, 0)$ and $(\tilde{f}(\tilde{t}), -1)$, with $\tilde{t} \neq s$. From this description, it is obvious that a lifted partial permutation has only finitely many crossings.

Crossings in the lifted partial permutation are of two kinds: either the two strands i and j are in the same G_m-orbit, or they are not. In the pong diagram, crossings of the first kind correspond to left or right extrema (i.e., whose x-coordinate is $\frac{1}{2}$ or $m - \frac{1}{2}$), while crossings of the second kind can occur either at these extrema, or they can occur in the interior. Indeed, we can read off the crossing number from the pong diagram as follows. The pong diagram contains a one-manifold away from finitely many points. In the interior, t strands can meet, and such a point contributes $\binom{t}{2}$ crossings; at the $x = \frac{1}{2}$ or $m - \frac{1}{2}$ walls, $2t$ line segments meet, and such points contribute t^2 to the crossing count. The validity of the formula can be readily seen by lifting the pong diagram. See Figure 5.

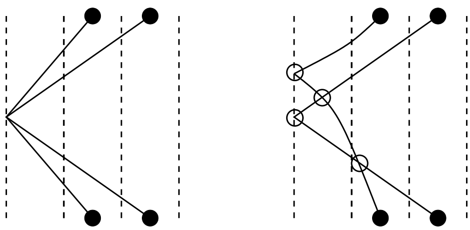


Figure 5. *Crossings at extrema.* The intersection at the wall in the left picture (corresponding to a quadruple points in the lifted pong diagram) is perturbed to give four crossings in the generic pong diagram in the right picture.

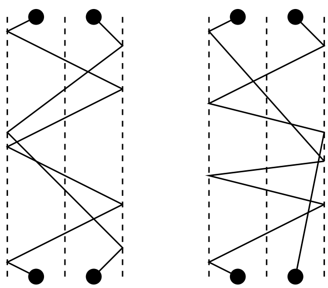


Figure 6. *Generic perturbations of pong diagrams.* The pong diagram on the left, which has 12 crossings, has a perturbation, shown on the right, which has 14 crossings.

Definition 4.8. Define the *crossing number* $\mathrm{cross}(f)$ to be the number of crossings in the lifted partial permutation f; i.e.,

$$\mathrm{cross}(f) = |\mathrm{Cross}(f)|.$$

Alternatively, we can consider a more general kind of lifted pong diagram, where the curves connecting $(\tilde{s}, 0)$ and $(\tilde{f}(\tilde{s}), -1)$ are no longer required to be straight, but we still require that the picture is G_m-equivariant. We can arrange for this picture to be generic, in the sense that there are no triple points. Equivariance gives rise to a picture in the quotient, which we think of as a perturbed pong diagram. The quotient of a generic pong diagram has no triple points; and no two strands intersect at $\{\frac{1}{2}\} \times [-1, 0]$ or $\{m - \frac{1}{2} \times [-1, 0]\}$. The number of crossings plus the number of (left and right) extrema in any sufficiently small generic perturbation of a pong diagram for (f, S) agrees with the $\mathrm{cross}(f, S)$. (Note that larger perturbations can give rise to additional crossings; see Figure 6.)

Pong data (S, f) and (T, g) with $T = Q \circ f(S)$ specify lifted partial permutations, whose composition is represented by $(S, \tilde{g} \circ f)$.

Lemma 4.9. *If (S, f) and (T, g) are two pong data, with $T = Q \circ f(S)$, then*

$$\mathfrak{w}(\tilde{g} \circ f) \leq \mathfrak{w}(g) + \mathfrak{w}(f), \tag{4.1}$$

$$\mathrm{cross}(\tilde{g} \circ f) \leq \mathrm{cross}(g) + \mathrm{cross}(f). \tag{4.2}$$

Moreover,

$$\mathfrak{w}(\tilde{g} \circ f) - \mathfrak{w}(g) - \mathfrak{w}(f) \in \mathbb{Z}^m \subset \left(\frac{1}{2}\mathbb{Z}\right)^m.$$

Proof. Fix $(i, j) \in (\widetilde{S} \times \widetilde{S})/G_m$. It is straightforward to verify the following:

- if $\langle i, j\rangle \notin \mathrm{Cross}(f)$, and $\langle \tilde{f}(i), \tilde{f}(j)\rangle \notin \mathrm{Cross}(g)$, then $\langle i, j\rangle \notin \mathrm{Cross}(\tilde{g} \circ f)$,
- if $\langle i, j\rangle \in \mathrm{Cross}(f)$, and $\langle \tilde{f}(i), \tilde{f}(j)\rangle \notin \mathrm{Cross}(g)$, then $\langle i, j\rangle \in \mathrm{Cross}(\tilde{g} \circ f)$,
- if $\langle i, j\rangle \notin \mathrm{Cross}(f)$, and $\langle \tilde{f}(i), \tilde{f}(j)\rangle \in \mathrm{Cross}(g)$, then $\langle i, j\rangle \in \mathrm{Cross}(\tilde{g} \circ f)$,
- if $\langle i, j\rangle \in \mathrm{Cross}(f)$, and $\langle \tilde{f}(i), \tilde{f}(j)\rangle \in \mathrm{Cross}(g)$, then $\langle i, j\rangle \notin \mathrm{Cross}(\tilde{g} \circ f)$.

Thus, if $\langle i, j\rangle \in \mathrm{Cross}(\tilde{g} \circ f)$, then exactly one of the following two conditions holds:

- $\langle i, j\rangle \in \mathrm{Cross}(f)$,
- $\langle \tilde{f}(i), \tilde{f}(j)\rangle \in \mathrm{Cross}(g)$.

This dichotomy gives an injection

$$\mathrm{Cross}(\tilde{g} \circ f) \to \mathrm{Cross}(g) \cup \mathrm{Cross}(f),$$

establishing inequality (4.2).

To see inequality (4.1), we argue similarly. For any $t \in \frac{1}{2} + \mathbb{Z}$, let

$$X_t(f) = \left\{ i \in \widetilde{S} \mid i < t < \tilde{f}(i) \text{ or } \tilde{f}(i) < t < i \right\} \tag{4.3}$$

so that

$$\mathfrak{w}_t(f) = \frac{1}{2} \# X_t(f).$$

It is straightforward to verify that the $i \in \mathbb{Z}$ can be partitioned into the following four types:

(T-1) if $i \in X_t(f)$ and $f(i) \in X_t(g)$, then $i \notin X_t(\tilde{g} \circ f)$,

(T-2) if $i \in X_t(f)$ and $f(i) \notin X_t(g)$, then $i \in X_t(\tilde{g} \circ f)$,

(T-3) if $i \notin X_t(f)$ and $f(i) \in X_t(g)$, then $i \in X_t(\tilde{g} \circ f)$,

(T-4) if $i \notin X_t(f)$ and $f(i) \notin X_t(g)$, then $i \notin X_t(\tilde{g} \circ f)$.

Inequality (4.1) follows. Moreover, each $i \in \mathbb{Z}$ of type (T-1) contributes 1 to $\mathfrak{w}_t(f) + \mathfrak{w}_t(g) - \mathfrak{w}_t(\tilde{g} \circ f)$, and all other $i \in \mathbb{Z}$ contribute 0. ∎

Definition 4.10. Let (S, f) be a pong data, and fix some crossing $\langle i, j\rangle$ for f. The *lifted resolution of f at $\langle i, j\rangle$* is the G_m-equivariant map $\tilde{f}_{\langle i,j\rangle}\colon \widetilde{S} \to \mathbb{Z}$ defined by

$$\tilde{f}_{\langle i,j\rangle}(k) = \begin{cases} \tilde{f}(k) & \text{if } [k] \neq [i], [j], \\ g \cdot \tilde{f}(j) & \text{if } k = g \cdot i, \\ g \cdot \tilde{f}(i) & \text{if } k = g \cdot j. \end{cases}$$

The *resolution of f at $\langle i, j\rangle$*, denoted $f_{\langle i,j\rangle}$, is the induced map $f_{\langle i,j\rangle} = \tilde{f}_{\langle i,j\rangle}/G_m$.

To see that $\tilde{f}_{\langle i,j\rangle}$ is indeed a lifted partial permutation, note that if $\bar{f}\colon S \to \mathbb{Z}/G_m$ denotes the map induced from f, then the map from $S \subset \mathbb{Z}/G_m \to \mathbb{Z}/G_m$ induced by $f_{\langle i,j\rangle}$ is obtained by pre-composing $\bar{f}$ with the automorphism of S that switches $[i]$ and $[j]$; i.e., it is also injective.

Lemma 4.11. *Let (S, f) be a lifted partial permutation, and $\langle i, j\rangle \in \mathrm{Cross}(f)$ be a crossing. Then,*

$$\mathfrak{w}(f_{\langle i,j\rangle}) \leq \mathfrak{w}(f), \tag{4.4}$$

$$\mathrm{cross}(f_{\langle i,j\rangle}) \leq \mathrm{cross}(f) - 1. \tag{4.5}$$

Moreover, $\mathfrak{w}(f) - \mathfrak{w}(f_{\langle i,j\rangle}) \in \mathbb{Z}^m \subset (\frac{1}{2}\mathbb{Z})^m$.

Proof. Clearly, $\langle k, \ell\rangle$ with $\{[k], [\ell]\} \cap \{[i], [j]\} = \emptyset$, $\langle k, \ell\rangle \in \mathrm{Cross}(f)$ if and only if $\langle k, \ell\rangle \in \mathrm{Cross}(f_{\langle i,j\rangle})$. Moreover, it is straightforward to verify the following:

- if $|\{\langle i, k\rangle, \langle j, k\rangle\} \cap \mathrm{Cross}(f)| = 0$, then $|\{\langle i, k\rangle, \langle j, k\rangle\} \cap \mathrm{Cross}(f_{\langle i,j\rangle})| = 0$,
- if $|\{\langle i, k\rangle, \langle j, k\rangle\} \cap \mathrm{Cross}(f)| = 1$, then $|\{\langle i, k\rangle, \langle j, k\rangle\} \cap \mathrm{Cross}(f_{\langle i,j\rangle})| = 1$,
- if $|\{\langle i, k\rangle, \langle j, k\rangle\} \cap \mathrm{Cross}(f)| = 2$, then $|\{\langle i, k\rangle, \langle j, k\rangle\} \cap \mathrm{Cross}(f_{\langle i,j\rangle})| = 0$.

Obviously, $\langle i, j\rangle \in \mathrm{Cross}(f)$ and $\langle i, j\rangle \notin \mathrm{Cross}(f_{\langle i,j\rangle})$. Thus,

$$\mathrm{cross}(f) - \mathrm{cross}(f_{\langle i,j\rangle}) = 1 + 2\#\big\{k \mid \langle i, k\rangle \text{ and } \langle j, k\rangle \in \mathrm{Cross}(f))\big\},$$

and inequality (4.5) follows.

Fix $t \in \frac{1}{2} + \mathbb{Z}$, and let $X_t(f)$ be as in equation (4.3). Given $\langle i, j\rangle \in \mathrm{Cross}(f)$, define a map $T\colon \mathbb{Z} \to \mathbb{Z}$ by

$$T(k) = \begin{cases} k & \text{if } [k] \notin \{[i], [j]\}, \\ g \cdot j & \text{if } k = g \cdot i \text{ for some } g \in G_m, \\ g \cdot i & \text{if } k = g \cdot j \text{ for some } g \in G_m. \end{cases}$$

Let

$$A = \big\{k \in X_t(f) \mid T(k) \in X_t(f), k < t < T(k) \text{ or } T(k) < t < k\big\}$$

and $B = X_t(f) \setminus A$. It is elementary to see that $B = X_t(f_{\langle i,j\rangle})$. To see this, note that $k \in B$ if t lies between k and $f(k)$, and one of the following three holds:

- $[k] \notin \{[i],[j]\}$, in which case $f(k) = f_{\langle i,j\rangle}(k)$,
- $k = g \cdot i$ for some $g \in G_m$, and t does not lie between $g \cdot i$ and $g \cdot j$. In this case, t lies between $g \cdot j$ and $g \cdot f(i)$, i.e., $t \in X_t(f_{\langle i,j\rangle})$,
- $k = g \cdot j$ for some $g \in G_m$, and t does not lie between $g \cdot i$ and $g \cdot i$. In this case, t lies between $g \cdot i$ and $g \cdot f(j)$, so once again, $t \in X_t(f_{\langle i,j\rangle})$.

Moreover,

$$\mathfrak{w}_t(f) - \mathfrak{w}_t(f_{\langle i,j\rangle}) = \frac{1}{2}|A|.$$

The map T is a fixed point free involution on A, showing that $\mathfrak{w}_{[t]}(f) - \mathfrak{w}_t(f_{\langle i,j\rangle}) \in \mathbb{Z}^m$, as claimed. ■

Lemma 4.12. *Given any $x \in \mathrm{Cross}(f)$, there is an inclusion*

$$\Phi_x\colon \mathrm{Cross}(f_x) \to \mathrm{Cross}(f)$$

with the property that if $y = \Phi_x(y')$, then for some $x' \in \mathrm{Cross}(f_y)$, $x = \Phi_y(x')$; and

$$(f_x)_{y'} = (f_y)_{x'}.$$

Proof. Write $x = \langle i,j\rangle$ and $\langle k,\ell\rangle \in \mathrm{Cross}(f) \setminus \{x\}$.

If $\{[i],[j]\} \cap \{[k],[\ell]\} = \emptyset$, let $\Phi_x(\langle k,\ell\rangle) = \langle k,\ell\rangle$.

Otherwise, we can assume without loss of generality that $[j] = [\ell]$, i.e., $\ell = g \cdot j$. Replacing k by $g \cdot i$, we have a triple of integers (i,j,k) so that $\langle i,j\rangle = x$ and $\langle j,k\rangle = y$. In this case, let

$$\Phi_x(\langle i,j\rangle) = \begin{cases} \langle i,k\rangle & \text{if } k \text{ is between } i \text{ and } j, \\ \langle j,k\rangle & \text{otherwise.} \end{cases}$$

It is straightforward to check that Φ_x is a one-to-one correspondence, and that $(f_x)_{y'} = (f_y)_{x'}$, as claimed. ■

4.2. The pong algebra

The pong algebra $\mathcal{P}(m,k)$ is generated over $\mathbb{Z}[v_1,\dots,v_m]$ by lifted partial permutations on k letters in $\{1,\dots,m-1\}$. We extend $\mathfrak{w}(f)$ to a grading by $(\frac{1}{2}\mathbb{Z})^m$, so that the grading of v_i is the ith standard basis vector in $\mathbb{Z}^m$.

Definition 4.13. In view of Lemma 4.9 (equation (4.1)), if (S,f) and (T,g) are pong data with $T = \tilde{f}(\tilde{S})/G_m$, then there is some monomial $v(f,g) \in \mathbb{Z}[v_1,\dots,v_m]$ with the property that

$$\mathfrak{w}(f) + \mathfrak{w}(g) = \mathfrak{w}(v(f,g) \cdot (\tilde{g} \circ f)). \tag{4.6}$$

Similarly, if $\langle i, j\rangle \in \mathrm{Cross}(f)$, then there is a monomial $v(f, \langle i, j\rangle)$ with the property that

$$\mathfrak{w}(f) = \mathfrak{w}(v(f, \langle i, j\rangle) \cdot f_{\langle i,j\rangle}). \tag{4.7}$$

If (S, f) and (T, g) are two lifted partial permutations, we define $\mu_2(g, f) = 0$ unless $T = f(S)/G_m$ and $\mathrm{cross}(\tilde{g} \circ f) = \mathrm{cross}(g) + \mathrm{cross}(f)$, in which case we let

$$\mu_2(g, f) = v(f, g) \cdot \tilde{g} \circ f,$$

where $v(f, g)$ is as in equation (4.6).

Given a partial permutation (S, f), define the derivative $d(f)$ to be given by

$$\mu_1(S, f) = \sum_{\{c \in \mathrm{Cross}(f) | \mathrm{cross}(f_c) = \mathrm{cross}(f) - 1\}} v(f, c) \cdot f_c,$$

where $v(f, c)$ as in equation (4.7).

Proposition 4.14. *The set $\mathcal{P}(m, k)$ equipped with μ_1 and μ_2 is a differential graded algebra.*

Proof. Associativity of μ_2 follows from the associativity of the composition of lifted partial permutations.

The property that $\mu_1 \circ \mu_1 = 0$ is an easy consequence of Lemma 4.12.

A crossing $\langle i, j\rangle$ in $\tilde{f} \circ g$ is a G_m-orbit of $i < j$ with $\tilde{f} \circ \tilde{g}(i) > \tilde{f} \circ \tilde{g}(j)$. For each crossing, either $\tilde{g}(i) < \tilde{g}(j)$, in which case $\langle \tilde{g}(i), \tilde{g}(j)\rangle \in \mathrm{Cross}(f)$ or $\tilde{g}(i) > \tilde{g}(j)$, in which case $\langle i, j\rangle \in \mathrm{Cross}(g)$. In the first case,

$$(\tilde{f} \circ \tilde{g})_{\langle i,j\rangle} = \tilde{f}_{\langle \tilde{g}(i), \tilde{g}(j)\rangle} \circ \tilde{g};$$

while in the second case,

$$(\tilde{f} \circ \tilde{g})_{\langle i,j\rangle} = \tilde{f} \circ \tilde{g}_{\langle i,j\rangle}.$$

The Leibniz rule,

$$\mu_1 \circ \mu_2 = \mu_2 \circ (\mathrm{Id} \otimes \mu_1 + \mu_1 \otimes \mathrm{Id})$$

follows readily. ∎

We abbreviate $\partial a = \mu_1(a)$ and $a \cdot b = \mu_2(a, b)$.

For example, we have

$$\partial((1, -2), (2, 1)) = v_1 \cdot ((1, 3), (2, 1)) + v_2 \cdot ((1, 0), (2, 3)),$$
$$\partial((1, 3)(2, -3)) = ((1, 4), (2, -2)) + v_2((1, -3), (2, 3)).$$

(The pong diagram for $((1, -2), (2, 1))$ is shown in Figure 2; pong diagrams for the four other terms are illustrated in Figure 7.)

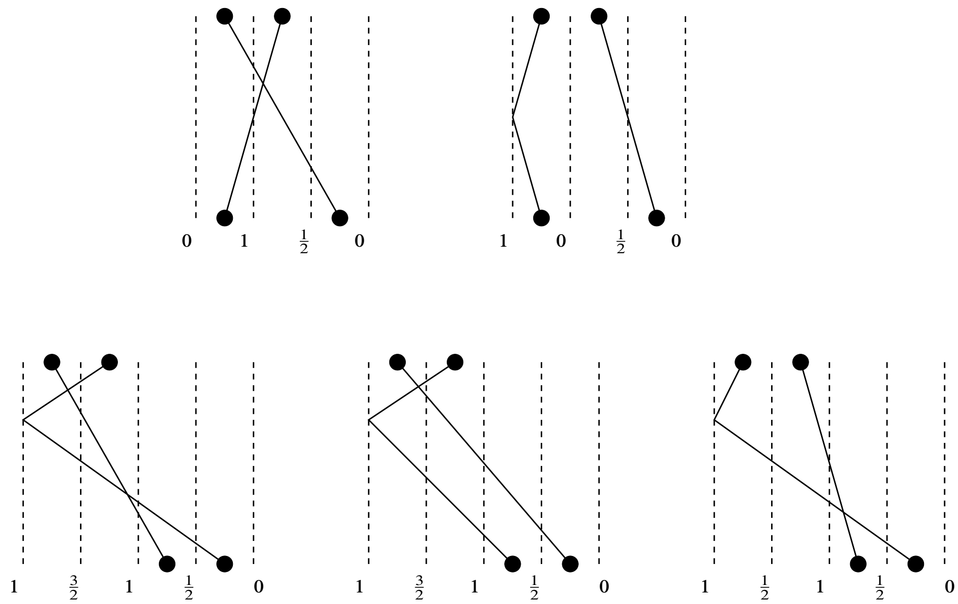


Figure 7. *Differentials of pong diagrams.* The pong diagram from Figure 2 has two crossings in it; the corresponding two terms in the differential are represented by the two pong diagrams in the top line: the first multiplied by v_1 and the second multiplied by v_2. The differential of pong diagram on the left of the second row has two non-zero terms in it (although it has 3 crossings): the middle picture, and the right picture, multiplied by v_2.

Informally, multiplication corresponds to stacking strands diagrams on top of each other, pulling the strands taut, and multiplying by a monomial in the v_i to preserve the weight vector.

When thinking of (S, f) as an element of $a \in \mathcal{P}(m,k)$, if $c \in \mathrm{Cross}(f)$, define

$$\partial_c(a) = \begin{cases} v(f,c)\cdot(f_c,S)v(f,c)\cdot(f_c,S) & \text{if } \mathrm{cross}(f_c) = \mathrm{cross}(f)-1, \\ 0 & \text{if } \mathrm{cross}(f_c) < \mathrm{cross}(f)-1, \end{cases} \tag{4.8}$$

so that

$$\partial(a) = \sum_{c\in\mathrm{Cross}(f)} \partial_c a.$$

4.3. Alexander and Maslov gradings

Idempotents in the pong algebra correspond to lifted partial permutations which send $S \subset \mathbb{Z}$ to itself via the identity map. In terms of their pong diagrams, these algebra elements are collections of k vertical segments. These $\binom{m-1}{k}$ elements generate the idempotent subalgebra $\mathfrak{k}$ of the pong algebra.

In Definition 4.6, the weight $\mathfrak{w}$ of a lifted partial permutation f was defined as a map from $(\frac{1}{2} + Z)/G_m$ to $\frac{1}{2}\mathbb{Z}$. We prefer to think of it henceforth as a vector $\vec{\mathfrak{w}}$, with components indexed by integers $i \in (1, \dots, m)$. With this normalization, $\mathfrak{w}_i$ corresponds to the value of the function $\mathfrak{w}$ at $i - 1/2$.

This vector $\vec{\mathfrak{w}}$ induces a grading on $\mathcal{P}(m, k)$ with values in $\frac{1}{2}\mathbb{Z}^m$, with the understanding that the element v_i has grading e_i (the ith basis vector in $\mathbb{Z}^m$). This grading is additive under multiplication; and differentiation preserves this grading.

Similarly, the function $\mathrm{cross}(f)$ on lifted partial permutations induces a grading on $\mathcal{P}(m, k)$ with values in $\mathbb{Z}$, with the understanding that v_i has grading 0. The grading is additive under multiplication, and differential drops it by 1. In the language of Definition 2.2, $\mathcal{P}(m, k)$ is a Maslov–Alexander bigraded algebra.

Sometimes, we find it convenient to work with the following renormalized $\mathbb{Z}$-grading $\mathbf{N}$ on $\mathcal{P}(m, k)$, specified by

$$\mathbf{N}(f) = \mathrm{cross}(f) - \left(2 \sum_{i=1}^{m} \mathfrak{w}_i(f)\right). \tag{4.9}$$

Definition 4.15. By analogy with $\mathcal{C}$, each element in $\mathcal{P}(m, k)$ corresponding to a lifted partial permutation is called a *pure algebra element*. Thus, the pong algebra is the free $\mathbb{F}[v_1, \dots, v_m]$-module generated by the pure algebra elements.

5. Generators of the pong algebra

Idempotents in the pong algebra correspond to lifted partial permutations which send $S \subset \mathbb{Z}$ to itself via the identity map. In terms of their pong diagrams, these algebra elements are collections of k vertical segments. These $\binom{m-1}{k}$ elements generate the idempotent subalgebra of the pong algebra.

Given a pair of integers $1 \le i < j \le m - 1$, let $X_{i,j}$ denote the algebra element consisting of two moving strands, which go from i to j and j to i, so that all positions ℓ with $i < \ell < j$ are occupied. This is the element corresponding to the sum of all pong data (S, f) with $|S| = k$, with the following further properties:

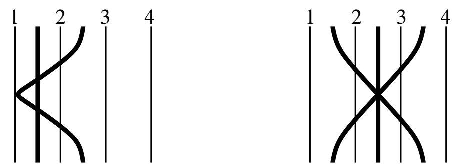
(X-1) $f(i) = j$, $f(j) = f(i)$,

(X-2) $\forall \ell \in S \setminus \{i, j\}$, $f(\ell) = \ell$,

(X-3) $\forall \ell$ with $i < \ell < j$, $\ell \in S$.

We let $\mathcal{X}_{i,j}$ denote the algebra element obtained by dropping Condition (X-3).

We extend to $i = 0$ or $j = m$ (but not both), so that $X_{0,j}$ has one strand from j to itself with one left cusp and $X_{i,m}$ has one strand from i to itself with one right



$X_{0,2}$ $X_{1,3}$

Figure 8. *The generators* $X_{i,j}$. These pictures are in $\mathcal{P}(4,2)$ and $\mathcal{P}(4,3)$, respectively.

cusp. See Figure 8 for some examples. The element $X_{0,j}$ corresponds to the sum of all pong data (S, f) with $|S| = k$, with the following further properties:

(X-1) $f(j) = r_{1/2}(j)$,

(X-2) $\forall \ell \in S \setminus \{j\}$, $f(\ell) = \ell$,

(X-3) $\forall \ell$ with $\ell < j$, $\ell \in S$.

The element $X_{i,m}$ is defined analogously, now using $r_{m+1/2}$ instead of $r_{1/2}$. We define $\mathcal{X}_{0,j}$ and $\mathcal{X}_{i,m}$ again dropping condition (X-3).

Given a pair of integers $1 \leq i < j \leq m-1$, let $R_{i,j}$ denote the algebra element consisting of one moving strand which goes from i to j, so that all positions ℓ with $i < \ell < j$ are occupied. This is the element corresponding to the sum of all pong data (S, f) with $|S| = k$, with the following further properties:

(R-1) $f(i) = j$,

(R-2) $\forall \ell \in S \setminus \{i\}$, $f(\ell) = \ell$,

(R-3) $\forall \ell$ with $i < \ell < j$, $\ell \in S$.

Again, we have elements $\mathcal{R}_{i,j}$ obtained by dropping condition (R-3). We define elements $L_{j,i}$, and $\mathcal{L}_{j,i}$, analogously. (In that case, the second condition above holds $\forall \ell \in S \setminus \{j\}$.)

Definition 5.1. An *atomic* element in $\mathcal{P}(m,k)$ is a non-zero element of the form $\mathbf{I}_{\mathbf{x}} \cdot X_{i,j}$, $\mathbf{I}_{\mathbf{x}} \cdot R_{i,j}$, and $\mathbf{I}_{\mathbf{x}} \cdot L_{j,i}$.

The terminology is justified by the following proposition.

Proposition 5.2. *Every pure algebra element in* $\mathcal{P}(m,k)$ *(i.e., those elements corresponding to lifted partial permutations) can be written as a product of idempotents and atomic generators.*

Proof. We find it convenient to construct factorizations into $\mathcal{X}_{i,j}$, $\mathcal{R}_{i,j}$, and $\mathcal{L}_{j,i}$. This suffices, since for any idempotent $\mathbf{x}$, the element $\mathbf{I}_{\mathbf{x}} \cdot \mathcal{X}_{i,j}$ can be written as a product of $\mathbf{I}_{\mathbf{x}}$ and elements of the form $X_{i',j'}$, $R_{i',j'}$, and $L_{i',j'}$. Similarly, $\mathbf{I}_{\mathbf{x}} \cdot \mathcal{R}_{i,j}$ can be written as a product of $\mathbf{I_x}$ and elements of the form $R_{i',j'}$; and $\mathbf{I_x} \cdot \mathcal{L}_{j,i}$ can be written as a product of $\mathbf{I_x}$ and elements of the form $L_{j',i'}$.

Suppose that $a = \mathbf{I_x} \cdot a \cdot \mathbf{I_y}$ for idempotent states $\mathbf{x}$ and $\mathbf{y}$. The factorization exists by induction on the total weight of a. Obviously, when the weight is zero, then a is an idempotent. For the inductive step, there are 3 cases.

Case 1. Suppose that for some $x_i \in \mathbf{x}$, the strand in a out of x_i moves to the right. Choose i maximal with this property. We divide this into two further subcases.

Case 1a. Suppose that there is some strand $j > i$ with the property that the strand out of x_j moves to the left. Choose j minimal with this property. If the strand out of x_i reaches x_j and the strand out of x_j reaches x_i, we can factor

$$a = \mathbf{I_x} \cdot \mathcal{X}_{x_i,x_j} \cdot b;$$

or if the strand out of x_i terminates at some $y \in \mathbf{x}$ with $y < x_j$, we can factor

$$a = \mathbf{I_x} \cdot \mathcal{R}_{x_i,y} \cdot b;$$

or if the strand out of x_j terminates at some $y \in \mathbf{x}$ with $x_i < y$, we can factor

$$a = \mathbf{I_x} \cdot \mathcal{L}_{x_j,y} \cdot b.$$

Case 1b. If there is no strand $j > i$ so that the strand out of x_j moves to the left, then there is a factorization $a = \mathbf{I_x} \cdot \mathcal{X}_{x_i,m} \cdot b$ or a factorization of $a = \mathbf{I_x} \cdot \mathcal{R}_{x_i,y} \cdot b$. The result now holds by induction on the total weight.

Case 2. All strands in a are stationary. This is clearly an idempotent.

Case 3. There is some strand in a that starts out moving to the left, but none that starts moving to the right. Take the minimal i so that there is a strand in a starting at x_i which moves to the left. If the strand bounces back past x_i, then we can factor

$$a = \mathbf{I_x} \cdot \mathcal{X}_{0,x_i} \cdot b,$$

or the strand terminates at some $y < x_i$, and we can factor

$$a = \mathbf{I_x} \cdot \mathcal{L}_{x_i,y} \cdot b. \qquad \blacksquare$$

We have the following characterization of the atomic generators.

Proposition 5.3. *For any non-zero homogeneous, non-idempotent element $a \in \mathcal{P}(m,k)$,*

$$2\mathfrak{w}(a) - \mathrm{cr}(a) \geq 1,$$

with equality precisely when a is atomic.

Proof. Let $\mathrm{gr}(a) = 2\mathfrak{w}(a) - \mathrm{cr}(a)$. Clearly, $\mathfrak{w}$ is additive under multiplication, in the sense that if a and b are pure algebra elements with $a \cdot b \neq 0$, then $\mathfrak{w}(a \cdot b) = \mathfrak{w}(a) + \mathfrak{w}(b)$. By construction, cr is also additive under multiplication. It is easy to see that for all atomic elements a, $\mathrm{gr}(a) = 1$. Thus, from Proposition 5.2, it follows that that $\mathrm{gr}(a) \geq 1$ for all pure algebra elements, with $\mathrm{gr}(a) = 1$ if and only if a is atomic. Having proved the proposition for pure algebra elements, the result for all algebra elements follows readily from the fact that for any homogenous element a and $i \in \{1, \dots, m\}$. $\mathrm{gr}(v_i \cdot a) = 2 + \mathrm{gr}(a)$. ∎

We can describe the differential on the pong algebra in terms of its action on atomic generators. Specifically, we have the following lemma.

Lemma 5.4. *The differentials of atomic generators are specified as follows. Given $i < j$, we have that*

$$dR_{i,j} = \sum_{i<\ell<j} R_{\ell,j} \cdot R_{i,\ell}, \tag{5.1}$$

$$dL_{j,i} = \sum_{i<\ell<j} L_{\ell,i} \cdot L_{j,\ell}. \tag{5.2}$$

Moreover, if $i \in \{0, \dots, m-1\}$, then

$$dX_{i,i+1} = U_{i+1},$$

while if $i + 1 < j$, then

$$dX_{i,j} = \sum_{i<\ell<j} X_{\ell,j} \cdot X_{i,\ell} + X_{i,\ell} \cdot X_{\ell,j}.$$

Proof. This follows immediately from the definition of the boundary map. ∎

6. Homology of the pong algebra

Consider the pong algebra $\mathcal{P}(m,k)$, over $\mathbb{F}[v_1, \dots, v_m]$. The aim of the present section is to compute its homology.

Let

$$\Omega = \sum_{0<i<m} [\mathcal{X}_{0,i}, \mathcal{X}_{i,m}].$$

Lemma 6.1. *Ω is an element with degree $2k$, $\mathfrak{w}_i(\Omega) = 1$ for all $i = 1, \dots, m$, and $d\Omega = 0$.*

Proof. This is straightforward. ∎

Theorem 6.2. *Fix $k < m$. The homology of $H_*(\mathcal{P}(m,k))$ is isomorphic to a polynomial algebra in Ω over $\mathcal{C}(m,m-k-1)$; i.e.,*

$$H_*(\mathcal{P}(m,k)) \cong \mathcal{C}(m,m-k-1)[\Omega].$$

We will also sometimes consider $\mathcal{P}'(m,k)$, which is obtained by adjoining to $\mathcal{P}(m,k)$ an element X with $dX = \Omega$.

6.1. A quotient

As a stepping-stone, we will find it convenient to consider the quotient $Q(m,k)$ of $\mathcal{P}(m,k)$ by the ideal generated by all the v_i, and the analogous $Q'(m,k)$ obtained by adjoining X with $dX = \Omega$. We will typically abbreviate $Q = Q(m,k)$ and $Q' = Q'(m,k)$.

For fixed weight vector $w = (w_1, \dots, w_m)$, let $Q(w) \subset Q$ and $Q'(w) \subset Q'$ be the subcomplexes with weight w; and let $Q(\mathbf{x},\mathbf{y};w) = \mathbf{I_x} \cdot Q(w) \cdot \mathbf{I_y}$ and $Q'(\mathbf{x},\mathbf{y};w) = \mathbf{I_x} \cdot Q'(w) \cdot \mathbf{I_y}$. Note that $Q(\mathbf{x},\mathbf{y};w)$ has a further grading

$$Q_*(\mathbf{x},\mathbf{y};w) = \bigoplus_{d \geq 0} Q_d(\mathbf{x},\mathbf{y};w),$$

where d counts the number of crossings.

To state the homology of $Q(m,k)$, it helps to introduce some notions.

Definition 6.3. A *compatible triple* $(\mathbf{x},\mathbf{y},w)$ consists of a vector $w = (w_1, \dots, w_m) \in (\frac{1}{2}\mathbb{Z})^m$ and two idempotent states $\mathbf{x}$ and $\mathbf{y}$, satisfying the condition that for each $i = 1, \dots, m$,

$$2w_i + \#(x_t < i) + \#(y_t < i)$$

is an even integer.

Clearly, $Q(\mathbf{x},\mathbf{y};w) = 0$ unless $(\mathbf{x},\mathbf{y};w)$ is a compatible triple.

Theorem 6.4. *Given a compatible triple $(\mathbf{x},\mathbf{y};w)$, we have that $H_*(Q'(\mathbf{x},\mathbf{y};w)) = 0$ unless all of the following conditions hold.*

(Q-1) *For each $s = 1, \dots, k-1$, $\max(x_s, y_s) < \min(x_{s+1}, y_{s+1})$.*

(Q-2) *Each weight w_i satisfies the inequality $0 \leq w_i \leq 1$.*

If all of the above conditions holds, then $H_*(Q'(\mathbf{x},\mathbf{y};w)) \cong \mathbb{F}$.

The proof will occupy most of the rest of the present subsection.

For cases where $Q'(\mathbf{x},\mathbf{y};w)$ has trivial homology, it will be useful to use the following notion.

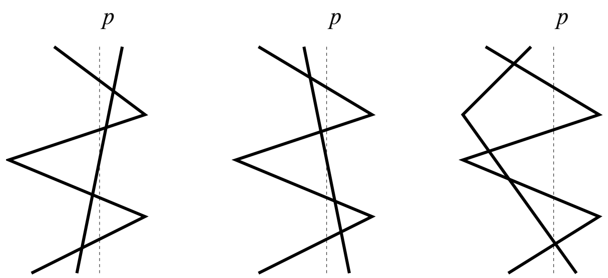


Figure 9. The generator on the left is not (right)-excessive (at p); the other two are.

Definition 6.5. Fix an integer p with $1 \leq p \leq m-1$. The algebra element a is called *(right-)excessive* at p if there are at least two non-left-moving lifted strands with inequivalent initial points that meet (i.e., cross, terminate at, or start at) p (see Figure 9). Equivalently, let $\mathcal{E}_p$ denote the set of (unlifted) strands in the algebra element a satisfying at least one of the following conditions:

- s is stationary at p,
- s starts out at p, moving to the right,
- s terminates at p, moving from the left,
- s crosses p from left to right.

Then, a is excessive at p precisely when $\mathcal{E}_p$ has at least two elements in it.

Lemma 6.6. *Fix $p \in \{1, \ldots, m-1\}$. If all the pure algebra generators in $Q_d(\mathbf{x}, \mathbf{y}; w)$ and $Q_{d-1}(\mathbf{x}, \mathbf{y}; w)$ are excessive, then $H_d(Q_*(\mathbf{x}, \mathbf{y}; w)) = 0$.*

Proof. Fix an excessive generator a in $Q'(\mathbf{x}, \mathbf{y}; w)$. By hypothesis, there must be at least two right-moving lifted strands s_1 and s_2 with inequivalent initial points that are either stationary at p or that meet p. Choose the strand s_1 so that its initial point i_1 (which is $\leq p$) is maximized. Choose s_2 so that its initial point i_2 is maximized among the remaining strands.

If s_1 and s_2 cross, let $h(a) = 0$. Otherwise, let $h(a) = a'$ be the pure algebra element obtained replacing the strands s_1 and s_2, which start at i_1 and i_2 and end at j_1 and j_2, by a strand s_1' from i_1 to j_2 and a strand s_2' from i_2 to j_1. Note that s_1' and s_2' intersect once.

Let $f(a) = i_1 + i_2$. Given two pure algebra elements, we write $a < b$ if $f(a) < f(b)$. Given two algebra elements a and b we write $a < b$ if we can write $a = \sum_i a_i$ and $b = \sum_i b_i$ as sums of pure algebra elements with $f(a_i) < f(b_j)$ for all i, j.

We claim that if $a \in Q_d(\mathbf{x}, \mathbf{y}; w)$,

$$\partial h(a) + h\partial(a) + a > a. \tag{6.1}$$

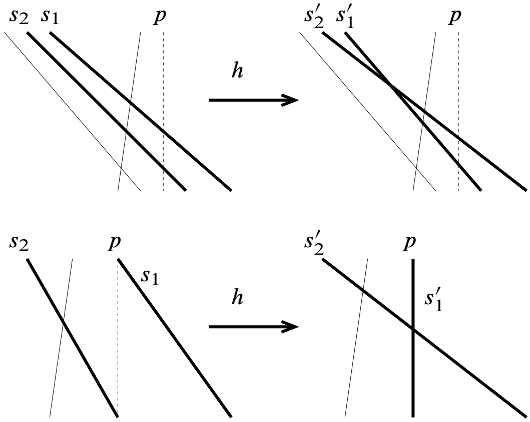


Figure 10. *Homotopy operator.* The dashed line represents position p. The bottom line represents a degenerate case, where the strand s'_1 is stationary.

Note that f is an integer-valued function. Since w is fixed, f is bounded below; also, f is bounded above by $2p-1$. It follows that

$$\mathrm{Id} + \partial h + h\partial \colon Q_d(\mathbf{x}, \mathbf{y}; w) \to Q_d(\mathbf{x}, \mathbf{y}; w)$$

is nilpotent. On the other hand, this nilpotent map is an isomorphism on homology, since it is chain homotopic to the identity; thus, the homology is trivial.

We verify equation (6.1) as follows. Suppose that a has no crossing between s_1 and s_2, so that $a' = h(a)$ is non-zero. Terms in $\partial h(a)$ correspond to various resolutions of the crossings in a'; terms in $h\partial(a)$ correspond to resolutions of the crossings in a.

We partition the set of crossings in a into the following types:

- the set A of crossings in a that involve one of the strands s_1 and s_2;
- the set B of crossings in a do not involve either s_1 or s_2 and which leave i_1 and i_2 unchanged;
- the set C of crossings in a that do not involve either s_1 or s_2 but which change i_1 and i_2. Such a crossing occurs between strands s_3 and s_4 with the following properties: s_3 crosses p and its initial point is smaller than i_2; and s_4 does not cross p and its initial point is greater than i_2.

See Figure 10 for illustrations. Let A', B', and C' be analogous crossings in a', with the understanding that A' does not include the distinguished crossing q'_0 between s'_1 and s'_2 (see Figure 11). The resolution of a' at q_0, of course, gives back a; i.e., $\partial_{q'_0} h(a) = a$.

There is a bijection between A and A' which gives a cancellation of terms in $\partial h(a)$ involving s_1 or s_2 with corresponding terms in A'. We construct this bijection

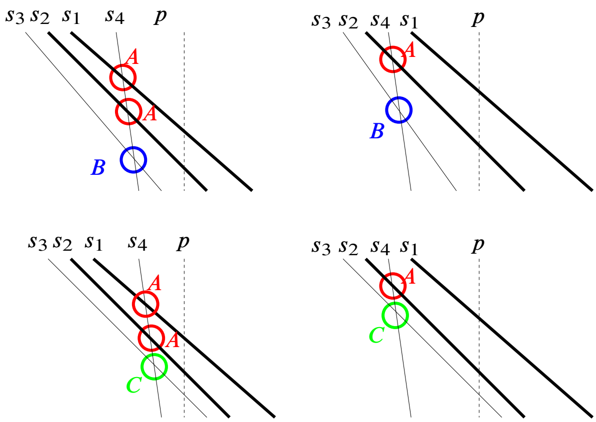


Figure 11. *Types of crossings.* The dashed lines represent the position p.

$\phi\colon A \to A'$ explicitly, as follows. Suppose there is some strand s_3 which meets $s_1 \cup s_2$, so that the intersections $s_3 \cap (s_1 \cup s_2)$ are of type A. There are two subcases. Either $s_3 \cap (s_1 \cup s_2)$ consists of one point, call it q; in which case $s_3 \cap (s_1' \cap s_2')$ consists of one point, and we define $\phi(q)$ to be that point. Or, $s_3 \cap (s_1 \cup s_2)$ consists of two points, denoted q_1 and q_2; and also $s_3 \cap (s_1 \cap s_2)$ consists of two points, as well, which we denote q_1' and q_2'. We can label $\{q_1, q_2\}$ so that the resolution of a at q_1 is in the kernel of h, while the resolution at q_2 is not. Similarly, we can label $\{q_1', q_2'\}$ so that the resolution of $h(a)$ at q_1' contains a double-crossing, but the resolution at q_2' does not. See Figure 12. Clearly,

$$\partial_{\phi(q)} \circ h(a) = h \circ \partial_q(a),$$

where ∂_q is as in equation (4.8). (Indeed, for $q = q_1$, both terms vanish.)

There is a similar one-to-one correspondence between elements in B and those in B', giving a further cancellation of terms $\partial_{q'} h(a)$ with those in $h \circ \partial_q(a)$ (with $q \in B$).

The remaining terms in $\partial \circ h(a)$ and $h \circ \partial(a)$ all correspond to crossings in C or C'.

Note that the only way a resolution at a crossing between strands other than s_1 and s_2 can change i_1 or i_2 is if that resolution properly increases f; i.e., all the terms c in $\partial h(a) + h\partial(a)$ corresponding to the resolutions at C or C' have $f(c) > f(a) = f(a')$. This completes the verification of equation (6.1) when s_1 and s_2 do not cross.

The formula where they do cross follows similarly. ■

Lemma 6.7. *If the condition (Q-1) from Theorem 6.4 is violated, then we have* $H(Q(\mathbf{x}, \mathbf{y}; w)) = 0$.

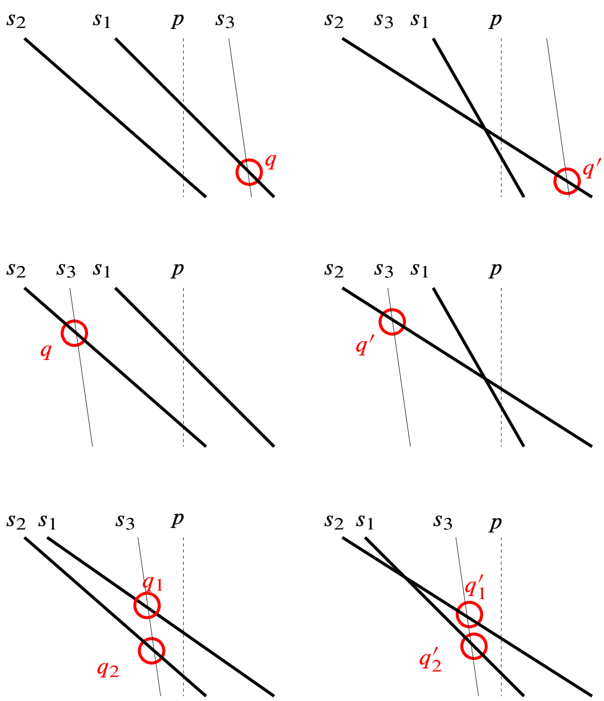


Figure 12. On the left, we have intersection points of type A, and on the right, we have their corresponding intersection points in A'. The top two rows represent cases where $|s_3 \cap (s_1 \cup s_2)| = 1$, while the third represents the case where $|s_3 \cap (s_1 \cup s_2)| = 2$.

Proof. Condition (Q-1) is equivalent to the following condition for all $p = 1, \dots, m-1$:

$$\big|\#\{x \in \mathbf{x} \mid x \leq p\} - \#\{y \in \mathbf{y} \mid y \leq p\}\big| \leq \begin{cases} 1 & \text{if } p \notin \mathbf{x} \cap \mathbf{y}, \\ 0 & \text{if } p \in \mathbf{x} \cap \mathbf{y}. \end{cases} \tag{6.2}$$

Let $\mathcal{E}_p$ denote the set of (unlifted) strands as in Definition 6.5. It is elementary to verify that

$$\#\{x \in \mathbf{x} \mid x \leq p\} - \#\{y \in \mathbf{y} \mid y < p\} \leq |\mathcal{E}|.$$

Thus, Lemma 6.6 shows that if

$$H_*(Q(\mathbf{x}, \mathbf{y}; w)) \neq 0,$$

then

$$\begin{aligned} \{x \in \mathbf{x} \mid x \leq p\} - \#\{y \in \mathbf{y} \mid y \leq p\} + \begin{Bmatrix} 1 & \text{if } p \in \mathbf{x} \cap \mathbf{y} \\ 0 & \text{otherwise} \end{Bmatrix} \\ \leq \#\{x \in \mathbf{x} \mid x \leq p\} - \#\{y \in \mathbf{y} \mid y < p\} \leq 1. \end{aligned}$$

There is an isomorphism of chain complexes $Q(\mathbf{x},\mathbf{y};w) \cong Q(r(\mathbf{x}),r(\mathbf{y}),r(w))$, where r is reflection of the interval. This gives the other bound

$$-\{x \in \mathbf{x} \mid x \leq p\} + \#\{y \in \mathbf{y} \mid y \leq p\} + \left\{\begin{matrix} 1 & \text{if } p \in \mathbf{x} \cap \mathbf{y} \\ 0 & \text{otherwise} \leq 1 \end{matrix}\right\} \leq 1$$

needed to complete the verification of equation (6.2). ■

Let r_p, resp., ℓ_p denote the number of lifted right-moving, resp., left-moving lifted strands that cross $p + \frac{1}{2}$.

Lemma 6.8. *Suppose that $a \in Q(\mathbf{x},\mathbf{y};w)$ represents a non-trivial homology class in $H_*(Q(\mathbf{x},\mathbf{y};w))$. Then, a is a sum of pure algebra elements with $|r_p - w_p| \leq \frac{1}{2}$ for all $p = 1,\dots,m$.*

Proof. Clearly, $r_p - \ell_p = \#(x \in \mathbf{x} < p) - \#(y \in \mathbf{y} < p)$. Lemma 6.7 implies that if $H_*(Q(\mathbf{x},\mathbf{y};w)) \neq 0$, then $r_p - \ell_p \leq 1$ for all p. The stated inequality follows from the fact that $r_p + \ell_p = 2w_p$. ■

Lemma 6.9. *Let w be a non-constant vector with the following properties:*

- *there is some integer $c > 0$ so that $w_i > c$ for some i,*
- $H_d(Q(w)) \neq 0$.

Then, for all i, $c \leq w_i$, and $d \geq 2kc$.

Proof. Let a be any representative of $Q_d(\mathbf{x},\mathbf{y};w)$. Choose p so that exactly one of w_{p-1} or w_p is greater than c. By symmetry, we can assume that there is a right-moving lifted strand s that either starts (when $w_p > w_{p-1}$) or terminates at p. Moreover, by Lemma 6.8, there are at least c additional lifted strands that cross position p from the left. If a fails to be excessive, which is inevitable by Lemma 6.6, all c of those lifted strands, and the additional strand s, must have equivalent starting points; i.e., there is an unlifted, right-moving strand that either starts or ends at p, and that covers the entire interval $[1,m]$ with multiplicity at least c. The bounds on d and c follow readily. ■

Definition 6.10. Let $(\mathbf{x},\mathbf{y},w)$ be a compatible triple, and suppose that there is some integer c so that $c \leq w_i \leq c+1$ for all $i = 1,\dots,m$. The *associated canonical cycle* is the element $z = z(\mathbf{x},\mathbf{y};w) \in Q(\mathbf{x},\mathbf{y};w)$ characterized by the following properties:

- $\mathfrak{w}_i(z) = w_i - c$,
- if s is any right-moving strand in z, then the terminal position of s is not the initial position of any other right-moving strand in z.

An element $z \in Q(\mathbf{x},\mathbf{y})$ is called a *canonical cycle* if there is a vector w for which z is the canonical cycle associated to $(\mathbf{x},\mathbf{y};w)$.

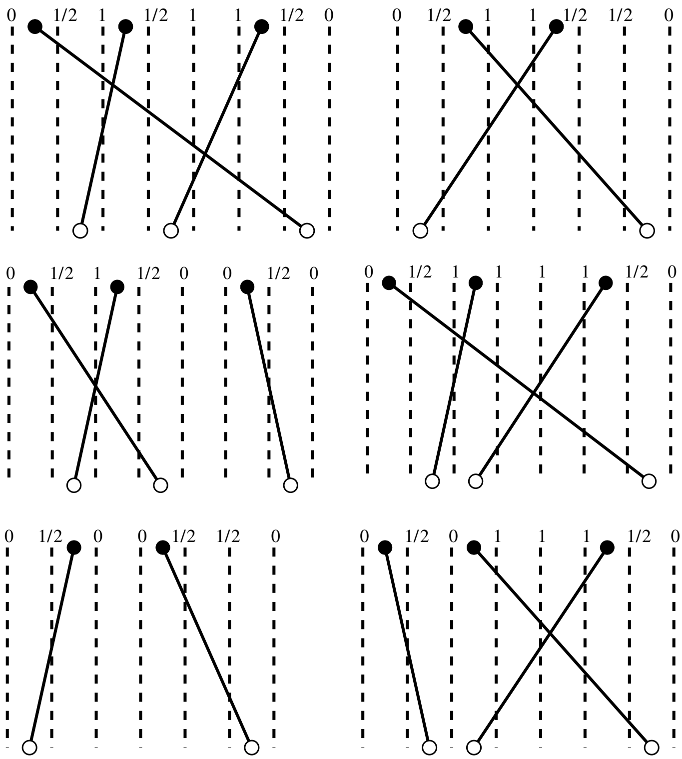


Figure 13. *Canonical cycles.* Here are six canonical cycles; the multiplicities are indicated at the dotted lines. (Note that there is another cycle homologous to the third on the top line with the same multiplicities and idempotents.) Cycles in the second row are obtained as resolutions of crossings on the first row.

For example, if all the components of w_i are integers, then z is the product of those $\mathcal{X}_{x_i,x_{i+1}}$ for which $w_{x_i} = c+1$, with the factors are arranged so that i is increasing. See also Figure 13.

Lemma 6.11. *Let $(\mathbf{x},\mathbf{y};w)$ be a compatible triple. Suppose that there is an integer c with the property that $c \leq w_i \leq c+1$ for all i and w_i is not constant. Let $z = z(\mathbf{x},\mathbf{y};w)$ be the associated canonical cycle. $Q'(\mathbf{x},\mathbf{y};w)$ is quasi-isomorphic to the subcomplex $(zQ')(\mathbf{x},\mathbf{y};w) = z\cdot Q'(\mathbf{y},\mathbf{y};(c,\dots,c)) \subset Q'(\mathbf{x},\mathbf{y};w)$. In particular, if w is a non-constant vector with $w_i \leq 1$ for all i, then $H(Q'(\mathbf{x},\mathbf{y};w))$ is one-dimensional.*

Proof. Fix some p so that $w_{p-1} = w_p = c+1$ and $p \in \mathbf{x}\cap\mathbf{y}$, consider the subspace $L_p \subset Q'(\mathbf{x},\mathbf{y};w)$ generated by pure algebra elements a with the property that the strand out of p is left-moving. It is easy to see that $L_p \subset Q'(\mathbf{x},\mathbf{y};w)$ is a subcomplex.

We claim that the inclusion map $L_p \to Q'(\mathbf{x},\mathbf{y};w)$ is a chain homotopy equivalence. To verify this, consider the map

$$h_p\colon Q'(\mathbf{x},\mathbf{y};w) \to Q'(\mathbf{x},\mathbf{y},w)$$

which is defined to vanish if a is not excessive; otherwise, $h_p(a)$ is as defined in the proof of Lemma 6.6. Observe that if the strand out of p in a is left-moving, then the strand out of p in $h_p(a)$ is also left-moving; i.e., $h_p(L_p) \subset L_p$. Observe also that the quotient $Q'(\mathbf{x},\mathbf{y};w)/L_p$ is generated by excessive elements: the strand out of p, the strand into p, and the c additional strands that cross position p cannot all coincide, for otherwise, we would get an inequality $w_i \geq c+1$, violating our hypotheses. It follows now from the proof of Lemma 6.6 that $H(Q'(\mathbf{x},\mathbf{y};w)/L_p) = 0$, verifying that $L_p \to Q'(\mathbf{x},\mathbf{y};w)$ is a quasi-isomorphism.

More generally, let

$$S = \big\{p \in \{1,\dots,m\} \mid w_{p-1} = w_p = c+1 \text{ and } p \in \mathbf{x} \cap \mathbf{y}\big\}.$$

Given any $T \subset S$, let $L_T \subset Q'(w)$ denote the intersection $\cap_{t\in T} L_t$. We claim that for any $p \in S \setminus T$, h_p maps L_T into L_T. (The homotopy operator h, introduces a crossing between two right-moving strands in the lift; these strands could start out left-moving or right-moving at their initial points in the projection; but the homotopy operator never changes left-moving to right-moving or vice versa. In a special case, for the strand through p, it might turn a right-moving strands into a stationary one; but here we are specifically considering $p \in S \setminus T$.) It follows that the inclusion $L_{\{p\}\cup T} \to L_T$ is a quasi-isomorphism. Thus, by induction, $L_S \subset Q'(\mathbf{x},\mathbf{y};w)$ is a quasi-isomorphism.

It remains to show that $L_S = (zQ')(w)$. This can be seen as follows. Choose p minimal so that $w_p = c+1$ and $p \in \mathbf{x}$. If $p = x_1$, the strand out of p moves to the left, so we can factor $a = z_0 \cdot a'$, where $z_0 = \mathcal{X}_{0,p}$. Assume that $p = x_i > x_1$. Let $x_{i+1} \in \mathbf{x}$ be the successor of $p = x_i \in \mathbf{x}$, and choose the maximal $y_j \in \mathbf{y}$ so that $x_i < y_j$. If $y_j < x_{i+1}$, we can write $a = z_0 \cdot a'$, where z_0 is the algebra element associated to a strand from x_i to y_j. If $y_j \geq x_{i+1}$, the strand out of $q = x_{i+1}$ necessarily moves to the left (because either $w_q > w_{q-1}$; or if $w_q = w_{q-1}$, but in that case $a \in L_q$). Thus, we can factor $a = z_0 \cdot a'$, where z_0 is the algebra element consisting of two strands, one of which goes (to the right) from x_i to x_{i+1}, and the other leaves x_{i+1}, going to the left, and terminating at y_{j-1}. In cases where the weight vector of $w' = \mathfrak{w}(a')$ is constant, we have the desired factorization. Otherwise, we can use the induction on the support of z, noting that $z(a) = z_0 \cdot z(a')$. ∎

Lemma 6.12. *Let w be a non-constant vector with the property that* $\max w_i > 1$*, then* $H(Q'(\mathbf{x},\mathbf{y};w)) = 0$.

Proof. By Lemma 6.9, we can assume that there is some integer c so that $c \leq w_i \leq c+1$; by hypothesis, $c > 0$. Let $z = z(\mathbf{x}, \mathbf{y}, w)$ be the canonical cycle. We wish to analyze the chain complex

$$(zQ')(\mathbf{x}, \mathbf{y}; w) = z \cdot Q'(\mathbf{y}, \mathbf{y}; w - w(z)).$$

Note that there is an integer c so that for all i, $(w - w(z))_i = c$. Let Q^0 denote the algebra $Q(k+1, k)$. There is an identification of $Q(\mathbf{y}, \mathbf{y}; w - w(z))$ with the portion of Q^0 whose support is the constant vector c with $k+1$ coordinates, which we write as $Q^0(c)$.

Indeed, let

$$L = \left\{ i \mid i \in \mathbf{y}, \mathfrak{w}_i(z) > 0 \right\} \quad \text{and} \quad R = \left\{ i \mid i \in \mathbf{y}, \mathfrak{w}_{i-1}(z) > 0, \mathfrak{w}_i(z) = 0 \right\}.$$

At each position $i \in L$, the strand out of z is left-moving; at each position $i \in R$, the strand out of z is right-moving.

We claim that $(zQ')(\mathbf{x}, \mathbf{y}; w)$ is the quotient complex of Q^0 by the differential right ideal generated by $X_{i,t}$ with $i \in L$ and $X_{s,j}$ with $j \in R$. i.e.,

$$z \cdot Q(\mathbf{y}, \mathbf{y}) \cong \left(\frac{Q^0}{\sum_{i \in L} X_{i,t} Q^0 + \sum_{j \in R} X_{s,j} Q^0} \right)_c. \tag{6.3}$$

Let d_0 be the dimension (number of crossings) in z. It follows from equation (6.3) that $(zQ(w))_d = 0$ if $d > d_0 + 2ck$; and $(zQ(w))_{d_0+2ck}$ is spanned by the elements $\{z \cdot (X_{0,i} \cdot X_{i,k+1})^c, z \cdot (X_{i,k+1} \cdot X_{0,i})^c\}_{0<i<k+1}$, with the understanding that

$$z \cdot (X_{0,i} \cdot X_{i,k+1})^c = 0$$

exactly when $i \in R$; and

$$z \cdot (X_{i,k+1} \cdot X_{0,i})^c = 0$$

exactly when $i \in L$. It follows that $H((zQ'(w))_d) = 0$ for all $d \geq d_0 + 2ck$.

It remains to show that $H((zQ(w))_d) = 0$ for all $d < d_0 + ck$. Since w is non-constant, there is some $p \in \mathbf{x} \cup \mathbf{y}$ with the property that

$$\min\{w_{p-1}, w_p\} \leq c < \max\{w_{p-1}, w_p\}. \tag{6.4}$$

Our verification is subdivided into four (very similar) cases, according to whether or not $p \in \mathbf{x}$ and whether or not $w_{p-1} < w_p$.

Suppose first that there is some $p \in \mathbf{x}$ so that $w_p > c$, $w_{p-1} < w_p$. There are c right-moving strands that cross position p; and an additional right-moving strand that leaves position p. If all the elements of $Q_d(\mathbf{x}, \mathbf{y}; w)$ are excessive at p, then $H_d(Q_*(\mathbf{x}, \mathbf{y}; w)) = 0$ by Lemma 6.6. If these elements are not all excessive, the (unlifted) strand leaving p covers all of the interval with multiplicity at least c. Thus, all k of the strands leaving z must cross this moving strand with multiplicity at least c, giving the inequality $d \geq d_0 + 2ck$, provided that there is some $p \in \mathbf{x}$ with $w_p > c$ and $w_{p-1} < w_p$.

The same argument applies when there is some $p \in \mathbf{x}$ so that $w_{p-1} > w_p$ and $w_{p-1} > c$, only now using the strand entering p (rather than the one leaving p).

Finally, there is no $p \in \mathbf{x}$ so that equation (6.4) holds, there must be such a $p \in \mathbf{y}$; and the result follows from symmetry. ∎

Lemma 6.13. *Suppose that there is a c so that $w_i = c$ for all i. Fix any idempotent state $\mathbf{x}$. Then,*

$$H(Q'(\mathbf{x},\mathbf{x};w)) = \begin{cases} \mathbb{F} & \text{if } c = 0 \text{ or } 1, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. The case $c = 0$ is obvious. Since w is constant, it follows that $\mathbf{x} = \mathbf{y}$. Let $p = x_1 \in \mathbf{x}$ be the smallest element, and let $L \subset Q' = Q'(\mathbf{x},\mathbf{y};w)$ be the subcomplex generated by all pure algebra elements for which the strand out of p is left-moving. Consider the short exact sequence

$$0 \to L(w) \to Q(w) \to (Q/L)(w) \to 0.$$

Observe $(Q/L)(w)$ is generated by excessive elements and the single non-excessive element $(\mathcal{X}_{p,m} \cdot \mathcal{X}_{0,p})^c$ (which in fact is the image of Ω under the quotient map). It follows at once that $H_*((Q/L)(w))$ is one-dimensional, generated by Ω^c. Thus, $H((Q'/L)(w)) = 0$.

As in the proof of Lemma 6.11, $L = \mathcal{X}_{0,p} \cdot Q$. We claim if w' is obtained from w by subtracting 1 from its first p coordinates, then the map $Q'(w') \to Q'(w)$ given by $a \mapsto \mathcal{X}_{0,p} \cdot a$ is an isomorphism of chain complexes. Lemma 6.12 gives the stated result when $c > 1$; while Lemma 6.11 gives the result when $c = 1$. Note that in this case, $z(\mathbf{x},\mathbf{x};w') = \mathcal{X}_{x_1,x_2} \cdots \mathcal{X}_{x_k,m}$, so the non-trivial homology class in $Q'(w)$ is represented by $\mathcal{X}_{0,x_1} \cdots \mathcal{X}_{x_k,m}$. ∎

Proof of Theorem 6.4. Condition (Q-1) was verified in Lemma 6.7; Condition (Q-2) is a combination of Lemma 6.9 (when the vector is non-constant) and Lemma 6.13 (when the vector is constant).

The fact that the homology has rank one in the remaining cases follows from Lemmas 6.13 and 6.13. ∎

Theorem 6.4 has the following quick corollary.

Corollary 6.14. *There is an isomorphism of algebras*

$$H_*(Q(m,m-1)) \cong \mathbb{F}[X_1,\dots,X_m,\Omega]/(X_i^2 = 0)_{i=1}^m. \tag{6.5}$$

Proof. Abbreviate $Q = Q(m, m-1)$ and $Q' = Q'(m, m-1)$. Clearly, X_i are homologically non-trivial elements in $H_*(Q')$. They satisfy $X_i^2 = 0$ (since they do so on the chain level). They also satisfy $[X_i][X_j] = [X_j] \cdot [X_i]$. When $j \neq i \pm 1$, the corresponding relation holds on the chain level; $[X_i][X_{i+1}] = [X_{i+1}][X_i]$, since

$$dX_{i-1,i+1} = X_i \cdot X_{i+1} + X_{i+1} X_i.$$

Thus, there is a tautological map of rings

$$\tau' : \mathbb{F}[X_1, \dots, X_m]/(X_i^2 = 0)_{i=1}^m \to H_*(Q').$$

If w is any weight vector with $\max(w_i) \leq 1$, then $Q'(w)$ is one-dimensional, by Theorem 6.4. (In the notation, we suppress idempotents now, since there is only one.) In fact, by Lemma 6.11 when w is non-constant, and Lemma 6.13 when w is constant, $H_*(Q'(w))$ is generated by the canonical cycle from Definition 6.10. Let w be a weight vector with $w_i \leq 1$. Let $I = \{i \in \{1, \dots, m\} \mid w_i = 1\}$. Let $\{n_1, \dots, n_\ell\}$ be the sequence of elements of I, arranged in decreasing order. It is straightforward to see that $\prod_{i=1}^{\ell} X_{n_i}$ is the canonical cycle of w.

Let $q : Q \to Q'$ denote the natural inclusion map. By construction, the cycles from $H_*(Q')$ come from cycles in $H_*(Q)$; thus q induces a surjection on homology. Thus, the long exact sequence of the mapping cone becomes the short exact sequence

$$0 \to H(Q) \xrightarrow{\Omega} H(Q) \to H(Q') \to 0.$$

The corollary follows readily. ∎

6.2. The Pong algebra

Given $\mathbf{x}$ and $\mathbf{y}$, let $Z(\mathbf{x}, \mathbf{y})$ denote the set of canonical cycles z for compatible triples $z(\mathbf{x}, \mathbf{y}; w)$, for various choices of w. Each canonical cycle is determined by its weight vector, which we can take to have $0 \leq w_i \leq 1$ for all i.

Definition 6.15. Let $(\mathbf{x}, \mathbf{y}; w)$ be a compatible triple, and suppose that $\mathbf{x}$ and $\mathbf{y}$ satisfy

$$\max(x_s, y_s) < \min(x_{s+1}, y_{s+1}). \tag{6.6}$$

The *selected weight vector of* $(\mathbf{x}, \mathbf{y}; w)$, denoted $s(z) \in \mathbb{Z}^{k+1}$, is given by the components

$$(w_0, w_{\max(x_1,y_1)}, w_{\max(x_2,y_2)}, \dots, w_{\max(x_k,y_k)}, w_m).$$

Lemma 6.16. *Fix two idempotent states* $\mathbf{x}$ *and* $\mathbf{y}$ *satisfying equation* (6.6)*, the map from* $Z(\mathbf{x}, \mathbf{y}) \to \mathbb{Z}^{k+1}$ *that maps* $z = z(\mathbf{x}, \mathbf{y}; w) \in Z(\mathbf{x}, \mathbf{y})$ *to the selected weight vector of* w *identifies* $Z(\mathbf{x}, \mathbf{y})$ *with the set of vectors* $\xi \in \mathbb{Z}^{k+1}$ *with* $\xi_i \in \{0, 1\}$*. Under this correspondence, the dimension of* z *is given by* $\xi_i \sigma_i$.

Proof. This is mostly straightforward. For the identification of the degree, note that the crossings in a canonical cycle $z = z(\mathbf{x}, \mathbf{y}; w)$ are in one-to-one correspondence with the non-zero components of the associated selected weight vector. ■

For $s = 1, \dots, k + 1$, let

$$V_s = \prod_{i=\max(x_{s-1}, y_{s-1})}^{\min(x_s, y_s)-1} v_i \in \mathbb{F}[v_1, \dots, v_m], \tag{6.7}$$

with the conventions that $x_0 = y_0 = 1$ and $x_{k+1} = y_{k+1} = m$.

Lemma 6.17. *The $\mathbb{F}[v_1, \dots, v_m]$-submodule of $P'(\mathbf{x}, \mathbf{y})$ generated by $Z(\mathbf{x}, \mathbf{y})$ is a subcomplex, which is quasi-isomorphic to $P'(\mathbf{x}, \mathbf{y})$. Moreover, its differential is specified by*

$$\partial \xi = \sum_{\{i | s_i = i\}} V_i \cdot (\xi - e_i),$$

where $e_i \in \mathbb{Z}^{k+1}$ is the ith basis vector.

Proof. Again, we use the fact that crossings in a canonical cycle z are in one-to-one correspondence with the non-zero components of the selected weight vector. It is also clear that resolving any of those crossings also gives a canonical cycle, times U_i. ■

Theorem 6.18. *The homology of $P'(\mathbf{x}, \mathbf{y}; w)$ is zero unless for each $s = 1, \dots, k - 1$,*

$$\max(x_s, y_s) < \min(x_{s+1}, y_{s+1}); \tag{6.8}$$

and in that case,

$$H_*(P'(\mathbf{x}, \mathbf{y}; w)) \cong \frac{\mathbb{F}[v_1, \dots, v_m]}{(V_1, \dots, V_{k+1})}. \tag{6.9}$$

Proof. If $\max(x_s, y_s) \geq \min(x_{s+1}, y_{s+1})$ for some s, then $H(Q'(\mathbf{x}, \mathbf{y})) = 0$ by Theorem 6.4. Now, $H(P'(\mathbf{x}, \mathbf{y}))$ has a filtration by U_i-powers, whose associated graded object is copies of $Q'(\mathbf{x}, \mathbf{y})$; so if the latter has trivial homology, then so does $H(P'(\mathbf{x}, \mathbf{y}))$.

Otherwise, the verification of equation (6.9) follows quickly from the model chain complex provided by Lemma 6.17. ■

Proof of Theorem 6.2. If $\mathbf{x}$ is an idempotent state (with k components), let

$$\mathbf{x}' = \{1, \dots, m - 1\} \setminus \mathbf{x}.$$

The condition that equation (6.8) holds for $s = 1, \dots, m - 1$ is equivalent to the condition that $\mathbf{x}'$ and $\mathbf{y}'$ are not too far, in the sense of Definition 3.2.

Moreover, the quotient by the ideal generated by the $(V_1, \dots, V_s)$ appearing in Theorem 6.18 corresponds exactly to the quotient described in Proposition 3.3, which gives $\mathbf{I}_{\mathbf{x}'} \cdot \mathcal{C}(m, m - k - 1) \cdot \mathbf{I}_{\mathbf{y}'}$. Thus, the theorem follows from Theorem 6.18. ■

7. A type *DD* bimodule

In Section 10, we establish a Koszul duality, identifying the cobar algebra of $\mathcal{C}(m,k)$ with the quotient of the pong algebra $\mathcal{Q}(m,k)$ (from equation (1.1)).

This duality is achieved by tensoring a type AA bimodule defined using holomorphic methods in Section 8 and a DD bimodule which is constructed using the following fairly straightforward algebraic considerations.

Let $\mathfrak{A}$ be the set of atomic generators for $\mathcal{Q}(m,k)$ (as in Definition 5.1). Let $f\colon \mathfrak{A} \to \mathcal{C}(m,k)$ be the map given by

$$f(L_{j,i}) = L_{i+1}\cdots L_j,\quad f(R_{i,j}) = R_j\cdots R_{i+1},\quad f(X_{i,j}\cdot \mathbf{I}_{\mathbf{x}}) = U_{i+1}\cdot\cdots U_j\cdot \cdot \mathbf{I}_{\mathbf{x}},$$

provided $X_{i,j}\cdot \mathbf{I}_{\mathbf{x}} \neq 0$; i.e., this is the map which maps an atomic generator to the element of $\mathcal{C}(m,k)$ with complementary left idempotent and the same weight.

In Section 5, we introduced certain atomic generators

$$\{X_{i,j}\}_{\substack{0\leq i<j\leq m\\ (i,j)\neq(0,m)}},\quad \{L_{j,i}\}_{1\leq i<j\leq m-1},\quad \{R_{i,j}\}_{1\leq i<j\leq m-1}.$$

These induce corresponding elements of $\mathcal{Q}(m,k)$.

Theorem 7.1. *There is a type DD bimodule* ${}^{\mathcal{C}(m,k)}\mathcal{X}^{\mathcal{Q}(m,k)}$ *with the following properties.*

- *$\mathcal{X}$ is isomorphic to $\mathcal{I}(m,k)$ as a $\mathcal{I}(m,k)$-bimodule; i.e., there is a set of generators of $\mathcal{X}$ which is in one-to-one correspondence with the idempotent states $\mathbf{x}$, $\gamma_{\mathbf{x}}$, so that*
$$(\mathbf{I}_{\mathbf{x}}\cdot\gamma_{\mathbf{x}}\cdot\mathbf{I}_{\mathbf{x}}) = \gamma_{\mathbf{x}}.$$
- *We have*
$$\delta^1(\gamma_{\mathbf{x}}) = \Bigg(\sum_{1\leq i<j\leq m-1} f(R_{i,j})\otimes\gamma_{\mathbf{x}}\otimes L_{j,i} + f(L_{j,i})\otimes\gamma_{\mathbf{x}}\otimes R_{i,j}\Bigg) + \Bigg(\sum_{0\leq i<j\leq m} f(X_{i,j})\otimes\gamma_{\mathbf{x}}\otimes X_{i,j}\Bigg). \tag{7.1}$$

More conceptually, in view of Proposition 5.3, we can interpret δ^1 as the sum over all atomic elements a of pong tensored with the element of $\mathcal{C}$ with the same weight as a.

Lemma 7.2. *If α and β are two atomic generators (for $\mathcal{Q}(m,k)$). Then, one of the following three properties holds:*

- $\alpha\cdot\beta = 0$,
- *There is another pair α' and β' of atomic generators with $\alpha\cdot\beta = \alpha'\cdot\beta'$,*

- *$\alpha \cdot \beta$ is of the form $R_{a,b} \cdot R_{b,c}$ or $L_{c,b} \cdot L_{b,a}$,*
- *$\alpha \cdot \beta$ is of the form $R_{b,c} \cdot R_{a,b}$, $L_{b,a} \cdot L_{c,b}$, $X_{a,b} \cdot X_{b,c}$, or $X_{b,c} \cdot X_{a,b}$.*

Proof. The atomic generators satisfy the following relations. For $a < b < c < d$,

$$\begin{aligned} X_{a,d} \cdot X_{b,c} &= X_{b,c} \cdot X_{a,d}, \\ X_{a,b} \cdot X_{c,d} &= X_{c,d} \cdot X_{a,b}. \end{aligned}$$

Also, for any $a \leq b \leq c$,

$$X_{a,b} \cdot X_{a,c} = X_{a,c} \cdot X_{a,b}, \tag{7.2}$$

$$X_{a,c} \cdot X_{b,c} = X_{b,c} \cdot X_{a,c}; \tag{7.3}$$

and indeed, both sides of equation (7.2) vanish if $0 < a$; similarly, both sides of equation (7.3) vanish if $c = m$. For $a < b < c < d$,

$$\begin{aligned} R_{a,b} \cdot R_{c,d} &= R_{c,d} \cdot R_{a,b}, \\ R_{a,c} \cdot R_{b,d} &= 0, \\ R_{b,d} \cdot R_{a,c} &= 0, \\ R_{a,d} \cdot R_{b,c} &= 0 = R_{b,c} \cdot R_{a,d}. \end{aligned}$$

($R_{a,c} \cdot R_{b,c} = 0$ because the juxtaposition contains a double-crossing; $R_{b,c} \cdot R_{a,c} = 0$ because of the idempotents). Corresponding identities hold with $L_{j,i}$ in place of $R_{i,j}$. For any $a \leq b \leq c$, we have the following:

$$\begin{aligned} R_{a,c} \cdot L_{b,a} &= X_{a,b} \cdot R_{b,c}, \\ L_{c,b} \cdot R_{a,c} &= R_{a,b} \cdot X_{b,c}, \\ R_{a,c} \cdot L_{b,a} &= X_{b,c} \cdot L_{b,a}, \\ R_{a,b} \cdot L_{c,a} &= L_{c,b} \cdot X_{a,b}. \end{aligned}$$

For any $a < b$, if $\xi_{a,b}, \eta_{a,b} \in \{X_{a,b}, R_{a,b}, L_{b,a}\}$, we have that

$$\xi_{a,b} \cdot \eta_{a,b} = 0.$$

(When ξ is of type R, this holds for idempotent reasons unless η is of type L, in which case it holds because of a double-crossing if $b > a + 1$. Moreover, $R_{a,a+1} \cdot L_{a+1,a} = U_{a+1}$ in $\mathcal{P}(m,k)$, therefore, the product vanishes in $Q(m,k)$.) ∎

Proof of Theorem 7.1. According to Lemma 7.2, the terms from $(\mu_2 \otimes \mathrm{Id} \otimes \mu_2) \circ (\mathrm{Id}_{\mathcal{C}} \otimes \delta^1 \otimes \mathrm{Id}_{\mathcal{P}}) \circ \delta^1$ have possibly non-zero terms of the following types:

- $\alpha \otimes \gamma_{\mathbf{x}} \otimes R_{a,b} \cdot R_{b,c}$; but in this case $\alpha = 0$,

- $\alpha \otimes \gamma_{\mathbf{x}} \otimes L_{c,b} \cdot L_{b,a}$; but in this case $a = 0$,
- terms of the form $\alpha \otimes \gamma_{\mathbf{x}} \otimes R_{b,c} \cdot R_{a,b}$,
- terms of the form $\alpha \otimes \gamma_{\mathbf{x}} \otimes L_{b,a} \cdot L_{c,b}$,
- terms of the form $\alpha \otimes \gamma_{\mathbf{x}} \otimes (X_{a,b} \cdot X_{b,c} + X_{b,c} \cdot X_{a,b})$.

These terms, in turn, are those that appear in $(\mathrm{Id} \otimes \mathrm{Id} \otimes \partial) \circ \delta^1$. (See Lemma 5.4.) ∎

7.1. Gradings

Endow $\mathcal{Q}$ with the $\mathbb{Z}$-grading $\mathbf{N}$ inherited from the $\mathbb{Z}$-grading on $\mathcal{P}(m,k)$ specified in equation (4.9).

Proposition 7.3. *Consider the type DD bimodule $X = {}^{\mathcal{C}}X^{\mathcal{Q}}$ constructed above. Suppose that there are generators x and y so that $\delta^1(x)$ contains $a \otimes y \otimes b$ with non-zero multiplicity. Then, $\mathfrak{w}(a) = \mathfrak{w}(b)$ and $\mathbf{N}(b) = -1$.*

Proof. This is straightforward from the definitions. ∎

In the language of Section 2.2, the above proposition states that X is a Maslov–Alexander graded type DD bimodule supported in grading 0, where $C = \mathcal{C}(m,k)$ is given the trivial $\mathbb{Z}$-grading and $\frac{1}{2}\mathbb{Z}^m$-grading specified by the weight vector; and $\mathcal{Q}$ is given the $\mathbb{Z}$-grading $\mathbf{N}$ and $\frac{1}{2}\mathbb{Z}^m$-grading given by -1 times the weight vector.

8. Holomorphic aspects

Fix integers m and k with $0 < k < m$.

We endow $\mathcal{C}(m,k)$ with a trivial grading—every element is supported in grading 0. Endow $\mathcal{Q}(m,k)$ with the renormalized grading $\mathbf{N}(a) = \#\mathrm{cross}(a) - 2(\sum_i \mathfrak{w}_i(a))$; while $\mathcal{C}(m,k)$ is thought of as having the trivial (i.e., vanishing) $\mathbb{Z}$-grading.

Our aim is to prove the following theorem.

Theorem 8.1. *There is a Maslov–Alexander bigraded, strictly unital type AA bimodule ${}_{\mathcal{Q}(m,k)}\mathcal{Y}_{\mathcal{C}(m,k)}$ with the following properties.*

(Y-1) *$\mathcal{Y}$ is isomorphic to $\mathcal{I}(m,k)$ as an $\mathcal{I}(m,k)$-bimodule; i.e., there is a set of generators of $\mathcal{Y}$ which is in one-to-one correspondence with the idempotent states $\mathbf{x}$, so that*
$$\mathbf{I}_{\mathbf{x}} \cdot Y_{\mathbf{x}} \cdot \mathbf{I}_{\mathbf{x}} = Y_{\mathbf{x}}.$$

(Y-2) *If $i \in \mathbf{x}$ and $i+1 \notin \mathbf{x}$, let $\mathbf{x}' = (\mathbf{x} \cup \{i+1\}) \setminus \{i\}$, then*
$$m_{1|1|1}(R_i, Y_{\mathbf{x}'}, L_i) = Y_{\mathbf{x}} \quad \textit{and} \quad m_{1|1|1}(L_i, Y_{\mathbf{x}}, R_i) = Y_{\mathbf{x}'}.$$

(Y-3) *If* $\{i, i+1\} \subset \mathbf{x}$, *then*

$$m_{1|1|1}(X_i, Y_{\mathbf{x}}, U_i) = Y_{\mathbf{x}}.$$

(Y-4) *If* $1 \in \mathbf{x}$, *then*

$$m_{1|1|1}(X_1, Y_{\mathbf{x}}, U_1) = Y_{\mathbf{x}}.$$

(Y-5) *If* $m-1 \in \mathbf{x}$, *then*

$$m_{1|1|1}(X_m, Y_{\mathbf{x}}, U_m) = Y_{\mathbf{x}}.$$

(Y-6) *The bimodule* $\mathcal{Y}$ *is supported in grading* $\mathrm{gr} = 0$ *and* $\vec{v} = 0$.

See equations (2.2) and (2.3) for the grading conventions on bimodules used here.

Remark 8.2. We have assumed throughout that $0 < k < m$. In the case where $k = 0$, $\mathcal{C}(m, 0) \cong \mathbb{F}$, and $U_i = 0$, so the above statement does not make any sense. Although the following discussion does work $k = m - 1$, in the applications, we will not need Theorem 8.1 in that case; see Section 12.1.

The above bimodule is constructed as follows. First, we construct bimodules associated to certain types of Heegaard diagrams by suitably adapting bordered methods (cf., [10, 20]). The bimodule is then associated to a simple "half-identity" Heegaard diagram (in the spirit of [8]). The verification of the properties from Theorem 8.1 is fairly routine.

Most of this section is devoted to importing bordered methods; the proof of Theorem 8.1 is given at the end of this section.

8.1. Lifted partial permutations and mirror-matched circles

Definition 8.3. Fix an integer $m \geq 2$. Let P be a set of $2m - 2$ points in the circle, equipped with a fixed-point free involution M (called a matching) that extends to an orientation-reversing involution of the circle. The data (S^1, P, M) is called a *mirror-matched circle* with $m - 1$ pairs.

Definition 8.4. A path $\rho\colon [0, 1] \to S^1$ is called *Reeb-like* if its local degrees away from $\rho(0)$ and $\rho(1)$ are non-negative. We view two Reeb-like paths as equivalent if they are homotopic relative to their endpoints. The initial point $\rho(0)$ is denoted ρ^- and the terminal point $\rho(1)$ is denoted ρ^+. A set $\boldsymbol{\rho}$ of Reeb-like paths is called *consistent* if for any two distinct $\rho_1, \rho_2 \in \boldsymbol{\rho}$, the set $\{\rho_1^-, \rho_2^-, M(\rho_1^-), M(\rho_2^-)\}$ consists of four (distinct) points; and also $\{\rho_1^+, \rho_2^+, M(\rho_2^+), M(\rho_2^+)\}$ consists of four points. (Note that if $\boldsymbol{\rho}$ consists of a single Reeb-like path, then it is automatically consistent.) Finally, if $\boldsymbol{\rho}$ is a set of Reeb-like paths, let $\boldsymbol{\rho}^-$, resp., $\boldsymbol{\rho}^+$ denote the set of initial, resp,. terminal points of each path in $\boldsymbol{\rho}$,

Definition 8.5. Let (S^1, P, M) be a mirror-matched circle with $m-1$ pairs. For each integer $1 \le k \le m-1$, an *M-consistent constraint packet* consists of a pair of subsets $S, T \subset P/M$ and a consistent set $\boldsymbol{\rho}$ of Reeb-like paths $\boldsymbol{\rho}$ with the following properties:

- $\boldsymbol{\rho}^-/M \subset S$,
- $\boldsymbol{\rho}^+/M \subset T$,
- $|S| = |T| = k$,
- $S \setminus (\rho^-/M) = T \setminus (\boldsymbol{\rho}^+/M)$.

We write an explicit model for a mirror-matched circle, as follows. The circle will be thought of as the quotient $\mathbb{R}/2m\mathbb{Z}$, the involution r is induced by the map $x \mapsto -x$ (thought of as in involution of $\mathbb{R}$), and

$$P = \left\{i + \frac{1}{2}\right\}_{i=-m}^{m-1}.$$

For the above parametrization of the mirror-matched circle, a set of homotopy classes of Reeb-like paths corresponds to a subset $S \subset P$ with $S \cap -S = \emptyset$, and a function $f\colon S \to \frac{1}{2} + \mathbb{Z}$, with the property that $s \le f(s)$ for all $s \in S$.

Let $(\widetilde{S}, \tilde{f})$ be a lifted partial permutation. There is a corresponding M-consistent constraint packet defined as follows:

- $S \subset \widetilde{S}/2m\mathbb{Z}$ is the image of those $\sigma \in \widetilde{S}$ with $\sigma \le \tilde{f}(\sigma)$,
- $T = \tilde{f}(\widetilde{S}/2m\mathbb{Z})$ consists of the equivalence class of those $\tilde{f}(\sigma) \in \mathbb{R}$ with the property that $\sigma \le \tilde{f}(\sigma)$.

Proposition 8.6. *The above map sets up a one-to-one correspondence between lifted partial permutations on k letters and M-consistent constraint packets.*

Proof. The argument is straightforward, amounting to the observation that $\tilde{f}(\sigma) < \sigma$ if and only if $\tilde{f}(-\sigma) > -\sigma$. ∎

On the level of pong diagrams, the above one-to-one correspondence can be described as follows. Start from a pong diagram for a lifted partial permutation, and reflect it once (to obtain two fundamental domains for the G_m action). Next, remove all the strands that are moving down and to the left. Finally, view the resulting picture as taking place on $S^1 \times [0, 1]$ (where $S^1 = \mathbb{R}/2m\mathbb{Z}$). See Figure 14.

Definition 8.7. If $\boldsymbol{\sigma}$ is an M-consistent constraint packet, and $|\boldsymbol{\sigma}| \le k$, we define the corresponding pong algebra element $a(\boldsymbol{\sigma})$ as follows. When $|\boldsymbol{\sigma}| = k$, let $a(\boldsymbol{\sigma})$ be the element of $\mathcal{P}(m, k)$ associated to the corresponding lifted partial permutation as in Proposition 8.6. When $|\boldsymbol{\sigma}| < k$, we sum over all lifted partial permutations obtained by extending by the identity map the given lifted partial permutations.

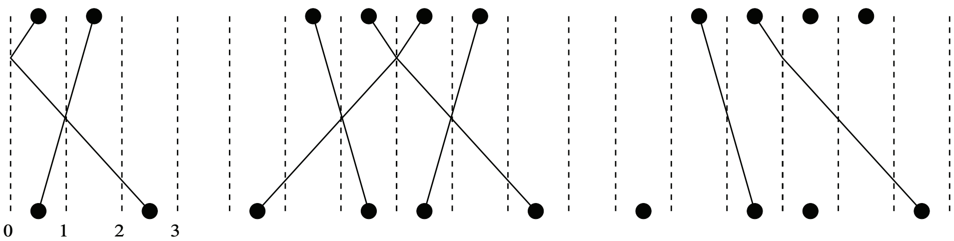


Figure 14. *Pong diagrams to Reeb-like chord packets.*

8.2. α-β bordered diagrams

Definition 8.8. An *α-β-bordered diagram $\mathcal{H}$* consists of the following data.

- An oriented surface Σ with genus g, whose boundary is decomposed as a disjoint union
$$\partial\Sigma = \partial_\alpha \coprod \partial_\beta.$$
- A collection $\{\alpha_1, \dots, \alpha_{m-1}\}$ of α-arcs whose boundaries lie on ∂_α.
- A collection $\{\beta_1, \dots, \beta_{m-1}\}$ of β-arcs whose boundaries lie on ∂_β.
- A collection $\{\alpha_i^c\}_{i=1}^g$ and $\{\beta_i^c\}_{i=1}^g$ of pairwise disjoint, embedded circles.
- The boundary ∂_α consists of a disjoint union of circles $Z_1, \dots, Z_m$, labeled so that α_i^c is a path from Z_i to Z_{i+1}.
- The boundary ∂_β consists of a single mirror-matched circle Z, where the matching is given by the endpoints of the β_i arcs.

This data is also required to satisfy the following compatibility condition. If we cut Σ along all the β-circles and arcs, we obtain m components $B_1 \cup \cdots \cup B_m$ with the following properties.

- The B_i has a boundary component β_{i-1} (unless $i = 1$, in which case this boundary component is empty) and another boundary component β_i (unless $i = m$, in which case this boundary component is empty).
- The boundary ∂B_i meets Z in one arc if $i \in \{1, m\}$, and it meets Z in two arcs otherwise.
- B_i meets exactly one component of ∂_α.

The boundary component ∂_β is the "pong-bordered" portion of the boundary of $\mathcal{H}$ mentioned in the introduction.

The key example of an α-β-bordered diagram $\mathcal{H}$—indeed, the only one we consider in this paper—is the *half-identity diagram $\mathcal{H}_m$*, indexed by the integer m, defined as follows. $\mathcal{H}_m$ has genus 0 and $m + 1$ boundary components, as pictured in Figure 15.

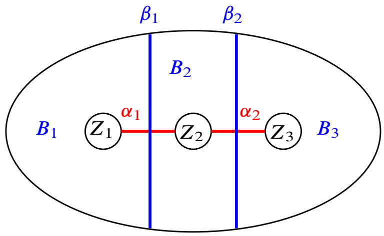


Figure 15. Half identity diagram with $m = 3$.

Definition 8.9. The *half-identity diagram* $\mathcal{H}$ is obtained from a genus 0 surface with $m + 1$ boundary components, one of which is ∂_β; and the remaining m boundary components are labeled $\partial^1_\alpha, \dots, \partial^m_\alpha$. Draw an arc α_i from ∂^i_α to ∂^{i+1}_α, and let β_i be an arc from ∂_β to itself that is disjoint from all α_j with $i \neq j$, intersecting α_i in a single transverse point.

It follows from the definitions that to each α-β-bordered diagram $\mathcal{H}$, there is a permutation σ on m letters with the property that B_i meets $Z_{\sigma(i)}$. For the half-identity, this permutation is the identity.

8.3. Heegaard states for bordered diagrams

Definition 8.10. Let $\mathcal{H}$ be an α-β-bordered Heegaard diagram as in Definition 8.8. Fix an integer $0 \leq k < m$. A *k-fold Heegaard state* consists of the following data.

- A k-element subset $S \subset \{1, \dots, m-1\}$.
- A k-element subset T of $\{1, \dots, m-1\}$.
- A $g + k$-tuple of points $\mathbf{x}$

satisfying the following conditions:

- $\mathbf{x} \subset (\bigcup_{s \in S} \alpha_s) \cup (\bigcup_{i=1}^g \alpha_i^c)$,
- $\mathbf{x} \subset (\bigcup_{t \in T} \beta_t) \cup (\bigcup_{i=1}^g \beta_i^c)$,
- each α-circle contains exactly one component of $\mathbf{x}$,
- each β-circle contains exactly one component of $\mathbf{x}$,
- for each $s \in S$, the corresponding α-arc α_s contains exactly one component of $\mathbf{x}$,
- each $t \in T$, the corresponding β-arc β_t contains exactly one component of $\mathbf{x}$.

Note that S and T are determined by $\mathbf{x}$. Let $\mathbf{S}(\mathcal{H})$ denote the set of Heegaard states of $\mathcal{H}$.

We associate to $\mathcal{H}$ a type AA-bimodule ${}_{\mathcal{Q}(m,k)}\widehat{\mathrm{CFPC}}(\mathcal{H})_{\mathcal{C}(m,k)}$.

As a $\mathbb{Z}/2\mathbb{Z}$-vector space, $\widehat{\mathrm{CFPC}}(\mathcal{H})$ is generated by Heegaard states. The $\mathcal{I}(m,k)$-bimodule structure is specified by

$$\mathbf{I}_S \cdot \mathbf{x} \cdot \mathbf{I}_T = \mathbf{x},$$

where the sets $S, T \subset \{1, \dots, m-1\}$ are as in Definition 8.10.

Actions on the module $\widehat{\mathrm{CFPC}}(\mathcal{H})$ are defined by counting holomorphic disks, modifying the definition from [20] slightly to take into account the ∂_β boundary. (See also [8, 9].) We pause our definition now to describe elements of the holomorphic theory, returning to the definition afterwards.

8.4. Pseudo-holomorphic curves

Following [7], we consider pseudo-holomorphic curves in $\Sigma \times [0,1] \times \mathbb{R} \to \Sigma$ with respect to an admissible almost-complex structure [20, Section 5].

We consider J-holomorphic curves

$$u\colon (\mathcal{S}, \partial\mathcal{S}) \to (\Sigma \times [0,1] \times \mathbb{R}, (\boldsymbol{\alpha} \times \{1\} \times \mathbb{R}) \cup (\boldsymbol{\beta} \cup \{0\} \times \mathbb{R})),$$

satisfying the asymptotic conditions of a pre-flowline (cf., [20, Definition 5.4]), with a few minor modifications: now there will not be any punctures marked by Reeb orbits, and there will be constraints on both $\partial_\alpha \mathcal{H}$ and $\partial_\beta \mathcal{H}$.

We outline the construction in a little more detail presently.

Definition 8.11. A *β-constraint packet* $\boldsymbol{\sigma}$ is a union of Reeb chords on $\partial_\beta \mathcal{H}$. The constraint packet is called *algebraic* if

- for any pair σ_a and σ_b of distinct chords in $\boldsymbol{\sigma}$, the two initial points σ_a^- and σ_b^- lie on distinct β-arcs,
- for any pair σ_a and σ_b of distinct chords, the two terminal points σ_a^+ and σ_b^+ lie on distinct β-arcs.

An *α-constraint packet* $\boldsymbol{\rho}$ is defined similarly. In that case, the constraint packet is called *algebraic* if the following third condition is also met.

- Any two distinct chords in $\boldsymbol{\rho}$ lie on distinct boundary components.

Definition 8.12. A *profile* is a triple $(\mathbf{x}, \vec{\boldsymbol{\rho}}, \vec{\boldsymbol{\sigma}})$ consisting of

- a Heegaard state $\mathbf{x}$,
- a sequence of α-constraint packets $\vec{\boldsymbol{\rho}} = \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_r$,
- a sequence of β-constraint packets $\vec{\boldsymbol{\sigma}} = \boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_s$.

A profile is called *boundary monotone*, in which case we also define its α-set, denoted $\alpha(\mathbf{x}, \vec{\boldsymbol{\rho}})$ (which is independent of the $\vec{\boldsymbol{\sigma}}$); and its β-set, denoted $\beta(\mathbf{x}, \vec{\boldsymbol{\sigma}})$, if the following conditions hold:

- $\alpha(\mathbf{x}) \supset \alpha(\boldsymbol{\sigma}_1^-)$,

- $\beta(\mathbf{x}) \supset \beta(\boldsymbol{\rho}_1^-)$,
- for $i = 1, \dots, s-1$, the initial points $\boldsymbol{\sigma}_i^-$ all lie on distinct α-arcs, the terminal points α_i^+ all lie on distinct β-arcs, and

$$\alpha(\boldsymbol{\sigma}_i^-) \subset \alpha(\mathbf{x}, \{\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_{i-1}\}),$$

  in this case, we define the α-set to be

$$\alpha(\mathbf{x}, \{\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_i\}) = \big(\alpha(\mathbf{x}, \{\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_{i-1}\}) \setminus \boldsymbol{\sigma}_i^-\big) \cup \boldsymbol{\sigma}_i^+,$$

- for $i = 1, \dots, r-1$, the initial points $\boldsymbol{\rho}_i^-$ all lie on distinct β-arcs, the terminal points $\boldsymbol{\rho}_i^+$ all lie on distinct β-arcs, and

$$\beta(\rho_i^-) \subset \beta(\mathbf{x}, \{\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_{i-1}\}),$$

  in this case, we define the β-set to be

$$\beta(\mathbf{x}, \{\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_i\}) = \big(\beta(\mathbf{x}, \{\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_{i-1}\}) \setminus \boldsymbol{\rho}_i^-\big) \cup \boldsymbol{\rho}_i^+.$$

Note that for a boundary monotone profile $(\mathbf{x}, (\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_s), (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_r))$, all the chord packets $\boldsymbol{\sigma}_i$ are algebraic, in the sense of Definition 8.11, and we can define corresponding algebra elements $a(\boldsymbol{\sigma}_i)$ as in Definition 8.7. The packets $\boldsymbol{\sigma}_i$, however, need not be algebraic in the sense of Definition 8.11 (though we will often require them to be). When they are, we can define algebra elements $a(\boldsymbol{\rho}_i)$ whose weights coincide with the Reeb chords.

The map from algebraic β-constraint packets to $\mathcal{C}(m,k)$ and the map from algebraic α-constraint packets to $\mathcal{Q}(m,k)$ are both denoted a; however, the distinction should be clear from the context. Also note that the map from algebraic chord packets to elements of $\mathcal{Q}(m,k)$ is injective; whereas for $\mathcal{C}(m,k)$, the corresponding map is not injective. For example, the algebra element $U_i \mathbf{I}_{\mathbf{x}}$ when $i-1, i \in \mathbf{x}$, can be represented by either length 1 chord which covers Z_i.

Definition 8.13. Fix an α-β bordered diagram $\mathcal{H}$ as in Definition 8.8, and an integer k with $0 \le k \le m-1$. A *decorated source* $\mathcal{S}$ is the following collection of data:

- a smooth oriented surface S with boundary and punctures on the boundary,
- a labeling of each puncture of $\mathcal{S}$ by one of $+$, $-$, or a Reeb chord on $\partial\mathcal{H}$.

Let $P(\mathcal{S})$, resp., $Q(\mathcal{S})$ denote the set of punctures marked by chords on $\partial_\alpha\mathcal{H}$, resp., $\partial_\beta\mathcal{H}$.

Definition 8.14. A *pre-flowline* is a map

$$u\colon (\mathcal{S}, \partial\mathcal{S}) \to (\Sigma \times [0,1] \times \mathbb{R}, (\boldsymbol{\alpha} \times \{1\} \times \mathbb{R}) \cup (\boldsymbol{\beta} \times \{0\} \times \mathbb{R}))$$

subject to the constraints:

($\mathcal{M}$-1) $u\colon \mathcal{S} \to \Sigma \times [0,1] \times \mathbb{R}$ is proper,

($\mathcal{M}$-2) u extends to a proper map $\bar{u}\colon \overline{\mathcal{S}}' \to \overline{\Sigma} \times [0,1] \times \mathbb{R}$, where $\overline{\mathcal{S}}'$ is obtained from $\mathcal{S}$ by filling in the punctures labeled by Reeb chords,

($\mathcal{M}$-3) $\pi_{\mathbb{D}} \circ u$ is a d-fold branched cover, where $d = g + k$,

($\mathcal{M}$-4) At each $-$-puncture q of $\mathcal{S}$, $\lim_{z \mapsto q}(t \circ u)(z) = -\infty$,

($\mathcal{M}$-5) At each $+$-puncture q of $\mathcal{S}$, $\lim_{z \mapsto q}(t \circ u)(z) = +\infty$,

($\mathcal{M}$-6) For each $p \in P(\mathcal{S}) \cup Q(\mathcal{S})$, $\lim_{z \to q}(\pi_\Sigma \circ u)(z)$ is the Reeb chord ρ labeling p,

($\mathcal{M}$-7) There are generalized Heegaard states $\mathbf{x}$ and $\mathbf{y}$ with the property that as $t \mapsto -\infty$, $\pi_\Sigma \circ u$ is asymptotic to $\mathbf{x}$ and as $t \mapsto +\infty$, $\pi_\Sigma \circ u$ is asymptotic to $\mathbf{y}$,

($\mathcal{M}$-8) For each $t \in \mathbb{R}$ and $i = 1, \dots, g$, $u^{-1}(\beta_i^c \times \{0\} \times \{t\})$ consists of exactly one point; and also $u^{-1}(\alpha_i^c \times \{1\} \times \{t\})$ consists of exactly one point,

($\mathcal{M}$-9) For each $t \in \mathbb{R}$ and $i = 1, \dots, m$, $u^{-1}(\alpha_i \times \{1\} \times \{t\})$ consists of at most one point; and also $u^{-1}(\beta_i \times \{0\} \times \{t\})$ consists of at most one point.

If u is a pre-flowline, the *associated profile* is the data consisting of the initial state $\mathbf{x}$, and the sequences $\vec{\boldsymbol{\sigma}}$, resp., $\vec{\boldsymbol{\rho}}$, consisting of the images of the $P(\mathcal{S})$, resp., $Q(\mathcal{S})$, ordered according to the value of $t \circ u$.

Fix the following data:

- $\mathbf{x}, \mathbf{y} \in \mathbf{S}(\mathcal{H})$,
- compatible sequences of constraint packets $\vec{\boldsymbol{\rho}}$ and $\vec{\boldsymbol{\sigma}}$,
- a homology class class $B \in \pi_2(\mathbf{x}, \mathbf{y})$,

and let $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \mathcal{S}, \vec{\boldsymbol{\rho}}, \vec{\boldsymbol{\sigma}})$ be the moduli space of holomorphic representatives of B whose asymptotics are as specified by the profile $(\mathbf{x}, \vec{\boldsymbol{\rho}}, \vec{\boldsymbol{\sigma}})$.

8.5. A digression on the index

We collect here properties of the index of holomorphic curves. This discussion is mostly an adaptation of [10, Section 5.7.1].

Following [10] (see also [20, Section 7]), for a packet $\boldsymbol{\rho}$ of Reeb chords, let $\iota(\boldsymbol{\rho}) = \mathrm{inv}(\boldsymbol{\rho}) - m([\boldsymbol{\rho}], \boldsymbol{\rho}^-)$, $\mathrm{inv}(\boldsymbol{\rho})$ denotes the minimal number of crossings between the various chords (i.e., this coincides with $\mathrm{inv}(\boldsymbol{\rho})$ when $\boldsymbol{\rho}$ represents a lifted partial permutation on a mirror matched circle), and $m([\boldsymbol{\rho}], \boldsymbol{\rho}^-)$ denotes the sum over all the initial points p of the chords $\boldsymbol{\rho}^-$ of the average local multiplicity of $[\boldsymbol{\rho}]$ near p. For example, if the packet consists of the single chord $\boldsymbol{\rho} = \{\rho\}$, then $-\iota(\boldsymbol{\rho})$ is the total weight of the chord.

Suppose that $\mathbf{x}$ and $\mathbf{y}$ are two states, and $(B, \vec{\rho} = (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_r), \vec{\boldsymbol{\sigma}} = (\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_s))$ is strongly boundary monotone. Let $|\vec{\rho}| = r$ and $|\vec{\boldsymbol{\sigma}}| = s$. Define

$$\chi_{\mathrm{emb}}(B, \vec{\boldsymbol{\rho}}, \vec{\boldsymbol{\sigma}}) = d + e(B) - n_{\mathbf{x}}(B) - n_{\mathbf{y}}(B) - \iota(\vec{\boldsymbol{\rho}}) - \iota(\vec{\boldsymbol{\sigma}}), \tag{8.1}$$

$$\mathrm{ind}(B, \mathbf{x}, \mathbf{y}; \vec{\boldsymbol{\rho}}) = e(B) + n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B) + |\vec{\boldsymbol{\rho}}| + |\vec{\boldsymbol{\sigma}}| + \iota(\vec{\boldsymbol{\rho}}) + \iota(\vec{\boldsymbol{\sigma}}), \tag{8.2}$$

where $\iota(\vec{\boldsymbol{\rho}}) = \sum_{i=1}^{\ell} \iota(\boldsymbol{\rho}_i)$.

Remark 8.15. Observe that constraint packets $\boldsymbol{\rho}_i$ in $\vec{\boldsymbol{\rho}} = (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_r)$ that we consider here consist entirely of chords; whereas in [20, Definition 7.9], the formulas allowed for additional orbit constraints. Also, when comparing with the formulas from [20], bear in mind that the Euler measure there is taken on the compactification of the Heegaard surface; this is why the total weight on the boundary appears there.

The following proposition is proved in [10, Proposition 5.69] [7, Proposition 4.2]; compare also [20, Proposition 7.11].

Proposition 8.16. *If $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \mathcal{S}, \vec{\boldsymbol{\rho}}, \vec{\boldsymbol{\sigma}})$ is represented by some pseudo-holomorphic representative u, then $\chi(\mathcal{S}) = \chi_{\mathrm{emb}}(B)$ if and only if u is embedded. In this case, the expected dimension of the moduli space is given by* $\mathrm{ind}(B, \mathbf{x}, \mathbf{y}, \vec{\boldsymbol{\rho}}, \vec{\boldsymbol{\sigma}})$. *Moreover, if a strongly monotone moduli space has a non-embedded holomorphic representative, then its expected dimension is* $\leq \mathrm{ind}(B, \mathbf{x}, \mathbf{y}\vec{\boldsymbol{\rho}}, \vec{\boldsymbol{\sigma}}) - 2$.

Proposition 8.17. *If $(\mathbf{x}, \vec{\boldsymbol{\rho}}, \vec{\boldsymbol{\sigma}})$ is a profile with the property that each of the $\boldsymbol{\sigma}_i$ and $\boldsymbol{\rho}_i$ are algebraic, then the profile is boundary monotone if and only if*

$$a(\boldsymbol{\rho}_s) \otimes \cdots \otimes a(\boldsymbol{\rho}_1) \otimes \mathbf{x} \otimes a(\boldsymbol{\sigma}_1) \otimes \cdots \otimes a(\boldsymbol{\sigma}_r) \neq 0$$

in $\mathcal{Q}(m,k)^{\otimes s} \otimes \mathcal{Y}(\mathcal{H}) \otimes \mathcal{C}(m,k)^{\otimes r}$, where all tensor products are taken over $\mathcal{I}(m,k)$. Moreover, in this case,

$$e(B) + n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B) = \mathrm{ind}(B, \vec{\boldsymbol{\rho}}, \vec{\boldsymbol{\sigma}}) - r - s + \iota(\vec{\boldsymbol{\rho}}) + \iota(\vec{\boldsymbol{\sigma}}).$$

Proof. This follows as in [20, Lemma 8.7]. The equation is a straightforward consequence of equation (8.2). ∎

8.6. Codimension one ends

Following [10, Theorem 5.61], we name various kinds of codimension one ends of this moduli space.

- A *two-story end* corresponds to a decomposition

  $$\mathcal{M}^{B_1}(\mathbf{x}, \mathbf{p}; \mathcal{S}_1; \vec{\boldsymbol{\rho}}_1, \vec{\boldsymbol{\sigma}}_1) \times \mathcal{M}^{B_2}(\mathbf{x}, \mathbf{p}; \mathcal{S}_2; \vec{\boldsymbol{\rho}}_2, \vec{\boldsymbol{\sigma}}_2),$$

  where $\mathbf{p} \in \mathbf{S}(\mathcal{H})$, $B_1 \in \pi_2(\mathbf{x}, \mathbf{p})$, $B_2 \in \pi_2(\mathbf{p}, \mathbf{y})$, $B = B_1 + B_2$.

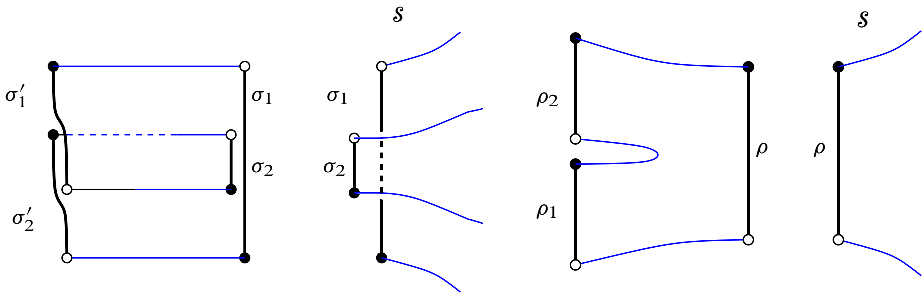


Figure 16. *Some codimension one ends.* On the left, we have illustrated an odd shuffle curve end; on the right, a join curve end.

- An *odd shuffle curve end* is an element of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \mathcal{S}', \vec{\boldsymbol{\rho}}', \vec{\boldsymbol{\sigma}}')$ obtained by picking two punctures q_1 and q_2 whose corresponding Reeb chords σ_1 and σ_2 (on $\partial_\beta \mathcal{H}$) are constrained in the same packet and nested. Here, $\mathcal{S}'$ is obtained by pre-gluing an odd shuffle component with punctures q_1' and q_2', labeled by interleaved Reeb chords σ_1' and σ_2'. Here, $\mathbf{P}'$ is obtained from $\mathbf{P}$ by replacing the chords σ_1 and σ_2 by σ_1' and σ_2'; while $\mathbf{Q}' = \mathbf{Q}$. (Note that there is no space on the cylinders attached to $\partial_\alpha \mathcal{H}$ for shuffle curves, as in [20].) See the left of Figure 16 for a schematic.
- A *join curve end* $\mathcal{M}(\mathbf{x}, \mathbf{y}; \mathcal{S}', \vec{\boldsymbol{\rho}}', \vec{\boldsymbol{\sigma}}')$ is determined by choosing a Reeb chord label ρ (which can be on $\partial_\alpha \mathcal{H}$ or $\partial_\beta \mathcal{H}$) for one of the punctures q of $\mathcal{S}$ and a decomposition $\sigma = \sigma_1 \uplus \sigma_2$. The decorated source $\mathcal{S}'$ has two new punctures q_1 and q_2, so that $\mathcal{S}$ is obtained from $\mathcal{S}'$ by pregluing a join component to $\mathcal{S}'$ at the punctures q_1 and q_2. The sequences of constraint packets for $\vec{\boldsymbol{\rho}}'$ and $\vec{\boldsymbol{\sigma}}'$ are obtained from $\vec{\boldsymbol{\rho}}$ and $\vec{\boldsymbol{\sigma}}$ by replacing a single copy of ρ by the two chords ρ_1 and ρ_2. See the right of Figure 16 for a schematic.
- A *collision level* is an element of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \mathcal{S}', \vec{\boldsymbol{\rho}}', \vec{\boldsymbol{\sigma}}')$ obtained by contracting the arcs on $\partial \mathcal{S}$ that connect punctures whose labels lie in two consecutive packets ($\boldsymbol{\rho}_i$ and $\boldsymbol{\rho}_{i+1}$ of $\vec{\boldsymbol{\rho}}'$ or $\boldsymbol{\sigma}_i$ and $\boldsymbol{\sigma}_{i+1}$ of $\vec{\boldsymbol{\sigma}}$), and replacing the constraint packet by the one obtained by forming the join of those two constraint packets.

Theorem 8.18. *If* $(\mathbf{x}, \vec{\boldsymbol{\rho}}, \vec{\boldsymbol{\sigma}})$ *is strongly boundary monotone and* $B \in \pi_2(\mathbf{x}, \mathbf{y})$*. Fix* $\mathcal{S}$ *so that* $\operatorname{ind}(B, \mathcal{S}, \vec{\boldsymbol{\rho}}, \vec{\boldsymbol{\sigma}}) = 2$*. Then, the total number of ends of* $\mathcal{M} = \mathcal{M}^B(\mathbf{x}, \mathbf{y}; \mathcal{S}; \vec{\boldsymbol{\rho}}, \vec{\boldsymbol{\sigma}})$ *of the following kinds is even:*

- *two-story ends of* $\mathcal{M}$*,*
- *join curve ends of* $\mathcal{M}$*,*
- *odd shuffle curve ends on* $\mathcal{M}$*,*
- *collision of levels of* i *and* $i+1$*, in which case the packets in the colliding levels are weakly composable.*

Proof. This follows as in [10, Theorem 5.61], though a few remarks are in order. The codimension-one phenomena on the β-side are those listed above, as in [10, Theorem 5.61]. As in that case, there are also possibly even shuffle-chord ends, but those cancel in pairs. (See [10, Proposition 5.42].)

There are also possibly codimension-one phenomena on the α-side, which are enumerated in Theorem [20, Theorem 7.17]. In that statement, there were additional curves we did not consider here—the "orbit curve ends" from [20, Theorem 7.17]. These do not appear, however, since the constraint packets we consider here do not contain Reeb orbits.

Boundary degenerations appearing as limiting objects cannot be ruled out using the same logic as in [10]: the β-arcs are not homologically linearly independent. However, our hypotheses on $\mathcal{H}$ (Definition 8.8) ensure that all the regions which are obtained by cutting along β-arcs (i.e., which support β-boundary degenerations) also meet the α-boundary. Thus, β-boundary degenerations occur in moduli spaces which contain closed Reeb orbits on that α-side; but these are not moduli spaces we consider presently. A simpler argument also rules out α-boundary degenerations. ∎

8.7. Actions on the bimodule

Let $\mathcal{H}$ be an α-β-bordered diagram, and let ${}_{\mathcal{Q}(m,k)}\widehat{\mathrm{CFPC}}(\mathcal{H})_{\mathcal{C}(m,k)}$ be its associated $\mathcal{I}(m,k)$-bimodule. Fix a sequence $\vec{a} = a_1, \dots, a_r$ of lifted partial permutations representing algebra elements in $\mathcal{Q}(m,k)$ and a sequence

$$\vec{b} = b_1, \dots, b_s \in \mathcal{C}(m,k)$$

of pure algebra elements. Write

$$\vec{a} \otimes \mathbf{x} \otimes \vec{b} = a_1 \otimes \cdots \otimes a_r \otimes \mathbf{x} \otimes b_1 \otimes \cdots \otimes b_s.$$

If $\vec{a} \otimes \mathbf{x} \otimes \vec{b} = 0$, define

$$m_{r|1|s}(a_1 \otimes \cdots \otimes a_r \otimes \mathbf{x} \otimes b_1 \otimes \cdots \otimes \otimes b_s) = 0.$$

Otherwise, if $\vec{\boldsymbol{\rho}} = \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_r$ and $\vec{\boldsymbol{\sigma}} = \boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_s$ are two sequences of constraint packets, define

$$a(\vec{\boldsymbol{\rho}}) = a(\boldsymbol{\rho}_1) \otimes \cdots \otimes a(\boldsymbol{\rho}_r) \quad \text{and} \quad a(\vec{\boldsymbol{\sigma}}) = a(\boldsymbol{\sigma}_1) \otimes \cdots \otimes a(\boldsymbol{\sigma}).$$

Let $(\mathbf{x}, (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_r), (\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_s))$ be the corresponding profile with

$$a(\boldsymbol{\rho}_1) \otimes \cdots \otimes a(\boldsymbol{\rho}_r) \otimes \mathbf{x} \otimes a(\boldsymbol{\sigma}_1) \otimes \cdots \otimes a(\boldsymbol{\sigma}_s) = a_1 \otimes \cdots \otimes a_r \otimes \mathbf{x} \otimes b_1 \otimes \cdots \otimes b_s,$$

and let

$$m_{r|1|s}(a_1,\dots,a_r,\mathbf{x},b_1,\dots,b_s) = \sum_{\substack{\{\mathbf{y}\in\mathfrak{S}(\mathcal{H}),\vec{\rho},\vec{\sigma}\,|\,\mathrm{ind}(B,\mathbf{x},\mathbf{y},\vec{\rho},\vec{\sigma})=1 \\ a(\vec{\rho})\otimes\mathbf{x}\otimes a(\vec{\sigma})=\vec{a}\otimes\mathbf{x}\otimes\vec{b}\}}} \#\left(\frac{\mathcal{M}^B(\mathbf{x},\mathbf{y};\vec{\rho},\vec{\sigma})}{\mathbb{R}}\right)\cdot\mathbf{y}.$$

Theorem 8.19. *If $\mathcal{H}$ is an α-β bordered diagram, then ${}_{\mathcal{Q}(m,k)}\widehat{\mathrm{CFPC}}(\mathcal{H})_{\mathcal{C}(m,k)}$ is a type AA bimodule.*

Proof. The $\mathcal{A}_\infty$ relations are a reinterpretation of Theorem 8.18. See also [20, Theorem 8.1], which in turn is modelled on [10, Proposition 7.12]. ∎

8.8. The dualizing bimodule

We now turn to the proof of Theorem 8.1.

First, we establish its grading properties.

Proposition 8.20. *The dualizing bimodule is graded, in the sense that equation (2.2) holds.*

Proof. It suffices to prove that for any boundary monotone profile $(\mathbf{x},\vec{\rho},\vec{\sigma})$ and compatible domain B, we have that

$$\mathfrak{w}_i(\partial_\alpha B) = \mathfrak{w}_i(\partial_\beta B) \tag{8.3}$$

and also

$$\mathrm{ind}(B,\mathbf{x},\mathbf{y},\vec{\rho},\vec{\sigma}) = |\vec{\rho}| + |\vec{\sigma}| + \sum_{i=1}^{|\vec{\rho}|}\mathbf{N}(a(\vec{\rho})). \tag{8.4}$$

Equation (8.3) in fact holds for any domain B. It is a consequence of the fact equation (8.3) holds for each elementary domain. In particular, it follows that

$$\mathfrak{w}(\vec{\rho}) = \mathfrak{w}(\vec{\sigma}) = \sum_{i=1}^{|\vec{\rho}|}\mathfrak{w}(a(\rho_i)).$$

Equation (8.4) is equivalent to the condition that for any domain $(B,\vec{\rho},\vec{\sigma})$,

$$e(B) + n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B) + \iota(\vec{\rho}) + \iota(\vec{\sigma}) = \sum_{i=1}^{|\vec{\rho}|}\mathbf{N}(a(\rho_i)).$$

Since each of the various chords in the chord packets $\boldsymbol{\sigma}_i$ lies on different boundary components (this is the "algebraic" hypothesis from Definition 8.11), it is easy to see that

$$\iota(\vec{\boldsymbol{\sigma}}) = -\mathfrak{w}(\vec{\boldsymbol{\rho}}).$$

Let C denote the total count of left- and right-walls hit among all of the pong elements $a(\boldsymbol{\rho}_1), \ldots, a(\boldsymbol{\rho}_r)$. We claim that $e(B) = -\mathfrak{w}(\vec{\boldsymbol{\rho}}) + \frac{C}{2}$, This is a straightforward computation: cut the diagram $\mathcal{H}$ along all the α- and β-arcs. The result will consist of $2m-2$ elementary domains with $e = -1/2$, meeting the chords U_1, L_1, R_1, L_2, $R_2, \ldots, L_{m-2}$, R_{m-2}, and U_{m-1}.

Thus, it remains to verify that

$$n_{\mathbf{x}} + n_{\mathbf{y}} + \iota(\vec{\boldsymbol{\rho}}) = -\frac{C}{2} + \sum_{i=1}^{|\vec{\boldsymbol{\rho}}|} \operatorname{cross}(a(\boldsymbol{\rho}_i)). \tag{8.5}$$

We prove this by decomposing the domain $(B, \vec{\boldsymbol{\rho}}, \vec{\boldsymbol{\sigma}})$ into elementary pieces, corresponding to the decomposition of pong elements from Proposition 5.2.

Consider the Reeb chords in the first packet $\boldsymbol{\rho}_1$.

Case 1. If there is a right-moving Reeb chord, consider the rightmost, right-moving Reeb chord ξ_0.

Case 1a. Suppose that there is also a left-moving Reeb chord η_0 to the right of ξ_0^-, and choose the leftmost such strand. If the terminal points of the two strands cross (on the interval), we can factor off the two oppositely moving chords in their own packet $\boldsymbol{\rho}_0$. There is a corresponding decomposition

$$(B, \mathbf{x}, \mathbf{y}, \{\boldsymbol{\rho}_0, \boldsymbol{\rho}_1', \boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_r\}) = (B_0, \mathbf{x}, x', \{\boldsymbol{\rho}_0\})\#(B', \mathbf{x}', \mathbf{y}, \{\boldsymbol{\rho}_1', \boldsymbol{\rho}_2, \ldots, \boldsymbol{\rho}_r\}).$$

(We have suppressed here the chord packets appearing on ∂_α, as they do not affect either side of the equation. Note that $\boldsymbol{\rho}_0$ contains a portion of ξ_0 and η_0.) If ξ_0 terminates before the initial point of η_0, we can factor off ξ_0,

$$(B, \mathbf{x}, \mathbf{y}, \{\boldsymbol{\rho}_0, \boldsymbol{\rho}_1', \boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_r\}) = (B_0, \mathbf{x}, \mathbf{x}', \{\{\boldsymbol{\rho}_0\}\})\#(B, \mathbf{x}', \mathbf{y}, \{\boldsymbol{\rho}_1', \boldsymbol{\rho}_2, \ldots, \boldsymbol{\rho}_r\}).$$

Case 1b. There is no left-moving strand starting to the right of the ξ_0-. In that case, we can factor

$$(B, \mathbf{x}, \mathbf{y}, \{\boldsymbol{\rho}_0, \boldsymbol{\rho}_1', \boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_r\}) = (B, \mathbf{x}, \mathbf{x}', \{\boldsymbol{\rho}_0\})\#(B, \mathbf{x}', \mathbf{y}, \{\boldsymbol{\rho}_1', \boldsymbol{\rho}_2, \ldots, \boldsymbol{\rho}_r\}),$$

where $\boldsymbol{\rho}_0$ consists of a single chord, which is a portion of ξ_0.

Case 2. All strands are stationary.

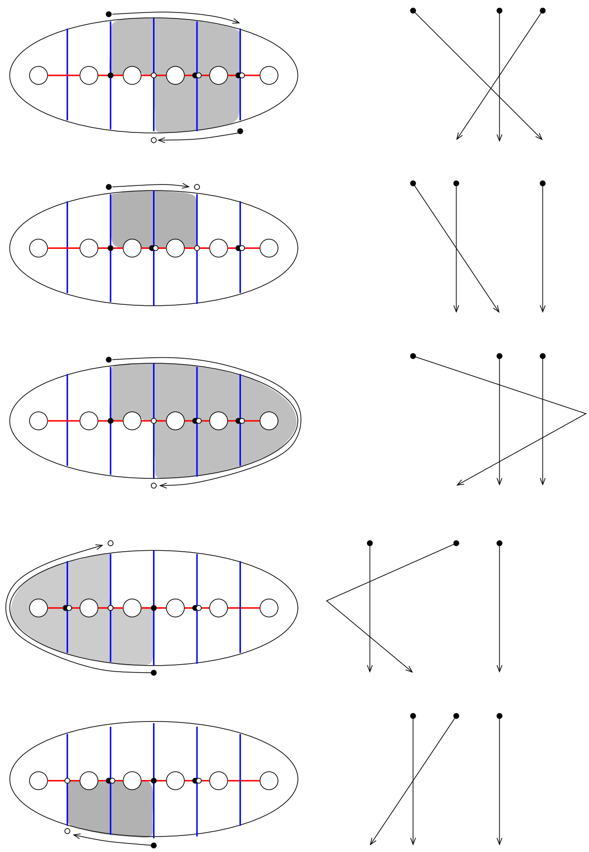


Figure 17. Decomposing the domain.

Case 3. There are no right-moving strands. There are two cases: either the leftmost left-moving strand terminates before hitting the wall, or it hits the wall. See Figure 17 for illustrations. (We have drawn the five non-trivial examples of initial domain B_0 in the order they appear in the text above.)

In all cases, we have decomposed our domain as a juxtaposition of simpler pieces $B = B_0 \# B'$, the first of which has a single chord packet on ∂_β. Verifying equation (8.5) for the domains B_0 is straightforward, and hence, the equation holds in general, by induction. ∎

Proof of Theorem 8.1. Let $\mathcal{H}_m$ denote the genus 0 α-β-bordered diagram as in Definition 8.9. The module $\mathcal{Y}$ is the associated bimodule $\widehat{\mathrm{CFPC}}(\mathcal{H})$. It is an $\mathcal{A}_\infty$ bimodule by Theorem 8.19. Property (Y-1) is straightforward. Bigons representing the actions from Properties (Y-2), (Y-4), (Y-5) are straightforward to see; compare Figure 18.

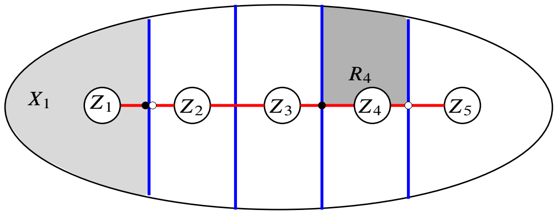


Figure 18. *Bigons representing actions.* Consider $\mathcal{H}_5$ with $k = 2$. The black dots represent a generator, corresponding to the idempotent state $\mathbf{x} = (1, 3)$, and the white dots represent $\mathbf{y} = (1, 4)$. The darkly shaded bigon represents the action $m_{1|1|1}(L_4, \mathbf{x}, R_4) = \mathbf{y}$; the lightly shaded one represents $m_{1|1|1}(X_1, \mathbf{x}, U_1) = \mathbf{x}$.

Property (Y-3) is verified as follows. Consider the moduli space $\mathfrak{M}^1$ of curves with the following properties:

- $\mathcal{S}$ has genus zero, two boundary punctures labeled by $+$, and two boundary punctures labeled by $-$, which divide the boundary into two α-boundaries and two β-boundaries,
- $\mathcal{S}$ has a single puncture on its α-boundary, which is labeled with the Reeb chord $R_i L_i$,
- $\mathcal{S}$ has two punctures q_1 and q_2 on the β-boundary, which are labeled R_i and L_i (on $\partial^{\partial_\beta}$).

There is another moduli space $\mathfrak{M}^2$ defined as above, only now using the chord $L_i R_i$.

Boundary monotone elements of $\mathfrak{M}^1 \cup \mathfrak{M}^2$ are precisely those elements for which $t(q_1) = t(q_2)$.

The moduli space $\mathfrak{M}^1$ can be concretely understood as an interval, parameterized by a cut parameter: cut in along α_i and use the Riemann mapping theorem to obtain a quadrilateral. Each quadrilateral has a single non-trivial involution, which in turn is equivalent to a branched cover of the disk, to obtain the desired element of $\mathfrak{M}^1$.

The moduli space $\mathfrak{M}^1$ has a join curve end, where the constraint packet on Z_i is the set $\{L_i, R_i\}$. Explicitly, this is realized as the cut along α_{i+1} extends from β_i to Z_i. Let u_0 denote the main component of this join curve. Note that this consists of a union of two bigons which meet along their α-boundary at a puncture. For generic choices of complex structure, $t(u_0(q_1)) \neq t(u_0(q_2))$.

This moduli space has another end as the cut parameter goes to 0. The limiting object in this case is a union of two bigons which meet at a single point on the β-boundary, so that both punctures q_1 and q_2 occur on the same bigon (separated by the node). Let u_1 denote the bigon component containing q_1 and q_2. Clearly, $t(u_1(q_1)) > t(u_1(q_2))$.

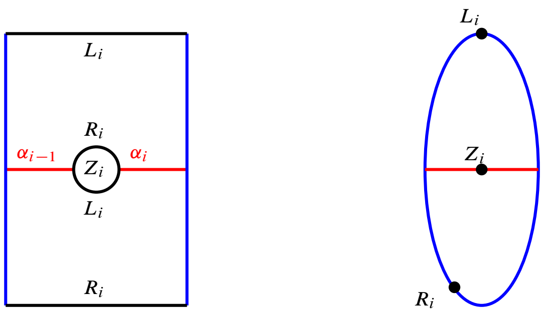


Figure 19. *The region B_i.* The region responsible for the action (Y-3). On the left is a combinatorial representation; on the right is a conformally accurate model (with cylindrical ends/punctures replacing boundary).

The moduli space $\mathfrak{M}^2$ also has a join curve end, whose main component is u_0 (i.e., it is the main component of the join curve end in $\mathfrak{M}^1$). It also has another end where the cut parameter goes to 0, which is identified with a union of two bigons, one of which contains both punctures q_1 and q_2. Let u_2 denote this bigon. In this case, $t(u_2(q_1)) < t(u_2(q_2))$. See Figures 19 and 20.

It follows that if we let $\mathfrak{M}$ denote the union of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ glued along their join curve end, then the map $u \mapsto t(u(q_1)) - t(u(q_2))$ defines a proper map of degree 1 near 0. In particular, the preimage of 0 has an odd number of points; but this count specifies the $Y_{\mathbf{x}}$-component in the action $m_{1|1|1}(X_i, Y_{\mathbf{x}}, U_i)$. ∎

9. The *DD* bimodule is quasi-invertible

We abbreviate $\mathcal{C} = \mathcal{C}(m,k)$ and $\mathcal{Q} = \mathcal{Q}(m,k)$. Our aim here is to prove the following.

Theorem 9.1. *The bimodules ${}^{\mathcal{C}}\mathcal{X}^{\mathcal{Q}}$ from Section 7 and ${}_{\mathcal{Q}}\mathcal{Y}_{\mathcal{C}}$ from Section 8 are quasi-inverses of one another.*

This will be a key step in computing the cobar algebra of $\mathcal{C}$ in the next section. The proof will occupy the rest of this section.

As a preliminary step, we must establish the boundedness assumptions required to ensure that ${}_{\mathcal{Q}}M^{\mathcal{C}} = {}_{\mathcal{C}}\mathcal{Y}_{\mathcal{Q}} \boxtimes {}^{\mathcal{Q}}\mathcal{X}^{\mathcal{C}}$ exist; i.e., we must prove that the structure maps in the tensor products are finite.

To this end, fix an input string $a_1, \dots, a_p$ in $\mathcal{Q}$, and let $c = \sum_{j=1}^{p} \mathfrak{w}(a_j)$. The following two statements hold.

- If $m_{p|1|q}(a_1 \otimes \cdots \otimes a_p \otimes x \otimes b_1 \otimes \cdots \otimes b_q) \neq 0$, then $c = \sum_{j=1}^{q} \mathfrak{w}_i(b_j)$. (This follows from Property (Y-6) from Theorem 8.1.)
- If $\delta^1(\mathbf{1}) = b \otimes \mathbf{1} \otimes e$, then $\mathfrak{w}(b) \geq \frac{1}{2}$.

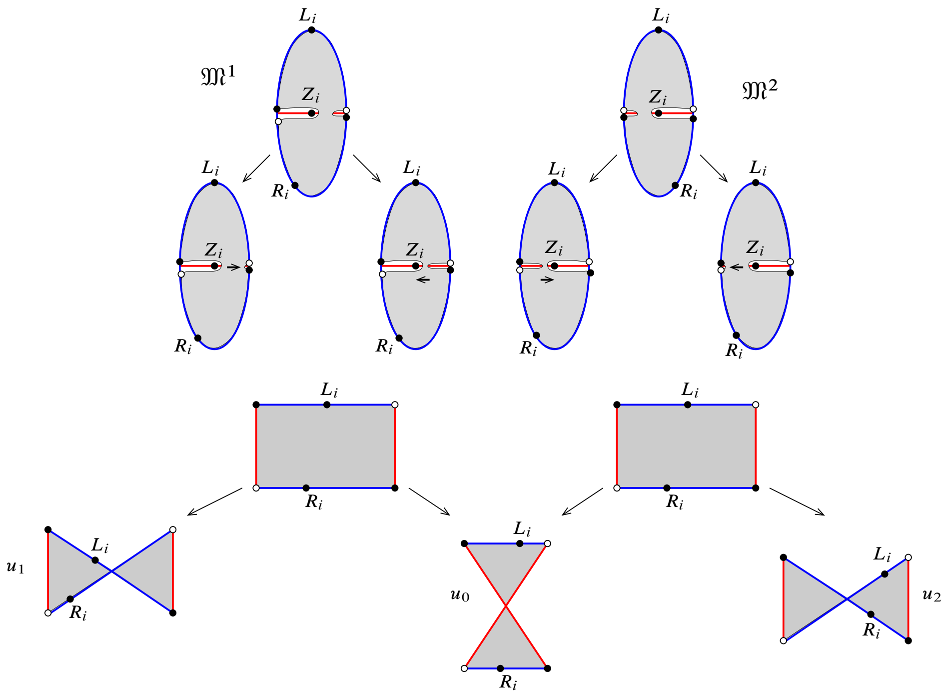


Figure 20. *Verifying the action* (Y-3). The moduli spaces $\mathfrak{M}^1$ and $\mathfrak{M}^2$ are represented by the cuts indicated in the first line of pictures; the conformal models of the domain curves are shown in the second line of pictures.

Thus, if the sum of the weights of the inputs is c, then the number of type DD operations which can pair non-trivially is bounded above by $2c$.

Consider now the DA bimodule

$$M = {}_{\mathcal{Q}}\mathcal{Y}_{\mathcal{C}} \boxtimes {}^{\mathcal{C}}\mathcal{X}^{\mathcal{Q}}.$$

Its generators correspond to basic idempotents, since the same is true for $\mathcal{Y}$ and $\mathcal{X}$. Moreover, the Maslov–Alexander multi-graded bimodule on the tensor factors induces one on the tensor product; i.e., if $s, t \in M$, and $n \geq 0$, and there is a non-trivial action

$$\delta^1_{n+1}(a_1, \dots, a_n, s) = t \otimes b,$$

then, for all $i = 1, \dots, m$,

$$\mathfrak{w}_i(a_1) + \cdots + \mathfrak{w}_i(a_n) = \mathfrak{w}_i(b) \tag{9.1}$$

and

$$\mathrm{cross}(a_1) + \cdots + \mathrm{cross}(a_n) - n + 1 = \mathrm{cross}(b). \tag{9.2}$$

Lemma 9.2. *There are two Maslov–Alexander bigraded $\mathcal{A}_\infty$ homomorphisms*

$$\phi\colon \mathcal{Q} \to \mathcal{Q} \quad \textit{and} \quad \psi\colon \mathcal{C} \to \mathcal{C}$$

so that

$$_{\mathcal{Q}}[\phi]^{\mathcal{Q}} = {}_{\mathcal{Q}}\mathcal{Y}_{\mathcal{C}} \boxtimes {}^{\mathcal{C}}\mathcal{X}^{\mathcal{Q}} \quad \textit{and} \quad {}^{\mathcal{C}}[\psi]_{\mathcal{C}} = {}^{\mathcal{C}}\mathcal{X}^{\mathcal{Q}} \boxtimes {}_{\mathcal{Q}}\mathcal{Y}_{\mathcal{C}}.$$

Proof. Consider $_{\mathcal{Q}}M^{\mathcal{Q}} = {}_{\mathcal{Q}}\mathcal{Y}_{\mathcal{C}} \boxtimes^{\mathcal{C}} \mathcal{X}^{\mathcal{Q}}$. From equation (9.1), it follows easily that δ^1_1 vanishes. By Lemma 2.4 (which is [9, Lemma 2.2.50]), it follows that there is a Maslov–Alexander multi-graded $\mathcal{A}_\infty$ homomorphism $\phi\colon \mathcal{Q} \to \mathcal{Q}$ so that $_{\mathcal{Q}}[\phi]^{\mathcal{Q}} \cong {}_{\mathcal{Q}}M^{\mathcal{Q}}$.

The analogous statements for $[\psi]$ follow similarly (except that in this case, we replace the grading cross on $\mathcal{Q}$ with the trivial grading on $\mathcal{C}$). ∎

Lemma 9.3. *The map ψ from Lemma 9.2 is the identity map; i.e.,*

$${}^{\mathcal{C}}\mathcal{X}^{\mathcal{Q}} \boxtimes {}_{\mathcal{Q}}\mathcal{Y}_{\mathcal{C}} = {}^{\mathcal{C}}\mathrm{Id}_{\mathcal{C}}.$$

Proof. In principle, $\psi = \{\psi_n\colon C^{\otimes n} \to C\}_{n=1}^{\infty}$. Gradings ensure that $\psi_n = 0$ for $n > 1$. (Instead of equation (9.2), we are now using the trivial grading on the algebra; and the analogue of that equation simply ensures that $n = 1$.) By Theorem 8.1, we have

$$\begin{aligned}
\psi(L_i) &= L_i \quad \forall i = 1, \dots, m, \\
\psi(R_i) &= R_i \quad \forall i = 0, \dots, m-1, \\
\psi(U_i) &= U_i \quad \forall i = 0, \dots, m.
\end{aligned}$$

Since $\mathcal{C}$ is generated (over the idempotent algebra) by the L_i, R_i, and U_i as above, it follows that ψ is the identity map. ∎

It is easy enough to see that $\psi \circ \phi\colon \mathcal{C} \to \mathcal{C}$ is the identity map. To see that $\phi \circ \psi\colon \mathcal{Q} \to \mathcal{Q}$ induces the identity map on homology, we use the following proposition. Recall the generating set of atomic generators $\mathbf{I}_{\mathbf{x}} \cdot X_{i,j}$, $\mathbf{I}_{\mathbf{x}} \cdot R_{i,j}$ and $\mathbf{I}_{\mathbf{x}} \cdot L_{j,i}$ (Definition 5.1). For $i = 1, \dots, m$, let X_i denote $X_{i-1,i}$.

Proposition 9.4. *If $f\colon \mathcal{Q}(m,k) \to \mathcal{Q}(m,k)$ is an $\mathcal{A}_\infty$ homomorphism with the following properties:*

- $f(X_{i-1,i}) = X_{i-1,i}$ *for all* $i = 1, \dots, m$,
- $f(L_{i,i-1}) = L_{i,i-1}$ *for all* $i = 2, \dots, m$,
- $f(R_{i,i+1}) = R_{i,i+1}$ *for all* $i = 1, \dots, m-1$,

then, f induces the identity map on homology.

Proposition 9.4 is proved in Section 9.1. We prove Theorem 1.2, assuming this result, presently.

Proposition 9.4 $\Rightarrow$ *Theorem* 9.1. From properties (Y-3), (Y-4), and (Y-5), it follows that $\phi(X_i) = U_i$ and $\psi(U_i) = X_i$. Thus, $\psi \circ \phi(X_i) = X_i$. Thus, Proposition 9.4 implies that $\psi \circ \phi \simeq \mathrm{Id}$. It is straightforward to see that $\phi \circ \psi = \mathrm{Id}$. ∎

9.1. On the homology of $Q(m, k)$

To complete the proof of Theorem 9.1, it remains to verify Proposition 9.4. We start with the case where $k = m - 1$.

We wish to glean information from the $\mathcal{A}_\infty$ structure on $H(Q(m, m-1))$. To this end, let $\phi\colon H(Q(m, m-1)) \to Q(m, m-1)$ be an $\mathcal{A}_\infty$ homomorphism. Given $T \subset \{1, \dots, m\}$, let $\mathfrak{S}(T)$ denote the set of one-to-one correspondences $\tau\colon \{1, \dots, \ell\} \to T$. Let

$$\Phi(T) = \sum_{\tau \in \mathfrak{S}(T)} \phi_\ell(X_{\tau(1)}, \dots, X_{\tau(\ell)}).$$

Lemma 9.5. *Let $\phi\colon H(Q(m, m-1)) \to Q(m, m-1)$ be a graded $\mathcal{A}_\infty$ homomorphism with $\phi(X_i) = X_i$, and fix a subset $T \subset \{1, \dots, m\}$. If T is not an interval, then $\Phi(T) = 0$. If T is an interval from i to j for some $i < j$, with $(i, j) \neq (1, m)$, then $\Phi(T) = X_{i,j}$.*

Proof. We consider the sum of the $\mathcal{A}_\infty$ relations on the homomorphism ϕ, where we sum over all permutations of the elements X_i with $i \in T$:

- for any sequence of distinct integers $i_1 < \dots < i_\ell$ with $\ell < m$, gradings ensure that $\mu_\ell(X_{i_1}, \dots, X_{i_\ell}) = 0$,
- for $1 \le i < j \le m$, we have that $[X_i][X_j] = [X_j][X_i]$ in $H(Q(m, m-1))$. There are two cases: if $j = i + 1$, the homology is supplied by $X_{i-1,i+1}$. If $j > i + 1$, then $X_i \cdot X_j = X_j \cdot X_i$.

It follows that the $\mathcal{A}_\infty$ relation shows that for any T,

$$d\Phi(T) = \sum_{T = T_1 \coprod T_2} \Phi(T_1) \cdot \Phi(T_2).$$

By the inductive hypothesis, if $T = \{i, \dots, j\}$ is an interval (different from $(1, m)$), then

$$d\Phi(T) = \sum_{\{k \mid i < k < j\}} X_{k,j} \cdot X_{i,k}.$$

It follows that $\Phi(T)$ is non-zero. Now, $2\mathfrak{w}(\Phi(T)) = \mathrm{cr}(\Phi(T)) + 1$, so by Proposition 5.3, we can conclude that $\Phi(T) = X_{i,j}$. ∎

The above lemma has the following easy consequence.

Proposition 9.6. *For any induced $\mathcal{A}_\infty$ structure on $\mathbb{F}[X_1, \dots, X_m, \Omega]/(X_i^2 = 0)_{i=1}^m$ induced from the isomorphism of equation (6.5), we have that*

$$\sum_{\sigma \in \mathfrak{S}_m} \mu_m(X_{\sigma(1)}, \dots, X_{\sigma(m)}) = \Omega.$$

Proof. The induced $\mathcal{A}_\infty$ structure on $H(Q(m, m-1))$ can be computed from the homological perturbation lemma [5]. Let us assume that any $\mathcal{A}_\infty$ homomorphism $\phi\colon H(Q(m, m-1)) \to Q(m, m-1)$ which induces the identity map on homology. Combining the $\mathcal{A}_\infty$ relation on ϕ with Lemma 9.5, we find that

$$\begin{aligned}
&\phi_1\Bigg(\sum_{\sigma \in \mathfrak{S}_m} \mu_m(X_{\sigma(1)}, \dots, X_{\sigma(m)})\Bigg) \\
&\sim \sum_{0<\ell<m} \big(\Phi(\{0, \dots, \ell\}) \cdot \Phi(\{\ell+1, \dots, m\}) + \Phi(\{\ell+1, \dots, m\}) \cdot \Phi(\{0, \dots, \ell\})\big) \\
&= \sum_{0<\ell<m} X_{0,\ell} \cdot X_{\ell,m} + X_{\ell,m} + \cdot X_{0,\ell} \\
&= \Omega.
\end{aligned}$$

Since ϕ_1 induces the identity map on homology, the result follows. ■

We can now prove Proposition 9.4 when $m = k$.

Lemma 9.7. *Let $\phi'\colon H(Q(m, m-1)) \to H(Q(m, m-1))$ be a graded $\mathcal{A}_\infty$ homomorphism with $\phi'([X_i]) = [X_i]$. Then, ϕ'_1 induces the identity map on homology.*

Proof. Define

$$\Phi'(T) = \sum_{\tau \in \mathfrak{S}(T)} \phi'_\ell([X_{\tau(1)}], \dots, [X_{\tau(\ell)}]).$$

As in the proof of Proposition 9.6, the $\mathcal{A}_\infty$ relation on ϕ' and Lemma 9.5 ensure that

$$\phi'_1\Bigg(\sum_{\sigma \in \mathfrak{S}_m} \mu_m(X_{\sigma(1)}, \dots, X_{\sigma(m)})\Bigg) = \Omega.$$

But by Proposition 9.6, the left-hand side is $\phi'_1(\Omega)$, so we conclude that the map on homology induced by ϕ'_1 fixes Ω, in addition to the $[X_i]_{i=1}^m$. But ϕ_1 is an algebra map, and $H_*(Q(m, m-1))$ is generated by $[X_i]$ and Ω; so we can conclude that ϕ'_1 fixes all of $H_*(Q(m, m-1))$. ■

We consider next the case of $Q(m, k)$ with $k < m-1$.

Lemma 9.8. *The ring $H(Q)$ is generated by the elements L_i, R_i, Ω, and the cycles of the form $\mathcal{X}_{i,j}$.*

Proof. Consider $Q(\mathbf{x},\mathbf{y};w)$. Suppose $\mathbf{x}=\mathbf{y}$. Then, there is an isomorphism of rings $\mathbf{I}_{\mathbf{x}}\cdot Q(m,k)\cdot\mathbf{I}_{\mathbf{x}}\cong Q(k+1,k)$, which identifies the generators $X_{i,j}$ of $Q(k+1,k)$ with $\mathcal{X}x_i, x_j$ in $Q(m,k)$. Thus, the statement follows from Corollary 6.14.

Suppose that $\mathbf{x}\neq\mathbf{y}$. Let $a\in H(Q(\mathbf{x},\mathbf{y};w))$ be non-zero. We prove that a can be factored as above, by induction on the maximum of w_i. Suppose that the maximum is ≤ 1, but w is non-constant. Then, such an element a can be easily seen to be factored as a product of L_i, R_i and $\mathcal{X}_{i,j}$. (This factorization is actually constructed in the proof of Lemma 6.11.) For the inductive step, suppose that w is a vector whose maximum is greater than 1. Then, by Theorem 6.4, we have that that $H(Q'(\mathbf{x},\mathbf{y};w))=0$. It follows from the long exact sequence of the mapping cone that $a\in\Omega\cdot H(Q(\mathbf{x},\mathbf{y};w-\vec{1}))$, and the result follows from the inductive hypothesis. ∎

Proof of Proposition 9.4. For any idempotent $\mathbf{x}$, there is a ring isomorphism

$$\mathbf{I}_{\mathbf{x}}\cdot Q(m,k)\cdot\mathbf{I}_{\mathbf{x}}\cong Q(k+1,k),$$

identifying $\mathcal{X}_{x_i,x_j}$ with some $X_{i,j}$, It follows now from Lemma 9.7 that if ϕ fixes all the $\mathcal{X}_{i,j}$, then in fact ϕ fixes the entire subring $\sum_{\mathbf{x}}\mathbf{I}_{\mathbf{x}}\cdot Q(m,k)\mathbf{I}_{\mathbf{x}}$. Note that Ω lies in this subring. Now, by Lemma 9.8, ϕ fixes the generators of $H(Q)$; and hence it must fix all of $H(Q)$. ∎

With the verification of Proposition 9.4 in place, the proof of Theorem 9.1 is complete.

10. Identifying the cobar algebra

Choose integers m and k with $0<k<m$. The aim of this section is to describe the cobar algebra of $\mathcal{C}(m,k)$ in terms of the pong algebra. Before doing this, we begin with a few general remarks about the cobar algebra of an associative algebra.

Let A be an augmented algebra, as in Definition 2.1. The *big cobar algebra* of A is a graded algebra, defined as follows. As a $\mathfrak{k}$-$\mathfrak{k}$ bimodule, it is given by

$$\mathbf{Cobar}(A)=\bigoplus_{n=0}^{\infty}\mathrm{Hom}(A_+^{\otimes n},\mathfrak{k}).$$

Multiplication is obtained by juxtaposing tensor summand. The differential is induced by μ_2^*. The summand $\mathrm{Hom}(A_+^{\otimes n},\mathfrak{k})$ is thought of as living in grading $-n$; and we call this induced grading the *length grading*.

We specialize to the case of $A=\mathcal{C}(m,k)$. As before, cobar algebra has a $\mathbb{Z}$-valued grading, defined so that $\mathrm{Hom}(A_+^{\otimes n},\mathfrak{k})$ has grading $-n$. We also have an Alexander grading with values in $(\frac{1}{2}\mathbb{Z})^m$ which is induced from the sums of the weights of the algebra elements in $\mathcal{C}(m,k)$. For example in $\mathcal{C}(4,2)$, the element $U_2^* \otimes U_4^*$ has homological grading -1 and Alexander multi-grading $(0,1,0,1)$. Moreover, its differential is the element

$$R_2^* \otimes L_2^* \otimes U_4^* + L_2^* \otimes R_2^* \otimes U_4^*.$$

Let

$$\mathrm{Cobar}(C)_{n,\vec{w}} = \mathrm{Hom}([C_+^{\otimes n}]_{\vec{w}}, \mathfrak{k}).$$

The *Maslov–Alexander graded cobar algebra*, or simply, the cobar algebra, is defined by

$$\mathrm{Cobar}(C) = \bigoplus_{n,\vec{w}} \mathrm{Cobar}(C)_{n,\vec{w}}.$$

(The big cobar algebra, by contrast, would be an infinite direct product, indexed by the various choices of weight vector.)

The algebra $Q(m,k)$ has a $\mathbb{Z}$-grading $\mathbf{N}$ as specified in equation (4.9). The differential preserves the weight vector, with values in $\frac{1}{2}\mathbb{Z}^m$.

Our aim here is to prove the following theorem.

Theorem 10.1. *Choose the case* $0 < k < m$. *There is a graded quasi-isomorphism* $\mathrm{Cobar}(\mathcal{C}(m,k))$ *to* $Q(m,k)$*, which respects the Alexander gradings, and which identifies the length grading on* Cobar *with* $\mathbb{Z}$*-grading* $\mathbf{N}$ *on* $Q(m,k)$.

Example 10.2. Consider the case $k=1$ and $m=2$. Then, $C = \mathcal{C}(2,1) \cong \mathbb{F}[U_1,U_2]/U_1U_2 = 0$. It is easy to see that the algebra $H(\mathrm{Cobar}(C))$ is generated by U_1^* and U_2^*. Moreover, there are relations $[U_1^*] \otimes [U_1^*] = 0$ and $[U_2^*] \otimes [U_2^*] = 0$. Thus, $H(\mathrm{Cobar}(C))$ is isomorphic to $Q(m,1)$, under an isomorphism that identifies $[U_1^*]$ and $X_{0,1}$ and $[U_2^*]$ with $X_{1,2}$.

As usual, we abbreviate $C = \mathcal{C}(m,k)$, $Q = Q(m,k)$.

Theorem 10.1 will follow from Koszul duality, in the form described in [8], with a suitable adaptation of some of the finiteness conditions from [12]. The proof follows the strategy of [8, Proposition 8.11]. Formally, the construction there used two quasi-inverse bimodules between C and $\mathrm{Cobar}(C)$, ${}^{\mathrm{Cobar}(C)}\mathbb{K}^C$ and ${}_{\mathrm{Cobar}(C)}\mathbb{L}_C$. The type AA bimodule ${}_{\mathrm{Cobar}(C)}\mathbb{L}_C$, called the *Kronecker module*, is defined as follows. Its underlying $\mathfrak{k}$-bimodule has rank one, and it is equipped with the following operations. All $m_{p|1|n} = 0$ except for the two unital actions

$$m_{1|1|0}(\mathbf{1}, x) = x \quad \text{and} \quad m_{0|1|1}(x, \mathbf{1}) = x,$$

and actions $m_{p|1|n}$ with $p = 1$, specified for $\gamma \in \mathrm{Cobar}(A)_n$ and $a_1 \otimes \cdots \otimes a_n \in (A_+)^n$ by

$$m_{1,1,n}(\gamma \otimes \mathbf{1} \otimes (a_1 \otimes \cdots \otimes a_n)) = \gamma(a_1 \otimes \cdots \otimes a_n).$$

We would also like a bimodule ${}^{C}\mathbb{K}^{\mathrm{Cobar}(C)}$ has generators corresponding to idempotents (which we suppress), and

$$\delta^1(\mathbf{1}) = \sum_c c \otimes c^*,$$

where the sum is taken over the basic generators c of C. Unlike [8], this sum is not finite, since the algebra is infinitely generated; rather the differential could be interpreted as an element of some completion of $C \otimes \mathrm{Cobar}(C)$; compare [12]. Rather than pursuing this perspective, we reformulate the proof of Theorem 10.1 without explicitly referring to ${}^{C}\mathbb{K}^{\mathrm{Cobar}(C)}$.

To this end, abbreviate the differential of the type DD bimodule from Section 7 as $\delta^1(\mathbf{1}) = c \otimes q$, so the DD bimodule relation takes the form

$$c \otimes dq + (c \cdot c) \otimes (q \cdot q) = 0.$$

We construct a DG algebra homomorphism $\Phi\colon \mathrm{Cobar}(C) \to Q$ as follows. Given $\gamma \in \mathrm{Cobar}_n(C)$, let

$$\Phi(\gamma) = \gamma(\overbrace{c \otimes \cdots \otimes c}^{n}) \cdot q^n. \tag{10.1}$$

Lemma 10.3. *Equation* (10.1) *defines a DG algebra homomorphism $\Phi\colon \mathrm{Cobar}(C) \to Q$, where the $\mathbb{Z}$-grading on $\mathrm{Cobar}(C)$ is given by the length grading, and the $\mathbb{Z}$-grading on Q is $\mathbf{N} = \mathrm{cross} - 2\bar{\mathfrak{w}}$. Viewing Φ as an A_∞ homomorphism with $\Phi_i = 0$, the associated type DA bimodule ${}_{\mathrm{Cobar}(C)}[\Phi]^{Q}$ satisfies*

$${}_{\mathrm{Cobar}(C)}[\Phi]^{Q} \simeq {}_{\mathrm{Cobar}(C)}\mathbb{L}_C \boxtimes {}^{C}\mathbb{K}^{Q}.$$

Proof. To see that Φ respects the $\mathbb{Z}$-gradings, note that if γ is homogeneous and $\gamma(\overbrace{c \otimes \cdots \otimes c}^{n}) \neq 0$, then the length grading of γ is $-n$. By Proposition 7.3, $\mathbf{N}(q) = -1$, so $\mathbf{N}(q^n) = -n$.

The other properties are similarly straightforward. ∎

We describe next a way of turning (certain) type AA bimodules into type DA bimodules, where the type D side is a cobar algebra. In the interest of clarity, we state the next lemma in slightly more generality than we need. Specifically, we explained earlier how to give $\mathrm{Cobar}(C)$ a Maslov–Alexander grading, whose $\mathbb{Z}$-component is specified by the length grading. Note that the algebra C has a trivial $\mathbb{Z}$-grading. By contrast, if B has a non-trivial $\mathbb{Z}$-grading,

$$B = \bigoplus_{d \in \mathbb{Z}, \vec{w} \in \frac{1}{2}\mathbb{Z}^m} B^{d,\vec{w}},$$

we endow $\mathrm{Cobar}(C)$ with a $\mathbb{Z}$-grading so that the summand

$$\left(B_+^{d_1,\vec{w}_1} \otimes \cdots \otimes B_+^{d_n,\vec{w}_n}\right)^* \subset \mathrm{Cobar}(C)$$

has $\mathbb{Z}$-grading $\mathrm{gr}^{\mathrm{Cobar}(B)}$ given by $d_1 + \cdots + d_n - n$.

Lemma 10.4. *Let ${}_AM_B$ be a Maslov–Alexander graded bimodule of rank one, whose generator, which we denote by* $\mathbf{1}$*, is supported in vanishing the Maslov–Alexander bigrading. There is an associated (Maslov–Alexander-graded) type DA bimodule* ${}_AK(M)^{\mathrm{Cobar}(B)}$*, characterized by*

$$\left\langle \delta^1_{p+1}(a_1,\dots,a_p,\mathbf{1}), b_1,\dots,b_q\right\rangle = m_{p|1|q}(a_1,\dots,a_p,\mathbf{1},b_1,\dots,b_q).$$

Proof. Note that for each fixed sequence $(a_1,\dots,a_p)$, the grading properties of M ensure that there is an upper bound on q so that there is a sequence $(b_1,\dots,b_q)$ with $m_{p|1|q}(a_1,\dots,a_p,\mathbf{1},b_1,\dots,b_q) \neq 0$. Thus, δ^1_{p+1} lands in $\mathrm{Cobar}(B)$.

The DA bimodule relation on ${}_AK(M)^{\mathrm{Cobar}(B)}$ is a straightforward consequence of the AA bimodule relation on M. The grading properties also follow easily. For example, if $m_{p|1|q}(a_1 \otimes \cdots \otimes a_p \otimes \mathbf{1} \otimes b_1 \otimes \cdots \otimes b_q) \neq 0$, then by equation (2.2),

$$\sum_{i=1}^{p}(\mathrm{gr}^A(a_i)) = \left(-q - \sum_{i=1}^{q}\mathrm{gr}^B(b_i)\right) + 1 - p = \mathrm{gr}^{\mathrm{Cobar}(B)}((b_1 \otimes \cdots \otimes b_q)^*) + 1 - p;$$

i.e., ${}_AK(M)^{\mathrm{Cobar}(B)}$ is $\mathbb{Z}$-graded. ∎

Lemma 10.5. *The above assignment ${}_AM_B \Rightarrow {}_AK(M)^{\mathrm{Cobar}(B)}$ satisfies the following properties.*

(K-1) *If A, B, and C are all over the same ring of idempotents, and ${}_AM^B$ is a type DA bimodule of rank one, ${}_BN_C$ is a type AA bimodule of rank one, then*
$$K({}_AM^B \boxtimes {}_BN_C) \simeq {}_AM^B \boxtimes {}_BK(N)^{\mathrm{Cobar}(C)}.$$

(K-2) *If ${}_{\mathrm{Cobar}(C)}\mathbb{L}_C$ is the Kronecker module, then ${}_{\mathrm{Cobar}(C)}K(\mathbb{L})^{\mathrm{Cobar}(C)}$ is the identity bimodule.*

(K-3) *If ${}_AY_B$ is a type AA bimodule, then*
$${}_AK(Y)^{\mathrm{Cobar}(B)} \boxtimes {}_{\mathrm{Cobar}(B)}\mathbb{L}_B = {}_AY_B.$$

(K-4) *If ${}_AM_B$ has the property that $m_{0|1|r} = 0$ for all $r \geq 0$, then there is an A_∞ homomorphism $\Psi\colon A \to \mathrm{Cobar}(B)$ so that*
$${}_A[\Psi]^{\mathrm{Cobar}(B)} = K(M).$$

(K-5) *If $M \sim M'$, then $K(M) \sim K(M')$.*

Proof. The statements are all easy consequences of the definitions. ■

Remark 10.6. When B_+ is nilpotent, then ${}^B\mathbb{K}^{\mathrm{Cobar}}$ as in [8] is a type DD bimodule, and ${}_AK(M)^{\mathrm{Cobar}(B)}$ agrees with the tensor product ${}_AM_B \boxtimes {}^B\mathbb{K}^{\mathrm{Cobar}(B)}$. In that case, the first property can then be thought of as associativity of $\boxtimes$, and both Properties (K-2) and (K-3) are manifestations of the fact that $\mathbb{L}$ and $\mathbb{K}$ are inverses.

Define $\Psi\colon Q \to \mathrm{Cobar}(C)$ by

$$ {}_Q[\Psi]^{\mathrm{Cobar}(C)} = K({}_Q\mathcal{Y}_C). $$

This definition is valid by Property (K-4), and the grading properties of $\mathcal{Y}$, which ensure that $m_{0|1|r} \equiv 0$ for all $r \geq 0$.

Proof of Theorem 10.1. Combining Lemmas 10.3, 10.5 Property (K-1), associativity of $\boxtimes$ (as in [9, Proposition 2.3.15]), the hypothesis that X and Y are quasi-inverses, and Lemma 10.5 Property (K-2), we see that

$$ \begin{aligned} [\Phi \circ \Psi] &= ({}_{\mathrm{Cobar}(C)}\mathbb{L}_C \boxtimes {}^C\mathcal{X}^Q) \boxtimes K({}_Q\mathcal{Y}_C) \\ &= K(({}_{\mathrm{Cobar}(C)}\mathbb{L}_C \boxtimes {}^C\mathcal{X}^Q) \boxtimes {}_A\mathcal{Y}_C) \\ &\simeq K({}_{\mathrm{Cobar}(C)}\mathbb{L}_C \boxtimes ({}^C\mathcal{X}^Q \boxtimes {}_A\mathcal{Y}_C)) \\ &\simeq K({}_{\mathrm{Cobar}(C)}\mathbb{L}_C) \\ &= {}_{\mathrm{Cobar}(C)}[\mathrm{Id}]^{\mathrm{Cobar}(C)}. \end{aligned} $$

Similarly, combining Lemma 10.3, associativity of $\boxtimes$ (now in the easier case of type AA bimodules; see [9, Lemma 2.3.14]), Property (K-3), and the hypotheses on X and Y, we find that

$$ \begin{aligned} [\Psi \circ \Phi] &= K({}_Q\mathcal{Y}_C) \boxtimes ({}_{\mathrm{Cobar}(C)}\mathbb{L}_C \boxtimes {}^C\mathcal{X}^Q) \\ &= (K({}_Q\mathcal{Y}_C) \boxtimes_{\mathrm{Cobar}(C)} \mathbb{L}_C) \boxtimes {}^C\mathcal{X}^Q \\ &= {}_Q\mathcal{Y}_C \boxtimes {}^C\mathcal{X}^Q \\ &\simeq {}_Q\mathrm{Id}^Q. \end{aligned} $$

It follows that Φ and Ψ are quasi-isomorphisms (see [8, Lemma 8.6]). ■

11. Computing the $\mathcal{A}_\infty$ action on the homology of $\mathcal{P}(m,k)$

Consider $\mathcal{C}(m,k)$ with $1 \leq k \leq m-1$. In the applications to knot Floer homology, we will consider $m = 2n$ and $k = n$ (although we could have alternatively considered $m = 2n$ and $k = n-1$).

Recall that $\mathcal{C}(m,k)$ is equipped with a weight grading

$$\mathfrak{w} = (\mathfrak{w}_1, \dots, \mathfrak{w}_m) \in \frac{1}{2}\mathbb{Z}^m.$$

There is also very uninteresting homological grading on $\mathcal{C}(m,k)$, defined so that all $c \in \mathcal{C}(m,k)$ have $\mathrm{gr}(c) = 0$.

We extend this data to the polynomial algebra $\mathcal{C}(m,k)[t]$, with the convention that $\mathfrak{w}_i(t) = 1$ for all i; and $\mathrm{gr}(t) = 2m - 2k - 2$. This is consistent with the existence of a non-trivial μ_{2m-2k}.

Definition 11.1. For a *graded $\mathcal{A}_\infty$ structure* on $\mathcal{C}(m,k)[t]$, we require that for each operation μ_ℓ, we have that

$$\mathfrak{w}(\mu_\ell(a_1, \dots, a_\ell)) = \sum_{i=1}^{\ell} \mathfrak{w}(a_i),$$

$$\mathrm{gr}(\mu_\ell(a_1, \dots, a_\ell)) = \ell - 2 + \sum_{i=1}^{\ell} \mathrm{gr}(a_i).$$

Remark 11.2. The above grading conventions are chosen so that the differential μ_1 on a differential graded algebra drops grading by one.

Theorem 11.3. *There is a graded $\mathcal{A}_\infty$ structure on $\mathcal{C}(m,k)[t]$, extending the natural algebra structure on $\mathcal{C}(m,k)$, and with a non-trivial μ_{2m-2k} operation. Moreover, any two such extensions are quasi-isomorphic as graded $\mathcal{A}_\infty$ algebras.*

Remark 11.4. The above result is not true when $k = 0$: in that case, $\mathcal{C}(m,0) \cong \mathbb{F}$, and there is no room for an μ_{2m}-operation.

An explicit realization of the $\mathcal{A}_\infty$-algebra characterized above is supplied by the homology of a pong algebra; see Theorem 12.1 below.

Theorem 11.3 is proved using Hochschild cohomology. A small model for the Hochschild cohomology can be obtained using a duality theorem, proved in Section 10. The relevant computation is then given in Section 11.

11.1. Deformation theory of graded algebras

We review here the deformation theory of graded algebras used in our proof of Theorem 1.1. We follow here the perspective from [12], with a somewhat simplified exposition, owing to the fact that we are here in a $\mathbb{Z}$-graded setting.

Let $A = A_*$ be a graded algebra—i.e., an algebra equipped with a graded multiplication. Denote that $\mathbb{Z}$-grading by gr. (We assume for simplicity that $\mu_1 = 0$; the case we will be interested in is a polynomial algebra over $\mathcal{C}(m,k)$.) The *Hochschild* *complex of the $\mathbb{Z}$-graded graded algebra* is a bigraded chain complex associated to A. $\mathbf{HC}^{n,d}(A)$ consists of $\mathfrak{k}-\mathfrak{k}$ bimodule maps

$$\phi\colon A_+^{\otimes n} \to A,$$

(where A_+ is as in Definition 2.1) with

$$\mathrm{gr}(\phi(a_1,\dots,a_n)) = d + n - 1 + \Big(\sum_i \mathrm{gr}(a_i)\Big). \tag{11.1}$$

The differential of the complex

$$\partial\colon \mathbf{HC}^{n,d} \to \mathbf{HC}^{n+1,d-1}$$

is defined by

$$\begin{aligned}\partial\phi(a_1,\dots,a_{n+1}) &= a_1\cdot\phi(a_2,\dots,a_{n+1}) + \phi(a_1,\dots,a_n)\cdot a_{n+1}\\ &\quad+\sum_{i=1}^{n}\phi(a_1,\dots,a_i\cdot a_{i+1},\dots,a_{n+1}).\end{aligned}$$

The (bigraded) homology of this complex is the *Hochschild cohomology*

$$\mathbf{HH}^{*,*} = \bigoplus_{n,d}\mathbf{HH}^{n,d}.$$

An $\mathcal{A}_n$ *algebra* is A equipped with operations

$$\mu_\ell\colon A_+^{\otimes\ell} \to A \in \mathbf{HC}^{\ell,-1}(A)$$

for $\ell \le n$, satisfying the $\mathcal{A}_\infty$ relations with up to $n+1$ inputs.

Proposition 11.5. *Each $\mathcal{A}_n$-algebra structure on A can be extended to an $\mathcal{A}_\infty$-algebra structure if $\mathbf{HH}^{\ell+2,-2}(A) = 0$ for all $\ell \ge n$. The set of isomorphism classes of $\mathcal{A}_{n+1}$ extensions of a given $\mathcal{A}_n$ structure is parameterized by $\mathbf{HH}^{n+1,-1}(A)$. Indeed, if $\mathbf{HH}^{\ell+1,-1}(A) = 0$ for all $\ell \ge n$, there is a unique isomorphism class of $\mathcal{A}_\infty$-algebra structure extending any given $\mathcal{A}_n$ structure on A.*

Proof. This is well-known. A proof is given in [12, Corollary 5.25]. ∎

We will be interested in $\mathcal{A}_\infty$ operations that preserve a further Alexander grading $\mathfrak{w}$ on A (with values in $(\frac{1}{2}\mathbb{Z})^m$); i.e.,

$$\mathfrak{w}(\mu_\ell(a_1\otimes\cdots\otimes a_\ell)) = \sum_{i=1}^{\ell}\mathfrak{w}(a_i). \tag{11.2}$$

To this end, we consider the subcomplex of $\mathrm{HC} \subset \mathbf{HC}$ consisting of ϕ satisfying equation (11.2) (with ϕ_n in place of μ_n) in addition to equation (11.1). Its homology HH is the Hochschild homology, which we will use henceforth.

The graded analogue of Proposition 11.5 is the following.

Proposition 11.6. *Each graded $\mathcal{A}_n$-algebra structure can be extended to a graded $\mathcal{A}_\infty$ structure if $\mathrm{HH}^{\ell+2,-2}(A) = 0$ for all $\ell \geq n$. Moreover, the set of isomorphism classes of $\mathcal{A}_{n+1}$ extensions of a given $\mathcal{A}_n$ structure is parameterized by $\mathrm{HH}^{n+1,-1}(A)$. Indeed, if $\mathrm{HH}^{\ell+1,-1}(A) = 0$ for all $\ell \geq n$, there is a unique isomorphism class of $\mathcal{A}_\infty$-algebra extending any given $\mathcal{A}_n$-algebra structure on A.*

We give the graded Hochschild complex the following explicit description. Consider the following graded algebra associated to A:

$$\mathcal{E}^* = \bigoplus_{n=1}^{\infty} \mathcal{E}^n,$$

where

$$\mathcal{E}^n = \prod_{\{s \in \mathbf{z}, \vec{w} \in |(\frac{1}{2}\mathbb{Z})^m\}} [t^s A]_{\vec{w}} \otimes_{\mathfrak{k}\otimes\mathfrak{k}} [A_+^{\otimes n}]^*_{\vec{w}}.$$

The tensor product here is shorthand for the condition that we are considering the span of

$$b \otimes (a_1 \otimes \cdots \otimes a_n)^* \in [A]_{\vec{w}} \otimes [A_+^{\otimes n}]^*_{\vec{w}}$$

so that the left idempotent of b agrees with the right idempotent of $(a_1 \otimes \cdots \otimes a_n)^*$; and the right idempotent of b agrees with the left idempotent of $(a_1 \otimes \cdots \otimes a_n)^* = a_n^* \otimes \cdots \otimes a_n^*$.

The cobar differential induces a differential (which could be denoted $\mathrm{Id}_A \otimes \partial$; but instead, we simplify notation a little)

$$\partial \colon \mathcal{E}^n \to \mathcal{E}^{n+1}.$$

There is an element $E \in \mathcal{E}$ given by

$$E = \sum_e e \otimes e^*,$$

where the sum is taken over a homogeneous generating set of A. (In the case we consider, $A = \mathcal{C}(m,k)$, and the sum is taken over all pure algebra generators e.)

Let $D \colon \mathcal{E}^n \to \mathcal{E}^{n+1}$ be the map defined by

$$D(x) = \partial x + E \cdot x + x \cdot E.$$

Lemma 11.7. *The map D defined above is a differential on $\mathcal{E}$. Moreover, there is an isomorphism between the graded Hochschild complex and $\mathcal{E}$, equipped with the differential D.*

Proof. This is straightforward. ∎

11.2. Computing in small models

Fix m and k, and abbreviate $\mathcal{C}(m,k)$ by C and $\mathcal{Q}(m,k)$ by Q.

Consider the algebra

$$\mathcal{S} = \bigoplus_{\mathbf{x},\mathbf{y}} \prod_{\{\vec{w}\in(\frac{1}{2}\mathbb{Z})^m\}} (\mathbf{I}_{\mathbf{x}} \cdot C[t] \cdot \mathbf{I}_{\mathbf{y}})_{\vec{w}} \otimes (\mathbf{I}_{\mathbf{x}} \cdot Q \cdot \mathbf{I}_{\mathbf{y}})_{\vec{w}}.$$

In words, $\mathcal{S}$ is generated by sequences (indexed by weight vectors) of elements $t^s\alpha \otimes b$, where α is a pure algebra element in C, b is a pure algebra element in Q,

- $\mathfrak{w}_i(\alpha) + s = \mathfrak{w}_i(b)$ for $i = 1, \dots, m$,
- the left idempotents of α and b agree,
- the right idempotents of α and b agree.

The algebra $\mathcal{S} = \mathcal{S}^{n,d}$ inherits a bigrading where each pure element $t^s \cdot a \otimes b \in \mathcal{S}$ has

$$n = 2\bar{\mathfrak{w}}(b) - \mathrm{cr}(b) \quad \text{and} \quad d = 1 - 2\bar{\mathfrak{w}} + \mathrm{cr}(b) + 2s(m-k-1).$$

(As we will see below, in Proposition 11.8, $\mathcal{S}$ is a small model for the bigraded Hochschild complex; compare also [12, Definition 5.41].)

Turn the DD bimodule ${}^{C}\mathcal{X}^{Q}$ into a left bimodule over $C \otimes Q^{\mathrm{op}} \cong C \otimes Q$. The differential δ^1 corresponds to the projection onto $\mathcal{S}$ of the element of $C \otimes Q$ given by

$$\begin{aligned} S = {} & \Bigg(\sum_{1\leq i<j\leq m-1} f(R_{i,j}) \otimes R_{i,j} + f(L_{j,i}) \otimes L_{j,i} \Bigg) \\ & + \Bigg(\sum_{0\leq i<j\leq m} f(X_{i,j}) \otimes X_{i,j} \Bigg). \end{aligned}$$

(Compare (7.1). Note that under the identification $Q^{\mathrm{op}} \cong Q$, the elements $R_{i,j}^{\mathrm{op}}$ are identified with $L_{j,i}$.)

Let $D^s \colon \mathcal{S} \to \mathcal{S}$ be the endomorphism defined by

$$D^s(x) = (\mathrm{Id} \otimes \partial_Q)(x) + S \cdot x + x \cdot S.$$

Note that, like the Hochschild differential, D^s respects the bigradings in the sense that $D^s \colon \mathcal{S}^{n,d} \to \mathcal{S}^{n+1,d-1}$.

Proposition 11.8. *There is a bigraded quasi-isomorphism $\mathcal{S}^{*,*} \simeq \mathrm{HC}^{*,*}$.*

Proof. Let $\Phi\colon \mathrm{Cobar}(C) \to Q$ be the algebra quasi-isomorphism from Theorem 10.1. (See Lemma 10.3.) There is an induced algebra quasi-isomorphism

$$\widetilde{\Phi}\colon \mathcal{E} \to \mathcal{S}.$$

Since $\Phi_i = 0$ for all $i > 1$ (Lemma 10.3), it follows that Φ, and hence, $\widetilde{\Phi}$ is a homomorphism of DG algebras. Write $S = c \otimes q$ for $c \in \mathcal{C}$ and $q \in Q$. We have that

$$\widetilde{\Phi}(E) = \sum_e e \otimes \Phi(e^*) = \sum_e e \otimes \langle e^*, c\rangle \otimes q = c \otimes q = S.$$

It follows at once that $\widetilde{\Phi} \circ D = D^s \circ \widetilde{\Phi}$; i.e., $\widetilde{\Phi}$ induces a bigraded quasi-isomorphism from $\mathrm{HC}^{*,*}$ (in view of Lemma 11.7) to $\mathcal{S}^{*,*}$. ∎

Lemma 11.9. *For the bigraded chain complex $\mathcal{E} = \mathcal{E}^{*,*}$, we have that $H(\mathcal{E}^{n,-1}) = 0$, except when $n = 2$ or $n = 2m - 2k$. Moreover, if $H(\mathbf{I_x} \cdot \mathcal{E}^{2m-2k,-1} \cdot \mathbf{I_y}) \neq 0$, then $\mathbf{x} = \mathbf{y}$, and indeed,*

$$H(\mathbf{I_x} \cdot \mathcal{E}^{2m-2k,-1} \cdot \mathbf{I_x}) = \begin{cases} \mathbb{F} & \text{if } k > 1, \\ \mathbb{F} \oplus \mathbb{F} & \text{if } k = 1. \end{cases} \tag{11.3}$$

Also, if $H(\mathcal{E}^{n,-2}) \neq 0$, then $n \in \{3, 2m - 2k + 1\}$.

Proof. Let $\vec{w}$ be a weight vector, and choose $s = \min w_i$. Let $w_i' = w_i - s$. The only non-zero element of $C[t]$ with weight $\vec{w}$ is $t^s a$, where $a \in C$ has weight $\vec{w}'$.

Assume first that $k > 1$. By Theorem 6.4, if $b \in H(\mathbf{I_x} \cdot Q \cdot \mathbf{I_y})$ is a homologically non-trivial element with weight $\vec{w}$, then $b = \Omega^s \cdot b'$, where $b' \in H(\mathbf{I_x} \cdot Q \cdot \mathbf{I_y})$ or, in the special case where $\vec{w}$ is constant, there is another possibility of $b = \Omega^{s-1} \cdot b'$, where b' is the homologically non-trivial element with $\mathrm{cross}(b') = k + 1$ and $\mathfrak{w}_i(b') = 1$. We call the latter type of element *special*, and the other types *generic*.

Thus, a special element has the form $(t^s, \Omega^{s-1} \cdot b')$ with $s \geq 1$. The bigrading of a special element is given by

$$(n, d) = (2ms - k(2s - 1) - 1, 2 - k - 2s).$$

When $d = -2$, we have $k = 2$ and $s = 1$, so $n = 2m - 3 = 2m - 2k + 1$.

Next, we compute the bigrading (n, d) of the element (at^s, b). In the generic case,

$$d = 1 - (2\bar{\mathfrak{w}}(b) - \mathrm{cross}(b)) + 2(m - k - 1)s = -(2\bar{\mathfrak{w}}(b') - \mathrm{cross}(b')) - 2s + 1.$$

So, if $d = -1$ (in the generic case), then $2\bar{\mathfrak{w}}(b') - \mathrm{cross}(b') = 2 - 2s$. Since for any algebra element b', $2\bar{\mathfrak{w}}(b') - \mathrm{cross}(b') \geq 0$ (Proposition 5.3), we have that $s = 0$

or 1. If $s = 0$, then $b = b'$, $n = 2$, $d = -1$. If $s = 1$, then $2\bar{\mathfrak{w}}(b) - \mathrm{cross}(b) = 0$, so b' is an idempotent (again, by Proposition 5.3), $b = \mathbf{I}_{\mathbf{x}} \cdot \Omega$ (and, in particular, $\mathbf{x} = \mathbf{y}$), and $n = 2m - 2k$.

If $d = -2$, then $2\bar{\mathfrak{w}}(b') - \mathrm{cross}(b') = 3 - 2s$ and $s = 0$ or 1. If $s = 1$, then $1 = 2\bar{\mathfrak{w}}(b') - \mathrm{cross}(b')$, and $n = 2m - 2k + 1$. If $s = 0$, then $n = 3$.

When $k = 1$, Theorem 6.4 once again proves that $b = \Omega^s \cdot b'$, where $b' \in H(\mathbf{I}_{\mathbf{x}} \cdot \mathcal{Q} \cdot \mathbf{I}_{\mathbf{y}})$, provided that $\vec{w}$ is a non-constant vector. Thus, when $\vec{w}$ is non-constant, the above computations establish all of the claims of the lemma, except equation (11.3).

When $k = 1$, Theorem 6.4 implies that for each $s \geq 1$, $H(\mathbf{I}_{\mathbf{x}} \cdot \mathcal{Q}(\mathbf{x}, \mathbf{x}; (s, \dots, s)) \cdot \mathbf{I}_{\mathbf{x}})$ is two-dimensional, spanned by Ω_+^s and Ω_-^s, where $\Omega_+ = X_{x,m}, \cdot X_{0,x}$ and $\Omega_- = X_{0,x} \cdot X_{x,m}$ are the two components of Ω. The bigrading of (t^s, Ω_+^s) is given by

$$(n, d) = (2s(m-1), 1 - 2s(m-1) + 2s(m-2)) = (2s(m-1), 1 - 2s).$$

Again, $d = -1$ implies that $s = 1$, verifying equation (11.3); the two homology generators are (t, Ω_+) and (t, Ω_-). ■

Definition 11.10. Let $\mathbf{x} = \{x_i\}_{i=1}^k$ and $\mathbf{y} = \{y_i\}_{i=1}^k$ be two idempotent states. We say that $\mathbf{x} \leq \mathbf{y}$ if for all $t \in \{1, \dots, k\}$, $x_t \leq y_t$. We say that $\mathbf{y}$ is an *immediate successor* of $\mathbf{x}$ if there is some $s \in \{1, \dots, k\}$ so that $x_t = y_t$ for all $t \neq s$ and $y_s = x_s + 1$; i.e., $\mathbf{I}_{\mathbf{y}} \cdot L_s \cdot \mathbf{I}_{\mathbf{x}} \neq 0$. We also say that $\mathbf{x}$ is an *immediate predecessor* of $\mathbf{y}$.

Lemma 11.11. *Let S be a non-empty set of idempotent states. Suppose that if $\mathbf{x} \in S$, then every immediate successor and every immediate predecessor of $\mathbf{x}$ is in S, then S is the set of all idempotent states.*

Proof. The fact that S is closed under immediate successors implies that for all $\mathbf{x} \in S$, if $\mathbf{x} \leq \mathbf{y}$, then $\mathbf{y} \in S$. The fact that it is closed under immediate predecessors implies that if $\mathbf{x} \in S$ and $\mathbf{y} \leq \mathbf{x}$, then $\mathbf{y} \in S$. Since there is a unique maximal and unique minimal idempotent state, the result follows. ■

Lemma 11.12. *The homology $\mathrm{sHH}^{*,*}$ of $\mathcal{S}^{*,*}$ has $\mathrm{sHH}^{n,-1} = 0$ unless $n = 2$ or $n = 2m - 2k$; $\mathrm{sHH}^{n,-2} = 0$ unless $n = 3$ or $n = 2m - 2k + 1$. Moreover, $\mathrm{sHH}^{2m-2k,-1} \cong \mathbb{F}$.*

Proof. Consider the chain complex $\mathcal{S}$, filtered by the weight of b in (a, b). The homology of the associated graded object is $H(\mathcal{E})$. The vanishing statements in the lemma now follow.

It remains to compute $\mathrm{sHH}^{2m-2k,-1}$. We start with the case $k > 1$. To this end, we compute the next differential in the spectral sequence associated to the filtration; i.e., the map

$$d_1 \colon H(\mathcal{E}^{2m-2k,-1}) \to H(\mathcal{E})^{2m-2k-1,-2}$$

induced by D^s. By Lemma 11.9, $H(\mathcal{E}^{2m-2k,-1})$ has rank $\binom{m}{k}$, with generators of the form $(t\mathbf{I}_{\mathbf{x}}, \Omega\cdot\mathbf{I}_{\mathbf{x}})$, uniquely determined by the idempotent states $\mathbf{x}$.

The map d_1 is specified by

$$\begin{aligned} d_1(\mathbf{I}_{\mathbf{x}}\cdot t, \Omega\cdot\mathbf{I}_{\mathbf{x}}) = (L\cdot\mathbf{I}_{\mathbf{x}}\cdot t, L\cdot\Omega\cdot\mathbf{I}_{\mathbf{x}}) + (\mathbf{I}_{\mathbf{x}}\cdot L\cdot\mathbf{I}_{\mathbf{x}}, \Omega\cdot\mathbf{I}_{\mathbf{x}}\cdot L)\\ + (R\cdot\mathbf{I}_{\mathbf{x}}\cdot t, R\cdot\Omega\cdot\mathbf{I}_{\mathbf{x}}) + (\mathbf{I}_{\mathbf{x}}\cdot R\cdot\mathbf{I}_{\mathbf{x}}, \Omega\cdot\mathbf{I}_{\mathbf{x}}\cdot R). \end{aligned}$$

Clearly,

$$(t,\Omega) = \sum_{\mathbf{x}} (\mathbf{x}\cdot t, \Omega\cdot\mathbf{I}_{\mathbf{x}})$$

is in the kernel of d_1. Moreover, if there is a non-empty S so that $\sum_{\mathbf{x}\in S}(t,\Omega\cdot\mathbf{I}_{\mathbf{x}})$ in the kernel of d_1, then the form of d_1 implies that S is closed under immediate successors and predecessors. From Lemma 11.11, it follows that S consists of all idempotent states; i.e., (t,Ω) is the only element of the kernel of d_1. The fact that (t,Ω) is not in the image of D is an easy consequence of the form of D^s, and the fact that Ω maximizes cross among all elements with weight vector identically 1. It follows readily that $\mathrm{sHH}^{2m-2k,-1}\cong\mathbb{F}$, when $k>1$.

For the case $k=1$, we have the following explicit description of $\mathcal{S}^{2m-2,-1}$ and $\mathcal{S}^{2m-2,-2}$, and the differential. As in the proof of Lemma 11.9, the space $\mathcal{S}^{2m-2,-1}$ has dimension $2m-2$, with basis given by $\{(t, \mathcal{X}_{0,i}\cdot\mathcal{X}_{i,m})\}_{i=1,m-1}$ and $\{(t,\mathcal{X}_{i,m}\cdot\mathcal{X}_{0,i})\}_{i=1,m-1}$. We abbreviate these generators $\{\Omega_i^-\}_{i=1}^{m-1}$ and $\{\Omega_i^+\}_{i=1}^{m-1}$. (Note that Lemma 11.9 states that $H(\mathbf{I}_{\mathbf{x}}\cdot\mathcal{E}^{2m-2,-1}\cdot\mathbf{I}_{\mathbf{x}})$ is two-dimensional; but the proof in fact shows that $\mathbf{I}_{\mathbf{x}}\cdot\mathcal{E}^{2m-2,-1}\cdot\mathbf{I}_{\mathbf{x}}$ is two-dimensional.)

This follows as in the proof of Lemma 11.9: the generators in bigrading $(2m-1,-2)$ have the form $(tb',\Omega\cdot b')$, where $2\bar{\mathfrak{w}}(b')-\mathrm{cross}(b)=1$. Explicitly, the space $\mathcal{S}^{2m-1,-2}$ has dimension $2m-2$, with basis given by

$$\begin{aligned} &\{(tL_i, L_i\cdot\mathcal{X}_{0,i-1}\mathcal{X}_{i-1,m})\}_{i=2}^{m-1}\\ &\quad\cup\{(tR_i, R_i\cdot\mathcal{X}_{i+1,m}\mathcal{X}_{0,i+1})\}_{i=1}^{m-2}\\ &\quad\cup\{(tU_1, \mathcal{X}_{0,1}\cdot\mathcal{X}_{1,m}\cdot\mathcal{X}_{0,1}), (tU_m, \mathcal{X}_{m-1,m}\cdot\mathcal{X}_{0,m-1}\cdot\mathcal{X}_{m-1,m})\}. \end{aligned}$$

We abbreviate basis vectors from these sets $\{L_i\}_{i=2}^{m-1}$, $\{R_i\}_{i=1}^{m-2}$, and $\{U_1,U_m\}$, respectively. The differential has the form

$$D^s\Omega_i^- = \begin{cases} U_1+L_2 & \text{for } i=1,\\ L_i+L_{i+1} & \text{for } 1<i<m-1,\\ L_{m-1}+U_m & \text{for } i=m-1,\end{cases}$$

$$D^s\Omega_i^+ = \begin{cases} U_1+R_2 & \text{for } i=1,\\ R_i+R_{i+1} & \text{for } 1<i<m-1,\\ R_{m-1}+U_m & \text{for } i=m-1.\end{cases}$$

It follows at once, that the kernel of D^s (when $k = 1$) is one-dimensional, spanned by

$$\Omega = \sum_{i=1}^{m-1} \Omega_i^- + \Omega_i^+. \qquad \blacksquare$$

Remark 11.13. It follows from the above proof that the cycle $(t, \Omega) \in \mathcal{S}^{2m-2k,-1}$, in fact generates $\mathrm{sHH}^{2m-2k,-1}$.

Proof of Theorem 11.3. Any Hochschild cochain in $\mathrm{HC}^{2m-2k,-1}(\mathcal{C})$ determines an A_{2m-2k}-algebra structure on $\mathcal{C}[t]$. By Proposition 11.8 and Lemma 11.12. since $\mathrm{HH}^{n,-2} = 0$ for $n \geq 2m - 2k + 2$; so by Proposition 11.6, this A_{2m-2k}-algebra structure can be extended to an A_∞ structure. Moreover, since $\mathrm{HH}^{n+1,-1} = 0$ for all $n \geq 2m - 2k$, this A_∞ structure is unique up to isomorphism.

We claim that there is a unique non-trivial possibility for the operation μ_{2m-2k} giving $\mathcal{C}[t]$ the structure of an A_{2m-2k}-algebra. Such an operation is necessarily a cocycle in $\mathrm{HC}^{2m-2k,-1}$. Since $\mathrm{HH}^{2m-2k,-1} \cong \mathbb{F}$ by Proposition 11.8 and Lemma 11.12, we conclude that any two such operations μ_{2m-2k} and μ'_{2m-2k} are cohomologous. But $\mathrm{HC}^{2m-2k-1,0} = 0$; which, in turn, is an easy consequence of equation (11.1) and the fact the grading of any homogeneous element of $C[t]$ is a multiple of $2m - 2k - 2$. Thus, $\mu_{2m-2k} = \mu'_{2m-2k}$. $\blacksquare$

12. The $\mathcal{A}_\infty$ structure on the homology of the pong algebra

The aim of this section is to determine the homology of $\mathcal{P}(m, m-k-1)$ with its A_∞ structure, as follows.

Theorem 12.1. *For integers $0 < k < m - 1$, the homology of $\mathcal{P}(m, m-k-1)$ with its induced A_∞ structure is quasi-isomorphic to the A_∞ algebra from Theorem* 11.3.

Proof. By Theorem 6.2, we have

$$H_*(\mathcal{P}(m, m-k-1)) \cong H_*(\mathcal{C}(m,k)[\Omega]),$$

where $\mathrm{gr}(\Omega) = 2m - 2k - 2$. Thus, by Theorem 11.3, it suffices to compute a single non-zero μ_{2m-2k} action. To this end, consider the algebra sequence

$$(\mathbf{I_x} \cdot a_1, \dots, a_{2m-2k}) = (\mathbf{I_x} \cdot v_1 \cdots v_k, L_{k+1}, \dots, L_{m-1}, v_m, R_{m-1}, \dots, R_{k+1})$$

with starting idempotent $\mathbf{x} = \{k + 1, \dots, m - 1\}$. Note that in $\mathcal{C}(m,k)$, this corresponds to the idempotent $\{1, \dots, k\}$ and the corresponding algebra sequence,

$$(v_1 \cdots v_k, R_{k+1}, \dots, R_{m-1}, v_m, L_{m-1}, \dots, L_{k+1}).$$

Each a_i is a cycle; and indeed, each a_i represents a homologically non-trivial element. We claim that

$$\mu_{2m-2k}(a_1,\dots,a_{2m-2k}) = \Omega\cdot\mathbf{I}_{\mathbf{x}}.$$

We consider first the case where $k=1$, after introducing some notation. Suppose that $a_i*\cdots*a_j$ be an element that satisfies

$$d(a_i*\cdots*a_j) = \sum_{i<k<j}(a_i*\cdots*a_k)\cdot(a_{k+1}*\cdots*a_j).$$

It is straightforward to check the following identities (in the relevant idempotents):

$$\begin{aligned}
v_1*L_2*\cdots*L_i &= \mathcal{L}_{i,1}X_{0,1},\\
L_{i+1}*\cdots*L_{m-1}v_m R_{m-1}*\cdots*R_2 &= \mathcal{R}_{1,i}X_{i,m},\\
v_m*R_{m-1}*\cdots*R_2 &= \mathcal{R}_{1,m-1}X_{m-1,m},\\
v_1*L_2*\cdots*L_{m-1}*v_m &= X_{m-1,m}\mathcal{L}_{m-1,1}X_{0,1},\\
v_1*L_2\cdots*L_{m-1}*v_m*R_{m-1}*\cdots*R_i &= X_{i-1,m}\mathcal{L}_{i-1,1}X_{0,1},\\
R_{i-1}*\cdots*R_2 &= \mathcal{R}_{1,i-1}.
\end{aligned}$$

(See Example 12.2.)

It follows readily that

$$\mu_{2m-2}(\mathbf{I}_{\mathbf{x}}\cdot v_1, L_2,\dots,L_{m-1},v_m,R_{m-1},\dots,R_2) = \Omega\cdot\mathbf{I}_{\mathbf{x}}.$$

The general case follows by substituting $v_1\cdots v_k$ for v_1, and shifting indices. ∎

Example 12.2. Consider $\mathcal{P}(4,2)$. We have shown the chains used in the computation of $\mu_6(v_1,L_2,L_3,v_4,R_3,R_2)$ in Figure 21. Observe that

$$\begin{aligned}
&\mu_6(v_1,L_2,L_3,v_4,R_3,R_2)\\
&\quad=(v_1*L_2)\cdot(L_3*v_4*R_3*R_2)+(v_1*L_2*L_3)\cdot(v_4*R_3*R_2)\\
&\qquad+(v_1*L_2*L_3*v_4)\cdot(R_3*R_2)+(v_1*L_2*L_3*v_4*R_3)\cdot R_2\\
&\quad=\Omega.
\end{aligned}$$

Together, Theorems 6.2 and 12.1 give Theorem 1.1.

12.1. Degenerate cases

The reader may wonder to what extent Theorem 1.1 holds when $k=0$ or $k=m-1$.

When $k=m-1$, we have that

$$\begin{aligned}
\mathcal{C}(m,m-1) &\cong \mathbb{F}[v_1,\dots,v_m]/(v_1\cdots v_m),\\
\mathcal{P}(m,0) = H(\mathcal{P}(m,0)) &\cong \mathbb{F}[v_1,\dots,v_m].
\end{aligned}$$

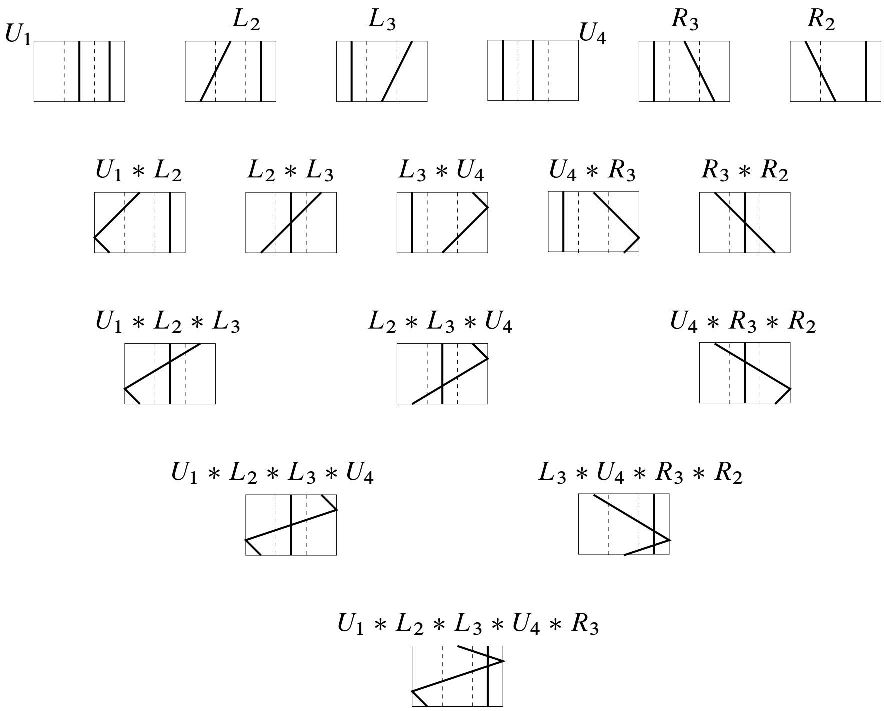


Figure 21. *Computing* μ_6. In $H_*(\mathcal{P}(4,2))$, we have shown steps in the verification that $\mu_6(v_1, L_2, L_3, v_4, R_3, R_2) = \Omega$.

Thus, $H_*(\mathcal{P}(m,0)) \cong \mathcal{C}(m,m-1)[t]$, provided we introduce to $\mathcal{C}(m,m-1)[t]$ new μ_2 operations, such as $\mu_2(v_1 \cdots v_i, v_{i+1} \cdots v_m) = t$ for all $1 < i < m$.

When $k = 0$, we have $\mathcal{C}(m,0) \cong \mathbb{F}$, while $H_*(\mathcal{P}(m,m-1)) \cong \mathbb{F}[\Omega]$. Thus, Theorem 1.1 holds on additively (i.e., Theorem 6.2 holds), but there cannot be a non-trivial μ_{2m}-action.

Note that Theorem 1.2 applies when $k = m - 1$. When $k = 0$, the theorem is also true, but it is a tautology: in that case, $Q(m,0) \cong \mathbb{F}$, $\mathcal{C}(m,k) = 0$, so that $\operatorname{Cobar}(\mathcal{C}(m,k)) \cong \mathbb{F}$.

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