



Robin Ming Chen · Samuel Walsh · Miles H. Wheeler

Global bifurcation for monotone fronts of elliptic equations

Received November 15, 2022

Abstract. We present two results on global continuation of monotone front-type solutions to elliptic PDEs posed on infinite cylinders. This is done under quite general assumptions, and in particular applies even to fully nonlinear equations as well as quasilinear problems with transmission boundary conditions. Our approach is rooted in the analytic global bifurcation theory of Dancer (1973) and Buffoni–Toland (2003), but extending it to unbounded domains requires contending with new potential limiting behavior relating to loss of compactness. We obtain an exhaustive set of alternatives for the global behavior of the solution curve that is sharp, with each possibility having a direct analogue in the bifurcation theory of second-order ODEs.

As a major application of the general theory, we construct global families of internal hydrodynamic bores. These are traveling front solutions of the full two-phase Euler equation in two dimensions. The fluids are confined to a channel that is bounded above and below by rigid walls, with incompressible and irrotational flow in each layer. Small-amplitude fronts for this system have been obtained by several authors. We give the first large-amplitude result in the form of continuous curves of elevation and depression bores. Following the elevation curve to its extreme, we find waves whose interfaces either overturn (develop a vertical tangent) or become exceptionally singular in that the flow in both layers degenerates at a single point on the boundary. For the curve of depression waves, we prove that either the interface overturns or it comes into contact with the upper wall.

Keywords: global bifurcation, water waves, internal waves, free boundary problem.

Contents

1. Introduction	5194
2. Abstract global continuation	5206

Robin Ming Chen: Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA; mingchen@pitt.edu

Samuel Walsh: Department of Mathematics, University of Missouri, Columbia, MO 65211, USA; walshsa@missouri.edu

Miles H. Wheeler: Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK; mw2319@bath.ac.uk

Mathematics Subject Classification 2020: 35B32 (primary); 35J66, 35Q31, 76B03, 76B55, 35R35 (secondary).

3. Application to variational and transmission problems	5218
4. Large fronts for semilinear Robin problems	5221
5. Large-amplitude hydrodynamic bores	5232
A. Principal eigenvalues and the spectral condition	5257
B. Quoted results	5265
References	5268

1. Introduction

Let $\Omega = \mathbb{R} \times \Omega'$ be an unbounded cylinder whose base $\Omega' \subset \mathbb{R}^{n-1}$ is bounded. For simplicity, assume that Ω is connected with a $C^{k+2+\alpha}$ boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$ for a fixed $k \geq 0$ and $\alpha \in (0, 1)$ and such that $\Gamma_0 \cap \Gamma_1 = \emptyset$. Note that $\Gamma_0 = \mathbb{R} \times \Gamma'_0$ and $\Gamma_1 = \mathbb{R} \times \Gamma'_1$ for some $\Gamma'_0, \Gamma'_1 \subset \partial\Omega'$. Points in Ω will be denoted (x, y) , where $x \in \mathbb{R}$ and $y \in \Omega'$.

We will study nonlinear PDEs set on Ω taking the quite general form

$$\begin{cases} \mathcal{F}(y, u, \nabla u, D^2 u, \Lambda) = 0 & \text{in } \Omega, \\ \mathcal{G}(y, u, \nabla u, \Lambda) = 0 & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_0, \end{cases} \quad (1.1)$$

where $\Lambda \in \mathbb{R}^m$ is a collection of parameters, and $\mathcal{F}$ and $\mathcal{G}$ are regular enough that the map

$$\begin{aligned} \mathcal{V} &\rightarrow C^{k+\alpha}(\overline{\Omega'}) \times C^{k+1+\alpha}(\Gamma'_1), \\ (z, \xi, r, \Lambda) &\mapsto (\mathcal{F}(\cdot, z, \xi, r, \Lambda), \mathcal{G}(\cdot, z, \xi, \Lambda)), \end{aligned} \quad \text{is real-analytic,} \quad (1.2)$$

for some open set $\mathcal{V} \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^{n \times n} \times \mathbb{R}^m$. We assume that (1.1) is uniformly elliptic with a uniformly oblique boundary condition on Γ_1 : there exist $c_1, c_2 > 0$ such that

$$\mathcal{F}_{r_{ij}}(\cdot, z, \xi, r, \Lambda) \eta_i \eta_j \geq c_1 |\eta|^2 \quad \text{on } \overline{\Omega}, \quad \mathcal{G}_{\xi i}(\cdot, z, \xi, \Lambda) v_i > c_2 \quad \text{on } \Gamma'_1 \quad (1.3)$$

for all $(z, \xi, r, \Lambda) \in \mathcal{V}$ and $\eta \in \mathbb{R}^n$. Here $v = v(y)$ is the outward-pointing normal to Ω at $(x, y) \in \Gamma_1$.

We call a solution (u, Λ) of (1.1) a *front* if $u \in C_b^{k+2+\alpha}(\overline{\Omega})$ has well-defined pointwise limits as $x \rightarrow -\infty$ and $x \rightarrow \infty$. Anticipating applications to water waves, we call these limits the upstream and downstream states, respectively. From the structure of the equation, one can further infer that $\partial^\beta u$ will have uniform limits as $x \rightarrow \pm\infty$ for any $|\beta| \leq k+2$, and hence that the upstream and downstream states are x -independent solutions of (1.1). We say a front is *monotone* if $\partial_x u \leq 0$ (or $\partial_x u \geq 0$) in Ω , and *strictly monotone* if $\partial_x u < 0$ (or $\partial_x u > 0$) in $\Omega \cup \Gamma_1$. Fronts are studied in a vast array of physical settings, including the spread of invasive species or alleles in biology, and, more broadly, phase transitions in reaction-diffusion equations.

Our purpose in this work is two-fold. First, we develop a systematic approach to constructing large monotone fronts through analytic global bifurcation theory. For this, substantial new analysis is needed to overcome a host of issues stemming from the unboundedness of the domain. Ultimately, we obtain a list of alternatives for the limiting behavior

of the bifurcation curve that is sharp, with each alternative having an analogue for second-order ODEs. At the same time, we impose only minimal conditions on the structure of the equation, making the resulting machinery quite robust.

The paper's second part concerns a longstanding open problem in water waves. Using the general theory, we construct many curves of large-amplitude hydrodynamic bores of elevation and depression. These are, respectively, strictly increasing and strictly decreasing front solutions of the full two-phase free boundary Euler equations. Of special significance is that, in the limit along the curve of elevation bores, the waves overturn (the free surface develops a vertical tangent) or else develop a highly degenerate singularity. Overhanging steady gravity water waves were observed numerically nearly 40 years ago, but a proof of their existence continues to be one of the most sought-after results in the field. Following the curve of depression bores, we find that the free boundary either overturns or contacts the upper wall. In the latter case, the flow is expected to approach a *gravity current* – a type of traveling wave that has been investigated extensively in fluid mechanics, both experimentally and computationally [60]. While formal analytical studies date back at least to the famous work of von Kármán [67] in 1940, gravity currents have never been constructed rigorously.

We begin in the next section by describing the global continuation theorems in the general setting. The application to water waves is then discussed in Section 1.3.

1.1. Statement of abstract results

To simplify the notation, we introduce the spaces

$$\mathcal{X} := \{u \in C^{k+2+\alpha}(\bar{\Omega}) : u|_{\Gamma_0} = 0\}, \quad \mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2 := C^{k+\alpha}(\bar{\Omega}) \times C^{k+1+\alpha}(\Gamma_1).$$

Note that elements of $\mathcal{X}$ and $\mathcal{Y}$ are *locally* Hölder continuous. By convention, we say $u_n \rightarrow u$ in $C_{\text{loc}}^{\ell+\beta}$ provided $u_n \rightarrow u$ in $C^{\ell+\beta}$ on any compact subset of $\bar{\Omega}$. On the other hand, we denote by $\mathcal{X}_b$ and $\mathcal{Y}_b$ the Banach spaces of functions whose corresponding Hölder norms are finite. Finally, let $\mathcal{X}'$ be the subspace of $\mathcal{X}$ consisting of functions that are independent of x , and likewise for $\mathcal{Y}'$.

The PDE (1.1) can then be rewritten as the abstract operator equation

$$\mathcal{F}(u, \Lambda) = 0,$$

and from (1.2) it follows that $\mathcal{F}$ is a real-analytic mapping $\mathcal{X}_b \times \mathbb{R}^m \supset \mathcal{U} \rightarrow \mathcal{Y}_b$ for some open set $\mathcal{U}$ determined by $\mathcal{V}$.

For a monotone front (u, Λ) , we define the *transversal linearized operator at $x = \pm\infty$* to be the bounded linear mapping

$$\mathcal{L}'_{\pm}(u, \Lambda) : \mathcal{X}' \rightarrow \mathcal{Y}', \quad w \mapsto \lim_{x \rightarrow \pm\infty} \mathcal{F}_u(u, \Lambda)w. \quad (1.4)$$

Note that these limits exist by the discussion above, and in particular $\mathcal{L}'_{\pm}(u, \Lambda)$ is a linear elliptic operator with mixed boundary conditions posed on Ω' . One can then show that $\mathcal{L}'_{\pm}(u, \Lambda)$ has a principal eigenvalue that we will denote by $\sigma_0^{\pm}(u, \Lambda)$; see Appendix A.1.

Suppose first that we are given a single strictly monotone front (u, Λ) . As the system (1.1) is invariant under translation in x , a simple elliptic regularity argument shows that $\partial_x u$ lies in $\ker \mathcal{F}_u(u, \Lambda)$. Let us assume that the kernel is exactly one-dimensional:

$$\ker \mathcal{F}_u(u, \Lambda) = \text{span} \{\partial_x u\}. \quad (\text{H1})$$

It is well known that the Fredholm properties of $\mathcal{F}_u(u, \Lambda)$ are determined by the limiting linearized operators upstream and downstream (see, for example, [65, 66]). In this work, we focus on the situation where

$$\sigma_0^-(u, \Lambda), \sigma_0^+(u, \Lambda) < 0. \quad (\text{H2})$$

Loosely speaking, this corresponds to the upstream and downstream states being spectrally stable in a sense to be discussed shortly.

We will show that (H2) implies in particular that $\mathcal{F}_u(u, \Lambda)$ is Fredholm index 0. Generically, then, one expects the zero-set of $\mathcal{F}$ to be locally a curve provided that the parameter space is two-dimensional ($m = 2$), though the solutions on it need not be monotone nor even fronts. The next theorem says something much stronger: there exists a *global* curve of strictly monotone fronts. This curve is maximal in a certain sense among all (locally) analytic curves containing (u, Λ) , and its limiting behavior is characterized by a set of four alternatives, all of which are realizable.

Theorem 1.1 (Global implicit function theorem). *Consider the elliptic PDE (1.1) with two parameters $\Lambda = (\lambda, \mu)$. Suppose that $(u_0, \lambda_0, \mu_0) \in \mathcal{U}$ is a strictly monotone front solution to (1.1) satisfying the nondegeneracy condition (H1), the spectral condition (H2), and the transversality condition*

$$\mathcal{F}_\mu(u_0, \lambda_0, \mu_0) \notin \text{rng } \mathcal{F}_u(u_0, \lambda_0, \mu_0). \quad (\text{H3})$$

Then there exists a global curve $\mathcal{C} \subset \mathcal{U}$ of strictly monotone front solutions with the parameterization

$$\mathcal{C} = \{(u(s), \lambda(s), \mu(s)) : s \in \mathbb{R}\} \subset \mathcal{F}^{-1}(0)$$

for some continuous $\mathbb{R} \ni s \mapsto (u(s), \lambda(s), \mu(s)) \in \mathcal{U}$ with $(u(0), \lambda(0), \mu(0)) = (u_0, \lambda_0, \mu_0)$.

(a) (Alternatives) *As $s \rightarrow \infty$, one of four alternatives must occur:*

(A1) (Blowup) *We have*

$$N(s) := \|u(s)\|_{\mathcal{X}} + |\Lambda(s)| + \frac{1}{\text{dist}((u(s), \Lambda(s)), \partial\mathcal{U})} \rightarrow \infty. \quad (1.5)$$

(A2) (Heteroclinic degeneracy) *There exist a sequence $s_n \rightarrow \infty$ and a sequence $x_n \rightarrow \pm\infty$ with*

$$(u(s_n)(\cdot + x_n, \cdot), \Lambda(s_n)) \rightarrow (u_*, \Lambda_*) \text{ in } C_{\text{loc}}^{k+2}(\bar{\Omega}) \times \mathbb{R}^2$$

for some monotone front solution $(u_, \Lambda_*) \in \mathcal{U}$, but the three limiting states*

$$\lim_{x \rightarrow \mp\infty} u_*(x, \cdot), \quad \lim_{n \rightarrow \infty} \lim_{x \rightarrow \infty} u(s_n)(x, \cdot), \quad \lim_{n \rightarrow \infty} \lim_{x \rightarrow -\infty} u(s_n)(x, \cdot),$$

are all distinct.

(A3) (Spectral degeneracy) *There exists a sequence $s_n \rightarrow \infty$ with $\sup_n N(s_n) < \infty$ such that*

$$\sigma_0^-(u(s_n), \Lambda(s_n)) \rightarrow 0 \quad \text{or} \quad \sigma_0^+(u(s_n), \Lambda(s_n)) \rightarrow 0.$$

(A4) (Loop) *$\mathcal{C}$ is a closed loop in the sense that $s \mapsto (u(s), \Lambda(s))$ is T -periodic for some $T > 0$.*

- (b) *One of the above alternatives must also occur as $s \rightarrow -\infty$. If in either of these limits the loop alternative (A4) happens, then clearly it happens in both limits.*
- (c) (Analyticity) *At each parameter value $s \in \mathbb{R}$, $\mathcal{C}$ admits a local real-analytic reparameterization.*
- (d) (Maximality) *If $\mathcal{J} \subset \mathcal{U}$ is any locally real-analytic curve of monotone front solutions to (1.1) that contains (u_0, Λ_0) and along which (H2) holds, then $\mathcal{J} \subset \mathcal{C}$.*

Global implicit function theorems have been used by many authors, including for instance very recent work [32] on pairs of rotating vortex patches. Since a preprint of the present article first appeared, Theorem 1.1 above has been adapted to show the existence of global curves of solutions to the Boussinesq $abcd$ system [17]. We remark that, at first glance, one might be concerned that the loop alternative (A4) occurs with $\mathcal{C}$ being just a family of translates of u_0 . In fact, this degenerate scenario is prevented in our construction through the use of a functional (2.5) that breaks the translation symmetry.

The next theorem addresses the related problem of continuing a given “local” curve $\mathcal{C}_{\text{loc}}$ of strictly monotone front solutions. It is most natural in this setting to consider the one-parameter case ($m = 1$), so we write $\Lambda = \lambda \in \mathbb{R}$. Usually, one obtains $\mathcal{C}_{\text{loc}}$ through a preliminary local bifurcation argument. A common scenario on unbounded domains is that $\mathcal{C}_{\text{loc}}$ originates from an x -independent solution to (1.1) that is singular in the sense that the linearized operator there fails to be Fredholm. With that in mind, suppose that $\mathcal{C}_{\text{loc}}$ admits the C^0 parameterization

$$\mathcal{C}_{\text{loc}} = \{(u(\varepsilon), \lambda(\varepsilon)) : 0 < \varepsilon < \varepsilon_0\} \subset \mathcal{U},$$

where

$$\begin{aligned} (u(\varepsilon), \lambda(\varepsilon)) &\rightarrow (u_0, \lambda_0) \in \mathcal{U} \quad \text{as } \varepsilon \rightarrow 0+, \\ \sigma_0^+(u_0, \lambda_0) &= 0 \quad \text{or} \quad \sigma_0^-(u_0, \lambda_0) = 0. \end{aligned} \tag{H4}$$

Our main global bifurcation result is then the following.

Theorem 1.2 (Global bifurcation). *Consider the elliptic PDE (1.1) with a single parameter $\Lambda = \lambda$. Let $\mathcal{C}_{\text{loc}}$ be a curve of strictly monotone front solutions bifurcating from a singular point as in (H4). Assume that at each $(u, \lambda) \in \mathcal{C}_{\text{loc}}$, the nondegeneracy (H1) and the spectral (H2) conditions hold.*

Then, possibly after translation, $\mathcal{C}_{\text{loc}}$ is contained in a global curve of strictly monotone front solutions $\mathcal{C} \subset \mathcal{U}$, parameterized as

$$\mathcal{C} := \{(u(s), \lambda(s)) : 0 < s < \infty\} \subset \mathcal{F}^{-1}(0)$$

for some continuous $\mathbb{R}_+ \ni s \mapsto (u(s), \lambda(s)) \in \mathcal{U}$ with the properties enumerated below:

- (a) As $s \rightarrow \infty$ either the blowup (A1), heteroclinic degeneracy (A2), or spectral degeneracy (A3) alternative occurs.
- (b) The curve $\mathcal{C}$ is locally real-analytic and maximal in the sense of Theorem 1.1 (d).
- (c) For all s sufficiently large, $(u(s), \lambda(s)) \notin \mathcal{C}_{\text{loc}}$. In particular, $\mathcal{C}$ is not a closed loop.

Some extended discussion of the assumptions and conclusions of Theorems 1.1 and 1.2 is given below.

On the hypotheses. In developing this theory we have endeavored to make no structural hypotheses on the system beyond analyticity (1.2) and ellipticity (1.3). Moreover, we do not impose compactness requirements on $\mathcal{F}$ except at the given front or local curve. This contrasts dramatically with the analytic global bifurcation theory in [13], for example, where the zero-set $\mathcal{F}^{-1}(0)$ must be locally compact and the restriction of $\mathcal{F}$ to it must be Fredholm index 0. Degree-theoretic global bifurcation typically makes the even stronger assumption that $\mathcal{F}$ is Fredholm index 0 and locally proper throughout its domain of definition (see, for example, [56]). These hypotheses are reasonable for elliptic PDEs set on a compact region; for classical solutions, they usually follow from Schauder theory.

On unbounded domains, however, it becomes a major analytical challenge to prove that a nonlinear elliptic operator is locally proper. Indeed, as we see in alternative (A2), the global curve may not be locally pre-compact. Nor can Fredholmness be taken for granted: as hinted at by (H4), we often wish to continue curves that bifurcate from singular points where 0 is in the essential spectrum of $\mathcal{F}_u$. Moreover, the Fredholm index is determined by the spectral properties of the linearized operators at infinity, which will in principle change in physically meaningful ways as we traverse $\mathcal{C}$. These considerations argue strongly that the loss of compactness be treated as an *alternative*. This decision shifts the difficulty from verifying properness and Fredholmness to classifying qualitatively how they might fail.

To motivate the spectral condition (H2), it is instructive to look at two specific classes of equation. Consider first the semilinear Robin problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u_y - u + g(u, \Lambda) = 0 & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_0, \end{cases} \quad (1.6)$$

set on the infinite cylinder $\Omega := \mathbb{R} \times (0, 1)$ with upper boundary $\Gamma_1 := \{y = 1\}$ and lower boundary $\Gamma_0 := \{y = 0\}$. Here $g = g(z, \Lambda)$ is a smooth nonlinearity with one or two parameters. This equation is *reversible* (invariant under reflection in x) and *variational*

(its solutions are formally critical points of a certain functional). It can be rewritten as the infinite-dimensional spatial Hamiltonian system

$$\partial_x \begin{pmatrix} u \\ v \end{pmatrix} = J \delta \mathcal{H}, \quad J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (1.7)$$

where $v := \partial_x u$, and the Hamiltonian $\mathcal{H}$ is given by

$$\mathcal{H}(u, v; x) := \int_0^1 \left(\frac{u_y^2 - v^2}{2} - uu_y + g(u, \Lambda)u_y \right) dy.$$

Suppose now that (u, Λ) is a monotone front solution to (1.6) with limiting states $U_{\pm}$. For the Hamiltonian formulation (1.7), this corresponds to a heteroclinic connection between the rest points $(U_-, 0)$ and $(U_+, 0)$. It is easily seen that $\sigma \in \mathbb{C}$ is an eigenvalue of $J \delta^2 \mathcal{H}(U_{\pm})$ if and only if σ^2 is an eigenvalue of $-\mathcal{L}'_{\pm}(u, \Lambda)$. Thus the spectral condition (H2) is equivalent to the hyperbolicity of the equilibria for the spatial dynamical system.

As a second example, consider the time-dependent reaction-diffusion equation

$$\partial_t u - \Delta u + b(y, \Lambda) \partial_x u = f(u, \Lambda), \quad (1.8)$$

where $f = f(z, \Lambda)$ is a smooth, parameter-dependent semilinear term. For discussion purposes, let $\Gamma_1 = \emptyset$ so that the boundary conditions are simply homogeneous Dirichlet. A traveling front with wave speed $c \in \mathbb{R}$ will then satisfy the elliptic PDE

$$\Delta u + (c - b(y, \Lambda)) \partial_x u + f(u, \Lambda) = 0. \quad (1.9)$$

Berestycki and Nirenberg [11] give a rather comprehensive treatment of this problem in the case that b and f are independent of Λ and c serves as the parameter. This assumption, however, considerably simplifies the global behavior as the set of x -independent solutions and the corresponding limiting transversal linearized operators will be fixed.

Let (u, Λ) be a monotone front solution to (1.9) with U_- and U_+ its upstream and downstream states. As they are necessarily independent of x , $U_{\pm}$ are both stationary solutions of the time-dependent problem (1.8). The limiting linearized operator at $x = \pm\infty$ is

$$\mathcal{L}_{\pm}(u, \Lambda) = \Delta + (c - b(y, \Lambda)) \partial_x + f_z(U_{\pm}, \Lambda).$$

In Proposition A.4, we prove that the spectral assumption (H2) is equivalent to the essential spectrum of $\mathcal{L}_{\pm}$ being properly contained in the left complex half-plane $\mathbb{C}_-$. Thus both the upstream and downstream states are spectrally stable as steady state solutions of (1.8).

As this reasoning shows, for reaction-diffusion equations of the form (1.9), our theory is tailored to so-called *bistable* or *Allen–Cahn-type nonlinearities*. They are referred to as “Type C” by Berestycki and Nirenberg [11], who also impose (H2) in several of their results regarding this case. By exploiting more fully the structure of the equation – especially the damping effect of the first-order term – they are able to treat other classes

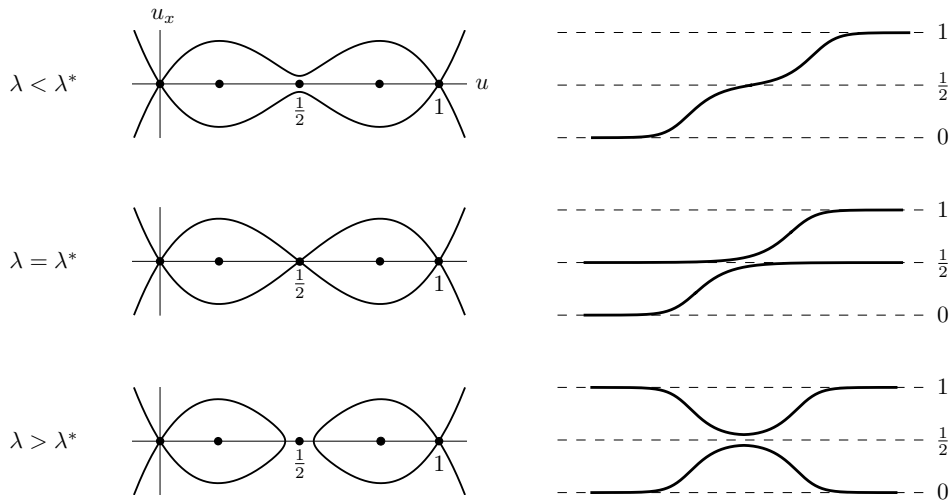


Fig. 1. An ODE $u_{xx} = f(u, \lambda)$ which experiences a heteroclinic degeneracy (A2). The left shows phase portraits for fixed values of λ , and the graphs on the right are selected heteroclinic/homoclinic solutions as functions of the independent variable x . For $\lambda < \lambda^*$, there is an increasing heteroclinic orbit connecting the constant solutions $u = 0$ and $u = 1$. At $\lambda = \lambda^*$, however, this orbit degenerates into two, one connecting $u = 0$ to $u = 1/2$, and another connecting $u = 1/2$ to $u = 1$. For $\lambda > \lambda^*$, there are instead only homoclinic orbits connecting $u = 0$ to itself and similarly for $u = 1$.

of nonlinearity as well. However, these arguments will not hold in the general context of (1.1). Indeed, one cannot expect monotonicity to persist without (H2). We note that the first-order term in (1.9) plays a similarly important role in the existence and multiplicity results of Bakker, van den Berg, and Vandervorst [8], which are based on topological methods from dynamical systems.

On the alternatives. Let us now discuss in somewhat more detail the limiting behavior along the curve. Blowup (A1) is commonly thought of as the most desirable alternative as it indicates that $\mathcal{C}$ includes arbitrarily large solutions or else limits to some pathological behavior characterized by the boundary of $\mathcal{U}$.

The intuition for the heteroclinic degeneracy alternative (A2) is best explained in terms of bifurcation of ODEs. Consider a scenario where a heteroclinic orbit between two equilibria breaks down and a new heteroclinic is born that connects one of them to an intermediate rest point as in Figure 1. This sort of breakdown can also occur for the PDE (1.1), where the upstream and downstream states play the role of the equilibria in the original heteroclinic orbit, and the intermediate equilibrium is a distinct x -independent solution. The sequence of translations in (A2) shifts this intermediate state off to $\pm\infty$, so that locally we have convergence to a new front. There is a related phenomenon for solitary waves (that is, homoclinic orbits) wherein the solution broadens into an infinitely long “table top.” This has been observed numerically in [62], for example.

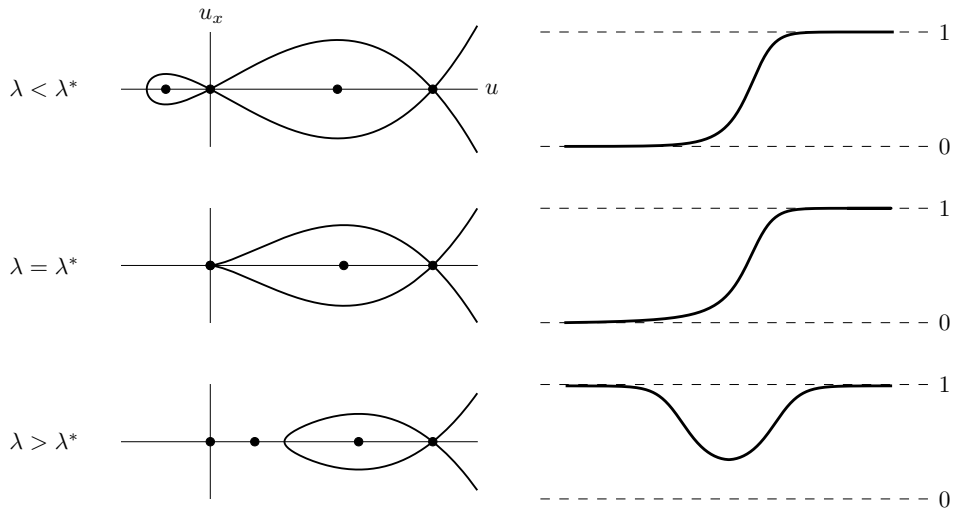


Fig. 2. An ODE $u_{xx} = f(u, \lambda)$ which experiences a spectral degeneracy (A3). The left shows phase portraits for fixed values of λ , and the graphs on the right are selected heteroclinic/homoclinic solutions as functions of the independent variable x . For $\lambda < \lambda^*$, there is an increasing heteroclinic orbit connecting the constant solutions $u = 0$ and $u = 1$. This orbit persists for $\lambda = \lambda^*$, but the Jacobian matrix at $u = 0$ ceases to be invertible and the orbit no longer decays exponentially as $x \rightarrow -\infty$. For $\lambda > \lambda^*$, the heteroclinic orbit degenerates into a homoclinic orbit to $u = 1$, while $u = 0$ becomes a center.

Next, consider the spectral degeneracy alternative (A3). As mentioned above, (H2) is equivalent to the essential spectrum of the limiting linearized operators being properly contained in $\mathbb{C}_-$. By standard elliptic theory, the principal eigenvalues are real and lie strictly to the right of the rest of the spectrum of $\mathcal{L}'_{\pm}$. Thus spectral degeneracy indicates resonance: the essential spectrum of the linearized problem upstream or downstream moves through the origin. This results in a loss of semi-Fredholmness and potentially relative compactness of the zero-set. For reaction-diffusion equations, it corresponds to the onset of “essential instability” [58, 59]. To continue any further would require a detailed study of this limit and the resulting linearized problem, perhaps using center manifold reduction techniques. There is little hope of successfully carrying out such an argument without making additional structural hypotheses. Even for ODEs, heteroclinic orbits may cease to exist beyond spectral degeneracy; see Figure 2.

While the statements of alternatives (A2) and (A3) appear somewhat complicated, often they can be drastically simplified. This is especially true if there are conserved quantities for the problem, such as the Hamiltonian $\mathcal{H}$ in (1.6), and if the set of x -independent solutions can be completely characterized. One then obtains a finite set of conditions on Λ that are necessary for a heteroclinic condition to exist; these we call the *conjugate flow equations* in reference to Benjamin’s seminal work [10]. They play a central role in both of our applications.

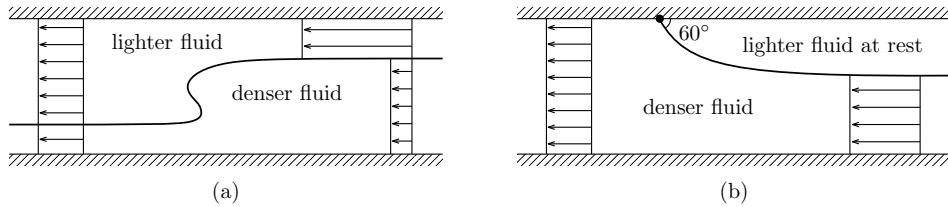


Fig. 3. Limiting configurations along the families of monotone bores from Theorem 5.1. The strictly increasing bores may overturn, meaning that the interface between the lighter and denser fluids develops a vertical tangent. Depicted in (a) is the expected state sometime after this has occurred. On the other hand, the strictly decreasing bores either overturn or else the distance between the interface and upper wall limits to 0. The latter case is illustrated in (b). Both formal analysis [67] and numerical computations [28] suggest that the free boundary will meet the wall at an angle of 60° .

Finally, let us note that there is an important (though subtle) distinction to be made regarding (A4). We consider $\mathcal{C}$ to be a closed loop if it has a global C^0 parameterization that is periodic and locally real-analytic. Clearly, this cannot be true for $\mathcal{C}$ in Theorem 1.2 due to (H4). However, it can happen that $\mathcal{C}$ *reconnects* to the singular point (u_0, λ_0) as $s \rightarrow \infty$; this possibility is captured by the spectral degeneracy alternative (A3). One can imagine this occurring, for instance, if there are multiple local bifurcation curves branching from the same singular point – a scenario that can be ruled out with a complete account of the monotone solutions nearby. For the applications presented in this paper, we accomplish such a characterization via center manifold reduction.

1.2. Large fronts for semilinear Robin problems

We present two applications of the general theory. The first is a detailed study of the semilinear Robin problem (1.6) for a class of nonlinearities g related to double well potentials. A similar family of equations was studied by Rabinowitz [57] through global variational techniques. Using a center manifold reduction, we construct a local curve of perturbative solutions. The hypotheses of Theorem 1.2 are then verified, furnishing a global curve of strictly monotone fronts. By proving uniform bounds and exploiting the Hamiltonian structure of the problem, we are able to link the qualitative properties of g to each of the alternatives (A1), (A2), and (A3). See Theorem 4.1 and Proposition 4.7 for precise statements of these results.

1.3. Large-amplitude bores

Waves in salt water bodies are frequently stratified: they exhibit large nearly homogeneous density regions that are separated by much thinner regions, called *pycnoclines*, where the density varies rapidly. It is then reasonable to treat the water as two immiscible fluids, each with constant density and governed by the incompressible Euler equations. The pycnocline accordingly becomes a surface of discontinuity for the density and a free

boundary dividing the layers. Since the 1960s, oceanographers have known that massive *internal waves* can propagate along this interface, and even remain coherent over long distances. While this phenomenon has been studied extensively in geophysics, the results are mostly restricted to linear or weakly nonlinear model equations that do not adequately capture the features of very large internal waves [37]. Front-type solutions in this context are called (smooth) *hydrodynamic bores*. They are quintessentially a product of stratification in the sense that traveling fronts in homogeneous density water do not exist [69].

To focus on the motion of the internal interface, we suppose that the water is bounded above and below by flat rigid boundaries. We also take the velocity field in each layer to be irrotational and assume there are no (horizontal) stagnation points. Under these assumptions, the system can be transformed into a quasilinear elliptic PDE with a transmission boundary condition. The first mathematical construction of heteroclinic solutions for this problem is due to Amick and Turner [7], who found small-amplitude bores in a neighborhood of the trivial state where the free boundary is flat. Later, Mielke [52], Makarenko [48], and Chen, Walsh, and Wheeler [19] obtained similar results using alternative methods.

In this paper, we give the first existence theory for genuinely large-amplitude bores; see Theorem 5.1. By means of Theorem 1.2, the local curve of solutions constructed in [19] is continued globally. Careful consideration of the conjugate flow equations enables us to rule out heteroclinic degeneracy (A2) and spectral degeneracy (A3), leaving only blowup (A1). Then, through a priori bounds, we prove that (A1) leads inexorably to stagnation: as we follow the global curve, we encounter solutions where at some point the horizontal velocity comes arbitrarily close to the speed of the wave itself. This type of limiting behavior is well known in the water waves literature. Most famously, the family of Stokes waves terminates at an “extreme wave” with a stagnation point at its crest [5]. Unlike Stokes waves, the limiting form of the interface along our solution curves is not a corner. Instead, using a novel free boundary regularity argument, we prove that the elevation bores either overturn (develop a vertical tangent) or become singular in a certain sense to be made clear shortly. On the other hand, the depression bores will limit either to an overturning front or one whose interface meets the upper wall. See Figure 3 and the precise statements in Theorem 5.2 as well as Remarks 5.3 and 5.4.

To put this into context, recall that a steady water wave is said to be *overhanging* if its free boundary elevation is multi-valued. Note that these are traveling wave solutions, so the interface will persist in this unusual configuration for all time. With surface tension but absent gravity (pure capillary waves) explicit solutions of this type were discovered by Crapper [22]. For different physical regimes, Akers, Ambrose, and Wright [1] and Córdoba, Enciso, and Grubic [21] constructed overhanging periodic capillary-gravity waves using the (local) implicit function theorem at a Crapper wave to introduce a small amount of gravity. By contrast, Ambrose, Strauss, and Wright [4] obtained periodic capillary-gravity internal waves through global bifurcation; numerical computations by the same authors found that some of these solutions are overhanging.

All of this theory relies crucially on capillarity. However, a series of remarkable numerical results in the 1980s predicted the existence of overhanging gravity waves

[55, 62]. Rather than due to surface tension, they are able to maintain their shape due to vorticity, either in the form of a background current or a vortex sheet, as is the case for internal waves. Since that time, constructing overhanging gravity waves has been among the largest open problems in water waves.

Constantin, Strauss, and Vărvăruță [20] made significant progress in this direction, proving a global bifurcation result for two-dimensional periodic gravity water waves with constant vorticity in a setting that permits overhanging. Subsequent computations by Dyachenko and Hur [30, 31] offer overwhelming numerical evidence that this family does indeed contain waves that overhang. Unfortunately, no rigorous proof is available as they are unable to exclude the possibility that a corner forms prior to overturning. Indeed, this scenario is also seen numerically for some parameter regimes. The central issue is that it is exceedingly difficult to track the qualitative features of the waves as one follows the global curve. This is particularly true when working with the Babenko-like formulation adopted in [20]. Similarly, an earlier work by Sun [61] treats periodic internal gravity waves in a two-fluid system where each layer is infinitely deep. He derives an integral equation formulation in the spirit of Nekrasov that allows overhanging waves, but he does not guarantee that they are present on the global bifurcation curve. Recent numerics [49] suggest that these waves do not in fact overturn. Other global bifurcation results for waves which may be overhanging have been established by Haziot [35] for linear stratification and constant vorticity, by Haziot and Wheeler [36] for solitary waves with constant vorticity, and by Wahlén and Weber [68] for waves with general vorticity.

Very recently, Hur and Wheeler [40] (see also [39]) found a family of explicit overturning waves with constant vorticity, infinite depth, zero gravity, and zero surface tension. Perhaps surprisingly, the surfaces of these waves are exactly the same as for Crapper's pure capillary waves, although of course the flow pattern beneath the wave is completely different. Hur and Wheeler [41] then perturbed these waves to allow for a small amount of gravity, in the same spirit as [1, 21] for capillary-gravity waves. While this rigorously confirms the existence of overturning gravity waves, many related questions remain unresolved, not least about the waves on the global bifurcation curves in [20].

Our global bifurcation argument is conducted using the Dubreil–Jacotin formulation, which forbids us from having stagnation points. However, as compensation, it is much simpler to detect overturning in these variables. For the elevation bores, we are able to eliminate the possibility of the interface meeting the walls. The only remaining obstruction to a definitive overturning result is excluding a degenerate scenario where both phases limit to stagnation at exactly the same point on the free boundary. This is the “singularity” mentioned above. In fact, double stagnation would occur at the point on the interface whose elevation coincides with the amplitude of the wave of greatest height in the one-fluid case. As far as we are aware, none of the numerical studies of this system report seeing this occur. Indeed, the paper of Dias and Vanden-Broeck [28] treats exactly our problem, and they find that the elevation bores invariably overturn. On that basis, we conjecture that the singularity alternative can be eliminated with further analysis; see Remark 5.3 for additional comments. If one can confirm overturning occurs, then in principle one can reformulate the problem in the style of Constantin et al. to continue into the

overhanging regime. At the same time, we wish to emphasize that the global bifurcation theory needed to construct bores is significantly more subtle than that for periodic waves for all the reasons discussed in Section 1.1.

1.4. Idea of the proof

We conclude the section by outlining the proof of Theorems 1.1 and 1.2 and its main novelties. The majority of the argument is contained in Section 2. We begin by fixing an appropriate functional-analytic framework for studying fronts. This is more subtle than it first appears, since we must avoid overreliance on the structural features of the equation. For example, it is a common practice in reaction-diffusion equations to establish convergence rates for the upstream and downstream limits (this is an integral component of Berestycki and Nirenberg's treatment [11, Section 3]). However, this would be very technically challenging to carry out in the general case of (1.1). Another reasonable sounding idea is to subtract a background front, so that the resulting functions vanish at infinity. Unfortunately, that introduces x -dependence into the underlying equation, which is clearly undesirable. Our approach is to instead work in a space $\mathcal{X}_\infty$ of Hölder continuous functions with well-defined limits at infinity. While unconventional for PDEs, this turns out to be a very natural setting for analyzing fronts in the general case.

In Section 2.2, a maximum principle argument is used to confirm that, as long as spectral degeneracy (A3) does not occur, strict monotonicity is both an open and closed property in a relative topology on $\mathcal{F}^{-1}(0)$. As a consequence, strict monotonicity holds on any connected subset of the zero-set containing (u, Λ) or $\mathcal{C}_{\text{loc}}$. On the other hand, in Section 2.3, we show that the zero-set is relatively pre-compact as long as the solutions it contains are strictly monotonic and heteroclinic degeneracy (A2) does not occur.

By exploiting monotonicity and the definition of $\mathcal{X}_\infty$, we are able to eliminate in a rather elegant way the kernel direction generated by the translation invariance. This is accomplished by means of a functional $\mathcal{C}$ that compares the value of u at a near-field point to the average of its values upstream and downstream at the same height. The kernel of $\mathcal{C}$ thus contains precisely one translate of any monotone front. Following that approach, we introduce a “bordered problem” whose linearization at a monotone front solution is Fredholm index 0. Through the global bifurcation theory in [18], we can then continue the zero-set of the augmented nonlinear operator to obtain a global curve that satisfies a larger set of alternatives. Tying these threads together, we complete the proofs of Theorems 1.1 and 1.2 in Section 2.5.

In Section 3 we look at the special case where (1.1) has a variational structure. We prove that for such problems, the translation invariance in x generates a conserved quantity, which can then be used to analyze the conjugate flows. We also discuss how Theorems 1.1 and 1.2 can be extended to handle problems with transmission boundary conditions.

We then turn to the applications. Global bifurcation for the semilinear Robin problem is undertaken in Section 4, while Section 5 is devoted to the hydrodynamic bore problem.

Lastly, two appendices are included. In Appendix A, we discuss the principal eigenvalues of elliptic operators on bounded domains. Some facts from the literature are recalled, and we prove a new result for transmission problems that is needed for the internal waves application. We then establish the relationship between (H2) and the essential spectrum of the linearized operators at infinity. Appendix B contains a version of the global bifurcation theory in [18] reframed in the way most convenient for the present paper.

2. Abstract global continuation

2.1. Spaces

The first task is to set down the functional-analytic framework. We want to work in a closed subspace of $\mathcal{X}_b$ that enforces front-type behavior as $x \rightarrow \pm\infty$, and then choose an appropriate codomain so that restricting to these spaces does not alter the Fredholm properties of the mapping. Our strategy is to consider spaces of Hölder continuous functions that have well-defined upstream and downstream limits; this is essentially the largest subspace of $\mathcal{X}_b$ containing all fronts. We then define a linear functional $\mathcal{C}$ on these spaces which enables us to distinguish between different translates of a strictly monotone front.

We begin by defining

$$\mathcal{X}_\infty := \left\{ u \in \mathcal{X}_b : \lim_{x \rightarrow -\infty} \partial^\beta u \text{ and } \lim_{x \rightarrow \infty} \partial^\beta u \text{ exist for all } |\beta| \leq k+2 \right\}, \quad (2.1)$$

where all of the above limits are uniform in y . Notice that as a consequence of boundedness, we know moreover that if $u \in \mathcal{X}_\infty$, then $\partial^\beta \partial_x u \rightarrow 0$ uniformly in y as $x \rightarrow \pm\infty$ for all multi-indices $|\beta| \leq k+1$. For the target space, we take

$$\mathcal{Y}_\infty := \left\{ (f_1, f_2) \in \mathcal{Y}_b : \lim_{x \rightarrow -\infty} \partial^\beta f_1 \text{ and } \lim_{x \rightarrow \infty} \partial^\beta f_1 \text{ exist for all } |\beta| \leq k, \right. \\ \left. \lim_{x \rightarrow -\infty} \partial_\tau^\beta f_2 \text{ and } \lim_{x \rightarrow \infty} \partial_\tau^\beta f_2 \text{ exist for all } |\beta| \leq k+1 \right\}, \quad (2.2)$$

where ∂_τ^β denotes a generic tangential derivative of order β along Γ_1 . Finally, let

$$\mathcal{U}_\infty := \{(u, \Lambda) \in \mathcal{U} : u \in \mathcal{X}_\infty\}.$$

The following two lemmas are immediate consequences of these definitions.

Lemma 2.1. $\mathcal{X}_\infty$ and $\mathcal{Y}_\infty$ are closed subspaces of $\mathcal{X}_b$ and $\mathcal{Y}_b$, respectively, and hence are Banach spaces.

Lemma 2.2. $\mathcal{F}(\mathcal{U}_\infty) \subset \mathcal{Y}_\infty$ and $\mathcal{F}$ is real-analytic as a mapping $\mathcal{U}_\infty \rightarrow \mathcal{Y}_\infty$.

The choice of $\mathcal{X}_\infty$ is justified by the next lemma.

Lemma 2.3. If $(u, \Lambda) \in \mathcal{U}$ is a front solution to $\mathcal{F}(u, \Lambda) = 0$, then there exist $U_\pm \in C^{k+2+\alpha}(\Omega')$ such that

$$\lim_{x \rightarrow \pm\infty} \partial^\beta u(x, \cdot) = \partial^\beta U_\pm \text{ for all } |\beta| \leq k+2 \quad (2.3)$$

uniformly in y , and consequently $u \in \mathcal{X}_\infty$ and $\mathcal{F}(U_\pm, \Lambda) = 0$.

Proof. Since (u, Λ) is a front, there exist $U_{\pm} = U_{\pm}(y)$ such that $u(x, y) \rightarrow U_{\pm}(y)$ as $x \rightarrow \pm\infty$ for each $y \in \Omega'$. Consider the doubly infinite sequence of translates $\{u_{\tau} = u(\cdot + \tau, \cdot)\}_{\tau \in \mathbb{Z}}$. Then (u_{τ}, Λ) is a front solution for each τ , and $\{u_{\tau}\}$ is uniformly bounded in $C_b^{k+2+\alpha}$. It follows that, possibly after passing to a subsequence, there exists $u_{\pm} \in \mathcal{X}_b$ such that $u_{\tau} \rightarrow u_{\pm}$ in $C_{\text{loc}}^{k+2+\alpha/2}$ as $\tau \rightarrow \pm\infty$. Clearly, we must have $u_{\pm} = U_{\pm}$, and hence $(U_{\pm}, \Lambda)$ is an x -independent solution of (1.1).

Now, let $v_{\tau} := u_{\tau} - U_{+}$ and consider the limit $\tau \rightarrow \infty$ on the bounded set $\Omega_2 := (-2, 2) \times \Omega'$. For τ sufficiently large, the translation invariance of (1.1) and the fact that v_{τ} is the difference of two solutions implies that v_{τ} satisfies a linear elliptic PDE

$$\begin{cases} a_{\tau}^{ij}(x, y) \partial_i \partial_j v_{\tau} + b_{\tau}^i(x, y) \partial_i v_{\tau} + c_{\tau}(x, y) v_{\tau} = 0 & \text{in } \Omega_2, \\ \beta_{\tau}^i(x, y) \partial_i v_{\tau} + \gamma_{\tau}(x, y) v_{\tau} = 0 & \text{on } \Gamma_1 \cap \partial\Omega_2, \\ v_{\tau} = 0 & \text{on } \Gamma_0 \cap \partial\Omega_2, \end{cases} \quad (2.4)$$

where we are using the summation convention with the shorthand $\nabla_{(x,y)} =: (\partial_1, \dots, \partial_n)$. The coefficients are defined in terms of the convex combinations $u_{\tau}^{(s)} := su_{\tau} + (1-s)U_{+}$ by the integrals

$$\begin{aligned} a_{\tau}^{ij} &:= \int_0^1 \mathcal{F}_{r^{ij}}(y, u_{\tau}^{(s)}, \nabla u_{\tau}^{(s)}, D^2 u_{\tau}^{(s)}, \Lambda) ds, \\ b_{\tau}^i &:= \int_0^1 \mathcal{F}_{\xi i}(y, u_{\tau}^{(s)}, \nabla u_{\tau}^{(s)}, D^2 u_{\tau}^{(s)}, \Lambda) ds, \\ c_{\tau} &:= \int_0^1 \mathcal{F}_z(y, u_{\tau}^{(s)}, \nabla u_{\tau}^{(s)}, D^2 u_{\tau}^{(s)}, \Lambda) ds, \quad \beta_{\tau}^i := \int_0^1 \mathcal{G}_{\xi i}(y, u_{\tau}^{(s)}, \nabla u_{\tau}^{(s)}, \Lambda) ds, \\ \gamma_{\tau} &:= \int_0^1 \mathcal{G}_z(y, u_{\tau}^{(s)}, \nabla u_{\tau}^{(s)}, \Lambda) ds. \end{aligned}$$

These integrands are well defined for τ sufficiently large because

$$(u_{\tau}^{(s)}, \nabla u_{\tau}^{(s)}, D^2 u_{\tau}^{(s)}) \rightarrow (U_{+}, \nabla U_{+}, D^2 U_{+}) \quad \text{as } \tau \rightarrow \infty,$$

uniformly in $s \in [0, 1]$ and $(x, y) \in \Omega_2$. It is easy to see that the uniform bounds on u_{τ} and U_{+} in $C_b^{k+2+\alpha}(\overline{\Omega})$ imply that the $C^{k+\alpha}$ norms of $a_{\tau}^{ij}, b_{\tau}^i, c_{\tau}$ as well as the $C^{k+1+\alpha}$ norms of $\beta_{\tau}^i, \gamma_{\tau}$ are uniformly bounded. As (2.4) is uniformly elliptic with a uniformly oblique boundary condition, we have a (linear) Schauder estimate

$$\|v_{\tau}\|_{C^{k+2+\alpha}(\Omega_1)} \leq C \|v_{\tau}\|_{C^0(\Omega_2)},$$

where the constant $C > 0$ is independent of τ and $\Omega_1 := (-1, 1) \times \Omega'$. Since $v_{\tau} = u_{\tau} - U_{+} \rightarrow 0$ in $C_b^0(\overline{\Omega_2})$, we conclude that $v_{\tau} \rightarrow 0$ in $C_b^{k+2+\alpha}(\overline{\Omega_1})$. Recalling the definition, this shows that

$$\partial^{\beta} u(x, \cdot) \rightarrow \partial^{\beta} U_{+} \quad \text{for all } |\beta| \leq k+2,$$

as $x \rightarrow \infty$, uniformly in y . Redefining $v_{\tau} := u_{\tau} - U_{-}$ and running the same argument for the limit $\tau \rightarrow -\infty$, we obtain (2.3). The fact that $u \in \mathcal{X}_{\infty}$ is then clear from the definition in (2.1). ■

Clearly, translation in x is an isometry on $\mathcal{X}_\infty$. When necessary, we will kill this symmetry by introducing the bounded linear functional

$$\mathcal{C}: \mathcal{X}_\infty \rightarrow \mathbb{R}, \quad \mathcal{C}u := u(0, y_0) - \frac{1}{2} \left(\lim_{x \rightarrow \infty} u(x, y_0) + \lim_{x \rightarrow -\infty} u(x, y_0) \right), \quad (2.5)$$

where y_0 is an arbitrary but fixed point in $\Omega' \cup \Gamma'_1$. If (u, Λ) is a strictly monotone front, then exactly one translate of u lies in the kernel of $\mathcal{C}$, and moreover $\mathcal{C}u_x = u_x(0, y_0) \neq 0$.

2.2. Monotonicity

An important component of the continuation argument is that the solutions along the global curve inherit the strict monotonicity of the solution at the bifurcation point, in the case of Theorem 1.1, or along the local curve in the setting of Theorem 1.2. We accomplish this by showing that monotonicity is both an open and closed property in an appropriate relative topology on $\mathcal{F}^{-1}(0)$ so long as the spectral condition (H2) holds. The main tools will be the maximum principle and Hopf boundary-point lemma; see Lemma B.2.

In particular, we will use many times the idea of “quasilinearizing” the equation. Because the PDE (1.1) is translation invariant in the x -direction, it follows that, for any solution $(u, \Lambda) \in \mathcal{U}$, $\partial_x u$ is in the kernel of the linearized operator at (u, λ) . This operator $\mathcal{L} := \mathcal{F}_u(u, \Lambda)$ takes the form

$$\begin{aligned} \mathcal{L}_1 v &:= a^{ij}(x, y) \partial_i \partial_j v + b^i(x, y) \partial_i v + c(x, y) v, \\ \mathcal{L}_2 v &:= (\beta^i(x, y) \partial_i v + \gamma(x, y) v)|_{\Gamma_1}, \end{aligned} \quad (2.6)$$

where $a^{ij}, b^i, c \in C_b^{k+\alpha}(\bar{\Omega})$, $\beta^i, \gamma \in C_b^{k+1+\alpha}(\Gamma_1)$ and, in light of (1.3), $\mathcal{L}_1$ is a uniformly elliptic operator, while $\mathcal{L}_2$ is a uniformly oblique boundary operator. Elliptic regularity ensures that $\partial_x u \in C^{k+2+\alpha}(\bar{\Omega})$. If $u \in \mathcal{X}_\infty$, then the coefficients have well-defined limits as $x \rightarrow \pm\infty$, and $\mathcal{L}$ limits to a linearized operator at $x = \pm\infty$, which we denote by

$$\begin{aligned} \mathcal{L}_{1\pm} v &:= a_\pm^{ij}(y) \partial_i \partial_j v + b_\pm^i(y) \partial_i v + c_\pm(y) v, \\ \mathcal{L}_{2\pm} v &:= (\beta_\pm^i(y) \partial_i v + \gamma_\pm(y) v)|_{\Gamma_1}. \end{aligned} \quad (2.7)$$

The next lemma gives a crucial implication of the spectral condition (H2): the existence of a comparison function $\varphi_\pm$ that permits us to apply the maximum principle in a neighborhood of $x = \pm\infty$. Without loss of generality, here and in the remainder of the section, we present the argument for the decreasing case.

Lemma 2.4 (Comparison function). *Let $(u, \Lambda) \in \mathcal{U}_\infty$ satisfying (H2) be given. For all $\delta > 0$ sufficiently small, there exists $\varphi_\pm \in C^{k+2+\alpha}(\bar{\Omega}')$ with $\varphi_\pm > 0$ on $\bar{\Omega}'$ and such that*

$$\begin{cases} (\mathcal{L}'_{1\pm} + \delta)\varphi_\pm = 0 & \text{in } \Omega', \\ \mathcal{L}'_{2\pm}\varphi_\pm = 1 & \text{on } \Gamma'_1, \\ \varphi_\pm = 1 & \text{on } \Gamma'_0. \end{cases} \quad (2.8)$$

Proof. By continuity, the operator $(\mathcal{L}'_{1\pm} + \delta, \mathcal{L}'_{2\pm})$ has a strictly negative principal eigenvalue for $\delta > 0$ sufficiently small; see Lemma A.2. We may therefore define $\varphi_{\pm} \in C^{k+2+\alpha}(\overline{\Omega'})$ to be the (unique) solution to (2.8), which in particular ensures $\varphi_{\pm} > 0$ on Γ'_0 and $\mathcal{L}'_{2\pm}\varphi_{\pm} \geq 0$. The unique solvability of (2.8) follows from [47, Theorem 4.1] and a standard homotopy argument similar to the proof of Lemma A.5. The negativity of the principal eigenvalue then enables us to appeal to Theorem A.1(c) to conclude further that $\varphi_{\pm} > 0$ on all of $\overline{\Omega'}$. ■

Lemma 2.5 (Asymptotic monotonicity). *For $M > 0$, denote*

$$\Omega_+^M := (M, \infty) \times \Omega', \quad \Omega_-^M := (-\infty, -M) \times \Omega',$$

and let $\Gamma_{0\pm}^M$ and $\Gamma_{1\pm}^M$ be the corresponding boundary components of $\Omega_{\pm}^M$. For any $(\bar{u}, \bar{\Lambda}) \in \mathcal{F}^{-1}(0) \cap \mathcal{U}_{\infty}$ satisfying (H2), there exist $\varepsilon > 0$ and $M > 0$ such that any other $(u, \Lambda) \in \mathcal{F}^{-1}(0) \cap \mathcal{U}_{\infty}$ with

$$\|u - \bar{u}\|_{C^2(\Omega_{\pm}^M)} + |\Lambda - \bar{\Lambda}| < \varepsilon \quad \text{and} \quad \partial_x u \leq 0 \text{ on } \{x = \pm M\},$$

satisfies

$$\partial_x u = 0 \quad \text{in } \overline{\Omega_{\pm}^M} \quad \text{or} \quad \partial_x u < 0 \quad \text{in } \Omega_{\pm}^M \cup \Gamma_{1\pm}^M.$$

Proof. Let $(\bar{u}, \bar{\Lambda})$ be given as above. By hypothesis (H2) and Lemma 2.4, there exists a comparison function $\bar{\varphi}_{\pm}$ as in (2.8). After perhaps shrinking $\varepsilon > 0$, (u, Λ) also satisfies (H2), and the associated comparison function $\varphi_{\pm}$ (with the same δ) has $\varphi_{\pm} \rightarrow \bar{\varphi}_{\pm}$ in $C^2(\overline{\Omega'})$ as $\varepsilon \rightarrow 0$. Taking $u_x =: v\varphi_{\pm}$, we see that $v \in C_b^{k+2+\alpha}(\overline{\Omega_{\pm}^M})$, vanishes in the limit $x \rightarrow \pm\infty$, and solves

$$\begin{cases} a^{ij}(x, y)\partial_i\partial_j v + \left(\frac{a^{ij}(x, y)\partial_j\varphi_{\pm}}{\varphi_{\pm}} + b^i(x, y) \right) \partial_i v + \frac{\mathcal{L}_1\varphi_{\pm}}{\varphi_{\pm}} v = 0 & \text{in } \Omega_{\pm}^M, \\ \beta^i(x, y)\partial_i v + \frac{\mathcal{L}_2\varphi_{\pm}}{\varphi_{\pm}} v = 0 & \text{on } \Gamma_{1\pm}^M, \\ v = 0 & \text{on } \Gamma_{0\pm}^M. \end{cases} \quad (2.9)$$

By choosing $\varepsilon > 0$ sufficiently small and $M > 0$ sufficiently large, we can make

$$\left\| \frac{\mathcal{L}_1\varphi_{\pm}}{\varphi_{\pm}} - \frac{\bar{\mathcal{L}}'_{1\pm}\bar{\varphi}_{\pm}}{\bar{\varphi}_{\pm}} \right\|_{C^0(\Omega_{\pm}^M)}, \quad \left\| \frac{\mathcal{L}_2\varphi_{\pm}}{\varphi_{\pm}} - \frac{\bar{\mathcal{L}}'_{2\pm}\bar{\varphi}_{\pm}}{\bar{\varphi}_{\pm}} \right\|_{C^0(\Gamma_{1\pm}^M)}$$

arbitrarily small. In particular, since

$$\frac{\bar{\mathcal{L}}'_{1\pm}\bar{\varphi}_{\pm}}{\bar{\varphi}_{\pm}} = -\delta < 0, \quad \frac{\bar{\mathcal{L}}'_{2\pm}\bar{\varphi}_{\pm}}{\bar{\varphi}_{\pm}} = \frac{1}{\bar{\varphi}_{\pm}} > 0,$$

we can pick ε and M so that the zeroth order coefficients in (2.9) have the strict signs

$$\frac{\mathcal{L}_1\varphi_{\pm}}{\varphi_{\pm}} < 0 \quad \text{on } \Omega_{\pm}^M, \quad \frac{\mathcal{L}_2\varphi_{\pm}}{\varphi_{\pm}} > 0 \quad \text{on } \Gamma_{1\pm}^M. \quad (2.10)$$

We can now conclude the proof using the maximum principle and the Hopf lemma as follows. By assumption, v vanishes on $\Gamma_{0\pm}^M$ and in the limit $x \rightarrow \pm\infty$. Moreover, from the positivity of $\varphi_{\pm}$, we also know that $v \leq 0$ on $\{x = \pm M\}$. So assume for the sake of contradiction that v achieves a nonnegative maximum at some point $(x_0, y_0) \in \Omega_{\pm}^M \cup \Gamma_{1\pm}^M$. Thanks to the first inequality in (2.10), if $(x_0, y_0) \in \Omega_{\pm}^M$ then v must vanish identically. Similarly, if $(x_0, y_0) \in \Gamma_{1\pm}^M$, then the Hopf boundary-point lemma implies that either v vanishes identically or else

$$0 < \beta^i \partial_i v = -\frac{\mathcal{L}_2 \varphi_{\pm}}{\varphi_{\pm}} v \quad \text{at } (x_0, y_0),$$

which contradicts $v(x_0, y_0) \geq 0$ and the second inequality in (2.10). The statement now follows by recalling the relation between v and $\partial_x u$. ■

Lemma 2.6 (Open property). *Let $(\bar{u}, \bar{\Lambda}) \in \mathcal{U}_{\infty}$ be a strictly monotone front solution to (1.1) satisfying (H2). There exists $\delta = \delta(\bar{u}, \bar{\Lambda}) > 0$ such that any solution $(u, \lambda) \in \mathcal{U}_{\infty}$ of (1.1) with*

$$\|u - \bar{u}\|_{C^2(\Omega)} + |\Lambda - \bar{\Lambda}| < \delta$$

is also strictly monotone.

Proof. Choose $\varepsilon > 0$ and $M > 0$ as in Lemma 2.5. First, we consider the situation on the finite cylinder $(-2M, 2M) \times \Omega'$. By assumption, $\partial_x \bar{u} < 0$ in $\Omega \cup \Gamma_1$. Moreover, since $\partial_x \bar{u}$ vanishes on Γ_0 , the Hopf boundary-point lemma applied to the elliptic equation $\bar{\mathcal{L}}_1 \partial_x u = 0$ yields

$$v \cdot \nabla \partial_x \bar{u} \geq c_2 > 0 \quad \text{on } [-2M, 2M] \times \Gamma'_0$$

for some $c_2 = c_2(M)$. Choosing $\delta > 0$ sufficiently small we can therefore ensure that

$$\partial_x u < 0 \quad \text{on } [-2M, 2M] \times (\Omega' \cup \Gamma'_1).$$

In particular, this also implies that $\partial_x u < 0$ on $\{x = \pm M\}$. Shrinking δ further if necessary so that $\delta < \varepsilon$, Lemma 2.5 then furnishes the strict monotonicity of u on the tail regions $\Omega_{\pm}^M \cup \Gamma_{1\pm}^M$. ■

We also need an analogue of Lemma 2.6 for limiting states.

Lemma 2.7 (Open property for limiting states). *Let $(\bar{u}, \bar{\Lambda}) \in \mathcal{U}_{\infty}$ be a front solution to (1.1) satisfying (H2), and let $\bar{U}$ be one of its limiting states. Then all solutions (u, Λ) in an open neighborhood of $(\bar{U}, \bar{\Lambda})$ in $\mathcal{F}^{-1}(0) \cap \mathcal{U}$ depend only the transverse variable y .*

Proof. By Lemma A.5, the spectral condition (H2) implies that the linearized operator $\mathcal{F}_u(\bar{U}, \bar{\Lambda})$ is invertible $\mathcal{X}_b \rightarrow \mathcal{Y}_b$. Applying the analytic implicit function theorem, there is therefore an open neighborhood of $(\bar{U}, \bar{\Lambda})$ in $\mathcal{F}^{-1}(0) \cap \mathcal{U}$ which can be expressed as a graph $\mathcal{M} = \{(u(\Lambda), \Lambda) : \Lambda \in \mathcal{I}\}$, where $\mathcal{I}$ is a neighborhood of Λ in $\mathbb{R}^m$ and $u: \mathcal{I} \rightarrow \mathcal{X}_b$ is analytic.

Now let $\mathcal{F}': \mathcal{U}' \rightarrow \mathcal{Y}'$ be the restriction of $\mathcal{F}$ to functions of $y \in \Omega'$ alone, written $\mathcal{F}' = \mathcal{F}'(U, \Lambda)$. As in the proof of Lemma 2.4, (H2) also implies that the linearized operator $\mathcal{F}'_U(\bar{U}, \bar{\Lambda})$ is invertible. Thus there is an open neighborhood of $(\bar{U}, \bar{\Lambda})$ in $\mathcal{F}^{-1}(0) \cap \mathcal{U}'$ which is a graph $\mathcal{M}' = \{(U(\Lambda), \Lambda) : \Lambda \in \mathcal{I}'\}$. Perhaps after shrinking $\mathcal{I}$, uniqueness forces $\mathcal{M} \subset \mathcal{M}'$, and the proof is complete. ■

Lemma 2.8 (Closedness). *Suppose that $\{(u_n, \Lambda_n)\} \subset \mathcal{U}_\infty$ is a sequence of strictly monotone front solutions of (1.1). If $(u_n, \Lambda_n) \rightarrow (u, \Lambda)$ in $\mathcal{X}_b \times \mathbb{R}$ for some $(u, \Lambda) \in \mathcal{U}_\infty$ satisfying (H2), then (u, Λ) is strictly monotone.*

Proof. First, observe that $\mathcal{X}_\infty$ is closed under the $C^{k+2+\alpha}$ norm, and hence $(u, \Lambda) \in \mathcal{X}_\infty \times \mathbb{R}$. Moreover, the continuity of $\mathcal{F}$ ensures that (u, Λ) is also a solution.

Suppose that u_x does not vanish identically. It is immediately clear that

$$u_x \leq 0 \quad \text{in } \bar{\Omega},$$

so suppose for the sake of contradiction that u_x vanishes at some point $(x_0, y_0) \in \Omega \cup \Gamma_1$. Since $\mathcal{L}_1 u_x = 0$, the strong maximum principle implies that $(x_0, y_0) \notin \Omega$. Unlike in the proof of Lemma 2.5 above, we can ignore the sign of the zeroth-order coefficient c because the maximum in question is 0. On the other hand, if $(x_0, y_0) \in \Gamma_1$, then the Hopf boundary-point lemma and the uniform obliqueness of $\mathcal{L}_2$ yield $\beta^i \partial_i u_x > 0$ there. But this is a contradiction since $\mathcal{L}_2 u_x = 0$ forces $\beta^i \partial_i u_x = -\gamma u_x = 0$ at (x_0, y_0) .

It remains to consider the case where u_x vanishes identically. Then u depends only on the transverse variable y , and so we can apply Lemma 2.7 with $\bar{u} = \bar{U} = u$ to see that, for n sufficiently large, u_n also only depends on y , contradicting the assumed strict monotonicity. ■

Combining the above lemmas, we arrive at the main result of this subsection.

Theorem 2.9 (Monotonicity). *Suppose that $\mathcal{K} \subset \mathcal{F}^{-1}(0) \cap \mathcal{U}_\infty$ is a connected set that contains a strictly decreasing (or strictly increasing) front. If (H2) holds on $\mathcal{K}$, then every element of $\mathcal{K}$ is a strictly decreasing (or strictly increasing) front.*

2.3. Compactness or heteroclinic degeneracy

With the monotonicity properties in hand, we next characterize the loss of compactness scenario. Previously, this was done for homoclinic solutions (solitary waves) in [18, Lemma 6.3]. The adaptation of these ideas to the heteroclinic setting is somewhat subtle, and involves the linear functional $\mathcal{C}$ introduced in Section 2.1.

Lemma 2.10 (Compactness or heteroclinic degeneracy). *Suppose that there exists a uniformly bounded sequence $\{(u_n, \Lambda_n)\} \subset \mathcal{U}_\infty$ of strictly monotone front solutions to (1.1) with $\mathcal{C}u_n = 0$. Then either*

- (i) *we can extract a subsequence such that $(u_n, \Lambda_n) \rightarrow (u, \Lambda)$ in $C_b^{k+2+\alpha}(\bar{\Omega}) \times \mathbb{R}^m$, where (u, Λ) is a monotone front solution to (1.1); or*

(ii) *there exists a sequence $x_n \rightarrow \pm\infty$ such that, after extracting a subsequence,*

$$(u_n(\cdot + x_n, \cdot), \Lambda_n) \rightarrow (u_*, \Lambda_*) \quad \text{in } C_{\text{loc}}^{k+2}(\bar{\Omega}) \times \mathbb{R}^m$$

for some $(u_, \Lambda_*) \in C_b^{k+2+\alpha}(\bar{\Omega}) \times \mathbb{R}$ that is a monotone front solution of (1.1), but the three limiting states*

$$\lim_{x \rightarrow \mp\infty} u_*(x, \cdot), \quad \lim_{n \rightarrow \infty} \lim_{x \rightarrow \infty} u_n(x, \cdot), \quad \lim_{n \rightarrow \infty} \lim_{x \rightarrow -\infty} u_n(x, \cdot),$$

are all distinct.

Indeed, we will see from the proof that the limiting states in (ii) are strictly ordered.

Proof. Extracting a subsequence, we can assume that the fronts are all increasing or decreasing. We give the proof in the decreasing case, the other case being completely analogous.

Throughout the proof, let $U_{n\pm} \in C^{k+2+\alpha}(\bar{\Omega}')$ denote the limiting states of u_n . Passing to a subsequence, we can assume that $U_{n\pm} \rightarrow U_{\pm}$ in $C^{k+2}(\bar{\Omega}')$ and $\Lambda_n \rightarrow \Lambda_* \in \mathbb{R}^m$, for some $(U_{\pm}, \Lambda_*) \in C^{k+2+\alpha}(\bar{\Omega}') \times \mathbb{R}^m$. Arguing in almost exactly the same way as in the proof of [18, Lemma 6.3], we can show that if

$$\lim_{x \rightarrow \pm\infty} \sup_n \|u_n(x, \cdot) - U_{n\pm}\|_{C^0(\Omega')} = 0,$$

then the compactness alternative (i) holds. So assume to the contrary that there exists $\varepsilon > 0$ and a sequence of points $(x_n, y_n) \in \Omega$ with $x_n \rightarrow \pm\infty$ such that

$$|u_n(x_n, y_n) - U_{n\pm}(y_n)| > \varepsilon. \quad (2.11)$$

As $\bar{\Omega}'$ is compact, passing to a subsequence we can assume that $y_n \rightarrow y_* \in \bar{\Omega}'$. Suppose that $x_n \rightarrow \infty$; the case $x_n \rightarrow -\infty$ follows from an almost identical argument. The uniform boundedness of $\{u_n\}$ in $C^{k+2+\alpha}(\bar{\Omega})$ allows us to extract a subsequence with

$$\tilde{u}_n := u_n(\cdot + x_n, \cdot) \rightarrow u_* \quad \text{in } C_{\text{loc}}^{k+2}(\bar{\Omega})$$

for some $u_* \in C_b^{k+2+\alpha}(\bar{\Omega})$. Thus (u_*, Λ_*) is a solution of (1.1) which is monotone in that $\partial_x u_* \leq 0$. This monotonicity implies the existence of pointwise limits $U_{*\pm}$ as $x \rightarrow \pm\infty$, which by Lemma 2.3 satisfy $U_{*\pm} \in C^{k+2+\alpha}(\bar{\Omega}')$ and $\mathcal{F}(U_{*\pm}, \Lambda_*) = 0$. Since $U_{n-} \geq \tilde{u}_n \geq U_{n+}$, we conclude that these four limiting states have the nonstrict ordering

$$U_- \geq U_{*-} \geq U_{*+} \geq U_+. \quad (2.12)$$

It remains to show that the three states U_{*-}, U_-, U_+ are distinct.

Let U_1, U_2 be any two of the limiting states in (2.12) with $U_1 \geq U_2$. Since $\mathcal{F}(U_1, \Lambda) = \mathcal{F}(U_2, \Lambda) = 0$ by continuity, arguing as in the proof of Lemma 2.3 we find that the difference $v := U_2 - U_1 \leq 0$ satisfies an elliptic equation of the form (2.4), except that the coefficients depend only on the transverse variable y . Applying the strong maximum

principle and the Hopf boundary-point lemma as in the proof of Lemma 2.8, we conclude that either $U_1 \equiv U_2$ or $U_1 > U_2$ on $\Omega' \cup \Gamma'_1$.

First consider the pair U_{*-}, U_+ . The assumption (2.11) implies $\tilde{u}_n(0, y_n) > U_{n+}(y_n) + \varepsilon$. Taking limits yields

$$U_{*-}(y_*) \geq u_*(0, y_*) > U_+(y_*),$$

and hence by the above argument $U_{*-} > U_+$ on $\Omega' \cup \Gamma'_1$. Plugging this into (2.12) implies that $U_- > U_+$ on $\Omega' \cup \Gamma'_1$.

Finally, consider the pair U_-, U_{*-} . Here we will need the assumption $u_n \in \ker \mathcal{C}$, i.e. at the fixed point $y_0 \in \Omega' \cup \Gamma'_1$ we have

$$u_n(0, y_0) = \frac{1}{2}(U_{n+}(y_0) + U_{n-}(y_0)).$$

Fix $x \in \mathbb{R}$. For n sufficiently large we have $-x_n < x$, and hence by monotonicity,

$$\tilde{u}_n(x, y_0) < \tilde{u}_n(-x_n, y_0) = u_n(0, y_0) = \frac{1}{2}(U_{n+}(y_0) + U_{n-}(y_0)).$$

Letting $n \rightarrow \infty$ yields

$$u_*(x, y_0) \leq \frac{1}{2}(U_+(y_0) + U_-(y_0)), \quad (2.13)$$

so that upon letting $x \rightarrow -\infty$ we recover

$$U_{*-}(y_0) \leq \frac{1}{2}(U_+(y_0) + U_-(y_0)) < U_-(y_0), \quad (2.14)$$

where the last inequality follows from $U_- > U_+$ in $\Omega' \cup \Gamma'_1$. Thus we must have

$$U_- > U_{*-} > U_+ \quad \text{on } \Omega' \cup \Gamma'_1.$$

In particular, these three functions are distinct and the proof is complete. ■

2.4. Fredholm properties

In this section, we establish some necessary facts about the Fredholm index of $\mathcal{F}_u(u, \Lambda)$. First, we consider it as a mapping between the larger spaces $\mathcal{X}_b \rightarrow \mathcal{Y}_b$.

Lemma 2.11 (Fredholm on $\mathcal{X}_b$). *Suppose that $(u, \Lambda) \in \mathcal{U}_\infty$ satisfies the spectral hypothesis (H2). Then $\mathcal{F}_u(u, \Lambda)$ is Fredholm index 0 as a mapping $\mathcal{X}_b \rightarrow \mathcal{Y}_b$.*

As the proof of this result relies on some general facts about principal eigenvalues, we postpone it to Appendix A.2.

It remains now to understand how this translates to properties of $\mathcal{F}_u(u, \Lambda)$ as a mapping $\mathcal{X}_\infty \rightarrow \mathcal{Y}_\infty$. Since $\mathcal{X}_\infty$ is a closed subspace, we know that if (u, Λ) satisfies (H2), then $\mathcal{F}_u(u, \Lambda): \mathcal{X}_\infty \rightarrow \mathcal{Y}_\infty$ is locally proper, i.e. semi-Fredholm with a finite-dimensional kernel. To characterize the range, we will need the lemma below. It is quite similar to [70, Lemma A.10].

Lemma 2.12. *Let $(u, \Lambda) \in \mathcal{U}_\infty$ satisfying (H2) be given. If $\mathcal{F}_u(u, \Lambda)w = (f_1, f_2) \in \mathcal{Y}_\infty$, for some $w \in \mathcal{X}_b$, then in fact $w \in \mathcal{X}_\infty$.*

Proof. Suppose on the contrary that there exists some $w \in \mathcal{X}_b \setminus \mathcal{X}_\infty$ such that $\mathcal{F}_u(u, \Lambda)w =: (f_1, f_2) \in \mathcal{Y}_\infty$. To obtain a contradiction, it suffices to show that all of the limits as $x \rightarrow \pm\infty$ behave as in the definition (2.1) of $\mathcal{X}_\infty$. We will only present the argument for $|\beta| \leq 2$; the rest follows by differentiating the equation in the usual way.

Assume that $\lim_{x \rightarrow \infty} \partial^\beta w$ does not exist for some $|\beta| \leq 2$. Then there is a $\delta > 0$, $y_* \in \overline{\Omega}'$, and two sequences $x_{1n}, x_{2n} \rightarrow \infty$ with

$$|\partial^\beta w(x_{1n}, y_*) - \partial^\beta w(x_{2n}, y_*)| \geq \delta.$$

Consider now the shifted functions

$$\begin{aligned} v_n &:= w(\cdot + x_{1n}, \cdot) - w(\cdot + x_{2n}, \cdot), \\ g_{in} &:= f_i(\cdot + x_{1n}, \cdot) - f_i(\cdot + x_{2n}, \cdot) \quad \text{for } i = 1, 2, \end{aligned}$$

and shifted linear operators

$$\mathcal{L}_n := \mathcal{F}_u(u(\cdot + x_{1n}, \cdot), \Lambda).$$

The uniform bound of v_n in $C^{k+2+\alpha}$ implies that we can extract a subsequence such that $v_n \rightarrow v$ in $C_{\text{loc}}^{k+2}(\overline{\Omega})$ for some $v \in \mathcal{X}_b$. Since $(f_1, f_2) \in \mathcal{Y}_\infty$, we see that $g_{1n} \rightarrow 0$ in $C_{\text{loc}}^k(\overline{\Omega})$, and $g_{2n} \rightarrow 0$ in $C_{\text{loc}}^{k+1}(\Gamma_1)$. Thus

$$\mathcal{L}_n v_n = (g_{1n}, g_{2n}) \rightarrow 0 \quad \text{in } C_{\text{loc}}^k(\overline{\Omega}) \times C_{\text{loc}}^{k+1}(\Gamma_1),$$

from which it follows that $\mathcal{L}_+ v = 0$. As $\sigma_0^+(u, \Lambda) < 0$, Lemma A.5 forces $v = 0$. On the other hand, we find that

$$|\partial^\beta v(0, y_*)| = \lim_{n \rightarrow \infty} |\partial^\beta w(x_{1n}, y_*) - \partial^\beta w(x_{2n}, y_*)| \geq \delta,$$

which is a contradiction. Arguing similarly for $x \rightarrow -\infty$, we conclude that $\partial^\beta w$ has well-defined limits as $x \rightarrow \pm\infty$ for all $|\beta| \leq k + 2$. Finally, because $w \in \mathcal{X}_b$ this also implies that $\partial^\beta \partial_x w$ vanishes as $x \rightarrow \infty$ for all $|\beta| \leq k + 1$. ■

Corollary 2.13. *Let $(u, \Lambda) \in \mathcal{U}_\infty$ be given such that (H2) holds. Then $\mathcal{F}_u(u, \Lambda)$ is Fredholm index 0 as a mapping $\mathcal{X}_\infty \rightarrow \mathcal{Y}_\infty$.*

Proof. We use the same pair of homotopies as in the proof of Lemma 2.11 in Appendix A.2, connecting the operator $\mathcal{L} := \mathcal{F}_u(u, \Lambda)$ to the operator $\mathcal{T} := (\Delta - \sigma, \partial_\nu)$, where $\sigma > 0$ is a constant and ∂_ν is the outward-pointing normal derivative. The remainder of the argument is very similar to arguments in [70, Appendix A]. By the proof of Lemma 2.11, the operators along these homotopies are locally proper $\mathcal{X}_b \rightarrow \mathcal{Y}_b$. Since $\mathcal{X}_\infty \subset \mathcal{X}_b$ and $\mathcal{Y}_\infty \subset \mathcal{Y}_b$ are closed subspaces, these operators are then also locally proper, and hence semi-Fredholm, as mappings $\mathcal{X}_\infty \rightarrow \mathcal{Y}_\infty$. By the continuity of the index, it therefore suffices to show that $\mathcal{T}$ is invertible, and hence Fredholm index 0, as

a mapping $\mathcal{X}_\infty \rightarrow \mathcal{Y}_\infty$. From the proof of Lemma 2.11, we know that $\mathcal{T}$ is invertible $\mathcal{X}_b \rightarrow \mathcal{Y}_b$, which immediately implies that it is injective $\mathcal{X}_\infty \rightarrow \mathcal{Y}_\infty$. Moreover, it is easy to see that Lemma 2.12 continues to hold with the same proof if $\mathcal{F}_u(u, \Lambda)$ replaced by $\mathcal{T}$. This and the surjectivity of $\mathcal{T}: \mathcal{X}_b \rightarrow \mathcal{Y}_b$ implies surjectivity of $\mathcal{T}: \mathcal{X}_\infty \rightarrow \mathcal{Y}_\infty$, which completes the proof. $\blacksquare$

2.5. Proof of the main theorems

We begin with Theorem 1.1, the global implicit function theorem. Let (u_0, Λ_0) be a strict monotone front solution satisfying (H1)–(H3). By Lemma 2.3, we have $u_0 \in \mathcal{X}_\infty$. Then, in light of (H2) and Corollary 2.13, the linearized operator $\mathcal{F}_u(u_0, \Lambda_0)$ is Fredholm index 0 as a mapping $\mathcal{X}_\infty \rightarrow \mathcal{Y}_\infty$. However, it is not an isomorphism due to the kernel direction generated by the translation invariance in x , and so we cannot apply Theorem B.1 directly.

Instead, we study a certain bordered problem: set $w := (u, \mu)$ and consider the new nonlinear operator

$$\mathcal{G}: (\mathcal{X}_\infty \times \mathbb{R}) \times \mathbb{R} \supset \mathcal{W} \rightarrow \mathcal{Y}_\infty \times \mathbb{R}$$

given by

$$\mathcal{G}(w, \lambda) := \begin{pmatrix} \mathcal{F}(u, \lambda, \mu) \\ \mathcal{C}_u \end{pmatrix}, \quad (2.15)$$

where $\mathcal{W}$ is an open set derived from $\mathcal{U}_\infty$ in the obvious way, and $\mathcal{C}: \mathcal{X} \rightarrow \mathbb{R}$ is the bounded linear functional defined in (2.5). Note that here we are adopting the convention that whenever $(w, \lambda) \in \mathcal{W}$, it is understood that u denotes the component of w in $\mathcal{X}_\infty$. Without loss of generality, assume that $\mathcal{C}u_0 = 0$. Appealing to Theorem B.1, we then obtain a preliminary global bifurcation curve.

Lemma 2.14 (Global IFT for $\mathcal{G}$). *There exists a global curve $\mathcal{K} \subset \mathcal{W}$, parameterized as*

$$\mathcal{K} = \{(w(s), \lambda(s)) : s \in \mathbb{R}\} \subset \mathcal{G}^{-1}(0)$$

for some continuous $\mathbb{R} \ni s \mapsto (w(s), \lambda(s)) \in \mathcal{W}$ with $(w(0), \lambda(0)) = (w_0, \lambda_0)$. Moreover:

- (a) In each of the limits $s \rightarrow \infty$ and $s \rightarrow -\infty$, one of the alternatives (A1'), (A2'), (A3'), or (A4') of Theorem B.1 is realized. If in either direction (A4') happens, then clearly it happens in both.
- (b) At each point, $\mathcal{K}$ admits a local real-analytic reparameterization.
- (c) The curve $\mathcal{K}$ is maximal in the sense that if $\mathcal{J} \subset \mathcal{G}^{-1}(0)$ is another locally real-analytic curve of monotone front solutions along which (H2) holds, and (w_0, Λ_0) is in $\mathcal{J}$, then $\mathcal{J} \subset \mathcal{K}$.

Proof. Because $\mathcal{C}$ is a bounded functional on $\mathcal{X}_\infty$, it follows from Corollary 2.13 and Fredholm bordering (see Lemma B.4) that

$$\mathcal{G}_w(w_0, \lambda_0) = \begin{pmatrix} \mathcal{F}_u(u_0, \lambda_0, \mu_0) & \mathcal{F}_\mu(u_0, \lambda_0, \mu_0) \\ \mathcal{C} & 0 \end{pmatrix}$$

is Fredholm index 0 as a mapping $\mathcal{X}_\infty \times \mathbb{R} \rightarrow \mathcal{Y}_\infty \times \mathbb{R}$. We claim moreover that it is injective and hence an isomorphism. Let $\dot{w} = (\dot{u}, \dot{\mu})$ be an element of $\ker \mathcal{G}_w(w_0, \lambda_0)$. Then from (2.15), we have

$$\mathcal{F}_u(u_0, \Lambda_0)\dot{u} + \mathcal{F}_\mu(u_0, \Lambda_0)\dot{\mu} = 0,$$

which by the transversality assumption (H3) implies that $\dot{\mu} = 0$ and $\dot{u} \in \ker \mathcal{F}_u(u_0, \Lambda_0)$. Hypothesis (H1) requires that the kernel of $\mathcal{F}_u(u_0, \lambda_0, \mu_0)$ be generated by $\partial_x u_0$. However,

$$\mathcal{C}\partial_x u_0 = \partial_x u_0(0, y_0) \neq 0,$$

since u is a strictly monotone front. Therefore, we must have $\dot{u} = 0$, which confirms that $\ker \mathcal{G}_w(w_0, \lambda_0)$ is trivial. The invertibility of $\mathcal{G}_w(w_0, \lambda_0)$ follows.

Taking $\mathcal{W} := \mathcal{X}_\infty \times \mathbb{R}$ and $\mathcal{Z} := \mathcal{Y}_\infty \times \mathbb{R}$, the above argument implies that $\mathcal{G}$ satisfies the hypotheses (B.1) of Theorem B.1. As an immediate consequence, there exists a global curve $\mathcal{K} \subset \mathcal{G}^{-1}(0)$ of solutions exhibiting the properties claimed in (a)–(c). ■

Finally, we complete the proof of Theorem 1.1 by translating the global theory for the bordered problem back to the original equation (1.1). In doing so, we will use the results of Sections 2.2 and 2.3 to confirm that strict monotonicity is preserved along the curve, which will then allow us to characterize the loss of compactness (A2') and Fredholmness alternatives (A3') in terms of heteroclinic degeneracy (A2) and spectral degeneracy (A3).

Proof of Theorem 1.1. In Lemma 2.14, we constructed a global curve $\mathcal{K}$ of solutions to the Fredholm bordered problem. Naturally, this implies the existence of a corresponding C^0 curve

$$\mathcal{C} = \{(u(s), \Lambda(s)) : s \in \mathbb{R}\} \subset \mathcal{F}^{-1}(0) \cap \mathcal{U}_\infty$$

of solutions to (1.1).

Observe that the uniformity of the limits in the definition of the space $\mathcal{X}_\infty$ and the continuity of the principal eigenvalue given by Lemma A.2 together ensure that $s \mapsto \sigma_0^\pm(u(s), \Lambda(s))$ is continuous. Perhaps making a C^0 reparameterization, we can therefore assume that (H2) holds along $\mathcal{C}$. Since $\mathcal{C}$ is connected and contains (u_0, Λ_0) , Theorem 2.9 guarantees that every $(u(s), \Lambda(s))$ is strictly monotone.

Consider now the behavior as $s \rightarrow \infty$. Assume that the heteroclinic degeneracy alternative (A2) does not occur. Since the solutions on $\mathcal{C}$ are strictly monotone, we conclude from Lemma 2.10 that $\mathcal{C}$ is locally pre-compact in $\mathcal{X}_\infty \times \mathbb{R}^2$. The curve $\mathcal{K}$ is likewise locally pre-compact, and hence the loss of compactness alternative (A2') cannot occur for it.

Suppose further that the spectral degeneracy alternative (A3) does not occur. Then if $s_n \rightarrow \infty$ with $\sup_n N(s_n) < \infty$ and $(u(s_n), \Lambda(s_n))$ converging to $(u_*, \Lambda_*) \in \mathcal{U}_\infty$, then the spectral condition (H2) must be satisfied at this limit. By Corollary 2.13, the linearized operator $\mathcal{F}_u(u_*, \Lambda_*) : \mathcal{X}_\infty \rightarrow \mathcal{Y}_\infty$ is therefore Fredholm index 0. Thus, along the curve $\mathcal{K}$, the loss of Fredholmness alternative (A3') cannot happen.

Recalling Lemma 2.14, this winnows the possible limiting behavior of $\mathcal{K}$ to either (A1') or (A4'). For $\mathcal{C}$, these lead to the blowup alternative (A1) or closed loop (A4), respectively. This proves part (a), and applying the same reasoning to the limit $s \rightarrow -\infty$ yields part (b). Part (c) is given directly by Lemma 2.14(b). Finally, if $\mathcal{J}$ is a curve of monotone front solutions as in part (d), then by Lemma 2.3 it lies in $\mathcal{U}_\infty$. The result then follows from Lemma 2.14(c) and the proof of the theorem is complete. ■

Next we turn to Theorem 1.2 on global continuation of a local curve of strictly monotone fronts.

Proof of Theorem 1.2. Let $\mathcal{C}_{\text{loc}}$ be given as in the statement of the theorem. Recalling Lemma 2.3, we know that $u(\varepsilon) \in \mathcal{X}_\infty$ for each $\varepsilon \in (0, \varepsilon_0)$. Fix any $\varepsilon_1 \in (0, \varepsilon_0)$ and set $(u_1, \lambda_1) := (u(\varepsilon_1), \lambda(\varepsilon_1))$. Then there is a unique $\tau_1 \in \mathbb{R}$ such that $\mathcal{C}u_1(\cdot - \tau_1, \cdot) = 0$, where $\mathcal{C}$ is the functional in (2.5). Let

$$c(\varepsilon, \tau) := \mathcal{C}[u(\varepsilon)(\cdot - \tau, \cdot)] \quad \text{for } (\varepsilon, \tau) \in (0, \varepsilon_0) \times \mathbb{R}.$$

By construction, $c(\varepsilon_1, \tau_1) = 0$, and we compute that

$$\partial_\tau c(\varepsilon_1, \tau_1) = -\partial_x u(\varepsilon)(-\tau_1, y_0) \neq 0.$$

The implicit function theorem furnishes a C^0 curve $\tau = \tau(\varepsilon)$, defined for $|\varepsilon - \varepsilon_1| \ll 1$, so that $c(\varepsilon, \tau(\varepsilon)) = 0$. Repeating this argument at each parameter value ensures that there exists an extension of $\tau(\varepsilon)$ to all of $(0, \varepsilon_0)$. We will identify $\mathcal{C}_{\text{loc}}$ with its image under this family of translations, so that $\mathcal{C}$ vanishes along it. Naturally, the hypotheses (H1), (H2), and (H4) are unaffected.

We claim moreover that this curve is locally real-analytic. To see this, fix again any $\varepsilon_1 \in (0, \varepsilon_0)$ and put $(u_1, \lambda_1) := (u(\varepsilon_1), \lambda(\varepsilon_1))$. As argued before, hypothesis (H2) and Corollary 2.13 imply that $\mathcal{F}_u(u_1, \lambda_1)$ is Fredholm index 0 as a map $\mathcal{X}_\infty \rightarrow \mathcal{Y}_\infty$. In view of (H1), there exists $\chi_1 \in \mathcal{Y}_\infty \setminus \{0\}$ such that

$$\mathcal{Y}_\infty = \text{rng } \mathcal{F}_u(u_1, \lambda_1) \oplus \mathbb{R}\chi_1.$$

Consider the two-parameter bordered problem

$$\mathcal{G}(w, \lambda) = 0,$$

where, as in the proof of Theorem 1.1, we are writing $w = (u, \mu) \in \mathcal{X}_\infty \times \mathbb{R}$, but now define

$$\mathcal{G}: (\mathcal{X}_\infty \times \mathbb{R}) \times \mathbb{R} \supset \mathcal{W} \rightarrow \mathcal{Y}_\infty \times \mathbb{R}, \quad (u, \lambda, \mu) \mapsto \begin{pmatrix} \mathcal{F}(u, \lambda) + \mu\chi_1 \\ \mathcal{C}u \end{pmatrix},$$

with the open set $\mathcal{W}$ derived from $\mathcal{U}_\infty$ in the same way. The local curve $\mathcal{C}_{\text{loc}}$ trivially lifts to a local C^0 curve of solutions to the bordered problem

$$\mathcal{K}_{\text{loc}} := \{(w(\varepsilon), \lambda(\varepsilon)) : \varepsilon \in (0, \varepsilon_0)\} \subset \mathcal{G}^{-1}(0),$$

where $w(\varepsilon) := (u(\varepsilon), 0)$. Clearly $\mathcal{G}$ is real-analytic, and letting $w_1 = (u_1, 0)$, we compute that

$$\mathcal{G}_w(w_1, \lambda_1) = \begin{pmatrix} \mathcal{F}_u(u_1, \lambda_1) & \chi_1 \\ \mathcal{C} & 0 \end{pmatrix}.$$

By Fredholm bordering, this is a Fredholm index 0 mapping $\mathcal{X}_\infty \times \mathbb{R} \rightarrow \mathcal{Y}_\infty \times \mathbb{R}$. Moreover, if $(\dot{w}, \dot{\mu})$ is in its kernel, then

$$\mathcal{F}_u(u_1, \lambda_1)\dot{u} + \dot{\mu}\chi_1 = 0,$$

and hence $\dot{\mu} = 0$, $\dot{u} \in \text{span}\{\partial_x u_1\}$, and $\mathcal{C}\dot{u} = 0$. But u_1 is a strictly monotone front, and so this forces $\dot{u} = 0$. Thus $\mathcal{G}_w(w_1, \lambda_1)$ is an isomorphism. The analytic implicit function theorem implies that the zero-set of $\mathcal{G}$ locally consists of a curve through (w_1, λ_1) that is real-analytic, and, by uniqueness, coincides with $\mathcal{K}_{\text{loc}}$. This confirms that at every point, $\mathcal{K}_{\text{loc}}$ (and hence $\mathcal{C}_{\text{loc}}$) admits a local real-analytic reparameterization. In fact, the same argument also shows that there can be no secondary bifurcation points along $\mathcal{C}_{\text{loc}}$; it locally comprises the complete zero-set of $\mathcal{F}$.

Fixing $\varepsilon_1 \in (0, \varepsilon_0)$ once more and defining $\mathcal{G}$ as above, we also can apply Theorem B.1 to the bordered problem at (w_1, λ_1) , yielding a global curve $\mathcal{K} \subset \mathcal{G}^{-1}(0)$. As it is maximal in $\mathcal{W}$, we must have $\mathcal{K}_{\text{loc}} \subset \mathcal{K}$. But note that $\mu = 0$ for each $(w, \lambda) \in \mathcal{K}_{\text{loc}}$, and so local analyticity implies that $\mu = 0$ on the entirety of $\mathcal{K}$. Projecting away this trivial component gives the desired global curve $\mathcal{C} \subset \mathcal{F}^{-1}(0)$ extending $\mathcal{C}_{\text{loc}}$.

In light of hypothesis (H4), we can reparameterize the curve so that

$$(u(s), \lambda(s)) \rightarrow (u_0, \lambda_0) \quad \text{as } s \rightarrow 0+,$$

where (u_0, λ_0) does not satisfy (H2). This in particular means that $\mathcal{C}$ (and hence $\mathcal{K}$) is not a closed loop. Part (a) now follows exactly as in the poof of Theorem 1.1. The local real-analyticity and maximality of $\mathcal{C}$ is likewise a consequence of Theorem B.1. Finally, (c) can be established as in the proof of [18, Theorem 6.1 (c)]. ■

3. Application to variational and transmission problems

3.1. Invariant quantities and conjugate flows for variational problems

In many applications, equation (1.1) formally arises as a variational problem

$$\delta \int_{\Omega} \mathcal{L}(y, u, \nabla u, \Lambda) \, dx \, dy = 0, \tag{3.1}$$

where the variations are taken with respect to u vanishing on Γ_0 , and $\mathcal{L}$ is regular enough that the mapping

$$\mathcal{W} \rightarrow C_b^{k+2+\alpha}(\overline{\Omega}'), \quad (z, \xi, \Lambda) \mapsto \mathcal{L}(\cdot, z, \xi, \Lambda), \tag{3.2}$$

is real-analytic for some open set $\mathcal{W} \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ related to $\mathcal{U}$. In this case (1.1) has the form

$$\begin{cases} \nabla \cdot \mathcal{L}_\xi(y, u, \nabla u, \Lambda) - \mathcal{L}_z(y, u, \nabla u, \Lambda) = 0 & \text{in } \Omega, \\ \nu \cdot \mathcal{L}_\xi(y, u, \nabla u, \Lambda) = 0 & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_0, \end{cases} \quad (3.3)$$

where as always $\nu = \nu(y)$ is the outward normal to Γ_1 .

The variational principle (3.1) is invariant under translations in x , and so one expects a corresponding conserved quantity, which is the content of the following lemma.

Lemma 3.1 (Conserved quantity). *If $(u, \Lambda) \in \mathcal{U}$ solves (3.3), then the integral*

$$\mathcal{H}(u, \Lambda; x) := \int_{\Omega'} (\mathcal{L}(y, u, \nabla u, \Lambda) - \mathcal{L}_{\xi_1}(y, u, \nabla u, \Lambda)u_x) dy$$

is a constant independent of x .

Proof. First differentiating under the integral and then using (3.3) to eliminate $\partial_x \mathcal{L}_\xi$, we find that

$$\begin{aligned} \frac{d}{dx} \mathcal{H} &= \int_{\Omega'} (\mathcal{L}_z u_x + \mathcal{L}_{\xi_1} u_{xx} + \mathcal{L}_{\xi'} \cdot \nabla_y u_x - (\partial_x \mathcal{L}_{\xi_1})u_x - \mathcal{L}_{\xi_1} u_{xx}) dy \\ &= \int_{\Omega'} ((\nabla_y \cdot \mathcal{L}_{\xi'})u_x + \mathcal{L}_{\xi'} \cdot \nabla_y u_x) dy \\ &= \int_{\Omega'} \nabla_y \cdot (\mathcal{L}_{\xi'} u_x) dy = \int_{\Gamma'_0 \cup \Gamma'_1} (\nu \cdot \mathcal{L}_\xi)u_x dS(y). \end{aligned}$$

Since $\nu \cdot \mathcal{L}_\xi$ vanishes on Γ_1 while u and hence u_x vanish on Γ_0 , we obtain $\partial_x \mathcal{H} = 0$ as desired. ■

Sending $x \rightarrow \pm\infty$ for a front solution, we obtain the following corollary.

Corollary 3.2. *Suppose that $(u, \Lambda) \in \mathcal{U}_\infty$ solves $\mathcal{F}(u, \Lambda) = 0$, and let $U_\pm$ be the limits of u as $x \rightarrow \pm\infty$. Then $U_\pm \in C^{k+2}(\Omega')$ and*

$$(C1) \quad \mathcal{F}(U_\pm, \Lambda) = 0,$$

$$(C2) \quad \mathcal{H}(U_+, \Lambda) = \mathcal{H}(U_-, \Lambda).$$

Proof. The regularity of $U_\pm$ and (C1) were already shown in Lemma 2.3. The remaining statement (C2) follows at once from Lemma 3.1:

$$\mathcal{H}(U_+, \Lambda) = \lim_{x \rightarrow \infty} \mathcal{H}(u, \Lambda; x) = \lim_{x \rightarrow -\infty} \mathcal{H}(u, \Lambda; x) = \mathcal{H}(U_-, \Lambda). \quad \blacksquare$$

Definition 3.3 (Conjugate flows). For a fixed parameter value Λ , we say that two distinct functions $U_\pm \in C^{k+2}(\Omega')$ are *conjugate* (or *conjugate flows*) if they satisfy (C1) and (C2).

The terminology “conjugate flows” comes from steady hydrodynamics; see [10]. By studying the conjugate flow problem in our applications below, we will be able to rule out heteroclinic degeneracy (A2) using the following lemma.

Lemma 3.4 (Triple conjugacy). *Suppose that the PDE (1.1) has the variational structure (3.3). Then in Lemma 2.10(ii) the three distinct limiting states*

$$\lim_{x \rightarrow -\infty} u_*(x, \cdot), \quad \lim_{n \rightarrow \infty} \lim_{x \rightarrow \infty} u_n(x, \cdot), \quad \lim_{n \rightarrow \infty} \lim_{x \rightarrow -\infty} u_n(x, \cdot),$$

are also conjugate in the sense of Definition 3.3.

Proof. As in the proof of Lemma 2.10, the limiting states U above all lie in $C^{k+2+\alpha}(\overline{\Omega'})$ and solve $\mathcal{F}(U, \Lambda_*) = 0$. Moreover, by Lemma 3.1 they all have the same value of the conserved quantity $\mathcal{H}$,

$$\mathcal{H}(U, \Lambda_*) = \lim_{n \rightarrow \infty} \mathcal{H}(u_n, \Lambda_n; 0). \quad \blacksquare$$

3.2. Transmission conditions

Finally, with only superficial modifications, the above global bifurcation theory applies to a broad class of quasilinear transmission problems. Suppose that $\Omega = \Omega_1 \cup \Omega_2$ is a (slitted) cylinder with $\Omega_1 \cap \Omega_2 = \emptyset$. Let $\Gamma_1 := \partial\Omega_1 \cap \partial\Omega_2 \neq \emptyset$ be their common boundary and set $\Gamma_0 := \partial\Omega \setminus \Gamma_1$. We require that Γ_0 and Γ_1 are $C^{k+2+\alpha}$ and disjoint. Set

$$\begin{aligned} \mathcal{F}(y, u, \nabla u, D^2 u, \Lambda) &:= \nabla \cdot \mathcal{A}(y, u, \nabla u, \Lambda) + \mathcal{B}(y, u, \Lambda), \\ \mathcal{G}(y, u, \nabla u, \Lambda) &:= \llbracket -v \cdot \mathcal{A}(y, u, \nabla u, \Lambda) \rrbracket + \mathcal{E}(y, u, \Lambda), \end{aligned} \quad (3.4)$$

where $\mathcal{A}, \mathcal{B}, \mathcal{E}$ are such that the same regularity (1.2) and ellipticity conditions (1.3) hold, where the function spaces are naturally adapted to the current case. The main novelty is the introduction of $\llbracket \cdot \rrbracket = (\cdot)_2 - (\cdot)_1$, which denotes the jump of a quantity across the interface Γ_1 from Ω_1 to Ω_2 . As before, we express (3.4) as the abstract operator equation

$$\mathcal{F}(u, \Lambda) = 0,$$

where $\mathcal{F}: \mathcal{X}_b \times \mathbb{R}^m \supset \mathcal{U} \rightarrow \mathcal{Y}_b$ is real-analytic, and the basic spaces are redefined to be

$$\begin{aligned} \mathcal{X} &:= \{u \in C^0(\overline{\Omega}) \cap C^{k+2+\alpha}(\overline{\Omega_1}) \cap C^{k+2+\alpha}(\overline{\Omega_2}) : u|_{\Gamma_0} = 0\}, \\ \mathcal{Y} &= \mathcal{Y}_1 \times \mathcal{Y}_2 := (C^{k+\alpha}(\overline{\Omega_1}) \cap C^{k+\alpha}(\overline{\Omega_2})) \times C^{k+1+\alpha}(\Gamma_1), \end{aligned} \quad (3.5)$$

with $\mathcal{X}_b, \mathcal{X}_\infty, \mathcal{Y}_b$, and $\mathcal{Y}_\infty$ given analogously.

Corollary 3.5 (Global continuation for transmission problems). *Consider the quasilinear elliptic transmission problem (3.4).*

- (a) (Global IFT) *If there are two parameters $\Lambda = (\lambda, \mu)$ and $(u, \lambda, \mu) \in \mathcal{U}$ is a strictly monotone front solution satisfying (H1)–(H3), then there exists a global curve $\mathcal{C}$ of strictly monotone front solutions to (3.4) such that properties (a)–(d) of Theorem 1.1 hold.*
- (b) (Global bifurcation) *If there is one parameter $\Lambda = \lambda$ and $\mathcal{C}_{\text{loc}} \subset \mathcal{U}$ is a local curve of strictly monotone front solutions satisfying (H1), (H2), and (H4), then there exists a global curve $\mathcal{C}$ of strictly monotone front solutions extending $\mathcal{C}_{\text{loc}}$ and exhibiting properties (a)–(c) of Theorem 1.2.*

Proof. We will prove this result by verifying the results of Sections 2.2–2.4 in the current setting of (3.4) and with the spaces (3.5).

Note first that the linearization of (3.4) at $u \in \mathcal{X}_\infty$ results in an elliptic operator of the form of (A.13), which allows one to characterize the maximum principle and Hopf boundary-point lemma in the same way as before. Therefore the spectral condition (H2) together with Corollary A.7 guarantees the existence of a positive comparison function as in Lemma 2.4. The same “quasilinearization” technique (using Lemma A.6 in place of the one-phase Hopf lemma) leads to an asymptotic monotonicity result of the form Lemma 2.5. It is then immediate that Theorem 2.9 holds for the transmission problem.

The characterization of the loss of compactness, namely Lemma 2.10, relies only on Schauder estimates and so it follows in exactly the same way.

Lastly, we show in Appendix A.3 that all the prerequisites for the Fredholm theory lemmas in Section 2.4 still hold true in the setting of (3.4). The arguments in Section 2.5 now apply verbatim, and we obtain the desired results. ■

4. Large fronts for semilinear Robin problems

In this section, we study a simple class of semilinear elliptic equations that is nonetheless rich enough to illustrate all of the alternatives in the global bifurcation theory. Specifically, consider the PDE

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u_y - u + g(u, \lambda) = 0 & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_0, \end{cases} \quad (4.1)$$

set on the infinite cylinder $\Omega := \mathbb{R} \times (0, 1)$ with upper boundary $\Gamma_1 := \{y = 1\}$ and lower boundary $\Gamma_0 := \{y = 0\}$. Here $g = g(z, \lambda)$ is assumed to be real-analytic in its arguments and λ is a real parameter.

Under certain natural hypotheses on g , we are able to construct global curves of solutions to (4.1). By tailoring the choice of nonlinearity, we can in fact control precisely which of the limiting scenarios in Theorem 1.2 (a) occurs, which shows that they are *sharp*.

Typically, it is very difficult to infer the behavior of solutions along a global bifurcation curve. What makes it possible here is an illuminating analogy between the conjugate flow problem (C1)–(C2) for the PDE (4.1) and classical finite-dimensional Hamiltonian mechanics. Recall that the ODE

$$\ddot{q} = -\nabla V(q) \quad (4.2)$$

has conserved energy $\frac{1}{2}|\dot{q}|^2 + V(q)$. Clearly, the critical points of the potential V are rest points for the system. Likewise, a heteroclinic orbit can connect two rest points only if they are on the same level set of V .

We first observe that the PDE (4.1) is of the variational form (3.3) with

$$\mathcal{L}(y, z, \xi, \eta, \lambda) = \frac{\xi^2 + \eta^2}{2} - z\eta + g(z, \lambda)\eta.$$

Hence by Lemma 3.1 there is an “energy”

$$\begin{aligned}\mathcal{H}(u, \lambda; x) &:= \int_0^1 (\mathcal{L}(y, u, u_x, u_y, \lambda) - \mathcal{L}_\xi(y, u, u_x, u_y, \lambda) u_x) dy \\ &:= \int_0^1 \left(\frac{u_y^2 - u_x^2}{2} - u u_y + g(u, \lambda) u_y \right) dy\end{aligned}$$

that will be independent of x for any solution (u, λ) . Introducing the primitive $G = G(z, \lambda)$ of $g(\cdot, \lambda)$ vanishing when $z = 0$, and recalling the Dirichlet condition on $\Gamma_0 = \{y = 0\}$, we can integrate the second two terms above to obtain the alternative formula

$$\mathcal{H}(u, \lambda; x) = \frac{1}{2} \int_0^1 (u_y^2(x, y) - u_x^2(x, y)) dy - \frac{1}{2} u^2(x, 1) + G(u(x, 1), \lambda). \quad (4.3)$$

We can then use $\mathcal{H}$ to characterize the conjugate flows in the sense of Definition 3.3. It is easy to confirm that $U = U(y)$ solves (4.1) if and only if $U = ry$ for r a root of $g(\cdot, \lambda)$. The corresponding energy (4.3) for U is simply

$$\mathcal{H}(U, \lambda) = G(r, \lambda). \quad (4.4)$$

Thus the conjugate flows for (4.1) have an easy visual interpretation: they are linear functions whose slopes are (distinct) critical points of G that lie on the same level set. In that sense, G plays an analogous role to the potential V in the Hamiltonian system (4.2).

With that in mind, we now impose some structural assumptions on g and G . As we wish to construct heteroclinic orbits, suppose that

$$g(0, \lambda) = 0, \quad g(\lambda, \lambda) = 0 \quad \text{for all } \lambda \in \mathbb{R}, \quad (4.5)$$

and

$$\begin{aligned}G(0, \lambda) &= G(\lambda, \lambda) = 0 && \text{for all } \lambda \in \mathbb{R}, \\ G_{rr}(0, \lambda), G_{rr}(\lambda, \lambda) &> 0 && \text{for all } \lambda \in \mathcal{O} \setminus \{0\}, \\ G(\cdot, \lambda)^{-1}(0) \cap g(\cdot, \lambda)^{-1}(0) &= \{0, \lambda\} && \text{for all } \lambda \in \mathcal{O},\end{aligned} \quad (4.6)$$

for some neighborhood $\mathcal{O} \subset \mathbb{R}$ of the origin. Note that (4.5) ensures that 0 and λy are x -independent solutions for all λ , while the first line of (4.6) says that they are conjugate. As we show in Lemma 4.4, the convexity of G asked for in (4.6) is equivalent to the spectral assumption (H2), while the remaining condition is related to heteroclinic degeneracy (A2).

We can now state the main result. As in the previous section, choose $\alpha \in (0, 1)$ and take

$$\mathcal{X} := \{u \in C^{2+\alpha}(\bar{\Omega}) : u|_{\Gamma_0} = 0\}, \quad \mathcal{Y} := C^\alpha(\bar{\Omega}) \times C^{1+\alpha}(\mathbb{R}),$$

with the spaces $\mathcal{X}_b, \mathcal{Y}_b, \mathcal{X}_\infty, \mathcal{Y}_\infty$ defined accordingly. Let $\mathcal{F}: \mathcal{X}_b \times \mathbb{R}^2 \rightarrow \mathcal{Y}_b$ be the nonlinear operator corresponding to the PDE (4.1).

Theorem 4.1 (Global bifurcation). *Consider the semilinear elliptic problem (4.1). Assume that the structural conditions (4.5), (4.6) hold, and that*

$$g_{zz\lambda}(0, 0) < 0. \quad (4.7)$$

Then there exist global continuous curves $\mathcal{C}^\pm$ of monotone front solutions to (4.1) lying in $\mathcal{X}_\infty$.

- (a) *On $\mathcal{C}^+$, all the fronts are strictly increasing while on $\mathcal{C}^-$ they are strictly decreasing.*
- (b) *As one follows $\mathcal{C}^\pm$, one of the three alternatives in Theorem 1.2 (a) must occur.*
- (c) *Moreover, the curves $\mathcal{C}^\pm$ leave a small neighborhood of $(0, 0)$ in $\mathcal{X}_\infty \times \mathbb{R}$ and do not re-enter.*

The proof of this theorem represents a fairly concise application of Theorem 1.2. In Section 4.1, we begin the process by constructing local curves $\mathcal{C}_{\text{loc}}^\pm$. The hypotheses of the general theory are then verified in Section 4.2, furnishing Theorem 4.1. Finally, in Section 4.3, we discuss the realizability of the alternatives.

4.1. Small-amplitude theory

As the first step, we will prove the existence of small-amplitude monotone front solutions to (4.1). The mechanical intuition makes it clear how to proceed. At $\lambda = 0$, G has a unique critical point corresponding to the trivial solution $u = 0$. As λ moves to the left or right, a second rest point develops that will be conjugate to 0. We therefore seek a local curve of heteroclinics solutions parameterized by λ that bifurcate at $\lambda = 0$ from the trivial solution.

Theorem 4.2 (Small-amplitude fronts). *Under the hypotheses of Theorem 4.1, there is a C^0 curve $\mathcal{C}_{\text{loc}}$ of solutions (u, λ) to (4.1) that admits the parameterization*

$$\mathcal{C}_{\text{loc}} = \{(u(\lambda), \lambda) : |\lambda| < \varepsilon_0\} \subset \mathcal{X}_\infty \times \mathcal{O}$$

for some $\varepsilon_0 > 0$. Moreover, the following statements hold along $\mathcal{C}_{\text{loc}}$:

- (a) (Asymptotics) *The solutions have leading form expressions given by*

$$u(\lambda)(x, y) = \frac{\lambda}{2}(1 + \tanh(\kappa_1|\lambda|x))y + O(\lambda^2) \quad \text{in } \mathcal{X}_b \quad (4.8)$$

for an explicit positive constant κ_1 given in (4.14).

- (b) (Monotonicity) *If $\lambda > 0$, then $(u(\lambda), \lambda)$ is a strictly increasing front, while for $\lambda < 0$, it is a strictly decreasing front.*
- (c) (Uniqueness) *In a neighborhood of $(0, 0)$ in $\mathcal{X}_\infty \times \mathbb{R}$, $\mathcal{C}_{\text{loc}}$ comprises all monotone fronts (up to translation).*
- (d) (Kernel) *For all $0 < |\lambda| < \varepsilon_0$, the kernel of $\mathcal{F}_u(u(\lambda), \lambda)$ is one-dimensional and generated by $\partial_x u(\lambda)$.*

Our approach is based on the center manifold reduction theory recently introduced in [19], which is more convenient here than traditional spatial dynamics methods. First, it is formulated in Hölder spaces as required by the global theory. As we shall see shortly, it also allows us to compute the reduced equation through a power series expansion that is particularly simple for the present problem. It also gives us the freedom to choose the projection onto the kernel of the linearized operator, which is useful in establishing the monotonicity of solutions.

Notation. In order to construct the center manifold, it will be necessary to (temporarily) expand our function spaces to include solutions exhibiting some growth at infinity. For $v \in \mathbb{R}$, we define the exponentially weighted Hölder space

$$C_v^{k+\alpha}(\bar{\Omega}) := \{u \in C^{k+\alpha}(\bar{\Omega}) : \|u\|_{C_v^{k+\alpha}(\Omega)} < \infty\},$$

where

$$\|u\|_{C_v^{k+\alpha}(\Omega)} := \sum_{|\beta| \leq k} \|\operatorname{sech}(vx) \partial^\beta u\|_{C^0(\Omega)} + \sum_{|\beta|=k} \|\operatorname{sech}(vx) |\partial^\beta u|_\alpha\|_{C^0(\Omega)},$$

and $|\cdot|_\alpha$ is the local Hölder seminorm

$$|u|_\alpha(x, y) := \sup_{(\tilde{x}, \tilde{y}) \in B_1(x, y) \cap \Omega} \frac{|u(\tilde{x}, \tilde{y}) - u(x, y)|}{|(\tilde{x} - x, \tilde{y} - y)|^\alpha}.$$

Let $\mathcal{X}_v$ and $\mathcal{Y}_v$ denote the corresponding versions of $\mathcal{X}_b$ and $\mathcal{Y}_b$.

Center manifold reduction. Using (4.5) we compute that the linearized transversal operator at $(u, \lambda) = (0, 0)$ is given by

$$\mathcal{L}'_-: \mathcal{X}' \rightarrow \mathcal{Y}', \quad w \mapsto \begin{pmatrix} w_{yy} \\ (w_y - w)|_{\Gamma_1} \end{pmatrix}.$$

It is easily confirmed that 0 is a simple eigenvalue of $\mathcal{L}'_-$ with corresponding eigenfunction $\varphi_0 = \varphi_0(y) := y$, while the remainder of the spectrum is strictly negative. As a consequence, if $\mathcal{F}_u(0, 0)$ is viewed as a mapping $\mathcal{X}_v \rightarrow \mathcal{Y}_v$ for $0 < v \ll 1$; its null space is two-dimensional and characterized by

$$\ker(\mathcal{F}_u(0, 0): \mathcal{X}_v \rightarrow \mathcal{Y}_v) = \{(A + Bx)\varphi_0 \in \mathcal{X}_v : (A, B) \in \mathbb{R}^2\}.$$

A convenient projection $\mathcal{Q}$ onto this kernel is found by setting (A, B) to be (u, u_x) evaluated at the chosen point $(0, 1) \in \Gamma_1$:

$$\mathcal{Q}u := (u(0, 1) + u_x(0, 1)x)\varphi_0(y).$$

We have now verified the hypotheses of the center manifold reduction theorem [19, Theorem 1.1], the conclusions of which are recorded below.

Lemma 4.3 (Center manifold). *There exists $\nu > 0$, neighborhoods $\mathcal{N} \subset \mathcal{X}_b \times \mathcal{O}$ and $N \subset \mathbb{R}^3$ and a coordinate map $\Psi = \Psi(A, B, \lambda)$ satisfying*

$$\Psi \in C^4(\mathbb{R}^3, \mathcal{X}_\nu), \quad \Psi(0, 0, \lambda) = 0 \quad \text{for all } \lambda, \quad \Psi_A(0, 0, 0) = \Psi_B(0, 0, 0) = 0,$$

such that the following hold:

- (a) *Suppose that $(u, 0, \lambda) \in \mathcal{N}$ solves (4.1). Then $v := u(\cdot, 1)$ solves the second-order ODE*

$$v'' = f(v, v', \lambda), \quad (4.9)$$

where $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ is the C^4 mapping

$$f(A, B, \lambda) := \left. \frac{d^2}{dx^2} \right|_{x=0} \Psi(A, B, \lambda)(x, 0).$$

- (b) *Conversely, if v satisfies the ODE (4.9) and $(v(x), v'(x), \lambda) \in N$ for all x , then $v = u(\cdot, 1)$ for a solution $(u, \lambda) \in \mathcal{N}$ of the PDE (4.1). Moreover,*

$$u(x + \tau, y) = (\mathcal{Q}u)(\tau, y) + \Psi(v(x), v'(x), \lambda)(\tau, y) \quad \text{for all } \tau \in \mathbb{R}.$$

The next step is to compute f and the corresponding reduced equation (4.9) for the problem at hand. Our approach relies on [19, Theorem 1.6], which states that Ψ admits the Taylor expansion

$$\Psi(A, B, \lambda) = \sum_{\substack{2 \leq i+j+k \leq 3 \\ i+j \geq 1}} \Psi_{ijk} A^i B^j \lambda^k + O((|A| + |B|)(|A| + |B| + |\lambda|)^3) \quad \text{in } \mathcal{X}_\nu.$$

Here the coefficients Ψ_{ijk} are the unique functions in $\mathcal{X}_\nu$ with $\mathcal{Q}\Psi_{ijk} = 0$ and

$$\partial_A^i \partial_B^j \partial_\lambda^k |_{(A, B, \lambda)=0} \mathcal{F}((A + Bx)\varphi_0 + \Psi(A, B, \lambda), \lambda) = 0 \quad \text{for all } i + j + k \leq 3,$$

where the derivatives are taken in the formal Gâteaux sense.

This leads to a hierarchy of linear equations of the general form

$$\begin{cases} \mathcal{F}_u(0, 0)\Psi_{ijk} = R_{ijk}, \\ \mathcal{Q}\Psi_{ijk} = 0, \end{cases}$$

for some $R_{ijk} \in \mathcal{Y}_\mu$ that are explicit given previously computed terms; for more discussion, see [19, Section 4]. By Fredholm theory [19, Lemma 2.3], these problems are uniquely solvable in $\mathcal{X}_\nu$. Following this procedure, we ultimately find that

$$f(A, B, \lambda) = 3g_{12}\lambda^2 A + 3g_{21}\lambda A^2 + 3g_{30}A^3 + r(A, B, \lambda), \quad (4.10)$$

where $g_{\ell m} := (\partial_z^\ell \partial_\lambda^m g)(0, 0)/(\ell!m!)$, and $r \in C^3$ is a remainder term with

$$r(A, B, \lambda) = O(|A|(|A| + |B|^{1/2} + |\lambda|)^3 + |B|(|A| + |B|^{1/2} + |\lambda|)^2).$$

Notice that we do not see any B dependence in the (expected) leading order part of f ; this is a consequence of the invariance of (4.1) under reflection in x . Indeed, that symmetry actually guarantees that f is even in B .

The coefficients in (4.10) are in fact related to one another due to the structural assumptions on g and G in (4.5) and (4.6). Differentiating the identity $g(\lambda, \lambda) = 0$ three times and evaluating at $\lambda = 0$ gives

$$g_{zzz} + 3g_{zz\lambda} + 3g_{z\lambda\lambda} = 0 \quad \text{at } (0, 0).$$

Similarly, differentiating the equation $G(\lambda, \lambda) = 0$ four times, and recalling that $G_z = g$, we find that

$$g_{zzz} + 4g_{zz\lambda} + 6g_{z\lambda\lambda} = 0 \quad \text{at } (0, 0).$$

Together, these imply that

$$g_{12} = -\frac{1}{3}g_{21}, \quad g_{30} = -\frac{2}{3}g_{21}. \quad (4.11)$$

Taking f and neglecting the remainder term gives the following truncated reduced equation on the center manifold:

$$v_{xx}^0 = g_{21}(-\lambda^2 v^0 + 3\lambda(v^0)^2 - 2(v^0)^3). \quad (4.12)$$

Under the assumption (4.7), the above ODE has the explicit heteroclinic orbit

$$v^0(x) = \lambda \frac{1 + \tanh(\kappa_1 |\lambda| x)}{2}, \quad (4.13)$$

where

$$\kappa_1 := |g_{21}|^{1/2}/2. \quad (4.14)$$

It is also important to confirm that the conserved quantity $\mathcal{H}$ for the full PDE has an analogue on the center manifold. Setting

$$h(A, B, \lambda) := \mathcal{H}((A + Bx)\varphi_0(y) + \Psi(A, B, \lambda); x),$$

we indeed see that, for any solution v of the reduced ODE (4.10), $h(v, v', \lambda)$ is independent of x . Expanding as before, we find that

$$h(A, B, \lambda) = \frac{g_{30}}{4}A^4 + \frac{g_{21}}{3}\lambda A^3 + \frac{g_{12}}{2}\lambda^2 A^2 - \frac{1}{6}B^2 + \tilde{r}(A, B, \lambda), \quad (4.15)$$

for a C^4 remainder

$$\tilde{r}(A, B, \lambda) = O(|A|(|A| + |B|^{1/2} + |\lambda|)^4 + |B|(|A| + |B|^{1/2} + |\lambda|)^3).$$

We are now prepared to prove the existence of small-amplitude monotone front solutions to (4.1).

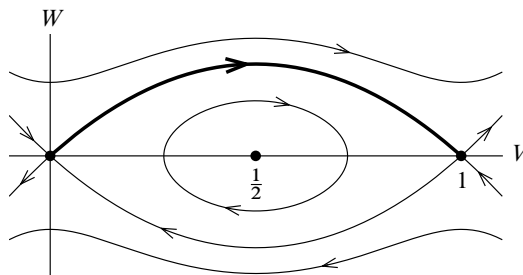


Fig. 4. Phase portrait for the rescaled reduced ODE (4.16) at $\lambda = 0$. The thick curve is the heteroclinic orbit (V^0, W^0) connecting the rest points $(0, 0)$ and $(1, 0)$.

Proof of Theorem 4.2. Working in the rescaled variables

$$x =: |\lambda|^{-1}X, \quad v(x) =: \lambda V(X), \quad v_x(x) =: \lambda|\lambda|W(X),$$

the (full) reduced ODE (4.9) can be recast as the planar system

$$\begin{cases} V_X = W, \\ W_X = 3g_{12}V + 3g_{21}V^2 + 3g_{30}V^3 + R(V, W, \lambda), \end{cases} \quad (4.16)$$

where the remainder term is

$$R(V, W, \lambda) = O(|\lambda|(|V| + |W|)).$$

When $\lambda = 0$, (4.16) has the solution

$$V^0(X) := \frac{1 + \tanh(\kappa_1 X)}{2}, \quad W^0(X) := \kappa_1 \frac{\operatorname{sech}^2(\kappa_1 X)}{2}$$

shown in Figure 4, which corresponds to the rescaling of v^0 . We must now show that this solution persists when $|\lambda| > 0$. For that, we will need to make use of the conserved quantity

$$H(V, W, \lambda) := \frac{1}{2}W^2 + \frac{g_{21}}{2}V^2 - g_{21}V^3 + \frac{g_{21}}{2}V^4 + \tilde{R}(V, W, \lambda),$$

which results from composing h with the scaling and using the relations (4.11). One checks that the error term satisfies

$$\tilde{R}(V, W, \lambda) = O(|\lambda|(|V| + |W|)).$$

At $\lambda = 0$, the system has rest points $(0, 0)$ and $(1, 0)$. They are connected via (V^0, W^0) and lie on the level set $\{H(\cdot, \cdot, 0) = 0\}$. For $|\lambda| > 0$, our assumptions (4.5)–(4.6) imply that $(0, 0)$ and $(1, 0)$ continue to be rest points and to lie on the level set $\{H(\cdot, \cdot, \lambda) = 0\}$. The existence of a heteroclinic orbit (V^λ, W^λ) between $(0, 0)$ and $(1, 0)$ then follows

from the nondegeneracy of this level set. Undoing the scaling gives the leading order asymptotics stated in (4.8), proving part (a).

Next, consider the monotonicity claimed in part (b). Analyzing the phase portrait of the reduced ODE, we easily verify that

$$(\operatorname{sgn} \lambda) \partial_x u(\lambda)(\cdot, 1) = (\operatorname{sgn} \lambda) \partial_x v(\lambda) > 0.$$

But $\partial_x u(\lambda)$ is harmonic in Ω and vanishes on Γ_0 as well in the limits $x \rightarrow \pm\infty$. The maximum principle then tells us that $(\operatorname{sgn} \lambda) \partial_x u(\lambda) > 0$ in $\Omega \cup \Gamma_1$, meaning $u(\lambda)$ is strictly monotone.

To show (c), first note that for $\lambda = 0$, the conjugate flow analysis above together with (4.6) forces all fronts to vanish identically. For $\lambda > 0$, we likewise find that the only state conjugate to 0 is λy , corresponding to the rest points $(0, 0)$ and $(1, 0)$ of (4.16). Consider a monotone front $(u, \lambda) \in \mathcal{N}$ with $u \in \mathcal{X}_\infty$. Then the corresponding orbit (V, W) of (4.16) must satisfy $0 \leq V \leq \lambda$ and $W \geq 0$. Since the phase portrait of (4.16) is qualitatively the same as in Figure 4, we conclude that the only possibility is that u is a translate of $u(\lambda)$. The argument for $\lambda < 0$ is similar.

Finally, to prove (d) we will use [19, Theorem 1.6] which reduces the issue to the center manifold. Specifically, this result tells us that $\dot{u} \in \ker \mathcal{F}_u(u, 0, \lambda)$ only if $\dot{v} := \dot{u}(\cdot, 1)$ solves the linearized reduced equation

$$\dot{v}'' = \nabla_{(A,B)} f(v, v', \lambda) \cdot (\dot{v}, \dot{v}'),$$

where $v = u(\lambda)(\cdot, 1)$. Let $(\dot{V}, \dot{W})$ be the corresponding rescaled quantities, which solve a nonautonomous planar system of the form

$$\begin{pmatrix} \dot{V}_X \\ \dot{W}_X \end{pmatrix} = \mathcal{M}(X) \begin{pmatrix} \dot{V} \\ \dot{W} \end{pmatrix}.$$

Using the expansion of f in (4.10) and the relations (4.11), we find that

$$\lim_{X \rightarrow \pm\infty} \mathcal{M}(X) = \begin{pmatrix} 0 & 1 \\ |g_{21}| + O(\lambda) & O(\lambda) \end{pmatrix}.$$

Thus $\mathcal{M}$ is strictly hyperbolic upstream and downstream with one negative and one positive eigenvalue. By a familiar dynamical systems argument, this implies that there cannot be two linearly independent solutions of the reduced linearized problem that are uniformly bounded. We may then conclude that the kernel of $\mathcal{F}_u(u(\lambda), \lambda): \mathcal{X}_b \rightarrow \mathcal{Y}_b$ is indeed generated by $\partial_x u(\lambda)$. ■

4.2. Large fronts for the semilinear problem

Now that we have local curves of small-amplitude fronts, we seek to continue them globally using Theorem 1.2. For that, it will be necessary to understand the linearized problem at an arbitrary front.

Letting $(u, \lambda) \in \mathcal{X}_\infty \times \mathbb{R}$ be given, a simple computation shows that

$$\begin{aligned}\mathcal{F}_u(u, \lambda)\dot{u} &= \begin{pmatrix} \Delta \dot{u} \\ (\dot{u}_y - \dot{u} + g_z(u, \lambda)\dot{u})|_{\Gamma_1} \end{pmatrix}, \\ \mathcal{F}_\lambda(u, \lambda)\dot{\lambda} &= \begin{pmatrix} 0 \\ g_\lambda(u|_{\Gamma_1}, \lambda)\dot{\lambda} \end{pmatrix}.\end{aligned}$$

Now suppose (u, λ) is a front-type solution to (4.1). Our characterization of the conjugate flows (4.4) ensures that $\lim_{x \rightarrow \infty} (u - ry) = 0$ for some root r of $g(\cdot, \lambda)$. The transversal linearized operators upstream and downstream are therefore

$$\mathcal{L}'_-(u, \lambda)\dot{u} = \begin{pmatrix} \dot{u}_{yy} \\ (\dot{u}_y - \dot{u} + g_z(0, \lambda)\dot{u})|_{\Gamma_1} \end{pmatrix}, \quad \mathcal{L}'_+(u, \lambda)\dot{u} = \begin{pmatrix} \dot{u}_{yy} \\ (\dot{u}_y - \dot{u} + g_z(r, \lambda)\dot{u})|_{\Gamma_1} \end{pmatrix}.$$

It is readily seen that $\xi^2 \geq 0$ is in the spectrum of $\mathcal{L}'_-(u, \lambda)$ if and only if

$$1 = \frac{\tanh \xi}{\xi}(1 - g_z(0, \lambda)),$$

which is possible if and only if $g_z(0, \lambda) \leq 0$. Likewise, $\xi^2 \geq 0$ is an eigenvalue of $\mathcal{L}'_+(u, \lambda)$ provided that

$$1 = \frac{\tanh \xi}{\xi}(1 - g_z(r, \lambda)).$$

Thus the spectral degeneracy alternative (A3) is equivalent to the loss of strict convexity of G at the corresponding critical point. In summary, we have proved the following.

Lemma 4.4 (Convexity condition). *Let $(u, \lambda) \in \mathcal{X}_\infty \times \mathbb{R}$ be a monotone front solution of (4.1) with $\lim_{x \rightarrow \infty} u_y = r$. Then $\sigma_0^-(u, \lambda) < 0$ if and only if $G_{zz}(0, \lambda) > 0$, and $\sigma_0^+(u, \lambda) < 0$ if and only if $G_{zz}(r, \lambda) > 0$.*

The existence of the global bifurcation curves now follows easily.

Proof of Theorem 4.1. Let

$$\mathcal{C}_{\text{loc}}^\pm := \{(u(\lambda), \lambda) \in \mathcal{C}_{\text{loc}} : 0 < |\lambda| < \varepsilon, \pm \lambda > 0\}.$$

Then by Theorem 4.2 (b), solutions on $\mathcal{C}_{\text{loc}}^+$ are strictly increasing fronts, while those on $\mathcal{C}_{\text{loc}}^-$ are strictly decreasing fronts.

Moreover, the structural assumptions on G in (4.6) and Lemma 4.4 imply that the spectral condition (H2) holds at each front (u, λ) with $\lambda \in \mathcal{O} \setminus \{0\}$. In particular, this includes all solutions on the local curves. On the other hand, $G_{zz}(0, 0) = 0$ means that the bifurcation point $(u_0, \lambda_0) = (0, 0)$ satisfies (H4). We have already confirmed in Theorem 4.2 (d) that the kernel assumption (H1) is satisfied. Theorem 1.2 may therefore be applied to $\mathcal{C}_{\text{loc}}^\pm$ yielding the global curves $\mathcal{C}^\pm$ satisfying (a) and (b). Finally, (c) follows from Theorem 4.2 (c). ■

4.3. Realization of the alternatives

In this subsection, we show that the alternatives for the global curve given in Theorem 4.1 are essentially sharp. Indeed, one of the most appealing features of the PDE (4.1) is that it is simple to connect each of the scenarios above with qualitative properties of G .

As a first step, we establish some basic uniform regularity results that more precisely characterize which quantities are unbounded in the event that the blowup alternative (A1) occurs.

Lemma 4.5 (Uniform regularity). *If $(u, \lambda) \in \mathcal{X}_\infty \times \mathbb{R}$ is a monotone front solution to (4.1) with $|\lambda| < M$, then*

$$\|u\|_{C^{2+\alpha}(\Omega)} \leq C_1 \|u\|_{L^\infty(\Omega)} \leq C_2, \quad (4.17)$$

where $C_1, C_2 > 0$ depend only on M and

$$R := \sup \{ |r| : (r, \lambda') \in g^{-1}(0) \cap G^{-1}(0), |\lambda'| < M \}.$$

Proof. Throughout the proof, C denotes a generic positive constant depending only on M and R . Because $u \in \mathcal{X}_\infty$, it has well-defined upstream and downstream states that are constrained by λ . Indeed, monotonicity implies

$$\|u\|_{L^\infty} \leq R,$$

which proves the second inequality in (4.17).

It remains to show that one can control the full $C^{2+\alpha}$ norm of u in terms of $\|u\|_{L^\infty}$. Towards that end, note that u solves

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u_y = h & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_0, \end{cases} \quad (4.18)$$

where

$$h := u - g(u, \lambda) \in \mathcal{X}_\infty, \quad \|h\|_{L^\infty} \leq C \|u\|_{L^\infty}.$$

It follows from standard elliptic estimates for the Dirichlet problem that, for any set $\Omega_0 \subset\subset \Omega \cup \Gamma_0$,

$$\|u\|_{C^{2+\alpha}(\Omega_0)} < C. \quad (4.19)$$

In particular, C above is independent of axial translation, so we uniformly control u in $C^{2+\alpha}$ away from Γ_1 by $C \|u\|_{L^\infty(\Omega)}$.

Next, let $(x_0, 1) \in \Gamma_1$ be given and consider the half-balls

$$\Omega_{1/4} := \Omega \cap B_{1/4}((x_0, 1)) \subset \Omega_{1/2} := \Omega \cap B_{1/2}((x_0, 1)).$$

If we can prove that the bound (4.19) holds with $\Omega_{1/4}$ in place of Ω_0 , then the conclusion of the lemma is immediate. To accomplish that, we will reflect the problem (4.18) and use interior estimates. Denote by $\tilde{\Omega}_{1/4}$ and $\tilde{\Omega}_{1/2}$ the balls $B_{1/4}$ and $B_{1/2}$ centered at $(x_0, 1)$,

respectively, and define $\tilde{u}$ to be the even extension of u over Γ_1 . Thus $\tilde{u} \in H^1(\tilde{\Omega}_{1/2})$ is a weak solution to

$$\Delta \tilde{u} = 2\partial_y \tilde{h} \quad \text{in } \tilde{\Omega}_{1/2}$$

with $\tilde{h} := h(x)\chi_{\{y>1\}} \in L^\infty(\tilde{\Omega}_{1/2})$. Using the De Giorgi–Nash-type result [33, Theorem 8.24], we conclude that for any $p > 2$, there exists $\tilde{\alpha} \in (0, 1)$ such that

$$\|u\|_{C^{\tilde{\alpha}}(\Omega_{1/4})} \leq \|\tilde{u}\|_{C^{\tilde{\alpha}}(\tilde{\Omega}_{1/4})} \lesssim \|\tilde{u}\|_{L^2(\tilde{\Omega}_{1/2})} + \|\tilde{h}\|_{L^p(\tilde{\Omega}_{1/2})} \leq C\|u\|_{L^\infty(\Omega_{1/2})}.$$

Therefore, u is uniformly bounded in $C^{\tilde{\alpha}}$ on the segment of Γ_1 lying in $\tilde{\Omega}_{1/4}$. Because h is real-analytic, the desired bound (4.19) for the half-ball $\Omega_{1/4}$ is now a consequence of Schauder estimates for (4.18) and a standard bootstrapping argument. This completes the proof. ■

Remark 4.6. A similar argument yields versions of this result whenever the PDE (1.1) is semilinear. The key point above is to control $\|u\|_{C^\alpha}$ by $\|u\|_{L^\infty}$ and $|\lambda|$. For divergence form semilinear operators (on quite rough domains), this follows from [53, Proposition 3.6].

Now we are able to give scenarios in which each of the three alternatives in Theorem 4.1 can be expected; these are illustrated in Figure 5. In particular, we can guarantee that the unboundedness alternative (A1) and spectral degeneracy alternative (A3) happen for some explicit classes of G . We also provide a necessary condition for heteroclinic degeneracy (A2).

Proposition 4.7 (Realization). *Let $\mathcal{C}^\pm$ be the global bifurcation curve furnished by Theorem 4.1.*

(a) *If G satisfies (4.6) for all $\lambda \in \mathbb{R}$, then the unboundedness alternative (A1) must occur. In particular,*

$$|\lambda(s)| \rightarrow \infty \quad \text{as } s \rightarrow \pm\infty. \quad (4.20)$$

(b) *Heteroclinic degeneracy (A2) can occur only if there exists $r \neq 0, \lambda_*$ with $g(r, \lambda_*) = 0$ and $G(r, \lambda_*) = 0$.*

(c) *Suppose that there exist bounded open sets $\mathcal{N}_2 \supset \supset \mathcal{N}_1 \supset \mathcal{O}$ such that*

$$G(\cdot, \lambda)^{-1}(0) \cap g(\cdot, \lambda)^{-1}(0) = \{0, \lambda\} \quad \text{for all } \lambda \in \mathcal{N}_2$$

and G satisfies (4.6) on $\mathcal{N}_1$, but

$$\min \{G_{zz}(0, \lambda), G_{zz}(\lambda, \lambda)\} \rightarrow 0 \quad \text{as } \lambda \rightarrow \partial\mathcal{N}_1.$$

Then necessarily spectral degeneracy (A3) occurs.

Proof. First we consider part (b). Suppose that the heteroclinic degeneracy does indeed occur as in (A2), in which case there are three distinct limiting states

$$\lim_{x \rightarrow \mp\infty} u_*(x, \cdot) =: r_* y, \quad \lim_{n \rightarrow \infty} \lim_{x \rightarrow \infty} u(s_n)(x, \cdot) = \lambda_* y, \quad \lim_{n \rightarrow \infty} \lim_{x \rightarrow -\infty} u(s_n)(x, \cdot) = 0,$$

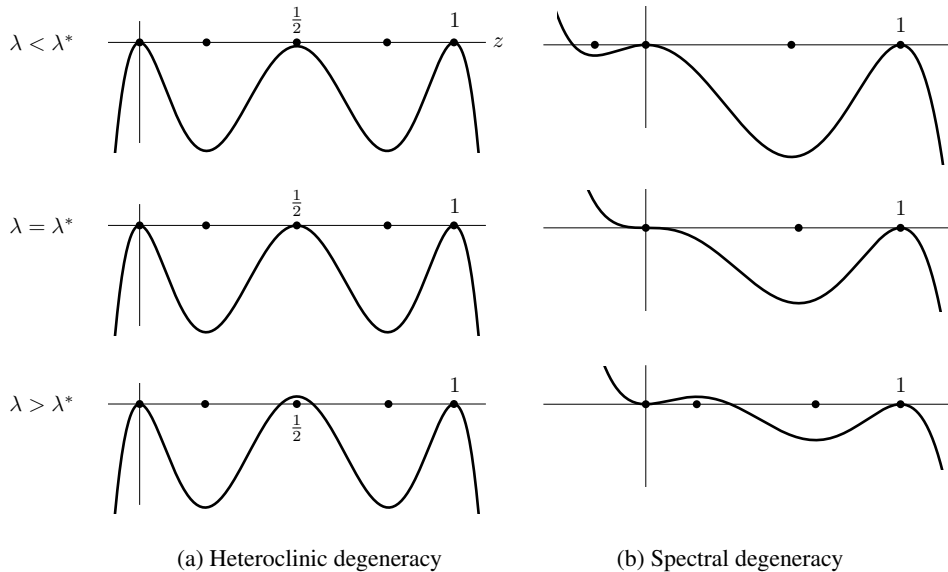


Fig. 5. The curves above depict the graph of $-G(\cdot, \lambda)$ for two different families of potential and at a subcritical ($\lambda < \lambda^*$), critical ($\lambda = \lambda^*$), and supercritical ($\lambda > \lambda^*$) parameter value. In (a), as λ passes through λ^* , a second critical point at the energy level 0 forms, which permits heteroclinic degeneracy. The corresponding orbits of the ODE $\ddot{u} = G_z(u, \lambda)$ are shown in Figure 1. In (b), at $\lambda = \lambda^*$, $G(\cdot, \lambda)$ loses strict convexity at 0 implying the onset of spectral degeneracy. The corresponding orbits of the ODE $\ddot{u} = G_z(u, \lambda)$ are shown in Figure 2.

all of which are x -independent solutions of (4.1). By Lemma 3.4, they are all mutually conjugate, and so in addition to $g(r_*, \lambda_*) = g(0, \lambda_*) = 0$ we have $G(r_*, \lambda_*) = G(0, \lambda_*) = 0$. The statement follows.

Next, suppose that G satisfies (4.6) globally. Lemma 4.4 implies that spectral degeneracy (A3) can only happen if $\mathcal{C}^\pm$ approaches $(0, 0)$ as $s \rightarrow \pm\infty$. But this is ruled out by Theorem 4.1 (c). From part (b), we see that the heteroclinic degeneracy alternative is likewise impossible. Thus blowup must occur. In view of Lemma 4.5, unboundedness of $\mathcal{C}^\pm$ in $\mathcal{X}_b \times \mathbb{R}$ can happen only if $|\lambda(s)| \rightarrow \infty$.

Finally, assume that G is given as in part (c). As the set $\mathcal{N}_1$ is bounded, we know by the previous paragraph that $\|u(s)\|_{L^\infty}$ (and hence $\|u(s)\|_{\mathcal{X}}$) is uniformly bounded so long as $\lambda(s) \in \mathcal{N}_1$. Likewise, the unboundedness alternative cannot occur if $\lambda(s) \subset \subset \mathcal{N}_2$ for all $|s| > 0$, so there must exist some $|s_0| < \infty$ such that $\lambda(s) \rightarrow \partial\mathcal{N}_1$ as $s \rightarrow s_0 \mp$. In light of Lemma 4.4, this leads to spectral degeneracy (A3). ■

5. Large-amplitude hydrodynamic bores

In this section, we apply our abstract results to the classical problem of bores traveling along the interface between two fluids of different densities, which can be formulated

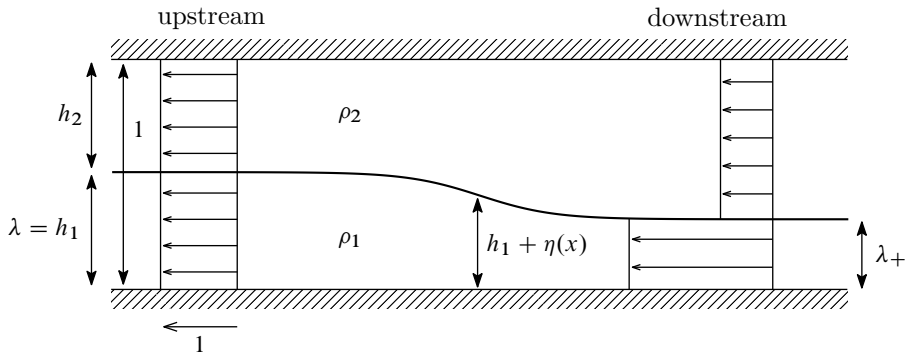


Fig. 6. A monotone bore in a two-layer system.

as the quasilinear transmission problem (5.9) below. While this system is much more complicated than (4.1), we will show that it enjoys many of the same qualitative features. In particular, thanks to its variational structure, it is possible to rule out both heteroclinic degeneracy (A2) and spectral degeneracy (A3) via careful study of the conjugate flow problem. On the other hand, since the system is quasilinear rather than semilinear, the analysis of the blowup alternative (A1) becomes quite delicate.

Consider a stably stratified configuration in which a lighter fluid with constant density $\rho_2 > 0$ lies atop a heavier fluid with constant density $\rho_1 > \rho_2$. Working in a reference frame moving with the wave, we assume that the interface between the two layers as well as the fluid velocity fields are independent of time. Both for simplicity and because it is the setting with the most applied interest, suppose that the fluid velocity tends to some constant value $(-c, 0)$ in the upstream limit $x \rightarrow -\infty$, where $c > 0$ is interpreted as wave speed. Letting $h_1, h_2 > 0$ be the upstream thicknesses of the two layers and $g > 0$ be the constant acceleration due to gravity, the dimensionless Froude number

$$F := \frac{c}{\sqrt{g(h_1 + h_2)}}$$

measures the relative importance of inertial and gravitational effects. Switching to dimensionless units with $h_1 + h_2$ as the length scale and c as the velocity scale, we are left with the three dimensionless parameters F , ρ_2/ρ_1 , and h_1 .

The origin of coordinates is chosen so that the interface between the two layers is $y = \eta(x)$ where $\eta \rightarrow 0$ as $x \rightarrow -\infty$. The bottom boundary of the channel is therefore $y = -h_1$, and the upper boundary is $y = h_2 = 1 - h_1$. We denote the upper fluid domain in these variables by $\mathcal{D}_2$, the lower fluid by $\mathcal{D}_1$, and set $\mathcal{D} := \mathcal{D}_1 \cup \mathcal{D}_2$. Finally, we write $\mathcal{S}$ for the internal interface.

Requiring the flow to be irrotational and incompressible in each layer, the velocity field can be expressed as $(\partial_y \psi, -\partial_x \psi)$ for some *stream function* ψ satisfying

$$\Delta \psi = 0 \quad \text{in } \mathcal{D}. \quad (5.1a)$$

The stream function is constant along the interface as it is a material surface, and we normalize this constant to be zero:

$$\psi = 0 \quad \text{on } \mathcal{S}. \quad (5.1b)$$

In particular, ψ is continuous across the interface. Our assumptions upstream can be written as

$$\nabla \psi \rightarrow (0, -1), \quad \eta \rightarrow 0 \quad \text{as } x \rightarrow -\infty. \quad (5.1c)$$

The rigid boundaries $y = -h_1$ and $y = h_2$ are also material surfaces, and hence level curves of the stream function. Calculating these values using (5.1b) and (5.1c), we find

$$\begin{aligned} \psi &= h_1 & \text{on } y = -h_1, \\ \psi &= -h_2 & \text{on } y = h_2. \end{aligned} \quad (5.1d)$$

Finally, the dynamic boundary condition on $y = \eta(x)$ asserts the continuity of the pressure,

$$\frac{1}{2} \llbracket \rho |\nabla \psi|^2 \rrbracket + \frac{\llbracket \rho \rrbracket}{F^2} y = \frac{\llbracket \rho \rrbracket}{2} \quad \text{on } \mathcal{S}. \quad (5.1e)$$

As in Section 3.2, $\llbracket \cdot \rrbracket = (\cdot)_2 - (\cdot)_1$ denotes the jump of a quantity across the interface $y = \eta(x)$. For general traveling waves, (5.1e) would contain an undetermined Bernoulli constant, but in our case we have calculated this constant explicitly using (5.1c).

For more background on (5.1) and related problems, see Section 1.3 in the introduction or the lengthier discussion in [18, Section 1.2]. Our first theorem is the following global bifurcation result for monotone bores.

Theorem 5.1 (Large monotone bores). *Fix an $\alpha \in (0, 1)$ and densities $0 < \rho_2 < \rho_1$. There exist global C^0 curves*

$$\mathcal{C}^\pm = \{(\psi(s), \eta(s), \lambda(s)) : \pm s \in (0, \infty)\}$$

of solutions to the internal wave problem (5.1) with $h_1 = \lambda(s)$, $h_2 = 1 - h_1$, and F given by (5.10). They enjoy the Hölder regularity

$$\psi(s) \in C_b^{2+\alpha}(\overline{\mathcal{D}_1(s)}) \cap C_b^{2+\alpha}(\overline{\mathcal{D}_2(s)}) \cap C_b^0(\overline{\mathcal{D}(s)}), \quad \eta(s) \in C_b^{2+\alpha}(\mathbb{R}),$$

where $\mathcal{D}(s)$ denotes the fluid domain corresponding to $\eta(s)$ and $\lambda(s)$.

(a) (Strict monotonicity) *Each solution on $\mathcal{C}^\pm$ is a strictly monotone bore:*

$$\begin{aligned} \pm \partial_x \eta(s) &< 0 \quad \text{on } \mathbb{R}, \\ \pm \psi_x &< 0, \\ \partial_y \psi(s) &< 0 \quad \text{in } \overline{\mathcal{D}(s)}. \end{aligned} \quad (5.2)$$

(b) (Stagnation limit) *Following $\mathcal{C}^\pm$, we encounter waves that are arbitrarily close to having a horizontal stagnation point on the internal interface:*

$$\lim_{s \rightarrow \pm\infty} \sup_{\mathcal{S}(s)} \partial_y \psi_i(s) = 0 \quad \text{for } i = 1 \text{ or } 2. \quad (5.3)$$

(c) (Laminar origin) Both $\mathcal{C}^-$ and $\mathcal{C}^+$ emanate from the same laminar solution in that

$$\eta(s) \rightarrow 0, \quad \nabla \psi(s) \rightarrow (0, -1), \quad \lambda(s) \rightarrow \lambda_+ \pm \quad \text{as } s \rightarrow 0 \pm,$$

where λ_+ is the constant (5.16) below.

The next result characterizes the limiting form of the profile along $\mathcal{C}^\pm$.

Theorem 5.2 (Limiting interface). *Consider the behavior of the profile as we traverse $\mathcal{C}^\pm$.*

(a) (Overturning or singularity) *In the limit along $\mathcal{C}^-$, either the interface overturns in that*

$$\limsup_{s \rightarrow -\infty} \|\partial_x \eta(s)\|_{L^\infty(\mathbb{R})} = \infty, \quad (5.4)$$

or it becomes singular in that we can extract a translated subsequence

$$\eta(s) \rightarrow \eta^* \in \text{Lip}(\mathbb{R}) \quad \text{in } C_{\text{loc}}^\varepsilon \text{ for all } \varepsilon \in (0, 1)$$

such that $\{y < \eta^(x)\}$ simultaneously fails to satisfy both an interior sphere and exterior sphere condition at a single point on its boundary.*

(b) (Overturning or contact) *Following $\mathcal{C}^+$, either the interface overturns or it comes into contact with the upper wall:*

$$\limsup_{s \rightarrow \infty} \lambda(s) = 1 \quad \text{or} \quad \limsup_{s \rightarrow \infty} \|\partial_x \eta(s)\|_{L^\infty(\mathbb{R})} = \infty. \quad (5.5)$$

Remark 5.3. The singularity alternative along $\mathcal{C}^-$ is considerably more exotic than a corner or cusp, as both of these would satisfy exterior sphere conditions. In fact, the singular point will be one where the flow in *both* layers approaches stagnation. Observe that the dynamic condition (5.1e) can be rearranged as

$$\frac{1}{2} \llbracket \rho |\nabla \psi|^2 \rrbracket = -\frac{\llbracket \rho \rrbracket}{F^2} \left(y - \frac{F^2}{2} \right).$$

For elevation waves, $y > 0$ along $\mathcal{S}$, so the right-hand side above vanishes at the (unique) point on the interface with height $y = F^2/2$. This highly degenerate scenario is outside the scope of current free boundary regularity theory. Indeed, the one-phase case was only recently considered by Varvaruca and Weiss [64], and several open questions regarding it remain. We strongly believe, however, that through a more detailed analysis the singularity alternative can be eliminated. This will be the subject of future work.

On the other hand, it is interesting to note that clearly the height $y = F^2/2$ is special. For example, in the numerical paper [50], Malakov and Sharipov compute a family of overhanging solitary internal waves. Roughly speaking, the free surface profile for the near-limiting waves resembles a mushroom or capital Ω . They observe that the point where the free boundary folds back occurs exactly at $y = F^2/2$, and the flow there is close to stagnation in both layers.

Remark 5.4. The numerical results in [28] suggest that along $\mathcal{C}^+$, the interface always hits the wall rather than overturning. Formally, one expects that in this scenario the entire upper layer will become stagnant, resulting in what is known as a gravity current. Formal computations by von Kármán [67] indicate that, at the point of contact, the interface makes a precise 60° angle with the wall (see also the paper of Benjamin [9]). In a very real sense, this is the analogue of the Stokes conjecture in the context of internal bores. We hope to address it as well in a forthcoming paper.

5.1. Reformulation

To fix the domain, we now switch to the new coordinates

$$q = x, \quad p = -\psi/h, \quad (5.6)$$

where h denotes the piecewise constant function which is h_1 in the lower layer and h_2 in the upper layer. This is sometimes called the Dubreil–Jacotin or partial hodograph transform. It is valid whenever there are no horizontal stagnation points:

$$\sup_{\mathcal{D}} \psi_y < 0. \quad (5.7)$$

The image of $\mathcal{D}$ under this change of variables is the slitted cylinder

$$\Omega = \Omega_1 \cup \Omega_2, \quad \Omega_1 := \mathbb{R} \times (-1, 0), \quad \Omega_2 := \mathbb{R} \times (0, 1).$$

In keeping with the general theory, denote by $\Gamma_0 := \{p = -1\} \cup \{p = 1\}$ the corresponding lower and upper boundaries, and let $\Gamma_1 := \{p = 0\}$ be the image of the internal interface $\mathcal{S}$. As a new dependent variable, we consider

$$y = y(q, p),$$

which measures the vertical deflection of the streamlines (level sets of ψ) relative to their upstream heights. This transforms (5.1) into

$$\left(-\frac{1+y_q^2}{2y_p^2}\right)_p + \left(\frac{y_q}{y_p}\right)_q = 0 \quad \text{for } 0 < |p| < 1. \quad (5.8a)$$

$$\frac{1}{2} \left[\left[\rho h^2 \frac{1+y_q^2}{y_p^2} \right] \right] + \frac{[\![\rho]\!]}{F^2} y = \frac{[\![\rho]\!]}{2} \quad \text{on } p = 0, \quad (5.8b)$$

$$y = -h_1 \quad \text{on } p = -1, \quad (5.8c)$$

$$y = h_2 \quad \text{on } p = 1, \quad (5.8d)$$

together with the asymptotic condition

$$y \rightarrow hp \quad \text{as } q \rightarrow -\infty. \quad (5.8e)$$

The upstream state in (5.1c) corresponds to $y = hp$, and so to conform to the conventions in the previous sections we introduce the difference

$$u(q, p) := y(q, p) - hp.$$

While u often denotes the horizontal fluid velocity in the fluid literature, we emphasize that it has no such connotation here. In terms of u , (5.8) becomes

$$\left(-\frac{1+u_q^2}{2(h+u_p)^2}\right)_p + \left(\frac{u_q}{h+u_p}\right)_q = 0 \quad \text{in } \Omega, \quad (5.9a)$$

$$-\frac{1}{2} \left[\rho h^2 \frac{1+u_q^2}{(h+u_p)^2} \right] - \frac{[\![\rho]\!]}{F^2} u + \frac{[\![\rho]\!]}{2} = 0 \quad \text{on } \Gamma_1, \quad (5.9b)$$

$$u = 0 \quad \text{on } \Gamma_0, \quad (5.9c)$$

with the asymptotic conditions

$$u \rightarrow 0 \quad \text{as } x \rightarrow -\infty. \quad (5.9d)$$

In what follows we will hold the densities $0 < \rho_2 < \rho_1$ as fixed constants, and allow the layer thicknesses h_1, h_2 to vary subject to the constraint $h_1 + h_2 = 1$. To conform with the notation of the previous sections, we therefore write

$$h_1 = \lambda, \quad h_2 = 1 - \lambda,$$

where $\lambda \in (0, 1)$ is our parameter. As we will see below in Section 5.3, in order to have a nontrivial solution with distinct limits as $u \rightarrow \pm\infty$, the Froude number F must be given by

$$F^2 = \frac{\sqrt{\rho_1} - \sqrt{\rho_2}}{\sqrt{\rho_1} + \sqrt{\rho_2}}, \quad (5.10)$$

independently of λ .

An elementary calculation shows that

$$u_q = -\psi_x/\psi_y, \quad h + u_p = -h/\psi_y. \quad (5.11)$$

Hence, u_q measures the slope of the relative velocity field, while $h + u_p$ is inversely proportional to the relative horizontal velocity. The absence of horizontal stagnation (5.7) is equivalent to the requirement that

$$\sup_{\Omega} (h + u_p) < \infty. \quad (5.12)$$

Following the conventions for transmission problems in Section 3.2, fix an $\alpha \in (0, 1)$ and let

$$\mathcal{X} := \{u \in C^{2+\alpha}(\overline{\Omega_1}) \cap C^{2+\alpha}(\overline{\Omega_2}) \cap C^0(\overline{\Omega}) : u|_{\Gamma_0} = 0\},$$

$$\mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2 := (C^\alpha(\overline{\Omega_1}) \cap C^\alpha(\overline{\Omega_2})) \times C^{1+\alpha}(\Gamma_1),$$

with $\mathcal{X}_b$, $\mathcal{X}_\infty$, $\mathcal{Y}_b$, and $\mathcal{Y}_\infty$ defined accordingly. For each $\delta > 0$, take

$$\mathcal{U}^\delta := \left\{ (u, \lambda) \in \mathcal{X}_b \times (0, 1) : \inf_{\Omega} (u_p + h) > \delta, \quad \frac{1}{\lambda} + \frac{1}{1-\lambda} < \frac{1}{\delta} \right\}, \quad (5.13)$$

so that the nonlinear operator $\mathcal{F}$ corresponding to (5.9) viewed as a map $\mathcal{X}_b \times \mathbb{R} \supset \mathcal{U}^\delta \rightarrow \mathcal{Y}_b$ satisfies the real-analyticity (1.2) and uniform ellipticity and obliqueness requirements (1.3). This collection is nested, so we write $\mathcal{U} := \bigcup_{\delta>0} \mathcal{U}^\delta$, and denote by $\mathcal{U}_\infty$ the set of $(u, \lambda) \in \mathcal{U}$ for which $u \in \mathcal{X}_\infty$. Note that the sets $\mathcal{U}^\delta$ are open, as are $\mathcal{U}$ and $\mathcal{U}_\infty$.

5.2. Variational structure and weak formulations

The problem (5.9) has a variational structure (3.1) with

$$\mathcal{L}(p, z, \xi, \lambda) = \rho \left[h^2 \frac{1 + \xi_1^2}{2(h + \xi_2)^2} + \frac{1}{2} - \frac{1}{F^2}(hp + z) \right] (h + \xi_2).$$

In particular, (5.9a)–(5.9c) can be written as

$$\begin{cases} \nabla \cdot \mathcal{L}_\xi(p, u, \nabla u, \lambda) - \mathcal{L}_z(p, u, \nabla u, \lambda) = 0 & \text{in } \Omega, \\ \llbracket \mathcal{L}_{\xi_2}(p, u, \nabla u, \lambda) \rrbracket = 0 & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_0. \end{cases} \quad (5.14)$$

Using an analogue of Lemma 3.1, we see that (5.9) has the conserved quantity

$$\mathcal{H}(u, \lambda; q) := \int_{-1}^1 \rho \left(h^2 \frac{1 - u_q^2}{2(h + u_p)^2} + \frac{1}{2} - \frac{1}{F^2}(hp + u) \right) (h + u_p) dp,$$

as can be verified by a direct calculation.

Notice that the first two lines of (5.14) can be interpreted as $\nabla \cdot \mathcal{L}_\xi - \mathcal{L}_z = 0$ holding in a weak sense. A related weak formulation is

$$\begin{cases} \nabla \cdot \left(\rho \nabla f(\nabla u, h) - \frac{\rho}{F^2} u \mathbf{e}_2 \right) + \frac{\rho}{F^2} u_p = 0 & \text{on } \Omega \cup \Gamma_1, \\ u = 0 & \text{on } \Gamma_0, \end{cases} \quad (5.15)$$

where the function $f: \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$f(\xi, a) = \frac{a^2 \xi_1^2 + \xi_2^2}{2(a + \xi_2)}.$$

Here and in what follows, the gradient $\nabla_\xi f$ and Hessian $D_\xi^2 f$ will simply be denoted by ∇f and $D^2 f$, respectively.

5.3. Conjugate flows

As in Section 4.2, our ability to refine the alternatives in Theorem 1.2 depends in a large part on the fact that the conjugate flow problem (C1)–(C2) is explicitly solvable. This fact is well known in the literature on internal waves; see for instance [43, Appendix A].

Lemma 5.5. *Fix $\lambda \in (0, 1)$. Then there is a flow U_+ conjugate to $U_- \equiv 0$ in the sense of Definition 3.3 if and only if the Froude number F is given by (5.10) and $\lambda \neq \lambda_+$ where*

$$\lambda_+ := \frac{\sqrt{\rho_1}}{\sqrt{\rho_1} + \sqrt{\rho_2}}. \quad (5.16)$$

Moreover, in this case U_+ is unique and given by

$$U_+(p) = (\lambda_+ - \lambda)(1 - |p|). \quad (5.17)$$

Proof. From (5.9a), (5.9c), and the continuity of U , we see that (C1) forces (5.17) for some $\lambda_+ \in (0, 1) \setminus \{\lambda\}$, which can be interpreted physically as the downstream depth of the lower layer. Plugging into (5.9b), we conclude that (C1) is satisfied if and only if λ_+ satisfies

$$\frac{1}{2} \left(\rho_2 \frac{(1-\lambda)^2}{(1-\lambda_+)^2} - \rho_1 \frac{\lambda^2}{\lambda_+^2} \right) + \frac{\rho_2 - \rho_1}{F^2} (\lambda - \lambda_+) = \frac{\rho_2 - \rho_1}{2}. \quad (5.18)$$

On the other hand,

$$\begin{aligned} \mathcal{H}(U, \lambda) &= \lambda_+ \rho_1 \int_{-1}^0 \left(\frac{\lambda^2}{2\lambda_+^2} + \frac{1}{2} - \frac{(\lambda_+ - \lambda)(1+p) + \lambda p}{F^2} \right) dp \\ &\quad + (1 - \lambda_+) \rho_2 \int_0^1 \left(\frac{(1-\lambda)^2}{2(1-\lambda_+)^2} + \frac{1}{2} - \frac{(\lambda_+ - \lambda)(1-p) + (1-\lambda)p}{F^2} \right) dp \\ &= \lambda_+ \rho_1 \left(\frac{\lambda^2}{2\lambda_+^2} + \frac{1}{2} + \frac{2\lambda - \lambda_+}{2F^2} \right) + (1 - \lambda_+) \rho_2 \left(\frac{(1-\lambda)^2}{2(1-\lambda_+)^2} + \frac{1}{2} + \frac{2\lambda - \lambda_+ - 1}{2F^2} \right). \end{aligned}$$

Setting $\lambda_+ = \lambda$ to calculate $\mathcal{H}(0, \lambda)$, we see that (C2) is equivalent to

$$\begin{aligned} \lambda_+ \rho_1 \left(\frac{\lambda^2}{2\lambda_+^2} + \frac{1}{2} + \frac{2\lambda - \lambda_+}{2F^2} \right) + (1 - \lambda_+) \rho_2 \left(\frac{(1-\lambda)^2}{2(1-\lambda_+)^2} + \frac{1}{2} + \frac{2\lambda - \lambda_+ - 1}{2F^2} \right) \\ = \lambda \rho_1 \left(1 + \frac{\lambda}{2F^2} \right) + (1 - \lambda) \rho_2 \left(1 + \frac{\lambda - 1}{2F^2} \right). \end{aligned} \quad (5.19)$$

Eliminating F^2 between (5.18) and (5.19), we discover that the only possible value of $\lambda_+ \in (0, 1) \setminus \{\lambda\}$ is (5.16). Substituting (5.16) into (5.18) then yields (5.10). ■

Lemma 5.6 (Spectral nondegeneracy). *Suppose that the Froude number F is given by (5.10), and let $(u, \lambda) \in \mathcal{U}_\infty$ be a monotone front solution of $\mathcal{F}(u, \lambda) = 0$. If $\lambda \neq \lambda_+$, then the spectral assumption (H2) holds, while $\lambda = \lambda_+$ instead implies $\sigma_0^-(u, \lambda) = 0$.*

Proof. Letting $U_\pm$ denote the limits of u as $x \rightarrow \pm\infty$, the operators $\mathcal{L}'_\pm(u, \lambda)$ are given by

$$\begin{aligned} \mathcal{L}'_{\pm 1}(u, \lambda)w &= \left(\frac{w_p}{(h + \partial_p U_\pm)^3} \right)_p, \\ \mathcal{L}'_{\pm 2}(u, \lambda)w &= \left[\rho h^2 \frac{w_p}{(h + \partial_p U_\pm)^3} \right] - \frac{[\rho]}{F^2} w. \end{aligned}$$

The upstream limit U_- is identically 0 since $u \in \mathcal{X}_\infty$, so let us first consider

$$\mathcal{L}'_{-1}(u, \lambda)w = \frac{w_{pp}}{h^3}, \quad \mathcal{L}'_{-2}(u, \lambda)w = \left[\frac{\rho w_p}{h} \right] - \frac{[\rho]}{F^2} w.$$

Setting $w = 1 - |p|$, we have $\mathcal{L}'_{+1}(u, \lambda)w = 0$ and

$$\mathcal{L}'_{-2}(u, \lambda)w = -\frac{\rho_2}{h_2} - \frac{\rho_1}{h_1} - \frac{\rho_2 - \rho_1}{F^2} = -\frac{\rho_2}{1-\lambda} - \frac{\rho_1}{\lambda} + (\sqrt{\rho_1} + \sqrt{\rho_2})^2.$$

Optimizing over $\lambda \in (0, 1)$ we see that $\mathcal{L}'_{-2}(u, \lambda)w \leq 0$ with equality only if $\lambda = \lambda_+$. For $\lambda \neq \lambda_+$, w is therefore a positive strict supersolution in the sense of Appendix A.3, implying that the principal eigenvalue $\sigma_0^-(u, \lambda)$ of $\mathcal{L}'_-(u, \lambda)$ is strictly negative as desired. For $\lambda = \lambda_+$, on the other hand, w is a positive eigenfunction of $\mathcal{L}'_-(u, \lambda)$ with eigenvalue 0, and hence $\sigma_0^-(u, \lambda) = 0$.

If $u \equiv 0$ then the argument for $\mathcal{L}'_+(u, \lambda)$ is identical, so suppose that $u \not\equiv 0$. Then Lemma 5.5 forces U_+ to be given by (5.17). Letting h_+ be the function which is λ_+ in the lower layer and $1 - \lambda_+$ in the upper layer, we find

$$\mathcal{L}'_{+1}(u, \lambda)w = \frac{w_{pp}}{h_+^3}, \quad \mathcal{L}'_{+2}(u, \lambda)w = \left[\frac{\rho h^2 w_p}{h_+^3} \right] - \frac{[\![\rho]\!]}{F^2} w.$$

Once more taking $w = 1 - |p|$, we have $\mathcal{L}'_{+1}(u, \lambda)w = 0$. The boundary term is

$$\mathcal{L}'_{+2}(u, \lambda)w = -\frac{(1-\lambda)^2 \rho_2}{(1-\lambda_+)^3} - \frac{\lambda^2 \rho_1}{\lambda_+^3} + (\sqrt{\rho_1} + \sqrt{\rho_2})^2.$$

Optimizing over $\lambda \in (0, 1)$ we again find that $\mathcal{L}'_{+2}(u, \lambda)w \geq 0$ with equality if and only if $\lambda = \lambda_+$. Arguing as above we conclude that $\sigma_0^+(u, \lambda) \leq 0$ with equality if and only if $\lambda = \lambda_+$. ■

5.4. Small-amplitude bores

We next recall the small-amplitude existence theory from [19, Section 7]. That paper treated the more general setting where the vorticity in the upper layer is constant, not necessarily zero. This substantially complicates the conjugate flow equations, as well as potentially allowing for critical layers (lines of stagnation points) in the bulk. As one consequence, the authors were forced to use a much less elegant change of variables to fix the fluid domain. For the layer-wise irrotational case (5.1a), however, translating between these two coordinate systems is tedious but rather straightforward. The result is then as follows.

Theorem 5.7 (Small-amplitude bores). *Fix densities $0 < \rho_2 < \rho_1$, and let the Froude number F be given by (5.10). There exists a C^0 curve $\mathcal{C}_{\text{loc}}$ of solutions to the internal wave problem (5.9) that admits the parameterization*

$$\mathcal{C}_{\text{loc}} = \{(u(\varepsilon), \lambda(\varepsilon)) : |\varepsilon| < \varepsilon_0\} \subset \mathcal{X}_\infty \times (0, 1)$$

for some $\varepsilon_0 > 0$. Moreover, the following statements hold along $\mathcal{C}_{\text{loc}}$:

- (a) (Asymptotics) *The height function $u(\varepsilon)$ and upstream depth ratio $\lambda(\varepsilon)$ have leading form expressions given by*

$$\begin{aligned} u(\varepsilon)(q, p) &= -\frac{\varepsilon}{2}(1 + \tanh(\kappa_1 |\varepsilon| q))(1 - |p|) + O(\varepsilon^2) \quad \text{in } \mathcal{X}_b, \\ \lambda(\varepsilon) &= \lambda_+ + \varepsilon = \frac{\sqrt{\rho_1}}{\sqrt{\rho_1} + \sqrt{\rho_2}} + \varepsilon, \end{aligned} \tag{5.20}$$

for the explicit positive constant

$$\kappa_1^2 := \frac{3(\sqrt{\rho_1} + \sqrt{\rho_2})^4}{4\rho_1(\rho_1 - \sqrt{\rho_1}\sqrt{\rho_2} + \rho_2)}.$$

- (b) (Monotonicity) If $\varepsilon > 0$, then $(u(\varepsilon), \lambda(\varepsilon))$ is a strictly decreasing front, while for $\varepsilon < 0$, it is a strictly increasing front.
- (c) (Uniqueness) In a neighborhood of $(0, \lambda_+)$ in $\mathcal{X}_\infty \times \mathbb{R}$, $\mathcal{C}_{\text{loc}}$ comprises all monotone fronts (up to translation).
- (d) (Kernel) For all $0 < |\varepsilon| < \varepsilon_0$, the kernel of $\mathcal{F}_u(u(\varepsilon), \lambda(\varepsilon))$ is one-dimensional and generated by $\partial_x u(\varepsilon)$.

Proof. The existence of $\mathcal{C}_{\text{loc}}$ and the asymptotics in (5.20) are essentially given by [19, Theorem 7.1, Corollary 7.3]. For that reason, we will only sketch the argument. In fact, the proof is remarkably similar to that of Theorem 4.2: one applies a center manifold reduction to obtain a planar system whose bounded solutions give rise to solutions of the height equation (5.9). This ODE at leading order coincides with (4.12), only the coefficients involve ρ , λ , and F . The conjugate flow analysis in Lemma 5.5 identifies the correct parameter regime in which the truncated reduced ODE admits a heteroclinic orbit, which is then shown to persist for the full reduced ODE using a conserved quantity derived from the flow force $\mathcal{H}$.

The monotonicity claimed in part (b) is a consequence of [19, Remark 7.9] and the maximum principle. In particular, the center manifold reduction argument gives directly $\pm(\partial_x u)(\varepsilon)(\cdot, 0) < 0$ for $\pm\varepsilon > 0$. To see that the rest of the streamlines are likewise monotone, we argue as in [19, Theorem 7.8 (a)].

Part (c) follows exactly as in the proof of Theorem 4.2 (c). Verifying that the kernel along $\mathcal{C}_{\text{loc}}$ is one-dimensional can likewise be accomplished using [19, Theorem 1.6] to reduce the question to the linearized reduced equation. By the same argument as in Theorem 4.2, it is readily seen that this planar system cannot have two linearly independent bounded solutions, and hence (H1) holds. ■

Remark 5.8. It bears mentioning that the truncated reduced ODE on the center manifold is exactly the stationary extended Korteweg–de Vries equation (eKdV). This is a nonlinear dispersive PDE that has been derived previously as a model for bores in a certain long wave scaling (see [37] and the references therein).

5.5. Velocity bounds

We now establish some a priori estimates for solutions of the internal wave problem. These are crucial to the proof of Theorem 5.1 as they will eventually allow us to conclude that the blowup alternative implies the stagnation limit (5.3). The first task is to derive L^∞ control of the velocity $(\partial_y \psi, -\partial_x \psi)$; this will ensure that the height equation (5.9) is uniformly elliptic along the global curve.

In the case of constant density rotational waves [63] or continuously stratified waves [18], one can show that a modified pressure enjoys a maximum principle and hence

has a lower bound. This can then be translated to an upper bound on the magnitude of the relative velocity through Bernoulli's law. For two-layer flows, however, it is not clear whether a similar approach is feasible due to the transmission condition. We therefore follow the strategy of [6] to first derive a local H^1 estimate on the stream function, and then use the classical monotonicity formula of Alt–Caffarelli–Friedman [2] in conjunction with Bernoulli's law to produce the bound for the velocity. Because our nondimensionalization differs from that in [6], we cannot simply quote their result. Indeed, the precise dependence on parameters is crucially important, so it is necessary to carefully re-prove the theorem in order to understand the effects of the new scaling.

Lemma 5.9. *Let (ψ, η, λ) be a solution of (5.1) with $\psi_y < 0$ and $\lambda \in (0, 1)$. Then, for all $m \in \mathbb{R}$,*

$$\int_{m-1}^{m+1} \int_{-\lambda}^{1-\lambda} |\nabla \psi|^2 dy dx \leq C_0, \quad (5.21)$$

where $C_0 = C_0(\rho, F) > 0$.

Proof. As $\psi_y < 0$, we can use the (q, p) variables to write

$$\int_{m-1}^{m+1} \int_{-\lambda}^{1-\lambda} |\nabla \psi|^2 dy dx = \int_{m-1}^{m+1} \int_{-1}^1 h^2 \frac{1 + u_q^2}{h + u_p} dp dq. \quad (5.22)$$

A second consequence is that $u_p > -h$, and hence $|u| < 1$.

For any $m \in \mathbb{R}$, consider a cutoff function $\zeta = \zeta(q) \in C_c^\infty([m-2, m+2])$ with $0 \leq \zeta \leq 1$ and $\zeta \equiv 1$ on $[m-1, m+1]$. Multiplying (5.15) by $\zeta^2 u$ and integrating over Ω , we get

$$\begin{aligned} & \iint_{\Omega} \frac{\rho h^2 \zeta^2}{2} \frac{2h + u_p}{(h + u_p)^2} \left(u_q^2 + \frac{u_p^2}{h^2} \right) dq dp \\ &= - \iint_{\Omega} 2\zeta \zeta' \rho h^2 \frac{u u_q}{h + u_p} dq dp + \frac{1 - \rho_2}{F^2} \int_{\Gamma_1} \zeta^2 u^2 dq \\ &\leq \epsilon \iint_{\Omega} \rho h^3 \zeta^2 \frac{u^2 u_q^2}{(h + u_p)^2} dq dp + \frac{1}{\epsilon} \int_{m-2}^{m+2} \rho h (\zeta')^2 dq + \frac{4(1 - \rho_2)}{F^2}, \end{aligned} \quad (5.23)$$

where in the last line we have used Young's inequality and the fact $|u| < 1$. Choosing $\epsilon = 1/2$ in each layer, this gives the bound

$$\frac{1}{2} \int_{m-1}^{m+1} \int_{-1}^1 \rho \frac{2h + u_p}{(h + u_p)^2} u_p^2 dp dq + \frac{1}{2} \int_{m-1}^{m+1} \int_{-1}^1 \rho h^2 \frac{u_q^2}{h + u_p} dp dq \leq C(\rho, F). \quad (5.24)$$

On the other hand,

$$\frac{1}{h + u_p} \leq \begin{cases} \frac{2}{h} & \text{when } u_p \geq -h/2, \\ \frac{4}{h^2} \frac{u_p^2}{h + u_p} & \text{otherwise,} \end{cases}$$

and therefore

$$\int_{m-1}^{m+1} \int_{-1}^1 \frac{h^2}{h+u_p} dp dq \leq C(\rho, F).$$

Substituting the above estimate and (5.24) into (5.22) yields (5.21). $\blacksquare$

Theorem 5.10 (Bounds on velocity). *Let $(u, \lambda) \in \mathcal{X}_\infty \times (\delta, 1 - \delta)$ be a strictly monotone front solution of (5.9) for some $\delta > 0$. Then*

$$\left\| \frac{1}{h+u_p} \right\|_{L^\infty(\Omega)} \leq C \quad (5.25)$$

and

$$\left\| \frac{u_q}{h+u_p} \right\|_{L^\infty(\Omega)} \leq C, \quad (5.26)$$

with $C = C(\delta, \rho, F) > 0$.

Remark 5.11. Observe that, in light of (5.11), this leads directly to a bound on the corresponding velocity field

$$\|\nabla \psi\|_{L^\infty(\mathcal{D})} \leq C$$

with $C = C(\delta, \rho, F) > 0$.

Proof of Theorem 5.10. Differentiating (5.9a) and then applying the maximum principle, we know that u_p attains its minimum either on $\partial\Omega$ or in the limits $q \rightarrow \pm\infty$. As $u \in \mathcal{X}_\infty$ is a monotone front, it does not change sign in Ω ; for definiteness, say u is strictly decreasing so that $u < 0$ in $\Omega \cup \Gamma_1$. By Lemma 5.5, this implies that

$$\lim_{q \rightarrow -\infty} u_p = 0, \quad \lim_{q \rightarrow \infty} u_p = U'_+ = (\lambda - \lambda_+) \operatorname{sgn} p,$$

where $\lambda_+ < \lambda$. Because u vanishes on Γ_0 , the Hopf boundary-point lemma tells us that $u_{1p} < 0$ on $\{p = -1\}$ and $u_{2p} > 0$ on $\{p = 1\}$. Thus the minimum of u_p is negative and is attained on $\{p = -1\}$, $\{p = 0\}$, or in the downstream limit.

Likewise, differentiating (5.9a) in q , we see that u_{1q} obeys the maximum principle. Since $u_q < 0$ in $\Omega \cup \Gamma_1$ and $u_{1q} = 0$ on $\{p = -1\}$, the Hopf boundary-point lemma implies that $u_{1pq} < 0$ on $\{p = -1\}$. Therefore

$$\min_{\Gamma_0} u_p = U'_+(-1) = \lambda_+ - \lambda.$$

In other words, both the minimums of u_p on Γ_0 and in the downstream limit are controlled by δ and ρ .

The above argument and the relationship between $\nabla \psi$ and ∇u given by (5.11) imply that the estimate (5.25) will follow from an upper bound of $\partial_y \psi_1$ and $\partial_y \psi_2$ on $\mathcal{S}$, whereas the estimate (5.26) is equivalent to an upper bound of ψ_x in $\mathcal{D}$. But ψ_x is harmonic in each layer, and so it attains its maximum on $\partial\mathcal{D}$. As it vanishes identically on the upper and lower boundaries as well as in the limits $q \rightarrow \pm\infty$, we need only estimate it on $\mathcal{S}$.

Consider now the pseudo-stream function $\tilde{\psi} := \sqrt{\rho}\psi$. Obviously, it suffices to control $|\nabla \tilde{\psi}|$ on $\mathcal{S}$. Following [6, Theorem 6.8], let a point $(x, y) \in \mathcal{S}$ be given and denote by B_r the ball of radius r centered there. We apply the monotonicity result of [2, Lemma 5.1] to conclude that the function

$$\phi(r) := \left(\frac{1}{r^2} \iint_{B_r \cap \mathcal{D}_1} |\nabla \tilde{\psi}|^2 dx dy \right) \left(\frac{1}{r^2} \iint_{B_r \cap \mathcal{D}_2} |\nabla \tilde{\psi}|^2 dx dy \right)$$

is increasing in r for $0 < r < a := \text{dist}(\mathcal{S}, \partial \bar{\mathcal{D}})$. The regularity of $\tilde{\psi}$ implies that $\phi(r) \rightarrow \phi(0)$ as $r \rightarrow 0$, and hence by Lemma 5.9,

$$\phi(0) = \frac{\pi^2}{4} |\nabla \tilde{\psi}_1(x, y)|^2 |\nabla \tilde{\psi}_2(x, y)|^2 < \phi(a) \leq C. \quad (5.27)$$

Recall that by the Bernoulli condition (5.1e) we have

$$\frac{1}{2} (|\nabla \tilde{\psi}_2|^2 - |\nabla \tilde{\psi}_1|^2) = \frac{\rho - 1}{2} \left(1 - \frac{2}{F^2} \eta \right) \quad \text{on } \mathcal{S}.$$

Since $\lambda_+ < \eta < 0$, this combined with (5.27) implies that $|\nabla \tilde{\psi}|$ is bounded by C . ■

Remark 5.12. From the above theorem we see that $\|u_q\|_{L^\infty(\Omega)} \leq C \|h + u_p\|_{L^\infty(\Omega)}$. Moreover, because we can always write

$$u(q, p) = \begin{cases} \int_{-1}^p u_p(q, p') dp' & \text{in } \Omega_1, \\ -\int_p^1 u_p(q, p') dp' & \text{in } \Omega_2, \end{cases}$$

it follows that

$$\|u\|_{L^\infty(\Omega)} \leq \|u_p\|_{L^\infty(\Omega)}, \quad \|u\|_{C^1(\Omega)} \leq C(1 + \|u_p\|_{C^0(\Omega)}).$$

5.6. Uniform regularity

The purpose of this section is to show that $\|u\|_{\mathcal{X}}$ is controlled by $\|u_p\|_{L^\infty(\Omega)}$ together with lower bounds on λ and $1 - \lambda$. When we construct the global bifurcation curve, this will allow us to conclude that blowup in norm coincides with either the stagnation limit or the interface coming into contact with the walls. Arguments of this kind are well known, for example, in the constant density and continuously stratified settings, where the system for the height function is elliptic with oblique boundary condition. Here, however, we are dealing with a transmission boundary condition. Instead of applying a Schauder-type argument, we will again follow the idea of Amick–Turner [6] and appeal to weak solution theory for elliptic equations of divergence form. For this, we quote the following theorem of Meyers, adjusted slightly to fit in our setting.

Theorem 5.13 (Meyers, [51]). *Let $\mathcal{D} \subset \mathbb{R}^2$ be a smooth bounded domain. Consider the equation*

$$\nabla \cdot (\mathbf{A} \nabla u) = \nabla \cdot G + g \quad \text{in } \mathcal{D}, \quad u = 0 \quad \text{on } \partial \mathcal{D}, \quad (5.28)$$

where the matrix $\mathbf{A} = \mathbf{A}(x)$ has measurable coefficients and satisfies $c_1 \mathbf{I} \leq \mathbf{A} \leq \frac{1}{c_1} \mathbf{I}$ for some $c_1 > 0$, where $\mathbf{I}$ is the 2×2 identity matrix. Then there exists some $r = r(c_1) > 2$ such that, for any $G \in L^r(\mathcal{D})$ and $g \in L^2(\mathcal{D})$, (5.28) admits a unique solution $u \in W_0^{1,r}(\mathcal{D})$, which obeys the estimate

$$\|\nabla u\|_{L^r(\mathcal{D})} \leq C(\|G\|_{L^r(\mathcal{D})} + \|g\|_{L^2(\mathcal{D})}),$$

where $C = C(\mathcal{D}, c_1, r) > 0$.

Our strategy will be to repeatedly differentiate (5.15) with respect to q and then apply Theorem 5.13 to obtain $W^{1,r}$ estimates on successively higher-order q derivative of u . Eventually, this will lead to sufficient uniform Hölder regularity of the trace $u|_{\Gamma_1}$ so that Schauder theory for the Dirichlet problem in Ω_i furnishes the desired result. Unlike Amick and Turner, we only know that u and its derivatives are locally integrable, so these estimates must be carried out with additional care.

Now for a fixed $\delta > 0$, we consider solutions (u, λ) of the height equation satisfying

$$\inf_{\Omega}(h + u_p) > \delta, \quad \|u\|_{C^1(\Omega)} + \frac{1}{h_1} + \frac{1}{h_2} < \frac{1}{\delta}, \quad (5.29)$$

where recall that $h_1 = \lambda$ and $h_2 = 1 - \lambda$. It is easy to see that under condition (5.29), we have uniform ellipticity:

$$D^2 f(\nabla u, h) \geq \frac{\delta^5}{4(1 + \delta)^3} \mathbf{I} =: c_1 \mathbf{I}, \quad |D^2 f(\nabla u, h)| \lesssim \frac{1}{\delta^5}. \quad (5.30)$$

Throughout this section, we will denote $\Omega_{m,k} := [m - k, m + k] \times (-1, 1)$ for $m \in \mathbb{R}$ and $k > 0$. It is important to note that these are connected domains that contain a portion of the internal interface Γ_1 .

Lemma 5.14. *Let (u, h) be a solution of the height equation (5.9) satisfying (5.29). Then there exists an $r = r(\delta) > 2$ such that for any $m \in \mathbb{R}$,*

$$\|\nabla v\|_{L^2(\Omega_{m,2})} \leq C \|\nabla u\|_{L^2(\Omega_{m,3})} \leq C, \quad (5.31)$$

$$\|\nabla u\|_{L^r(\Omega_{m,1})} \leq C, \quad (5.32)$$

$$\|\nabla v\|_{L^r(\Omega_{m,1})} \leq C, \quad (5.33)$$

where $v := u_q$ and $C = C(\delta) > 0$.

Proof. Consider a cutoff function $\tilde{\zeta} \in C_c^\infty([m - 4, m + 4])$ with $\tilde{\zeta} \equiv 1$ on $[m - 3, m + 3]$ and $0 \leq \tilde{\zeta} \leq 1$. Multiplying the equation by $\tilde{\zeta}^2 u$ and integrating by parts as in (5.23) yields

$$\begin{aligned} \iint_{\Omega} \frac{\rho h^2 \tilde{\zeta}^2}{2} \frac{2h + u_p}{(h + u_p)^2} \left(u_q^2 + \frac{u_p^2}{h^2} \right) dq dp \\ = - \iint_{\Omega} 2\tilde{\zeta} \tilde{\zeta}' \rho h^2 \frac{u u_q}{h + u_p} dq dp + \frac{2}{F^2} \iint_{\Omega} \rho \tilde{\zeta}^2 u u_p dq dp. \end{aligned}$$

Condition (5.29) guarantees that $\frac{2h+u_p}{(h+u_p)^2} \geq C(\delta) > 0$. Applying Young's inequality we conclude that

$$\|\nabla u\|_{L^2(\Omega_{m,3})} \leq C \|u\|_{L^2(\Omega_{m,4})},$$

and hence from Remark 5.12 the second estimate in (5.31) holds.

Now, differentiating (5.9) in q , we obtain the following equation for $v := u_q$:

$$\begin{cases} \nabla \cdot \left(\rho D^2 f(\nabla u, h) \nabla v - \frac{\rho}{F^2} v \mathbf{e}_2 \right) + \frac{\rho}{F^2} v_p = 0 & \text{in } \Omega \cup \Gamma_1, \\ v = 0 & \text{on } \Gamma_0, \end{cases} \quad (5.34)$$

where $\mathbf{e}_2 := (0, 1)^T$. Consider another cutoff function $\tilde{\zeta} \in C_c^\infty([m-3, m+3])$ with $\tilde{\zeta} \equiv 1$ on $[m-2, m+2]$ and $0 \leq \tilde{\zeta} \leq 1$. Testing (5.34) against the function $\tilde{\zeta}^2 v$, we have

$$\begin{aligned} \iint_{\Omega} \rho \tilde{\zeta}^2 \nabla v \cdot (D^2 f(\nabla u, h) \nabla v) dq dp &= \iint_{\Omega} \rho (\tilde{\zeta}^2)' v (D^2 f(\nabla u, h) \nabla v) \cdot \mathbf{e}_1 dq dp \\ &+ \frac{2}{F^2} \iint_{\Omega} \tilde{\zeta}^2 v v_p dq dp. \end{aligned} \quad (5.35)$$

Thus a use of Young's inequality and (5.30) leads to

$$\|\nabla v\|_{L^2(\Omega_{m,2})} \leq C \|v\|_{L^2(\Omega_{m,3})},$$

and hence the first estimate of (5.31).

Next, we will use Theorem 5.13 to derive the $\dot{W}^{1,r}$ estimates asserted in (5.32) and (5.33). Rewrite (5.15) as

$$\begin{cases} \nabla \cdot \left(\mathcal{A}(q, p, h) \nabla u - \frac{\rho}{F^2} u \mathbf{e}_2 \right) + \frac{\rho}{F^2} u_p = 0 & \text{in } \Omega \cup \Gamma_1 \\ u = 0 & \text{on } \Gamma_0, \end{cases}$$

where

$$\mathcal{A}(q, p, h) := \rho \int_0^1 D^2 f(t \nabla u(q, p), h) dt.$$

Since if (5.29) holds for functions w_1 and w_2 , it holds for all convex combinations of w_1 and w_2 , we know that $c_1 \mathbf{I} \leq \mathcal{A} \leq \frac{1}{c_1} \mathbf{I}$ for $c_1 = O(\delta^5)$ as in (5.30). Let $\bar{u} := \zeta u$ for ζ a cutoff function as in the proof of Lemma 5.9 and consider a domain $\mathcal{D}$ with a smooth boundary and $\Omega_{m,2} \subset \mathcal{D} \subset \Omega_{m,3}$. Then $\bar{u}$ satisfies

$$\begin{cases} \nabla \cdot (\mathcal{A} \nabla \bar{u}) = \nabla \cdot \left(u \mathcal{A} \nabla \zeta + \frac{\rho}{F^2} \zeta u \mathbf{e}_2 \right) + \nabla \zeta \cdot (\mathcal{A} \nabla u) - \frac{\rho}{F^2} \zeta u_p & \text{in } \mathcal{D}, \\ \bar{u} = 0 & \text{on } \partial \mathcal{D}. \end{cases}$$

Hence Theorem 5.13 and translation invariance imply

$$\|\nabla \bar{u}\|_{L^r(\mathcal{D})} \leq C \left(\left\| u \mathcal{A} \nabla \zeta + \frac{\rho}{F^2} \zeta u \mathbf{e}_2 \right\|_{L^r(\mathcal{D})} + \left\| \nabla \zeta \cdot (\mathcal{A} \nabla u) - \frac{\rho}{F^2} \zeta u_p \right\|_{L^2(\mathcal{D})} \right) \leq C,$$

where $r = r(\delta) > 2$ and $C = C(\delta)$. We have therefore proved (5.32). Applying a similar argument to (5.34) likewise gives (5.33). ■

If we further differentiate (5.34), we can obtain an L^r bound on the gradient of $\tilde{v} := v_q$. The equation for $\tilde{v}$ is

$$\begin{cases} \partial_i(\rho f^{ij} \partial_j \tilde{v}) + \partial_i(\rho f^{ijk} \partial_j v \partial_k v) - \partial_2 \left(\frac{\rho}{F^2} \tilde{v} \right) + \frac{\rho}{F^2} \partial_2 \tilde{v} = 0 & \text{in } \Omega \cup \Gamma_1, \\ \tilde{v} = 0 & \text{on } \Gamma_0, \end{cases} \quad (5.36)$$

where $\partial_1 = \partial_q$, $\partial_2 = \partial_p$ and we are using the shorthand

$$f^{ij} := (\partial_{\xi_i} \partial_{\xi_j} f)(\nabla u, h), \quad f^{ijk} := (\partial_{\xi_i} \partial_{\xi_j} \partial_{\xi_k} f)(\nabla u, h).$$

As before, we can test against a cutoff function $\zeta = \zeta(q)$ to obtain a local version for $\tilde{w} := \zeta \tilde{v}$ that reads

$$\nabla \cdot (\rho D^2 f \nabla \tilde{w}) = \nabla \cdot \tilde{G} + g \quad \text{in } \mathcal{D}, \quad (5.37a)$$

for a smooth domain $\Omega_{m,2} \subset \mathcal{D} \subset \Omega_{m,3}$. Here, the forcing terms are given explicitly by

$$\tilde{G} = \rho \tilde{v} D^2 f \nabla \zeta - \zeta G, \quad g = \rho \nabla \zeta \cdot D^2 f \nabla \tilde{v} + \nabla \zeta \cdot G - \frac{\rho}{F^2} \zeta \partial_2 \tilde{v}, \quad (5.37b)$$

where

$$G = \rho \left(f^{1jk} \partial_j v \partial_k v \right) - \frac{\rho}{F^2} \tilde{v} \mathbf{e}_2. \quad (5.37c)$$

Therefore, in order to apply Theorem 5.13, we need to ensure that $\nabla \tilde{v} \in L^2$ and $\nabla v \in L^s$ for some $s > 2$ sufficiently large.

Lemma 5.15. *Let (u, h) be a solution of (5.9) that satisfies (5.29). Then there exist an $r = r(\delta) > 2$ and a constant $C = C(\delta)$ such that, for any $m \in \mathbb{R}$,*

$$\|\nabla v\|_{L^4(\Omega_{m,1})}^4 \leq C(\|\nabla v_q\|_{L^2(\Omega_{m,1})}^{4-r} + 1), \quad (5.38)$$

where $v = u_q$.

Proof. Throughout the proof, let $C > 0$ denote a generic positive constant depending only on δ and $\|u_p\|_{C^0}$. Recall that v satisfies (5.34). From (5.31) and (5.33) we know that there exists some $r = r(\delta) > 2$ such that

$$\|v\|_{W^{1,\tilde{r}}(\Omega_{m,1})} \leq C \|u_p\|_{C^0(\Omega)} \leq C \quad (5.39)$$

for all $\tilde{r} \in [2, r]$ and $m \in \mathbb{R}$. Thus (5.38) holds if $r \geq 4$. So we assume that $r < 4$.

By the Gagliardo–Nirenberg inequality we have, for any $g \in H^1(\Omega_{m,k})$,

$$\|g\|_{L^4(\Omega_{m,k})} \lesssim \|\nabla g\|_{L^2(\Omega_{m,k})}^{1-\theta/4} \|g\|_{L^s(\Omega_{m,k})}^{\theta/4} \quad (5.40)$$

for all $\theta \in [1, 4]$. In particular, applying (5.40) with $\theta = 2$ to (5.39), we infer that

$$\|v\|_{L^4(\Omega_{m,1})}^4 \lesssim \|\nabla v\|_{L^2(\Omega_{m,1})}^2 \|v\|_{L^2(\Omega_{m,1})}^2 \leq C \|v\|_{L^2(\Omega_{m,2})}^4 \leq C \|u_p\|_{C^0(\Omega)}^4. \quad (5.41)$$

Now, the first equation of (5.34) can be rewritten as

$$\partial_p w = -\rho(f^{111}\tilde{v}^2 + (f^{112} + f^{121})v_p\tilde{v} + f^{122}v_p^2 + f^{1i}\partial_i\tilde{v}) - \frac{\rho}{F^2}v_p,$$

where

$$w := \rho f^{2i}\partial_i v - \frac{\rho}{F^2}v \quad \text{and} \quad \tilde{v} := v_q. \quad (5.42)$$

Moreover, from the definition of w we have

$$\partial_q w = \rho(f^{211}\tilde{v}^2 + (f^{212} + f^{221})v_p\tilde{v} + f^{122}v_p^2 + f^{2i}\partial_i\tilde{v}) - \frac{\rho}{F^2}v_q.$$

It follows from this and (5.39) that

$$\begin{aligned} \|w\|_{L^{\tilde{r}}(\Omega_{m,1})} &\leq C, \\ \|\nabla w\|_{L^2(\Omega_{m,1})}^2 &\leq C(\|\nabla\tilde{v}\|_{L^2(\Omega_{m,1})}^2 + \|\nabla v\|_{L^2(\Omega_{m,1})}^2 + \|v_q\|_{L^4(\Omega_{m,1})}^4 + \|v_p\|_{L^4(\Omega_{m,1})}^4). \end{aligned} \quad (5.43)$$

Hence from the definition (5.42) of w , (5.30), (5.41) and the Gagliardo–Nirenberg inequality (5.40) with $\theta = r$, we have

$$\begin{aligned} \|v_p\|_{L^4(\Omega_{m,1})}^4 &\leq C(\|w\|_{L^4(\Omega_{m,1})}^4 + \|v_q\|_{L^4(\Omega_{m,1})}^4 + \|v\|_{L^4(\Omega_{m,1})}^4) \\ &\leq C(\|\nabla w\|_{L^2(\Omega_{m,1})}^{4-r} \|w\|_{L^r(\Omega_{m,1})}^r + \|v_q\|_{L^4(\Omega_{m,1})}^4 + 1) \\ &\leq C((\|\nabla\tilde{v}\|_{L^2(\Omega_{m,1})}^2 + \|v_q\|_{L^4(\Omega_{m,1})}^4 + \|v_p\|_{L^4(\Omega_{m,1})}^4)^{2-r/2} + \|v_q\|_{L^4(\Omega_{m,1})}^4 + 1). \end{aligned}$$

Since $r > 2$, the above bound yields

$$\|v_p\|_{L^4(\Omega_{m,1})}^4 \leq C(\|\nabla v_q\|_{L^2(\Omega_{m,1})}^{4-r} + \|v_q\|_{L^4(\Omega_{m,1})}^4 + 1).$$

Applying (5.40) to v_q with $\theta = r$ and using (5.39), we see that

$$\|v_q\|_{L^4(\Omega_{m,1})}^4 \lesssim \|\nabla v_q\|_{L^2(\Omega_{m,1})}^{4-r} \|v_q\|_{L^r(\Omega_{m,1})}^r \leq C\|\nabla v_q\|_{L^2(\Omega_{m,1})}^{4-r}.$$

Therefore

$$\|v_p\|_{L^4(\Omega_{m,1})}^4 \leq C(\|\nabla v_q\|_{L^2(\Omega_{m,1})}^{4-r} + 1),$$

which implies (5.38). ■

Lemma 5.16. *Let (u, h) be a solution of (5.9) satisfying (5.29). Then there exists a constant $C = C(\delta)$ such that for any $m \in \mathbb{R}$,*

$$\|\nabla v_q\|_{L^2(\Omega_{m,1})} \leq C, \quad (5.44)$$

where $v = u_q$.

Proof. Recall that the function $\tilde{v} := v_q$ solves (5.36). We now test (5.34) against $\zeta^2\tilde{v}$ with the cutoff function ζ given as in Lemma 5.9. Using the ellipticity condition (5.30),

Young's inequality, (5.31) and (5.38), we discover that

$$\begin{aligned}
 c_1 \iint_{\Omega} \rho \zeta^2 |\nabla \tilde{v}|^2 dq dp &\leq \iint_{\Omega} \rho \zeta^2 f^{ij} \partial_j \tilde{v} \partial_i \tilde{v} dq dp \\
 &= - \iint_{\Omega} \rho \zeta^2 f^{ijk} \partial_j v \partial_k v \partial_i \tilde{v} dq dp - \iint_{\Omega} \rho (\zeta^2)' \tilde{v} f^{1j} \partial_j \tilde{v} dq dp \\
 &\quad - \iint_{\Omega} \rho (\zeta^2)' \tilde{v} f^{1jk} \partial_j v \partial_k v dq dp + \iint_{\Omega} \frac{2\rho}{F^2} \zeta^2 \tilde{v} \tilde{v}_p dq dp \\
 &\leq \frac{c_1}{2} \iint_{\Omega} \rho \zeta^2 |\nabla \tilde{v}|^2 dq dp + C(\|\nabla v\|_{L^4(\Omega_{m,2})}^4 + \|\tilde{v}\|_{L^2(\Omega_{m,2})}^2),
 \end{aligned}$$

which implies that

$$\|\nabla \tilde{v}\|_{L^2(\Omega_{m,1})}^2 \leq C(\|\nabla \tilde{v}\|_{L^2(\Omega_{m,2})}^{4-r} + 1).$$

Taking the supremum over all m , we find that

$$\sup_m \|\nabla \tilde{v}\|_{L^2(\Omega_{m,1})}^2 \leq C \left(\sup_m \|\nabla \tilde{v}\|_{L^2(\Omega_{m,2})}^{4-r} + 1 \right) \lesssim C \left(\sup_m \|\nabla \tilde{v}\|_{L^2(\Omega_{m,1})}^{4-r} + 1 \right).$$

Finally, because $r > 2$, this gives

$$\sup_m \|\nabla \tilde{v}\|_{L^2(\Omega_{m,1})} \leq C,$$

proving (5.44). ■

Corollary 5.17. *Let (u, h) be a solution to (5.9) that satisfies (5.29). Then, for all $s \geq 2$, there exists a constant $C = C(\delta, s)$ such that, for any $m \in \mathbb{R}$,*

$$\|\nabla v\|_{L^s(\Omega_{m,1})} \leq C. \quad (5.45)$$

Proof. From (5.38), (5.44) and (5.43) we know that

$$\|\nabla v\|_{L^4(\Omega_{m,1})}^4 + \|\nabla w\|_{L^2(\Omega_{m,1})}^2 \leq C,$$

where w is given in (5.42). Sobolev embedding then implies that $w \in L^s(\Omega_{m,1})$ for all $s \geq 2$. Similarly, from (5.39) and (5.44) we know that $v, v_q \in L^s(\Omega_{m,1})$. In this way, (5.45) follows from the fact that f^{22} is bounded from below by a positive constant depending on δ . ■

Now we can apply the elliptic regularity result of Theorem 5.13 to v_q , which gives the following.

Lemma 5.18. *Let (u, h) be a solution of (5.9) satisfying (5.29). Then there exist some $s_* = s_*(\delta_*) > 2$ and a constant $C = C(\delta)$ such that, for any $m \in \mathbb{R}$,*

$$\|\nabla v_q\|_{L^{s_*}(\Omega_{m,1})} \leq C. \quad (5.46)$$

Proof. Recall that we obtained an equation (5.37) governing a cutoff version of $\tilde{v} = v_q$ on a smooth subdomain $\Omega_{m,2} \subset \mathcal{D} \subset \Omega_{m,3}$. From (5.45) we know that $G \in L^s(\Omega_{m,2})$ for all $s \geq 2$, and hence $\tilde{G} \in L^s(\mathcal{D})$. Also (5.44) implies that $g \in L^2(\mathcal{D})$. Therefore, (5.46) is a direct consequence of Theorem 5.13. ■

Repeating the same argument as in Lemmas 5.15–5.18 one can actually obtain

Corollary 5.19. *Let (u, h) be a solution of (5.9) satisfying (5.29). Then for any integer $k \geq 2$, there exist some $s_* = s_*(k, \delta) > 2$ and a constant $C = C(k, \delta)$ such that, for any $m \in \mathbb{R}$,*

$$\|\nabla \partial_q^k u\|_{L^{s_*}(\Omega_{m,1})} \leq C. \quad (5.47)$$

Recalling that $\Gamma_1 \subset \bigcup_m \Omega_{m,1}$, the above result and Morrey's inequality ensure that the trace $u|_{\Gamma_1}$ can be bootstrapped to arbitrarily high Hölder regularity. This is the key to proving our main result of the subsection.

Theorem 5.20 (Uniform regularity and analyticity). *Let (u, h) be a solution to (5.9) satisfying (5.29). Then for any integer $k \geq 2$ and any $\alpha \in (0, 1)$, there is a constant $C = C(k, \delta) > 0$ such that $u \in C^{k+\alpha}(\overline{\Omega_i})$ and*

$$\|u\|_{C^{k+\alpha}(\overline{\Omega_i})} \leq C.$$

Moreover, u is real-analytic in $\overline{\Omega_i}$.

Proof. From Lemma 5.18 and Sobolev embedding we know that the trace $u|_{\Gamma_1}$ is of class $C^{k+\alpha}$. Furthermore, the conditions of [33, Theorem 6.19] are met and hence the uniform regularity follows from standard Schauder theory.

The analyticity can be inferred from the classical result of [42]. More specifically, returning to the stream function formulation (5.1) one may apply [42, Theorem 3.2] to obtain analyticity. ■

5.7. Global bifurcation

We now have all the necessary tools for the proof of our existence theorem for large-amplitude bores.

Proof of Theorem 5.1. In Theorem 5.7, it was shown that there exists a local curve $\mathcal{C}_{\text{loc}}$ of solutions to the internal wave problem, and the kernel condition (H1) holds along it. Let $\mathcal{C}_{\text{loc}}^\pm$ denote the segment corresponding to the parameter values $\pm\epsilon \in (0, \epsilon_0)$. By construction, both $\mathcal{C}_{\text{loc}}^-$ and $\mathcal{C}_{\text{loc}}^+$ emanate from the trivial solution $(0, \lambda_+)$, so that (H4) follows from Lemma 5.6. It is also clear that they lie in $\mathcal{U}_\infty$ and $\mathcal{U}^\delta$ for all $0 < \delta \ll 1$. Moreover, we confirmed in Lemma 5.6 that the spectral assumption (H2) holds along $\mathcal{C}_{\text{loc}}^\pm$.

Applying Corollary 3.5 to $\mathcal{C}_{\text{loc}}^\pm$ with $\mathcal{F}|_{\mathcal{U}^\delta}$ gives a global curve $\mathcal{C}_\delta^\pm \subset \mathcal{U}^\delta$ of strictly monotone bore solutions. By maximality, they are nested, and hence their union $\mathcal{C}^\pm := \bigcup_{\delta>0} \mathcal{C}_\delta^\pm \subset \mathcal{U}_\infty$ is also a C^0 curve of strictly monotone fronts. Like the local curves, the fronts on $\mathcal{C}^+$ are strictly decreasing while those on $\mathcal{C}^-$ are strictly increasing. Expressed

in Eulerian variables through (5.11), this gives (5.2), proving part (a). We already concluded that part (c) holds, so it remains to prove (b).

With that in mind, consider the limiting behavior along $\mathcal{C}^+$; an identical argument will apply to $\mathcal{C}^-$. Thanks to our characterization in Lemma 5.5 of the flows that are conjugate to 0, Corollary 3.2 implies that every solution along $(u(s), \lambda(s)) \in \mathcal{C}^+$ has

$$\lim_{q \rightarrow \infty} u(s)(q, p) = (\lambda_+ - \lambda(s))(1 - |p|),$$

where λ_+ is given in (5.16). This shows that $\liminf_s \lambda(s) \geq \lambda_+$, as otherwise strict monotonicity would fail along $\mathcal{C}^+$. It also rules out heteroclinic degeneracy (A2). Indeed, if (A2) were to occur, then by Lemma 3.4 the three flows

$$\begin{aligned} \lim_{q \rightarrow \mp \infty} u_*(q, \cdot), \quad \lim_{n \rightarrow \infty} \lim_{q \rightarrow \infty} u(s_n)(q, \cdot) &= (\lambda_+ - \lambda_*)(1 - |p|), \\ \lim_{n \rightarrow \infty} \lim_{q \rightarrow -\infty} u(s_n)(q, \cdot) &= 0, \end{aligned}$$

must all be conjugate and distinct, contradicting Lemma 5.5. Likewise, Lemma 5.6 and Theorem 5.7 (c) ensure that the spectral degeneracy alternative (A3) does not happen. Thus we are left only with blowup (A1).

Suppose first that the internal interface stays bounded away from the walls:

$$\limsup_{s \rightarrow \infty} \lambda(s) < 1.$$

Note that we already have a uniform lower bound $\lambda(s) > \lambda_+ > 0$. By Theorem 5.10, it follows that $\mathcal{C}^+ \subset \mathcal{U}^\delta$ for some $\delta > 0$. Recalling the definition of $N(s)$ in (1.5), this implies that $\|u(s)\|_{\mathcal{X}} \rightarrow \infty$ as $s \rightarrow \infty$. In light of Theorem 5.20, this can occur only if $\|\partial_p u(s)\|_{C^0(\Omega)} \rightarrow \infty$.

To see that this leads to stagnation on the interface (5.3), let $\psi(s)$ be the corresponding stream functions and $\mathcal{D}(s)$ the fluid domain. Translated to Eulerian variables using (5.11), the blowup alternative becomes

$$\lim_{s \rightarrow \infty} \left\| \frac{1}{\partial_y \psi(s)} \right\|_{C^0(\mathcal{D}(s))} = \infty.$$

Because $\partial_y \psi(s)$ is harmonic in $\mathcal{D}(s)$, it cannot attain its minimum in the interior of either layer. Also, $\partial_x \psi(s)$ vanishes identically on the upper and lower boundaries, and so we have $\partial_y^2 \psi(s) = -\partial_x^2 \psi(s) = 0$ along them as well. The Hopf boundary-point lemma therefore implies that $\partial_y \psi(s)$ does not attain its minimum or maximum on the walls. But the upstream limit of $\partial_y \psi(s)$ is simply -1 , while the downstream limit is bounded uniformly away from 0 in terms of $1 - \lambda(s)$. We must then have

$$\inf_{\mathcal{D}(s)} |\partial_y \psi(s)| = \inf_{\mathcal{S}(s)} |\partial_y \psi(s)| = - \sup_{\mathcal{S}(s)} \partial_y \psi(s) \rightarrow 0 \quad \text{as } s \rightarrow \infty.$$

Assume instead that $\lambda(s) \rightarrow 1$. Then from (5.1d) we have

$$h_2 = 1 - \lambda(s) = - \int_{\eta(s)(x)}^{1-\lambda(s)} \partial_y \psi(s)(x, y) dy \quad \text{for all } x \in \mathbb{R},$$

and thus

$$\inf_{\eta(s)(x) < y < 1 - \lambda(s)} |\partial_y \psi(s)(x, y)| \leq \frac{1 - \lambda(s)}{1 - \lambda(s) - \eta(s)(x)},$$

which in particular means that a stagnation point develops on the interface. The proof of the theorem is therefore complete. ■

5.8. Limiting interfaces

We now turn to the proof of Theorem 5.2 on the limiting behavior of the free surface profile along the curves $\mathcal{C}^\pm$.

Lower bound for elevation bores. An important step in proving Theorem 5.2(a) is to ensure that along $\mathcal{C}^-$, the interface does not come into contact with the lower boundary. Throughout the subsection we will suppose that the interface does not overturn, meaning that $\eta(s)$ is uniformly bounded in Lipschitz norm:

$$\limsup_{s \rightarrow -\infty} \|\partial_x \eta(s)\|_{L^\infty(\mathbb{R})} = M < \infty. \quad (5.48)$$

It follows that $|\nabla \psi_i(s)| \lesssim_M |\partial_y \psi_i(s)|$ on $\mathcal{S}(s)$, and thus in the stagnation limit (5.3), the full gradient vanishes.

Observe also that because each $\eta(s)$ is increasing and vanishes upstream, there exists a unique $x(s) \in \mathbb{R}$ such that $\eta(s)(x(s)) = \sigma$, where $\sigma := \min\{F^2/4, \lambda_+/4\}$. The uniform Lipschitz bound (5.48) then gives

$$\sigma/2 < \eta(s) < \sigma \quad \text{on } (x(s) - L, x(s)) \quad (5.49)$$

for $L := \sigma/(2M)$.

Since $0 < \lambda(s) < \lambda_+$, we may pass to a subsequence along which $\lambda(s) \rightarrow \lambda^* \in [0, \lambda_+]$. The case $\lambda^* = 0$ indicates the lower layer has collapsed, while the fact $\lambda^* \leq \lambda_+$ is a consequence of the preservation of monotonicity.

Consider now the translated family of profiles

$$\tilde{\eta}(s) := \eta(s)(\cdot + x(s) - L/2)$$

with domain $\mathcal{I} := (-L/2, L/2)$. As it is uniformly bounded in $\text{Lip}(\bar{\mathcal{I}})$, we can extract a subsequence converging in C^ε for all $\varepsilon \in (0, 1)$ to some $\eta^* \in \text{Lip}(\bar{\mathcal{I}})$. By (5.49), the entire family, $\tilde{\eta}(s)$ is uniformly positive on $\mathcal{I}$.

Denote $\mathcal{S}(s) := \{(x, \tilde{\eta}(s)(x)) : x \in \mathcal{I}\}$, $\mathcal{D}(s) := \mathcal{I} \times (-\lambda(s), 1 - \lambda(s))$, and let $\tilde{\psi}_i(s)$ be the corresponding translated stream functions. From Lemma 5.9, we know that $\nabla \tilde{\psi}(s)$ is bounded uniformly in $L^2(\mathcal{D}(s))$. As in the proof of Theorem 5.10, the Alt–Caffarelli–Friedman monotonicity formula (5.27) and dynamic condition (5.1e) together give

$$\limsup_{s \rightarrow \infty} \|v \cdot \nabla \tilde{\psi}_i(s)\|_{L^\infty(\mathcal{S}(s))} \lesssim 1,$$

where ν is the outward unit normal to $\mathcal{D}_1(s)$ along $\mathcal{S}(s)$. Since each $\mathcal{S}(s)$ is C^2 , this leads to uniform Lipschitz control of $\tilde{\psi}_i(s)$ in a neighborhood $\mathcal{N}$ of $\mathcal{S}(s)$; see, for example, [14, Lemma 11.19]. On the other hand, $\tilde{\psi}(s)$ satisfies homogeneous Neumann conditions on the top and bottom of $\mathcal{D}$, so Schauder theory gives uniform bounds on $\tilde{\psi}(s)$ in $C^{2+\alpha}(\mathcal{D}(s) \setminus \mathcal{N})$.

Now, let $\bar{\psi}_1(s) \in \text{Lip}(\bar{\mathcal{I}} \times [-1, 1])$ be the function found by extending $\tilde{\psi}_1(s)$ by zero above $\mathcal{S}(s)$ and by $\lambda(s)$ below $y = -\lambda(s)$. Letting $\bar{\psi}_2(s)$ be defined likewise, the above argument show that $\bar{\psi}_i(s)$ is uniformly Lipschitz on $\mathcal{D}(s)$. Then along a subsequence we have

$$\begin{aligned} \bar{\psi}_i(s) &\xrightarrow{C^\varepsilon} \psi_i^* \in \text{Lip}(\bar{\mathcal{D}}) \quad \text{for all } \varepsilon \in (0, 1), \\ \nabla \bar{\psi}_i(s) &\longrightarrow \nabla \psi_i^* \quad \text{weakly-* in } L^\infty(\mathcal{D}), \end{aligned} \quad (5.50)$$

where $\mathcal{D} := \mathcal{I} \times (-\lambda^*, 1 - \lambda^*)$. Denote

$$\begin{aligned} \psi^* &:= \psi_1^* + \psi_2^*, & \mathcal{S} &:= \{y = \eta^*(x)\} \cap \mathcal{D}, \\ \mathcal{D}_1 &:= \{y < \eta^*(x)\} \cap \mathcal{D}, & \mathcal{D}_2 &:= \{y > \eta^*(x)\} \cap \mathcal{D}. \end{aligned} \quad (5.51)$$

Lemma 5.21. *Assume that along $\mathcal{C}^-$ the interface does not overturn, i.e. (5.48) holds. Let ψ^* and the sets $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{S}$ be as in (5.51). Then*

$$\psi^* \in C^{2+\alpha}(\mathcal{D}_1) \cap C^{2+\alpha}(\mathcal{D}_2), \quad \psi^* < 0 \quad \text{in } \mathcal{D}_2.$$

If $\lambda^ = 0$, then $\psi^* = 0$ in $\overline{\mathcal{D}_1}$. On the other hand, if $\lambda^* > 0$, then $\psi^* > 0$ in $\mathcal{D}_1$.*

Proof. The interior regularity is obvious: $\tilde{\psi}(s)$ is harmonic in each layer, and thus ψ^* is harmonic in $\mathcal{D} \setminus \mathcal{S}$. Since $\tilde{\psi}(s) = \lambda(s) - 1 < \lambda_+ - 1 < 0$ on the top boundary of $\mathcal{D}(s)$, we must have $\psi_2^* < 0$ on the upper boundary of $\mathcal{D}_2$. Because ψ_2^* vanishes on $\mathcal{S}$, the strong maximum principle implies that $\psi^* < 0$ in the upper layer.

If $\lambda^* > 0$, the same argument ensures that $\psi^* > 0$ in $\mathcal{D}_1$. On the other hand, if $\lambda^* = 0$, then because $\tilde{\psi}(s) = \lambda(s)$ on the lower boundary of $\mathcal{D}(s)$, ψ^* vanishes on both the top and bottom of $\mathcal{D}_1$. But, by continuity $\psi_y^* \leq 0$ in $\mathcal{D}_1$, and so we must have $\psi^* = 0$ in $\overline{\mathcal{D}_1}$. ■

Lemma 5.22 (Lower bound). *In the limit along $\mathcal{C}^-$, if the interface does not overturn, i.e. (5.48) holds, then*

$$\liminf_{s \rightarrow -\infty} \lambda(s) > 0.$$

Proof. Seeking a contradiction, suppose that (5.48) holds but along some subsequence $\lambda(s) \rightarrow 0$. From Lemma 5.21, we see that by passing to a further subsequence we have $\bar{\psi}_1(s) \rightarrow 0$ in $C^\varepsilon(\bar{\mathcal{D}})$ for all $\varepsilon \in (0, 1)$, and so in view of (5.50), $\nabla \bar{\psi}_1(s) \rightarrow 0$ weakly-* in $L^\infty(\mathcal{D})$.

Let ξ be a smooth nonnegative test function supported in a sufficiently small neighborhood of $\mathcal{S}$ in $\mathcal{I} \times (-1, 1)$. Then

$$\int_{\mathcal{F}(s)} \nu \cdot \nabla \bar{\psi}_1(s) \xi \, dS = \int_{\mathcal{D}(s)} \nabla \bar{\psi}_1(s) \cdot \nabla \xi \, dx \, dy \rightarrow 0 \quad \text{as } s \rightarrow \infty,$$

where $\tilde{\mathcal{D}}(s)$ is the translated fluid domain determined by $\tilde{\eta}$, dS is surface measure, and ν is the outward unit normal to $\tilde{\mathcal{D}}_1(s)$. Since $\nu \cdot \nabla \tilde{\psi}_1(s) = \nu \cdot \nabla \tilde{\psi}_1(s)$ is strictly negative on $\mathcal{S}(s)$ and ξ is nonnegative, we infer that, passing to a further subsequence,

$$\nu \cdot \nabla \tilde{\psi}_1(s)|_{y=\tilde{\eta}(s)(x)} \rightarrow 0 \quad \text{a.e. on } \mathcal{I}.$$

But then the Bernoulli condition (5.1e) leads to the inequality

$$\tilde{\eta}(s) > \frac{F^2}{2} + \frac{F^2 \rho_1}{2\llbracket \rho \rrbracket} |\nabla \tilde{\psi}_1(s)|^2 > \frac{F^2}{4} \quad \text{a.e. on } \mathcal{I}, \text{ for } s \gg 1,$$

which contradicts the upper bound on $\tilde{\eta}(s)$ in (5.49). ■

$C^{1+\varepsilon}$ regularity at regular boundary points and the proof of Theorem 5.2. The next (and last) step of the argument is to show that if overturning does not occur and the interface remains bounded away from the walls, then we can extract a subsequence converging to a limiting wave defined near the stagnation point. In fact, this will be a solution in the viscosity sense of a two-phase free boundary problem. Through the classical work of Caffarelli [15], we will prove that the interface is smooth enough to apply the Hopf boundary-point lemma, and thus obtain a contradiction since there is a stagnation point.

For completeness, let us first recall the following definition.

Definition 5.23. Let $\mathcal{D} \subset \mathbb{R}^2$ be a smooth bounded domain. We say that $w \in C^0(\mathcal{D})$ is a (weak) viscosity solution to the elliptic free boundary problem

$$\begin{cases} \Delta w = 0 & \text{in } \{w > 0\}, \\ \Delta w = 0 & \text{in } \text{int } \{w \leq 0\}, \\ \partial_\mu w^+ = g(\partial_\mu w^-, X) & \text{on } \partial\{w > 0\}, \end{cases} \quad (5.52)$$

provided it satisfies the interior equations in the usual viscosity sense, and the boundary condition is interpreted as follows:

- If at $X_0 \in \partial\{w > 0\}$ there is a tangent ball $B \subset \{w > 0\}$ and $a, b \geq 0$ such that

$$\begin{aligned} w^+ &\geq a \langle \cdot - X_0, \mu \rangle^+ + o(|\cdot - X_0|) \quad \text{in } B, \\ w^- &\leq b \langle \cdot - X_0, \mu \rangle^- + o(|\cdot - X_0|) \quad \text{in } B^c, \end{aligned}$$

with equality on nontangential domains, then $a \leq g(b, X_0)$. Here μ is normal to B at X_0 pointing into $\{w > 0\}$ and the superscripts $\pm$ denote the positive and negative parts.

- If at $X_0 \in \partial\{w > 0\}$ there is a tangent ball $B \subset \text{int } \{w \leq 0\}$ and $a, b \geq 0$ such that

$$\begin{aligned} w^- &\geq b \langle \cdot - X_0, \mu \rangle^- + o(|\cdot - X_0|) \quad \text{in } B, \\ w^+ &\leq a \langle \cdot - X_0, \mu \rangle^+ + o(|\cdot - X_0|) \quad \text{in } B^c, \end{aligned}$$

with equality on nontangential domains and μ defined as before, then $a \geq g(b, X_0)$.

Note that there are several equivalent characterizations of viscosity solutions; see [14, 27] for background and further discussion. For our purposes, we need only the following two facts, both of which are well-known consequences of the definition. First, any classical solution is a viscosity solution, and second, if $\{w_n\}$ is a sequence of solutions converging uniformly to w , then w is a viscosity solution.

Our interest in viscosity solutions lies in the next theorem, which is essentially the main result of [15] (with a later refinement in [16]). It can also be obtained using the techniques of De Silva [26].

Theorem 5.24 (Caffarelli). *Assume that $w \in C^0(\mathcal{D})$ solves the free boundary problem (5.52) in the weak viscosity sense of Definition 5.23 and that in a neighborhood of $X_0 \in \partial\{w > 0\}$, the free boundary is the graph of a Lipschitz function. Assume that g is smooth; $g(0, \cdot)$ is uniformly positive near X_0 and $g(\cdot, X)$ is strictly increasing; and, for some $N \gg 1$, $t \mapsto t^{-N}g(t, X)$ is strictly decreasing. Then in a small neighborhood of X_0 , the free boundary is $C^{1+\varepsilon}$ for some $\varepsilon \in (0, 1)$.*

With that in mind, let us now show that if the alternatives in Theorem 5.2 do not occur, then we can extract a limiting bore that satisfies the equations locally in a viscosity sense. In what follows, we suppose that $(x(s), y(s)) \in \mathcal{S}(s)$ is a family of boundary points along which the stagnation limit (5.3) occurs as $s \rightarrow \pm\infty$. First, we look at the family of depression bores.

Lemma 5.25 (Depression limiting problem). *If along $\mathcal{C}^+$, neither alternative in (5.5) occurs, then we can extract translated subsequences so that in the limit $s \rightarrow \infty$,*

$$\begin{aligned} (x(s), y(s)) &\longrightarrow (0, y^*), \\ \lambda(s) &\longrightarrow \lambda^* \in [\lambda_+, 1), \\ \eta(s) &\xrightarrow{C^\varepsilon} \eta^* \in \text{Lip}(\bar{\mathcal{I}}) \quad \text{for all } \varepsilon \in (0, 1), \\ \bar{\psi}_i(s) &\xrightarrow{C^\varepsilon} \psi_i^* \in \text{Lip}(\bar{\mathcal{D}}) \quad \text{for all } \varepsilon \in (0, 1), \\ \nabla \bar{\psi}_i(s) &\longrightarrow \nabla \psi_i^* \quad \text{weakly-* in } L^\infty(\mathcal{D}), \end{aligned} \tag{5.53}$$

where $\mathcal{I}$ is a sufficiently small open interval containing 0 and $\mathcal{D} := \mathcal{I} \times (-\lambda^*, 1 - \lambda^*)$. Moreover, ψ^* is a viscosity solution of the free boundary problem (5.52) for the function

$$g(\partial_\mu w^-, y) := \frac{1}{\rho_1} \left(\rho_2 |\partial_\mu w^-|^2 + \frac{2\llbracket \rho \rrbracket}{F^2} y - \llbracket \rho \rrbracket \right)^{1/2}. \tag{5.54}$$

Proof. By hypothesis, $\limsup \lambda(s) < 1$ and so we can extract a subsequence converging to $\lambda^* \in [\lambda_+, 1)$. Repeatedly translating $x(s)$ to the origin and arguing as in the proof of Lemma 5.21, the rest of the limits in (5.53) follow easily. Moreover, we see that

$$\psi^* > 0 \quad \text{in } \mathcal{D}_1 := \{y < \eta^*(x)\} \cap \mathcal{D}, \quad \psi^* < 0 \quad \text{in } \mathcal{D}_2 := \{y > \eta^*(x)\} \cap \mathcal{D},$$

so that $\partial\{\psi^* > 0\} = \{y = \eta^*(x)\}$. By the Hopf lemma, we know that the inward normal derivative $\mu \cdot \nabla \psi_1(s)$ is strictly positive on $\mathcal{S}(s)$. Observe also that along $\mathcal{C}^+$,

$$\eta(s) < \lim_{x \rightarrow \infty} \eta(s) = \lambda_+ - \lambda(s) < 0.$$

This justifies rearranging the Bernoulli condition (5.1e) as

$$\partial_\mu \psi_1(s) = g(\partial_\mu \psi_2(s), y) \quad \text{on } \mathcal{S}(s) = \partial\{\psi_1(s) > 0\}$$

with g given by (5.54). Thus, ψ^* is the uniform limit of viscosity solutions to the free boundary problem, and so it too is a viscosity solution. ■

The situation for $\mathcal{C}^-$ is nearly the same except that we must be wary of the case $y^* = F^2/2$ as this permits double stagnation.

Lemma 5.26 (Elevation limiting problem). *If along $\mathcal{C}^-$, the interface does not overturn, then in the limit $s \rightarrow -\infty$ we can extract a convergent subsequence as in (5.53) except that now $\lambda^* \in (0, \lambda_+]$. If $y^* < F^2/2$, then perhaps after shrinking $\mathcal{I}$, ψ^* is a viscosity solution of the free boundary problem (5.52) for g as in (5.54). If instead $y^* > F^2/2$, then $-\psi^*$ is a viscosity solution taking*

$$g(\partial_\mu w^-, y) := \frac{1}{\rho_2} \left(\rho_1 |\partial_\mu w^-|^2 - \frac{2\llbracket \rho \rrbracket}{F^2} y + \llbracket \rho \rrbracket \right)^{1/2}. \quad (5.55)$$

Proof. Assuming that overturning (5.4) does not occur, we have by Lemma 5.22 that $\liminf \lambda(s) > 0$. The existence of the subsequential limits (5.53) follows exactly as before. It remains only to consider the Bernoulli condition. If $y^* < F^2/2$, then in a neighborhood $(0, y^*)$ the radicand of g in (5.54) is strictly positive. On the other hand, if $y^* > F^2/2$, then we must solve (5.1e) for $\partial_\mu \psi_2(s)$ in terms of y and $\partial_\mu \psi_1(s)$. This results in the function g defined in (5.55). ■

Applying Theorem 5.24, we arrive at the following immediate corollary of these lemmas.

Corollary 5.27 ($C^{1+\varepsilon}$ regularity). *Consider the regularity of the limiting profiles along $\mathcal{C}^\pm$.*

- (a) *In the setting of Lemma 5.25, the limiting profile η^* along $\mathcal{C}^+$ is of class $C^{1+\varepsilon}$ for some $\varepsilon \in (0, 1)$ on a neighborhood of 0.*
- (b) *The same statement holds in the setting of Lemma 5.26 for the limiting profile along $\mathcal{C}^-$ provided $y^* \neq F^2/2$.*

We are at last ready to prove the main result on the limiting behavior of the interface.

Proof of Theorem 5.2. First consider the statement in part (a). Seeking a contradiction, suppose that (5.4) does not occur. Then Lemma 5.26 gives us the existence of a limiting profile η^* and stream function $\psi^* = \psi_1^* + \psi_2^*$.

If $y^* \neq F^2/2$, we know by Corollary 5.27 (b) that $\eta^* \in C^{1+\varepsilon}$ in a neighborhood of 0, and thus ψ_1^* and ψ_2^* are $C^{1+\varepsilon}$ up to that portion of the boundary. This is a contradiction: by construction, $(0, y^*)$ is the limiting stagnation point and so one of $|\nabla \bar{\psi}_i|$ must vanish there, which cannot be true in view of the Hopf boundary-point lemma. Note that this requires a sharper version than the classical result; see Lemma B.2.

On the other hand, if $y^* = F^2/2$, then from the Bernoulli condition we see that *both* phases limit to stagnation at $(0, y^*)$, that is,

$$\nabla \psi_1(s)(x(s), y(s)), \nabla \psi_2(s)(x(s), y(s)) \rightarrow 0 \quad \text{as } s \rightarrow -\infty.$$

Thus $\{y < \eta^*\}$ satisfies neither an interior nor an exterior sphere condition at $(0, y^*)$ as otherwise the Hopf lemma would be violated. We have therefore proved that either overturning (5.4) occurs, or else the free boundary becomes singular.

The argument for part (b) is essentially the same: assuming that (5.5) does not occur, we may apply Lemma 5.25 and Corollary 5.27 (a) leading to a contradiction with the Hopf lemma. ■

Appendix A. Principal eigenvalues and the spectral condition

A.1. Principal eigenvalues

In this subsection, we recall some important information about the principal eigenvalue of linear elliptic operators on bounded domains. We will work on the base of the cylinder Ω' which has boundary components Γ'_0 and Γ'_1 . Note that from the beginning of Section 1 on the setup of the problem, these boundary components are both $C^{k+2+\alpha}$.

Let $\mathfrak{L} : C^{k+2+\alpha}(\overline{\Omega}') \rightarrow C^{k+\alpha}(\overline{\Omega}') \times C^{k+1+\alpha}(\Gamma'_1)$ be given by

$$\begin{aligned} \mathfrak{L}_1 v &:= \alpha^{ij}(y) \partial_i \partial_j v + \mathfrak{b}^i(y) \partial_i v + c(y)v, \\ \mathfrak{L}_2 v &:= (\mathfrak{d}^i(y) \partial_i v + e(y)v)|_{\Gamma'_1}, \end{aligned} \tag{A.1}$$

where the coefficients satisfy

$$\alpha^{ij} = \alpha^{ji}, \quad \alpha^{ij}, \mathfrak{b}^i, c \in C^{k+\alpha}(\overline{\Omega}'), \quad \mathfrak{d}^i, e \in C^{k+1+\alpha}(\Gamma'_1),$$

the interior operator is uniformly elliptic, and the boundary condition on Γ'_1 is uniformly oblique. Notice here that the summation is over $1 \leq i, j \leq n-1$ and $\partial_i = \partial_{y_i}$. One should imagine $\mathfrak{L}$ as standing in for the limiting transversal operator $\mathcal{L}'_{\pm}$ elsewhere in the paper.

Call $v \in C^2(\Omega') \cap C^1(\overline{\Omega}')$ a *supersolution* of $\mathfrak{L}$ provided that

$$\begin{cases} \mathfrak{L}_1 v \leq 0 & \text{in } \Omega', \\ \mathfrak{L}_2 v \geq 0 & \text{on } \Gamma'_1, \\ v \geq 0 & \text{on } \Gamma'_0, \end{cases} \tag{A.2}$$

and a *strict supersolution* if at least one of these inequalities is strict. The following theorem ensures the existence of a principal eigenvalue of $\mathfrak{L}$, and states that it is negative precisely when $\mathfrak{L}$ obeys the maximum principle and Hopf boundary-point lemma.

Theorem A.1. Consider a linear elliptic operator $\mathfrak{L}$ of the form (A.1).

(a) There exists a unique $\sigma \in \mathbb{C}$ such that the spectral problem

$$\begin{cases} \mathfrak{L}_1 \varphi = \sigma \varphi & \text{in } \Omega', \\ \mathfrak{L}_2 \varphi = 0 & \text{on } \Gamma'_1, \\ \varphi = 0 & \text{on } \Gamma'_0, \end{cases} \quad (\text{A.3})$$

has a solution $\varphi_0 \in C^{k+2+\alpha}(\overline{\Omega'})$ satisfying

$$\varphi_0 > 0 \quad \text{on } \Omega' \cup \Gamma'_1, \quad v \cdot \nabla \varphi_0 < 0 \quad \text{on } \Gamma'_0.$$

We call σ the principal eigenvalue of $\mathfrak{L}$ and denote it $\sigma_0(\mathfrak{L})$.

(b) $\sigma_0(\mathfrak{L})$ is always simple and real. Moreover, for any eigenvalue ξ of $\mathfrak{L}$, we have $\operatorname{Re} \xi \leq \sigma_0(\mathfrak{L})$ with equality if and only if $\xi = \sigma_0(\mathfrak{L})$.

(c) The following are equivalent:

- (i) $\sigma_0(\mathfrak{L}) < 0$;
- (ii) $\mathfrak{L}$ possesses a positive strict supersolution;
- (iii) if $v \in C^{2+\alpha}(\overline{\Omega'})$ is a nonzero supersolution, then

$$v > 0 \quad \text{on } \Omega' \cup \Gamma'_1, \quad v \cdot \nabla v < 0 \quad \text{on } \Gamma'_0 \cap v^{-1}(0).$$

Proof. Parts (a) and (b) are essentially given by Amann [3, Theorem 12.1]. We are assuming more smoothness on the domain and coefficients, which by a straightforward elliptic regularity argument allows us to upgrade φ_0 to $C^{k+2+\alpha}$. Part (c) is due to López-Gómez [47, Theorem 6.1]. ■

The next lemma is an obvious consequence of the above theory, recorded for the convenience of the reader.

Lemma A.2. For all $\varepsilon > 0$ and linear operators $\mathfrak{L}$ of the form (A.1), there exists $\delta > 0$ such that for operators $\tilde{\mathfrak{L}}$ of the same form whose coefficients satisfy

$$\|\alpha^{ij} - \tilde{\alpha}^{ij}\|_{C^{k+\alpha}}, \|\mathfrak{b}^i - \tilde{\mathfrak{b}}^i\|_{C^{k+\alpha}}, \|c - \tilde{c}\|_{C^{k+\alpha}}, \|\mathfrak{d}^i - \tilde{\mathfrak{d}}^i\|_{C^{k+1+\alpha}}, \|e - \tilde{e}\|_{C^{k+1+\alpha}} < \delta,$$

we have

$$|\sigma_0(\mathfrak{L}) - \sigma_0(\tilde{\mathfrak{L}})| < \varepsilon.$$

Proof. Recall from [47, Section 6], say, that the principal eigenvalue is obtained by applying the Krein–Rutman theorem to the resolvent operator associated to the eigenvalue problem (A.3). That is, set

$$\mathfrak{R}: C^{k+1+\alpha}(\overline{\Omega'}) \ni f \mapsto v \in C^{k+2+\alpha}(\overline{\Omega'}) \hookrightarrow C^{k+1+\alpha}(\overline{\Omega'}),$$

where v is the unique solution to

$$\begin{cases} \mathfrak{L}_1 v - \sigma v = f & \text{in } \Omega', \\ \mathfrak{L}_2 v = 0 & \text{on } \Gamma'_1, \\ v = 0 & \text{on } \Gamma'_0, \end{cases}$$

and $\sigma > 0$ is chosen large enough so that the above inverse exists. Then $\sigma_0(\mathfrak{L})$ is determined by σ and the spectral radius of $\mathfrak{R}$. Both of these can be controlled uniformly in terms of the coefficients of $\mathfrak{L}$ according to Schauder theory, and so the lemma follows. ■

A.2. Spectral degeneracy and the essential spectrum

Recall that if $\mathscr{W}$ and $\mathscr{Z}$ are Banach spaces, a mapping $\mathcal{G}: \mathscr{W} \rightarrow \mathscr{Z}$ is said to be *locally proper* provided that $\mathcal{G}^{-1}(\mathcal{K}) \cap \mathcal{D}$ is compact for any compact subset $\mathcal{K} \subset \mathscr{Z}$ and closed and bounded set $\mathcal{D} \subset \mathscr{W}$. For a bounded linear operator, this is equivalent to being semi-Fredholm with a finite-dimensional kernel. Likewise, for a bounded linear operator $\mathcal{A}: \mathscr{X}_b \rightarrow \mathscr{Y}_b$, we define the *essential spectrum* to be the set

$$\sigma_{\text{ess}}(\mathcal{A}) := \{\xi \in \mathbb{C} : (\mathcal{A}_1 - \xi, \mathcal{A}_2) \text{ is not locally proper } \mathscr{X}_b \rightarrow \mathscr{Y}_b\}. \quad (\text{A.4})$$

In particular, if $0 \notin \sigma_{\text{ess}}(\mathcal{A})$, then $\mathcal{A}$ is semi-Fredholm with index $\text{ind } \mathcal{A} < \infty$.

Consider now the linear elliptic operator $\mathcal{L}: \mathscr{X}_b \rightarrow \mathscr{Y}_b$ given by

$$\begin{aligned} \mathcal{L}_1 v &:= a^{ij}(x, y) \partial_i \partial_j v + b^i(x, y) \partial_i v + c(x, y) v, \\ \mathcal{L}_2 v &:= (\beta^i(x, y) \partial_i v + \gamma(x, y) v)|_{\Gamma_1}, \end{aligned} \quad (\text{A.5})$$

with

$$a^{ij} = a^{ji}, \quad a^{ij}, b^i, c \in C_b^{k+\alpha}(\bar{\Omega}), \quad \beta_i, \gamma \in C_b^{k+1+\alpha}(\Gamma_1),$$

such that $\mathcal{L}_1$ is uniformly elliptic and $\mathcal{L}_2$ is uniformly oblique. We assume that these coefficients have well-defined limits

$$\begin{aligned} a^{ij}(x, \cdot) &\rightarrow a_{\pm}^{ij}, \quad b^i(x, \cdot) \rightarrow b_{\pm}^i, \quad c(x, \cdot) \rightarrow c_{\pm} \\ \beta^i(x, \cdot) &\rightarrow \beta_{\pm}^i, \quad \gamma(x, \cdot) \rightarrow \gamma_{\pm} \end{aligned} \quad \text{as } x \rightarrow \pm\infty \quad (\text{A.6})$$

where $a_{\pm}^{ij}, b_{\pm}^i, c_{\pm} \in C^{k-2+\alpha}(\bar{\Omega}')$ and $\beta_{\pm}^i, \gamma_{\pm} \in C^{k-1+\alpha}(\Gamma'_1)$. Let $\mathcal{L}_{\pm}$ denote the corresponding limiting operator and $\mathcal{L}'_{\pm}$ the transversal limiting operator at $x = \pm\infty$. Note in particular that $\mathcal{L}'_{\pm}$ will be of the form (A.1), and hence by Theorem A.1 it has a principal eigenvalue.

The injectivity of these limiting operators is in fact equivalent to the local properness of $\mathcal{L}$, as the following well-known result shows. See, for example, [65] or [69, Lemma A.7]. We note in passing that the proof of [69, Lemma A.5] incorrectly cites [33, Theorem 9.19]. Thankfully, this can be resolved by substituting [45, Theorem 5.54], as was pointed out to us by one of the authors of [46].

Lemma A.3 (Local properness). *The elliptic operator $\mathcal{L}$ is locally proper $\mathcal{X}_b \rightarrow \mathcal{Y}_b$ if and only both $\mathcal{L}_+$ and $\mathcal{L}_-$ are injective $\mathcal{X}_b \rightarrow \mathcal{Y}_b$.*

It is then possible to characterize the spectral condition (H2) through the essential spectrum.

Proposition A.4 (Principal eigenvalues and essential spectrum). *Consider the linear elliptic operator $\mathcal{L}$ with corresponding limiting transversal operators $\mathcal{L}'_{\pm}$. We have*

$$\sigma_{\text{ess}}(\mathcal{L}) \subset \mathbb{C}_- \quad \text{if and only if} \quad \sigma_0(\mathcal{L}'_-), \sigma_0(\mathcal{L}'_+) < 0, \quad (\text{A.7})$$

where $\mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ is the left half-plane.

Proof. For a fixed $m > 0$, consider the periodized cylindrical domain

$$\Omega^{(m)} := \mathbb{T}_m \times \Omega', \quad \text{where} \quad \mathbb{T}_m := \mathbb{R}/\frac{2\pi}{m}\mathbb{Z}.$$

Denote by $\mathcal{L}_{\pm}^{(m)}$ the restriction of $\mathcal{L}_{\pm}$ to the subspace of $\mathcal{X}_b$ consisting of functions which are $2\pi/m$ -periodic in x . By classical theory, the spectrum of $\mathcal{L}_{\pm}^{(m)}$ consists of discrete eigenvalues with no finite accumulation points. Moreover, each of these operators has a principal eigenvalue that, abusing notation slightly, we denote

$$\sigma_0(\mathcal{L}_{\pm}^{(m)}) \in \mathbb{R}.$$

By definition, the principal eigenvalue is the unique eigenvalue for which the corresponding eigenfunction is positive on $\Omega^{(m)} \cup \Gamma_1^{(m)}$. On the other hand, any eigenfunction of $\mathcal{L}_{\pm}^{(m)}$ is also an eigenfunction for $\mathcal{L}_{\pm}^{(m)}$. Therefore,

$$\sigma_0(\mathcal{L}_{\pm}^{(m)}) = \sigma_0(\mathcal{L}'_{\pm}) \quad \text{for all } m > 0. \quad (\text{A.8})$$

In Lemma A.3, we argued that $\xi \in \sigma_{\text{ess}}(\mathcal{L})$ if and only if there exists a nontrivial solution $v_{\pm} \in \mathcal{X}_b$ (complexified in the usual way) to

$$\begin{cases} \mathcal{L}_{1\pm} v_{\pm} = \xi v_{\pm} & \text{in } \Omega, \\ \mathcal{L}_{2\pm} v_{\pm} = 0 & \text{on } \Gamma_1. \end{cases}$$

The coefficients above are independent of x , and so via Fourier analysis, it is clear that we can take $v_{\pm}$ to be either independent of x or else $2\pi/m$ -periodic in x for some $m > 0$. In the former case, ξ will be an eigenvalue of $\mathcal{L}'_{\pm}$, whereas in the latter ξ will be in the spectrum of $\mathcal{L}_{\pm}^{(m)}$. From this and (A.8) we conclude that

$$\max \{\operatorname{Re} \xi : \xi \in \sigma_{\text{ess}}(\mathcal{L})\} = \max \{\sigma_0(\mathcal{L}'_-), \sigma_0(\mathcal{L}'_+)\}.$$

The statement in (A.7) follows. ■

Lemma A.5 (Asymptotic invertibility). *If $\mathcal{L}$ satisfies $\sigma_0(\mathcal{L}'_-), \sigma_0(\mathcal{L}'_+) < 0$, then $\mathcal{L}_-$ and $\mathcal{L}_+$ are invertible $\mathcal{X}_b \rightarrow \mathcal{Y}_b$.*

Proof. Consider first the elliptic operator $(\mathcal{L}_{1\pm} - \sigma, \mathcal{L}_{2\pm})$ for $\sigma \gg 1$ to be determined. Let $f \in \mathcal{Y}_b$ be given and set $p := n/(1 - \alpha)$. Then clearly

$$\begin{aligned} \sup_{m \in \mathbb{Z}} (\|f_1\|_{W^{k,p}(\Omega_m)} + \|f_2\|_{W^{k+1-1/p,p}(\Gamma_{1m})}) \\ \lesssim \|f_1\|_{C^{k+\alpha}(\Omega)} + \|f_2\|_{C^{k+1+\alpha}(\Gamma_1)} = \|f\|_{\mathcal{Y}}, \end{aligned}$$

where $\Omega_m := (m, m+1) \times \Omega'$ and $\Gamma_{1m} := (m, m+1) \times \Gamma'_1$. The leftmost quantity corresponds to the F_∞ norm in the notation of [65]. We may then apply [65, Theorem 7.5] to conclude that, for σ sufficiently large, there exists a unique $u \in W_{\text{loc}}^{k+2,p}(\Omega)$ such that

$$\begin{cases} \mathcal{L}_{1\pm}u - \sigma u = f_1 & \text{in } \Omega, \\ \mathcal{L}_{2\pm}u = f_2 & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_0, \end{cases}$$

with the boundary conditions satisfied in the trace sense. Moreover, the same result states that u obeys a “uniformly local” Sobolev regularity estimate

$$\sup_{m \in \mathbb{Z}} \|u\|_{W^{k+2,p}(\Omega_m)} \lesssim \sup_{m \in \mathbb{Z}} (\|f_1\|_{W^{k,p}(\Omega_m)} + \|f_2\|_{W^{k+1-1/p,p}(\Gamma_{1m})}),$$

which by Morrey’s inequality leads to the Hölder norm bound

$$\|u\|_{C^{k+1+\alpha}(\Omega)} \lesssim \|f\|_{\mathcal{Y}}.$$

A straightforward Schauder theory argument allows us to infer the improved local Hölder regularity $u \in \mathcal{X}$. Taken together with the uniform C^0 bounds on u in terms of $\|f\|_{\mathcal{Y}}$, this implies that $u \in \mathcal{X}_b$; see, for instance, [69, Lemma A.1].

The above reasoning shows that $(\mathcal{L}_{1\pm} - \sigma, \mathcal{L}_{2\pm})$ is invertible $\mathcal{X}_b \rightarrow \mathcal{Y}_b$ if $\sigma \gg 1$. On the other hand, because $\sigma_0(\mathcal{L}'_-), \sigma_0(\mathcal{L}'_+) < 0$, we find that $(\mathcal{L}_{1\pm} - t\sigma, \mathcal{L}_{2\pm})$ is semi-Fredholm as a mapping between these spaces for all $t \in [0, 1]$. Continuity of the index then implies $\text{ind } \mathcal{L}_\pm = 0$. The proof is now complete as the hypothesis also ensures that $\mathcal{L}_\pm$ is injective. ■

With these tools in hand, we can now provide the proof of Lemma 2.11 on the Fredholm properties of the linearized operator $\mathcal{F}_u(u, \Lambda)$ at a front in $u \in \mathcal{X}_b$.

Proof of Lemma 2.11. Fix a solution $(u, \Lambda) \in \mathcal{F}^{-1}(0) \cap \mathcal{U}_\infty$ and let $\mathcal{L} := \mathcal{F}_u(u, \Lambda)$. Clearly, $\mathcal{L}$ is of the form (A.5), and since $u \in \mathcal{X}_\infty$, the coefficients have well-defined far-field limits as in (A.6), with limiting operators $\mathcal{L}_\pm$. By hypothesis, the corresponding transversal limiting operators $\mathcal{L}'_\pm$ satisfy (H2) and so Lemma A.5 shows that $\mathcal{L}_\pm$ is invertible $\mathcal{X}_b \rightarrow \mathcal{Y}_b$. In particular, this means that $\mathcal{L}: \mathcal{X}_b \rightarrow \mathcal{Y}_b$ is semi-Fredholm in light of Lemma A.3. We will calculate its index $\text{ind } \mathcal{L} < \infty$ using a pair of homotopies.

Let $\sigma > 0$ be a constant to be determined, and consider the one-parameter family of operators

$$(\mathcal{L}_1 - t\sigma, \mathcal{L}_2): \mathcal{X}_b \rightarrow \mathcal{Y}_b \quad \text{for } t \in [0, 1].$$

Each of these operators is uniformly elliptic and uniformly oblique, and has well-defined limiting transversal operators at $x = \pm\infty$. As $t\sigma \geq 0$, (H2) guarantees that the corre-

sponding principal eigenvalues are all strictly negative. Thus, arguing as above, the entire family is semi-Fredholm. By the continuity of the index, the operators at the endpoints $t = 0$ and $t = 1$ therefore have the same index, that is, $\text{ind } \mathcal{L} = \text{ind}(\mathcal{L}_1 - \sigma, \mathcal{L}_2)$.

Next we consider the one-parameter family of operators

$$(t\Delta + (1-t)\mathcal{L}_1 - \sigma, t\partial_\nu + (1-t)\mathcal{L}_2): \mathcal{X}_b \rightarrow \mathcal{Y}_b \quad \text{for } t \in [0, 1], \quad (\text{A.9})$$

where $\partial_\nu = \nu \cdot \nabla$ is the outward-pointing normal derivative. We easily check that each of these operators is uniformly elliptic and uniformly oblique. By Lemma A.2, the principal eigenvalues of the limiting transversal operators

$$\begin{aligned} \sigma_0(t\Delta' + (1-t)\mathcal{L}'_1 - \sigma, t\partial_\nu + (1-t)\mathcal{L}'_2) \\ = \sigma_0(t\Delta' + (1-t)\mathcal{L}'_1, t\partial_\nu + (1-t)\mathcal{L}'_2) - \sigma \end{aligned} \quad (\text{A.10})$$

depend continuously on $t \in [0, 1]$. Choosing $\sigma > 0$ sufficiently large, we can therefore guarantee that the principal eigenvalues (A.10) are strictly negative, and hence the entire family of operators (A.9) is semi-Fredholm. Appealing to the continuity of the index for a second time, we deduce that

$$\text{ind } \mathcal{L} = \text{ind}(\mathcal{L}_1 - \sigma, \mathcal{L}_2) = \text{ind}(\Delta - \sigma, \partial_\nu) = 0,$$

where the last equality follows, for instance, from Lemma A.5. ■

A.3. Principal eigenvalues and spectral degeneracy for transmission problems

For our application to internal waves, we require analogous results for transmission boundary conditions. We start with the problem posed on the transversal domain $\Omega' = \Omega'_1 \cup \Omega'_2$ which satisfies $\Omega'_1 \cap \Omega'_2 = \emptyset$, $\partial\Omega'_1 \cap \partial\Omega'_2 = \Gamma'_1$, and $\Gamma'_0 = \partial\Omega' \setminus \Gamma'_1$. The operator in (A.1) now becomes

$$\begin{aligned} \mathfrak{L}_1 v &:= \partial_i(\alpha^{ij}(y)\partial_j v + \alpha^i(y)v) + \mathfrak{b}^i(y)\partial_i v + \mathfrak{c}(y)v, \\ \mathfrak{L}_2 v &:= \llbracket -\nu_i \alpha^{ij}(y)\partial_j v \rrbracket + \llbracket -\nu_i \alpha^i(y) \rrbracket v|_{\Gamma'_1}, \end{aligned} \quad (\text{A.11})$$

where $\nu = (\nu_1, \dots, \nu_{n-1})$ is the normal vector field on Γ'_1 pointing outward from Ω'_1 . The coefficients satisfy the following regularity conditions:

$$\alpha^{ij} = \alpha^{ji}, \quad \alpha^{ij}, \alpha^i \in C^{k+1+\alpha}(\overline{\Omega'_1}) \cap C^{k+1+\alpha}(\overline{\Omega'_2}), \quad \mathfrak{b}^i, \mathfrak{c} \in C^{k+\alpha}(\overline{\Omega'_1}) \cap C^{k+\alpha}(\overline{\Omega'_2}).$$

Therefore, from the uniform ellipticity of $\mathfrak{L}$, we know that $\nu_j \alpha^{ij}(y)$ is an outward-pointing vector field for Ω'_1 along Γ'_1 .

We define a supersolution of $\mathfrak{L}$ in (A.11) to be a function $v \in C(\overline{\Omega'}) \cap C^1(\overline{\Omega'_i}) \cap C^2(\Omega'_i)$ for $i = 1, 2$ satisfying (A.2). As before, v is called a strict supersolution if any of these inequalities are strict.

Since the boundary components of Ω' are $C^{k+2+\alpha}$, it is straightforward to verify that when $\mathfrak{c} \leq 0$ then $\mathfrak{L}$ satisfies the maximum principle, and the following Hopf lemma also holds true.

Lemma A.6. *Let $c \leq 0$ and $v \in C(\overline{\Omega'}) \cap C^2(\Omega'_1) \cap C^2(\Omega'_2)$ be nonconstant and satisfy*

$$\mathfrak{L}_1 v \geq 0 \quad \text{and} \quad v \leq m \quad \text{in } \Omega$$

for some $m \geq 0$, where $\mathfrak{L}$ is given in (A.11). Suppose that $u = m$ at some $x_0 \in \partial\Omega'$. Then

- (a) *if $x_0 \in \Gamma'_0$ then for any outward-pointing vector ξ we have $\xi \cdot \nabla v(x_0) > 0$;*
- (b) *if $x_0 \in \Gamma'_1$ then for any outward-pointing vector ξ of Ω'_1 we have $\llbracket \xi \cdot \nabla v(x_0) \rrbracket < 0$.*

With the help of the above discussion, one can adapt the argument in [47] to reproduce Theorem A.1 in the setting of transmission problems. This is the content of the next theorem which, though straightforward, appears to be new.

Corollary A.7 (Principal eigenvalues for transmission problems). *The conclusions of Theorem A.1 hold for the elliptic transmission operator $\mathfrak{L}$ defined in (A.11), with obvious modification on the regularity of functions. Moreover, the principal eigenvalue $\sigma_0(\mathfrak{L})$ depends continuously on the coefficients in the same form as in Lemma A.2.*

Proof. The statements corresponding to Theorem A.1 (a, b) still follow largely from Amann [3] with a few small adjustments. Observe first that the $C^{2+\alpha}$ regularity of the domain ensures the existence of a function $\psi \in C(\overline{\Omega'}) \cap C^{2+\alpha}(\overline{\Omega'_i})$ and a constant $\gamma > 0$ such that

$$\llbracket -v_j \alpha^{ij}(y) \partial_i \psi \rrbracket \geq \gamma \quad \text{on } \Gamma'_1.$$

Indeed, in a neighborhood Γ'_1 , it works to take $\psi := \text{dist}(\cdot, \Gamma'_1)$, and then extend smoothly to $\overline{\Omega}$. The unique solvability of the spectral problem (A.3) for the transmission operator can now be inferred from the existence of ψ , [44, Theorem 1.2], and an argument similar to [47, Theorem 4.1].

Next consider part (c). One can then easily check that

$$h \text{ is a strictly positive supersolution of } \mathfrak{L} \implies \begin{array}{l} \text{if } v \text{ is a supersolution of } \mathfrak{L}, \\ \text{then } v/h \text{ is a supersolution of } \mathfrak{L}^h, \end{array} \quad (\text{A.12})$$

where $\mathfrak{L}^h$ is the operator defined by

$$\begin{aligned} \mathfrak{L}_1^h w &:= \partial_i (\alpha^{ij}(y) \partial_j w + \alpha^i(y) w) + \left(b^i(y) + \frac{2}{h} \alpha^{ij}(y) \partial_j h \right) \partial_i w + \frac{\mathfrak{L}_1 h}{h} w \\ \mathfrak{L}_2^h w &:= \llbracket -v_j \alpha^{ij}(y) \partial_i w \rrbracket + \left(\frac{\mathfrak{L}_2 h}{h} w \right) \Big|_{\Gamma'_1}. \end{aligned}$$

Indeed, if we take $w := v/h$, then $w \in C(\overline{\Omega'}) \cap C^1(\overline{\Omega'_i}) \cap C^2(\Omega'_i)$, and $\mathfrak{L}v = h\mathfrak{L}^h w$. Since $h > 0$ and is a supersolution of $\mathfrak{L}$, we have

$$\mathfrak{L}_1^h w \leq 0, \quad \frac{\mathfrak{L}_1 h}{h} \leq 0 \quad \text{in } \Omega'; \quad \mathfrak{L}_2^h w \geq 0, \quad \frac{\mathfrak{L}_2 h}{h} \geq 0 \quad \text{on } \Gamma'_1,$$

implying that w is a supersolution of $\mathfrak{L}^h$.

Now, it follows from (A.12) and Lemma A.6 that the classification result [47, Theorem 3.1] generalizes to the transmission case. Together with the regularity assumption on the domain, this further implies a result as in [47, Theorem 5.2], and so ultimately we see that the conclusions of [47, Theorem 6.2] hold for the transmission problem. This leads precisely to the statements in (c). Finally, the continuity of $\sigma_0(\mathcal{L})$ is a consequence of Schauder estimates for transmission problems (see, for example, [44, Theorem 1.2]) and the Krein–Rutman theorem. ■

Having established the theory for the principal eigenvalues for the transversal problem, next we consider the problem on the full domain $\Omega = \Omega_1 \cup \Omega_2$, where recall that $\Omega_1 \cap \Omega_2 = \emptyset$, $\partial\Omega_1 \cap \partial\Omega_2 = \Gamma_1$, and $\Gamma_0 = \partial\Omega \setminus \Gamma_1$. The linear operator $\mathcal{L}: \mathcal{X}_b \rightarrow \mathcal{Y}_b$ is given by

$$\begin{aligned}\mathcal{L}_1 v &:= \partial_i (a^{ij}(x, y) \partial_j v + a^i(x, y) v) + b^i(x, y) \partial_i v + c(x, y) v, \\ \mathcal{L}_2 v &:= \llbracket -v_i a^{ij} \partial_j v \rrbracket + \llbracket -v_i a^i \rrbracket v,\end{aligned}\tag{A.13}$$

with

$$\begin{aligned}a^{ij} &= a^{ji}, \quad a^{ij}, a^i \in C_b^{k+1+\alpha}(\overline{\Omega_1}) \cap C^{k+1+\alpha}(\overline{\Omega_2}), \\ b^i, c &\in C_b^{k+\alpha}(\overline{\Omega_1}) \cap C_b^{k+\alpha}(\overline{\Omega_2}),\end{aligned}$$

such that $\mathcal{L}_1$ is uniformly elliptic. Here the spaces $\mathcal{X}_b$ and $\mathcal{Y}_b$ are naturally defined as in Section 3.2. The limits of the coefficients are

$$a^{ij}(x, \cdot) \rightarrow a_{\pm}^{ij}, \quad a^i(x, \cdot) \rightarrow a_{\pm}^i, \quad b^i(x, \cdot) \rightarrow b_{\pm}^i, \quad c(x, \cdot) \rightarrow c_{\pm} \quad \text{as } x \rightarrow \pm\infty\tag{A.14}$$

where $a_{\pm}^{ij}, a_{\pm}^i \in C^{k-1+\alpha}(\overline{\Omega'_1}) \cap C^{k-1+\alpha}(\overline{\Omega'_2})$, $b_{\pm}^i, c_{\pm} \in C^{k-2+\alpha}(\overline{\Omega'_1}) \cap C^{k-2+\alpha}(\overline{\Omega'_2})$. Following Section A.2, we introduce the limiting operator $\mathcal{L}_{\pm}$, and the transversal limiting operator $\mathcal{L}'_{\pm}$ which is of the form (A.11). From Corollary A.7 we know that $\sigma_0(\mathcal{L}'_{\pm})$ exist.

Note that Lemma A.3 and Proposition A.4 still hold for $\mathcal{L}$ in (A.13). Moreover, the same Fredholm property of $\mathcal{F}_u(u, \Lambda)$ for a transmission problem $\mathcal{F}$ can be proved provided the asymptotic invertibility of $\mathcal{L}$.

Lemma A.8 (Asymptotic invertibility for transmission operator). *Let $\mathcal{L}$ be given as in (A.13). If $\mathcal{L}$ satisfies $\sigma_0(\mathcal{L}'_-), \sigma_0(\mathcal{L}'_+) < 0$, then $\mathcal{L}_-$ and $\mathcal{L}_+$ are invertible $\mathcal{X}_b \rightarrow \mathcal{Y}_b$.*

Proof. Following the same idea as in Lemma A.5, for $\sigma \gg 1$, we consider the solvability of

$$\begin{cases} \mathcal{L}_{1\pm} u - \sigma u = f_1 & \text{in } \Omega, \\ \mathcal{L}_{2\pm} u = f_2 & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_0. \end{cases}$$

Note that one may find $g = (g^1, \dots, g^n) \in (C_b^{k+1+\alpha}(\overline{\Omega_1}) \cap C^{k+1+\alpha}(\overline{\Omega_2}))^n$ with

$\llbracket -v_i g_i \rrbracket = f_2$, and thus the above problem can be written as

$$\begin{cases} \mathcal{L}_{1\pm} u - \sigma u = f + \partial_i g^i & \text{in } \Omega \cup \Gamma_1, \\ \mathcal{L}_{2\pm} u = \llbracket -v_i g^i \rrbracket & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_0, \end{cases} \quad (\text{A.15})$$

where $f = f_1 - \partial_i g^i$. The advantage of (A.15) is that it suggests the appropriate weak formulation of the problem. More precisely, let $p := n/(1 - \alpha)$, and call $u \in W_0^{1,p}(\Omega \cup \Gamma_1)$ a weak solution to (A.15) provided it satisfies

$$\int_{\Omega} (-a^{ij} \partial_i u \partial_j \varphi - a^j u \partial_j \varphi + b^i \partial_x u \varphi + cu \varphi) dx dy = \int_{\Omega} (f \varphi - g^j \partial_j \varphi) dx dy$$

for any $\varphi \in W_0^{1,p'}(\Omega \cup \Gamma_1)$. Defining Ω_m and Γ_{1m} as in the proof of Lemma A.5, the local solvability of (A.15) in $W^{1,p}(\Omega_m)$ for σ sufficiently large can thus be obtained from [29, Theorem 5]. Following the strategy of [65, Theorem 7.5], we conclude that there exists a unique solution $u \in W_{\text{loc}}^{1,p}(\Omega)$ of (A.15). Using a standard partition of unity argument and local coordinate charts, it suffices to assume that Γ_{1m} lies in a hyperplane. Letting τ be a tangent vector and differentiating the equation, one easily obtains $W^{1,2}$ estimates for $\tau \cdot \nabla u$. Theorem 5.13 then allows us to control $\tau \cdot \nabla u$ in $W^{1,p}$, and so we have the $W^{2,p}$ bound

$$\|u\|_{W^{2,p}(\tilde{\Omega}_m^i)} \lesssim \|f_1\|_{W^{1,p}(\Omega_m^i)} + \|f_2\|_{W^{1-1/p,p}(\Gamma_{1m})} + \|u\|_{W^{1,2}(\Omega_m)},$$

where $\Omega_m^i = (m, m+1) \times \Omega'_i$ and $\tilde{\Omega}_m^i = (m+1/4, m+3/4) \times \Omega'_i$ for $i = 1, 2$. The last term on the right-hand side can be dropped by the $W^{1,p}$ estimate of [29, Theorem 5]. Repeating this process yields

$$\sup_{m \in \mathbb{Z}} \|u\|_{W^{k+2,p}(\tilde{\Omega}_m^i)} \lesssim \sup_{m \in \mathbb{Z}} (\|f_1\|_{W^{k,p}(\Omega_m^i)} + \|f_2\|_{W^{k+1-1/p,p}(\Gamma_{1m})}).$$

The rest of the argument follows the same way as in the proof of Lemma A.5. ■

Appendix B. Quoted results

Analytic global bifurcation theory was first introduced by Dancer [23, 25] in the late 1970s, and then refined and popularized by Buffoni and Toland [13]. As mentioned in the introduction, these results were developed with an eye towards problems on bounded domains, and thus took as given the fact that the nonlinear operator is Fredholm index 0 and that the solution set is locally compact. In [18], the authors offer a variant of the classical theory that removes those assumptions at the cost of additional alternatives along the global curve. Though this was stated as a global continuation theorem similar to Theorem 1.2, it can be easily reconfigured as the following global implicit function theorem.

Theorem B.1. Let $\mathcal{W}$ and $\mathcal{Z}$ be Banach spaces, and $\mathcal{W} \subset \mathcal{W} \times \mathbb{R}$ an open set containing a point (w_0, λ_0) . Suppose that $\mathcal{G}: \mathcal{W} \rightarrow \mathcal{Z}$ is real-analytic and satisfies

$$\mathcal{G}(w_0, \lambda_0) = 0, \quad \mathcal{G}_w(w_0, \lambda_0): \mathcal{W} \rightarrow \mathcal{Z} \text{ is an isomorphism.} \quad (\text{B.1})$$

Then there exists a curve $\mathcal{K}$ that admits the global C^0 parameterization

$$\mathcal{K} = \{(w(s), \lambda(s)) : s \in \mathbb{R}\} \subset \mathcal{G}^{-1}(0) \cap \mathcal{W},$$

and satisfies the following:

(a) At each $s \in \mathbb{R}$, the linearized operator $\mathcal{G}_w(w(s), \lambda(s)): \mathcal{W} \rightarrow \mathcal{Z}$ is Fredholm index 0.

(b) One of the following alternatives holds as $s \rightarrow \infty$ and $s \rightarrow -\infty$.

(A1') (Blowup) We have

$$N(s) := \|w(s)\|_{\mathcal{W}} + |\lambda(s)| + \frac{1}{\text{dist}((w(s), \lambda(s)), \partial \mathcal{W})} \rightarrow \infty. \quad (\text{B.2})$$

(A2') (Loss of compactness) There exists a sequence $s_n \rightarrow \pm\infty$ such that $\sup_n N(s_n) < \infty$, but $(w(s_n), \lambda(s_n))$ has no convergent subsequence in $\mathcal{W} \times \mathbb{R}$.

(A3') (Loss of Fredholmness) There exists a sequence $s_n \rightarrow \pm\infty$ such that $\sup_n N(s_n) < \infty$ and $(w(s_n), \lambda(s_n)) \rightarrow (w_*, \lambda_*) \in \mathcal{W}$ in $\mathcal{W} \times \mathbb{R}$, but $\mathcal{G}_w(w_*, \lambda_*)$ is not Fredholm index 0.

(A4') (Closed loop) There exists $T > 0$ such that $(w(s + T), \lambda(s + T)) = (w(s), \lambda(s))$ for all $s \in (0, \infty)$.

(c) Near each point $(w(s_0), \lambda(s_0)) \in \mathcal{K}$, we can locally reparameterize $\mathcal{K}$ so that $s \mapsto (w(s), \lambda(s))$ is real-analytic.

(d) The curve $\mathcal{K}$ is maximal in the sense that if $\mathcal{J} \subset \mathcal{G}^{-1}(0) \cap \mathcal{W}$ is a locally real-analytic curve containing (w_0, λ_0) and along which $\mathcal{G}_w$ is Fredholm index 0, then $\mathcal{J} \subset \mathcal{K}$.

Proof. This result follows from a straightforward adaptation of [18, Theorem 6.1], which is in turn based on [13, Theorem 9.1.1], and so we only provide a sketch of the details that differ. Following [13], we call a (maximal) connected component of the set

$$\mathcal{A} := \{(w, \lambda) \in \mathcal{W} : \mathcal{G}(w, \lambda) = 0, \mathcal{G}_w(w, \lambda) \text{ is an isomorphism } \mathcal{W} \rightarrow \mathcal{Z}\}$$

a *distinguished arc*. By (B.1) and the analytic implicit function theorem, there exists a distinguished arc A_0 containing (w_0, λ_0) . Moreover, it admits the parameterization

$$A_0 = \{(w(s), \lambda(s)) : s \in (-1, 1)\},$$

with $s \mapsto (w(s), \lambda(s))$ real-analytic $(-1, 1) \rightarrow \mathcal{W} \times \mathbb{R}$, and $(w(0), \lambda(0)) = (w_0, \lambda_0)$. Consider now the limit $s \rightarrow 1$. Through the same argument as in [18, Theorem 6.1], we conclude that one of the alternatives (A1')–(A4') occurs, or else A_0 connects to another distinguished arc A_1 . In the former case, taking $\mathcal{K} = A_0$ completes the proof. In the latter,

we can continue inductively by applying the same logic to (a suitably reparameterized) $A_0 \cup A_1$, and so on. From here the proof is exactly as before, except there is the additional possibility of a closed loop since (w_0, λ_0) is in the interior of $\mathcal{W}$. This gives us the alternatives claimed in part (b). Because $\mathcal{K}$ is connected, the above construction also shows that part (a) holds (in fact, the linearized operator is invertible except at the endpoints of the distinguished arcs). In the interior of a distinguished arc, the analytic reparameterization asked for in part (c) is an immediate consequence of the analytic implicit function theorem. At an endpoint where two arcs meet, it follows from a deep result characterizing the structure of analytic varieties; see [13, Chapter 7] and [13, proof of Theorem 9.1.1(d)]. Likewise, the maximality claimed in part (d) is a consequence of the construction and in particular the uniqueness of analytic continuation. ■

The next result we quote is the celebrated Hopf boundary-point lemma, which relates the geometric properties of the boundary to the nondegeneracy of solutions to elliptic PDE set there. In its classical form, the regularity requirement on the domain amounts to an interior ball condition [38, 54], but this can be relaxed to $C^{1+\alpha}$ [34] and below.

Lemma B.2 (Hopf boundary-point lemma). *Let $\Omega \subset \mathbb{R}^n$ be a connected, open set (possibly unbounded) and consider the second-order operator $\mathcal{L}$ given by*

$$\mathcal{L} := a^{ij}(x)\partial_i\partial_j + b^i(x)\partial_i + c(x), \quad (\text{B.3})$$

where we are using the summation convention with $\partial_i := \partial_{x_i}$, the coefficients satisfy

$$a^{ij} = a^{ji}, \quad a^{ij}, b^i, c \in L^\infty(\Omega),$$

and $\mathcal{L}$ is uniformly elliptic. Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ be a classical solution of $\mathcal{L}u = 0$ in Ω .

- (i) (Interior ball) Suppose that u attains its maximum value on $\bar{\Omega}$ at a point $x_0 \in \partial\Omega$ for which there exists an open ball $B \subset \Omega$ with $\bar{B} \cap \partial\Omega = \{x_0\}$. Assume that either $c \leq 0$ in Ω , or else $u(x_0) = 0$. Let ν be the outward unit normal to Ω at x_0 . Then u is a constant function or

$$\liminf_{t \rightarrow 0^+} \frac{u(x_0 + t\mu) - u(x_0)}{t} < 0,$$

where μ is an arbitrary vector such that $\mu \cdot \nu < 0$.

- (ii) ($C^{1+\alpha}$ boundary) Suppose that Ω is $C^{1+\alpha}$ for some $\alpha \in (0, 1)$ and u attains its maximum value on $\bar{\Omega}$ at a point $x_0 \in \partial\Omega$. Assume that either $c \leq 0$ in Ω , or else $u(x_0) = 0$. Then the same result as in (i) holds.

Remark B.3. If $u \in C^1(\Omega \cup \{x_0\})$, then the above result states that $\nu \cdot \nabla u(x_0) > 0$ for any outward-pointing vector ν .

Finally, for completeness, we recall the following standard fact about bordering of Fredholm operators; see, for example, [12, Lemma 2.3].

Lemma B.4 (Fredholm bordering). *Let $\mathcal{W}$ and $\mathcal{Z}$ be Banach spaces and suppose that $\mathcal{A}: \mathcal{W} \rightarrow \mathcal{Z}$, $\mathcal{B}: \mathbb{R}^m \rightarrow \mathcal{Z}$, $\mathcal{C}: \mathcal{W} \rightarrow \mathbb{R}^n$, and $\mathcal{D}: \mathbb{R}^m \rightarrow \mathbb{R}^n$ are bounded linear mappings. If in addition $\mathcal{A}$ is Fredholm index k , then the operator matrix*

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}: \mathcal{W} \times \mathbb{R}^m \rightarrow \mathcal{Z} \times \mathbb{R}^n$$

is Fredholm with index $k + m - n$.

Acknowledgments. The authors wish to thank the Erwin Schrödinger Institute for Mathematics and Physics, University of Vienna, for their hospitality. A portion of this work was completed during a Research-in-Teams Program generously supported by the ESI. Finally, the authors thank the anonymous referees whose comments led to several improvements in the paper.

Funding. The research of RMC is supported in part by the NSF through DMS-1907584. The research of SW is supported in part by the NSF through DMS-1812436.

References

- [1] Akers, B. F., Ambrose, D. M., Wright, J. D.: [Gravity perturbed Crapper waves](#). Proc. Roy. Soc. London Ser. A Math. Phys. Engrg. Sci. **470**, article no. 20130526, 14 pp. (2014) Zbl [1371.76038](#) MR [3145074](#)
- [2] Alt, H. W., Caffarelli, L. A., Friedman, A.: [Variational problems with two phases and their free boundaries](#). Trans. Amer. Math. Soc. **282**, 431–461 (1984) Zbl [0844.35137](#) MR [0732100](#)
- [3] Amann, H.: [Dual semigroups and second order linear elliptic boundary value problems](#). Israel J. Math. **45**, 225–254 (1983) Zbl [0535.35017](#) MR [0719122](#)
- [4] Ambrose, D. M., Strauss, W. A., Wright, J. D.: [Global bifurcation theory for periodic traveling interfacial gravity-capillary waves](#). Ann. Inst. H. Poincaré C Anal. Non Linéaire **33**, 1081–1101 (2016) Zbl [1383.35145](#) MR [3519533](#)
- [5] Amick, C. J., Fraenkel, L. E., Toland, J. F.: [On the Stokes conjecture for the wave of extreme form](#). Acta Math. **148**, 193–214 (1982) Zbl [0495.76021](#) MR [0666110](#)
- [6] Amick, C. J., Turner, R. E. L.: [A global theory of internal solitary waves in two-fluid systems](#). Trans. Amer. Math. Soc. **298**, 431–484 (1986) Zbl [0631.35029](#) MR [0860375](#)
- [7] Amick, C. J., Turner, R. E. L.: [Small internal waves in two-fluid systems](#). Arch. Rational Mech. Anal. **108**, 111–139 (1989) Zbl [0681.76103](#) MR [1011554](#)
- [8] Bakker, B., van den Berg, J. B., Vandervorst, R.: [A Floer homology approach to traveling waves in reaction-diffusion equations on cylinders](#). SIAM J. Appl. Dynam. Systems **17**, 2634–2706 (2018) Zbl [1404.35136](#) MR [3876855](#)
- [9] Benjamin, T. B.: [Gravity currents and related phenomena](#). J. Fluid Mech. **31**, 209–248 (1968) Zbl [0169.28503](#)
- [10] Benjamin, T. B.: [A unified theory of conjugate flows](#). Philos. Trans. Roy. Soc. London Ser. A **269**, 587–643 (1971) Zbl [0226.76037](#) MR [0446075](#)
- [11] Berestycki, H., Nirenberg, L.: [Travelling fronts in cylinders](#). Ann. Inst. H. Poincaré C Anal. Non Linéaire **9**, 497–572 (1992) Zbl [0799.35073](#) MR [1191008](#)
- [12] Beyn, W.-J.: [The numerical computation of connecting orbits in dynamical systems](#). IMA J. Numer. Anal. **10**, 379–405 (1990) Zbl [0706.65080](#) MR [1068199](#)
- [13] Buffoni, B., Toland, J.: [Analytic theory of global bifurcation](#). Princeton Ser. Appl. Math., Princeton University Press, Princeton, NJ (2003) Zbl [1021.47044](#) MR [1956130](#)

- [14] Caffarelli, L., Salsa, S.: [A geometric approach to free boundary problems](#). Grad. Stud. Math. 68, American Mathematical Society, Providence, RI (2005) Zbl 1083.35001 MR 2145284
- [15] Caffarelli, L. A.: [A Harnack inequality approach to the regularity of free boundaries. I. Lipschitz free boundaries are \$C^{1,\alpha}\$](#) . Rev. Mat. Iberoamer. **3**, 139–162 (1987) Zbl 0676.35085 MR 0990856
- [16] Caffarelli, L. A.: [A Harnack inequality approach to the regularity of free boundaries. II. Flat free boundaries are Lipschitz](#). Comm. Pure Appl. Math. **42**, 55–78 (1989) Zbl 0676.35086 MR 0973745
- [17] Chen, R. M., Jin, J.: [Global bifurcation of solitary waves to the Boussinesq \$abcd\$ system](#). J. Differential Equations **310**, 235–263 (2022) Zbl 1515.76029 MR 4355919
- [18] Chen, R. M., Walsh, S., Wheeler, M. H.: [Existence and qualitative theory for stratified solitary water waves](#). Ann. Inst. H. Poincaré C Anal. Non Linéaire **35**, 517–576 (2018) Zbl 1408.35135 MR 3765551
- [19] Chen, R. M., Walsh, S., Wheeler, M. H.: [Center manifolds without a phase space for quasilinear problems in elasticity, biology, and hydrodynamics](#). Nonlinearity **35**, 1927–1985 (2022) Zbl 1485.35210 MR 4399497
- [20] Constantin, A., Strauss, W., Vărvăruță, E.: [Global bifurcation of steady gravity water waves with critical layers](#). Acta Math. **217**, 195–262 (2016) Zbl 1375.35294 MR 3689941
- [21] Córdoba, D., Enciso, A., Grubic, N.: [On the existence of stationary splash singularities for the Euler equations](#). Adv. Math. **288**, 922–941 (2016) Zbl 1331.35267 MR 3436402
- [22] Crapper, G. D.: [An exact solution for progressive capillary waves of arbitrary amplitude](#). J. Fluid Mech. **2**, 532–540 (1957) Zbl 0079.19304 MR 0091075
- [23] Dancer, E. N.: [Bifurcation theory for analytic operators](#). Proc. London Math. Soc. (3) **26**, 359–384 (1973) Zbl 0252.47069 MR 0322615
- [24] Dancer, E. N.: [Global solution branches for positive mappings](#). Arch. Ration. Mech. Anal. **52**, 181–192 (1973) Zbl 0275.47043 MR 0353077
- [25] Dancer, E. N.: [Global structure of the solutions of non-linear real analytic eigenvalue problems](#). Proc. London Math. Soc. (3) **27**, 747–765 (1973) Zbl 0268.47065 MR 0375019
- [26] De Silva, D.: [Free boundary regularity for a problem with right hand side](#). Interfaces Free Bound. **13**, 223–238 (2011) Zbl 1219.35372 MR 2813524
- [27] De Silva, D., Ferrari, F., Salsa, S.: [Recent progresses on elliptic two-phase free boundary problems](#). Discrete Contin. Dynam. Systems **39**, 6961–6978 (2019) Zbl 1425.35238 MR 4026176
- [28] Dias, F., Vanden-Broeck, J.-M.: [On internal fronts](#). J. Fluid Mech. **479**, 145–154 (2003) Zbl 1163.76369 MR 2011822
- [29] Dong, H., Kim, D.: [Elliptic equations in divergence form with partially BMO coefficients](#). Arch. Ration. Mech. Anal. **196**, 25–70 (2010) Zbl 1206.35249 MR 2601069
- [30] Dyachenko, S. A., Hur, V. M.: [Stokes waves with constant vorticity: folds, gaps and fluid bubbles](#). J. Fluid Mech. **878**, 502–521 (2019) Zbl 1430.76073 MR 4010456
- [31] Dyachenko, S. A., Hur, V. M.: [Stokes waves with constant vorticity: I. Numerical computation](#). Stud. Appl. Math. **142**, 162–189 (2019) Zbl 1420.35208 MR 3915685
- [32] García, C., Haziot, S. V.: [Global bifurcation for corotating and counter-rotating vortex pairs](#). Comm. Math. Phys. **402**, 1167–1204 (2023) Zbl 1529.35353 MR 4627319
- [33] Gilbarg, D., Trudinger, N. S.: [Elliptic partial differential equations of second order](#). Classics in Mathematics, Springer, Berlin (2001) Zbl 1042.35002 MR 1814364
- [34] Giraud, G.: [Problèmes de valeurs à la frontière relatifs à certaines données discontinues](#). Bull. Soc. Math. France **61**, 1–54 (1933) Zbl 59.0479.02 MR 1504997
- [35] Haziot, S. V.: [Stratified large-amplitude steady periodic water waves with critical layers](#). Comm. Math. Phys. **381**, 765–797 (2021) Zbl 1456.76021 MR 4207457

- [36] Haziot, S. V., Wheeler, M. H.: [Large-amplitude steady solitary water waves with constant vorticity](#). Arch. Ration. Mech. Anal. **247**, article no. 27, 49 pp. (2023) Zbl [1528.76014](#) MR [4565802](#)
- [37] Helfrich, K. R., Melville, W. K.: [Long nonlinear internal waves](#). In: Annual review of fluid mechanics. Vol. 38, Annual Reviews, Palo Alto, CA, 395–425 (2006) Zbl [1098.76018](#) MR [2206980](#)
- [38] Hopf, E.: [A remark on linear elliptic differential equations of second order](#). Proc. Amer. Math. Soc. **3**, 791–793 (1952) Zbl [0048.07802](#) MR [0050126](#)
- [39] Hur, V. M., Vanden-Broeck, J.-M.: [A new application of Crapper’s exact solution to waves in constant vorticity flows](#). Eur. J. Mech. B Fluids **83**, 190–194 (2020) Zbl [1477.76017](#) MR [4102022](#)
- [40] Hur, V. M., Wheeler, M. H.: [Exact free surfaces in constant vorticity flows](#). J. Fluid Mech. **896**, article no. R1, 10 pp. (2020) Zbl [1460.76138](#) MR [4111654](#)
- [41] Hur, V. M., Wheeler, M. H.: [Overhanging and touching waves in constant vorticity flows](#). J. Differential Equations **338**, 572–590 (2022) Zbl [1505.76009](#) MR [4473791](#)
- [42] Kinderlehrer, D., Nirenberg, L., Spruck, J.: [Regularity in elliptic free boundary problems](#). J. Anal. Math. **34**, 86–119 (1979) (1978) Zbl [0402.35045](#) MR [0531272](#)
- [43] Laget, O., Dias, F.: [Numerical computation of capillary-gravity interfacial solitary waves](#). J. Fluid Mech. **349**, 221–251 (1997) Zbl [0897.76011](#) MR [1480072](#)
- [44] Le, H.: [Elliptic equations with transmission and Wentzell boundary conditions and an application to steady water waves in the presence of wind](#). Discrete Contin. Dynam. Systems **38**, 3357–3385 (2018) Zbl [1398.35038](#) MR [3809086](#)
- [45] Lieberman, G. M.: [Oblique derivative problems for elliptic equations](#). World Scientific Publishing, Hackensack, NJ (2013) Zbl [1273.35006](#) MR [3059278](#)
- [46] Lisai, S., Wilkinson, M.: [Smooth solutions of the surface semi-geostrophic equations](#). Calc. Var. Partial Differential Equations **59**, article no. 16, 31 pp. (2020) Zbl [1431.35133](#) MR [4040625](#)
- [47] López-Gómez, J.: [Classifying smooth supersolutions for a general class of elliptic boundary value problems](#). Adv. Differential Equations **8**, 1025–1042 (2003) Zbl [1290.35059](#) MR [1989288](#)
- [48] Makarenko, N. I.: [Smooth bore in a two-layer fluid](#). In: Free boundary problems in continuum mechanics (Novosibirsk, 1991), Internat. Ser. Numer. Math. 106, Birkhäuser, Basel, 195–204 (1992) Zbl [0818.35088](#) MR [1229538](#)
- [49] Maklakov, D. V.: [A note on the existence of pure gravity waves at the interface of two fluids](#). Phys. D **401**, article no. 132157, 5 pp. (2020) Zbl [1453.76024](#) MR [4034682](#)
- [50] Maklakov, D. V., Sharipov, R. R.: [Almost limiting configurations of steady interfacial overhanging gravity waves](#). J. Fluid Mech. **856**, 673–708 (2018) Zbl [1415.76107](#) MR [3862620](#)
- [51] Meyers, N. G.: [An \$L^p\$ -estimate for the gradient of solutions of second order elliptic divergence equations](#). Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) **17**, 189–206 (1963) Zbl [0127.31904](#) MR [0159110](#)
- [52] Mielke, A.: [Homoclinic and heteroclinic solutions in two-phase flow](#). In: Proceedings of the IUTAM/ISIMM Symposium on Structure and Dynamics of Nonlinear Waves in Fluids (Hanover, 1994), Adv. Ser. Nonlinear Dynam. 7, World Scientific Publishing, River Edge, NJ, 353–362 (1995) Zbl [0872.76021](#) MR [1685879](#)
- [53] Nittka, R.: [Regularity of solutions of linear second order elliptic and parabolic boundary value problems on Lipschitz domains](#). J. Differential Equations **251**, 860–880 (2011) Zbl [1225.35077](#) MR [2812574](#)
- [54] Oleĭnik, O. A.: [On properties of solutions of certain boundary problems for equations of elliptic type](#). Mat. Sbornik N.S. **30(72)**, 695–702 (1952) (in Russian) Zbl [0046.10401](#) MR [0050125](#)

- [55] Pullin, D. I., Grimshaw, R. H. J.: [Finite-amplitude solitary waves at the interface between two homogeneous fluids](#). *Phys. Fluids* **31**, 3550–3559 (1988) Zbl [0656.76083](#) MR [0973153](#)
- [56] Rabinowitz, P. H.: [Some global results for nonlinear eigenvalue problems](#). *J. Funct. Anal.* **7**, 487–513 (1971) Zbl [0212.16504](#) MR [0301587](#)
- [57] Rabinowitz, P. H.: [Solutions of heteroclinic type for some classes of semilinear elliptic partial differential equations](#). *J. Math. Sci. Univ. Tokyo* **1**, 525–550 (1994) Zbl [0823.35062](#) MR [1322690](#)
- [58] Sandstede, B., Scheel, A.: [Essential instability of pulses and bifurcations to modulated travelling waves](#). *Proc. Roy. Soc. Edinburgh Sect. A* **129**, 1263–1290 (1999) Zbl [0956.35065](#) MR [1728529](#)
- [59] Sandstede, B., Scheel, A.: [Essential instabilities of fronts: bifurcation, and bifurcation failure](#). *Dynam. Systems* **16**, 1–28 (2001) Zbl [1055.37069](#) MR [1835905](#)
- [60] Simpson, J. E.: [Gravity currents in the laboratory, atmosphere, and ocean](#). *Annual Rev. Fluid Mech.* **14**, 213–234 (1982) Zbl [0488.76107](#)
- [61] Sun, S. M.: [Existence of large amplitude periodic waves in two-fluid flows of infinite depth](#). *SIAM J. Math. Anal.* **32**, 1014–1031 (2001) Zbl [0980.35133](#) MR [1828316](#)
- [62] Turner, R. E. L., Vanden-Broeck, J.-M.: [Broadening of interfacial solitary waves](#). *Phys. Fluids* **31**, 2486–2490 (1988) Zbl [0649.76008](#)
- [63] Varvaruca, E.: [On the existence of extreme waves and the Stokes conjecture with vorticity](#). *J. Differential Equations* **246**, 4043–4076 (2009) Zbl [1162.76011](#) MR [2514735](#)
- [64] Varvaruca, E., Weiss, G. S.: [A geometric approach to generalized Stokes conjectures](#). *Acta Math.* **206**, 363–403 (2011) Zbl [1238.35194](#) MR [2810856](#)
- [65] Volpert, V.: [Elliptic partial differential equations. Volume 1: Fredholm theory of elliptic problems in unbounded domains](#). *Monogr. Math.* 101, Birkhäuser/Springer Basel, Basel (2011) Zbl [1222.35002](#) MR [2778694](#)
- [66] Volpert, V., Volpert, A.: [Properness and topological degree for general elliptic operators](#). *Abstr. Appl. Anal.* **2003**, 129–181 Zbl [1030.35038](#) MR [1980315](#)
- [67] von Kármán, T.: [The engineer grapples with non-linear problems](#). *Bull. Amer. Math. Soc.* **46**, 615–683 (1940) Zbl [66.0985.04](#) MR [0003131](#)
- [68] Wahlén, E., Weber, J.: [Large-amplitude steady gravity water waves with general vorticity and critical layers](#). *Duke Math. J.* **173**, 2197–2258 (2024) Zbl [07918162](#) MR [4780521](#)
- [69] Wheeler, M. H.: [Large-amplitude solitary water waves with vorticity](#). *SIAM J. Math. Anal.* **45**, 2937–2994 (2013) Zbl [1282.76078](#) MR [3106477](#)
- [70] Wheeler, M. H.: [Solitary water waves of large amplitude generated by surface pressure](#). *Arch. Ration. Mech. Anal.* **218**, 1131–1187 (2015) Zbl [1457.35043](#) MR [3375547](#)