



Kihyun Kim

Rigidity of smooth finite-time blow-up for equivariant self-dual Chern–Simons–Schrödinger equation

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Abstract. We consider the long-time dynamics for the self-dual Chern–Simons–Schrödinger (CSS) equation within equivariant symmetry. The self-dual CSS equation is a self-dual L^2 -critical equation having pseudoconformal invariance and solitons. In this paper, we show that any m -equivariant, $m \geq 1$, H^3 finite-time blow-up solution to the self-dual CSS equation is a pseudoconformal blow-up solution. More precisely, such a solution decomposes into the sum of one modulated soliton, which contracts at the pseudoconformal rate $\lambda(t) \sim T - t$, and a radiation. Applying the pseudoconformal transform in reverse, we also obtain a refined soliton resolution theorem for m -equivariant, $m \geq 1$, sufficiently regular and localized solutions: any such solutions blow up in the pseudoconformal regime, scatter (to 0), or scatter to a modulated soliton with some fixed scale and phase. To our knowledge, this is the first result on the full classification of the dynamics of arbitrary smooth and spatially decaying solutions in the class of nonlinear Schrödinger equations which are not known to be completely integrable. Our analysis not only builds upon the previous works, especially the soliton resolution theorem by the author, Kwon, and Oh, but also refines all steps of the arguments typically employed in the forward construction of blow-up dynamics. The key feature of the proof is that we can identify the singular and regular parts of any H^3 finite-time blow-up solutions, such that the evolution of the singular part is governed by a universal modulation dynamics while the regular part is kept H^3 -bounded even up to the blow-up time. As a byproduct, we also observe that the asymptotic profile has a universal singular structure.

Keywords: Chern–Simons–Schrödinger equation, self-duality, blow-up, rigidity, soliton resolution.

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Kihyun Kim: IHÉS, 35 route de Chartres, 91440 Bures-sur-Yvette, France; Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University, 1 Gwanak-ro, Gwanak-gu, 08826 Seoul, South Korea; kihyun.kim@snu.ac.kr

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1. Introduction

1.1. Self-dual Chern–Simons–Schrödinger equation

We consider the self-dual Chern–Simons–Schrödinger equation within equivariant symmetry. This equation reads

$$i(\partial_t + iA_t[u])u + \partial_{rr}u + \frac{1}{r}\partial_ru - \left(\frac{m + A_\theta[u]}{r}\right)^2 u + |u|^2u = 0, \quad (\text{CSS})$$

where $m \in \mathbb{Z}$ (called *equivariance index*), $u: [0, T) \times (0, \infty) \rightarrow \mathbb{C}$, and the connection components $A_t[u]$ and $A_\theta[u]$ are given by the formulas

$$A_t[u] = -\int_r^\infty (m + A_\theta[u])|u|^2 \frac{dr'}{r'}, \quad A_\theta[u] = -\frac{1}{2} \int_0^r |u|^2 r' dr'.$$

The (full non-equivariant) covariant Chern–Simons–Schrödinger equation,

$$\begin{cases} \mathbf{D}_t \phi = i(\mathbf{D}_1 \mathbf{D}_1 + \mathbf{D}_2 \mathbf{D}_2) \phi + i g |\phi|^2 \phi, \\ F_{t1} = -\text{Im}(\bar{\phi} \mathbf{D}_2 \phi), \\ F_{t2} = \text{Im}(\bar{\phi} \mathbf{D}_1 \phi), \\ F_{12} = -\frac{1}{2} |\phi|^2, \end{cases} \quad (1.1)$$

with the coupling constant $g > 0$, the scalar field $\phi: [0, T) \times \mathbb{R}^2 \rightarrow \mathbb{C}$, the covariant derivatives $\mathbf{D}_\alpha = \partial_\alpha + iA_\alpha$, and the curvature components $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, $\alpha, \beta \in \{t, 1, 2\}$, was introduced by the physicists Jackiw and Pi [16] as a gauged 2D nonlinear Schrödinger equation that admits *explicit static self-dual solitons* within the particular choice of the coupling constant $g = 1$. This shows a stark contrast with the 2D cubic nonlinear Schrödinger equation

$$i\partial_t \psi + \Delta \psi + |\psi|^2 \psi = 0, \quad \psi: [0, T) \times \mathbb{R}^2 \rightarrow \mathbb{C}, \quad (\text{NLS})$$

where the equation does not have static (time-independent) finite energy solutions and no explicit soliton formulas are known. Model (CSS) under consideration is derived from the covariant system (1.1) after choosing $g = 1$, fixing the Coulomb gauge condition, and imposing *equivariant symmetry* on the scalar field ϕ ,

$$\phi(t, x) = u(t, r)e^{im\theta}, \quad m \in \mathbb{Z},$$

where (r, θ) denotes the polar coordinates on $\mathbb{R}^2$. We refer to [8, 15–18] for more physical contents of the Chern–Simons–Schrödinger equation (1.1) and to the introduction of [27, 35] for the derivation of (CSS).

Recent studies on the long-term dynamics of (CSS) make this model more interesting. To be further explained in this introduction, two notable features are *rotational instability* and *soliton resolution*. Rotational instability is an instability mechanism of some finite-time blow-up solutions and soliton resolution asserts a strong rigidity of the asymptotic behavior of solutions. The present paper continues our study on soliton resolution and our main result gives a complete description of the asymptotic behavior of m -equivariant, $m \geq 1$, smooth (and spatially decaying) solutions to (CSS).

We continue our discussion on the model. Equation (CSS) enjoys various symmetries and conservation laws analogous to (NLS). Among the most basic symmetries are the time translation $(u(t, r) \rightarrow u(t + t_0, r), t_0 \in \mathbb{R})$ and the phase rotation symmetries $(u(t, r) \rightarrow e^{i\gamma_0} u(t, r), \gamma_0 \in \mathbb{R}/2\pi\mathbb{Z})$. Associated to these are the conservation of *energy* and *mass*¹

$$E[u] := \int \frac{1}{2} |\partial_r u|^2 + \frac{1}{2} \left(\frac{m + A_\theta[u]}{r} \right)^2 |u|^2 - \frac{1}{4} |u|^4, \quad (1.2)$$

$$M[u] := \int |u|^2, \quad (1.3)$$

where we denoted $\int f(r) = 2\pi \int f(r) r dr$. Equation (CSS) moreover admits a *Hamiltonian* structure

$$\partial_t u = -i \nabla E[u], \quad (1.4)$$

where ∇ (acting on a functional) is the Fréchet derivative with respect to the real inner product $(u, v)_r = \int \operatorname{Re}(\bar{u}v)$. Of particular importance in this work are the L^2 -scaling

¹The physical interpretation of the quantity $M[u]$ is the *total charge*, but in this paper we shall call it mass following the widespread convention for NLS.

symmetry and the pseudoconformal symmetry; if $u(t, r)$ is a solution to (CSS), then the functions u_λ and $\mathcal{C}u$ also solve (CSS),

$$u_\lambda(t, r) := \frac{1}{\lambda} u\left(\frac{t}{\lambda^2}, \frac{r}{\lambda}\right) \quad \forall \lambda > 0, \quad (1.5)$$

$$[\mathcal{C}u](t, r) := \frac{1}{|t|} u\left(-\frac{1}{t}, \frac{r}{|t|}\right) e^{ir^2/(4t)} \quad \forall t \neq 0. \quad (1.6)$$

Associated to (1.5) and (1.6) are the *virial identities*

$$\partial_t \int r^2 |u|^2 = 4 \int \operatorname{Im}(\bar{u} \cdot r \partial_r u), \quad \partial_t \int \operatorname{Im}(\bar{u} \cdot r \partial_r u) = 4E[u].$$

We note that the covariant system (1.1) also shares very similar symmetries and conservation laws with (NLS).

As alluded to above, a notable feature of (CSS) distinguished from (NLS) is *self-duality*. Let us introduce the (covariant) *Cauchy–Riemann operator*

$$\mathbf{D}_u f := \partial_r f - \frac{m + A_\theta[u]}{r} f,$$

which indeed corresponds to the radial part of the operator $\mathbf{D}_+ := \mathbf{D}_1 + i\mathbf{D}_2$. Now the energy functional (1.2) can be written into the self-dual form

$$E[u] = \int \frac{1}{2} |\mathbf{D}_u u|^2. \quad (1.7)$$

We call the nonlinear operator $u \mapsto \mathbf{D}_u u$ the *Bogomol’nyi operator*. Due to (1.4) and (1.7), any static solutions to (CSS) are given by solutions (Jackiw–Pi vortices) to the *Bogomol’nyi equation*

$$\mathbf{D}_Q Q = 0. \quad (1.8)$$

Since (1.8) is a first-order (nonlocal) equation, we say that the equation is *self-dual*. In fact, there is a covariant version of the Bogomol’nyi equation, and Jackiw–Pi [15] found a transform of the covariant Bogomol’nyi equation into the Liouville equation, which is completely integrable. Solving (1.8), we obtain a unique (up to symmetries) *explicit* m -equivariant static solution for any $m \geq 0$,

$$Q(r) = \sqrt{8}(m+1) \frac{r^m}{1 + r^{2m+2}}, \quad m \geq 0. \quad (1.9)$$

There is no finite energy solution to (1.8) when $m < 0$.² Applying the pseudoconformal transform (1.6) to Q , we also obtain an explicit finite-time blow-up solution ($m \geq 0$)

$$S(t, r) := \frac{1}{|t|} Q\left(\frac{r}{|t|}\right) e^{-ir^2/(4|t|)}, \quad t < 0,$$

²This fact can be proved either by ODE analysis or by showing the coercivity of energy [29].

which is another fundamental object in the long-term dynamics for (CSS). Due to the spatial decay $Q(r) \sim r^{-(m+2)}$, $S(t)$ has finite energy (i.e., $S(t)$ belongs to H^1) if and only if $m \geq 1$.

Note that we have suppressed the m -dependences in various notations such as $A_t[u]$, $E[u]$, $\mathbf{D}_u$, Q , and $S(t)$. We will keep suppressing these m -dependences throughout this article for the simplicity of notations.

1.2. Long-term dynamics for the Chern–Simons–Schrödinger equation

Let us begin with some previous works on the Cauchy problem of the covariant Chern–Simons–Schrödinger equation (1.1). The local well-posedness has been studied by many authors [1, 14, 34, 36]. Lim [34] proved the H^1 well-posedness under Coulomb gauge and Liu–Smith–Tataru [36] proved the small data H^{0+} well-posedness under the heat gauge. These results still miss the critical L^2 -space. There are also results on long-term dynamics such as the finite-time blow-up for negative energy solutions (necessarily $g > 1$) with a formal derivation of the log-log rate [1, 2] and small data scattering [49].

We turn to the long-term dynamics for (CSS). First, it is easy to see that (CSS) is well-posed in the critical space L^2 because there is no derivative nonlinearity within equivariance and one can use a standard contraction argument with the $L^4_{t,x}$ -Strichartz estimates [35, Section 2]. In particular, any small L^2 -solutions exist globally and scatter both forwards and backwards in time.

For large solutions, the ground state Q provides a natural threshold for the non-scattering dynamics. Indeed, Liu–Smith [35] proved the *subthreshold theorem*: for $m \geq 0$, any m -equivariant L^2 -solutions u with $M[u] < M[Q]$ exist globally and scatter. Following their proof, the same should be true for any negative-equivariant (i.e., $m < 0$) solutions without any restriction on mass; see also [29]. Recently, Li–Liu [33] considered the threshold dynamics ($M[u] = M[Q]$, necessarily $m \geq 0$) of H^1_m -solutions and proved that Q and $S(t)$ are the only non-scattering solutions up to symmetries.³

Recently, author, Kwon, and Oh studied finite-time blow-up dynamics of (CSS) with an eye towards the dynamics above threshold. We remark that the finite-time blow-up dynamics is closely related to the dynamics of non-scattering global solutions through the pseudoconformal transform.

In [26–28, 30], the authors studied the dynamics of finite-time blow-up solutions to (CSS) of the form

$$u(t, r) - \frac{e^{i\gamma(t)}}{\lambda(t)} Q\left(\frac{r}{\lambda(t)}\right) \rightarrow z^*(r) \quad \text{in } L^2 \text{ as } t \rightarrow T. \quad (1.10)$$

The authors in [26, 27] considered (CSS) when $m \geq 1$ and studied *pseudoconformal blow-up solutions*, namely, the solutions of form (1.10) with $\frac{\lambda(t)}{T-t} \rightarrow \ell \in (0, \infty)$ and $\gamma(t) \rightarrow \gamma^* \in \mathbb{R}/2\pi\mathbb{Z}$. They in particular constructed pseudoconformal blow-up solutions

³When $m = 0$, $S(t)$ does not have finite energy due to the slow spatial decay of Q .

and uncovered their instability mechanism (*rotational instability*); see Remark 1.12 for more discussions. On the other hand, when $m = 0$, $S(t) \notin H^1$, so even the existence of finite energy blow-up solutions is not trivial. In this case, the authors in [28, 30] constructed finite energy finite-time blow-up solutions with the rates $\lambda(t) \sim \frac{T-t}{|\log(T-t)|^2}$ or $\lambda(t) \sim \frac{(T-t)^p}{|\log(T-t)|}$ for any $p > 1$, where the former regime can arise from a *smooth* compactly supported initial data.

With the above blow-up solutions at hand, it is natural to ask the following:

(Q1) Is there any blow-up scenario other than one-bubble blow-up?

(Q2) Can we classify all possible dynamics of scale and phase?

The first question was answered for finite energy solutions by the author, Kwon, and Oh [29] (see also Theorem 1.1 below). The result roughly says that any H_m^1 finite-time blow-up solutions admit decomposition (1.10) with certain upper bounds on $\lambda(t)$. By pseudoconformal symmetry, one further obtains *soliton resolution* for well-localized regular data: every such solution to (CSS) decomposes into at most one soliton and a radiation. A remarkable fact is that (CSS) does not admit multi-soliton configurations and this turns out to be a consequence of two distinguished features of (CSS): *self-duality* and *non-locality*. We refer to Remarks 1.2 and 1.3 for more explanations.

The present paper concerns the second question, namely, the classification of the dynamics of scale and phase.

1.3. Main results

The main result of this paper is that any m -equivariant, $m \geq 1$, H^3 finite-time blow-up solutions to (CSS) must be pseudoconformal blow-up solutions and hence this gives a complete classification of the dynamics of scale and phase. Via the pseudoconformal transform, we also give the full description of the dynamics of any m -equivariant, $m \geq 1$, $H^{3,3}$ -solutions.

Let us briefly introduce some important notations. We denote the modulated soliton by

$$Q_{\lambda,\gamma}(r) := \frac{e^{i\gamma}}{\lambda} Q\left(\frac{r}{\lambda}\right), \quad \lambda \in (0, \infty), \quad \gamma \in \mathbb{R}/2\pi\mathbb{Z}.$$

For $k \in \mathbb{N}$, we denote by H_m^k (resp. $H_m^{k,k}$), the space of m -equivariant $H^k(\mathbb{R}^2)$ (resp. $H^{k,k}(\mathbb{R}^2)$) functions. Here, $f \in H^{k,k}(\mathbb{R}^2)$ means that $\langle x \rangle^\ell f \in H^{k-\ell}(\mathbb{R}^2)$ for all $0 \leq \ell \leq k$. See Section 2.1 for more pieces of notations.

We begin by recalling the soliton resolution theorem of [29] for H_m^1 finite-time blow-up solutions.

Theorem 1.1 (Soliton resolution for H_m^1 finite-time blow-up solutions [29]).

- When $m < 0$, there are no H_m^1 finite-time blow-up solutions.
- When $m \geq 0$, if u is an H_m^1 -solution to (CSS) that blows up forwards in time at $T < +\infty$, then $u(t)$ admits the decomposition

$$u(t, \cdot) - Q_{\lambda(t), \gamma(t)} \rightarrow z^* \quad \text{in } L^2 \text{ as } t \rightarrow T, \quad (1.11)$$

for some $\lambda(t) \in (0, \infty)$, $\gamma(t) \in \mathbb{R}/2\pi\mathbb{Z}$, and $z^* \in L^2$ with the following properties:

- (Further regularity of the asymptotic profile) We have

$$\partial_r z^*, \frac{1}{r} z^* \in L^2. \quad (1.12)$$

- (Bound on the blow-up speed) As $t \rightarrow T$, we have

$$\lambda(t) \lesssim_{M[u]} \sqrt{E[u]}(T - t). \quad (1.13)$$

When $m = 0$, we further have an improved bound

$$\lambda(t) \lesssim_{M[u]} \frac{\sqrt{E[u]}(T - t)}{|\log(T - t)|^{1/2}}. \quad (1.14)$$

Remark 1.2 (Exterior defocusing nature). A remarkable fact in the decomposition (1.11) is that only one soliton appears. This is a consequence of the *self-duality* and *nonlocality*, which render (CSS) *defocusing* (in the form of strict positivity of energy) after extracting out the soliton profile. Hence no multi-soliton configurations appear. See the nonlinear coercivity of energy in [29] or Lemma 4.2 of this paper.

Remark 1.3 (Soliton resolution). Via pseudoconformal transform, we also obtain soliton resolution for $H_m^{1,1}$ -solutions to (CSS), which says that at most one soliton can appear in the resolution. Soliton resolution, in a more general context, says that (under a genericity assumption) every solution decomposes into the sum of decoupled solitons and a radiation. This has been known for a wide range of completely integrable equations. However, it is mostly open for non-integrable equations with exception of some nonlinear wave equations within symmetry, such as energy-critical radial nonlinear wave equation and equivariant wave maps [5, 9–11, 21, 22]. In these equations, multi-solitons are known to exist [19]. Thus the previously mentioned exterior defocusing nature is special to (CSS).

From now on, we consider the $m \geq 1$ case. Pseudoconformal blow-up solutions constructed in [26, 27] saturate the upper bound (1.13) for $\lambda(t)$, but the question of whether an exotic blow-up speed other than the pseudoconformal one exists or not was left open. Our main result claims that such an exotic regime does not exist and the blow-up must be pseudoconformal if one assumes H^3 -data.

Theorem 1.4 (Rigidity of H_m^3 finite-time blow-up when $m \geq 1$). *Let $m \geq 1$. Let u be an H_m^3 -solution to (CSS) which blows up forwards in time at $T < +\infty$. Then, the following holds:*

- (Rigidity of blow-up rates) In decomposition (1.11), we can choose

$$\lambda(t) = \ell \cdot (T - t) \quad \text{and} \quad \gamma(t) = \gamma^* \quad (1.15)$$

for some $\ell = \ell(u) \in (0, \infty)$ and $\gamma^* = \gamma^*(u) \in \mathbb{R}/2\pi\mathbb{Z}$.

- (Structure of the asymptotic profile) We can decompose

$$z^* = z_{\text{sg}}^* + z_{\text{rg}}^*$$

for some $z_{\text{sg}}^*, z_{\text{rg}}^* \in L^2$ such that the singular part z_{sg}^* is given by

$$z_{\text{sg}}^*(r) = \begin{cases} -e^{i\gamma^*} \ell^2 \frac{\sqrt{2}}{8} \chi(r) \cdot r & \text{if } m = 1, \\ e^{i\gamma^*} \ell^3 \frac{\sqrt{2}}{64} i \chi(r) \cdot r^2 & \text{if } m = 2, \\ 0 & \text{if } m \geq 3, \end{cases}$$

where $\chi(r)$ is a smooth cutoff function near $r = 0$ (see Section 2.1), and the regular part z_{rg}^* satisfies

$$|z_{\text{rg}}^*|_{-3} := \sup_{0 \leq k \leq 3} |r^{-(3-k)} \partial_r^k z_{\text{rg}}^*| \in L^2. \quad (1.16)$$

In particular, when $m \in \{1, 2\}$, $z^*(r)$ satisfies the near-origin asymptotics

$$z^*(r) = qr^m + o_{r \rightarrow 0}(r^2) \quad (1.17)$$

for some nonzero constant $q = q(u) \neq 0$.

The above theorem can also be viewed as a refined soliton resolution theorem for H_m^3 finite-time blow-up solutions. As a standard application of pseudoconformal symmetry, we obtain a refined soliton resolution theorem for arbitrary $H_m^{3,3}$ -solutions.

Corollary 1.5 (Soliton resolution for equivariant $H_m^{3,3}$ -data when $m \geq 1$). *Let $m \geq 1$. Let u be an $H_m^{3,3}$ -solution to (CSS).*

- (Finite-time blow-up) *If u blows up forwards in time at $T < +\infty$, then u satisfies the properties as in Theorem 1.4.*
- (Global solutions) *If u exists globally forwards in time, then either u scatters forwards in time in L^2 , or u scatters to Q_{λ^*, γ^*} forwards in time in L^2 for some $\lambda^* \in (0, \infty)$ and $\gamma^* \in \mathbb{R}/2\pi\mathbb{Z}$.*

In view of time reversal symmetry, the analogous statements for Theorem 1.4 and Corollary 1.5 hold for backward-in-time evolutions.

Remark 1.6 (Rigidity of long-term dynamics). Theorem 1.4 completely classifies the possible dynamics of the scale and phase for H_m^3 finite-time blow-up solutions. In view of pseudoconformal symmetry, even the long-term dynamics of arbitrary $H_m^{3,3}$ solutions is rigid as stated in Corollary 1.5. Moreover, all the scenarios in Corollary 1.5 indeed occur, thanks to the blow-up construction in [26].

Our result might be compared to the first part of the series of works [37–39] by Martel, Merle, and Raphaël, who considered the near-soliton dynamics for the critical generalized Korteweg–de Vries (gKdV) equation

$$\partial_t u + \partial_x(\partial_{xx} u + u^5) = 0, \quad u: [0, T) \times \mathbb{R} \rightarrow \mathbb{R}. \quad (\text{gKdV})$$

The result can be roughly stated as follows: (i) if an H^1 -solution u with strong spatial decay on the right starts from an initial data H^1 -close to the ground state Q , then u blows up in finite time with $\lambda(t) \sim T - t$, u exits the soliton tube,⁴ or u scatters to a modulated Q with some fixed scale and velocity; (ii) moreover, the former two regimes are stable.

We remark that there is no near-soliton assumption in Corollary 1.5. For (CSS) with $m \geq 1$, the equation however allows complex-valued functions and has additional phase rotation invariance. In contrast to the gKdV case, the (pseudoconformal) blow-up regime is believed to be unstable [27]. Thus the ‘scattering to 0’ regime is the only stable regime for (CSS). See however Remark 1.12 for rotational instability of pseudoconformal blow-up solutions, which we believe is an interesting nonlinear dynamics that generically occur.

Remark 1.7 (Stabilization of scale and phase for global $H_m^{3,3}$ -solutions). Although the soliton resolution theorem for global non-scattering $H_m^{1,1}$ -solutions [29] only says that $u(t) - Q_{\lambda(t), \gamma(t)} - e^{it\Delta} u^* \rightarrow 0$ with $\lambda(t) \lesssim 1$, an interesting fact from Corollary 1.5 is that $\lambda(t)$ and $\gamma(t)$ stabilize for $H_m^{3,3}$ -solutions when $m \geq 1$.

We believe that the same result might be false without spatial decay assumptions. A similar statement in [37] for the critical gKdV assumes a fast spatial decay on the right of solutions, and it is even shown in [39] that a slow spatial decay on the right might lead to both finite-time and infinite-time blow-up. Moreover, for 2-equivariant harmonic map heat flow, which is closely related to 1-equivariant (CSS) (see the discussion in Section 2.3.3), exotic behaviors such as infinite-time blow-up, oscillation, or relaxation can appear for solutions having slow spatial decay [12]. It would be interesting to construct exotic solutions having smooth but slowly decaying data.

Remark 1.8 (Comparison with the mass-critical NLS). Let us begin with some general comparisons between (NLS) and (CSS). As alluded to above, (CSS) and (NLS) share almost the same conservation laws and symmetries. Moreover, the (sub-)threshold theorem for (NLS) holds the same [7, 24, 25, 32] as in the case of (CSS) [33, 35].

However, the dynamics above threshold rather shows a drastic difference between (NLS) and (CSS) cases. We note that standing waves for (NLS) have a nonzero speed of phase rotation and have the form $e^{it} R(x)$ up to symmetry. The profile R solves $\Delta R - R + R^3 = 0$ or equivalently $\nabla E_{\text{NLS}}[R] + \nabla M[R] = 0$ with the NLS energy $E_{\text{NLS}}[\psi] = \frac{1}{2} \int |\nabla \psi|^2 - \frac{1}{4} \int |\psi|^4$. Notice that $\nabla E[Q] = 0$ for (CSS). The difference between these soliton equations ultimately leads to differences in underlying linearized operators and modulation dynamics.

As studied in a series of works [40–44, 51] by Merle and Raphaël, (NLS) has a *stable* almost self-similar blow-up regime $\lambda(t) \sim \sqrt{2\pi(T-t)}(\log|\log(T-t)|)^{-1}$, called the log-log blow-up. This regime essentially arises from negative energy solutions and hence does not appear in the case of (CSS) [29] (see (1.13)). Nevertheless, the log-log regime is formally expected for the focusing non-self-dual CSS ($g > 1$), which allows negative energy solutions.

⁴If the minimal mass blow-up solution to critical gKdV scatters backwards in time, then u scatters to zero [38].

The pseudoconformal blow-up solutions constructed in [26, 27] can be viewed as (CSS)-analogs of Bourgain–Wang solutions [3] to (NLS). However, their instability mechanisms, studied in [47] for (NLS) and in [27] for (CSS), turn out to be completely different. While Bourgain–Wang solutions to (NLS) arise as the border of log-log blow-up solutions and global scattering solutions, generic perturbations of pseudoconformal blow-up solutions to (CSS) become global scattering solutions and rather exhibit rotational instability; see Remark 1.12 below.

To directly compare our main theorem with the case of (NLS), let us briefly recall a result in [51]: if an H^1 -solution $\psi(t)$ to (NLS) with slightly supercritical mass (i.e., $M[\psi] - M[R] \ll 1$) blows up in finite time, then either $\psi(t)$ blows up in the log-log regime or $\lambda(t) \lesssim \sqrt{E[u]}(T - t)$. The latter regime might be compared with (1.13) and our Theorem 1.4 improves (1.13) to rigidity (1.15) in the case of (CSS) when $m \geq 1$. The question of whether this upper bound for (NLS) can be replaced by rigidity (1.15) for $\lambda(t)$ or not is a long-standing open problem.⁵

Remark 1.9 (Structure of the asymptotic profile). Theorem 1.4 provides a more precise understanding of the structure of z^* for H_m^3 finite-time blow-up solutions compared to the H^1 -regularity of z^* for H_m^1 blow-up solutions of Theorem 1.1. We note that the choice of χ itself is not important; only the $r \rightarrow 0$ behavior of z^* has meaning.

When $m \in \{1, 2\}$, $z^*(r)$ always has a nontrivial singular part z_{sg}^* . Note that this is not a contradiction to the existence of the minimal mass blow-up solution $S(t)$ (which has zero asymptotic profile) because $S(t) \notin H_m^3$ when $m \in \{1, 2\}$. We may also write $z^* \notin H_{-m-2}^3$ when $m \in \{1, 2\}$,⁶ thus the asymptotic profile does not necessarily have the same regularity as the solution $u(t)$. Such an observation is already well known in various contexts and the regularity of the asymptotic profile is closely related to the blow-up speed; we refer to the discussions in [23, 37, 43, 45, 52].

An interesting fact in Theorem 1.4 is that the singular part of z^* has a universal structure (1.17). See also [37, 43]. In particular, an asymptotic profile of the form $z^*(r) \approx qr^\nu$ with $\nu \in (1, 2)$ cannot be achieved from H_m^3 finite-time blow-up solutions if $m \geq 1$. It would be very interesting to fully understand the singular structure of the asymptotic profile arising from smooth initial data (say H_m^∞).

Our decomposition of z^* is tied to the decomposition $u = P^\# + \varepsilon^\#$ of the full solution $u(t)$ to be used in our modulation analysis (see Remark 1.10).

Remark 1.10 (Structure of the full solution). The key observation of this paper is that we can realize the decomposition $u = P^\# + \varepsilon^\#$ (see (1.18) below) such that the evolution of the singular, blow-up part $P^\#$ is governed by a universal modulation dynamics while *the regular part $\varepsilon^\#$ remains H^3 -bounded* (see also (1.19)) up to the blow-up time. In the limit

⁵We do not expect the convergence of the phase $\gamma(t)$ for (NLS) because Bourgain–Wang solutions do not have this property.

⁶Due to the nonlocality of (CSS), the dynamics outside of the soliton region is similar to the dynamics of $-(m+2)$ -equivariant (CSS), where $-(m+2) = m + A_\theta[Q](+\infty)$. See [27, 29].

$t \rightarrow T$, the regular part $\varepsilon^\sharp$ converges to z_{rg}^* and the blow-up part $P^\sharp$ detects z_{sg}^* , leading to the structure of z^* .

The H^3 -boundedness of $\varepsilon^\sharp$ is not limited to finite-time blow-up solutions. For any H_m^3 -solutions u lying in a neighborhood of modulated solitons (i.e., $Q_{\lambda,\gamma}$), we show in Corollary 4.17 that the H^3 -norm of $\varepsilon^\sharp$ cannot blow up in finite time. In this sense, we essentially prove that the regular parts $\varepsilon^\sharp$ of nearby solutions are *uniformly* H^3 -bounded on any fixed finite-time interval.

To prove the H^3 -boundedness, we not only refine the modified profile construction but also introduce a new energy-Morawetz functional to control $\varepsilon^\sharp$ forwards in time. Several novelties regarding the proof are explained in the strategy below (Section 1.4).

Remark 1.11 (Other equivariance). When $m < 0$, there is no H^1 finite-time blow-up as in Theorem 1.1, so the $m = 0$ case is the interesting one to consider.

When $m = 0$, there are no finite energy pseudoconformal blow-up solutions [29] (or see (1.14)). Recall that $S(t)$ has infinite energy when $m = 0$ due to the slow spatial decay of Q . Instead, the authors in [28] construct smooth (finite energy) finite-time blow-up solutions with the blow-up rate $\lambda(t) \sim \frac{T-t}{|\log(T-t)|^2}$.

We do not even expect the uniqueness of blow-up rates for solutions with smooth data. In view of the blow-up constructions for example in [6, 13, 54] for some critical parabolic equations and in [4, 46] for some energy-supercritical dispersive equations, we expect that (countably) infinitely many blow-up rates of the form $\{\frac{(T-t)^{c_\ell}}{|\log(T-t)|^{d_\ell}}\}_{\ell \in \mathbb{N}}$ can arise from smooth initial data. Identifying such admissible blow-up rates and proving rigidity in this context are of great interest. Note that rigidity of this type is obtained for some energy-supercritical semi-linear heat equations within radial symmetry [48] using the intersection comparison argument, which is not available for dispersive equations.

If one further allows rougher blow-up solutions, then the blow-up dynamics is no more rigid. In fact, there is a continuum of possible blow-up rates for H_0^1 -solutions [30]; for any $p > 1$, there is an H_0^1 finite-time blow-up solution such that $\lambda(t) \sim \frac{|t|^p}{|\log|t||}$. Let us note that (see Section 2.3.3) the dynamics of m -equivariant (CSS) is closely related to the dynamics of $(m+1)$ -equivariant wave maps and Schrödinger maps, and such a continuum of blow-up rates have already been constructed in these equations within 1-equivariance [23, 31, 50].

We finally note that the stabilization of scale for global-in-time solutions is *always* false for solutions with $H_0^{1,1}$ initial data. Indeed, it is proved in [29] that $\lambda(t) \lesssim (\log t)^{-1/2}$. This does not contradict the existence of the static solution Q because Q does not belong to $H_0^{1,1}$ when $m = 0$.

Remark 1.12 (Rotational instability). By Theorem 1.4 (when $m \geq 1$), pseudoconformal blow-up is the only blow-up regime allowed for smooth initial data but it is in fact non-generic and unstable. Thus the dynamics of solutions near these blow-up solutions is generic and of interest.

In [27], the authors constructed a curve of solutions $\{u^{(\eta)}\}_{\eta \in [0, \eta^*]}$ such that the scale and phase parameters approximately evolve as (3.3). In other words, while $u^{(0)}$ is a pseudoconformal blow-up solution, $u^{(\eta)}$ with $\eta \neq 0$ concentrates only up to the spatial scale $|\eta|$,

then exhibits a quick phase rotation by the fixed angle $\operatorname{sgn}(\eta)(m+1)\pi$ within a time interval of length $\sim|\eta|$, and then spreads out like a backward-in-time evolution of pseudoconformal blow-up solutions (*rotational instability*). On the other hand, a codimension one set of initial data leading to pseudoconformal blow-up is constructed in [26]. These two observations naturally suggest the following *rotational instability conjecture*: there is a codimension one manifold $\mathcal{M}$ of smooth initial data u_0 such that (i) if $u_0 \in \mathcal{M}$, then $u(t)$ blows up in finite time (and hence is a pseudoconformal blow-up) and (ii) if $u_0 \notin \mathcal{M}$ but is close to $\mathcal{M}$, then $u(t)$ does not blow up and rather exhibits rotational instability.

Rotational instability seems to be not limited to (CSS) but rather *universal*. The work [57] provides some formal computations and numerical simulations for a quick spatial rotation by the angle π for the 1-equivariant Landau–Lifshitz–Gilbert equation. See also [45]. The same is expected for the critical wave maps. We finally note that the H^3 -boundedness of the regular part of solution as in Remark 1.10 not only holds for finite-time blow-up solutions but also for nearby solutions. We believe that this observation plays a crucial role for the resolution of the rotational instability conjecture.

1.4. Strategy of the proof

We use the notations collected in Section 2.1. As Corollary 1.5 is a direct consequence of pseudoconformal symmetry and Theorem 1.4, we only sketch the proof of Theorem 1.4.

Let u be an H_m^3 -solution to (CSS) that blows up forwards in time at $T < +\infty$. Recall from [29] (or Remark 2.13 of this paper) that after renormalization $u(t)$ eventually enters the $\dot{H}_m^1$ -vicinity of the soliton

$$\inf_{y \in \mathbb{R}/2\pi\mathbb{Z}} \|\hat{\lambda}(t)e^{-iy}u(t, \hat{\lambda}(t)\cdot) - Q\|_{\dot{H}_m^1} \rightarrow 0 \quad \text{as } t \rightarrow T, \quad \text{where } \hat{\lambda}(t) = \frac{\|Q\|_{\dot{H}_m^1}}{\|u(t)\|_{\dot{H}_m^1}}.$$

We note that there is no L^2 -closeness because we allow u to have an arbitrarily large mass.

We study the near-soliton dynamics of (CSS) using modulation analysis. As in the previous work [26], we will write $u(t)$ in the form

$$u(t, r) = \frac{e^{iy(t)}}{\lambda(t)} [P(\cdot; b(t), \eta(t)) + \varepsilon(t, \cdot)] \left(\frac{r}{\lambda(t)} \right) =: [P + \varepsilon]^\#(t, r) \quad (1.18)$$

for some additional modulation parameters b, η and some modified profile $P = P(y; b, \eta)$ with $P(\cdot; 0, 0) = Q$. Note that the finite-time blow-up assumption only says that $\lambda(t) \rightarrow 0$ as $t \rightarrow T$.

Our key ingredient for the proof of the refined descriptions in Theorem 1.4 will be the proof of H^3 -boundedness of $\varepsilon^\#(t)$ up to the blow-up time. To be more precise, we will in fact prove the boundedness of $\varepsilon^\#(t)$ in a stronger topology, which in particular guarantees the boundedness of (see (2.2) for the notation $|\cdot|_{-3}$)

$$\|\mathbf{1}_{r \geq \lambda(t)} |\varepsilon^\#(t)|_{-3}\|_{L^2}. \quad (1.19)$$

Taking the limit $t \rightarrow T$ gives regularity (1.16) of z_{rg}^* . Note that the modified profile P should also be chosen carefully because we a posteriori know from Theorem 1.4 that $P^\#$

should detect the remaining part of z^* , namely, z_{sg}^* in the limit $t \rightarrow T$. We also note that the H^3 -boundedness of $\varepsilon^\sharp$ implies the rigidity of blow-up rates; see Step 5 of this strategy.

As can be seen in the strategy below, our approach shares a similar spirit with the forward construction of blow-up dynamics as in [37, 45, 46, 52, 56] and for (CSS), in [26, 28] ($m \geq 1$ and $m = 0$, respectively). However, we need further improvements in all parts of the proof compared to the blow-up construction in [26]. The main reason is that, since we are dealing with arbitrary finite-time blow-up solutions, we *cannot assume dynamical controls* such as

$$b > 0 \quad \text{and} \quad |\eta| \ll b \quad (1.20)$$

(meaning that λ is shrinking and the instability mode is turned off, respectively), which can be safely assumed in the blow-up construction of [26]. We also note that the H^3 -boundedness of $\varepsilon^\sharp(t)$ is not even necessary there. In this sense, our analysis here focuses more on the general near-soliton dynamics as in [37]. See also [42, 51, 55] for the mass-critical NLS.

1. *Conjugated variables and setup for the modulation analysis.* As in [28], we view (CSS) as a system of equations for nonlinearly conjugated variables u , $u_1 := \mathbf{D}_u u$, and $u_2 := A_u \mathbf{D}_u u$, where $A_u = \mathbf{D}_u - \frac{1}{r}$. Being u_1 and u_2 naturally defined (see (2.4) below), evolution equations (2.6) and (2.7) for u_1 and u_2 still have simple forms.

For the modulation analysis, we will decompose u , u_1 , u_2 of the form

$$\begin{cases} u(t, r) = \frac{e^{i\gamma(t)}}{\lambda(t)} [P(\cdot; b(t), \eta(t)) + \varepsilon(t, \cdot)] \left(\frac{r}{\lambda(t)} \right), \\ u_1(t, r) = \frac{e^{i\gamma(t)}}{\lambda^2(t)} [P_1(\cdot; b(t), \eta(t)) + \varepsilon_1(t, \cdot)] \left(\frac{r}{\lambda(t)} \right), \\ u_2(t, r) = \frac{e^{i\gamma(t)}}{\lambda^3(t)} [P_2(\cdot; b(t), \eta(t)) + \varepsilon_2(t, \cdot)] \left(\frac{r}{\lambda(t)} \right), \end{cases} \quad (1.21)$$

for some modified profiles P , P_1 , P_2 to be constructed. Four orthogonality conditions are imposed either on ε or ε_1 to fix the decomposition. To analyze the flow, we introduce the renormalized coordinates (s, y) as well as the renormalized variable w ,

$$\frac{ds}{dt} = \frac{1}{\lambda^2(t)}, \quad y = \frac{r}{\lambda(t)}, \quad w(s, y) = \lambda e^{-i\gamma} u(t, \lambda y)|_{t=t(s)}.$$

The associated conjugated variables are $w_1 := \mathbf{D}_w w$ and $w_2 := A_w \mathbf{D}_w w$. The system of evolution equations for w , w_1 , w_2 can be found in (2.8)–(2.10).

One notable advantage of using the system of equations for the nonlinearly conjugated variables w , w_1 , w_2 is that we can separate the roles of each evolution equation. For example, the w -equation will only be used to detect the modulation equations for λ and γ , which allows simpler (or, rougher) ansatz for P compared to P_1 or P_2 . The w_1 -equation will be used to detect the modulation equations for b and η . In fact, these equations become more apparent in the w_1 -equation rather than the w -equation itself, allowing the

authors in [28] to capture the crucial logarithmic corrections in the b_s - and η_s -equations. Finally, the w_2 -equation will be used in the final energy estimates for the remainder ε . The use of the w_2 -equation not only simplifies the structure of the nonlinearity; it is also the w_2 -equation, whose linearized Hamiltonian $\tilde{H}_Q$ has a *repulsive* property (see Section 2.3.3). This enables the authors in [28] to perform the modified energy method for the variable ε_2 . We refer to [28] for more discussions on this approach.

2. Construction of the modified profiles. In [26, 27], when $m \geq 1$, the authors introduced a modified profile, which is constructed in a nonlinear way and suggests the formal modulation equations

$$\frac{\lambda_s}{\lambda} + b = 0, \quad \gamma_s \approx (m+1)\eta, \quad b_s + b^2 + \eta^2 = 0, \quad \eta_s = 0. \quad (1.22)$$

The $\eta = 0$ case corresponds to pseudoconformal blow-up and its instability mechanism can be dictated by varying η ; see Section 3.1 for more details.

However, when $m \in \{1, 2\}$, that profile turns out to be insufficient to address the H^3 -boundedness of $\varepsilon^\sharp$ and we need a new profile ansatz. We perform the tail computations as in [45, 46, 52]. It turns out that when $m = 1$, there are nontrivial *cubic corrections* to the b_s - and η_s -modulation equations, say

$$b_s + b^2 + \eta^2 + p_3^{(b)} = 0 \quad \text{and} \quad \eta_s + p_3^{(\eta)} = 0,$$

and the new profiles deviate from the old ones at the cubic order. When $m = 2$, the same happens at the quartic order.

Another interesting observation is that *the cutoff radius must be*

$$y \sim B_1 := (b^2 + \eta^2)^{-1/2}, \quad (1.23)$$

which corresponds to $r \sim 1$ in the original space variable r . Moreover, in the original spacetime variables, our modified profile $P^\sharp(t)$ converges to the singular part z_{sg}^* as $t \rightarrow T$. This is a sharp contrast to the use of almost self-similar scales typically employed in many of the prior blow-up constructions such as [26, 45, 46, 52].

3. Key a priori bounds from nonlinear coercivity of energy. Despite the lack of dynamical controls (1.20), we are able to obtain the following *size controls*:

$$\|\varepsilon\|_{L^2} \lesssim_{M[u]} 1 \quad \text{and} \quad |b| + |\eta| + \|\varepsilon\|_{\dot{H}_m^1} \lesssim_{M[u]} \lambda \sqrt{E[u]} =: \mu. \quad (1.24)$$

The former estimate is clear from mass conservation, but the latter, which we call the *nonlinear coercivity of energy*, is not obvious. In a simpler context where b and η are absent, this estimate was proved in [29] as a remarkable consequence of the *self-duality* and *nonlocality* of our problem, which are two distinguished features of (CSS). We refer to [29] or Remark 4.4 below for more explanations. Recalling its proof in [29], it is not difficult to extend the nonlinear coercivity in the presence of b - and η -modulations. See Lemma 4.2 below.

4. *Energy-Morawetz functional and uniform H^3 -bound.* We propagate the H^3 -norm boundedness of $\varepsilon^\sharp$, which is the heart of the proof of Theorem 1.4. The basic strategy will be the *energy method with repulsivity*, the idea being that the non-perturbative but localized error terms appearing in the energy estimates can be handled by the monotonicity from repulsivity. This strategy has successfully addressed many of the forward constructions of blow-up dynamics, for example in [45, 46, 52, 56] and for (CSS) in [26, 28].

Again, the difficulty is that we cannot assume dynamical controls such as (1.20). However, we have the *size control* (1.24) and our main assertion will be that this is sufficient to propagate the H^3 -boundedness of $\varepsilon^\sharp$. We should mention here that in the work [37] on the critical gKdV, the authors were able to design monotonicity formulas that are sufficient to conclude the rigidity of the near-soliton dynamics even without the size control (1.24). Instead, they used the local virial estimate and properties specific (e.g., Kato-type smoothing) to gKdV, where the latter seems difficult to be implemented in the Schrödinger setting.

We propose a new energy functional $\mathcal{F}$ that propagates the control $\|\varepsilon^\sharp\|_{\dot{H}^3 \cap \dot{H}^2} \lesssim 1$ on any finite-time intervals only under the size control (1.24). As mentioned before, we look at the ε_2 -variable, whose associated linearized Hamiltonian $\tilde{H}_Q$ has a *repulsive* character. The functional $\mathcal{F}$ takes the form

$$\mathcal{F} = \frac{(\varepsilon_2, \tilde{H}_Q \varepsilon_2)_r}{\mu^6} - A \frac{(i\varepsilon_2, \psi' \partial_y \varepsilon_2)_r}{\mu^5} + A^2 \frac{\|\varepsilon_2\|_{L^2}^2}{\mu^4} \quad (1.25)$$

for some large constant $A > 1$ and some well-chosen function $\psi' = \psi'(y)$ of Morawetz-type (see Lemma 2.1). Note that $\mathcal{F}$ is coercive in the sense that $\mathcal{F} \lesssim 1$ implies $\|\varepsilon_2\|_{\dot{H}^1} \lesssim \mu^3$ and $\|\varepsilon_2\|_{L^2} \lesssim \mu^2$, which in turn give $\|\varepsilon\|_{\dot{H}^3} \lesssim \mu^3$ and $\|\varepsilon\|_{\dot{H}^2} \lesssim \mu^2$ using coercivity estimates. By scaling, we have the H^3 -boundedness $\|\varepsilon^\sharp\|_{\dot{H}^3 \cap \dot{H}^2} \lesssim 1$. Thus our goal is to propagate the control $\mathcal{F} \lesssim 1$.

Propagation of the control $\mathcal{F} \lesssim 1$ in this work crucially utilizes the assumption $m \geq 1$ as well as the size control (1.24). Indeed, the *Morawetz-type repulsivity* (Lemma 2.1), i.e., the positivity of the commutator $[\tilde{H}_Q, \Lambda_\psi]$ for some choice of $0 \leq \psi' \leq 1$, where $\Lambda_\psi = \psi' \partial_y + \frac{\Delta \psi}{2}$, heavily relies on $m \geq 1$. Moreover, $|b| + |\eta| \lesssim \mu$ of (1.24) is used to dominate the non-perturbative (but localized) error terms arising in $\partial_s \left\{ \frac{(\varepsilon_2, \tilde{H}_Q \varepsilon_2)_r}{\mu^6} \right\}$ by the Morawetz monotonicity from $\partial_s \left\{ \frac{(i\varepsilon_2, \psi' \partial_y \varepsilon_2)_r}{\mu^5} \right\}$. Furthermore, $\|\varepsilon\|_{\dot{H}_m^1} \lesssim \mu$ of (1.24) is used to make pure ε -terms perturbative; the contribution of such pure ε -terms arises in the form $\partial_s \mathcal{F} \lesssim \mu^2(1 + \mathcal{F})$, which gives $\partial_t \mathcal{F} \lesssim 1 + \mathcal{F}$, which in turn yields $\mathcal{F}(t) \lesssim 1 + \mathcal{F}(t_0)$ on any finite-time interval by Grönwall's inequality. More technical aspects of this energy estimate will be explained in Section 4.5.

As a result of the boundedness of $\mathcal{F}$, we roughly have

$$\|\varepsilon\|_{\dot{H}^k} \lesssim \lambda^k, \quad 0 \leq k \leq 3. \quad (1.26)$$

5. *Rigidity of the blow-up rates.* In order to prove rigidity of the blow-up rate, an important role is played by the quantity

$$\frac{\beta^2}{\lambda^2} = \frac{b^2 + \eta^2}{\lambda^2}.$$

This corresponds to the energy of the blow-up part $P^\sharp$ and is conserved under the formal modulation ODEs (1.22). Note that by (1.24) we only know $\frac{\beta}{\lambda} \lesssim 1$ at this stage. Our goal is to show the reverse inequality. In fact, we show that

$$\frac{\beta}{\lambda} \rightarrow \ell \neq 0 \quad \text{as } t \rightarrow T. \quad (1.27)$$

We note that $\ell \neq 0$ is important because if $\ell = 0$ one can consider an exotic blow-up regime $\lambda(t) \sim (T - t)^\nu$ and $b(t) \sim (T - t)^{2\nu-1}$ with $\nu > 1$. We also note that energy conservation of the full solution $u(t)$ is useless because the $\varepsilon^\sharp$ -part is nontrivial.⁷ Instead, we look at the evolution of the quantity $\frac{\beta}{\lambda} + \lambda$ and obtain (5.20), i.e.,

$$\left(\frac{\beta}{\lambda} + \lambda\right)_t \lesssim \left(1 + \frac{\|\varepsilon\|_{\text{loc}}^2}{\lambda^4}\right) \left(\frac{\beta}{\lambda} + \lambda\right)$$

for some spatially localized norm $\|\cdot\|_{\text{loc}}$. By choosing $\|\cdot\|_{\text{loc}} = \|\cdot\|_{\dot{H}^3}$ and applying the uniform H^3 -bound (1.26), we obtain $(\frac{\beta}{\lambda} + \lambda)_t \lesssim \frac{\beta}{\lambda} + \lambda$. By Grönwall's inequality, $\frac{\beta}{\lambda} + \lambda$ cannot converge to zero in finite time. See also Remark 5.4. Since $\lambda \rightarrow 0$ on the other hand, we obtain (1.27). Once we have $\frac{\beta}{\lambda} \rightarrow \ell \neq 0$ at hand, it is not difficult to conclude $\lambda(t) \approx \ell \cdot (T - t)$ and $\gamma(t) \rightarrow \gamma^*$. Let us finally note that there is a small technical issue when m is small and we need to introduce corrections $\hat{b}, \hat{\eta}$ of b, η to run the previous argument.

6. Structure of the asymptotic profile. As the existence of the asymptotic profile z^* is a consequence of Theorem 1.1, we focus on the structure of z^* . When $m = 1$, the quadratic term of $P^\sharp$ resembles $[\chi_{B_1} b^2 \gamma]^\sharp = \chi_{\lambda/\beta}(r) (\frac{b}{\lambda})^2 r$, which gives a nontrivial singular part z_{sg}^* in the limit $t \rightarrow T$ by choice (1.23) of the cutoff and (1.27). Similarly, when $m = 2$, the cubic term of $P^\sharp$ converges to z_{sg}^* . Bound (1.16) follows from taking the $t \rightarrow T$ limit of H^3 -boundedness (1.19).

Organization of the paper. In Section 2, we collect notations and preliminaries for our analysis. In Section 3, we construct the modified profiles P, P_1, P_2 . In Section 4, we perform modulation analysis of the near soliton dynamics and prove the H^3 -boundedness of $\varepsilon^\sharp$, which is the heart of the paper. Finally, in Section 5, we prove Theorem 1.4. For the proof of Corollary 1.5, see Remark 5.5.

2. Preliminaries

2.1. Notations

For $A \in \mathbb{C}$ and $B \geq 0$, we use the standard asymptotic notation $A \lesssim B$ or $A = O(B)$ if there is a constant $C > 0$ such that $|A| \leq CB$. $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$. The

⁷In [55], where the authors consider minimal mass blow-up solutions, (1.27) easily follows from energy conservation and the improved smallness $\|\varepsilon^\sharp\|_{H^1} \rightarrow 0$. Let us note however that this smallness estimate is nontrivial. Nevertheless, $\|\varepsilon^\sharp\|_{H^1} \rightarrow 0$ does not hold in our context.

dependences of C are specified by subscripts, e.g., $A \lesssim_E B \Leftrightarrow A = O_E(B) \Leftrightarrow |A| \leq C(E)B$. In this paper, any dependences on the equivariance index m will be omitted.

We also use the little o -notation such as $o_{\alpha^* \rightarrow 0}(1)$ or $o_{t \rightarrow T}(1)$, which denotes a quantity that can be made arbitrarily small by taking $\alpha^* \rightarrow 0$ or $t \rightarrow T$, respectively. Similarly, we use the notation $\delta_M(\alpha^*)$ to denote a quantity that goes to 0 as $\alpha^* \rightarrow 0$ but may depend on M (thus may not be uniform in M).

We also use the notations $\langle x \rangle$, $\log_+ x$, $\log_- x$ defined by

$$\langle x \rangle := (|x|^2 + 1)^{1/2}, \quad \log_+ x := \max\{\log x, 0\}, \quad \log_- x := \max\{-\log x, 0\}.$$

We let $\chi = \chi(x)$ be a smooth spherically symmetric cutoff function such that $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$. For $A > 0$, we define its rescaled version by $\chi_A(x) := \chi(\frac{x}{A})$. We also define $\chi_{\geq A} := 1 - \chi_A$.

We mainly work with equivariant functions on $\mathbb{R}^2$, say $\phi: \mathbb{R}^2 \rightarrow \mathbb{C}$, or equivalently their radial part $u: \mathbb{R}_+ \rightarrow \mathbb{C}$ with $\phi(x) = u(r)e^{im\theta}$, where $\mathbb{R}_+ := (0, \infty)$ and $x_1 + ix_2 = re^{i\theta}$. We denote by $\partial_+ = \partial_1 + i\partial_2$ the Cauchy–Riemann operator, which acts on m -equivariant function via the formula

$$\partial_+[u(r)e^{im\theta}] = \left[\left(\partial_r - \frac{m}{r} \right) u \right](r) e^{i(m+1)\theta}. \quad (2.1)$$

We denote by $\Delta^{(m)} = \partial_{rr} + \frac{1}{r}\partial_r - \frac{m^2}{r^2}$ the Laplacian acting on m -equivariant functions.

The integral symbol $\int$ means

$$\int = \int_{\mathbb{R}^2} dx = 2\pi \int r dr.$$

For complex-valued functions f and g , we define their *real* inner product by

$$(f, g)_r := \int \operatorname{Re}(\bar{f}g).$$

For a real-valued functional F and a function u , we denote by $\nabla F[u]$ the *functional derivative* of F at u under this real inner product.

For $s \in \mathbb{R}$, denote by Λ_s the generator of $\dot{H}^s$ -scaling

$$\Lambda_s := r\partial_r + 1 - s, \quad \Lambda := \Lambda_0 = r\partial_r + 1.$$

Note that Λ is formally anti-symmetric. We also use a truncated version of Λ ; for a spherically symmetric function $\psi = \psi(x)$ on $\mathbb{R}^2$, we define the operator

$$\Lambda_\psi := \psi' \partial_r + \frac{\Delta \psi}{2},$$

where $\psi' = \partial_r \psi$ and $\Delta \psi = \partial_{rr} \psi + \frac{1}{r} \partial_r \psi$. Note that Λ_ψ is formally anti-symmetric.

For $k \in \mathbb{N}$, we define

$$\begin{aligned} |f|_k &:= \max\{|f|, |r\partial_r f|, \dots, |(r\partial_r)^k f|\}, \\ |f|_{-k} &:= \max\left\{|\partial_r^k f|, \left|\frac{1}{r}\partial_r^{k-1} f\right|, \dots, \left|\frac{1}{r^k} f\right|\right\}. \end{aligned} \quad (2.2)$$

We note that $|f|_k \sim r^k |f|_{-k}$. The following Leibniz rules hold:

$$|fg|_k \lesssim |f|_k |g|_k, \quad |fg|_{-k} \lesssim |f|_k |g|_{-k}.$$

Given a scaling parameter $\lambda \in \mathbb{R}_+$, phase rotation parameter $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$, and a function f , we write

$$f_{\lambda,\gamma}(r) := \frac{e^{i\gamma}}{\lambda} f\left(\frac{r}{\lambda}\right).$$

When λ and γ are clear from the context, we will also denote the above by $f^\sharp$ as in [27], that is,

$$f^\sharp := f_{\lambda,\gamma}.$$

Similarly, we define the $\flat$ -operation as

$$g^\flat := g_{1/\lambda, -\gamma}.$$

When a time-dependent scaling parameter $\lambda(t)$ is given, we define rescaled spacetime variables s, y by the relations

$$\frac{ds}{dt} = \frac{1}{\lambda^2(t)} \quad \text{and} \quad y = \frac{r}{\lambda(t)}. \quad (2.3)$$

The raising operation $\sharp$ converts a function $f = f(y)$ to a function of r : $f^\sharp = f^\sharp(r)$. The lowering operator $\flat$ does the reverse. In the modulation analysis in this paper, the dynamical parameters such as λ, γ, b, η are functions of either the variable t or s under $\frac{ds}{dt} = \frac{1}{\lambda^2}$.

We collect the definitions of various linear operators. Let $A_\theta[\psi_1, \psi_2]$ be the polarization of $A_\theta[\psi]$:

$$A_\theta[\psi_1, \psi_2] = -\frac{1}{2} \int_0^y \operatorname{Re}(\bar{\psi}_1 \psi_2) y' dy'.$$

We will often use the first-order operators and their formal L^2 -adjoints:

$$\begin{aligned} \mathbf{D}_w &= \partial_y - \frac{1}{y}(m + A_\theta[w]), & \mathbf{D}_w^* &= -\partial_y - \frac{1}{y}(m + 1 + A_\theta[w]), \\ L_w &= \mathbf{D}_w - \frac{2}{y}A_\theta[w, \cdot], & L_w^* &= \mathbf{D}_w^* + w \int_y^\infty \operatorname{Re}(\bar{w} \cdot) dy', \\ A_w &= \mathbf{D}_w - \frac{1}{y}, & A_w^* &= \mathbf{D}_w^* - \frac{1}{y}. \end{aligned}$$

The second-order operators of particular importance are

$$\begin{aligned} \mathcal{L}_w &= \nabla^2 E[w], \quad \text{i.e., the Hessian of } E, \\ H_w &= -\partial_{yy} - \frac{1}{y}\partial_y + \frac{1}{y^2}(m + 1 + A_\theta[w])^2 - \frac{1}{2}|w|^2 = A_w^* A_w, \\ \tilde{H}_w &= -\partial_{yy} - \frac{1}{y}\partial_y + \frac{1}{y^2}(m + 2 + A_\theta[w])^2 + \frac{1}{2}|w|^2 = A_w A_w^*. \end{aligned}$$

See Section 2.3 for more explanations of these linear operators.

The relevant function spaces such as $\dot{H}_m^k$, $\dot{\mathcal{H}}_m^k$, and $\dot{V}_m^k$, and their norms will be defined in Section 2.4.

2.2. Conjugation identities

In this subsection, we record the evolution equations for u , $u_1 = \mathbf{D}_u u$, and $u_2 = A_u \mathbf{D}_u u$, where we recall $\mathbf{D}_u = \partial_r - \frac{1}{r}(m + A_\theta[u])$ and $A_u = \mathbf{D}_u - \frac{1}{r}$. As mentioned in the strategy (Section 1.4), we will view (CSS) as a system of equations for u , u_1 , u_2 .

First, we note that u , u_1 , and u_2 are realized as m -, $(m+1)$ -, and $(m+2)$ -equivariant functions. This is because the covariant Cauchy–Riemann operator $\mathbf{D}_+ = \mathbf{D}_1 + i\mathbf{D}_2$ shifts the equivariance index by 1 (cf. (2.1)) and u , u_1 , u_2 are identified as

$$\begin{cases} \phi(x) = u(r)e^{im\theta}, \\ [\mathbf{D}_+ \phi](x) = [\mathbf{D}_u u](r)e^{i(m+1)\theta} = u_1(r)e^{i(m+1)\theta}, \\ [\mathbf{D}_+ \mathbf{D}_+ \phi](x) = [A_u \mathbf{D}_u u](r)e^{i(m+2)\theta} = u_2(r)e^{i(m+2)\theta}. \end{cases} \quad (2.4)$$

Now, we record the evolution equations for u , u_1 , and u_2 . The following can be derived using identification (2.4) and the curvature relations of (1.1):

$$\partial_t u + i\mathbf{D}_u^* u_1 + \left(\int_r^\infty \operatorname{Re}(\bar{u}u_1) dr' \right) iu = 0, \quad (2.5)$$

$$\partial_t u_1 + iA_u^* u_2 + \left(\int_r^\infty \operatorname{Re}(\bar{u}u_1) dr' \right) iu_1 = 0, \quad (2.6)$$

$$\partial_t u_2 + i\tilde{H}_u u_2 + \left(\int_r^\infty \operatorname{Re}(\bar{u}u_1) dr' \right) iu_2 - i\bar{u}u_1^2 = 0. \quad (2.7)$$

See [28, Section 2] for more details.

When (C^1) parameter curves $\lambda(t)$ and $\gamma(t)$ are given, using the renormalized coordinates (s, y) defined in (2.3), we consider the renormalized variables w , w_1 , w_2 of u , u_1 , u_2 ,

$$\begin{aligned} w(s, y) &:= \lambda e^{-i\gamma} u(t, \lambda r)|_{t=t(s)}, \\ w_1(s, y) &:= \lambda^2 e^{-i\gamma} u_1(t, \lambda r)|_{t=t(s)} = \mathbf{D}_w w, \\ w_2(s, y) &:= \lambda^3 e^{-i\gamma} u_2(t, \lambda r)|_{t=t(s)} = A_w \mathbf{D}_w w. \end{aligned}$$

These variables solve the following evolution equations:

$$\left(\partial_s - \frac{\lambda_s}{\lambda} \Lambda + \gamma_s i \right) w + i\mathbf{D}_w^* w_1 + \left(\int_y^\infty \operatorname{Re}(\bar{w}w_1) dy' \right) iw = 0, \quad (2.8)$$

$$\left(\partial_s - \frac{\lambda_s}{\lambda} \Lambda_{-1} + \gamma_s i \right) w_1 + iA_w^* w_2 + \left(\int_y^\infty \operatorname{Re}(\bar{w}w_1) dy' \right) iw_1 = 0, \quad (2.9)$$

$$\left(\partial_s - \frac{\lambda_s}{\lambda} \Lambda_{-2} + \gamma_s i \right) w_2 + i\tilde{H}_w w_2 + \left(\int_y^\infty \operatorname{Re}(\bar{w}w_1) dy' \right) iw_2 - i\bar{w}w_1^2 = 0. \quad (2.10)$$

2.3. Linearization

We quickly record the linearization of (CSS) around Q . For a more detailed exposition, see the corresponding sections of [26–28]. We note that the discussion in this subsection formally assumes the decomposition

$$u(t, r) = Q(r) + \varepsilon(t, r)$$

for small ε and hence we will identify $t = s$ and $r = y$. One may consider the situation where $\lambda(t) \equiv 1$ and $\gamma(t) \equiv 0$.

2.3.1. Linearization of the Bogomol'nyi operator. We first linearize the Bogomol'nyi operator $w \mapsto \mathbf{D}_w w$. We can write

$$\mathbf{D}_{w+\varepsilon}(w + \varepsilon) = \mathbf{D}_w w + L_w \varepsilon + N_w(\varepsilon), \quad (2.11)$$

where

$$\begin{aligned} L_w \varepsilon &:= \mathbf{D}_w \varepsilon - \frac{2}{y} A_\theta[w, \varepsilon]w, \\ N_w(\varepsilon) &:= -\frac{2}{y} A_\theta[w, \varepsilon]\varepsilon - \frac{1}{y} A_\theta[\varepsilon]w, \end{aligned}$$

and $A_\theta[\psi_1, \psi_2]$ is defined through polarization

$$A_\theta[\psi_1, \psi_2] := -\frac{1}{2} \int_0^y \operatorname{Re}(\bar{\psi}_1 \psi_2) y' dy'.$$

The L^2 -adjoint L_w^* of L_w takes the form

$$L_w^* v = \mathbf{D}_w^* v + w \int_y^\infty \operatorname{Re}(\bar{w} v) dy'.$$

We remark that the operator L_w and its adjoint L_w^* are *only* $\mathbb{R}$ -linear. From (1.8) and (1.7), we have the following expansion for the energy:

$$E[Q + \varepsilon] = \frac{1}{2} \|L_Q \varepsilon\|_{L^2}^2 + \text{higher-order terms}. \quad (2.12)$$

2.3.2. Linearization of (CSS) and generalized kernel relations. Next, we linearize (CSS). Recall the Hamiltonian form (1.4),

$$\partial_t u + i \nabla E[u] = 0.$$

We decompose

$$\nabla E[w + \varepsilon] = \nabla E[w] + \mathcal{L}_w \varepsilon + R_w(\varepsilon), \quad (2.13)$$

where $\mathcal{L}_w \varepsilon$ collects the linear terms in ε and $R_w(\varepsilon)$ collects the remainders. Note that $\mathcal{L}_w$ is the Hessian of E , i.e.,

$$\nabla^2 E[w] = \mathcal{L}_w,$$

and hence $\mathcal{L}_w$ is formally symmetric with respect to the real inner product. If one recalls (1.7), we have $\nabla E[u] = L_u^* \mathbf{D}_u u = L_u^* u_1$, and hence (CSS) can also be written as (compare with (2.5))

$$i \partial_t u + L_u^* u_1 = 0.$$

Note that (2.13) and (2.11) yield the following full expression of $\mathcal{L}_w$

$$\begin{aligned} \mathcal{L}_w \varepsilon &= L_w^* L_w \varepsilon + \left(\frac{1}{y} \int_0^y \operatorname{Re}(\bar{w} \varepsilon) y' dy' \right) w_1 + w \int_y^\infty \operatorname{Re}(\bar{\varepsilon} w_1) dy' \\ &\quad + \varepsilon \int_y^\infty \operatorname{Re}(\bar{w} w_1) dy', \end{aligned}$$

where $w_1 = \mathbf{D}_w w$, but this full expression is not important in our analysis. We remark that the operator $\mathcal{L}_w$ is only $\mathbb{R}$ -linear. From $\mathbf{D}_Q Q = 0$, we observe the *self-dual factorization* of $i \mathcal{L}_Q$,⁸

$$i \mathcal{L}_Q = i L_Q^* L_Q.$$

Thus the linearization of (CSS) at Q is

$$\partial_t \varepsilon + i \mathcal{L}_Q \varepsilon \approx 0 \quad \text{or} \quad \partial_t \varepsilon + i L_Q^* L_Q \varepsilon \approx 0. \quad (2.14)$$

We briefly recall the formal generalized kernel relations of the linearized operator $i \mathcal{L}_Q$. We have

$$N_g(i \mathcal{L}_Q) = \operatorname{span}_{\mathbb{R}} \left\{ \Lambda Q, i Q, \frac{i}{4} y^2 Q, \rho \right\}$$

with the relations (see [27, Proposition 3.4])

$$\begin{cases} i \mathcal{L}_Q \left(i \frac{y^2}{4} Q \right) = \Lambda Q, & i \mathcal{L}_Q \rho = i Q, \\ i \mathcal{L}_Q (\Lambda Q) = 0, & i \mathcal{L}_Q (i Q) = 0, \end{cases} \quad (2.15)$$

where the existence of ρ is given in [27, Lemma 3.6]. In fact, we have

$$\begin{cases} L_Q \left(i \frac{y^2}{4} Q \right) = i \frac{y}{2} Q, & L_Q \rho = \frac{1}{2(m+1)} y Q, \\ L_Q^* \left(i \frac{y}{2} Q \right) = -i \Lambda Q, & L_Q^* \left(\frac{1}{2(m+1)} y Q \right) = Q, \\ L_Q (\Lambda Q) = 0, & L_Q (i Q) = 0. \end{cases} \quad (2.16)$$

One can uniquely determine ρ by requiring $\rho \lesssim y^2 Q$ and ρ can be explicitly constructed by the formula

$$\rho := {}^{(\text{out})} L_Q^{-1} \left(\frac{1}{2(m+1)} y Q \right), \quad (2.17)$$

where ${}^{(\text{out})} L_Q^{-1}$ denotes a formal right inverse of L_Q whose explicit formula is given in Lemma A.1.

⁸This identity was first observed by Lawrie, Oh, and Shahshahani in their unpublished note.

2.3.3. *Linear conjugation identities and Morawetz repulsivity.* Assume that $u = Q + \varepsilon$. Formula (2.11) says that the first conjugated variable $u_1 = \mathbf{D}_u u$ satisfies

$$u_1 = \mathbf{D}_u u \approx L_Q \varepsilon =: \hat{\varepsilon}_1.$$

This u_1 -equation (2.6) can be linearized as

$$\partial_t \hat{\varepsilon}_1 + i A_Q^* A_Q \hat{\varepsilon}_1 \approx 0. \quad (2.18)$$

In this regard, the operator H_w naturally arises

$$\begin{aligned} H_w &:= A_w^* A_w = -\partial_{yy} - \frac{1}{y} \partial_y + \frac{1}{y^2} V_w, \\ V_w &:= (m+1 + A_\theta[w])^2 - \frac{1}{2} y^2 |w|^2. \end{aligned}$$

On the other hand, if we simply take L_Q to (2.14), we also have

$$\partial_t \hat{\varepsilon}_1 + i L_Q i L_Q^* \hat{\varepsilon}_1 \approx 0.$$

Comparing this with (2.18) gives the *linear conjugation identity*

$$L_Q i L_Q^* = i A_Q^* A_Q = i H_Q.$$

The linear conjugation identity was first observed in [26]. While the linearized operator $\mathcal{L}_w$ is only $\mathbb{R}$ -linear and is nonlocal, H_w is now $\mathbb{C}$ -linear and local. Moreover, the kernels ΛQ and $i Q$ of $i \mathcal{L}_Q$ have been removed by taking L_Q to (2.14), and the generalized kernels $i \frac{y^2}{4} Q$ and $(m+1)\rho$ have become the kernels $i \frac{y}{2} Q$ and $\frac{y}{2} Q$ of $i H_Q$, thanks to (2.16) and $A_Q(yQ) = 0$. More interestingly, $H_Q = A_Q^* A_Q$ is exactly the same operator arising in the linearization of several critical geometric equations around harmonic maps within $k = (m+1)$ -equivariance, such as in [45, 52, 53, 56]. Therefore, at the linearized level, the dynamics of m -equivariant (CSS) is closely related to the dynamics of $(m+1)$ -equivariant Schrödinger maps. These earlier works motivated the authors in [26] to consider further linearly conjugated variable $\hat{\varepsilon}_2 = A_Q \hat{\varepsilon}_1 = A_Q L_Q \varepsilon$ below.

The further conjugated variable u_2 can be linearized as

$$u_2 = A_u \mathbf{D}_u u \approx A_Q \hat{\varepsilon}_1 =: \hat{\varepsilon}_2,$$

and the linearization of u_2 -equation (2.7) becomes

$$\partial_t \hat{\varepsilon}_2 + i A_Q A_Q^* \hat{\varepsilon}_2 \approx 0. \quad (2.19)$$

In this regard, we define the linear operator $\tilde{H}_w$ by

$$\tilde{H}_w := A_w A_w^* = -\partial_{yy} - \frac{1}{y} \partial_y + \frac{1}{y^2} \tilde{V}_w, \quad (2.20)$$

$$\tilde{V}_w := (m+2 + A_\theta[w])^2 + \frac{1}{2} y^2 |w|^2. \quad (2.21)$$

Note that the kernels iyQ and yQ of iH_Q have been removed by taking A_Q to (2.18) and the operator $\tilde{H}_Q$ has no kernels.

More importantly, as observed in [56] for the wave maps (and in [26] in the present context), $\tilde{H}_Q$ has a *repulsive* character,

$$\tilde{V}_Q \geq m^2 \quad \text{and} \quad -y\partial_y \tilde{V}_Q = y^2 Q^2 \geq 0. \quad (2.22)$$

One instance of the usefulness of this repulsivity is the following spacetime control of $\hat{e}_2$ obtained through a virial-type computation:

$$\begin{aligned} \frac{1}{2} \partial_t (\hat{e}_2, -i\Lambda \hat{e}_2)_r &= \left(\hat{e}_2, \frac{1}{2} [\tilde{H}_Q, \Lambda] \hat{e}_2 \right)_r = (\hat{e}_2, \tilde{H}_Q \hat{e}_2)_r + \left(\hat{e}_2, \frac{-\partial_y \tilde{V}}{2y} \hat{e}_2 \right)_r \\ &\geq (\hat{e}_2, \tilde{H}_Q \hat{e}_2)_r = \|A_Q^* \hat{e}_2\|_{L^2}^2. \end{aligned}$$

As mentioned in the introduction, repulsivity (2.22) has been used in many of the blow-up constructions. Note that the virial term $(\hat{e}_2, -i\Lambda \hat{e}_2)_r$ above is not bounded in the usual Sobolev spaces (e.g., for $\hat{e}_2 \in H^1$). In this work, we will use a *Morawetz-type* monotonicity as follows.

Lemma 2.1 (Morawetz-type repulsivity). *Let $m \geq 1$. There exists a smooth function $\psi: (0, \infty) \rightarrow \mathbb{R}$ with the following properties:*

- ψ admits a smooth radially symmetric extension on $\mathbb{R}^2$.
- (Bounds on ψ') Denote $\psi' := \partial_y \psi$. We have $0 \leq \psi' \leq 1$ and

$$y\partial_y \psi' \lesssim \langle y \rangle^{-1}. \quad (2.23)$$

- (Morawetz-type repulsivity) Recall the operator $\Lambda_\psi = \psi' \partial_y + \frac{\Delta \psi}{2}$. There exists $c > 0$ such that for any smooth $(m+2)$ -equivariant function f , we have

$$(\tilde{H}_Q f, \Lambda_\psi f)_r \geq c \int \left(\frac{1}{\langle y \rangle^2} |\partial_y f|^2 + \frac{1}{\langle y \rangle} \left| \frac{f}{y} \right|^2 \right) =: c \|f\|_{\dot{V}_{m+2}^{3/2}}^2. \quad (2.24)$$

Remark 2.2 (On $m \geq 1$). In the proof, we heavily rely on the assumption $m \geq 1$ to obtain the *nonnegativity* of $(\tilde{H}_Q f, \Lambda_\psi f)_r$. When $m = 0$, $\tilde{H}_Q$ is rather similar to the 2D Laplacian $-\Delta_0 = -\partial_{yy} - \frac{1}{y} \partial_y$ near the spatial infinity, and hence there are some subtleties to obtain a Morawetz estimate due to the zero resonance of Δ_0 . Note that the blow-up construction in [28] for the $m = 0$ case (and similarly in [45, 52] for critical wave maps and Schrödinger maps) rather constructs precise correction terms for the modified energy under the shrinking regime $\partial_t \lambda(t) < 0$, which we do not assume in the present context.

Remark 2.3. Bound (2.23) will be used in the energy estimates of our blow-up analysis (Proposition 4.12). As is well known, if one only considers a Morawetz estimate for the linear equation (2.19) without requiring bound (2.23), then the weight $\langle y \rangle^{-2}$ of (2.24) can be improved to $\langle y \rangle^{-1+\delta}$ by considering, for example, $\psi(y) = \langle y \rangle - \delta \langle y \rangle^{1-\delta}$ with small $\delta > 0$.

Proof of Lemma 2.1. We consider ψ of the form

$$\psi' = \frac{y}{\langle y \rangle} - \delta \frac{y}{\langle y \rangle^2} =: \hat{\psi}' + \delta \tilde{\psi}' \quad (2.25)$$

for some small $0 < \delta < 1$ to be chosen below. We may explicitly set $\psi(y) = \langle y \rangle - \frac{\delta}{2} \log(1 + y^2)$ and hence ψ admits a smooth radially symmetric extension on $\mathbb{R}^2$. Moreover, it is clear from (2.25) that ψ' satisfies $0 \leq \psi' \leq 1$ and $y \partial_y \psi' \lesssim \langle y \rangle^{-1}$.

We turn to the proof of monotonicity (2.24). Integrating by parts, one has

$$(\tilde{H}_Q f, \Lambda_\psi f)_r = \int \psi'' |\partial_y f|^2 + \int \left\{ \frac{\psi'}{y} \left(\tilde{V}_Q - \frac{y \partial_y \tilde{V}_Q}{2} \right) - \frac{y^2 \Delta^2 \psi}{4} \right\} \left| \frac{f}{y} \right|^2.$$

Applying $\psi' \geq 0$ and (2.22), we have

$$(\tilde{H}_Q f, \Lambda_\psi f)_r \geq \int \psi'' |\partial_y f|^2 + \int \left\{ \frac{\psi'}{y} - \frac{y^2 \Delta^2 \psi}{4} \right\} \left| \frac{f}{y} \right|^2. \quad (2.26)$$

On the one hand, we have

$$\hat{\psi}'' = \frac{1}{\langle y \rangle^3} \quad \text{and} \quad \frac{\hat{\psi}'}{y} - \frac{y^2 \Delta^2 \hat{\psi}}{4} = \frac{3y^6 + 4y^4 + 20y^2 + 4}{4\langle y \rangle^7} \gtrsim \frac{1}{\langle y \rangle}.$$

On the other hand, we have

$$\left| \tilde{\psi}'' - \frac{1}{\langle y \rangle^2} \right| \lesssim \frac{1}{\langle y \rangle^4} \quad \text{and} \quad \left| \frac{\tilde{\psi}'}{y} - \frac{y^2 \Delta^2 \tilde{\psi}}{4} \right| \lesssim \frac{1}{\langle y \rangle^2}.$$

Therefore, there exists some $0 < \delta < 1$ such that $\psi' = \hat{\psi}' + \delta \tilde{\psi}'$ satisfies

$$\psi'' \gtrsim \frac{1}{\langle y \rangle^2} \quad \text{and} \quad \frac{\psi'}{y} - \frac{y^2 \Delta^2 \psi}{4} \gtrsim \frac{1}{\langle y \rangle}. \quad (2.27)$$

Substituting (2.27) into (2.26) gives (2.24). $\blacksquare$

2.4. Adapted function spaces

In this subsection, we quickly recall the definition of equivariant Sobolev spaces and introduce adapted function spaces. For more explanations, see [26, 27].

Let $m \in \mathbb{Z}$. For $s \geq 0$, we denote by H_m^s and $\dot{H}_m^s$ the restriction of the usual Sobolev spaces $H^s(\mathbb{R}^2)$ and $\dot{H}^s(\mathbb{R}^2)$ on m -equivariant functions. For high equivariance indices, we have the generalized Hardy's inequality [27, Lemma A.7]: whenever $0 \leq k \leq |m|$, we have

$$\| |f|_{-k} \|_{L^2} \sim \|f\|_{\dot{H}_m^k}.$$

Specializing this to $k = 1$, we have the *Hardy–Sobolev inequality* [27, Lemma A.6]: whenever $|m| \geq 1$, we have

$$\|r^{-1} f\|_{L^2} + \|f\|_{L^\infty} \lesssim \|f\|_{\dot{H}_m^1}. \quad (2.28)$$

Note that in general $H_0^1 \hookrightarrow L^\infty$ is *false*. For $k \in \mathbb{N}$, the space $H_m^{k,k}$ is defined by the set of m -equivariant $H^{k,k}(\mathbb{R}^2)$ functions, where $f \in H^{k,k}(\mathbb{R}^2)$ means that $\langle x \rangle^\ell f \in H^{k-\ell}(\mathbb{R}^2)$ for all $0 \leq \ell \leq k$.

Let $m \geq 1$. We turn to the definitions of adapted function spaces. As mentioned before, we will need a priori bounds (such as (1.24)) from mass/energy conservation and perform an energy estimate for ε_2 to prove the H^3 -boundedness of $\varepsilon^\sharp$. For this purpose, we need to transfer the controls on $\varepsilon_2 \approx \hat{\varepsilon}_2 = A_Q L_Q \varepsilon$ and $\varepsilon_1 \approx \hat{\varepsilon}_1 = L_Q \varepsilon$ to those for ε . The controls on ε and hence the norms of the adapted function spaces will be captured by various (sub-)coercivity estimates for the linear operators A_Q and L_Q . Note that, as A_Q and L_Q have nontrivial kernels $\text{span}_{\mathbb{R}}\{yQ, iyQ\}$ and $\text{span}_{\mathbb{R}}\{\Lambda Q, iQ\}$, respectively, the coercivity estimates (Lemma 2.4 below) hold after excluding these kernel elements.

The following norms will be used in this paper. Recall from (2.4) that ε , ε_1 , ε_2 are naturally identified as m -, $(m+1)$ -, $(m+2)$ -equivariant functions. For ε_2 , we will use the norms

$$\|\varepsilon_2\|_{L^2}, \quad \|\varepsilon_2\|_{\dot{H}_{m+2}^1}, \quad \text{and} \quad \|\varepsilon_2\|_{\dot{V}_{m+2}^{3/2}},$$

where we recall the definition of the $\dot{V}_{m+2}^{3/2}$ -norm (see (2.24)),

$$\|\varepsilon_2\|_{\dot{V}_{m+2}^{3/2}}^2 := \left\| \frac{\partial_y \varepsilon_2}{\langle y \rangle} \right\|_{L^2}^2 + \left\| \frac{\varepsilon_2}{\langle y \rangle^{1/2} y} \right\|_{L^2}^2. \quad (2.29)$$

The first two norms are adapted to the coercivity of the energy functional $\mathcal{F}$ introduced in (1.25). The last norm captures the spatial part of the Morawetz estimate in Lemma 2.1.

For ε_1 , we will use the norms

$$\|\varepsilon_1\|_{L^2}, \quad \|\varepsilon_1\|_{\dot{H}_{m+1}^1}, \quad \|\varepsilon_1\|_{\dot{H}_{m+1}^2}, \quad \text{and} \quad \|\varepsilon_1\|_{\dot{V}_{m+1}^{5/2}},$$

where the last $\dot{V}_{m+1}^{5/2}$ -norm is defined by

$$\|\varepsilon_1\|_{\dot{V}_{m+1}^{5/2}} := \left\| \frac{\partial_{yy} \varepsilon_1}{\langle y \rangle} \right\|_{L^2}^2 + \left\| \frac{|\varepsilon_1| - 1}{\langle y \rangle^{1/2} y} \right\|_{L^2}^2.$$

The first norm, i.e., $\|\varepsilon_1\|_{L^2}$, is adapted to energy conservation (see (2.12)). The remaining high Sobolev norms are adapted to the L^2 , $\dot{H}_{m+2}^1$, and $\dot{V}_{m+2}^{3/2}$ controls on $\varepsilon_2 \approx A_Q \varepsilon_1$; see item (2) of Lemma 2.4 below.

Finally, for ε , we will use the norms

$$\|\varepsilon\|_{L^2}, \quad \|\varepsilon\|_{\dot{H}_m^1}, \quad \|\varepsilon\|_{\dot{\mathcal{H}}_m^2}, \quad \text{and} \quad \|\varepsilon\|_{\dot{\mathcal{H}}_m^3},$$

where the $\dot{\mathcal{H}}_m^2$ - and $\dot{\mathcal{H}}_m^3$ -norms are defined by

$$\|\varepsilon\|_{\dot{\mathcal{H}}_m^2} := \begin{cases} \|\varepsilon\|_{\dot{H}_m^2} & \text{if } m \geq 2, \\ \|\partial_{+\varepsilon}\|_{\dot{H}_2^1} + \|\partial_{yy} \varepsilon\|_{L^2} + \left\| \frac{|\varepsilon| - 1}{(\log_- y) y} \right\|_{L^2} & \text{if } m = 1 \end{cases}$$

and

$$\|\varepsilon\|_{\dot{\mathcal{H}}_m^3} := \begin{cases} \|\varepsilon|_{-3}\|_{L^2} \sim \|\varepsilon\|_{\dot{H}_m^3} & \text{if } m \geq 3, \\ \|\partial_+\varepsilon\|_{\dot{H}_3^2} + \|\partial_{yy}\varepsilon\|_{L^2} + \left\| \frac{|\varepsilon|_{-2}}{\langle \log_- y \rangle y} \right\|_{L^2} & \text{if } m = 2, \\ \|\partial_+\varepsilon\|_{\dot{H}_2^2} + \|\partial_{yy}\varepsilon|_{-1}\|_{L^2} + \left\| \frac{|\varepsilon|_{-1}}{\langle \log_- y \rangle \langle y \rangle y} \right\|_{L^2} & \text{if } m = 1. \end{cases}$$

Of course, the L^2 -norm of ε is adapted to mass conservation. The remaining higher-order norms are adapted to the L^2 , $\dot{H}_{m+1}^1$, and $\dot{H}_{m+1}^2$ controls on $\varepsilon_1 \approx L_Q \varepsilon$; see the last item of Lemma 2.4 below. When m is small, the above $\dot{\mathcal{H}}_m^k$ -norms can be viewed as slight modifications of the usual $\dot{H}_m^k$ -norms. However, these $\dot{\mathcal{H}}_m^k$ -norms control the scaling critical Hardy norms in the region $y \geq 1$,

$$\|\varepsilon\|_{\dot{\mathcal{H}}_m^2} \gtrsim \|\mathbf{1}_{y \geq 1} |\varepsilon|_{-2}\|_{L^2} \quad \text{and} \quad \|\varepsilon\|_{\dot{\mathcal{H}}_m^3} \gtrsim \|\mathbf{1}_{y \geq 1} |\varepsilon|_{-3}\|_{L^2}.$$

In this sense, the $\dot{\mathcal{H}}_m^k$ -norms are stronger than the $\dot{H}_m^k$ -norms when m is small.

The following lemma collects the coercivity estimates of L_Q and A_Q .

Lemma 2.4 (Linear coercivity estimates). *Let $m \geq 1$.*

(1) (Unconditional coercivity of $\tilde{H}_Q$) *We have*

$$(\varepsilon_2, \tilde{H}_Q \varepsilon_2)_r \geq \|\partial_y \varepsilon_2\|_{L^2}^2 + \left\| \frac{1}{y} \varepsilon_2 \right\|_{L^2}^2. \quad (2.30)$$

(2) (Coercivity of A_Q) *Suppose that⁹ $\tilde{\mathcal{Z}}_3, \tilde{\mathcal{Z}}_4 \in C_{c,m+1}^\infty$ are such that the 2×2 matrix $\{(a_{kj})\}_{k,j \in \{3,4\}}$ defined by $a_{k3} = (\tilde{\mathcal{Z}}_k, iyQ)_r$ and $a_{k4} = (\tilde{\mathcal{Z}}_k, yQ)_r$ has nonzero determinant. Then, we have coercivity estimates*

$$\|\varepsilon_1\|_{\dot{H}_{m+1}^1} \lesssim_{\tilde{\mathcal{Z}}_3, \tilde{\mathcal{Z}}_4} \|A_Q \varepsilon_1\|_{L^2} \lesssim \|\varepsilon_1\|_{\dot{H}_{m+1}^1} \quad \forall \varepsilon_1 \in \dot{H}_{m+1}^1 \cap \{\tilde{\mathcal{Z}}_3, \tilde{\mathcal{Z}}_4\}^\perp, \quad (2.31)$$

$$\|\varepsilon_1\|_{\dot{H}_{m+1}^2} \lesssim_{\tilde{\mathcal{Z}}_3, \tilde{\mathcal{Z}}_4} \|A_Q \varepsilon_1\|_{\dot{H}_{m+2}^1} \lesssim \|\varepsilon_1\|_{\dot{H}_{m+1}^2} \quad \forall \varepsilon_1 \in \dot{H}_{m+1}^2 \cap \{\tilde{\mathcal{Z}}_3, \tilde{\mathcal{Z}}_4\}^\perp, \quad (2.32)$$

$$\|\varepsilon_1\|_{\dot{V}_{m+1}^{5/2}} \lesssim_{\tilde{\mathcal{Z}}_3, \tilde{\mathcal{Z}}_4} \|A_Q \varepsilon_1\|_{\dot{V}_{m+2}^{3/2}} \lesssim \|\varepsilon_1\|_{\dot{V}_{m+1}^{5/2}} \quad \forall \varepsilon_1 \in \dot{V}_{m+1}^{5/2} \cap \{\tilde{\mathcal{Z}}_3, \tilde{\mathcal{Z}}_4\}^\perp, \quad (2.33)$$

where $\perp$ is taken with respect to the real inner product $(\cdot, \cdot)_r$.

(3) (Coercivity of L_Q) *Suppose $\mathcal{Z}_1, \mathcal{Z}_2 \in C_{c,m}^\infty$ are such that the matrix $\{(a_{kj})\}_{k,j \in \{1,2\}}$ defined by $a_{k1} = (\mathcal{Z}_k, \Lambda Q)_r$ and $a_{k2} = (\mathcal{Z}_k, iQ)_r$ has nonzero determinant. Then, we have coercivity estimates*

$$\|\varepsilon\|_{\dot{H}_m^1} \lesssim_{\mathcal{Z}_1, \mathcal{Z}_2} \|L_Q \varepsilon\|_{L^2} \lesssim \|\varepsilon\|_{\dot{H}_m^1} \quad \forall \varepsilon \in \dot{H}_m^1 \cap \{\mathcal{Z}_1, \mathcal{Z}_2\}^\perp, \quad (2.34)$$

$$\|\varepsilon\|_{\dot{\mathcal{H}}_m^2} \lesssim_{\mathcal{Z}_1, \mathcal{Z}_2} \|L_Q \varepsilon\|_{\dot{H}_{m+1}^1} \lesssim \|\varepsilon\|_{\dot{\mathcal{H}}_m^2} \quad \forall \varepsilon \in \dot{\mathcal{H}}_m^2 \cap \{\mathcal{Z}_1, \mathcal{Z}_2\}^\perp, \quad (2.35)$$

$$\|\varepsilon\|_{\dot{\mathcal{H}}_m^3} \lesssim_{\mathcal{Z}_1, \mathcal{Z}_2} \|L_Q \varepsilon\|_{\dot{H}_{m+1}^2} \lesssim \|\varepsilon\|_{\dot{\mathcal{H}}_m^3} \quad \forall \varepsilon \in \dot{\mathcal{H}}_m^3 \cap \{\mathcal{Z}_1, \mathcal{Z}_2\}^\perp, \quad (2.36)$$

where $\perp$ is taken with respect to the real inner product $(\cdot, \cdot)_r$.

⁹That is, $\tilde{\mathcal{Z}}_3$ and $\tilde{\mathcal{Z}}_4$ have smooth compactly supported $(m+1)$ -equivariant extensions on $\mathbb{R}^2$.

See Appendix B. Some of these estimates are already proved in [26].

Remark 2.5 (Choice of $\mathcal{Z}_1, \mathcal{Z}_2, \tilde{\mathcal{Z}}_3, \tilde{\mathcal{Z}}_4$). In this work, we fix the orthogonality profiles $\mathcal{Z}_1, \mathcal{Z}_2 \in C_{c,m}^\infty$ and $\tilde{\mathcal{Z}}_3, \tilde{\mathcal{Z}}_4 \in C_{c,m+1}^\infty$ satisfying the following properties. We require for $\mathcal{Z}_1, \mathcal{Z}_2 \in C_{c,m}^\infty$ that $\mathcal{Z}_1$ is real, $\mathcal{Z}_2$ is purely imaginary, and $\mathcal{Z}_1, \mathcal{Z}_2$ satisfy the transversality condition

$$(\Lambda Q, \mathcal{Z}_1)_r, (iQ, \mathcal{Z}_2)_r \neq 0 \quad (2.37)$$

and a gauge condition

$$\left(-i \frac{y^2}{4} Q, \mathcal{Z}_k\right)_r = (\rho, \mathcal{Z}_k)_r = 0 \quad \forall k \in \{1, 2\}. \quad (2.38)$$

Next, we fix $\tilde{\mathcal{Z}}_3, \tilde{\mathcal{Z}}_4 \in C_{c,m+1}^\infty$ such that

$$\tilde{\mathcal{Z}}_3 := -i \frac{1}{2} y Q \chi_{R_o} \quad \text{and} \quad \tilde{\mathcal{Z}}_4 := -\frac{1}{2} y Q \chi_{R_o}, \quad (2.39)$$

where $R_o > 1$ is some radius such that

$$\|\mathbf{1}_{[R_o, \infty)} y Q\|_{L^2} \leq \frac{1}{2} \|y Q\|_{L^2}, \quad (2.40)$$

and the supports of $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are contained in $y \leq 2R_o$. It is clear that Lemma 2.4 is applicable with this choice of orthogonality profiles.

2.5. Weighted L^∞ and integral estimates

In this subsection, we record weighted L^∞ and integral estimates, which will be very useful in our analysis.

Lemma 2.6 (Weighted L^∞ estimates). *For m -equivariant functions ε , we have*

$$\|\varepsilon\|_{L^\infty} \lesssim \|\varepsilon\|_{\dot{H}_m^1}, \quad \|\langle y \rangle^{-1} \varepsilon\|_{L^\infty} \lesssim \|\varepsilon\|_{\dot{H}_m^2}, \quad \|\langle y \rangle^{-2} \varepsilon\|_{L^\infty} \lesssim \|\varepsilon\|_{\dot{H}_m^3}. \quad (2.41)$$

For $(m+1)$ -equivariant functions ε_1 , we have

$$\begin{aligned} \|\varepsilon_1\|_{L^\infty} &\lesssim \|\varepsilon_1\|_{\dot{H}_{m+1}^1}, \quad \|\langle y \rangle^{-1} \varepsilon_1\|_{L^\infty} \lesssim \|\varepsilon_1\|_{\dot{H}_{m+1}^2}, \\ \|\langle y \rangle^{-3/4} \partial_y \varepsilon_1\|_{L^\infty} + \left\| \langle y \rangle^{-3/2} \frac{1}{y} \varepsilon_1 \right\|_{L^\infty} &\lesssim \|\varepsilon_1\|_{\dot{V}_{m+1}^{5/2}}. \end{aligned} \quad (2.42)$$

For $(m+2)$ -equivariant functions ε_2 , we have

$$\|\varepsilon_2\|_{L^\infty} \lesssim \|\varepsilon_2\|_{\dot{H}_{m+2}^1}, \quad \|\langle y \rangle^{-3/4} \varepsilon_2\|_{L^\infty} \lesssim \|\varepsilon_2\|_{\dot{V}_{m+1}^{3/2}}. \quad (2.43)$$

Remark 2.7. The controls in the region $y \lesssim 1$ are not sharp. It is also possible to control higher derivatives of ε . See, for example, [26, Appendix A]. We will not need such sharp controls in this paper.

Proof of Lemma 2.6. All the estimates easily follow from the fundamental theorem of calculus argument. We only prove the last inequality of (2.42), which is the most complicated, and omit the remaining proof. Integrating the inequalities

$$\begin{aligned}\partial_y(\langle y \rangle^{-3} y^{-2} |\varepsilon_1|^2) &\lesssim \langle y \rangle^{-3} y^{-2} \cdot |\varepsilon_1|_{-1} |\varepsilon_1|, \\ \partial_y(\langle y \rangle^{-3/2} |\partial_y \varepsilon_1|^2) &\lesssim \langle y \rangle^{-3/2} \cdot |\partial_y \varepsilon_1|_{-1} |\partial_y \varepsilon_1|\end{aligned}$$

backwards from the spatial infinity and using the Cauchy–Schwarz inequality, we get

$$\begin{aligned}\langle y \rangle^{-3} y^{-2} |\varepsilon_1|^2 &\lesssim \|\langle y \rangle^{-3/2} y^{-1} \cdot |\varepsilon_1|_{-1}\|_{L^2(ydy)} \|\langle y \rangle^{-3/2} y^{-1} \cdot \frac{1}{y} \varepsilon_1\|_{L^2(ydy)} \lesssim \|\varepsilon_1\|_{\dot{V}_{m+1}^{5/2}}^2, \\ \langle y \rangle^{-3/2} |\partial_y \varepsilon_1|^2 &\lesssim \|\langle y \rangle^{-1} \cdot |\partial_y \varepsilon_1|_{-1}\|_{L^2(ydy)} \|\langle y \rangle^{-1/2} \cdot \frac{1}{y} \partial_y \varepsilon_1\|_{L^2(ydy)} \lesssim \|\varepsilon_1\|_{\dot{V}_{m+1}^{5/2}}^2,\end{aligned}$$

which complete the proof of the last inequality of (2.42). $\blacksquare$

Lemma 2.8 (Mapping properties for integral operators [29]). *Let $1 \leq p, q \leq \infty$, and let $s \in [0, 2]$ be such that $(p, q, s) = (1, \infty, 0)$ or $\frac{1}{q} + 1 = \frac{1}{p} + \frac{s}{2}$ with $p > 1$. Then, we have*

$$\left\| \frac{1}{y^s} \int_0^y f(y') y' dy' \right\|_{L^q} \lesssim_p \|f\|_{L^p}. \quad (2.44)$$

Denoting by p' and q' the Hölder conjugates of p and q , respectively, we have the dual estimate

$$\left\| \int_y^\infty g(y') (y')^{1-s} dy' \right\|_{L^{p'}} \lesssim_p \|f\|_{L^{q'}}. \quad (2.45)$$

Remark 2.9. We will often use the above estimates when $p, q \in \{1, 2, \infty\}$.

Remark 2.10. If a nonnegative weight $w = w(y)$ is decreasing, then one can have a weighted version of (2.44). For example, we can have

$$\left\| \frac{w(y)}{y^s} \int_0^y f(y') y' dy' \right\|_{L^q} \leq \left\| \frac{1}{y^s} \int_0^y w(y') |f(y')| y' dy' \right\|_{L^q} \lesssim_p \|wf\|_{L^p}.$$

Similar argument works for (2.45) if w is increasing. We will implicitly use this simple argument in many of the estimates.

2.6. Proximity to soliton for small energy profiles

We recall a variational lemma from [29], which provides the starting point of our modulation analysis.

Lemma 2.11 (Proximity to Q for small energy profiles [29, Lemma 4.2]). *For any $M > 0$ and $\alpha^* > 0$, there exists $\delta > 0$ such that¹⁰ the following holds. Let $u \in H_m^1$ be a nonzero*

¹⁰Compared to [29, Lemma 4.2], we switched the roles of α^* and δ . This is because α^* in this paper will be used as a small parameter measuring the $\dot{H}_m^1$ -distance between the renormalized solution and the soliton Q .

profile satisfying $\|u\|_{L^2} \leq M$ and the small energy condition $\sqrt{E[u]} \leq \delta \|u\|_{\dot{H}_m^1}$. Then, there exists $\hat{\gamma} \in \mathbb{R}/2\pi\mathbb{Z}$ such that

$$\|e^{-i\hat{\gamma}} \hat{\lambda} u(\hat{\lambda} \cdot) - Q\|_{\dot{H}_m^1} < \alpha^*, \quad (2.46)$$

where $\hat{\lambda} = \frac{\|Q\|_{\dot{H}_m^1}}{\|u\|_{\dot{H}_m^1}}$.

Remark 2.12. Notice that we allow profiles having arbitrarily large mass and hence the use of the L^2 -norm is prohibited in (2.46). The above lemma is a consequence of the self-duality (1.7). More precisely, it is a consequence of the nonnegativity of energy and the uniqueness of solutions to the Bogomol'nyi equation (1.8) up to scaling and phase rotation invariances. For the mass-critical NLS, the analogous statement only holds under additional restriction on mass such as $\|\psi\|_{L^2} < \|R\|_{L^2} + \delta$ for the ground state R and in that case the L^2 -distance can also be made small.

Remark 2.13. Any H_m^1 finite-time blow-up solutions to (CSS) eventually enters the vicinity of modulated solitons. Indeed, if $u(t)$ is an H_m^1 -solution that blows up at $T < +\infty$, then $\|u(t)\|_{\dot{H}_m^1} \rightarrow \infty$ as $t \rightarrow T$ by the standard Cauchy theory. Therefore, for any $\delta > 0$ the small energy condition $\sqrt{E[u]} \leq \delta \|u\|_{\dot{H}_m^1}$ eventually holds by energy conservation. Note that $\|u(t)\|_{L^2}$ remains bounded by mass conservation.

3. Modified profiles

For the modulation analysis in the next section, we will decompose the full solution $u(t)$ of the form

$$u(t, r) = \frac{e^{i\gamma(t)}}{\lambda(t)} [P(\cdot; b(t), \eta(t)) + \varepsilon(t, \cdot)] \left(\frac{r}{\lambda(t)} \right).$$

(In fact, we decompose u , u_1 , u_2 as in (1.21).) In this section, we aim to construct the profiles $P(\cdot; b, \eta)$, which we call the *modified profiles*, and derive formal modulation equations for b and η . Note that the four parameters λ , γ , b , η correspond to the basis $\{\Lambda Q, iQ, -i\frac{\gamma^2}{4}Q, \rho\}$ of the formal generalized kernel of $i\mathcal{L}_Q$. The profile P will be chosen such that $P \approx Q$, $\partial_b P \approx -i\frac{\gamma^2}{4}Q$, and $\partial_\eta P \approx -(m+1)\rho$.

Basically, we will perform the tail computations in the spirit of [45, 46, 52] (and [28] for (CSS) when $m = 0$). We assume the *adiabatic ansatz*

$$\frac{\lambda_s}{\lambda} + b = 0 \quad \text{and} \quad \gamma_s \approx (m+1)\eta$$

and assume the expansions of b_s , η_s , and P , which roughly take the form

$$\begin{aligned} b_s &= \sum_{k+\ell \geq 2} c_{k,\ell}^{(b)} b^k \eta^\ell, & \eta_s &= \sum_{k+\ell \geq 2} c_{k,\ell}^{(\eta)} b^k \eta^\ell, \\ P &= Q - ib\frac{\gamma^2}{4}Q - (m+1)\eta\rho + \sum_{k+\ell \geq 2} b^k \eta^\ell T_{k,\ell} \end{aligned}$$

with some scalars $c_{k,\ell}^{(b)}$ and $c_{k,\ell}^{(\eta)}$, and profiles $T_{k,\ell} = T_{k,\ell}(y)$. By the tail computations, we seek for the choices of $c_{k,\ell}^{(b)}$ and $c_{k,\ell}^{(\eta)}$ such that $T_{k,\ell}$ grows the slowest.

We mention some observations in the profile construction of this work.

- In practice, we need to truncate the profiles P . In this work, we will truncate the profiles at the scale $y = B_1 = (b^2 + \eta^2)^{-1/2}$, which is much further than the self-similar scale $y = b^{-1/2}$, and *this scale turns out to be the only scale such that our analysis works for $m \in \{1, 2\}$* . This can be explained as follows. As mentioned in Section 1.4, our basic strategy for the rigidity theorem is to prove that the remainder part $\varepsilon^\#$ remains regular up to the blow-up time, or more specifically $\varepsilon^\#$ remains bounded in the H^3 -topology. Thus the blow-up part $P^\#$ should contain the singular part of the solution u , which turns out to be nontrivial when $m \in \{1, 2\}$ by Theorem 1.4. To recognize this from the blow-up part $P^\#$, the cutoff scale must be $r \sim 1$, which corresponds to $y \sim B_1$ in the modulation dynamics.
- When $m = 1$, there are nontrivial *cubic corrections* to the modulation equations of b and η . The profile ansatz from [26] as well as the modulation equations $b_s + b^2 + \eta^2 = 0$ and $\eta_s = 0$ are no longer correct; the tail computations suggest $b_s + b^2 + \eta^2 + p_3^{(b)} = 0$ and $\eta_s + p_3^{(\eta)} = 0$ instead, for some nonzero cubic polynomials $p_3^{(b)}$ and $p_3^{(\eta)}$. We believe in general that there are $(m + 2)$ -order corrections, which are tied to the spatial decay rate of Q as in [52], when $m \geq 1$.

Additionally, we will adopt the framework of [28]. Not only do we look at the variable w , we also look at the variables w_1 and w_2 ; thus in this section we will construct the profiles P , P_1 , P_2 adapted to the variables w , w_1 , w_2 . This strategy well separates the roles of each evolution equation. An important advantage in the profile construction is that the derivations of b_s - and η_s -modulation equations become much more direct at the level of the w_1 -equation. Another advantage is that P - and P_1 -ansatz do not need to be as precise as the P_2 -ansatz; it suffices to construct P , P_1 , and P_2 , up to quadratic, cubic, and quartic orders, respectively. At the same time, P_1 and P_2 enjoy degeneracies; the P_1 - and P_2 -expansions start from the linear and quadratic terms, respectively.

3.1. Old profile ansatz

In this subsection, we discuss the modified profiles as well as the modulation dynamics observed in [26, 27], from which pseudoconformal blow-up and rotational instability manifest.

In [26, 27], the authors studied the dynamics near pseudoconformal blow-up solutions. They used the profiles of the form

$$P_{\text{old}}(y; b, \eta) = \chi_B(y) e^{-iby^2/4} Q^{(\eta)}(y), \quad (3.1)$$

where $B = \infty$ (i.e., no cutoff) and $\eta \geq 0$ in [27], and $0 < b \ll 1$, $|\eta| \ll b$, and $B = b^{-1/2}$ in [26]. The profile $Q^{(\eta)}(y)$ is constructed in a way that it solves the *modified Bogomol'nyi equation* [27]

$$\mathbf{D}_{Q^{(m)}} Q^{(\eta)} = -\eta \frac{y}{2} Q^{(\eta)}$$

on $y \in (0, \infty)$ if $\eta \geq 0$ and $0 < y \ll |\eta|^{-1/2}$ if $\eta < 0$. This nonlinear construction of the profile effectively avoided the difficulties from nonlocal nonlinearities and yielded the formal modulation equations

$$\frac{\lambda_s}{\lambda} + b = 0, \quad \gamma_s \approx (m+1)\eta, \quad b_s + b^2 + \eta^2 = 0, \quad \eta_s = 0. \quad (3.2)$$

Typical solutions to (3.2), parametrized by $\eta_0 \in \mathbb{R}$, are given by

$$\begin{aligned} b(t) &= -t, \quad \eta(t) = \eta_0, \quad \lambda(t) = (t^2 + \eta_0^2)^{1/2}, \\ \gamma(t) &= \begin{cases} 0 & \text{if } \eta_0 = 0, \\ \operatorname{sgn}(\eta_0)(m+1) \tan^{-1}\left(\frac{t}{|\eta_0|}\right) & \text{if } \eta_0 \neq 0. \end{cases} \end{aligned} \quad (3.3)$$

When $\eta_0 = 0$, the solution represents the pseudoconformal blow-up and exhibits no phase rotation. When $\eta_0 \neq 0$, no matter how much small $|\eta_0|$ is, the solution does not blow up and rather exhibits a quick phase rotation (on the time interval of length $\sim |\eta_0|$) by the fixed amount of angle $(m+1)\pi$ (*rotational instability*).

Keeping dynamics (3.3) in mind, by varying $\eta_0 \geq 0$ and using the backward construction, the authors in [27] constructed one-parameter families of solutions exhibiting rotational instability. Using the forward construction and the shooting method for η , the authors in [26] constructed a codimension one set of smooth initial data leading to pseudoconformal blow-up.

In the present context, we deal with arbitrary H_m^3 finite-time blow-up solutions. Any hypotheses such as $b > 0$ (contracting scale) or $|\eta| \ll b$ (trapped regime) cannot be assumed; we need to allow any (small) values of b and η . A naive remedy would be to slightly modify definition (3.1) of P_{old} and consider instead

$$\tilde{P}_{\text{old}}(y; b, \eta) = \chi_{\delta B_0} e^{-iby^2/4} Q^{(\eta)}(y), \quad (3.4)$$

where $0 < \delta \ll 1$ and $B_0 = (b^2 + \eta^2)^{-1/4} =: \beta^{-1/2}$. When $m \geq 3$, this profile indeed works in our framework. However, it *no longer works* when $m \in \{1, 2\}$ because of the profile equation error Ψ (see (3.44) below for more precise meaning of Ψ) of size $\|\Psi\|_{\mathcal{H}_m^3} \gtrsim \beta^{m/2+3} \gtrsim \beta^4$, which is insufficient in view of Remark 3.5 below. Moreover, Ψ cannot be improved by adjusting the cutoff radius because of the pseudoconformal phase $e^{-iby^2/4}$ and the fact that $Q^{(\eta)}$ might not be defined for large y , e.g., $y \gg \delta B_0$.

A similar difficulty was faced by the authors in [28] for their blow-up construction in the $m = 0$ case. Motivated from the connection between the linearized (CSS) and the linearized wave maps or Schrödinger maps (see the discussion in Section 2.3.3) and the fact that the zero resonance of H_Q might lead to nontrivial logarithmic corrections to the modulation equations (as observed, for example, in [45, 52] for energy-critical geometric equations), the authors were able to carry out the tail computations at the level of the w_1 -equation and observe a *logarithmic correction* to the modulation equation $b_s + b^2 + \eta^2 = 0$. Note that several simplifications were possible because the trapped regime $|\eta| \ll b$ was assumed in [28].

In the present context, we further push this idea (i) in the $m \geq 1$ case and (ii) without the assumption $|\eta| \ll b$. It turns out that, when $m = 1$, there are nontrivial *cubic corrections* to the modulation equations $b_s + b^2 + \eta^2 = 0$ and $\eta_s = 0$ (see, e.g., (3.7)) and the cubic terms of P_1 and P_2 have polynomially improved spatial decays. Similarly, *quartic* terms have improved decays when $m = 2$. As b_s has the nontrivial quadratic term $b^2 + \eta^2$, our profile construction will involve very explicit computations. Moreover, the linearized operator H_Q for the w_1 -equation has a zero eigenfunction yQ , so the solvability condition must be taken into account when we invert H_Q .

3.2. Definition of unlocalized profiles

In this subsection, we define the unlocalized profiles $\hat{P}$, $\hat{P}_1$, $\hat{P}_2$ for w , w_1 , w_2 and also derive the formal modulation equations for b and η . We denote

$$\beta := \sqrt{b^2 + \eta^2} \quad \text{and} \quad \boldsymbol{\beta} := ib + \eta.$$

The unlocalized profiles will take the form

$$\begin{cases} \hat{P} = Q + T_1^{(0)} + T_2^{(0)} + T_3^{(0)}, \\ \hat{P}_1 = T_1^{(1)} + T_2^{(1)} + T_3^{(1)}, \\ \hat{P}_2 = T_2^{(2)} + T_3^{(2)} + T_4^{(2)}, \end{cases} \quad (3.5)$$

where each $T_j^{(k)}$ can be further decomposed as

$$T_j^{(k)}(y; b, \eta) = \sum_{j_1 + j_2 = j} b^{j_1} \eta^{j_2} T_{j_1, j_2}^{(k)}(y). \quad (3.6)$$

The quartic term $T_4^{(2)}$ will be chosen in the next subsection. Moreover, we will set

$$T_3^{(0)} = 0 \quad \text{if } m = 1 \text{ or } m \geq 3,$$

so the $\hat{P}$ -expansion is essentially quadratic. When $m = 1$, the formal modulation equations for b and η will be changed into

$$b_s + b^2 + \eta^2 + p_3^{(b)} = 0 \quad \text{and} \quad \eta_s + p_3^{(\eta)} = 0, \quad (3.7)$$

where $p_3^{(b)}$ and $p_3^{(\eta)}$ are real cubic polynomials satisfying (3.14).

Up to quadratic orders, the profiles $T_j^{(k)}$ are straightforward because the deviation from the old profile ansatz will appear (only when $m = 1$) from the cubic order. Thus the linear and quadratic profiles can be easily determined from the old profile ansatz (3.1), which can be viewed as $\hat{P}_1 = -\boldsymbol{\beta} \frac{y}{2} \hat{P}$ and similarly $\hat{P}_2 = -\boldsymbol{\beta} \frac{y}{2} \hat{P}_1$. Motivated by the generalized kernel relations (2.15) and (2.16), the linear terms are defined by

$$T_1^{(1)} := -\boldsymbol{\beta} \frac{y}{2} Q, \quad (3.8)$$

$$T_1^{(0)} := -ib \frac{y^2}{4} Q - (m+1)\eta\rho. \quad (3.9)$$

The quadratic terms are given by

$$T_2^{(2)} := \beta^2 \frac{y^2}{4} Q, \quad (3.10)$$

$$T_2^{(1)} := -\beta \frac{y}{2} T_1^{(0)}, \quad (3.11)$$

$$T_2^{(0)} := {}^{(\text{out})}L_Q^{-1} \left[T_2^{(1)} + \frac{2}{y} A_\theta[Q, T_1^{(0)}] T_1^{(0)} + \frac{1}{y} A_\theta[T_1^{(0)}] Q \right], \quad (3.12)$$

where we recall the polarization

$$A_\theta[\psi_1, \psi_2] = -\frac{1}{2} \int_0^y \text{Re}(\bar{\psi}_1 \psi_2) y' dy'$$

and the operator ${}^{(\text{out})}L_Q^{-1}$ from (A.4). A seemingly complicated definition of $T_2^{(0)}$ is motivated by the compatibility $\mathbf{D}_P P \approx P_1$.

We turn to the cubic terms of the profile expansions. We first state the definitions of these cubic profiles. First, we define $T_3^{(2)}$ by

$$T_3^{(2)} := \begin{cases} {}^{(\text{out})}(A_Q^*)^{-1} \left[\beta^2 A_Q^* \left(\frac{y^2}{4} T_1^{(0)} \right) - p_3^{(b)} \frac{y}{2} Q + p_3^{(\eta)} i \frac{y}{2} Q \right] & \text{if } m = 1, \\ \beta^2 \frac{y^2}{4} T_1^{(0)} & \text{if } m \geq 2, \end{cases} \quad (3.13)$$

where $p_3^{(b)}$ and $p_3^{(\eta)}$ are cubic homogeneous real polynomials of b and η such that

$$p_3^{(b)} - i p_3^{(\eta)} = \begin{cases} \frac{8\pi\beta^3}{\|yQ\|_{L^2}^2} & \text{if } m = 1, \\ 0 & \text{if } m \geq 2, \end{cases} \quad (3.14)$$

and ${}^{(\text{out})}(A_Q^*)^{-1}$ is the inverse of A_Q^* , whose explicit formula is given in (A.10). Next, we define $T_3^{(1)}$ using the compatibility $A_P P_1 \approx P_2$ (see Step 3 of the proof of Proposition 3.3 for the derivation),

$$T_3^{(1)} := {}^{(\text{out})}A_Q^{-1} \left[T_3^{(2)} - \beta (A_\theta[Q, T_2^{(0)}] Q + A_\theta[Q, T_1^{(0)}] T_1^{(0)} + \frac{1}{2} A_\theta[T_1^{(0)}] Q) \right], \quad (3.15)$$

where ${}^{(\text{out})}A_Q^{-1}$ denotes a formal right inverse of A_Q whose explicit formula is given in (A.9). Finally, we define $T_3^{(0)}$ by

$$T_3^{(0)} := \begin{cases} 0 & \text{if } m = 1 \text{ or } m \geq 3, \\ -\beta^3 \frac{y^6}{384} Q & \text{if } m = 2. \end{cases} \quad (3.16)$$

In particular, the profile $T_3^{(0)}$ is nonzero only if $m = 2$. It is introduced to capture the singular part z_{sg}^* of z^* when $m = 2$, not to improve the profile equation error Ψ for P .

The definition of $T_3^{(2)}$ can be derived from the tail computations. Note that when $m \geq 2$, the old profile ansatz (3.1) is still valid at the cubic order so we can easily expect

that $T_3^{(2)} = \beta^2 \frac{y^2}{4} T_1^{(0)}$. However, when $m = 1$, nontrivial cubic order modifications are necessary and the derivation of $T_3^{(2)}$ needs more involved computations. In the spirit of the tail computations, we substitute the formal modulation equations (3.7) with $p_3^{(b)}$ and $p_3^{(\eta)}$ unknown and $(w, w_1, w_2) = (P, P_1, P_2)$ into (2.9) to have¹¹

$$0 \approx (b^2 + \eta^2 + p_3^{(b)})i \partial_b P_1 + p_3^{(\eta)}i \partial_\eta P_1 - ib\Lambda_{-1}P_1 - (m+1)\eta P_1 - \left(\int_0^y \operatorname{Re}(\bar{P}P_1)dy' \right) P_1 + A_P^* P_2.$$

After a series of explicit computations (see the proof of (3.76) in Step 6 of the proof of Proposition 3.3), we can extract out the cubic terms of the above equation as

$$0 \approx p_3^{(b)} \frac{y}{2} Q - p_3^{(\eta)} i \frac{y}{2} Q + A_Q^* \left(T_3^{(2)} - \beta^2 \frac{y^2}{4} T_1^{(0)} \right). \quad (3.17)$$

We then require the following *solvability condition* for A_Q^* in order to have $T_3^{(2)}$ growing the slowest:

$$p_3^{(b)} \frac{y}{2} Q - p_3^{(\eta)} i \frac{y}{2} Q - A_Q^* \left(\beta^2 \frac{y^2}{4} T_1^{(0)} \right) \perp yQ. \quad (3.18)$$

When $m \geq 2$, fast spatial decay of yQ and $A_Q(yQ) = 0$ guarantee $A_Q^* \left(\beta^2 \frac{y^2}{4} T_1^{(0)} \right) \perp yQ$ and thus we set $p_3^{(b)} = p_3^{(\eta)} = 0$ and $T_3^{(2)} = \beta^2 \frac{y^2}{4} T_1^{(0)}$ as expected. When $m = 1$, the spatial decay $yQ \sim y^{-2}$ is *critical*, and the following inner product becomes *nonzero*:

$$\int_0^\infty A_Q^* \left(\beta^2 \frac{y^2}{4} T_1^{(0)} \right) \cdot yQ y dy = -\beta^2 \frac{y^2}{4} T_1^{(0)} \cdot yQ \cdot y \Big|_{y=0}^\infty = \beta^3 \lim_{y \rightarrow +\infty} \frac{1}{16} y^6 Q^2 = 2\beta^3,$$

where we used (3.19) in the middle equality. In order to ensure (3.18), this computation suggests us to define $p_3^{(b)}$ and $p_3^{(\eta)}$ such that

$$p_3^{(b)} - ip_3^{(\eta)} = \frac{2\beta^3}{\int_0^\infty \frac{y}{2} Q \cdot yQ y dy} = \frac{8\pi\beta^3}{\|yQ\|_{L^2}^2},$$

which is (3.14). With this choice of $p_3^{(b)}$ and $p_3^{(\eta)}$, we construct $T_3^{(2)}$ according to (3.17), giving definition (3.13).

We end this subsection with some useful pointwise estimates for the profiles $T_j^{(k)}$.

Lemma 3.1 (Pointwise estimates for $T_j^{(k)}$). *Recall the definitions of $T_j^{(k)}$'s above.*

- (Smooth profiles) All $T_j^{(k)}$'s have smooth $(m+k)$ -equivariant extensions on $\mathbb{R}^2$.
- (Large y -asymptotics) Let $y \geq 2$ and $k \in \mathbb{N}$. We have

$$\left| T_1^{(0)} + \beta \frac{y^2}{4} Q \right|_k \lesssim_k \beta Q \log y, \quad (3.19)$$

$$\left| T_2^{(0)} - \beta^2 \frac{y^4}{32} Q \right|_k + y \left| T_2^{(1)} - \beta^2 \frac{y^3}{8} Q \right|_k \lesssim_k \beta^2 y^2 Q \log y. \quad (3.20)$$

¹¹More precisely, one needs to substitute $\gamma_s = -(m+1)\eta - \int_0^\infty \operatorname{Re}(\bar{P}P_1)dy$ instead of $\eta_s = (m+1)\eta$. See definition (3.28) of **Mod** and Remark 3.6 below.

For the cubic profiles, we have

$$|T_3^{(1)}|_k + y|T_3^{(2)}|_k \lesssim_k \beta^3 y^3 Q \log y \quad \text{if } m = 1, \quad (3.21)$$

$$\left| T_3^{(1)} + \beta^3 \frac{y^5}{64} Q \right|_k + y \left| T_3^{(2)} + \beta^3 \frac{y^4}{16} Q \right|_k \lesssim_k \beta^3 y^3 Q \log y \quad \text{if } m \geq 2. \quad (3.22)$$

In particular, we have

$$|T_3^{(1)}|_k + y|T_3^{(2)}|_k \lesssim_k \beta^3 y. \quad (3.23)$$

The above asymptotics also hold after taking ∂_b or ∂_η to the left-hand side and removing one β from the right-hand side.

Remark 3.2. Typically the growth tail of $T_{j+1}^{(k)}$ is βy^2 times that of $T_j^{(k)}$. However, when $m = 1$, the growth tail of $T_3^{(k)}$ is only $\beta \log y$ times that of $T_2^{(k)}$. This polynomially improved decay is a consequence of the tail computations and is crucial to have the profile error estimates (3.47)–(3.48). Similarly, we will see in Proposition 3.3 that the same happens for the $T_4^{(2)}$ profile when $m = 2$.

Proof of Lemma 3.1. The fact that all $T_j^{(k)}$ have smooth $(m + k)$ -equivariant extensions easily follows from the smoothness assertions in Appendix A. Henceforth, we focus on the proof of large y -asymptotics.

Proof of (3.19). By (3.9) and (2.17), we have

$$T_1^{(0)} + \beta \frac{y^2}{4} Q = \eta \left\{ -^{(\text{out})} L_Q^{-1} \left(\frac{y}{2} Q \right) + \frac{y^2}{4} Q \right\}.$$

Using formula (A.4) of $^{(\text{out})} L_Q^{-1}$, we have

$$^{(\text{out})} L_Q^{-1} \left(\frac{y}{2} Q \right) = Q J_1 \int_0^y [\partial_y J_2] \frac{(y')^2}{2} dy' - Q J_2 \int_0^y [\partial_y J_1] \frac{(y')^2}{2} dy',$$

where J_1 and J_2 are defined in (A.1). We apply asymptotics (A.3) of J_1 and J_2 to the above. Note that the first term gives the leading order term $\frac{y^2}{4} Q$ with an error of size $O(Q)$, while the second term is of size $O(Q \log y)$. This yields (3.19).

Proof of (3.20). Recall from (3.11) that $T_2^{(1)} = -\beta \frac{y}{2} T_1^{(0)}$, so (3.20) for $T_2^{(1)}$ follows from (3.19). For $T_2^{(0)}$ (recall definition (3.12)), we note the estimates for $y \geq 2$,

$$\begin{aligned} \mathbf{1}_{y \geq 2} \left| T_2^{(1)} - \beta^2 \frac{y^3}{8} Q \right|_k &\lesssim_k \mathbf{1}_{y \geq 2} \beta^2 y Q \log y, \\ \mathbf{1}_{y \geq 2} \left| \frac{2}{y} A_\theta[Q, T_1^{(0)}] T_1^{(0)} + \frac{1}{y} A_\theta[T_1^{(0)}] Q \right|_k &\lesssim_k \mathbf{1}_{y \geq 2} \beta^2 y Q. \end{aligned}$$

Combining these estimates with (3.12) and (A.4) gives (3.20) for $T_2^{(0)}$.

Proof of (3.21). We first prove (3.21) for $T_3^{(2)}$. Recall formula (3.13),

$$T_3^{(2)} = ^{(\text{out})} (A_Q^*)^{-1} \left[\beta^2 A_Q^* \left(\frac{y^2}{4} T_1^{(0)} \right) - p_3^{(b)} \frac{y}{2} Q + p_3^{(\eta)} i \frac{y}{2} Q \right].$$

As $m = 1$, we have $A_Q^* \approx -\partial_y + \frac{1}{y}$ and $y^2 T_1^{(0)} \sim \beta y^4 Q \sim \beta y$, which result in the cancellation $(-\partial_y + \frac{1}{y})\beta y = 0$. Incorporating the error terms using (3.19), this cancellation is captured by the following estimate:

$$\mathbf{1}_{y \geq 2} \left| A_Q^* \left(\frac{y^2}{4} T_1^{(0)} \right) \right|_k \lesssim_k \mathbf{1}_{y \geq 2} \beta y Q \log y.$$

Thus we have

$$\mathbf{1}_{y \geq 2} \left| \beta^2 A_Q^* \left(\frac{y^2}{4} T_1^{(0)} \right) - p_3^{(b)} \frac{y}{2} Q + p_3^{(\eta)} i \frac{y}{2} Q \right|_k \lesssim_k \mathbf{1}_{y \geq 2} \beta^3 y Q \log y. \quad (3.24)$$

We now recall that $p_3^{(b)}$ and $p_3^{(\eta)}$ are chosen to satisfy (3.18). Thus we can apply formula (A.11),

$$T_3^{(2)} = \frac{1}{y^2 Q} \int_y^\infty y' Q \left[\beta^2 A_Q^* \left(\frac{(y')^2}{4} T_1^{(0)} \right) - p_3^{(b)} \frac{y'}{2} Q + p_3^{(\eta)} i \frac{y'}{2} Q \right] y' dy'.$$

Substituting (3.24) into the above formula gives (3.21) for $T_3^{(2)}$.

Next, we prove (3.21) for $T_3^{(1)}$. Combining the pointwise estimates

$$\begin{aligned} |T_3^{(2)}|_k &\lesssim_k \beta^3 y^2 Q \log y, \\ \left| \beta (A_\theta[Q, T_1^{(0)}] T_1^{(0)} + \frac{1}{2} A_\theta[T_1^{(0)}] Q) \right|_k &\lesssim_k \beta^3 y^2 Q \end{aligned}$$

with (3.15) and (A.9) gives (3.21) for $T_3^{(1)}$.

Proof of (3.22). As $m \geq 2$, we have $p_3^{(b)} = p_3^{(\eta)} = 0$ and $T_3^{(2)} = \beta^2 \frac{y^2}{4} T_1^{(0)}$. Thus (3.22) for $T_3^{(2)}$ easily follows from (3.19). Combining the pointwise estimates

$$\begin{aligned} \left| T_3^{(2)} + \beta^3 \frac{y^4}{16} Q \right|_k &\lesssim_k \beta^3 y^2 Q \log y, \\ \left| \beta (A_\theta[Q, T_1^{(0)}] T_1^{(0)} + \frac{1}{2} A_\theta[T_1^{(0)}] Q) \right|_k &\lesssim_k \beta^3 y^2 Q \end{aligned}$$

with (3.15) and (A.9) gives (3.22) for $T_3^{(1)}$.

Completion of the proof. Estimate (3.23) follows from (3.21) and (3.22). As all $T_j^{(k)}$'s are of form (3.6), all the estimates used in this proof holds after taking ∂_b or ∂_η in the left-hand side and removing one β from the right-hand side. This completes the proof of Lemma 3.1. $\blacksquare$

3.3. Localization and main proposition

Having defined the profiles up to cubic terms, we can state the precise form of the profiles P , P_1 , and P_2 including the cutoffs. Let

$$B_0 := \beta^{-1/2} \quad \text{and} \quad B_1 := \beta^{-1}.$$

We will use the modified profiles defined by (cf. (3.5))

$$P = Q + \chi_{B_1} \{T_1^{(0)} + T_2^{(0)} + T_3^{(0)}\}, \quad (3.25)$$

$$P_1 = \chi_{B_1} \{T_1^{(1)} + T_2^{(1)} + T_3^{(1)}\}, \quad (3.26)$$

$$P_2 = \chi_{B_1} \{T_2^{(2)} + T_3^{(2)}\} + \chi_{B_0} T_4^{(2)}, \quad (3.27)$$

where the quartic term $T_4^{(2)}$ of P_2 will be defined in the proof of Proposition 3.3 below. As mentioned at the beginning of Section 3, the cutoff radius B_1 is important in our analysis. See also Remark 3.4 below for more detailed explanations.

We now state the main proposition. Define the vectors

$$\mathbf{Mod} := \begin{pmatrix} \frac{\lambda_s}{\lambda} + b \\ \gamma_s + (m+1)\eta + \int_0^\infty \operatorname{Re}(\bar{P} P_1) dy \\ b_s + b^2 + \eta^2 + p_3^{(b)} \\ \eta_s + p_3^{(\eta)} \end{pmatrix}, \quad \mathbf{v}_k := \begin{pmatrix} \Lambda P_k \\ -i P_k \\ -\partial_b P_k \\ -\partial_\eta P_k \end{pmatrix}, \quad (3.28)$$

where we wrote $P_0 = P$ and $\mathbf{v}_0 = \mathbf{v}$.

Proposition 3.3 (Modified profiles). *Let $m \geq 1$ and $\beta \in (-\frac{1}{10}, \frac{1}{10})$. There exists a smooth profile*

$$T_4^{(2)}(y; b, \eta) = \sum_{i+j=4} b^i \eta^j T_{i,j}^{(2)}(y)$$

that admits a smooth $(m+2)$ -equivariant extension on $\mathbb{R}^2$ and satisfies the pointwise estimates

$$|T_4^{(2)}|_k \lesssim_k \begin{cases} \beta^4 y (\log y)^2 & \text{if } m = 1, \\ \beta^4 \log y & \text{if } m = 2, \\ \beta^4 y^{4-m} & \text{if } m \geq 3 \end{cases} \quad (3.29)$$

for any $k \in \mathbb{N}$ in the region $y \geq 2$, such that the following bounds hold.

- (Estimates for modulation vectors) For $\mathbf{v} = \mathbf{v}_0$, we have for any $R > 0$,

$$\mathbf{1}_{(0,R]} \{ |(\mathbf{v}_0)_1 - \Lambda Q|_k + |(\mathbf{v}_0)_2 + iQ|_k \} \lesssim_R \beta, \quad (3.30)$$

$$\mathbf{1}_{(0,R]} \left\{ \left| (\mathbf{v}_0)_3 - i \frac{y^2}{4} Q \right|_k + |(\mathbf{v}_0)_4 - (m+1)\rho|_k \right\} \lesssim_R \beta. \quad (3.31)$$

For $\mathbf{v}_1$, we have for any $k \in \mathbb{N}$,

$$|(\mathbf{v}_1)_1|_k + |(\mathbf{v}_1)_2|_k \lesssim_k \mathbf{1}_{(0,2]} \beta y^2 + \mathbf{1}_{[2,2B_1]} (\beta y^{-2} + \beta^2), \quad (3.32)$$

and

$$\left| (\mathbf{v}_1)_3 - \chi_{B_1} i \frac{y}{2} Q \right|_k + \left| (\mathbf{v}_1)_4 - \chi_{B_1} \frac{y}{2} Q \right|_k \lesssim_k \mathbf{1}_{(0,2]} \beta y^4 + \mathbf{1}_{[2,2B_1]} \beta. \quad (3.33)$$

For $\mathbf{v}_2$, we have for any $k \in \mathbb{N}$,

$$|(\mathbf{v}_2)_1|_k + |(\mathbf{v}_2)_2|_k \lesssim_k \mathbf{1}_{(0,2]} \beta^2 y^3 + \mathbf{1}_{[2,2B_1]} \beta^2 y^{-1}, \quad (3.34)$$

$$|(\mathbf{v}_2)_3|_k + |(\mathbf{v}_2)_4|_k \lesssim_k \mathbf{1}_{(0,2]} \beta y^3 + \mathbf{1}_{[2,2B_1]} \beta y^{-1}. \quad (3.35)$$

- (Compatibility) For $\mathbf{D}_P P$ and P_1 , we have

$$\|\mathbf{D}_P P - P_1\|_{L^2} \lesssim \beta, \quad (3.36)$$

$$\|\mathbf{D}_P P - P_1\|_{\dot{H}_{m+1}^1} \lesssim \beta^2, \quad (3.37)$$

$$\|\mathbf{D}_P P - P_1\|_{\dot{H}_{m+1}^2} \lesssim \beta^3. \quad (3.38)$$

For $A_P P_1$ and P_2 , we have

$$\|A_P P_1 - P_2\|_{L^2} \lesssim \beta^2, \quad (3.39)$$

$$\|A_P P_1 - P_2\|_{\dot{H}_{m+2}^1} \lesssim \beta^3, \quad (3.40)$$

$$\|A_P P_1 - P_2\|_{\dot{V}_{m+2}^{3/2}} \lesssim \beta^{7/2}. \quad (3.41)$$

- (Equation for P) We have

$$\begin{aligned} & \left(\partial_s - \frac{\lambda_s}{\lambda} \Lambda + \gamma_s i \right) P + i \mathbf{D}_P^* P_1 + i \left(\int_y^\infty \operatorname{Re}(\bar{P} P_1) dy' \right) P \\ & = -\mathbf{Mod} \cdot \mathbf{v} + i \Psi, \end{aligned} \quad (3.42)$$

where Ψ satisfies for any $R > 0$,

$$\mathbf{1}_{(0,R]} |\Psi| \lesssim_R \beta^3. \quad (3.43)$$

- (Equation for P_1) We have

$$\begin{aligned} & \left(\partial_s - \frac{\lambda_s}{\lambda} \Lambda_{-1} + \gamma_s i \right) P_1 + i A_P^* P_2 + i \left(\int_y^\infty \operatorname{Re}(\bar{P} P_1) dy' \right) P_1 \\ & = -\mathbf{Mod} \cdot \mathbf{v}_1 + i \Psi_1, \end{aligned} \quad (3.44)$$

where Ψ_1 satisfies

$$\left\| \frac{1}{y^2} \Psi_1 \right\|_{L^1} \lesssim \beta^4 |\log \beta|^3. \quad (3.45)$$

- (Equation for P_2) We have

$$\begin{aligned} & \left(\partial_s - \frac{\lambda_s}{\lambda} \Lambda_{-2} + \gamma_s i \right) P_2 + i \tilde{H}_P P_2 + i \left(\int_y^\infty \operatorname{Re}(\bar{P} P_1) dy' \right) P_2 - i \bar{P} P_1^2 \\ & = -\mathbf{Mod} \cdot \mathbf{v}_2 + i \Psi_2 \end{aligned} \quad (3.46)$$

with the bounds

$$\|\Psi_2\|_{L^2} \lesssim \beta^4 |\log \beta|^2, \quad (3.47)$$

$$\|\Psi_2\|_{\dot{H}_{m+2}^1} \lesssim \beta^{9/2} |\log \beta|^2. \quad (3.48)$$

Remark 3.4 (Cutoff radius B_1). As alluded to above, $B_1 = \beta^{-1}$ is the only cutoff radius such that our analysis works when $m \in \{1, 2\}$. According to our strategy, $P^\sharp = P_{\lambda, \gamma}$

must contain the singular part z_{sg}^* of the asymptotic profile z^* and hence the cutoff at the original scale should be $r \sim 1$. In the formal modulation dynamics (3.2), the quantity

$$\frac{b^2 + \eta^2}{\lambda^2}$$

is *conserved*, and hence $r \sim 1$ corresponds to $y = \frac{r}{\lambda} \sim (b^2 + \eta^2)^{-1/2} = B_1$.

When we estimate the profile errors Ψ_k , we crucially use the *special cancellation*

$$[(b^2 + \eta^2)\partial_b - by\partial_y]\chi_{B_1} = 0, \quad (3.49)$$

which corresponds to the obvious identity $\partial_t \chi_R = 0$ for fixed R in the original (t, r) -coordinates. If we use other cutoff radii, say $B = \beta^{-p}$ with $p \neq 1$, then there are nontrivial cutoff errors

$$\begin{aligned} \|[(b^2 + \eta^2)\partial_b - by\partial_y]\chi_B T_2^{(2)}\|_{L^2} &\gtrsim \left\| \mathbf{1}_{y \sim B} b \beta^2 \frac{1}{y} \right\|_{L^2} \gtrsim |b|^3 \quad \text{if } m = 1, \\ \|[(b^2 + \eta^2)\partial_b - by\partial_y]\chi_B T_3^{(2)}\|_{\dot{H}_{m+2}^1} &\gtrsim \left\| \mathbf{1}_{y \sim B} b \beta^3 \right\|_{\dot{H}_{m+2}^1} \gtrsim |b|^4 \quad \text{if } m = 2, \end{aligned}$$

violating the necessary bounds (3.50) below.

When $m \geq 3$, cancellation (3.49) is not necessary and even the old profile ansatz (3.4) (where the cutoff is placed at the self-similar scale) works. This is consistent with the statement of Theorem 1.4 that there is no H^3 -singular part of the asymptotic profile.

Remark 3.5 (Bounds for Ψ_k). As we will see in Section 4.5, it is necessary to obtain the bounds

$$\mathbf{1}_{y \lesssim 1} |\Psi| \lesssim \beta^{2+}, \quad \mathbf{1}_{y \lesssim 1} |\Psi_1| \lesssim \beta^{3+}, \quad \|\Psi_2\|_{L^2} \lesssim \beta^{3+}, \quad \|\Psi_2\|_{\dot{H}_{m+2}^1} \lesssim \beta^{4+} \quad (3.50)$$

for our main energy estimates (Proposition 4.12). The first two bounds contribute to the λ - and γ -modulation estimates and b - and η -modulation estimates, respectively. In the energy estimates, we will use the P_2 -equation, and degeneracy estimates (3.34)–(3.35) for $\mathbf{v}_2$ allow the Ψ and Ψ_1 bounds to be rougher than the Ψ_2 bound by the factors β^2 and β , respectively. This also explains why P and P_1 ansatzes do not need to be as precise as the P_2 ansatz.

By (3.50), it is necessary to have at least quadratic, cubic, and quartic expansions of P , P_1 , and P_2 , respectively. This in principle guarantees

$$\Psi = O(\beta^3), \quad \Psi_1 = O(\beta^4), \quad \Psi_2 = O(\beta^5) \quad \text{in a compact region } y \lesssim 1. \quad (3.51)$$

If m is sufficiently large, (3.51) remains valid even in various Sobolev norms due to the fast spatial decay of Q . Thus all the β -losses in (3.45), (3.47), and (3.48) compared to (3.51) come from the lack of spatial decay of Q .

Finally, let us note that we did not attempt to make (3.45), (3.47), and (3.48) sharp because we only need the bound such as (3.50). Some of these bounds can be improved logarithmically or even polynomially in β .

Remark 3.6 (Phase correction). The integral term $\int_0^\infty \operatorname{Re}(\bar{P} P_1) dy$ in the definition of $\mathbf{Mod}_2$ is employed to replace the $\int_y^\infty$ -integrals in equations (2.8)–(2.10) by the $-\int_0^y$ -integrals so that the resulting equations become more suited for the tail computations. This integral term can be understood as a *phase correction* from the nonlocal nonlinearity. Note that it is equal to $-2(m+1)\eta$ at the linear order due to

$$\int_0^\infty \operatorname{Re}(\bar{P} P_1) dy = -\eta \frac{1}{2} \int_0^\infty Q^2 y dy + O(\beta^2) = -2(m+1)\eta + O(\beta^2), \quad (3.52)$$

so we have $\mathbf{Mod}_2 \approx \gamma_s - (m+1)\eta$ as in (3.2).

Remark 3.7. In contrast to [26], we cannot simply bound P by Q because the higher-order terms in the expansion of P become dominant in the region $B_0 \ll y \lesssim B_1$. Thus we need to take care about the higher-order terms in many places of the proof. In several cases, the contribution of the highest order terms indeed *saturates* the error estimates.

Proof of Proposition 3.3. Step 0. Remarks and notations for the proof. In many parts of the proof, we will only focus on the estimates in the far region $y \geq 2$. The contribution of error terms from the region $y \leq 2$ does not disrupt the validity of the proposition. Note that we do not need to worry about singularity issues near the origin $y = 0$ because all the profiles $T_j^{(k)}$ have smooth equivariant extensions on $\mathbb{R}^2$.

To simplify the presentation in various pointwise estimates, let us introduce more notations. We denote $f = \mathcal{O}(g)$ to mean that

$$f = \mathcal{O}(g) \Leftrightarrow \mathbf{1}_{y \geq 2} |f|_k \lesssim_k g \quad \forall k \in \mathbb{N}.$$

For a nonnegative integer $j \geq 0$, a real number ℓ , and a function $f = f(y; b, \eta)$, we use the following notations:

$$f = \mathcal{I}_\ell^j \Leftrightarrow f = \sum_{j_1+j_2=j} b^{j_1} \eta^{j_2} f_{j_1, j_2}(y) \quad \text{for some } f_{j_1, j_2} = \mathcal{O}(y^\ell), \quad (3.53)$$

$$f = \mathcal{O}(\mathcal{I}_\ell^j) \Leftrightarrow f = \mathcal{O}(\beta^j y^\ell). \quad (3.54)$$

We can also add logarithmic divergences to the above notations; for a real number ℓ' , we use the notation $f = \mathcal{I}_{\ell, \ell'}^j$ if f can be expressed as in (3.53) for some¹² $f_{j_1, j_2} = \mathcal{O}(y^\ell (\log y)^{\ell'})$. We can similarly define $f = \mathcal{O}(\mathcal{I}_{\ell, \ell'}^j)$. We will also use this notation combined with various operations such as addition, subtraction, and integration. A useful fact is that

$$\int_0^y \mathcal{I}_\ell^j \frac{dy'}{y'} = \begin{cases} \mathcal{I}_\ell^j & \text{if } \ell > 0, \\ \mathcal{I}_{0,1}^j & \text{if } \ell = 0, \\ \mathcal{I}_0^j & \text{if } \ell < 0, \end{cases}$$

and similarly for $\int_0^y \mathcal{O}(\mathcal{I}_\ell^j) \frac{dy'}{y'}$ by taking $\mathcal{O}$. We will also frequently use the estimate $\mathcal{I}_{\ell+1}^{j+1} \leq \mathcal{I}_\ell^j$ in the region $y \leq 2B_1$.

¹²We use the $\mathcal{O}$ -notation in the region $y \geq 2$, so $(\log y)^{\ell'}$ is well defined.

Step 1. Estimates for the modulation vectors. The proof of (3.30)–(3.31) is immediate from expansion (3.25) as we allow R -dependences. The proof of (3.32)–(3.33) follows from $T_1^{(1)} = -\beta \frac{y}{2} Q$, (3.20), and (3.23). Finally, the proof of (3.34)–(3.35) follows from $T_2^{(2)} = \beta^2 \frac{y^2}{4} Q$, (3.23), and (3.29) with $B_0 = \beta^{-1/2}$.¹³

Step 2. Proof of compatibility estimates (3.36)–(3.38). We denote by $\mathcal{R}_c^{(0)}$ any function satisfying the bounds (cf. right-hand sides of (3.36)–(3.38))

$$\|\mathcal{R}_c^{(0)}\|_{L^2} \lesssim \beta, \quad \|\mathcal{R}_c^{(0)}\|_{\dot{H}_{m+1}^1} \lesssim \beta^2, \quad \|\mathcal{R}_c^{(0)}\|_{\dot{H}_{m+1}^2} \lesssim \beta^3. \quad (3.55)$$

We denote $f = \mathcal{R}_c^{(0)}$ if f satisfies estimates (3.55).

First claim. We claim that we can approximate $\mathbf{D}_P P$ by $\chi_{B_1} \mathbf{D}_{\hat{P}} \hat{P}$:

$$\mathbf{D}_P P = \chi_{B_1} \mathbf{D}_{\hat{P}} \hat{P} + \mathcal{R}_c^{(0)}, \quad (3.56)$$

where $\hat{P}$ is the unlocalized profile (3.5). By the definition of P and $\mathbf{D}_Q Q = 0$, we have (recall $\chi_{\gtrsim B_1} = 1 - \chi_{B_1}$)

$$\begin{aligned} \mathbf{D}_P P &= \chi_{B_1} \mathbf{D}_P \hat{P} + \chi_{\gtrsim B_1} \mathbf{D}_P Q + (\partial_y \chi_{B_1})(\hat{P} - Q) \\ &= \chi_{B_1} \mathbf{D}_{\hat{P}} \hat{P} + \chi_{B_1} (\mathbf{D}_P - \mathbf{D}_{\hat{P}}) \hat{P} + \chi_{\gtrsim B_1} \{(\mathbf{D}_P - \mathbf{D}_Q) Q\} + (\partial_y \chi_{B_1})\{\hat{P} - Q\}. \end{aligned}$$

Thus it suffices to show that the last three terms of the above formula belong to $\mathcal{R}_c^{(0)}$. Note that $\mathbf{D}_P - \mathbf{D}_{\hat{P}}$ and $\mathbf{D}_P - \mathbf{D}_Q$ can be identified as the functions $\frac{1}{y}(A_\theta[\hat{P}] - A_\theta[P])$ and $\frac{1}{y}(A_\theta[Q] - A_\theta[P])$, and we have the estimates

$$\begin{aligned} \chi_{B_1} (\mathbf{D}_P - \mathbf{D}_{\hat{P}}), \partial_y \chi_{B_1} &= \mathcal{O}(\mathbf{1}_{[B_1, 2B_1]} y^{-1}), \\ \chi_{\gtrsim B_1} (\mathbf{D}_P - \mathbf{D}_Q) &= \mathcal{O}(\mathbf{1}_{[B_1, \infty)} y^{-1}). \end{aligned}$$

These estimates give

$$\begin{aligned} \chi_{B_1} (\mathbf{D}_P - \mathbf{D}_{\hat{P}}) \hat{P} + (\partial_y \chi_{B_1})\{\hat{P} - Q\} &= \mathcal{O}(\mathbf{1}_{[B_1, 2B_1]} \beta y^{-1}) = \mathcal{R}_c^{(0)}, \\ \chi_{\gtrsim B_1} \{(\mathbf{D}_P - \mathbf{D}_Q) Q\} &= \mathcal{O}(\mathbf{1}_{[B_1, \infty)} y^{-4}) = \mathcal{R}_c^{(0)}, \end{aligned}$$

as desired. This completes the proof of claim (3.56).

Second claim. We claim that

$$\chi_{B_1} \mathbf{D}_{\hat{P}} \hat{P} = \chi_{B_1} \{\mathbf{D}_{\hat{P}-T_3^{(0)}}(\hat{P} - T_3^{(0)}) + \mathbf{D}_Q T_3^{(0)}\} + \mathcal{R}_c^{(0)}. \quad (3.57)$$

To show this, it suffices to show that

$$\chi_{B_1} \{(\mathbf{D}_{\hat{P}} - \mathbf{D}_Q) T_3^{(0)} + (\mathbf{D}_{\hat{P}} - \mathbf{D}_{\hat{P}-T_3^{(0)}})(\hat{P} - T_3^{(0)})\} = \mathcal{R}_c^{(0)}.$$

¹³Note that $T_4^{(2)}$ is not constructed yet, but it will be constructed in Step 8 below satisfying $T_4^{(2)} = J_{1,2}^4$.

As $T_3^{(0)} = 0$ for $m \neq 2$, we may assume $m = 2$. Now the above error estimate follows from

$$\begin{aligned}\chi_{B_1}(\mathbf{D}_{\hat{P}} - \mathbf{D}_Q)T_3^{(0)} &= \mathcal{O}\left(\mathbf{1}_{[2,2B_1]} \frac{1}{y} \int_0^y (\mathcal{J}_{-6}^1 + \mathcal{J}_4^6) y' dy' \cdot \mathcal{J}_2^3\right) \\ &= \mathcal{O}(\mathbf{1}_{[2,2B_1]} \{\mathcal{J}_1^4 + \mathcal{J}_7^9\}) = \mathcal{R}_c^{(0)}\end{aligned}$$

and

$$\begin{aligned}\chi_{B_1}(\mathbf{D}_{\hat{P}} - \mathbf{D}_{\hat{P}-T_3^{(0)}})(\hat{P} - T_3^{(0)}) &= \mathcal{O}\left(\mathbf{1}_{[2,2B_1]} \frac{1}{y} \int_0^y (\mathcal{J}_{-2}^3 + \mathcal{J}_4^6) y' dy' \cdot (\mathcal{J}_{-4}^0 + \mathcal{J}_0^2)\right) \\ &= \mathcal{O}(\mathbf{1}_{[2,2B_1]} \{\mathcal{J}_{-5,1}^3 + \mathcal{J}_{-1,1}^5 + \mathcal{J}_1^6 + \mathcal{J}_5^8\}) = \mathcal{R}_c^{(0)}.\end{aligned}$$

Note that the contribution of the highest order term $\frac{1}{y} A_\theta[T_3^{(0)}]T_3^{(0)}$ saturates bound (3.55).

Last claim. We claim that

$$\chi_{B_1} \mathbf{D}_{\hat{P}-T_3^{(0)}}(\hat{P} - T_3^{(0)}) = P_1 - \chi_{B_1} T_3^{(1)} + \mathcal{R}_c^{(0)}. \quad (3.58)$$

To see this, we expand $\mathbf{D}_{\hat{P}-T_3^{(0)}}(\hat{P} - T_3^{(0)})$ using (2.11):

$$\begin{aligned}\chi_{B_1} \mathbf{D}_{\hat{P}-T_3^{(0)}}(\hat{P} - T_3^{(0)}) &= \chi_{B_1} \{L_Q(T_1^{(0)} + T_2^{(0)}) + N_Q(T_1^{(0)} + T_2^{(0)})\} \\ &= \chi_{B_1} \left\{ L_Q T_1^{(0)} + (L_Q T_2^{(0)} - \frac{2}{y} A_\theta[Q, T_1^{(0)}]T_1^{(0)} - \frac{1}{y} A_\theta[T_1^{(0)}]Q) \right\} \\ &\quad + \mathcal{O}(\mathbf{1}_{[2,2B_1]} \{\mathcal{J}_0^3 + \mathcal{J}_{0,1}^4 + \mathcal{J}_2^5 + \mathcal{J}_4^6\}).\end{aligned} \quad (3.59)$$

Applying (2.16) for the linear term and (3.12) for the quadratic term, the previous formula continues as

$$\begin{aligned}\text{RHS of (3.59)} &= \chi_{B_1} \{T_1^{(1)} + T_2^{(1)}\} + \mathcal{O}(\mathbf{1}_{[2,2B_1]} \{\mathcal{J}_0^3 + \mathcal{J}_{0,1}^4 + \mathcal{J}_2^5 + \mathcal{J}_4^6\}) \\ &= P_1 - \chi_{B_1} T_3^{(1)} + \mathcal{R}_c^{(0)},\end{aligned}$$

completing the proof of claim (3.58). As in the previous paragraph, we note that the highest order term $\frac{1}{y} A_\theta[T_2^{(0)}]T_2^{(0)}$ saturates bound (3.55) when $m = 1$.

Completion of the proof of (3.36)–(3.38). Combining claims (3.56), (3.57), and (3.58), we have proved that

$$\mathbf{D}_P P - P_1 = \chi_{B_1} \{\mathbf{D}_Q T_3^{(0)} - T_3^{(1)}\} + \mathcal{R}_c^{(0)}.$$

When $m = 1$ or $m \geq 3$, we have $T_3^{(0)} = 0$ and $T_3^{(1)} = \mathcal{J}_{0,1}^3$ (see (3.21)–(3.22)) so the right-hand side belongs to $\mathcal{R}_c^{(0)}$. When $m = 2$, we have $T_3^{(1)} = \mathcal{J}_1^3$ so $\chi_{B_1} T_3^{(1)}$ does not satisfy the last bound of (3.55) logarithmically. Thus we introduced a nontrivial $T_3^{(0)} = -\beta^3 \frac{y^6}{384} Q$ in this case such that

$$\chi_{B_1} \{\mathbf{D}_Q T_3^{(0)} - T_3^{(1)}\} = \chi_{B_1} \left\{ -\beta^3 \frac{y^5}{64} Q - T_3^{(1)} \right\} = \mathcal{O}(\mathbf{1}_{[2,2B_1]} \beta^3 y^{-1} \log y) = \mathcal{R}_c^{(0)},$$

where we used (3.22). This completes the proof of (3.36)–(3.38).

Step 3. Proof of compatibility estimates (3.39)–(3.41). We proceed similarly as in the previous step. Denote by $\mathcal{R}_c^{(1)}$ any function satisfying the bounds (cf. (3.39)–(3.41))

$$\|\mathcal{R}_c^{(1)}\|_{L^2} \lesssim \beta^2, \quad \|\mathcal{R}_c^{(1)}\|_{\dot{H}_{m+2}^1} \lesssim \beta^3, \quad \|\mathcal{R}_c^{(1)}\|_{\dot{V}_{m+2}^{3/2}} \lesssim \beta^{7/2}. \quad (3.60)$$

First claim. We claim that

$$A_P P_1 = \chi_{B_1} A_{\hat{P}} \hat{P}_1 + \mathcal{R}_c^{(1)}. \quad (3.61)$$

To see this, we note that

$$A_P P_1 = \chi_{B_1} A_{\hat{P}} \hat{P}_1 + \chi_{B_1} (A_P - A_{\hat{P}}) \hat{P}_1 + (\partial_y \chi_{B_1}) \hat{P}_1.$$

Thus the claim follows from (using $\mathbf{1}_{[B_1, \infty)} |P_1|_k \lesssim_k \mathbf{1}_{[B_1, 2B_1]} \beta^2$)

$$\chi_{B_1} (A_P - A_{\hat{P}}) \hat{P}_1 + (\partial_y \chi_{B_1}) \hat{P}_1 = \mathcal{O}(\mathbf{1}_{[B_1, 2B_1]} \beta^2 y^{-1}) = \mathcal{R}_c^{(1)}.$$

Second claim. We claim that

$$\chi_{B_1} A_{\hat{P}} \hat{P}_1 = \chi_{B_1} A_{\hat{P}-T_3^{(0)}} \hat{P}_1 + \mathcal{R}_c^{(1)}. \quad (3.62)$$

To see this, as $T_3^{(0)} \neq 0$ only if $m = 2$, we may assume $m = 2$. Thus the claim follows from (using $\hat{P}_1 = \mathcal{O}(\mathcal{I}_{-3}^1 + \mathcal{I}_1^3)$)

$$\begin{aligned} \chi_{B_1} (A_{\hat{P}} - A_{\hat{P}-T_3^{(0)}}) \hat{P}_1 &= \mathcal{O}\left(\mathbf{1}_{[2, 2B_1]} \frac{1}{y} \int_0^y (\mathcal{I}_{-2}^3 + \mathcal{I}_4^6) y' dy' \cdot (\mathcal{I}_{-3}^1 + \mathcal{I}_1^3)\right) \\ &= \mathcal{O}(\mathbf{1}_{[2, 2B_1]} \{\mathcal{I}_{-4,1}^4 + \mathcal{I}_{0,1}^6 + \mathcal{I}_2^7 + \mathcal{I}_6^9\}) = \mathcal{R}_c^{(1)}. \end{aligned}$$

Note that the highest order term $-\frac{1}{y} A_\theta [T_3^{(0)}] T_3^{(1)}$ saturates bound (3.60) when $m = 2$.

Third claim. We claim that

$$\begin{aligned} \chi_{B_1} A_{\hat{P}-T_3^{(0)}} (T_1^{(1)} + T_2^{(1)}) &= \chi_{B_1} \left\{ \beta^2 \frac{y^2}{4} Q + \beta \left(A_\theta [Q, T_2^{(0)}] Q + A_\theta [Q, T_1^{(0)}] T_1^{(0)} + \frac{1}{2} A_\theta [T_1^{(0)}] Q \right) \right\} \\ &\quad + \mathcal{R}_c^{(1)}. \end{aligned} \quad (3.63)$$

Indeed, the claim follows from (cf. (3.59))

$$\begin{aligned} \chi_{B_1} A_{\hat{P}-T_3^{(0)}} (T_1^{(1)} + T_2^{(1)}) &= \chi_{B_1} \left\{ -\beta \frac{y}{2} \mathbf{D}_{\hat{P}-T_3^{(0)}} (Q + T_1^{(0)}) \right\} \\ &= \chi_{B_1} \left\{ -\beta \frac{y}{2} \left(L_Q T_1^{(0)} - \frac{2}{y} A_\theta [Q, T_2^{(0)}] Q \right. \right. \\ &\quad \left. \left. - \frac{2}{y} A_\theta [Q, T_1^{(0)}] T_1^{(0)} - \frac{1}{y} A_\theta [T_1^{(0)}] Q \right) \right\} \\ &\quad + \mathcal{O}(\mathbf{1}_{[2, 2B_1]} \{\mathcal{I}_{-1,1}^4 + \mathcal{I}_1^5 + \mathcal{I}_3^6\}) \end{aligned}$$

and applying $L_Q T_1^{(0)} = -\beta \frac{y}{2} Q$. Note that the term $-\frac{1}{y} A_\theta [T_2^{(0)}] T_2^{(1)}$ saturates bound (3.60) when $m = 1$.

Last claim. We claim that

$$\chi_{B_1} A_{\hat{P}-T_3^{(0)}} T_3^{(1)} = \chi_{B_1} A_Q T_3^{(1)} + \mathcal{R}_c^{(1)}. \quad (3.64)$$

This easily follows from (using $T_3^{(1)} = \mathcal{I}_1^3$)

$$\chi_{B_1} A_{\hat{P}-T_3^{(0)}} T_3^{(1)} = \chi_{B_1} A_Q T_3^{(1)} + \mathcal{O}(\mathbf{1}_{[2,2B_1]}\{\mathcal{I}_0^4 + \mathcal{I}_{0,1}^5 + \mathcal{I}_2^6 + \mathcal{I}_4^7\}).$$

Completion of the proof of (3.39)–(3.41). By claims (3.61), (3.62), (3.63), and (3.64), we have proved that

$$\begin{aligned} A_P P_1 = \chi_{B_1} \left\{ \beta^2 \frac{y^2}{4} Q \right\} + \chi_{B_1} \left\{ A_Q T_3^{(1)} + \beta(A_\theta[Q, T_2^{(0)}]Q + A_\theta[Q, T_1^{(0)}]T_1^{(0)} \right. \\ \left. + \frac{1}{2}A_\theta[T_1^{(0)}]Q) \right\} + \mathcal{R}_c^{(1)}. \end{aligned}$$

By the definition of $T_2^{(2)}$, $T_3^{(1)}$, P_2 , the above formula continues as

$$\text{RHS} = \chi_{B_1} \{T_2^{(2)} + T_3^{(2)}\} + \mathcal{R}_c^{(1)} = P_2 - \chi_{B_0} T_4^{(2)} + \mathcal{R}_c^{(2)}.$$

Since $T_4^{(2)} = \mathcal{I}_{1,2}^4$ and $B_0 = \beta^{-1/2}$ (much smaller than B_1), we have $\chi_{B_0} T_4^{(2)} = \mathcal{R}_c^{(2)}$, and hence the proof of (3.39)–(3.41) is completed.

Step 4. Proof of (3.43). We denote by $\mathcal{R}_0$ any function satisfying the bound (cf. the right-hand side of (3.43))

$$\mathbf{1}_{(0,R]} |\mathcal{R}_0| \lesssim_R \beta^3.$$

We need to estimate

$$\begin{aligned} \Psi = (b^2 + \eta^2 + p_3^{(b)})i\partial_b P + p_3^{(\eta)}i\partial_\eta P - ib\Lambda P - (m+1)\eta P \\ - \left(\int_0^y \text{Re}(\bar{P} P_1) dy' \right) P + \mathbf{D}_P^* P_1. \end{aligned}$$

Since the bounds are allowed to depend on R , we only need to show that Ψ consists of cubic or higher-order terms in b and η . In particular, we do not need to care about any growing tails in y and cutoffs. The computations in this step are similar to those in [26], where the modified profile (3.1) was used.

We begin the proof of (3.43). Since $P_1 = -\beta \frac{y}{2} P + \mathcal{R}_0$, we have

$$\begin{aligned} \left(\int_0^y \text{Re}(\bar{P} P_1) dy' \right) P = \eta A_\theta[P] P + \mathcal{R}_0, \\ \mathbf{D}_P^* P_1 = -\beta \frac{y}{2} \left(\mathbf{D}_P^* - \frac{1}{y} \right) P + \mathcal{R}_0. \end{aligned}$$

We also note that $p_3^{(b)}i\partial_b P + p_3^{(\eta)}i\partial_\eta P = \mathcal{R}_0$. Thus we have

$$\Psi = (b^2 + \eta^2)i\partial_b P - ib\Lambda P - \eta((m+1) + A_\theta[P])P - \beta \frac{y}{2} \left(\mathbf{D}_P^* - \frac{1}{y} \right) P + \mathcal{R}_0.$$

Applying to the above the operator identities

$$\begin{aligned}\Lambda + \frac{y}{2} \left(\mathbf{D}_P^* - \frac{1}{y} \right) &= \frac{y}{2} \mathbf{D}_P, \\ (m+1) + A_\theta[P] + \frac{y}{2} \left(\mathbf{D}_P^* - \frac{1}{y} \right) &= -\frac{y}{2} \mathbf{D}_P,\end{aligned}$$

we have

$$\Psi = (b^2 + \eta^2) i \partial_b P - (ib - \eta) \frac{y}{2} \mathbf{D}_P P + \mathcal{R}_0.$$

Applying $(b^2 + \eta^2) i \partial_b P = (b^2 + \eta^2) \frac{y^2}{4} P + \mathcal{R}_0$ and $\mathbf{D}_P P = P_1 + \mathcal{R}_0 = -\beta \frac{y}{2} P + \mathcal{R}_0$ (see (3.38)) to the above formula completes the proof of (3.43).

Step 5. Some preliminary claims for the proof of (3.45) and (3.47)–(3.48). In contrast to the proof of (3.43), we cannot simply replace P_1 by $-\beta \frac{y}{2} P$ when $m = 1$ because the cubic terms of P_1 and $-\beta \frac{y}{2} P$, namely, $T_3^{(1)}$ and $-\beta \frac{y}{2} T_2^{(0)}$, are not the same and $-\beta \frac{y}{2} T_2^{(0)}$ rather has a dangerous growing tail. However, we would still like to partially replace P_1 by $-\beta \frac{y}{2} P$ in order to handle the algebraic computations efficiently. For this purpose, it is convenient to introduce the notations

$$\tilde{P} := Q + T_1^{(0)} \quad \text{and} \quad \tilde{P}_1 := T_1^{(1)} + T_2^{(1)} \quad \text{so that} \quad \tilde{P}_1 = -\beta \frac{y}{2} \tilde{P}.$$

With these additional notations, we claim the following.

(i) *A_θ -like terms.* We claim

$$\begin{aligned}\chi_{B_1} A_\theta[\tilde{P}] &= \chi_{B_1} A_\theta[Q] + \chi_{B_1} \mathcal{J}_0^1 + \mathcal{O}(\mathbf{1}_{[2,2B_1]} \mathcal{J}_{0,1}^2) \\ &= \chi_{B_1} A_\theta[Q] + \mathcal{O}(\mathbf{1}_{[2,2B_1]} \mathcal{J}_0^1),\end{aligned}\tag{3.65}$$

$$\begin{aligned}\chi_{B_1} \int_0^y \operatorname{Re}(\bar{P} P_1) dy' &= \chi_{B_1} \{ \eta A_\theta[\tilde{P}] \} + \mathcal{O}(\mathbf{1}_{[2,2B_1]} \{ \mathcal{J}_{0,1}^3 + \mathcal{J}_2^4 \}) \\ &= \chi_{B_1} \{ \eta A_\theta[Q] \} + \chi_{B_1} \mathcal{J}_0^2 + \mathcal{O}(\mathbf{1}_{[2,2B_1]} \{ \mathcal{J}_{0,1}^3 + \mathcal{J}_2^4 \}) \\ &= \chi_{B_1} \{ -2(m+1)\eta \} + \chi_{B_1} \mathcal{J}_{-2}^1 + \mathcal{O}(\mathbf{1}_{[2,2B_1]} \mathcal{J}_0^2).\end{aligned}\tag{3.66}$$

Estimate (3.65) is obvious from (3.19). The first equality of (3.66) follows from (using $P_1 = \mathcal{O}(\mathcal{J}_{-2}^1 + \mathcal{J}_0^2)$ and $P_1 + \beta \frac{y}{2} \tilde{P} = \mathcal{O}(\mathcal{J}_1^3)$ in the region $y \leq 2B_1$)

$$\begin{aligned}\chi_{B_1} \int_0^y \operatorname{Re} \left(\bar{P} P_1 + \eta \frac{y'}{2} \tilde{P}^2 \right) dy' \\ &= \chi_{B_1} \int_0^y \operatorname{Re} \left[\overline{(P - \tilde{P})} P_1 + \tilde{P} \left(P_1 + \beta \frac{y'}{2} \tilde{P} \right) \right] dy' \\ &= \mathcal{O} \left(\mathbf{1}_{[2,2B_1]} \int_0^y \{ \mathcal{J}_1^2 (\mathcal{J}_{-2}^1 + \mathcal{J}_0^2) + (\mathcal{J}_{-3}^0 + \mathcal{J}_{-1}^1) \mathcal{J}_1^3 \} dy' \right) \\ &= \mathcal{O}(\mathbf{1}_{[2,2B_1]} \{ \mathcal{J}_{0,1}^3 + \mathcal{J}_2^4 \}).\end{aligned}$$

This estimate combined with (3.65) gives the second equality of (3.66). Further applying $A_\theta[Q] = -2(m+1) + \mathcal{J}_{-4}^0$ gives the last equality of (3.66).

(ii) *Difference estimates for A_P^** (similar bounds also hold for A_P). We claim

$$\begin{aligned}\chi_{B_1} A_P^* &= \chi_{B_1} A_{\tilde{P}}^* + \chi_{B_1} \mathcal{J}_{-1,1}^2 + \mathcal{O}(\mathbf{1}_{[2,2B_1]}\{\mathcal{J}_1^3 + \mathcal{J}_3^4\}) \\ &= \chi_{B_1} A_Q^* + \chi_{B_1} \mathcal{J}_{-1}^1 + \mathcal{O}(\mathbf{1}_{[2,2B_1]}\{\mathcal{J}_{-1,1}^2 + \mathcal{J}_3^4\}).\end{aligned}\quad (3.67)$$

This follows from

$$\begin{aligned}\chi_{B_1}\{A_P^* - A_{\tilde{P}}^*\} &= \chi_{B_1}\left\{-\frac{1}{y}(A_\theta[P] - A_\theta[\tilde{P}])\right\} \\ &= \chi_{B_1}\left\{\frac{1}{2y}\int_0^y \mathcal{J}_{-2}^2 y' dy'\right\} + \mathcal{O}\left(\mathbf{1}_{[2,2B_1]}\frac{1}{y}\int_0^y (\mathcal{J}_0^3 + \mathcal{J}_2^4)y' dy'\right) \\ &= \chi_{B_1} \mathcal{J}_{-1,1}^2 + \mathcal{O}(\mathbf{1}_{[2,2B_1]}\{\mathcal{J}_1^3 + \mathcal{J}_3^4\})\end{aligned}$$

and (3.65).

(iii) *Difference estimates for $\tilde{H}_P$* (and the similar bounds hold for H_P). We claim

$$\begin{aligned}\chi_{B_1} \tilde{H}_P &= \chi_{B_1} \tilde{H}_{\tilde{P}} + \chi_{B_1} \mathcal{J}_{-2,1}^2 + \mathcal{O}(\mathbf{1}_{[2,2B_1]}\{\mathcal{J}_0^3 + \mathcal{J}_2^4\}) \\ &= \chi_{B_1} \tilde{H}_Q + \chi_{B_1} \mathcal{J}_{-2}^1 + \mathcal{O}(\mathbf{1}_{[2,2B_1]}\{\mathcal{J}_{-2,1}^2 + \mathcal{J}_2^4\}).\end{aligned}\quad (3.68)$$

The first equality follows from the expression

$$\begin{aligned}\tilde{H}_P - \tilde{H}_{\tilde{P}} &= \frac{1}{y}(2m + 4 + A_\theta[P] + A_\theta[\tilde{P}]) \cdot \frac{1}{y}(A_\theta[P] - A_\theta[\tilde{P}]) \\ &\quad + \frac{1}{2}(|P|^2 - |\tilde{P}|^2),\end{aligned}$$

the estimate

$$2m + 4 + A_\theta[P] + A_\theta[\tilde{P}] = \mathcal{J}_0^0 + \mathcal{O}(\mathbf{1}_{[2,2B_1]}\beta)$$

in the region $y \leq 2B_1$, and the first part of (3.67). The second equality can be proved similarly, using the second part of (3.67) instead.

Step 6. Key claim for the proof of (3.45). We denote by $\mathcal{R}_1$ any function satisfying the bound (cf. the right-hand side of (3.45))

$$\left\|\frac{1}{y^2}\mathcal{R}_1\right\|_{L^1} \lesssim \beta^4 |\log \beta|^3. \quad (3.69)$$

The goal of this step is to prove the following:

$$\begin{aligned}&(b^2 + \eta^2 + p_3^{(b)})i\partial_b P_1 + p_3^{(\eta)}i\partial_\eta P_1 - ib\Lambda_{-1}P_1 - (m+1)\eta P_1 \\ &\quad - \left(\int_0^y \operatorname{Re}(\bar{P}P_1)dy'\right)P_1 + A_P^*(\chi_{B_1}\{T_2^{(2)} + T_3^{(2)}\}) \\ &= \mathcal{R}_1.\end{aligned}\quad (3.70)$$

Notice that the left-hand side of (3.70) is equal to $\Psi_1 - A_P^*P_2$, but the expression itself does not involve $T_4^{(2)}$ that has not been chosen yet.

To show (3.70), we write

LHS of (3.70)

$$\begin{aligned}
&= A_P^* (\chi_{B_1} \{T_2^{(2)} + T_3^{(2)}\}) \\
&\quad + \chi_{B_1} \left\{ \left[(b^2 + \eta^2) i \partial_b - i b \Lambda_{-1} - \eta(m+1) - \left(\int_0^y \operatorname{Re}(\bar{P} P_1) dy' \right) \right] \tilde{P}_1 \right\} \\
&\quad + \chi_{B_1} \left\{ \left[(b^2 + \eta^2) i \partial_b - i b \Lambda_{-1} - \eta(m+1) - \left(\int_0^y \operatorname{Re}(\bar{P} P_1) dy' \right) \right] T_3^{(1)} \right\} \\
&\quad + \{(b^2 + \eta^2) i \partial_b - i b y \partial_y\} \chi_{B_1} \hat{P}_1 \\
&\quad + [p_3^{(b)} i \partial_b + p_3^{(\eta)} i \partial_\eta] P_1.
\end{aligned} \tag{3.71}$$

In what follows, we show that each line of the right-hand side of (3.71) belongs to $\mathcal{R}_1$.

First line of the right-hand side of (3.71). We claim that

$$\begin{aligned}
&A_P^* (\chi_{B_1} \{T_2^{(2)} + T_3^{(2)}\}) \\
&= \chi_{B_1} \left\{ -\beta A_{\tilde{P}}^* \left(\frac{y}{2} \tilde{P}_1 \right) \right\} + \chi_{B_1} \left\{ A_Q^* (T_3^{(2)} - \beta^2 \frac{y^2}{4} T_1^{(0)}) \right\} + \mathcal{R}_1.
\end{aligned} \tag{3.72}$$

To see this, we start from writing

$$\begin{aligned}
A_P^* (\chi_{B_1} \{T_2^{(2)} + T_3^{(2)}\}) &= \chi_{B_1} \{A_{\tilde{P}}^* T_2^{(2)} + A_Q^* T_3^{(2)}\} \\
&\quad + \chi_{B_1} \{(A_P^* - A_{\tilde{P}}^*) T_2^{(2)} + (A_P^* - A_Q^*) T_3^{(2)}\} \\
&\quad - (\partial_y \chi_{B_1}) (T_2^{(2)} + T_3^{(2)}).
\end{aligned}$$

We then substitute $T_2^{(2)} = \beta^2 \frac{y^2}{4} Q = -\beta \frac{y}{2} \tilde{P}_1 - \beta^2 \frac{y^2}{4} T_1^{(0)}$ into the above to obtain

$$\begin{aligned}
A_P^* P_2 &= \chi_{B_1} \left\{ -\beta A_{\tilde{P}}^* \left(\frac{y}{2} \tilde{P}_1 \right) + A_Q^* \left(T_3^{(2)} - \beta^2 \frac{y^2}{4} T_1^{(0)} \right) \right\} \\
&\quad + \chi_{B_1} \left\{ (A_P^* - A_{\tilde{P}}^*) T_2^{(2)} + (A_P^* - A_Q^*) T_3^{(2)} - (A_{\tilde{P}}^* - A_Q^*) \left(\beta^2 \frac{y^2}{4} T_1^{(0)} \right) \right\} \\
&\quad - (\partial_y \chi_{B_1}) (T_2^{(2)} + T_3^{(2)}).
\end{aligned}$$

Hence it remains to show that all the terms in the last two lines belong to $\mathcal{R}_1$. Using (3.67), we have

$$\begin{aligned}
\chi_{B_1} \{(A_P^* - A_{\tilde{P}}^*) T_2^{(2)}\} - (\partial_y \chi_{B_1}) T_2^{(2)} &= \mathcal{O} \left(\left(\mathbf{1}_{[2,2B_1]} \{ \mathcal{J}_{-1,1}^2 + \mathcal{J}_3^4 \} + \mathbf{1}_{[B_1,2B_1]} \frac{1}{y} \right) \cdot \mathcal{J}_{-1}^2 \right) \\
&= \mathcal{O}(\mathbf{1}_{[2,2B_1]} \{ \mathcal{J}_{-2,1}^4 + \mathcal{J}_2^6 \} + \mathbf{1}_{[B_1,2B_1]} \mathcal{J}_{-2}^2) = \mathcal{R}_1.
\end{aligned}$$

Using (3.67) and (3.23), we have

$$\begin{aligned}
\chi_{B_1} \{(A_P^* - A_Q^*) T_3^{(2)}\} - (\partial_y \chi_{B_1}) T_3^{(2)} &= \mathcal{O} \left(\left(\mathbf{1}_{[2,2B_1]} \{ \mathcal{J}_{-1}^1 + \mathcal{J}_3^4 \} + \mathbf{1}_{[B_1,2B_1]} \frac{1}{y} \right) \cdot \mathcal{J}_0^3 \right) \\
&= \mathcal{O}(\mathbf{1}_{[2,2B_1]} \{ \mathcal{J}_{-1}^4 + \mathcal{J}_3^7 \} + \mathbf{1}_{[B_1,2B_1]} \mathcal{J}_{-1}^3) = \mathcal{R}_1.
\end{aligned}$$

Using (3.65) and (3.19), we have

$$\begin{aligned}\chi_{B_1} \left\{ (A_{\tilde{P}}^* - A_Q^*) \left(\beta^2 \frac{y^2}{4} T_1^{(0)} \right) \right\} &= \mathcal{O}(\mathbf{1}_{[2,2B_1]} \mathcal{J}_{-1}^1 \cdot \mathcal{J}_1^3) \\ &= \mathcal{O}(\mathbf{1}_{[2,2B_1]} \mathcal{J}_0^4) = \mathcal{R}_1.\end{aligned}$$

This completes the proof of claim (3.72).

Second line of the right-hand side of (3.71). We claim that

$$\chi_{B_1} \left(\int_0^y \operatorname{Re}(\bar{P} P_1) dy' \right) \tilde{P}_1 = \chi_{B_1} \{ \eta A_\theta[\tilde{P}] \tilde{P}_1 \} + \mathcal{R}_1.$$

Indeed, this follows from (3.66) that

$$\begin{aligned}\chi_{B_1} \left(\int_0^y \operatorname{Re}(\bar{P} P_1) dy' \right) \tilde{P}_1 &= \chi_{B_1} \{ \eta A_\theta[\tilde{P}] \tilde{P}_1 \} + \mathcal{O}(\mathbf{1}_{[2,2B_1]} (\mathcal{J}_{0,1}^3 + \mathcal{J}_2^4) (\mathcal{J}_{-2}^1 + \mathcal{J}_0^2)) \\ &= \chi_{B_1} \{ \eta A_\theta[\tilde{P}] \tilde{P}_1 \} + \mathcal{O}(\mathbf{1}_{[2,2B_1]} \{ \mathcal{J}_{-2,1}^4 + \mathcal{J}_{0,1}^5 + \mathcal{J}_2^6 \}) \\ &= \chi_{B_1} \{ \eta A_\theta[\tilde{P}] \tilde{P}_1 \} + \mathcal{R}_1.\end{aligned}$$

Third line of the right-hand side of (3.71). We claim that

$$\chi_{B_1} \left\{ \left[(b^2 + \eta^2) i \partial_b - i b \Lambda_{-1} - \eta(m+1) - \left(\int_0^y \operatorname{Re}(\bar{P} P_1) dy' \right) \right] T_3^{(1)} \right\} = \mathcal{R}_1. \quad (3.73)$$

We separate into two cases: (i) $m = 1$ or $m \geq 3$, and (ii) $m = 2$. In the first case, i.e., when $m = 1$ or $m \geq 3$, the square bracket is of size $\mathcal{O}(\beta)$ and we can use the spatial decay $T_3^{(1)} = \mathcal{J}_{0,1}^3$ (see (3.21)–(3.22)) to have

$$\begin{aligned}\chi_{B_1} \left\{ \left[(b^2 + \eta^2) i \partial_b - i b \Lambda_{-1} - \eta(m+1) - \left(\int_0^y \operatorname{Re}(\bar{P} P_1) dy' \right) \right] T_3^{(1)} \right\} \\ = \mathcal{O}(\mathbf{1}_{[2,2B_1]} \mathcal{J}_{0,1}^4).\end{aligned}$$

In the latter case, i.e., when $m = 2$, the bound $T_3^{(1)} = \mathcal{O}(\mathcal{J}_1^3)$ is insufficient. However, we can exploit some spatial cancellation using the spatial asymptotics (3.22). We handle the integral term in the square bracket using (3.66)

$$\begin{aligned}\chi_{B_1} \left\{ - \int_0^y \operatorname{Re}(\bar{P} P_1) dy' \right\} &= \chi_{B_1} \left\{ \eta \frac{1}{2} \int_0^y Q^2 y' dy' \right\} + \mathcal{O}(\mathbf{1}_{[2,2B_1]} \mathcal{J}_0^2) \\ &= \chi_{B_1} \{ \eta(2m+2) \} + \chi_{B_1} \mathcal{J}_{-2}^1 + \mathcal{O}(\mathbf{1}_{[2,2B_1]} \mathcal{J}_0^2),\end{aligned}$$

from which we obtain

$$\begin{aligned}\chi_{B_1} \left\{ \left[(b^2 + \eta^2) i \partial_b - i b \Lambda_{-1} - \eta(m+1) - \left(\int_0^y \operatorname{Re}(\bar{P} P_1) dy' \right) \right] T_3^{(1)} \right\} \\ = \chi_{B_1} \{ [(b^2 + \eta^2) i \partial_b - i b \Lambda_{-1} + \eta(m+1)] T_3^{(1)} \} + \mathcal{O}(\mathbf{1}_{[2,2B_1]} \{ \mathcal{J}_{-1}^4 + \mathcal{J}_1^5 \}).\end{aligned}$$

We apply (3.22) to the above. Observe that *the contribution of the tail* $\beta^3 y^{-(m-3)} = \beta^3 y$ of $T_3^{(1)}$ *cancels when* $m = 2$:

$$[(b^2 + \eta^2)i\partial_b - ib\Lambda_{-1} + \eta(m+1)]\beta^3 y = [(b^2 + \eta^2)(-3\beta^2) - 3ib\beta^3 + 3\eta\beta^3]y = 0.$$

Therefore, we have the improved spatial decay

$$\chi_{B_1}\{[(b^2 + \eta^2)i\partial_b - ib\Lambda_{-1} + \eta(m+1)]T_3^{(1)}\} = \mathcal{O}(\mathbf{1}_{[2,2B_1]}\mathcal{J}_{-1,1}^4) = \mathcal{R}_1.$$

This completes the proof of claim (3.73).

Fourth line of the right-hand side of (3.71). Thanks to the *special cancellation (3.49) of the cutoff*, we have

$$\{[(b^2 + \eta^2)i\partial_b - iy\partial_y]\chi_{B_1}\}\hat{P}_1 = 0.$$

Last line of the right-hand side of (3.71). We claim that

$$[p_3^{(b)}i\partial_b + p_3^{(\eta)}i\partial_\eta]P_1 = \chi_{B_1}\left\{p_3^{(b)}\frac{y}{2}Q - p_3^{(\eta)}i\frac{y}{2}Q\right\} + \mathcal{R}_1. \quad (3.74)$$

Indeed, this follows from

$$\begin{aligned} & [p_3^{(b)}i\partial_b + p_3^{(\eta)}i\partial_\eta]P_1 \\ &= \chi_{B_1}\left\{p_3^{(b)}\frac{y}{2}Q - p_3^{(\eta)}i\frac{y}{2}Q\right\} + \chi_{B_1}\left\{[p_3^{(b)}i\partial_b + p_3^{(\eta)}i\partial_\eta]\left(\hat{P}_1 + \beta\frac{y}{2}Q\right)\right\} \\ & \quad + \{[p_3^{(b)}i\partial_b + p_3^{(\eta)}i\partial_\eta]\chi_{B_1}\}\hat{P}_1 \\ &= \chi_{B_1}\left\{p_3^{(b)}\frac{y}{2}Q - p_3^{(\eta)}i\frac{y}{2}Q\right\} + \mathcal{O}(\mathbf{1}_{[2,2B_1]}\beta^4). \end{aligned}$$

Completion of the proof of (3.70). Substituting claims (3.72)–(3.74) into (3.71), we arrive at

$$\begin{aligned} \text{LHS of (3.70)} &= \chi_{B_1}\left\{-\beta A_{\tilde{P}}^*\left(\frac{y}{2}\tilde{P}_1\right)\right\} + \chi_{B_1}\left\{A_Q^*\left(T_3^{(2)} - \beta^2\frac{y^2}{4}T_1^{(0)}\right)\right\} \\ & \quad + \chi_{B_1}\{[(b^2 + \eta^2)i\partial_b - ib\Lambda_{-1} - \eta(m+1) + A_\theta[\tilde{P}]]\tilde{P}_1\} \\ & \quad + \chi_{B_1}\left\{p_3^{(b)}\frac{y}{2}Q - p_3^{(\eta)}i\frac{y}{2}Q\right\} + \mathcal{R}_1. \end{aligned}$$

Applying to the above the operator identities

$$\begin{aligned} A_{\tilde{P}}^*\left(\frac{y}{2}\cdot\right) + \Lambda_{-1} &= \frac{y}{2}A_{\tilde{P}} + 1, \\ A_{\tilde{P}}^*\left(\frac{y}{2}\cdot\right) + m + 1 + A_\theta[\tilde{P}] &= -\left(\frac{y}{2}A_{\tilde{P}} + 1\right), \end{aligned}$$

we obtain

$$\begin{aligned} \text{LHS of (3.70)} &= \chi_{B_1}\left\{\left[(b^2 + \eta^2)i\partial_b - (ib - \eta)\left(\frac{y}{2}A_{\tilde{P}} + 1\right)\right]\tilde{P}_1\right\} \\ & \quad + \chi_{B_1}\left\{p_3^{(b)}\frac{y}{2}Q - p_3^{(\eta)}i\frac{y}{2}Q + A_Q^*\left(T_3^{(2)} - \beta^2\frac{y^2}{4}T_1^{(0)}\right)\right\} \\ & \quad + \mathcal{R}_1. \end{aligned} \quad (3.75)$$

A direct computation using $\tilde{P}_1 = -\beta \frac{y}{2} \tilde{P}$, $T_{1,0}^{(0)} = -i \frac{y^2}{4} Q$, and $L_Q T_1^{(0)} + \beta \frac{y}{2} Q = 0$ gives

$$\begin{aligned} & \left[(b^2 + \eta^2) i \partial_b - (i b - \eta) \left(\frac{y}{2} A_{\tilde{P}} + 1 \right) \right] \tilde{P}_1 \\ &= (b^2 + \eta^2) \left(\frac{y}{2} \tilde{P} - i \beta \frac{y}{2} T_{1,0}^{(0)} - \left(\frac{y}{2} A_{\tilde{P}} + 1 \right) \left(\frac{y}{2} \tilde{P} \right) \right) \\ &= (b^2 + \eta^2) \left(-i \beta \frac{y}{2} T_{1,0}^{(0)} - \frac{y^2}{4} \mathbf{D}_{\tilde{P}} \tilde{P} \right) \\ &= -(b^2 + \eta^2) \frac{y^2}{4} \left(\beta \frac{y}{2} Q + L_Q T_1^{(0)} + N_Q(T_1^{(0)}) \right) = -(b^2 + \eta^2) \frac{y^2}{4} N_Q(T_1^{(0)}), \end{aligned}$$

which belongs to $\mathcal{R}_1$,

$$\begin{aligned} & \chi_{B_1} \left\{ -(b^2 + \eta^2) \frac{y^2}{4} N_Q(T_1^{(0)}) \right\} \\ &= \mathcal{O} \left(\mathbf{1}_{[2,2B_1]} \mathcal{J}_2^2 \cdot \left(\frac{1}{y} \int_0^y (\mathcal{J}_{-4}^1 + \mathcal{J}_{-2}^2) y' dy' \cdot \mathcal{J}_{-1}^1 + \frac{1}{y} \int_0^y \mathcal{J}_{-2}^2 y' dy' \cdot \mathcal{J}_{-3}^0 \right) \right) \\ &= \mathcal{O}(\mathbf{1}_{[2,2B_1]} \mathcal{J}_0^4) = \mathcal{R}_1. \end{aligned}$$

Therefore, (3.75) becomes

$$\text{LHS of (3.70)} = \chi_{B_1} \left\{ p_3^{(b)} \frac{y}{2} Q - p_3^{(\eta)} i \frac{y}{2} Q + A_Q^* \left(T_3^{(2)} - \beta^2 \frac{y^2}{4} T_1^{(0)} \right) \right\} + \mathcal{R}_1. \quad (3.76)$$

By the definition of $p_3^{(b)}$, $p_3^{(\eta)}$, and $T_3^{(2)}$ (see (3.13) and (3.14)), the curly bracket of (3.76) vanishes. This completes the proof of (3.70).

Step 7. Key claim for the construction of $T_4^{(2)}$ and the proof of (3.47)–(3.48). We denote by $\mathcal{R}_2$ any function satisfying the bounds (cf. right-hand sides of (3.47)–(3.48))

$$\|\mathcal{R}_2\|_{L_{m+2}^2} \lesssim \beta^4 |\log \beta|^2 \quad \text{and} \quad \|\mathcal{R}_2\|_{\dot{H}_{m+2}^1} \lesssim \beta^{9/2} |\log \beta|^2. \quad (3.77)$$

Denote by $\mathcal{A}_4^{(2)}$ any function $f(y; b, \eta)$ of the form $f = \sum_{i+j=4} b^i \eta^j f_{i,j}(y)$ such that each $f_{i,j}$ satisfies the pointwise bounds

$$|f_{i,j}|_k \lesssim_k \begin{cases} y^{-m} \log y & \text{if } m \in \{1, 2\}, \\ y^{-m+2} & \text{if } m \geq 3. \end{cases} \quad (3.78)$$

The goal of this step is to prove the following:

$$\begin{aligned} & \left[(b^2 + \eta^2 + p_3^{(b)}) i \partial_b + p_3^{(\eta)} i \partial_\eta - i b \Lambda_{-2} - (m+1) \eta - \left(\int_0^y \text{Re}(\bar{P} P_1) dy' \right) + \tilde{H}_P \right] \\ & \quad \times (\chi_{B_1} \{T_2^{(2)} + T_3^{(2)}\}) - \bar{P} P_1^2 = \chi_{B_1} F_4^{(2)} + \mathcal{R}_2 \end{aligned} \quad (3.79)$$

for some function $F_4^{(2)}$ of class $\mathcal{A}_4^{(2)}$. Notice that the left-hand side of (3.79) does not involve $T_4^{(2)}$ that has not been chosen yet.

To show (3.79), we write

LHS of (3.79)

$$\begin{aligned}
&= \tilde{H}_P(\chi_{B_1}\{T_2^{(2)} + T_3^{(2)}\}) - \bar{P}P_1^2 \\
&\quad + \chi_{B_1}\left\{\left[(b^2 + \eta^2)i\partial_b - ib\Lambda_{-2} - (m+1)\eta - \left(\int_0^y \operatorname{Re}(\bar{P}P_1)dy'\right)\right]T_2^{(2)}\right\} \\
&\quad + \chi_{B_1}\left\{\left[(b^2 + \eta^2)i\partial_b - ib\Lambda_{-2} - (m+1)\eta - \left(\int_0^y \operatorname{Re}(\bar{P}P_1)dy'\right)\right]T_3^{(2)}\right\} \\
&\quad + \{(b^2 + \eta^2)i\partial_b - iby\partial_y\}\chi_{B_1}\{T_2^{(2)} + T_3^{(2)}\} \\
&\quad + [p_3^{(b)}i\partial_b + p_3^{(\eta)}i\partial_\eta](\chi_{B_1}\{T_2^{(2)} + T_3^{(2)}\}). \tag{3.80}
\end{aligned}$$

In the following, we show that each line of the right-hand side of (3.80) takes the form $\chi_{B_1}\mathcal{A}_4^{(2)} + \mathcal{R}_2$.

First line of the right-hand side of (3.80). We claim that

$$\begin{aligned}
&\tilde{H}_P(\chi_{B_1}\{T_2^{(2)} + T_3^{(2)}\}) - \bar{P}P_1^2 \\
&\quad = \chi_{B_1}\left\{-\beta^3\left(A_Q^* - \frac{1}{y}\right)\left(\frac{y^3}{8}Q\right)\right\} + \chi_{B_1}\mathcal{A}_4^{(2)} + \mathcal{R}_2. \tag{3.81}
\end{aligned}$$

To see this, we first treat the cutoff errors

$$\tilde{H}_P(\chi_{B_1}\{T_2^{(2)} + T_3^{(2)}\}) - \bar{P}P_1^2 = \chi_{B_1}\tilde{H}_P(T_2^{(2)} + T_3^{(2)}) - \chi_{B_1}\tilde{P}\hat{P}_1^2 + \mathcal{O}(\mathbf{1}_{[B_1, 2B_1]}\mathcal{J}_{-3}^2).$$

The term $\chi_{B_1}\tilde{P}\hat{P}_1^2$ can be written as

$$\begin{aligned}
\chi_{B_1}\tilde{P}\hat{P}_1^2 &= \chi_{B_1}\tilde{P}\tilde{P}_1^2 + \chi_{B_1}\mathcal{J}_{-3m}^4 + \mathcal{O}(\mathbf{1}_{[2, 2B_1]}\{\mathcal{J}_{-1}^5 + \mathcal{J}_1^6\}) \\
&= \chi_{B_1}\left\{|\tilde{P}|^2\left(\beta^2\frac{y^2}{4}(Q + T_1^{(0)})\right)\right\} + \chi_{B_1}\mathcal{J}_{-3m}^4 + \mathcal{O}(\mathbf{1}_{[2, 2B_1]}\{\mathcal{J}_{-1}^5 + \mathcal{J}_1^6\}) \\
&= \chi_{B_1}\left\{|\tilde{P}|^2 \cdot T_2^{(2)} + Q^2 \cdot \beta^2\frac{y^2}{4}T_1^{(0)}\right\} + \chi_{B_1}\mathcal{A}_4^{(2)} + \mathcal{R}_2.
\end{aligned}$$

Next, we use (3.68), $T_2^{(2)} = \mathcal{J}_{-m}^2$, $T_3^{(2)} = \mathcal{J}_{-m+2}^3$ (when it contributes to the quartic term) and $T_3^{(2)} = \mathcal{J}_0^3$ (when it contributes to the remainder, see (3.23)) to have

$$\begin{aligned}
\chi_{B_1}\tilde{H}_PT_2^{(2)} &= \chi_{B_1}\tilde{H}_{\bar{P}}T_2^{(2)} + \chi_{B_1}\mathcal{J}_{-(m+2),1}^4 + \mathcal{O}(\mathbf{1}_{[2, 2B_1]}\{\mathcal{J}_{-1}^5 + \mathcal{J}_1^6\}), \\
\chi_{B_1}\tilde{H}_PT_3^{(2)} &= \chi_{B_1}\tilde{H}_QT_3^{(2)} + \chi_{B_1}\mathcal{J}_{-m}^4 + \mathcal{O}(\mathbf{1}_{[2, 2B_1]}\{\mathcal{J}_{-2,1}^5 + \mathcal{J}_2^7\}).
\end{aligned}$$

By the previous three formulas, we have proved that

$$\begin{aligned}
&\tilde{H}_P(\chi_{B_1}\{T_2^{(2)} + T_3^{(2)}\}) - \bar{P}P_1^2 \\
&\quad = \chi_{B_1}\left\{(\tilde{H}_{\bar{P}} - |\tilde{P}|^2)T_2^{(2)} + \left(\tilde{H}_QT_3^{(2)} - Q^2 \cdot \beta^2\frac{y^2}{4}T_1^{(0)}\right)\right\} \\
&\quad \quad + \chi_{B_1}\mathcal{A}_4^{(2)} + \mathcal{R}_2. \tag{3.82}
\end{aligned}$$

For the first term of the right-hand side of (3.82), using the operator identity

$$\tilde{H}_{\tilde{P}} - |\tilde{P}|^2 = \left(A_{\tilde{P}}^* - \frac{1}{y}\right)\left(A_{\tilde{P}} - \frac{1}{y}\right)$$

and the cancellation $(A_Q - \frac{1}{y})T_2^{(2)} = 0$, we have

$$\chi_{B_1}\{(\tilde{H}_{\tilde{P}} - |\tilde{P}|^2)T_2^{(2)}\} = \chi_{B_1}\left\{\left(A_{\tilde{P}}^* - \frac{1}{y}\right)(A_{\tilde{P}} - A_Q)T_2^{(2)}\right\}.$$

We decompose this into the cubic, quartic, and the remainder terms

$$\begin{aligned} & \chi_{B_1}\left\{\left(A_{\tilde{P}}^* - \frac{1}{y}\right)(A_{\tilde{P}} - A_Q)T_2^{(2)}\right\} \\ &= \chi_{B_1}\left\{\left[A_Q^* - \frac{1}{y}\right]\left(-\frac{2}{y}A_\theta[Q, T_1^{(0)}]T_2^{(2)}\right)\right\} \\ & \quad + \chi_{B_1}\left\{\left[A_Q^* - \frac{1}{y}\right]\left(\frac{1}{y}\int_0^y \mathfrak{J}_{-2}^2 y' dy' \cdot \mathfrak{J}_{-m}^2\right)\right. \\ & \quad + \frac{1}{y^2}\int_0^y \mathfrak{J}_{-4}^1 y' dy' \int_0^y \mathfrak{J}_{-4}^1 y' dy' \cdot \mathfrak{J}_{-m}^2\left\} \\ & \quad + \mathcal{O}\left(\mathbf{1}_{[2,2B_1]}\left(\frac{1}{y^2}\int_0^y (\mathfrak{J}_{-4}^1 + \mathfrak{J}_{-2}^2)y' dy' \int_0^y \mathfrak{J}_{-2}^2 y' dy'\right) \cdot \mathfrak{J}_{-1}^2\right) \\ &= \chi_{B_1}\left\{\beta^2\left[A_Q^* - \frac{1}{y}\right]\left(-\frac{y}{2}A_\theta[Q, T_1^{(0)}]Q\right)\right\} \\ & \quad + \chi_{B_1}\{\mathfrak{J}_{-m-2,1}^4\} + \mathcal{O}(\mathbf{1}_{[2,2B_1]}(\mathfrak{J}_{-3,1}^5 + \mathfrak{J}_{-3,2}^6)). \end{aligned}$$

For the second term of the right-hand side of (3.82), we use $\tilde{H}_Q T_3^{(2)} = \beta^2 \tilde{H}_Q (\frac{y^2}{4} T_1^{(0)})$ (see (3.13)) to have

$$\begin{aligned} \chi_{B_1}\left\{\tilde{H}_Q T_3^{(2)} - \beta^2 \frac{y^2}{4} Q^2 T_1^{(0)}\right\} &= \chi_{B_1}\left\{\beta^2[\tilde{H}_Q - Q^2]\left(\frac{y^2}{4} T_1^{(0)}\right)\right\} \\ &= \chi_{B_1}\left\{\beta^2\left(A_Q^* - \frac{1}{y}\right)\left(A_Q - \frac{1}{y}\right)\left(\frac{y^2}{4} T_1^{(0)}\right)\right\} \\ &= \chi_{B_1}\left\{\beta^2\left[A_Q^* - \frac{1}{y}\right]\left(\frac{y^2}{4} \mathbf{D}_Q T_1^{(0)}\right)\right\}. \end{aligned}$$

Substituting the previous three formulas into (3.82) gives

$$\begin{aligned} & \tilde{H}_P(\chi_{B_1}\{T_2^{(2)} + T_3^{(2)}\}) - \bar{P}P_1^2 \\ &= \chi_{B_1}\left\{\beta^2\left[A_Q^* - \frac{1}{y}\right]\left(\frac{y^2}{4} \mathbf{D}_Q T_1^{(0)} - \frac{y}{2}A_\theta[Q, T_1^{(0)}]Q\right)\right\} + \chi_{B_1}\mathcal{A}_4^{(2)} + \mathcal{R}_2. \end{aligned}$$

Applying the identity

$$\mathbf{D}_Q T_1^{(0)} - \frac{2}{y}A_\theta[Q, T_1^{(0)}]Q = L_Q T_1^{(0)} = -\beta \frac{y}{2}Q$$

completes the proof of claim (3.81).

Second line of the right-hand side of (3.80). We claim that

$$\begin{aligned} \chi_{B_1} \left\{ \left[(b^2 + \eta^2) i \partial_b - i b \Lambda_{-2} - (m+1)\eta - \left(\int_0^y \operatorname{Re}(\bar{P} P_1) dy' \right) \right] T_2^{(2)} \right\} \\ = \chi_{B_1} \left\{ -\beta^3 \partial_y \left(\frac{y^3}{4} Q \right) \right\} + \chi_{B_1} \mathcal{A}_4^{(2)} + \mathcal{R}_2. \end{aligned} \quad (3.83)$$

To see this, we rewrite the left-hand side of (3.83) as

$$\begin{aligned} \chi_{B_1} \left\{ \left[(b^2 + \eta^2) i \partial_b - i b \Lambda_{-2} - (m+1)\eta - \left(\int_0^y \operatorname{Re}(\bar{P} P_1) dy' \right) \right] T_2^{(2)} \right\} \\ = \chi_{B_1} \{ [(b^2 + \eta^2) i \partial_b - i b \Lambda_{-2} - \eta((m+1) + A_\theta[Q])] T_2^{(2)} \} \\ - \chi_{B_1} \left\{ \left(\int_0^y \operatorname{Re}(\bar{P} P_1) dy' - \eta A_\theta[Q] \right) T_2^{(2)} \right\}. \end{aligned}$$

For the first term, we use $T_2^{(2)} = \beta^2 \frac{y^2}{4} Q$ and $(m + A_\theta[Q])Q = y \partial_y Q$ (by (1.8)) to have the algebraic identity

$$\begin{aligned} [(b^2 + \eta^2) i \partial_b - i b \Lambda_{-2} - \eta((m+1) + A_\theta[Q])] T_2^{(2)} \\ = [2(ib - \eta) - i b \Lambda_{-2} - \eta(y \partial_y - 1)] \left(\beta^2 \frac{y^2}{4} Q \right) \\ = [-\beta(y \partial_y + 1)] \left(\beta^2 \frac{y^2}{4} Q \right) \\ = -\beta^3 \partial_y \left(\frac{y^3}{4} Q \right). \end{aligned}$$

For the second term, we use (3.66) and $T_2^{(2)} = \mathcal{J}_{-m}^2$ to have

$$\begin{aligned} \chi_{B_1} \left\{ \left(\int_0^y \operatorname{Re}(\bar{P} P_1) dy' - \eta A_\theta[Q] \right) T_2^{(2)} \right\} \\ = \chi_{B_1} \mathcal{J}_{-m}^4 + \mathcal{O}(\mathbf{1}_{[2,2B_1]}\{\mathcal{J}_{-m,1}^5 + \mathcal{J}_{-m+2}^6\}) = \chi_{B_1} \mathcal{A}_4^{(2)} + \mathcal{R}_2. \end{aligned}$$

This completes the proof of (3.83).

Third line of the right-hand side of (3.80). We claim that

$$\begin{aligned} \chi_{B_1} \left\{ \left[(b^2 + \eta^2) i \partial_b - i b \Lambda_{-2} - (m+1)\eta - \left(\int_0^y \operatorname{Re}(\bar{P} P_1) dy' \right) \right] T_3^{(2)} \right\} \\ = \chi_{B_1} \mathcal{A}_4^{(2)} + \mathcal{R}_2. \end{aligned} \quad (3.84)$$

Indeed, the proof of this claim is very similar to that of (3.73). Using $T_3^{(2)} = \mathcal{J}_0^3$ and (3.66) and proceeding as in the proof of (3.73), we have

$$\begin{aligned} \chi_{B_1} \left\{ \left[(b^2 + \eta^2) i \partial_b - i b \Lambda_{-2} - \eta(m+1) - \left(\int_0^y \operatorname{Re}(\bar{P} P_1) dy' \right) \right] T_3^{(2)} \right\} \\ = \chi_{B_1} \{ [(b^2 + \eta^2) i \partial_b - i b \Lambda_{-2} + \eta(m+1) + \mathcal{J}_{-2}^1 + \mathcal{O}(\mathcal{J}_0^2)] T_3^{(2)} \}. \end{aligned}$$

When $m = 1$, we apply the improved decay $T_3^{(2)} = \mathcal{J}_{-1,1}^3$ from (3.21). When $m \geq 3$, we apply the decay $T_3^{(2)} = \mathcal{J}_{-m+2}^3$. When $m = 2$, as in the proof of (3.73), there is a full cancellation of the contribution of the tail β^3 from $T_3^{(2)}$. Therefore, we have

$$\begin{aligned} \chi_{B_1} & \left\{ \left[(b^2 + \eta^2) i \partial_b - i b \Lambda_{-1} - \eta(m+1) - \left(\int_0^y \operatorname{Re}(\bar{P} P_1) dy' \right) \right] T_3^{(2)} \right\} \\ &= \begin{cases} \chi_{B_1} \mathcal{J}_0^1 \cdot \mathcal{J}_{-1,1}^3 + \mathcal{O}(\mathbf{1}_{[2,2B_1]} \mathcal{J}_0^2 \cdot \mathcal{J}_{-1,1}^3) & \text{if } m = 1, \\ \chi_{B_1} (\mathcal{J}_0^1 \cdot \mathcal{J}_{-2,1}^3 + \mathcal{J}_{-2}^1 \cdot \mathcal{J}_0^3) + \mathcal{O}(\mathbf{1}_{[2,2B_1]} \mathcal{J}_0^2 \cdot \mathcal{J}_0^3) & \text{if } m = 2, \\ \chi_{B_1} \mathcal{J}_0^1 \cdot \mathcal{J}_{-m+2}^3 + \mathcal{O}(\mathbf{1}_{[2,2B_1]} \mathcal{J}_0^2 \cdot \mathcal{J}_{-(m-2)}^3) & \text{if } m \geq 3 \end{cases} \\ &= \chi_{B_1} \mathcal{A}_4^{(2)} + \mathcal{R}_2. \end{aligned}$$

This completes the proof of (3.84).

Fourth line of the right-hand side of (3.80). Thanks to cancellation (3.49), we have

$$\{[(b^2 + \eta^2) i \partial_b - i b y \partial_y] \chi_{B_1}\} (T_2^{(2)} + T_3^{(2)}) = 0.$$

Last line of the right-hand side of (3.80). We claim that

$$[p_3^{(b)} i \partial_b + p_3^{(\eta)} i \partial_\eta] (\chi_{B_1} \{T_2^{(2)} + T_3^{(2)}\}) = \chi_{B_1} \mathcal{A}_4^{(2)} + \mathcal{R}_2. \quad (3.85)$$

Indeed,

$$\begin{aligned} & [p_3^{(b)} i \partial_b + p_3^{(\eta)} i \partial_\eta] (\chi_{B_1} \{T_2^{(2)} + T_3^{(2)}\}) \\ &= \chi_{B_1} \{ [p_3^{(b)} i \partial_b + p_3^{(\eta)} i \partial_\eta] T_2^{(2)} \} + \{ [p_3^{(b)} i \partial_b + p_3^{(\eta)} i \partial_\eta] \chi_{B_1} \} T_2^{(2)} \\ & \quad + [p_3^{(b)} i \partial_b + p_3^{(\eta)} i \partial_\eta] (\chi_{B_1} T_3^{(2)}) \\ &= \chi_{B_1} \mathcal{J}_{-m}^4 + \mathcal{O}(\mathbf{1}_{[2,2B_1]} \mathcal{J}_0^5 + \mathbf{1}_{[B_1,2B_1]} \mathcal{J}_{-m}^4) \\ &= \chi_{B_1} \mathcal{A}_4^{(2)} + \mathcal{R}_2. \end{aligned}$$

Completion of the proof of (3.79). Substituting claims (3.81) and (3.83)–(3.85) into (3.80), we obtain

$$\text{LHS of (3.79)} = \chi_{B_1} \left\{ -\beta^3 \partial_y \left(\frac{y^3}{4} \mathcal{Q} \right) - \beta^3 \left(A_Q^* - \frac{1}{y} \right) \left(\frac{y^3}{8} \mathcal{Q} \right) \right\} + \chi_{B_1} \mathcal{A}_4^{(2)} + \mathcal{R}_2.$$

Since $\partial_y \left(\frac{y^3}{4} \mathcal{Q} \right) + \left(A_Q^* - \frac{1}{y} \right) \left(\frac{y^3}{8} \mathcal{Q} \right) = \left(A_Q - \frac{2}{y} \right) \left(\frac{y^3}{8} \mathcal{Q} \right) = 0$, the curly bracket of the above formula vanishes. This completes the proof of (3.79).

Step 8. Construction of $T_4^{(2)}$ and the proof of (3.45) and (3.47)–(3.48).

Construction of $T_4^{(2)}$. Recall (3.79). From the expression of the left-hand side of (3.79), the function $F_4^{(2)}$ depends only on already defined functions $T_j^{(k)}$, $j \leq 3$. Now we define

$$T_4^{(2)} := \begin{cases} -^{(\text{out})} \tilde{H}_Q^{-1} F_4^{(2)} & \text{if } m = 1, \\ -^{(\text{in})} \tilde{H}_Q^{-1} F_4^{(2)} & \text{if } m \geq 2, \end{cases} \quad (3.86)$$

where $^{(\text{out})} \tilde{H}_Q^{-1}$ and $^{(\text{in})} \tilde{H}_Q^{-1}$ are defined in Lemma A.6. One can easily check bound (3.29) for $T_4^{(2)}$ using Lemma A.6.

Proof of (3.45). It suffices to estimate

$$\Psi_1 = A_P^*(\chi_{B_0} T_4^{(2)}) + \text{LHS of (3.70)}.$$

By claim (3.70) and recalling $\mathcal{R}_1$ from (3.69), it suffices to show $A_P^*(\chi_{B_0} T_4^{(2)}) = \mathcal{R}_1$. Using (3.29), this follows from

$$A_P^*(\chi_{B_0} T_4^{(2)}) = \mathcal{O}(\mathbf{1}_{[2,2B_0]} \mathcal{I}_{0,2}^4) = \mathcal{R}_1.$$

Note that this term saturates bound (3.45). This completes the proof of (3.45).

Proof of (3.47)–(3.48). It suffices to estimate

$$\begin{aligned} \Psi_2 = & \left[(b^2 + \eta^2 + p_3^{(b)})i\partial_b + p_3^{(\eta)}i\partial_\eta - ib\Lambda_{-2} - (m+1)\eta \right. \\ & \left. - \left(\int_0^y \text{Re}(\bar{P}P_1)dy' \right) + \tilde{H}_P \right] (\chi_{B_0} T_4^{(2)}) + \text{LHS of (3.79)}. \end{aligned}$$

By claim (3.79) and definition (3.86) of $T_4^{(2)}$, we see that

$$\begin{aligned} \Psi_2 = & \left[(b^2 + \eta^2 + p_3^{(b)})i\partial_b + p_3^{(\eta)}i\partial_\eta - ib\Lambda_{-2} - (m+1)\eta - \left(\int_0^y \text{Re}(\bar{P}P_1)dy' \right) \right] \\ & \times (\chi_{B_0} T_4^{(2)}) + \chi_{B_0} (\tilde{H}_P - \tilde{H}_Q) T_4^{(2)} + \left[\chi_{B_0}, \partial_{yy} + \frac{1}{y} \partial_y \right] T_4^{(2)} \\ & + (\chi_{B_1} - \chi_{B_0}) F_4^{(2)} + \mathcal{R}_2, \end{aligned}$$

where we recall $\mathcal{R}_2$ from (3.77). It suffices to show that all terms of the above formula belong to $\mathcal{R}_2$. For the first and second lines, as the terms in the square bracket are of size $\mathcal{O}(\beta)$, these lines are of size $\mathcal{O}(\mathbf{1}_{[2,2B_0]} \mathcal{I}_{1,2}^5)$, which belongs to $\mathcal{R}_2$ due to $B_0 = \beta^{-1/2}$. For the last line of the above formula, we use (3.68), (3.29), and (3.78) to have

$$\begin{aligned} \chi_{B_0} (\tilde{H}_P - \tilde{H}_Q) T_4^{(2)} &= \mathcal{O}(\mathbf{1}_{[2,2B_0]} \{\mathcal{I}_{-1,2}^5 + \mathcal{I}_{3,2}^8\}) = \mathcal{R}_2, \\ \left[\chi_{B_0}, \partial_{yy} + \frac{1}{y} \partial_y \right] T_4^{(2)} &= \mathcal{O}(\mathbf{1}_{[B_0,2B_0]} \mathcal{I}_{-1,2}^4) = \mathcal{R}_2, \\ (\chi_{B_1} - \chi_{B_0}) F_4^{(2)} &= \mathcal{R}_2. \end{aligned}$$

This completes the proof of (3.47)–(3.48). This finishes the proof of Proposition 3.3. $\blacksquare$

4. Modulation analysis

In this section, we study the dynamics of solutions near soliton using modulation analysis. Near-soliton dynamics is relevant to our ultimate goal (Theorem 1.4) because any H_m^1 finite-time blow-up solutions eventually enter a small neighborhood of Q after renormalization by Remark 2.13.

Using modulation analysis, our main goal of this section is the *(local-in-time) uniform H^3 -bound of the remainder part of the solution*. More precisely, using the modified profile $P(\cdot; b, \eta)$ constructed in the previous section, we show that an H_m^3 -solution $u(t)$ lying in a soliton tube (i.e., in a neighborhood of solitons $Q_{\lambda, \gamma}$) admits a decomposition

$$u(t, r) = \frac{e^{i\gamma(t)}}{\lambda(t)} [P(\cdot; b(t), \eta(t)) + \varepsilon(t, \cdot)] \left(\frac{r}{\lambda(t)} \right) = [P + \varepsilon]^\sharp$$

with the remainder $\varepsilon^\sharp(t)$ being H_m^3 -bounded on I . It should be remarked that we obtain this H^3 -boundedness without any dynamical hypotheses on the modulation parameters such as $b > 0$ (λ is contracting) or $|\eta| \ll b$ (rotational instability is turned off), which are typically assumed in the blow-up construction.

4.1. Decomposition of solutions

In this subsection, we explain how we fix the decomposition of $u(t)$. The decomposition will be fixed for each time t by imposing four (nonlinear) orthogonality conditions on the remainder $\varepsilon(t)$. The following is a fixed-time analysis and hence we omit the time variable t in this subsection.

Recall that we view (CSS) as system (2.5)–(2.7) of equations for u , $u_1 = \mathbf{D}_u u$, and $u_2 = A_u \mathbf{D}_u u$. As in the previous section, where we constructed the modified profiles P , P_1 , and P_2 for the renormalized variables w , w_1 , and w_2 , the decomposition here will take the form

$$\begin{aligned} w &= \lambda e^{-i\gamma} u(\lambda \cdot) = P(\cdot; b, \eta) + \varepsilon, \\ w_1 &= \mathbf{D}_w w = P_1(\cdot; b, \eta) + \varepsilon_1, \\ w_2 &= A_w w_1 = P_2(\cdot; b, \eta) + \varepsilon_2. \end{aligned}$$

To fix the decomposition, as in [28], we impose two orthogonality conditions on ε and the remaining two on ε_1 . More precisely, we use the orthogonality conditions

$$(\varepsilon, Z_1)_r = (\varepsilon, Z_2)_r = (\varepsilon_1, \tilde{Z}_3)_r = (\varepsilon_1, \tilde{Z}_4)_r = 0, \quad (4.1)$$

where $Z_1, Z_2 \in C_{c,m}^\infty$ and $\tilde{Z}_3, \tilde{Z}_4 \in C_{c,m+1}^\infty$ are chosen in Remark 2.5. The reason for using two ε_1 -orthogonality conditions is that the b - and η -modulation equations become more apparent in the w_1 -equation compared to the w -equation. We note that because $\varepsilon_1 \approx L_Q \varepsilon$ at the leading order, the last two orthogonality conditions of (4.1) can be viewed as slight modifications of ε -orthogonality conditions $(\varepsilon, L_Q^* \tilde{Z}_3)_r = (\varepsilon, L_Q^* \tilde{Z}_4)_r = 0$.

We now state the decomposition lemma. Denote an $\dot{H}_m^1$ -soliton tube by

$$\dot{\mathcal{T}}_{\alpha^*} := \{u \in \dot{H}_m^1 : \inf_{(\lambda, \gamma) \in \mathbb{R}_+ \times (\mathbb{R}/2\pi\mathbb{Z})} \|u_{1/\lambda, -\gamma} - Q\|_{\dot{H}_m^1} < \alpha^*\}.$$

As a standard application of the implicit function theorem, we obtain the following.

Lemma 4.1 (Decomposition). *There exist $\delta_0, \alpha_0 > 0$ such that any $u \in \tilde{\mathcal{T}}_{\alpha_0}$ admits the unique decomposition*

$$\begin{cases} u = \frac{e^{i\gamma}}{\lambda} [P(\cdot; b, \eta) + \varepsilon] \left(\frac{\cdot}{\lambda} \right), \\ u_1 = \mathbf{D}_u u = \frac{e^{i\gamma}}{\lambda^2} [P_1(\cdot; b, \eta) + \varepsilon_1] \left(\frac{\cdot}{\lambda} \right), \end{cases}$$

satisfying the orthogonality conditions

$$(\varepsilon, Z_1)_r = (\varepsilon, Z_2)_r = (\varepsilon_1, \tilde{Z}_3)_r = (\varepsilon_1, \tilde{Z}_4)_r = 0, \quad (4.1)$$

and smallness

$$\beta + \|\varepsilon\|_{\dot{H}_m^1} < \delta_0.$$

Moreover, if $u \in \tilde{\mathcal{T}}_{\alpha^*}$ with $\alpha^* < \alpha_0$, we have a qualitative smallness

$$\beta + \|\varepsilon\|_{\dot{H}_m^1} = o_{\alpha^* \rightarrow 0}(1). \quad (4.2)$$

Proof sketch. By the standard argument using the implicit function theorem, see [28, Lemma 5.2] for a closely related proof, it suffices to show that

$$\begin{aligned} & \begin{pmatrix} ((\mathbf{v})_1, Z_1)_r & ((\mathbf{v})_2, Z_1)_r & ((\mathbf{v})_3, Z_1)_r & ((\mathbf{v})_4, Z_1)_r \\ ((\mathbf{v})_1, Z_2)_r & ((\mathbf{v})_2, Z_2)_r & ((\mathbf{v})_3, Z_2)_r & ((\mathbf{v})_4, Z_2)_r \\ ((\mathbf{v})_1, \tilde{Z}_3)_r & ((\mathbf{v})_2, \tilde{Z}_3)_r & ((\mathbf{v})_3, \tilde{Z}_3)_r & ((\mathbf{v})_4, \tilde{Z}_3)_r \\ ((\mathbf{v})_1, \tilde{Z}_4)_r & ((\mathbf{v})_2, \tilde{Z}_4)_r & ((\mathbf{v})_3, \tilde{Z}_4)_r & ((\mathbf{v})_4, \tilde{Z}_4)_r \end{pmatrix} \\ &= \text{diag}\left((\Lambda Q, Z_1)_r, (-iQ, Z_2)_r, \left(i\frac{\gamma}{2}Q, \tilde{Z}_3\right)_r, \left(\frac{\gamma}{2}Q, \tilde{Z}_4\right)_r\right) + O(\beta). \end{aligned} \quad (4.3)$$

Note that the entries in the right-hand side of are nonzero due to (2.37) and (2.39).

For the proof of (4.3), by (3.30)–(3.33) and the fact that $Z_1, Z_2, \tilde{Z}_3, \tilde{Z}_4$ are compactly supported, we have

$$\begin{aligned} \text{LHS of (4.3)} &= \begin{pmatrix} (\Lambda Q, Z_1)_r & (-iQ, Z_1)_r & (i\frac{\gamma^2}{4}Q, Z_1)_r & ((m+1)\rho, Z_1)_r \\ (\Lambda Q, Z_2)_r & (-iQ, Z_2)_r & (i\frac{\gamma^2}{4}Q, Z_2)_r & ((m+1)\rho, Z_2)_r \\ 0 & 0 & (i\frac{\gamma}{2}Q, \tilde{Z}_3)_r & (\frac{\gamma}{2}Q, \tilde{Z}_3)_r \\ 0 & 0 & (i\frac{\gamma}{2}Q, \tilde{Z}_4)_r & (\frac{\gamma}{2}Q, \tilde{Z}_4)_r \end{pmatrix} \\ &+ O(\beta). \end{aligned}$$

Here, the upper right 2×2 matrix is in fact zero, thanks to (2.38). Moreover, as $Z_1, \tilde{Z}_4$ are real and $Z_2, \tilde{Z}_3$ are imaginary, the remaining off-diagonal entries vanish. This completes the proof of (4.3). For the remaining arguments for the proof of this lemma, we refer to [28, Lemma 5.2]. $\blacksquare$

4.2. Nonlinear coercivity of energy

For H_m^1 -solutions, a remarkable fact observed in [29] (in the context where b - and η -modulations are absent with only the first two orthogonality conditions of (4.1) assumed) is that the qualitative smallness $\|\varepsilon\|_{\dot{H}_m^1} = o_{\alpha^* \rightarrow 0}(1)$ can be improved to the *quantitative smallness*

$$\|\varepsilon\|_{\dot{H}_m^1} \lesssim_{M[u]} \lambda \sqrt{E[u]}, \quad (4.4)$$

which we call the *nonlinear coercivity of energy*. Note that (4.4) is a significant improvement over the qualitative smallness in the sense that (4.4) gives the *uniform H^1 -control* $\|\varepsilon^\sharp\|_{\dot{H}_m^1} \lesssim_{M[u]} \sqrt{E[u]}$. In [29], this estimate was at the heart of the proof of soliton resolution and in particular of the proof of the upper bound (1.13). We remark that (4.4) is a non-perturbative estimate. It is rather a consequence of the *self-duality* and *nonlocality* of our problem, which are two distinguished features of (CSS); see the introduction of [29] or Remark 4.4 below for more details.

In this subsection, we will obtain such a quantitative smallness estimate in the presence of b - and η -modulations, namely, the bound

$$|b| + |\eta| + \|\varepsilon\|_{\dot{H}_m^1} \lesssim_{M[u]} \lambda \sqrt{E[u]}.$$

This bound will be the key *a priori* bound allowing the energy estimates in Section 4.5.

Before we state the full result, let us introduce more pieces of notations. For $u \in H_m^1 \cap \tilde{\mathcal{T}}_{\alpha_0}$, hence u having finite energy and having a decomposition as in Lemma 4.1, let us define a *scaling invariant* quantity

$$\mu := \lambda \sqrt{E[u]}.$$

For $M, \alpha^* > 0$, we denote an H_m^1 -soliton tube by

$$\mathcal{T}_{\alpha^*, M} := \{u \in H_m^1 : u \in \tilde{\mathcal{T}}_{\alpha^*} \text{ and } M[u] \leq M\}. \quad (4.5)$$

Lemma 4.2 (Nonlinear coercivity of energy). *For any $M > 0$, the following holds if $\alpha^* > 0$ is sufficiently small (depending on M). For any $u \in \mathcal{T}_{\alpha^*, M}$, consider the decomposition of u as in Lemma 4.1.*

- (Rough bound from mass conservation) *We have*

$$\|\varepsilon\|_{L^2} \lesssim_M 1. \quad (4.6)$$

- (Small parameter μ) *We have*¹⁴

$$\mu = \delta_M(\alpha^*). \quad (4.7)$$

- (Nonlinear coercivity of energy) *We have*

$$\beta + \|\varepsilon\|_{\dot{H}_m^1} + \|\varepsilon_1\|_{L^2} \sim_M \mu. \quad (4.8)$$

¹⁴Recall the notation $\delta_M(\alpha^*)$ from Section 2.1.

Remark 4.3. In view of the uniqueness of zero energy solutions, we have $\mu = 0$ if and only if $u = Q_{\lambda, y}$. Thus it is natural to expect that μ is a small parameter.

Remark 4.4 (Self-duality, nonlocality, and nonlinear coercivity of energy). Following the discussion in the introduction of [29], we explain why (4.4) is a non-perturbative estimate and how the self-duality and nonlocality give (4.4).

By the *self-duality* of (1.7), (4.4) is implied by

$$\|\mathbf{D}_{Q+\varepsilon}(Q + \varepsilon)\| \gtrsim_M \|\varepsilon\|_{\dot{H}_m^1}. \quad (4.9)$$

Note that we have $\|\varepsilon\|_{\dot{H}_m^1} \ll 1$ from the qualitative smallness but $\|\varepsilon\|_{L^2}$ might be large. In the compact region $y \lesssim 1$, the $\dot{H}_m^1$ -smallness is enough to guarantee

$$\mathbf{D}_{Q+\varepsilon}(Q + \varepsilon) \approx L_Q \varepsilon \quad \text{for } y \lesssim 1$$

in view of (2.11). Applying the linear coercivity (2.34) gives (4.9) in the region $y \lesssim 1$. However, in the far region $y \gg 1$, the pure ε -term $\frac{1}{y} A_\theta[\varepsilon]\varepsilon$ of $\mathbf{D}_{Q+\varepsilon}(Q + \varepsilon)$ is *not perturbative* due to the optimality of the scaling-invariant estimate $\|\frac{1}{y} A_\theta[\varepsilon]\varepsilon\|_{L^2} \lesssim \|\varepsilon\|_{L^2}^2 \|\varepsilon\|_{\dot{H}_m^1}$ and the fact that $\|\varepsilon\|_{L^2}$ is not necessarily small. In this region, we only have

$$\mathbf{D}_{Q+\varepsilon}(Q + \varepsilon) \approx \left(\partial_y - \frac{1}{y} (m + A_\theta[Q] + A_\theta[\varepsilon]) \right) \varepsilon =: \mathbf{D}_{Q, \varepsilon} \varepsilon \quad \text{for } y \gg 1.$$

Now we use the *nonlocality* of the problem, particularly the fact that $m + A_\theta[Q] \approx -(m + 2)$ is negative. In view of (1.7), this says that the exterior energy of $Q + \varepsilon$ is similar to the exterior energy of ε for $-(m + 2)$ -equivariant (CSS). Thanks to the negativity of both $m + A_\theta[Q]$ and $A_\theta[\varepsilon]$ as well as the uniform bound $|A_\theta[\varepsilon]| \lesssim \|\varepsilon\|_{L^2}^2 \lesssim_M 1$, we can expect a Hardy-type inequality for the operator $\mathbf{D}_{Q, \varepsilon}$ as

$$\|\mathbf{1}_{y \sim 1} |f|_{-1}\|_{L^2} + \|\mathbf{1}_{y \gg 1} \mathbf{D}_{Q, \varepsilon} f\|_{L^2} \gtrsim_M \|\mathbf{1}_{y \gg 1} |f|_{-1}\|_{L^2}.$$

Substituting $f = \varepsilon$ into the above and using the control of ε in the region $y \sim 1$, we obtain the control of ε even in the region $y \gg 1$, completing (the sketch of) the proof of (4.9).

Proof of Lemma 4.2. In the proof of (4.8), we will use the inequality

$$\left\| L_Q \varepsilon - \frac{1}{y} A_\theta[\varepsilon]\varepsilon \right\|_{L^2} \gtrsim_M \|\varepsilon\|_{\dot{H}_m^1}, \quad (4.10)$$

which is essentially proved in Remark 4.4. For a rigorous proof, the readers may refer to [29, Lemma 4.4]. Note that the proof of $\|\varepsilon\|_{L^2} \lesssim_M 1$ is obvious from mass conservation. Henceforth, we show (4.7) and (4.8).

Step 1. A preliminary claim. In this step, we show that

$$\mu \sim \beta + \|\varepsilon_1\|_{L^2}. \quad (4.11)$$

To see this, using the self-dual form (1.7) of energy, we have

$$\mu \sim \|P_1 + \varepsilon_1\|_{L^2}.$$

Thus the $\lesssim$ -inequality of (4.11) easily follows from $\|P_1\|_{L^2} \lesssim \beta$ and the triangle inequality.

We turn to the more delicate proof of the $\gtrsim$ -inequality of (4.11). As before, we will consider the expression $\|P_1 + \varepsilon_1\|_{L^2}$. The higher-order terms of P_1 are *non-perturbative* in the region $y \sim B_1$, so we first use the inequality

$$\|P_1 + \varepsilon_1\|_{L^2} \geq \|\mathbf{1}_{(0,B_0]}(P_1 + \varepsilon_1)\|_{L^2}. \quad (4.12)$$

In this region, the higher-order terms of P_1 become perturbative (using (4.2)):

$$\left\| \mathbf{1}_{(0,B_0]} \left(P_1 + \beta \frac{y}{2} Q \right) \right\|_{L^2} \lesssim \|\mathbf{1}_{(0,B_0]} \beta^2\|_{L^2} \lesssim \beta^{3/2} = \delta(\alpha^*) \cdot \beta. \quad (4.13)$$

On the other hand, by the choice of $\tilde{Z}_3, \tilde{Z}_4$ (see (2.40)), we have

$$\begin{aligned} \left| \left(\mathbf{1}_{(0,B_0]} \beta \frac{y}{2} Q, \mathbf{1}_{(0,B_0]} \varepsilon_1 \right)_r \right| &= \left| \left(\beta (\mathbf{1}_{(0,B_0]} - \chi_{R_o}) \frac{y}{2} Q, \mathbf{1}_{(0,B_0]} \varepsilon_1 \right)_r \right| \\ &\leq \frac{1}{2} \left\| \mathbf{1}_{(0,B_0]} \beta \frac{y}{2} Q \right\|_{L^2} \|\mathbf{1}_{(0,B_0]} \varepsilon_1\|_{L^2}, \end{aligned}$$

and hence

$$\begin{aligned} &\left\| \mathbf{1}_{(0,B_0]} \left(\beta \frac{y}{2} Q + \varepsilon_1 \right) \right\|_{L^2}^2 \\ &= \left\| \mathbf{1}_{(0,B_0]} \beta \frac{y}{2} Q \right\|_{L^2}^2 + 2 \left(\mathbf{1}_{(0,B_0]} \beta \frac{y}{2} Q, \mathbf{1}_{(0,B_0]} \varepsilon_1 \right)_r + \|\mathbf{1}_{(0,B_0]} \varepsilon_1\|_{L^2}^2 \\ &\geq \left\| \mathbf{1}_{(0,B_0]} \beta \frac{y}{2} Q \right\|_{L^2}^2 - \left\| \mathbf{1}_{(0,B_0]} \beta \frac{y}{2} Q \right\|_{L^2} \|\mathbf{1}_{(0,B_0]} \varepsilon_1\|_{L^2} + \|\mathbf{1}_{(0,B_0]} \varepsilon_1\|_{L^2}^2 \\ &\geq \frac{1}{2} \left(\left\| \mathbf{1}_{(0,B_0]} \beta \frac{y}{2} Q \right\|_{L^2}^2 + \|\mathbf{1}_{(0,B_0]} \varepsilon_1\|_{L^2}^2 \right) \gtrsim \beta^2 + \|\mathbf{1}_{(0,B_0]} \varepsilon_1\|_{L^2}^2. \end{aligned} \quad (4.14)$$

By (4.12), (4.13), and (4.14), we have

$$\mu \sim \|P_1 + \varepsilon_1\|_{L^2} \gtrsim \beta + \|\mathbf{1}_{(0,B_0]} \varepsilon_1\|_{L^2}. \quad (4.15)$$

Since $\|\varepsilon_1\|_{L^2} \lesssim \|P_1\|_{L^2} + \mu \lesssim \beta + \mu \lesssim \mu$ (using $\beta \lesssim \mu$ from (4.15)), we can improve (4.15) to

$$\mu \gtrsim \beta + \|\varepsilon_1\|_{L^2}.$$

This completes the proof of claim (4.11).

Step 2. The quantity μ is a small parameter: Proof of (4.7). By (4.2) and (4.11), it suffices to show that

$$\|\varepsilon_1\|_{L^2} \lesssim_M \beta + \|\varepsilon\|_{\dot{H}_m^1}. \quad (4.16)$$

For the proof of (4.16), we start from the relation

$$\varepsilon_1 = w_1 - P_1 = (\mathbf{D}_P P - P_1) + (\mathbf{D}_w w - \mathbf{D}_P P), \quad (4.17)$$

take the L^2 -norm, and apply bound (3.36) to have

$$\|\varepsilon_1\|_{L^2} \lesssim \beta + \|\mathbf{D}_w w - \mathbf{D}_P P\|_{L^2}.$$

We then notice that $\mathbf{D}_w w - \mathbf{D}_P P$ is the sum of $(\partial_y - \frac{m}{y})\varepsilon$ and a linear combination of $\frac{1}{y}A_\theta[\psi_1, \psi_2]\psi_3$ with at least one $\psi_j = \varepsilon$. Using the estimate

$$\left\| \frac{1}{y}A_\theta[\psi_1, \psi_2]\psi_3 \right\|_{L^2} \lesssim \|\psi_{j_1}\|_{L^2} \|\psi_{j_2}\|_{L^2} \left\| \frac{1}{y}\psi_{j_3} \right\|_{L^2}$$

for any choice of j_1, j_2, j_3 , putting $\psi_{j_3} = \varepsilon$, and using $\|P\|_{L^2} + \|\varepsilon\|_{L^2} \lesssim_M 1$, we have

$$\left\| \frac{1}{y}A_\theta[\psi_1, \psi_2]\psi_3 \right\|_{L^2} \lesssim \left\| \left(\partial_y - \frac{m}{y} \right) \varepsilon \right\|_{L^2} + O_M \left(\left\| \frac{1}{y}\varepsilon \right\|_{L^2} \right) \lesssim_M \|\varepsilon\|_{\dot{H}_m^1},$$

completing the proof of claim (4.16).

Step 3. Nonlinear coercivity of energy: Proof of (4.8). By (4.11) and (4.16), it suffices to show that

$$\|\varepsilon\|_{\dot{H}_m^1} \lesssim_M \beta + \|\varepsilon_1\|_{L^2}. \quad (4.18)$$

For the proof of (4.18), we rewrite relation (4.17) as

$$L_Q \varepsilon - \frac{1}{y}A_\theta[\varepsilon]\varepsilon = \varepsilon_1 - (\mathbf{D}_P P - P_1) - \left\{ (L_P - L_Q)\varepsilon + N_P(\varepsilon) + \frac{1}{y}A_\theta[\varepsilon]\varepsilon \right\}.$$

In the above formula, we take the L^2 -norm, apply the nonlinear Hardy inequality (4.10) for the left-hand side and (3.36) for $\mathbf{D}_P P - P_1$ to have

$$\begin{aligned} \|\varepsilon\|_{\dot{H}_m^1} &\lesssim_M \left\| L_Q \varepsilon - \frac{1}{y}A_\theta[\varepsilon]\varepsilon \right\|_{L^2} \\ &\lesssim_M \|\varepsilon_1\|_{L^2} + \beta + \left\| (L_P - L_Q)\varepsilon + N_P(\varepsilon) + \frac{1}{y}A_\theta[\varepsilon]\varepsilon \right\|_{L^2}. \end{aligned} \quad (4.19)$$

The following estimate will be shown in the next paragraph:

$$\left\| (L_P - L_Q)\varepsilon + N_P(\varepsilon) + \frac{1}{y}A_\theta[\varepsilon]\varepsilon \right\|_{L^2} \lesssim O_M(\beta) + \delta_M(\alpha^*) \cdot \|\varepsilon\|_{\dot{H}_m^1}. \quad (4.20)$$

Substituting this estimate into (4.19), we have

$$\|\varepsilon\|_{\dot{H}_m^1} \lesssim O_M(\beta + \|\varepsilon_1\|_{L^2}) + \delta_M(\alpha^*) \cdot \|\varepsilon\|_{\dot{H}_m^1}.$$

Using the smallness of $\delta_M(\alpha^*)$, the last term of the right-hand side can be absorbed into the left-hand side, yielding claim (4.18) and hence (4.8).

It remains to show estimate (4.20), which is used in the proof of (4.18). We note that $(L_P - L_Q)\varepsilon + N_P(\varepsilon) + \frac{1}{y}A_\theta[\varepsilon]\varepsilon$ is a linear combination of $\frac{1}{y}A_\theta[\psi_1, \psi_2]\psi_3$, where either (for $(L_P - L_Q)\varepsilon$) exactly one $\psi_{j_1} = P - Q$, exactly one $\psi_{j_2} = \varepsilon$, and the other $\psi_{j_3} \in \{P, Q\}$ or (for $N_P(\varepsilon) + \frac{1}{y}A_\theta[\varepsilon]\varepsilon$) exactly two $\psi_{j_1} = \psi_{j_2} = \varepsilon$ and exactly one $\psi_{j_3} = P$. In what follows, we only estimate $\left\| \frac{1}{y}A_\theta[\psi_1, \psi_2]\psi_3 \right\|_{L^2}$ that can contribute to either $(L_P - L_Q)\varepsilon$ or $N_P(\varepsilon) + \frac{1}{y}A_\theta[\varepsilon]\varepsilon$.

(i) If $\psi_3 = \varepsilon$, then we use

$$|\psi_1 \psi_2| \lesssim |P - Q|(|P| + |Q|) + |P\varepsilon| \lesssim |P - Q|(|P| + |Q| + |\varepsilon|) + |Q\varepsilon|$$

to obtain

$$\begin{aligned} \left\| \frac{1}{y} A_\theta[\psi_1, \psi_2] \varepsilon \right\|_{L^2} &\lesssim \|P - Q\|_{L^\infty} \| |P| + Q + |\varepsilon| \|_{L^2} \|\varepsilon\|_{L^2} + \|Q\varepsilon\|_{L^1} \left\| \frac{1}{y} \varepsilon \right\|_{L^2} \\ &\lesssim \beta(1 + \|\varepsilon\|_{L^2}) \|\varepsilon\|_{L^2} + \left\| \frac{1}{y} \varepsilon \right\|_{L^2}^2 \lesssim O_M(\beta) + \delta_M(\alpha^*) \cdot \|\varepsilon\|_{\dot{H}_m^1}. \end{aligned}$$

(ii) If $\psi_3 \in \{P, Q\}$, then we use

$$|\psi_1 \psi_2| \lesssim (|P - Q| + |\varepsilon|) |\varepsilon| \quad \text{and} \quad |\psi_3| \lesssim Q + \mathbf{1}_{(0, 2B_1]} \beta^2 y^4 Q$$

to obtain (using (2.28) for $\|\varepsilon\|_{L^\infty}$)

$$\begin{aligned} \left\| \frac{1}{y} A_\theta[\psi_1, \psi_2] \psi_3 \right\|_{L^2} &\lesssim \left\| \frac{1}{y} \psi_1 \psi_2 \right\|_{L^2} \|yQ\|_{L^2} + \|\psi_1 \psi_2\|_{L^1} \|\mathbf{1}_{(0, 2B_1]} \beta^2 y^3 Q\|_{L^2} \\ &\lesssim (\beta + \|\varepsilon\|_{L^\infty}) \left\| \frac{1}{y} \varepsilon \right\|_{L^2} + (1 + \|\varepsilon\|_{L^2}) \|\varepsilon\|_{L^2} \beta \\ &\lesssim \delta_M(\alpha^*) \cdot \|\varepsilon\|_{\dot{H}_m^1} + O_M(\beta). \end{aligned}$$

(iii) If $\psi_3 = P - Q$, then we use

$$|\psi_1 \psi_2| \lesssim (|P| + Q) |\varepsilon|$$

to obtain

$$\left\| \frac{1}{y} A_\theta[\psi_1, \psi_2] (P - Q) \right\|_{L^2} \lesssim \left\| \frac{1}{y} (P - Q) \right\|_{L^2} \|\psi_1 \psi_2\|_{L^1} \lesssim \beta \|\varepsilon\|_{L^2} \lesssim_M \beta.$$

This completes the proof of (4.20). By the first paragraph of Step 3, this completes the proof of (4.8) and hence the proof of Lemma 4.2. $\blacksquare$

Remark 4.5 (Assumptions in the rest of this section). Until the end of this section, we always assume that our solution u has H_m^3 -regularity and belongs to $\mathcal{T}_{\alpha^*, M}$ for sufficiently small $\alpha^* > 0$, depending on M . The required smallness of $\alpha^* > 0$ may shrink in each of the claims below.

4.3. Hardy bounds for higher Sobolev and local norms

In later sections, we will ultimately do the energy estimates for the variable ε_2 . To close the energy estimates for ε_2 and to obtain useful information on the original solution $u = [P + \varepsilon]_{\lambda, \gamma}$ from the control of ε_2 , it is necessary to convert various Hardy-type bounds for ε_2 into those for ε and ε_1 . For this purpose, let us define

$$X_2 := \|\varepsilon_2\|_{L^2} + \mu^2, \tag{4.21}$$

$$X_3 := \|\varepsilon_2\|_{\dot{H}_{m+2}^1} + \mu X_2, \tag{4.22}$$

$$V_{7/2} := \|\varepsilon_2\|_{\dot{Y}_{m+2}^{3/2}} + \mu^{1/2} X_3. \tag{4.23}$$

Note that the $\dot{V}_{m+2}^{3/2}$ -norm defined in (2.29) was motivated by the lower bound of the Morawetz monotonicity (2.24). The goal of this subsection is to show that these quantities bound the Sobolev norms of ε and ε_1 with correct scalings.

Lemma 4.6 (Hardy bounds for higher Sobolev and local norms).

(1) For ε_1 , we have

$$\|\varepsilon_1\|_{\dot{H}_{m+1}^1} \lesssim_M X_2, \quad (4.24)$$

$$\|\varepsilon_1\|_{\dot{H}_{m+1}^2} \lesssim_M X_3, \quad (4.25)$$

$$\|\varepsilon_1\|_{\dot{V}_{m+1}^{5/2}} \lesssim_M V_{7/2}. \quad (4.26)$$

(2) For ε , we have

$$\|\varepsilon\|_{\dot{\mathcal{H}}_m^2} \lesssim_M X_2, \quad (4.27)$$

$$\|\varepsilon\|_{\dot{\mathcal{H}}_m^3} \lesssim_M X_3. \quad (4.28)$$

Remark 4.7 (Weighted L^∞ controls). One can also control various weighted L^∞ -norms on ε , ε_1 , ε_2 by combining Lemma 2.6 with the above lemma.

Proof of Lemma 4.6. Step 0. Remark on the order of the proof. To be more rigorous, we need to prove this lemma in the following order: (4.24), (4.27), (4.25), (4.28), (4.26). However, we will give the proof of (4.24)–(4.26) first and then give the proof of (4.27)–(4.28) in order to group similar proofs as possible.

Step 1. From ε_2 to ε_1 : Proof of (4.24)–(4.26). Rewrite $A_w w_1 = w_2$ as

$$A_Q \varepsilon_1 = \varepsilon_2 - (A_P P_1 - P_2) - \{(A_w - A_Q)\varepsilon_1 - (A_w - A_P)P_1\}. \quad (4.29)$$

We claim that

$$\|(A_w - A_Q)\varepsilon_1 - (A_w - A_P)P_1\|_{L^2} \lesssim_M \mu^2, \quad (4.30)$$

$$\|(A_w - A_Q)\varepsilon_1 - (A_w - A_P)P_1\|_{\dot{H}_{m+2}^1} \lesssim_M \mu X_2, \quad (4.31)$$

$$\|(A_w - A_Q)\varepsilon_1 - (A_w - A_P)P_1\|_{\dot{V}_{m+2}^{3/2}} \lesssim_M \mu^{1/2} X_3. \quad (4.32)$$

Let us first assume these claims and finish the proof of (4.24)–(4.26). Applying to (4.29) the linear coercivity (2.31)–(2.33) for $A_Q \varepsilon_1$, bounds (3.39)–(3.41) for $A_P P_1 - P_2$, claims (4.30)–(4.32) for $(A_w - A_Q)\varepsilon_1 - (A_w - A_P)P_1$, and $\beta \lesssim_M \mu$, we obtain

$$\|\varepsilon_1\|_{\dot{H}_{m+1}^1} \lesssim \|\varepsilon_2\|_{L^2} + \beta^2 + O_M(\mu^2) \lesssim_M X_2,$$

$$\|\varepsilon_1\|_{\dot{H}_{m+1}^2} \lesssim \|\varepsilon_2\|_{\dot{H}_{m+2}^1} + \beta^3 + O_M(\mu X_2) \lesssim_M X_3,$$

$$\|\varepsilon_1\|_{\dot{V}_{m+1}^{5/2}} \lesssim \|\varepsilon_2\|_{\dot{V}_{m+2}^{3/2}} + \beta^{7/2} + O_M(\mu^{1/2} X_3) \lesssim_M V_{7/2}.$$

This completes the proof of (4.24)–(4.26).

It remains to show claims (4.30)–(4.32). For the term $(A_w - A_Q)\varepsilon_1$, we note the pointwise estimates

$$\begin{aligned}(A_w - A_Q)\varepsilon_1 &\lesssim \frac{1}{y} \int_0^y ||w|^2 - Q^2| y' dy' \cdot |\varepsilon_1|, \\ |(A_w - A_Q)\varepsilon_1|_{-1} &\lesssim \frac{1}{y} \int_0^y ||w|^2 - Q^2| y' dy' \cdot |\varepsilon_1|_{-1} + ||w|^2 - Q^2| |\varepsilon_1|.\end{aligned}$$

Using (2.42) and $\|\langle y \rangle^{-1/2} \varepsilon_1\|_{L^\infty} \lesssim \|\langle y \rangle^{-1/2} |\varepsilon_1|_{-1}\|_{L^2}$, we have

$$\begin{aligned}\|(A_w - A_Q)\varepsilon_1\|_{L^2} &\lesssim ||w|^2 - Q^2\|_{L^2} \|\varepsilon_1\|_{L^2}, \\ \|(A_w - A_Q)\varepsilon_1\|_{\dot{H}_{m+1}^1} &\lesssim ||w|^2 - Q^2\|_{L^2} \|\varepsilon_1\|_{\dot{H}_{m+1}^1}, \\ \|(A_w - A_Q)\varepsilon_1\|_{\dot{V}_{m+1}^{3/2}} &\lesssim ||w|^2 - Q^2\|_{L^2} \|\langle y \rangle^{-1/2} |\varepsilon_1|_{-1}\|_{L^2}.\end{aligned}$$

Applying to the above the estimates

$$\begin{aligned}||w|^2 - Q^2\|_{L^2} &\lesssim_M \|w - Q\|_{L^\infty} \lesssim_M \mu, \\ \|y^{-1/2} |\varepsilon_1|_{-1}\|_{L^2} &\lesssim \|\varepsilon_1\|_{\dot{H}_{m+1}^1}^{1/2} \|\varepsilon_1\|_{\dot{H}_{m+1}^2}^{1/2},\end{aligned}$$

we have

$$\begin{aligned}\|(A_w - A_Q)\varepsilon_1\|_{L^2} &\lesssim_M \mu \|\varepsilon_1\|_{L^2} \lesssim_M \mu^2, \\ \|(A_w - A_Q)\varepsilon_1\|_{\dot{H}_{m+1}^1} &\lesssim_M \mu \|\varepsilon_1\|_{\dot{H}_{m+1}^1} \lesssim_M \mu X_2, \\ \|(A_w - A_Q)\varepsilon_1\|_{\dot{V}_{m+1}^{3/2}} &\lesssim_M \mu \|\varepsilon_1\|_{\dot{H}_{m+1}^1}^{1/2} \|\varepsilon_1\|_{\dot{H}_{m+1}^2}^{1/2} \lesssim_M \mu^{1/2} X_3.\end{aligned}$$

For the term $(A_w - A_P)P_1$, we use

$$\begin{aligned}||w|^2 - |P|^2\|_{L^2} &\lesssim_M \|\varepsilon\|_{L^\infty} \lesssim_M \mu, \\ \left\| \frac{1}{\langle y \rangle} (|w|^2 - |P|^2) \right\|_{L^2} &\lesssim_M \left\| \frac{1}{\langle y \rangle} Q\varepsilon \right\|_{L^2} + \| |P - Q| + |\varepsilon| \|_{L^\infty} \left\| \frac{1}{\langle y \rangle} \varepsilon \right\|_{L^2} \lesssim_M X_2, \\ \left\| \frac{1}{\langle y \rangle^{3/2}} (|w|^2 - |P|^2) \right\|_{L^2} &\lesssim_M \left\| \frac{1}{\langle y \rangle^{3/2}} Q\varepsilon \right\|_{L^2} + \| |P - Q| + |\varepsilon| \|_{L^\infty} \left\| \frac{1}{\langle y \rangle^{3/2}} \varepsilon \right\|_{L^2} \\ &\lesssim_M \|\varepsilon\|_{\dot{\mathcal{H}}_m^3} + \mu \|\varepsilon\|_{\dot{H}_m^1}^{1/2} \|\varepsilon\|_{\dot{\mathcal{H}}_m^2}^{1/2} \lesssim_M X_3 + \mu^{1/2} X_2,\end{aligned}$$

and

$$\|P_1\|_{L^2} + \|\langle y \rangle |P_1|_1\|_{L^\infty} \lesssim \beta \lesssim_M \mu$$

to have

$$\begin{aligned}\|(A_w - A_P)P_1\|_{L^2} &\lesssim ||w|^2 - |P|^2\|_{L^2} \|P_1\|_{L^2} \lesssim_M X_2, \\ \|(A_w - A_P)P_1\|_{\dot{H}_{m+1}^1} &\lesssim \left\| \frac{1}{\langle y \rangle} (|w|^2 - |P|^2) \right\|_{L^2} \|\langle y \rangle |P_1|_1\|_{L^\infty} \lesssim_M X_3, \\ \|(A_w - A_P)P_1\|_{\dot{V}_{m+1}^{3/2}} &\lesssim \left\| \frac{1}{\langle y \rangle^{3/2}} (|w|^2 - |P|^2) \right\|_{L^2} \|\langle y \rangle |P_1|_1\|_{L^\infty} \lesssim_M V^{7/2}.\end{aligned}$$

This completes the proof of claims (4.30)–(4.32).

Step 2. From ε_1 to ε : Proof of (4.27)–(4.28). Rewrite $\mathbf{D}_w w = w_1$ as

$$L_Q \varepsilon = \varepsilon_1 - (\mathbf{D}_P P - P_1) - \{(L_P - L_Q)\varepsilon + N_P(\varepsilon)\}. \quad (4.33)$$

We claim that

$$\|(L_P - L_Q)\varepsilon + N_P(\varepsilon)\|_{\dot{H}_{m+1}^1} \lesssim_M \mu^2, \quad (4.34)$$

$$\|(L_P - L_Q)\varepsilon + N_P(\varepsilon)\|_{\dot{H}_{m+1}^2} \lesssim_M \mu X_2. \quad (4.35)$$

Let us first assume these claims and finish the proof of (4.27)–(4.28). Applying to (4.33) the linear coercivity estimates (2.35)–(2.36) for $L_Q \varepsilon$, bounds (4.24)–(4.25) for ε_1 , estimates (3.37)–(3.38) for $\mathbf{D}_P P - P_1$, and claims (4.34)–(4.35) for $(L_P - L_Q)\varepsilon + N_P(\varepsilon)$, we obtain

$$\begin{aligned} \|\varepsilon\|_{\dot{H}_m^2} &\lesssim X_2 + \beta^2 + O_M(\mu^2) \lesssim_M X_2, \\ \|\varepsilon\|_{\dot{H}_m^3} &\lesssim X_3 + \beta^3 + O_M(\mu X_2) \lesssim_M X_3. \end{aligned}$$

This completes the proof of (4.27) and (4.28).

It only remains to prove claims (4.34)–(4.35). Recall that $(L_P - L_Q)\varepsilon + N_P(\varepsilon)$ is a linear combination of $\frac{1}{y} A_\theta[\psi_1, \psi_2]\psi_3$ for suitable choices of ψ_j 's in $\{P, Q, P - Q, \varepsilon\}$. Note the pointwise estimates (recall $\partial_+ = \partial_y - \frac{m}{y}$ from the notation section)

$$\begin{aligned} \left| \frac{1}{y} A_\theta[\psi_1, \psi_2]\psi_3 \right|_{-1} &\lesssim \frac{1}{y} |A_\theta[\psi_1, \psi_2]| |\psi_3|_{-1} + |\psi_1 \psi_2 \psi_3|, \\ \left| \frac{1}{y} A_\theta[\psi_1, \psi_2]\psi_3 \right|_{-2} &\lesssim \frac{1}{y} |A_\theta[\psi_1, \psi_2]| |\partial_+ \psi_3|_{-1} + \frac{1}{y^3} |A_\theta[\psi_1, \psi_2]| |\psi_3| \\ &\quad + |\psi_1 \psi_2 \psi_3|_{-1}. \end{aligned}$$

By symmetry, we may replace the last term $|\psi_1 \psi_2 \psi_3|_{-1}$ by $|\psi_1 \psi_2| |\psi_3|_{-1}$. Therefore, it suffices to show the estimates

$$\left\| \frac{1}{y} |A_\theta[\psi_1, \psi_2]| |\psi_3|_{-1} + |\psi_1 \psi_2 \psi_3| \right\|_{L^2} \lesssim_M \mu^2, \quad (4.36)$$

$$\left\| \frac{1}{y} |A_\theta[\psi_1, \psi_2]| \left(|\partial_+ \psi_3|_{-1} + \frac{1}{y^2} |\psi_3| \right) + |\psi_1 \psi_2| |\psi_3|_{-1} \right\|_{L^2} \lesssim_M \mu X_2 \quad (4.37)$$

for the choices of ψ_1, ψ_2, ψ_3 that can contribute to either $(L_P - L_Q)\varepsilon$ or $N_P(\varepsilon)$.

(i) $\psi_3 = \varepsilon$: we have

$$\text{LHS of (4.36)} \lesssim \|\psi_1 \psi_2\|_{L^2} (\|\varepsilon\|_{-1} + \|\varepsilon\|_{L^\infty}) \lesssim \|\psi_1 \psi_2\|_{L^2} \|\varepsilon\|_{\dot{H}_m^1},$$

$$\begin{aligned} \text{LHS of (4.37)} &\lesssim \left\| \frac{\langle y \rangle}{y} \psi_1 \psi_2 \right\|_{L^2} \left(\|\partial_+ \varepsilon\|_{-1} + \left\| \frac{y}{\langle y \rangle} |\varepsilon|_{-1} \right\|_{L^\infty} \right) \\ &\lesssim \left\| \frac{\langle y \rangle}{y} \psi_1 \psi_2 \right\|_{L^2} \|\varepsilon\|_{\dot{H}_m^2}. \end{aligned}$$

Since

$$\begin{aligned} \left\| \frac{\langle y \rangle}{y} \psi_1 \psi_2 \right\|_{L^2} &\lesssim \left\| \frac{\langle y \rangle}{y} \{ |P - Q|(|P| + |Q|) + (|P| + |\varepsilon|)|\varepsilon| \} \right\|_{L^2} \\ &\lesssim \|P - Q\|_{L^\infty} + \|\varepsilon\|_{L^\infty} \left(1 + \left\| \frac{\langle y \rangle}{y} \varepsilon \right\|_{L^2} \right) \lesssim_M \mu, \end{aligned}$$

we have

$$\text{LHS of (4.36)} \lesssim_M \mu \|\varepsilon\|_{\dot{H}_m^1} \lesssim_M \mu^2,$$

$$\text{LHS of (4.37)} \lesssim_M \mu \|\varepsilon\|_{\dot{\mathcal{H}}_m^2} \lesssim_M \mu X_2.$$

(ii) $\psi_3 \in \{P, Q\}$: we use $|P|_2 + |Q|_2 \lesssim \langle y \rangle^{-1}$ to have

$$\text{LHS of (4.36)} \lesssim \left\| \frac{1}{\langle y \rangle} \psi_1 \psi_2 \right\|_{L^2},$$

$$\text{LHS of (4.37)} \lesssim \left\| \frac{1}{y \langle y \rangle} \psi_1 \psi_2 \right\|_{L^2}.$$

Applying to the above the estimates

$$|\psi_1 \psi_2| \lesssim |P - Q||\varepsilon| + |\varepsilon|^2 \lesssim (\beta + |\varepsilon|)|\varepsilon| \lesssim_M \mu |\varepsilon|,$$

we obtain

$$\text{LHS of (4.36)} \lesssim_M \mu \left\| \frac{1}{\langle y \rangle} \varepsilon \right\|_{L^2} \lesssim_M \mu \|\varepsilon\|_{\dot{H}_m^1} \lesssim_M \mu^2,$$

$$\text{LHS of (4.37)} \lesssim_M \mu \left\| \frac{1}{y \langle y \rangle} \varepsilon \right\|_{L^2} \lesssim_M \mu \|\varepsilon\|_{\dot{\mathcal{H}}_m^2} \lesssim_M \mu X_2.$$

(iii) $\psi_3 = P - Q$: we use

$$\left\| \frac{1}{y} |P - Q|_2 \right\|_{L^2} + \| |P - Q|_1 \|_{L^\infty} \lesssim \beta$$

to have

$$\text{LHS of (4.36)} \lesssim \beta \|\psi_1 \psi_2\|_{L^2},$$

$$\text{LHS of (4.37)} \lesssim \beta \left\| \frac{1}{y} \psi_1 \psi_2 \right\|_{L^2}.$$

We then apply $|\psi_1 \psi_2| \lesssim (|P| + |Q| + |\varepsilon|)|\varepsilon| \lesssim (\langle y \rangle^{-1} + |\varepsilon|)|\varepsilon|$ to have

$$\text{LHS of (4.36)} \lesssim \beta \left(\left\| \frac{1}{\langle y \rangle} \varepsilon \right\|_{L^2} + \|\varepsilon\|_{L^2} \|\varepsilon\|_{L^\infty} \right) \lesssim_M \mu \|\varepsilon\|_{\dot{H}_m^1} \lesssim_M \mu^2,$$

$$\text{LHS of (4.37)} \lesssim \beta \left(\left\| \frac{1}{y \langle y \rangle} \varepsilon \right\|_{L^2} + \left\| \frac{1}{y} \varepsilon \right\|_{L^2} \|\varepsilon\|_{L^\infty} \right) \lesssim_M \mu (\|\varepsilon\|_{\dot{\mathcal{H}}_m^2} + \mu^2) \lesssim_M \mu X_2.$$

This completes the proof of (4.34)–(4.35). ■

4.4. First modulation estimates

As is standard, we differentiate the orthogonality conditions (4.1) in time to obtain preliminary controls on the dynamics of the modulation parameters. As mentioned before, the ε -orthogonality conditions of (4.1) (i.e., the first two of (4.1)) give controls on the λ - and γ -dynamics. The b - and η -modulation estimates are obtained from differentiating the ε_1 -orthogonality conditions. Note that the evolution equation for ε_1 (see (4.40) below) effectively detects the dynamics of b and η , thanks to the degeneracy $\Lambda P_1, i P_1 = O(\beta)$ for the first two components of $\mathbf{v}_1$.

Now we derive the evolution equations for ε and ε_1 in the renormalized variables (s, y) . To derive the equation for ε , we first recall the equations for w and P (see (3.28) for the definition of $\mathbf{Mod}$ and $\mathbf{v}$),

$$\left(\partial_s - \frac{\lambda_s}{\lambda} \Lambda + \gamma_s i\right) w + i \mathbf{D}_w^* w_1 + \left(\int_y^\infty \operatorname{Re}(\bar{w} w_1) dy'\right) i w = 0, \quad (2.8)$$

$$\left(\partial_s - \frac{\lambda_s}{\lambda} \Lambda + \gamma_s i\right) P + i \mathbf{D}_P^* P_1 + \left(\int_y^\infty \operatorname{Re}(\bar{P} P_1) dy'\right) i P = -\mathbf{Mod} \cdot \mathbf{v} + i \Psi. \quad (3.42)$$

Subtracting the second equation from the first, we obtain a preliminary equation for ε ,

$$\begin{aligned} &\left(\partial_s - \frac{\lambda_s}{\lambda} \Lambda + \gamma_s i\right) \varepsilon + i \mathbf{D}_w^* w_1 - i \mathbf{D}_P^* P_1 \\ &\quad + \left(\int_y^\infty \operatorname{Re}(\bar{w} w_1 - \bar{P} P_1) dy'\right) i P + \left(\int_y^\infty \operatorname{Re}(\bar{w} w_1) dy'\right) i \varepsilon = \mathbf{Mod} \cdot \mathbf{v} - i \Psi. \end{aligned}$$

The above form is however not suitable for our analysis because we need to incorporate the *phase correction* from the nonlocal nonlinearity. For this purpose, we introduce a new modulation vector $\widetilde{\mathbf{Mod}}$, whose components are defined by

$$\widetilde{\mathbf{Mod}}_j := \begin{cases} \gamma_s + (m+1)\eta + \int_0^\infty \operatorname{Re}(\bar{w} w_1) dy & \text{if } j = 2, \\ \mathbf{Mod}_j & \text{otherwise.} \end{cases} \quad (4.38)$$

Using this modulation vector, we obtain the equation for ε ,

$$\begin{aligned} &\left(\partial_s - \frac{\lambda_s}{\lambda} \Lambda + \gamma_s i\right) \varepsilon + i \mathbf{D}_w^* w_1 - i \mathbf{D}_P^* P_1 \\ &\quad - \left(\int_0^y \operatorname{Re}(\bar{w} w_1 - \bar{P} P_1) dy'\right) i P + \left(\int_y^\infty \operatorname{Re}(\bar{w} w_1) dy'\right) i \varepsilon \\ &\quad = \widetilde{\mathbf{Mod}} \cdot \mathbf{v} - i \Psi. \end{aligned} \quad (4.39)$$

In a similar fashion, using (2.9), (3.44), and (4.38), we obtain the equation for ε_1 :

$$\begin{aligned} &\left(\partial_s - \frac{\lambda_s}{\lambda} \Lambda_{-1} + \gamma_s i\right) \varepsilon_1 + i A_w^* w_2 - i A_P^* P_2 \\ &\quad - \left(\int_0^y \operatorname{Re}(\bar{w} w_1 - \bar{P} P_1) dy'\right) i P_1 + \left(\int_y^\infty \operatorname{Re}(\bar{w} w_1) dy'\right) i \varepsilon_1 \\ &\quad = \widetilde{\mathbf{Mod}} \cdot \mathbf{v}_1 - i \Psi_1. \end{aligned} \quad (4.40)$$

The following is the main proposition of this subsection.

Proposition 4.8 (First modulation estimates). *We have*

$$|\mathbf{Mod}_1| + |\widetilde{\mathbf{Mod}}_2| \lesssim_M X_3, \quad (4.41)$$

$$|\mathbf{Mod}_3| + |\mathbf{Mod}_4| \lesssim_M V_{7/2}. \quad (4.42)$$

In particular, we have rough estimates

$$\left| \frac{\lambda_s}{\lambda} \right| + |\gamma_s| \lesssim_M \mu, \quad (4.43)$$

$$|b_s + b^2 + \eta^2| + |\eta_s| \lesssim_M X_3. \quad (4.44)$$

Remark 4.9. We mention once again that we need to incorporate the phase correction to the blow-up part of the solution as expressed in the definition of $\mathbf{Mod}_2$. Such a phase correction arises from the A_t -potential and this fact was already observed in [26, 27]. Note that $\widetilde{\mathbf{Mod}}_2 \approx \gamma_s - (m+1)\eta$, as can be seen from the estimate

$$\begin{aligned} \int_0^\infty \operatorname{Re}(\bar{w}w_1)dy &= \int_0^\infty \operatorname{Re}(\bar{P}P_1)dy + O\left(\left\|\frac{1}{y}\varepsilon\right\|_{L^2}\|w_1\|_{L^2} + \|P\|_{L^2}\left\|\frac{1}{y}\varepsilon_1\right\|_{L^2}\right) \\ &= -2(m+1)\eta + O_M(\mu^2), \end{aligned} \quad (4.45)$$

where we used (3.52), (4.8), and (4.24).

Note that we require different bounds for the components of $\widetilde{\mathbf{Mod}}$; bound (4.41) is rougher than (4.42), but it is sufficient for our modulation analysis. This fact allows us to construct P only up to quadratic terms.

To obtain these modulation estimates, we differentiate the orthogonality conditions (4.1) in time and use the evolution equations for ε and ε_1 . In other words, we test (4.39) and (4.40) against $\tilde{Z}_1, \tilde{Z}_2$ and $\tilde{\tilde{Z}}_3, \tilde{\tilde{Z}}_4$, respectively. It will be convenient to rewrite (4.39) and (4.40) as

$$\widetilde{\mathbf{Mod}} \cdot \mathbf{v} = \left(\partial_s - \frac{\lambda_s}{\lambda} \Lambda + \gamma_s i \right) \varepsilon + \check{F}, \quad (4.46)$$

$$\widetilde{\mathbf{Mod}} \cdot \mathbf{v}_1 = \left(\partial_s - \frac{\lambda_s}{\lambda} \Lambda_{-1} + \gamma_s i \right) \varepsilon_1 + \check{F}_1, \quad (4.47)$$

where $\check{F}$ and $\check{F}_1$ are defined accordingly. The following lemma collects the estimates for $\check{F}$ and $\check{F}_1$ for the proof of Proposition 4.8. We relegate its proof to Section 4.6.

Lemma 4.10 (Structural lemma for the modulation estimates).

- (Rough low regularity estimates in compact region) We have

$$|(\check{F}, Z_1)_r| + |(\check{F}, Z_2)_r| + |(\check{F}_1, \tilde{\tilde{Z}}_3)_r| + |(\check{F}_1, \tilde{\tilde{Z}}_4)_r| \lesssim_M \mu. \quad (4.48)$$

- (Higher Sobolev estimates in compact region) We have

$$|(\check{F}, Z_1)_r| + |(\check{F}, Z_2)_r| \lesssim_M X_3, \quad (4.49)$$

$$|(\check{F}_1, \tilde{\tilde{Z}}_3)_r| + |(\check{F}_1, \tilde{\tilde{Z}}_4)_r| \lesssim_M V_{7/2}. \quad (4.50)$$

- (Testing against almost nullspace elements) We have

$$|(\check{F}_1, \chi_{\mu^{-1}} y Q)_r| + |(\check{F}_1, \chi_{\mu^{-1}} i y Q)_r| \lesssim_M \mu X_3 + \beta^4 |\log \beta|^3. \quad (4.51)$$

Remark 4.11. The rough low regularity estimates (4.48) will be used for the proof of (4.43). The higher Sobolev estimates (4.49) and (4.50) will be used for the proof of (4.41) and (4.42), respectively. The last estimate (4.51) will be used to obtain second modulation estimates (Lemma 5.2) in Section 5.1.

Proof of Proposition 4.8 assuming Lemma 4.10. Recall from Remark 2.5 that the orthogonality profiles $Z_1, Z_2, \tilde{Z}_3, \tilde{Z}_4$ are supported in $y \leq 2R_0$.

Step 1. Proof of (4.41) and (4.42). On the one hand, we rearrange the equation for ε as

$$\widetilde{\mathbf{Mod}} \cdot \{\mathbf{v} + (\Lambda \varepsilon, -i \varepsilon, 0, 0)^t\} = \partial_s \varepsilon + b \Lambda \varepsilon - (m+1) \eta i \varepsilon + F.$$

We take the inner product of the above formula with Z_k for $k \in \{1, 2\}$. Note that we have

$$\begin{aligned} (\Lambda \varepsilon, Z_k)_r &= -(\varepsilon, \Lambda Z_k)_r \lesssim \|\mathbf{1}_{(0, 2R_0]} \varepsilon\|_{L^\infty} \lesssim_M \min\{\mu, X_3\}, \\ (-i \varepsilon, Z_k)_r &= (\varepsilon, i Z_k)_r \lesssim \|\mathbf{1}_{(0, 2R_0]} \varepsilon\|_{L^\infty} \lesssim_M \min\{\mu, X_3\}, \\ (\partial_s \varepsilon, Z_k)_r &= \partial_s (\varepsilon, Z_k)_r = 0, \end{aligned}$$

where in the last equality we used the orthogonality condition (4.1). Combining the above formula with estimate (4.49) for F , we obtain

$$\left| \sum_{j=1}^4 \{((\mathbf{v})_j, Z_k)_r + O_M(\mu)\} \widetilde{\mathbf{Mod}}_j \right| \lesssim_M X_3. \quad (4.52)$$

On the other hand, we rearrange the equation for ε_1 as

$$\widetilde{\mathbf{Mod}} \cdot \{\mathbf{v} + (\Lambda_{-1} \varepsilon_1, -i \varepsilon_1, 0, 0)^t\} = \partial_s \varepsilon_1 + b \Lambda_{-1} \varepsilon_1 - (m+1) \eta i \varepsilon_1 + F_1.$$

We take the inner product of the above formula with $\tilde{Z}_k$ for $k \in \{3, 4\}$. Proceeding similarly as in the previous paragraph, but this time using (4.50) for F_1 , we obtain

$$\left| \sum_{j=1}^4 \{((\mathbf{v}_1)_j, \tilde{Z}_k)_r + O_M(\mu)\} \widetilde{\mathbf{Mod}}_j \right| \lesssim_M V_{7/2}. \quad (4.53)$$

Thanks to the almost diagonal matrix structure (4.3), we have

$$\begin{aligned} |\widetilde{\mathbf{Mod}}_1| + |\widetilde{\mathbf{Mod}}_2| &\lesssim_M \text{RHS of (4.52)} + \mu \cdot \text{RHS of (4.53)}, \\ |\widetilde{\mathbf{Mod}}_3| + |\widetilde{\mathbf{Mod}}_4| &\lesssim_M \text{RHS of (4.53)} + \mu \cdot \text{RHS of (4.52)}. \end{aligned}$$

Substituting estimates (4.52) and (4.53) into the above completes the proof of (4.41) and (4.42).

Step 2. Proof of (4.43) and (4.44). For the proof of (4.43), one follows the argument of the previous step, but uses the rougher estimate (4.48) instead of (4.49) and (4.50). This gives the estimate $|\widetilde{\text{Mod}}| \lesssim_M \mu$. Combining this with $\beta \lesssim_M \mu$ and (4.45), we obtain (4.43).

Estimate (4.44) easily follows from (4.42), $V_{7/2} \leq X_3$, and $|p_3^{(b)}| + |p_3^{(\eta)}| \lesssim \beta^3 \lesssim \mu^3$. This completes the proof. $\blacksquare$

4.5. Energy-Morawetz functional and uniform H^3 -bound

This subsection is devoted to the proof of the key assertion of this paper, namely, the uniform H^3 -bound of $\varepsilon^\sharp$. As mentioned in Section 1.4, the basic strategy here is the *energy method with repulsivity*, which has been successfully employed in the blow-up constructions such as in [4, 45, 46, 52, 56] and for (CSS) in [26, 28]. However, as we are considering arbitrary H_m^3 -solutions near modulated solitons, we need to control them without dynamical controls such as (1.20). Note that controlling the solutions without such dynamical assumptions is addressed in [42, 51, 55] for the mass-critical NLS and is given in a more complete picture in [37] for the critical gKdV. See also [20, 23] for energy-critical equivariant wave maps.

We introduce the energy functional

$$\mathcal{F} := \frac{(\varepsilon_2, \tilde{H}_Q \varepsilon_2)_r}{\mu^6} - A \frac{(i\varepsilon_2, \psi' \partial_y \varepsilon_2)_r}{\mu^5} + A^2 \frac{\|\varepsilon_2\|_{L^2}^2}{\mu^4} \quad (4.54)$$

with some large constant $A = A(M) \gg 1$ and a function ψ' given in Lemma 2.1, and claim that $\mathcal{F} \lesssim 1$ can be propagated on any finite-time intervals only under the size control (4.8). Since $0 \leq \psi' \leq 1$, it is clear that $\mathcal{F}$ is coercive in the ε_2 -variable, and hence it controls $\mu^{-3} \|\varepsilon\|_{\mathcal{H}_m^3}$ and $\mu^{-2} \|\varepsilon\|_{\mathcal{H}_m^2}$ by Lemma 4.6. Note that the powers of μ in (4.54) cannot be changed; see Remark 4.13.

Note that $\mu = 0$ if and only if u is the modulated soliton $Q_{\lambda, \gamma}$ so we will not consider this trivial case. The key results (4.75)–(4.76) of this subsection are nevertheless well defined even for this soliton solution.

In what follows, we explain how $\mathcal{F} \lesssim 1$ can be propagated. Our discussion will be sketchy and non-rigorous. We look at the structure of the ε_2 -equation, whose full expression can be found in (4.59)–(4.60) below. Here we only discuss a toy model

$$\left(\partial_s - \frac{\lambda_s}{\lambda} \Lambda_{-2} + \gamma s i \right) \varepsilon_2 + i \tilde{H}_Q \varepsilon_2 = i \beta V_{\text{loc}} \varepsilon_2 + i(\beta^2 + |\varepsilon|^2) \varepsilon_2 - i \Psi_2, \quad (4.55)$$

where V_{loc} is a smooth fast decaying potential and Ψ_2 is the inhomogeneous error term (see (3.46)). We will call the first term of the right-hand side of (4.55) a *local term*, and call the second term a *perturbative term*, whose meaning will be made clear below.

We first consider the contribution of the perturbative term, whose $\dot{H}^1$ -norm can be roughly estimated by

$$\|i(\beta^2 + |\varepsilon|^2) \varepsilon_2\|_{\dot{H}^1} \lesssim (\beta^2 + \|\varepsilon\|_{L^\infty}^2) \|\varepsilon_2\|_{\dot{H}^1} \lesssim \mu^2 \|\varepsilon_2\|_{\dot{H}^1}$$

and similarly for its L^2 -norm, where in the last inequality we applied the *size control* (4.8). In this spirit, the contribution of the perturbative terms in the energy estimates can be considered as

$$\partial_s \sqrt{\mathcal{F}} \lesssim \mu^2 \sqrt{\mathcal{F}}, \quad \text{or} \quad \partial_t \sqrt{\mathcal{F}} \lesssim \sqrt{\mathcal{F}}.$$

By Grönwall's inequality, this inequality propagates the control $\mathcal{F} \lesssim 1$ on any finite-time intervals.

In a similar fashion, the contribution of the Ψ_2 -term appears as

$$\partial_s \sqrt{\mathcal{F}} \lesssim \mu^{-3} \|\Psi_2\|_{\dot{H}_m^1} + \mu^{-2} \|\Psi_2\|_{L^2} = \mu^2 (\mu^{-5} \|\Psi_2\|_{\dot{H}_m^1} + \mu^{-4} \|\Psi_2\|_{L^2}).$$

To have $\sqrt{\mathcal{F}}$ bounded, in view of $\partial_t = \lambda^{-2} \partial_s$ and $\lambda \sim \mu$, we require that

$$\frac{\|\Psi_2\|_{\dot{H}_m^1}}{\lambda^5} + \frac{\|\Psi_2\|_{L^2}}{\lambda^4} \quad \text{is integrable in } t.$$

Considering the formal ODE solutions (3.3), this suggests us to have at least $\|\Psi_2\|_{\dot{H}_m^1} = O(\beta^{4+})$ and $\|\Psi_2\|_{L^2} = O(\beta^{3+})$ as stated in (3.50).

We turn to the contribution of the local term, whose treatment requires the *Morawetz repulsivity* in a similar spirit to [46, 56]. (The contribution of $-\frac{\lambda_s}{\lambda} \Lambda_{-2} \varepsilon_2$ in the left-hand side of (4.55) will also require the same treatment.) Recall that we do not have a dynamical control such as $b > 0$, but our claim was that the size control (4.8) is sufficient to run the energy method with repulsivity. We observe in the computation of $\partial_s \{\mu^{-6}(\varepsilon_2, \tilde{H}_Q \varepsilon_2)_r\}$ that the following local terms appear:

$$\frac{1}{\mu^6} \left\{ -\frac{\lambda_s}{\lambda} \int \frac{\partial_y \tilde{V}_Q}{2y} |\varepsilon_2|^2 + (A_Q^* \varepsilon_2, A_Q^* (i\beta V_{\text{loc}} \varepsilon_2))_r \right\}, \quad (4.56)$$

where the factorization $\tilde{H}_Q = A_Q A_Q^*$ is used for the second term. In view of $-\frac{\lambda_s}{\lambda} \approx b$, (4.56) might be estimated by $\mu^{-6} \beta \|\varepsilon_2\|_{\dot{H}^1}^2 \lesssim \beta \mathcal{F}$, and hence $\partial_t \mathcal{F} \lesssim \frac{\beta}{\lambda^2} \mathcal{F}$, but this estimate is not sufficient because $\frac{\beta}{\lambda^2}$ is not integrable in t . Thus term (4.56) is non-perturbative in this sense. However, if we recall that the potentials involved in (4.56) are *well localized* and the *size control* $\beta \lesssim \mu$ is at hand, (4.56) can be estimated by a more local expression

$$\mu^{-6} \beta \|\varepsilon_2\|_{\dot{V}_{m+2}^{3/2}}^2 \lesssim \mu^{-5} \|\varepsilon_2\|_{\dot{V}_{m+2}^{3/2}}^2,$$

which can be dominated by the monotonicity from the Morawetz term,

$$\frac{1}{2} \partial_s \{\mu^{-5} (i\varepsilon_2, \psi' \partial_y \varepsilon_2)_r\} \geq c \mu^{-5} \|\varepsilon_2\|_{\dot{V}_{m+2}^{3/2}}^2 + \text{lower-order terms}.$$

This motivates the Morawetz term in (4.54). Note that the last lowest order term of (4.54) is introduced to make $\mathcal{F}$ coercive.

We now state the key monotonicity estimate obtained from $\mathcal{F}$, which is the main result of this subsection. We will integrate this estimate in Corollary 4.17 below to obtain the uniform H^3 -bound of $\varepsilon^\#$.

Proposition 4.12 (Energy-Morawetz functional). *There exist large constants*

$$A = A(M) > 1 \quad \text{and} \quad K = K(M) > 1$$

such that the functional

$$\mathcal{F} := \frac{(\varepsilon_2, \tilde{H}_Q \varepsilon_2)_r}{\mu^6} - A \frac{(i\varepsilon_2, \psi' \partial_y \varepsilon_2)_r}{\mu^5} + A^2 \frac{\|\varepsilon_2\|_{L^2}^2}{\mu^4} \quad (4.54)$$

satisfies the following properties:

- (Coercivity)

$$\mathcal{F} \sim \frac{\|\varepsilon_2\|_{\dot{H}_{m+2}^1}^2}{\mu^6} + A^2 \frac{\|\varepsilon_2\|_{L^2}^2}{\mu^4}, \quad \text{and hence} \quad X_3 \sim_M \mu^3 \sqrt{1 + \mathcal{F}}. \quad (4.57)$$

- (Monotonicity)

$$\partial_s \mathcal{F} + \frac{\|\varepsilon_2\|_{V_{m+2}^{3/2}}^2}{\mu^5} \leq K \mu^2 \left\{ (1 + \mathcal{F}) + \sqrt{\mathcal{F}} \cdot \frac{\beta^{9/2} |\log \beta|^2}{\mu^5} \right\}. \quad (4.58)$$

Remark 4.13 (The scales). The scales in (4.54), i.e., the powers of μ , cannot be changed. The use of other powers will contradict the H^3 -boundedness of $\varepsilon^\sharp$, either in the shrinking regime or in the expanding regime. In this sense, we are crucially using the fact that the linearized Hamiltonian $\tilde{H}_Q$ defines a coercive bilinear form on a *homogeneous* Sobolev space $\dot{H}_{m+2}^1$.

Remark 4.14 (On $m \geq 1$). In the following proof, we heavily use the Morawetz monotonicity (Lemma 2.1) to bound the local terms. Hence the assumption $m \geq 1$ is crucially used in the proof. See also Remark 2.2.

We now record the full equation for ε_2 . As in Section 4.4, we subtract P_2 -equation (3.46) from w_2 -equation (2.10) and incorporate the phase correction (4.38) to obtain

$$\left(\partial_s - \frac{\lambda_s}{\lambda} \Lambda_{-2} + \gamma_s i \right) \varepsilon_2 + i \tilde{H}_Q \varepsilon_2 = F_2, \quad (4.59)$$

where

$$\begin{aligned} F_2 = & \widetilde{\mathbf{Mod}} \cdot \mathbf{v}_2 - i \Psi_2 + (i \bar{w} w_1^2 - i \bar{P} P_1^2) \\ & - (i \tilde{H}_w - i \tilde{H}_Q) \varepsilon_2 - (i \tilde{H}_w - i \tilde{H}_P) P_2 \\ & - \left(\int_y^\infty \operatorname{Re}(\bar{w} w_1) dy' \right) i \varepsilon_2 + \left(\int_0^y \operatorname{Re}(\bar{w} w_1 - \bar{P} P_1) dy' \right) i P_2. \end{aligned} \quad (4.60)$$

The following lemma collects several estimates for F_2 for the proof of Proposition 4.12. As illustrated before, we are basically decomposing the terms of F_2 into the sum of local and perturbative terms. The proof is relegated to Section 4.7.

Lemma 4.15 (Structural lemma for the energy estimate). *We can decompose*

$$F_2 = F_{2,1}^{\text{loc}} + F_{2,2}^{\text{loc}} + F_2^{\text{pert}}, \quad \text{where } F_{2,2}^{\text{loc}} := \sum_{j=3}^4 \mathbf{Mod}_j(\mathbf{v}_2)_j, \quad (4.61)$$

such that

$$\|F_{2,1}^{\text{loc}} + F_2^{\text{pert}}\|_{L^2} \lesssim_M \mu X_3 + \beta^4 |\log \beta|^2, \quad (4.62)$$

$$\|\langle y \rangle |F_{2,1}^{\text{loc}}|_{-1}\|_{L^2} \lesssim_M \mu V_{7/2}, \quad (4.63)$$

$$\| |F_2^{\text{pert}}|_{-1} \|_{L^2} \lesssim_M \mu^2 X_3 + \beta^{9/2} |\log \beta|^2, \quad (4.64)$$

and

$$\|y F_{2,2}^{\text{loc}}\|_{L^2} \lesssim_M X_3, \quad (4.65)$$

$$\|\langle y \rangle^{1/2} |F_{2,2}^{\text{loc}}|_1\|_{L^2} \lesssim_M \mu^{1/2} V_{7/2}, \quad (4.66)$$

$$\|\langle y \rangle^{1/2} y^{-1} |F_{2,2}^{\text{loc}}|_2\|_{L^2} \lesssim_M \mu V_{7/2}. \quad (4.67)$$

Remark 4.16. When $m = 1$, the modulation term $F_{2,2}^{\text{loc}} = \sum_{j=3}^4 \widetilde{\mathbf{Mod}}_j \cdot (\mathbf{v}_2)_j$ does not satisfy bound (4.63) due to the lack of spatial decay of $(\mathbf{v}_2)_j$. Thus we will directly estimate this contribution in the energy estimates.

Proof of Proposition 4.12 assuming Lemma 4.15.

Step 1. Proof of coercivity (4.57). This is obvious from (2.30), $|\psi'| \leq 1$, the Cauchy–Schwarz inequality, and (4.21)–(4.22).

Step 2. Algebraic computation of $\partial_s \mathcal{F}$. In Steps 2–3, we prove monotonicity (4.58). The goal of this step is to derive the algebraic identity

$$\begin{aligned} \frac{1}{2} \partial_s \mathcal{F} &= \frac{1}{\mu^6} \left\{ -\frac{\lambda_s}{\lambda} \int \frac{\partial_y \tilde{V}_Q}{2y} |\varepsilon_2|^2 + (A_Q^* \varepsilon_2, A_Q^* F_2)_r \right\} \\ &\quad - \frac{A}{\mu^5} \left\{ -\frac{\lambda_s}{2\lambda} (i \varepsilon_2, (y \partial_y \psi') \partial_y \varepsilon_2)_r + (\tilde{H}_Q \varepsilon_2, \Lambda_\psi \varepsilon_2)_r + (i F_2, \Lambda_\psi \varepsilon_2)_r \right\} \\ &\quad + \frac{A^2}{\mu^4} (\varepsilon_2, F_2)_r. \end{aligned} \quad (4.68)$$

First, we use $\frac{\mu_s}{\mu} = \frac{\lambda_s}{\lambda}$ and equation (4.59) for ε_2 to have

$$\begin{aligned} \frac{1}{2} \partial_s \left\{ \frac{(\varepsilon_2, \tilde{H}_Q \varepsilon_2)_r}{\mu^6} \right\} &= \frac{1}{\mu^6} \left(\left(\partial_s - 3 \frac{\lambda_s}{\lambda} \right) \varepsilon_2, \tilde{H}_Q \varepsilon_2 \right)_r \\ &= \frac{1}{\mu^6} \left(\frac{\lambda_s}{\lambda} y \partial_y \varepsilon_2 - \gamma_s i \varepsilon_2 - i \tilde{H}_Q \varepsilon_2 + F_2, \tilde{H}_Q \varepsilon_2 \right)_r. \end{aligned}$$

Using $\tilde{H}_Q = A_Q A_Q^* = -\partial_{yy} - \frac{1}{y} \partial_y + \frac{1}{y^2} \tilde{V}_Q$ (see (2.20)), we obtain

$$\frac{1}{2} \partial_s \left\{ \frac{(\varepsilon_2, \tilde{H}_Q \varepsilon_2)_r}{\mu^6} \right\} = \frac{1}{\mu^6} \left\{ -\frac{\lambda_s}{\lambda} \int \frac{\partial_y \tilde{V}_Q}{2y} |\varepsilon_2|^2 + (A_Q^* \varepsilon_2, A_Q^* F_2)_r \right\}. \quad (4.69)$$

Next, we similarly compute

$$\begin{aligned} & \frac{1}{2} \partial_s \left\{ -A \frac{(i \varepsilon_2, \psi' \partial_y \varepsilon_2)_r}{\mu^5} \right\} \\ &= -\frac{A}{2\mu^5} \left[\frac{\lambda_s}{\lambda} \{ (i \Lambda_{-2} \varepsilon_2, \psi' \partial_y \varepsilon_2)_r + (i \varepsilon_2, \psi' \partial_y \Lambda_{-2} \varepsilon_2)_r - 5(i \varepsilon_2, \psi' \partial_y \varepsilon_2)_r \} \right. \\ & \quad + \{ (\tilde{H}_Q \varepsilon_2, \psi' \partial_y \varepsilon_2)_r - (\varepsilon_2, \psi' \partial_y \tilde{H}_Q \varepsilon_2)_r \} \\ & \quad \left. + \{ (i F_2, \psi' \partial_y \varepsilon_2)_r + (i \varepsilon_2, \psi' \partial_y F_2)_r \} \right]. \end{aligned}$$

In the above formula, we integrate by parts the scaling term

$$\begin{aligned} (i \varepsilon_2, \psi' \partial_y \Lambda_{-2} \varepsilon_2)_r &= (i \varepsilon_2, \Lambda_{-3}(\psi' \partial_y \varepsilon_2) - (y \partial_y \psi') \partial_y \varepsilon_2)_r \\ &= -(i \Lambda_3 \varepsilon_2, \psi' \partial_y \varepsilon_2)_r - (i \varepsilon_2, (y \partial_y \psi') \partial_y \varepsilon_2)_r \end{aligned}$$

and apply the operator identity $\psi' \partial_y - \partial_y^* \psi' = 2\Lambda_\psi$ for the terms involving $\tilde{H}_Q$ and F_2 . This gives

$$\begin{aligned} & \frac{1}{2} \partial_s \left\{ -A \frac{(i \varepsilon_2, \psi' \partial_y \varepsilon_2)_r}{\mu^5} \right\} \\ &= -\frac{A}{\mu^5} \left\{ -\frac{\lambda_s}{2\lambda} (i \varepsilon_2, (y \partial_y \psi') \partial_y \varepsilon_2)_r + (\tilde{H}_Q \varepsilon_2, \Lambda_\psi \varepsilon_2)_r + (i F_2, \Lambda_\psi \varepsilon_2)_r \right\}. \end{aligned}$$

Finally, we have

$$\frac{1}{2} \partial_s \left\{ A^2 \frac{\|\varepsilon_2\|_{L^2}^2}{\mu^4} \right\} = \frac{A^2}{\mu^4} (\varepsilon_2, F_2)_r. \quad (4.70)$$

Summing up (4.69)–(4.70) completes the proof of claim (4.68).

Step 3. Proof of monotonicity (4.58). In this step, we prove (4.58) using computation (4.68).

First line of the right-hand side of (4.68). We claim that

$$\begin{aligned} & \frac{1}{\mu^6} \left\{ -\frac{\lambda_s}{\lambda} \int \frac{\partial_y \tilde{V}_Q}{2y} |\varepsilon_2|^2 + (A_Q^* \varepsilon_2, A_Q^* F_2)_r \right\} \\ & \leq C_1(M) \left\{ \frac{\|\varepsilon_2\|_{\dot{V}_{m+2}^{3/2}}^2}{\mu^5} + \mu^2 \left(1 + \mathcal{F} + \sqrt{\mathcal{F}} \cdot \frac{\beta^{9/2} |\log \beta|^2}{\mu^5} \right) \right\}, \end{aligned} \quad (4.71)$$

where $C_1(M)$ is some constant depending on M . We rewrite the left-hand side of (4.71) using decomposition (4.61) as

$$\begin{aligned} \text{LHS of (4.71)} &= \frac{1}{\mu^6} \left\{ -\frac{\lambda_s}{\lambda} \int \frac{\partial_y \tilde{V}_Q}{2y} |\varepsilon_2|^2 + (\varepsilon_2, \tilde{H}_Q F_{2,2}^{\text{loc}})_r \right. \\ & \quad \left. + (A_Q^* \varepsilon_2, A_Q^* (F_{2,1}^{\text{loc}} + F_2^{\text{pert}}))_r \right\}. \end{aligned}$$

For the first term (which is a local term), we use (4.43) to have

$$-\frac{\lambda_s}{\lambda} \int \frac{\partial_y \tilde{V}_Q}{2y} |\varepsilon_2|^2 = \frac{\lambda_s}{2\lambda} \int Q^2 |\varepsilon_2|^2 \lesssim_M \mu \|\varepsilon_2\|_{\dot{V}_{m+2}^{3/2}}^2.$$

For the second term, we use (4.67), (4.23), and (4.57) to have

$$\begin{aligned} (\varepsilon_2, \tilde{H}_Q F_{2,2}^{\text{loc}})_r &\lesssim \|\langle y \rangle^{-1/2} y^{-1} \varepsilon_2\|_{L^2} \|\langle y \rangle^{1/2} y^{-1} |F_{2,2}^{\text{loc}}|_2\|_{L^2} \\ &\lesssim_M \|\varepsilon_2\|_{\dot{V}_{m+2}^{3/2}} \cdot \mu V_{7/2} \\ &\lesssim_M \mu \|\varepsilon_2\|_{\dot{V}_{m+2}^{3/2}}^2 + \mu^2 (X_3)^2 \lesssim_M \mu \|\varepsilon_2\|_{\dot{V}_{m+2}^{3/2}}^2 + \mu^8 (1 + \mathcal{F}). \end{aligned}$$

For the third term, we use (4.63)–(4.64) to have

$$\begin{aligned} (A_Q^* \varepsilon_2, A_Q^* (F_{2,1}^{\text{loc}} + F_2^{\text{pert}}))_r &\lesssim \|\langle y \rangle^{-1} A_Q^* \varepsilon_2\|_{L^2} \|\langle y \rangle |F_{2,1}^{\text{loc}}|_{-1}\|_{L^2} + \|A_Q^* \varepsilon_2\|_{L^2} \|F_2^{\text{pert}}|_{-1}\|_{L^2} \\ &\lesssim_M \|\varepsilon_2\|_{\dot{V}_{m+2}^{3/2}} \cdot \mu V_{7/2} + \|\varepsilon_2\|_{\dot{H}_{m+2}^1} \cdot (\mu^2 X_3 + \beta^{9/2} |\log \beta|^2) \\ &\lesssim_M \mu (V_{7/2})^2 + \mu^2 (X_3)^2 + \mu^3 \sqrt{\mathcal{F}} \cdot \beta^{9/2} |\log \beta|^2 \\ &\lesssim_M \mu \|\varepsilon_2\|_{\dot{V}_{m+2}^{3/2}}^2 + \mu^8 (1 + \mathcal{F}) + \mu^3 \sqrt{\mathcal{F}} \cdot \beta^{9/2} |\log \beta|^2. \end{aligned}$$

This completes the proof of claim (4.71).

Second line of the right-hand side of (4.68). We claim

$$\begin{aligned} &-\frac{A}{\mu^5} \left\{ -\frac{\lambda_s}{2\lambda} (i\varepsilon_2, (y\partial_y \psi')\partial_y \varepsilon_2)_r + (\tilde{H}_Q \varepsilon_2, \Lambda_\psi \varepsilon_2)_r + (iF_2, \Lambda_\psi \varepsilon_2)_r \right\} \\ &\leq -\frac{cA}{2\mu^5} \|\varepsilon_2\|_{\dot{V}_{m+2}^{3/2}}^2 + C_2(M) A \mu^2 \left\{ 1 + \mathcal{F} + \sqrt{\mathcal{F}} \cdot \frac{\beta^4 |\log \beta|^{5/2}}{\mu^4} \right\}, \end{aligned} \quad (4.72)$$

where c is the constant in (2.24) and $C_2(M)$ is some constant depending on M . To see this, similarly as in the previous paragraph, we rewrite

$$\begin{aligned} \text{LHS of (4.72)} &= -\frac{A}{\mu^5} \left\{ (\tilde{H}_Q \varepsilon_2, \Lambda_\psi \varepsilon_2)_r - \frac{\lambda_s}{2\lambda} (i\varepsilon_2, (y\partial_y \psi')\partial_y \varepsilon_2)_r \right. \\ &\quad \left. - (\varepsilon_2, i\Lambda_\psi F_{2,2}^{\text{loc}})_r - (i\Lambda_\psi \varepsilon_2, F_{2,1}^{\text{loc}} + F_2^{\text{pert}})_r \right\}. \end{aligned}$$

We apply (2.24) to the first term of the left-hand side of (4.72) to obtain the *key monotonicity*

$$-\frac{A}{\mu^5} (\tilde{H}_Q \varepsilon_2, \Lambda_\psi \varepsilon_2)_r \leq -\frac{cA}{\mu^5} \|\varepsilon_2\|_{\dot{V}_{m+2}^{3/2}}^2.$$

The other terms of the left-hand side of (4.72) are error terms. Using (4.43) and $y\partial_y \psi' \lesssim \langle y \rangle^{-1}$ from (2.23), we have

$$-\frac{\lambda_s}{2\lambda} (i\varepsilon_2, (y\partial_y \psi')\partial_y \varepsilon_2)_r \lesssim_M \mu \|\varepsilon_2\|_{\dot{H}_{m+2}^1}^2 \lesssim_M \mu^7 \mathcal{F}.$$

Using (4.66) and (4.23), we have

$$\begin{aligned}
 (\varepsilon_2, i\Lambda_\psi F_{2,2}^{\text{loc}})_r &\lesssim \| \langle y \rangle^{-1/2} y^{-1} \varepsilon_2 \|_{L^2} \| \langle y \rangle^{1/2} |F_{2,2}^{\text{loc}}|_1 \|_{L^2} \\
 &\lesssim_M \| \varepsilon_2 \|_{\dot{V}_{m+2}^{3/2}} \cdot \mu^{1/2} V_{7/2} \lesssim_M \mu^{1/2} (\| \varepsilon_2 \|_{\dot{V}_{m+2}^{3/2}}^2 + \mu X_3^2) \\
 &\lesssim \delta_M(\alpha^*) (\| \varepsilon_2 \|_{\dot{V}_{m+2}^{3/2}}^2 + \mu^7 (1 + \mathcal{F})).
 \end{aligned}$$

Using (4.62), we have

$$\begin{aligned}
 (i\Lambda_\psi \varepsilon_2, F_{2,1}^{\text{loc}} + F_2^{\text{pert}})_r &\lesssim_M \| |\varepsilon_2|_{-1} \|_{L^2} \cdot (\mu X_3 + \beta^4 |\log \beta|^2) \\
 &\lesssim_M \mu^3 \sqrt{\mathcal{F}} \cdot (\mu^4 (\sqrt{\mathcal{F}} + 1) + \beta^4 |\log \beta|^2).
 \end{aligned}$$

Substituting the previous bounds into the left-hand side of (4.72) and using the fact that $c - \delta_M(\alpha^*) > \frac{1}{2}c$ complete the proof of claim (4.72).

Last line of the right-hand side of (4.68). We claim

$$\frac{A^2}{\mu^4} (\varepsilon_2, F_2)_r \leq C_3(M) A \mu^2 \left\{ \mathcal{F} + \sqrt{\mathcal{F}} \left(\frac{\beta^4 |\log \beta|^2}{\mu^4} + 1 \right) \right\}, \quad (4.73)$$

where $C_3(M)$ is some constant depending on M . Indeed, this follows from (4.62) and (4.57):

$$\begin{aligned}
 \frac{A^2}{\mu^4} (\varepsilon_2, F_2)_r &\lesssim_M \frac{A^2}{\mu^4} \| \varepsilon_2 \|_{L^2} \cdot (\mu X_3 + \beta^4 |\log \beta|^2) \\
 &\lesssim_M A \mu^2 \sqrt{\mathcal{F}} \left\{ (\sqrt{\mathcal{F}} + 1) + \frac{\beta^4 |\log \beta|^2}{\mu^4} \right\}.
 \end{aligned}$$

Completion of the proof of (4.58). Choose $A = A(M) > 0$ such that $\frac{1}{2}cA > C_1(M) + 1$. Substituting claims (4.71)–(4.73) into (4.68), we have

$$\begin{aligned}
 &\frac{1}{2} \partial_s \mathcal{F} + \frac{\| \varepsilon_2 \|_{\dot{V}_{m+2}^{3/2}}}{\mu^5} \\
 &\leq (C_2(M) + C_3(M)) A \mu^2 \left\{ 1 + \mathcal{F} + \sqrt{\mathcal{F}} \left(\frac{\beta^{9/2} |\log \beta|^2}{\mu^5} + \frac{\beta^4 |\log \beta|^2}{\mu^4} \right) \right\}.
 \end{aligned}$$

Using $\beta \lesssim_M \mu$ and $\mu \ll 1$, the above formula can be written more compactly as

$$\partial_s \mathcal{F} + \frac{\| \varepsilon_2 \|_{\dot{V}_{m+2}^{3/2}}}{\mu^5} \lesssim_M \mu^2 \left\{ 1 + \mathcal{F} + \sqrt{\mathcal{F}} \cdot \frac{\beta^{9/2} |\log \beta|^2}{\mu^5} \right\},$$

which is (4.58). This completes the proof. ■

Integrating the monotonicity estimate (4.58), we obtain the key (local-in-time) *uniform H^3 -bound* as well as a spacetime bound.

Corollary 4.17 (Integration of the monotonicity). *Let I be a bounded time interval either of the form $[t_0, t_1)$ or $[t_0, t_1]$ such that $u(t) \in \mathcal{T}_{\alpha^*, M}$ for all $t \in I$. Let $L > 0$ be such that $E[u](t_1 - t_0) \leq L$. Then, we have*

$$\begin{aligned} & \sup_{t \in I} \left(\frac{\|\varepsilon_2\|_{\dot{H}_{m+2}^1}^2}{\mu^6} + \frac{\|\varepsilon_2\|_{L^2}^2}{\mu^4} \right)(t) + \int_I \frac{\|\varepsilon_2\|_{\dot{V}_{m+2}^{3/2}}^2}{\mu^5} \frac{dt}{\lambda^2} \\ & \lesssim_{M,L} 1 + \left(\frac{\|\varepsilon_2\|_{\dot{H}_{m+2}^1}^2}{\mu^6} + \frac{\|\varepsilon_2\|_{L^2}^2}{\mu^4} \right)(t_0). \end{aligned} \quad (4.74)$$

In particular, we have local-in-time uniform H^3 -bounds

$$(\|\varepsilon\|_{\dot{\mathcal{H}}_m^3} + \|\varepsilon_1\|_{\dot{H}_{m+1}^2} + \|\varepsilon_2\|_{\dot{H}_{m+2}^1})(t) \lesssim_{M,L} \left(\frac{\lambda(t)}{\lambda(t_0)} \right)^3 X_3(t_0), \quad (4.75)$$

$$(\|\varepsilon\|_{\dot{\mathcal{H}}_m^2} + \|\varepsilon_1\|_{\dot{H}_{m+1}^1} + \|\varepsilon_2\|_{L^2})(t) \lesssim_{M,L} \left(\frac{\lambda(t)}{\lambda(t_0)} \right)^2 \sqrt{\mu X_3(t_0)}. \quad (4.76)$$

Remark 4.18 (Uniform H^3 -bound). As is clear from the implicit constants involved in bounds (4.75)–(4.76), we see that $\|\varepsilon^\#(t)\|_{\dot{H}_m^3}$ is *uniformly bounded* for all H^3 nearby solutions (due to the presence of M and X_3) on a *bounded time interval* (due to the presence of L). The uniform bounds (4.75)–(4.76) can also be viewed as persistence of regularity estimates, but the point is that the H^3 -regularity of $\varepsilon^\#$ persists regardless whether u blows up or not. We also note that, as can be seen in the proof below, the L -dependences of the implicit constants in (4.74) and (4.75) are exponential in view of Grönwall's inequality.

Remark 4.19. All the bounds in the above corollary are scaling invariant.

Remark 4.20. We will not use the spacetime bound (i.e., the second term of the left-hand side of (4.74)) in our proof of Theorem 1.4. However, we include it to emphasize that (4.74) is indeed a monotonicity estimate.

Proof of Corollary 4.17. Step 1. Fixed-time bound of (4.74). In this step, we prove for any $t \in I$,

$$\mathcal{F}(t) \lesssim_{M,L} \mathcal{F}(t_0) + 1. \quad (4.77)$$

To show this, we first ignore the spacetime term of (4.58) and divide the both sides by $\sqrt{1 + \mathcal{F}}$ to obtain

$$\partial_s \sqrt{1 + \mathcal{F}} \leq K \mu^2 \left\{ \sqrt{1 + \mathcal{F}} + \frac{\beta^{9/2} |\log \beta|^2}{\mu^5} \right\}. \quad (4.78)$$

To integrate the inhomogeneous term of the right-hand side, we use (4.41) and (4.44) to note the following inequality

$$\begin{aligned} \partial_s \left(\frac{b}{\mu^{3/4}} \right) &= \frac{1}{\mu^{3/4}} \left\{ -\left(\frac{1}{4} b^2 + \eta^2 \right) + (b_s + b^2 + \eta^2) - \frac{3}{4} b \left(\frac{\mu_s}{\mu} + b \right) \right\} \\ &\leq -\frac{\beta^2}{4\mu^{3/4}} + K' \mu^{3-3/4} \sqrt{1 + \mathcal{F}} \end{aligned} \quad (4.79)$$

for some constant $K' = K'(M)$. Summing the above formula with (4.78), we obtain

$$\partial_s \left(\sqrt{1 + \mathcal{F}} + \frac{b}{\mu^{3/4}} \right) \leq \left(K \frac{\beta^{9/2} |\log \beta|^2}{\mu^3} - \frac{\beta^2}{4\mu^{3/4}} \right) + (K\mu^2 + K'\mu^{3-3/4}) \sqrt{1 + \mathcal{F}}.$$

Since $\beta \lesssim_M \mu \ll 1$, we have $\frac{|b|}{\mu^{3/4}} \ll 1$ and $\frac{\beta^{9/2} |\log \beta|^2}{\mu^3} \ll \frac{\beta^2}{\mu^{3/4}}$. Thus the above formula can be simplified as

$$\partial_s \left(\sqrt{1 + \mathcal{F}} + \frac{b}{\mu^{3/4}} \right) \leq K'' \mu^2 \left(\sqrt{1 + \mathcal{F}} + \frac{b}{\mu^{3/4}} \right) \quad (4.80)$$

for some constant $K'' = K''(M)$. Now (4.77) follows from integrating (4.80) using $ds = \lambda^2 dt$ and $\mu = \lambda \sqrt{E[u]}$,

$$\begin{aligned} \sqrt{\mathcal{F}}(t) &\leq \left(\sqrt{1 + \mathcal{F}} + \frac{b}{\mu^{3/4}} \right)(t) + 1 \\ &\leq e^{K'' E[u](t-t_0)} \left(\sqrt{1 + \mathcal{F}} + \frac{b}{\mu^{3/4}} \right)(t_0) + 1 \\ &\lesssim_{M,L} \sqrt{1 + \mathcal{F}}(t_0). \end{aligned}$$

Step 2. Spacetime bound of (4.74). In this step, we show the spacetime bound of (4.74). We integrate (4.58), apply (4.77) to pointwisely bound $\mathcal{F}(t)$, and use $\mu = \lambda \sqrt{E[u]}$ to have

$$\begin{aligned} \int_I \frac{\|\varepsilon_2\|_{\dot{V}_{m+2}^{3/2}}^2}{\mu^5} \frac{dt}{\lambda^2} &\lesssim \sup_{t \in I} \mathcal{F}(t) + \int_I K \mu^2 \left\{ 1 + \mathcal{F} + \sqrt{\mathcal{F}} \cdot \frac{\beta^{9/2} |\log \beta|^2}{\mu^5} \right\} \frac{dt}{\lambda^2} \\ &\lesssim_{M,L} (\mathcal{F}(t_0) + 1) + \sqrt{1 + \mathcal{F}}(t_0) \int_{s(I)} \frac{\beta^{9/2} |\log \beta|^2}{\mu^3} ds. \quad (4.81) \end{aligned}$$

For the last integral, we note as a consequence of (4.79), $\mu^{1/4} = \delta_M(\alpha^*)$, and (4.77) that

$$\begin{aligned} \int_{s(I)} \frac{\beta^{9/2} |\log \beta|^2}{\mu^3} ds &\leq \int_{s(I)} \frac{\beta^2}{\mu^{3/4}} ds \\ &\lesssim \sup_{t \in I} \frac{|b(t)|}{\mu^{3/4}(t)} + \int_{s(I)} K' \mu^{3-3/4} \sqrt{1 + \mathcal{F}} ds \\ &\lesssim \delta_M(\alpha^*) + \delta_M(\alpha^*) \left(\sup_{t \in I} \sqrt{1 + \mathcal{F}}(t) \right) \int_{s(I)} \mu^2 ds \\ &\lesssim_{M,L} \sqrt{1 + \mathcal{F}}(t_0). \end{aligned}$$

Substituting this bound into (4.81) gives

$$\int_I \frac{\|\varepsilon_2\|_{\dot{V}_{m+2}^{3/2}}^2}{\mu^5} \frac{dt}{\lambda^2} \lesssim_{M,L} (\mathcal{F}(t_0) + 1),$$

which is the spacetime bound of (4.74). This completes the proof of (4.74).

Step 3. *Proof of the uniform bounds (4.75)–(4.76).* By (4.57) and (4.77), we have

$$X_3(t) \lesssim_{M,L} \mu^3(t)(1 + \sqrt{\mathcal{F}(t_0)}) \sim_{M,L} \frac{\mu^3(t)}{\mu^3(t_0)} X_3(t_0) = \frac{\lambda^3(t)}{\lambda^3(t_0)} X_3(t_0).$$

Combining the above with Lemma 4.6 gives (4.75).

We turn to the proof of (4.76). Interpolating (4.75) and (4.8), we have

$$\|\varepsilon_1\|_{\dot{H}_{m+1}^1} \lesssim \|\varepsilon_1\|_{L^2}^{1/2} \|\varepsilon_1\|_{\dot{H}_{m+1}^2}^{1/2} \lesssim_{M,L} \mu^{1/2}(t) \frac{\lambda^{3/2}(t)}{\lambda^{3/2}(t_0)} \sqrt{X_3(t_0)} = \frac{\lambda^2(t)}{\lambda^2(t_0)} \sqrt{\mu X_3(t_0)}.$$

This bound, combined with (4.29), (4.30), and (3.39), gives

$$\|\varepsilon_2\|_{L^2} \lesssim \|\varepsilon_1\|_{\dot{H}_{m+1}^1} + \beta^2 + O_M(\mu^2) \lesssim_{M,L} \frac{\lambda^2(t)}{\lambda^2(t_0)} \sqrt{\mu X_3(t_0)}.$$

This bound propagates for ε via (4.27). This completes the proof of (4.76) and hence the proof of Corollary 4.17. $\blacksquare$

4.6. Proof of the structural Lemma 4.10

Proof of Lemma 4.10. Step 1. Estimates for $\check{F}$. We show estimates (4.48) and (4.49) for $\check{F}$. In particular, we only need estimates restricted to the compact region $y \leq 2R_0$, where we recall R_0 from Remark 2.5. Recall that

$$\check{F} = i(\mathbf{D}_w^* w_1 - \mathbf{D}_P^* P_1) - \left(\int_0^y \operatorname{Re}(\bar{w} w_1 - \bar{P} P_1) dy' \right) iP + \left(\int_y^\infty \operatorname{Re}(\bar{w} w_1) dy' \right) i\varepsilon + i\Psi.$$

The term $i\mathbf{D}_w^ \varepsilon_1$ of $\check{F}$.* This term has a derivative, so we need to integrate by parts to have low regularity estimate. Since $Z_k \in C_{c,m}^\infty$, we have

$$(i\mathbf{D}_w^* \varepsilon_1, Z_k)_r = -(\varepsilon_1, i\mathbf{D}_w Z_k)_r \lesssim_M \|\mathbf{1}_{(0,2R_0]}\varepsilon_1\|_{L^2} \lesssim_M \min\{\mu, X_3\},$$

where we applied either (4.8) or (4.25) in the last inequality. This yields (4.48) and (4.49) for the term $-i\mathbf{D}_w^* \varepsilon_1$ of $\check{F}$.

Remaining terms of $\check{F}$. The remaining terms are given by

$$\begin{aligned} \check{F} - i\mathbf{D}_w^* \varepsilon_1 &= i(\mathbf{D}_w^* - \mathbf{D}_P^*)P_1 \\ &\quad - \left(\int_0^y \operatorname{Re}(\bar{w} w_1 - \bar{P} P_1) dy' \right) iP + \left(\int_y^\infty \operatorname{Re}(\bar{w} w_1) dy' \right) i\varepsilon + i\Psi. \end{aligned}$$

These terms are merely estimated in local L^2 -norms. We claim that

$$\|\mathbf{1}_{(0,2R_0]}(\check{F} + i\mathbf{D}_w^* \varepsilon_1)\|_{L^2} \lesssim_M \min\{\mu, X_3\}. \quad (4.82)$$

This claim immediately implies (4.48) and (4.49) for $\check{F}$ since $Z_k \in L^2$. We turn to the proof of (4.82). First, we have

$$\begin{aligned} \|\mathbf{1}_{(0,2R_0]}(\mathbf{D}_w^* - \mathbf{D}_P^*)P_1\|_{L^2} &\lesssim \|\mathbf{1}_{(0,2R_0]}(|w|^2 - |P|^2)\|_{L^2} \|\mathbf{1}_{(0,2R_0]}P_1\|_{L^2} \\ &\lesssim \beta \|\mathbf{1}_{(0,2R_0]}\varepsilon\|_{L^\infty} \|w + P\|_{L^2} \lesssim_M \mu \|\mathbf{1}_{(0,2R_0]}\varepsilon\|_{L^\infty}. \end{aligned}$$

Next, we have

$$\begin{aligned} \left\| \mathbf{1}_{(0,2R_0]} \left(\int_0^y \operatorname{Re}(\bar{w} w_1 - \bar{P} P_1) dy' \right) i P \right\|_{L^2} &\lesssim \left\| \mathbf{1}_{(0,2R_0]} \frac{1}{y} |\bar{w} w_1 - \bar{P} P_1| \right\|_{L^1} \left\| \mathbf{1}_{(0,2R_0]} P \right\|_{L^2} \\ &\lesssim_M \left\| \mathbf{1}_{(0,2R_0]} \frac{1}{y} \varepsilon \right\|_{L^2} + \left\| \mathbf{1}_{(0,2R_0]} \varepsilon_1 \right\|_{L^2} \end{aligned}$$

and

$$\begin{aligned} \left\| \mathbf{1}_{(0,2R_0]} \left(\int_y^\infty \operatorname{Re}(\bar{w} w_1) dy' \right) i \varepsilon \right\|_{L^2} &\lesssim \left\| \frac{1}{y} w \right\|_{L^2} \|w_1\|_{L^2} \left\| \mathbf{1}_{(0,2R_0]} \varepsilon \right\|_{L^2} \\ &\lesssim_M \mu \left\| \mathbf{1}_{(0,2R_0]} \varepsilon \right\|_{L^2}. \end{aligned}$$

Finally, by (3.43), we have

$$\left\| \mathbf{1}_{(0,2R_0]} \Psi \right\|_{L^2} \lesssim \beta^3 \lesssim_M \min\{\mu, X_3\}.$$

As all the norms in the above error estimates are restricted to the compact region $y \leq 2R_0$, we can apply either (4.8) (for the bound μ) or Lemma 4.6 combined with Remark 4.7 (for the bound X_3). This completes the proof of claim (4.82) and hence the proof of (4.48) and (4.49) for $\check{F}$.

Step 2. Estimates for $\check{F}_1$. We prove (4.48) for $\check{F}_1$, (4.50), and (4.51). Recall that

$$\begin{aligned} \check{F}_1 &= i(A_w^* w_2 - A_P^* P_2) \\ &\quad - \left(\int_0^y \operatorname{Re}(\bar{w} w_1 - \bar{P} P_1) dy' \right) i P_1 + \left(\int_y^\infty \operatorname{Re}(\bar{w} w_1) dy' \right) i \varepsilon_1 + i \Psi_1. \end{aligned}$$

The term $i(A_w^ w_2 - A_P^* P_2)$.* This term has derivatives and contains the main linear part $iA_Q^* \varepsilon_2$ of the ε_1 -evolution.

(i) *Proof of (4.48).* For the low regularity estimate (4.48), we rewrite

$$\begin{aligned} A_w^* w_2 - A_P^* P_2 &= H_w w_1 - H_P P_1 + A_P^* (A_P P_1 - P_2) \\ &= (H_w - H_P) P_1 + H_w \varepsilon_1 + A_P^* (A_P P_1 - P_2) \end{aligned}$$

and integrate by parts to avoid derivatives falling on ε_1 ,

$$\begin{aligned} (i(A_w^* w_2 - A_P^* P_2), \tilde{Z}_k)_r &= -(\varepsilon_1, iH_w \tilde{Z}_k)_r - ((H_w - H_P) P_1, i\tilde{Z}_k)_r \\ &\quad - (A_P P_1 - P_2, iA_P \tilde{Z}_k)_r. \end{aligned}$$

Since $\tilde{Z}_k \in C_{c,m+1}^\infty$ is compactly supported, we can simply use $P \lesssim 1$, $\varepsilon \lesssim 1$, $P_1 \lesssim \beta$, $A_P P_1 - P_2 \lesssim \beta^3$ in the region $y \leq 2R_0$ (cf. (3.40)). As a result, we have

$$\begin{aligned} (\varepsilon_1, iH_w \tilde{Z}_k)_r &\lesssim_M \left\| \mathbf{1}_{(0,2R_0]} \varepsilon_1 \right\|_{L^2} \lesssim_M \mu, \\ ((H_w - H_P) P_1, i\tilde{Z}_k)_r &\lesssim_M \beta \left\| \mathbf{1}_{(0,2R_0]} \varepsilon \right\|_{L^\infty} \lesssim_M \mu, \\ (A_P P_1 - P_2, iA_P \tilde{Z}_k)_r &\lesssim \beta^3 \lesssim_M \mu, \end{aligned}$$

which yield (4.48) for $i(A_w^* w_2 - A_P^* P_2)$.

(ii) *Proof of (4.50) and (4.51).* For the higher Sobolev estimates (4.50) and (4.51), we begin by rewriting

$$A_w^* w_2 - A_P^* P_2 = A_Q^* \varepsilon_2 + \{(A_w^* - A_Q^*) \varepsilon_2 + (A_w^* - A_P^*) P_2\}.$$

For the first term $A_Q^* \varepsilon_2$, using $A_Q(yQ) = 0$, we have

$$\begin{aligned} (iA_Q^* \varepsilon_2, \tilde{Z}_k)_r &= -(\varepsilon_2, iA_Q \tilde{Z}_k)_r \lesssim \|\mathbf{1}_{(0,2R_0]} \varepsilon_2\|_{L^2} \lesssim \|\varepsilon_2\|_{\dot{V}_{m+2}^{3/2}}, \\ (iA_Q^* \varepsilon_2, \chi_{\mu^{-1}} yQ)_r &= -(\varepsilon_2, (\partial_y \chi_{\mu^{-1}}) i yQ)_r \lesssim \|\mathbf{1}_{[\mu^{-1}, 2\mu^{-1}]} y^{-3} \varepsilon_2\|_{L^1} \lesssim \mu^2 \|\varepsilon_2\|_{L^2}, \end{aligned}$$

and similarly for the inner product $(iA_Q^* \varepsilon_2, \chi_{\mu^{-1}} i yQ)_r$. This shows (4.50) and (4.51) for the term $iA_Q^* \varepsilon_2$.

To show (4.50) and (4.51) for the remaining term, namely, $i\{(A_w^* - A_Q^*) \varepsilon_2 + (A_w^* - A_P^*) P_2\}$, we claim the following L^1 -bound:

$$\left\| \frac{1}{y^2} \{(A_w^* - A_Q^*) \varepsilon_2 + (A_w^* - A_P^*) P_2\} \right\|_{L^1} \lesssim_M \mu X_3. \quad (4.83)$$

This claim follows from the estimates

$$\begin{aligned} \left\| \frac{1}{y^2} (A_w^* - A_Q^*) \varepsilon_2 \right\|_{L^1} &\lesssim \left\| \frac{1}{y^2} \int_0^y \|w\|^2 - Q^2 |y'| dy' \right\|_{L^2} \left\| \frac{1}{y} \varepsilon_2 \right\|_{L^2} \\ &\lesssim \| \|w\|^2 - Q^2 \|_{L^2} \left\| \frac{1}{y} \varepsilon_2 \right\|_{L^2} \lesssim_M \|w - Q\|_{L^\infty} \|\varepsilon_2\|_{\dot{H}_{m+2}^1} \\ &\lesssim_M \mu X_3 \end{aligned}$$

and

$$\begin{aligned} \left\| \frac{1}{y^2} (A_w^* - A_P^*) P_2 \right\|_{L^1} &\lesssim \left\| \frac{1}{y^3 \langle y \rangle} \int_0^y \|w\|^2 - |P|^2 |y'| dy' \right\|_{L^1} \|\langle y \rangle P_2\|_{L^\infty} \\ &\lesssim \left\| \frac{1}{y \langle y \rangle} (\|w\|^2 - |P|^2) \right\|_{L^1} \beta^2 \lesssim_M \mu^2 \left\| \frac{1}{y \langle y \rangle} \varepsilon \right\|_{L^2} \\ &\lesssim_M \mu^2 X_2, \end{aligned}$$

where we used the Fubini theorem for the inequality

$$\left\| \frac{1}{y^3 \langle y \rangle} \int_0^y f y' dy' \right\|_{L^1} \lesssim \left\| \frac{1}{y \langle y \rangle} f \right\|_{L^1}.$$

Now, claim (4.83) implies (4.50) and (4.51) since $\|y^2 \tilde{Z}_k\|_{L^\infty} + \|y^2 \cdot \chi_{\mu^{-1}} yQ\|_{L^\infty} \lesssim 1$. Thus we have proved (4.50) and (4.51) for the term $i(A_w^* w_2 - A_P^* P_2)$.

Remaining terms of $\tilde{F}_1$. Recall that the remaining terms are given by

$$\begin{aligned} &\tilde{F}_1 - i(A_w^* w_2 - A_P^* P_2) \\ &= - \left(\int_0^y \operatorname{Re}(\bar{w} w_1 - \bar{P} P_1) dy' \right) i P_1 + \left(\int_y^\infty \operatorname{Re}(\bar{w} w_1) dy' \right) i \varepsilon_1 + i \Psi_1. \end{aligned} \quad (4.84)$$

(i) *Proof of (4.48).* Since $\tilde{Z}_k \in C_{c,m+1}^\infty$, estimate (4.48) easily follows from

$$\begin{aligned} \|\mathbf{1}_{(0,2R_0]} \text{RHS of (4.84)}\|_{L^2} &\lesssim \left\| \frac{1}{y} (|\bar{w}w_1| + |\bar{P}P_1|) \right\|_{L^1} \| |P_1| + |\varepsilon_1| \|_{L^2} + \|\mathbf{1}_{(0,2R_0]} \Psi_1\|_{L^2} \\ &\lesssim_M \mu^2 + \beta^4 |\log \beta|^3 \lesssim \mu, \end{aligned}$$

where we used (3.45) for the term Ψ_1 .

(ii) *Proof of (4.50) and (4.51).* Since

$$\|y^2 \tilde{Z}_k\|_{L^\infty} + \|y^2 \cdot \chi_{\mu^{-1}y} Q\|_{L^\infty} \lesssim 1 \quad \text{and} \quad \beta^4 |\log \beta|^3 \lesssim \mu^{7/2},$$

it suffices to show (cf. (4.83))

$$\left\| \frac{1}{y^2} \text{RHS of (4.84)} \right\|_{L^1} \lesssim_M \mu X_3. \quad (4.85)$$

For the first term of the right-hand side of (4.84), we use $\|P_1\|_{L^2} + \|\langle y \rangle P_1\|_{L^\infty} \lesssim \beta$ to have

$$\begin{aligned} &\left\| \frac{1}{y^2} \int_0^y \operatorname{Re}(\bar{w}w_1 - \bar{P}P_1) dy' \cdot P_1 \right\|_{L^1} \\ &\lesssim \left\| \frac{1}{y} (\bar{w}w_1 - \bar{P}P_1) \right\|_{L^2} \|P_1\|_{L^2} \\ &\lesssim \left(\|w\|_{L^2} \left\| \frac{1}{y} \varepsilon_1 \right\|_{L^\infty} + \left\| \frac{1}{y \langle y \rangle} \varepsilon \right\|_{L^2} \|\langle y \rangle P_1\|_{L^\infty} \right) \cdot \beta \\ &\lesssim_M \mu \|\varepsilon_1\|_{\dot{H}_{m+1}^2} + \mu^2 \|\varepsilon\|_{\dot{H}_m^2} \lesssim_M \mu X_3. \end{aligned}$$

For the second term, we have

$$\left\| \frac{1}{y^2} \int_y^\infty \operatorname{Re}(\bar{w}w_1) dy' \cdot \varepsilon_1 \right\|_{L^1} \lesssim \|w w_1\|_{L^1} \left\| \frac{1}{y^2} \varepsilon_1 \right\|_{L^2} \lesssim_M \|w_1\|_{L^2} \left\| \frac{1}{y^2} \varepsilon_1 \right\|_{L^2} \lesssim_M \mu X_3.$$

For the last term, by (3.45), we have

$$\left\| \frac{1}{y^2} \Psi_1 \right\|_{L^1} \lesssim \beta^4 |\log \beta|^3,$$

completing the proof of (4.85), and hence the proof of (4.50) and (4.51). $\blacksquare$

4.7. Proof of the structural Lemma 4.15

Proof of Lemma 4.15. We estimate each term of F_2 . Recall that

$$\begin{aligned} F_2 = & \widetilde{\mathbf{Mod}} \cdot \mathbf{v}_2 - i\Psi_2 + (i\bar{w}w_1^2 - i\bar{P}P_1^2) - (i\tilde{H}_w - i\tilde{H}_Q)\varepsilon_2 - (i\tilde{H}_w - i\tilde{H}_P)P_2 \\ & - \left(\int_y^\infty \operatorname{Re}(\bar{w}w_1) dy' \right) i\varepsilon_2 + \left(\int_0^y \operatorname{Re}(\bar{w}w_1 - \bar{P}P_1) dy' \right) iP_2. \end{aligned} \quad (4.60)$$

Modulation terms $\widetilde{\mathbf{Mod}}_j(\mathbf{v}_2)_j$. For $j \in \{1, 2\}$, we show that $\widetilde{\mathbf{Mod}}_j(\mathbf{v}_2)_j$ is a perturbative term. Thus we need to prove (4.62) and (4.64). These estimates easily follow from (4.41) and (3.34):

$$\begin{aligned} \|\widetilde{\mathbf{Mod}}_j\|(\mathbf{v}_2)_j\|_{L^2} &\lesssim_M X_3 \cdot \beta^2 |\log \beta|^{1/2} \lesssim_M \mu X_3, \\ \|\widetilde{\mathbf{Mod}}_j\|A_Q^*(\mathbf{v}_2)_j\|_{L^2} &\lesssim_M X_3 \cdot \beta^2 \lesssim_M \mu^2 X_3. \end{aligned}$$

For $j \in \{3, 4\}$, we need to prove (4.65)–(4.67). These estimates follow from (4.42), $V_{7/2} \leq X_3$, and (3.35):

$$\begin{aligned} \|\widetilde{\mathbf{Mod}}_j\|y|(\mathbf{v}_2)_j\|_{L^2} &\lesssim_M V_{7/2} \lesssim_M X_3, \\ \|\widetilde{\mathbf{Mod}}_j\| \|\langle y \rangle^{1/2} |(\mathbf{v}_2)_j\|_{L^2} &\lesssim_M V_{7/2} \cdot \beta^{1/2} \lesssim_M \mu^{1/2} V_{7/2}, \\ \|\widetilde{\mathbf{Mod}}_j\| \|\langle y \rangle^{1/2} y(\tilde{H}_Q \mathbf{v}_2)_j\|_{L^2} &\lesssim_M V_{7/2} \cdot \beta \lesssim_M \mu V_{7/2}. \end{aligned}$$

Inhomogeneous term $-i\Psi_2$. This term is perturbative; estimates (4.62) and (4.64) are immediate from (3.47) and (3.48).

Nonlinear term $-i(\tilde{H}_w - \tilde{H}_P)P_2$. Let

$$h_1 := |w|^2 - |P|^2 \quad \text{and} \quad h_2 := |w|^2 + |P|^2$$

be such that

$$\tilde{H}_w - \tilde{H}_P = \frac{1}{y^2} \left(2m + 4 - \frac{1}{2} \int_0^y h_2 y' dy' \right) \cdot \left(-\frac{1}{2} \int_0^y h_1 y' dy' \right) + \frac{1}{2} h_1.$$

We show that $-i(\tilde{H}_w - \tilde{H}_P)P_2$ is a perturbative term; thus F_2^{pert} collects this term. Taking $|\cdot|_{-1}$ to the above formula and using $\|h_2\|_{L^1} \lesssim_M 1$ and $|h_2| \lesssim |P|^2 + |\varepsilon|^2 \lesssim y^{-2} + |\varepsilon|^2$, we have

$$\begin{aligned} i(\tilde{H}_w - \tilde{H}_P)P_2 &\lesssim_M \left(\frac{1}{y^2} \int_0^y |h_1| y' dy' + |h_1| \right) \cdot |P_2|, \\ |i(\tilde{H}_w - \tilde{H}_P)P_2|_{-1} &\lesssim_M \left(\frac{1}{y^2} \int_0^y |h_1| y' dy' + |h_1| \right) \cdot |P_2|_{-1} + |(\partial_y h_1)P_2| \\ &\quad + \left(\frac{1}{y} \int_0^y |h_1| y' dy' \right) \cdot |\varepsilon^2 P_2|. \end{aligned}$$

We now estimate the L^2 -norms. First, using $|P_2|_{-1} \lesssim \beta^2 \langle y \rangle^{-2}$ and $\|P\|_{L^2} \lesssim 1$, we have for $\ell \in \{0, 1\}$

$$\begin{aligned} \left\| \left(\frac{1}{y^2} \int_0^y |h_1| y' dy' + |h_1| \right) \cdot |P_2|_{-\ell} \right\|_{L^2} &\lesssim \beta^2 \|\langle y \rangle^{-(1+\ell)} h_1\|_{L^2} \\ &\lesssim_M \mu^2 \|\langle y \rangle^{-(1+\ell)} \varepsilon\|_{L^\infty} \lesssim_M \mu^2 X_{2+\ell}, \end{aligned}$$

where we used (2.41) and (4.27)–(4.28). The case $\ell = 0$ gives (4.62). Next, we similarly estimate using $|P|_1 \lesssim \langle y \rangle^{-1}$ and $|P_2| \lesssim \beta^2 \langle y \rangle^{-1}$,

$$\begin{aligned} \|(\partial_y h_1)P_2\|_{L^2} &\lesssim \|\varepsilon|_{-1} \cdot (|P|_1 + |\varepsilon|)P_2\|_{L^2} \\ &\lesssim \beta^2 (\|\langle y \rangle^{-1} |\varepsilon|_{-1}\|_{L^2} \|\varepsilon\|_{L^\infty} + \|\langle y \rangle^{-2} |\varepsilon|_{-1}\|_{L^2}) \lesssim_M \mu^2 X_3. \end{aligned}$$

Finally, we have

$$\begin{aligned} \left\| \int_0^y |h_1| y' dy' \cdot \frac{1}{y} \varepsilon^2 P_2 \right\|_{L^2} &\lesssim \left\| \frac{1}{y^2} \int_0^y |h_1| y' dy' \right\|_{L^2} \|\varepsilon\|_{L^\infty}^2 \beta^2 \\ &\lesssim_M \beta^2 \|h_1\|_{L^2} \mu^2 \lesssim_M \mu^4 \|\varepsilon\|_{L^\infty} \lesssim_M \mu^5 \lesssim_M \mu^2 X_3. \end{aligned}$$

This completes the proof of (4.62)–(4.64) for the term $-i(\tilde{H}_w - \tilde{H}_P)P_2$.

Nonlinear term $-i(\tilde{H}_w - \tilde{H}_Q)\varepsilon_2$. Let

$$h_1 := |w|^2 - Q^2 \quad \text{and} \quad h_2 := |w|^2 + Q^2$$

so that

$$\tilde{H}_w - \tilde{H}_Q = \frac{1}{y^2} \left(2m + 4 - \frac{1}{2} \int_0^y h_2 y' dy' \right) \cdot \left(-\frac{1}{2} \int_0^y h_1 y' dy' \right) + \frac{1}{2} h_1.$$

We further decompose

$$h_1 = 2 \operatorname{Re}(\overline{(w - Q)}Q) + |w - Q|^2 =: h_1^{\text{loc}} + h_1^{\text{pert}} \quad \text{and} \quad h_2 = 2Q^2 + h_1$$

so that

$$\begin{aligned} \tilde{H}_w - \tilde{H}_Q &= \left\{ \frac{1}{y^2} \left(2m + 4 - \int_0^y Q^2 y' dy' \right) \cdot \left(-\frac{1}{2} \int_0^y h_1^{\text{loc}} y' dy' \right) + \frac{1}{2} h_1^{\text{loc}} \right\} \\ &\quad + \left\{ \frac{1}{y^2} \left(2m + 4 - \int_0^y Q^2 y' dy' \right) \cdot \left(-\frac{1}{2} \int_0^y h_1^{\text{pert}} y' dy' \right) + \frac{1}{2} h_1^{\text{pert}} \right\} \\ &\quad + \frac{1}{4y^2} \left(\int_0^y h_1 y' dy' \right)^2 \\ &=: V_{\text{loc}} + V_{\text{pert},1} + V_{\text{pert},2}. \end{aligned}$$

We claim that the contribution of V_{loc} is local and those of $V_{\text{pert},1}, V_{\text{pert},2}$ are perturbative.

In other words, $F_{2,1}^{\text{loc}}$ collects $-iV_{\text{loc}}\varepsilon_2$ and F_2^{pert} collects $-i(V_{\text{pert},1} + V_{\text{pert},2})\varepsilon_2$.

For $-iV_{\text{loc}}\varepsilon_2$, we use $\|w - Q\|_{\dot{H}_m^1} \lesssim_M \mu$ to have

$$\begin{aligned} \|iV_{\text{loc}}\varepsilon_2\|_{L^2} &\lesssim \left\| \left(\frac{1}{y^2} \int_0^y |h_1^{\text{loc}}| y' dy' + |h_1^{\text{loc}}| \right) \cdot \varepsilon_2 \right\|_{L^2} \\ &\lesssim \|h_1^{\text{loc}}\|_{L^2} \|\varepsilon_2\|_{L^\infty} \\ &\lesssim_M \|w - Q\|_{L^\infty} \|\varepsilon_2\|_{\dot{H}_{m+2}^1} \lesssim_M \mu X_3 \end{aligned}$$

and (further using (2.43) and $Q \lesssim \langle y \rangle^{-3}$)

$$\begin{aligned} \|\langle y \rangle |iV_{\text{loc}}\varepsilon_2|_{-1}\|_{L^2} &\lesssim \|\langle y \rangle (\partial_y h_1^{\text{loc}}) \varepsilon_2\|_{L^2} + \left\| \langle y \rangle \left(\frac{1}{y^2} \int_0^y |h_1^{\text{loc}}| y' dy' + |h_1^{\text{loc}}| \right) \cdot |\varepsilon_2|_{-1} \right\|_{L^2} \\ &\lesssim \|w - Q\|_{-1} \|\langle y \rangle^{-2} \varepsilon_2\|_{L^\infty} + \|w - Q\|_{L^\infty} \|\langle y \rangle^{-2} |\varepsilon_2|_{-1}\|_{L^2} \\ &\lesssim_M \mu \|\varepsilon_2\|_{\dot{V}_{m+2}^{3/2}} \lesssim_M \mu V_{7/2}. \end{aligned}$$

For $-iV_{\text{pert},1}\varepsilon_2$, we have

$$\begin{aligned} \|iV_{\text{pert},1}\varepsilon_2\|_{L^2} &\lesssim \left\| \left(\frac{1}{y^2} \int_0^y |h_1^{\text{pert}}| y' dy' + |h_1^{\text{pert}}| \right) \cdot \varepsilon_2 \right\|_{L^2} \lesssim \|w - Q\|_{L^\infty}^2 \|\varepsilon_2\|_{L^2} \\ &\lesssim_M \mu X_3, \end{aligned}$$

and similarly

$$\begin{aligned} \|iV_{\text{pert},1}\varepsilon_2|_{-1}\|_{L^2} &\lesssim \|(\partial_y h_1^{\text{pert}})\varepsilon_2\|_{L^2} + \left\| \left(\frac{1}{y^2} \int_0^y |h_1^{\text{pert}}| y' dy' + |h_1^{\text{pert}}| \right) \cdot |\varepsilon_2|_{-1} \right\|_{L^2} \\ &\lesssim \|\partial_y h_1^{\text{pert}}\|_{L^2} \|\varepsilon_2\|_{L^\infty} + \|h_1^{\text{pert}}\|_{L^\infty} \|\varepsilon_2|_{-1}\|_{L^2} \\ &\lesssim \|w - Q\|_{\dot{H}_m^1}^2 \|\varepsilon_2\|_{\dot{H}_{m+2}^1} \lesssim_M \mu^2 X_3. \end{aligned}$$

For $-iV_{\text{pert},2}\varepsilon_2$, we have

$$\begin{aligned} \|iV_{\text{pert},2}\varepsilon_2\|_{L^2} &\lesssim \left\| \left(\frac{1}{y} \int_0^y |h_1| y' dy' \right)^2 \cdot \varepsilon_2 \right\|_{L^2} \lesssim \|h_1\|_{L^2}^2 \|\varepsilon_2\|_{L^2} \\ &\lesssim_M \|w - Q\|_{L^\infty}^2 \|\varepsilon_2\|_{L^2} \lesssim_M \mu X_3, \end{aligned}$$

and similarly

$$\begin{aligned} \|iV_{\text{pert},2}\varepsilon_2|_{-1}\|_{L^2} &\lesssim \left\| \left(\frac{1}{y} \int_0^y |h_1| y' dy' \right)^2 \cdot |\varepsilon_2|_{-1} \right\|_{L^2} + \left\| \frac{1}{y} \int_0^y |h_1| y' dy' \cdot h_1 \varepsilon_2 \right\|_{L^2} \\ &\lesssim \|h_1\|_{L^2}^2 (\|\varepsilon_2|_{-1}\|_{L^2} + \|\varepsilon_2\|_{L^\infty}) \\ &\lesssim_M \|w - Q\|_{L^\infty}^2 \|\varepsilon_2\|_{\dot{H}_{m+1}^1} \lesssim_M \mu^2 X_3. \end{aligned}$$

This completes the proof of (4.62)–(4.64) for the term $-i(\tilde{H}_w - \tilde{H}_Q)\varepsilon_2$.

Nonlinear term $(\int_0^y \text{Re}(\bar{w}w_1 - \bar{P}P_1)dy')iP_2$. We show that this term is perturbative. Using $P \lesssim \langle y \rangle^{-1}$, $P_1 \lesssim \beta \langle y \rangle^{-1}$, $|P_2|_1 \lesssim \mathbf{1}_{(0,2B_1]} \beta^2 \langle y \rangle^{-1}$, and Lemma 4.6, we have

$$\begin{aligned} &\left\| \left(\int_0^y \text{Re}(\bar{w}w_1 - \bar{P}P_1)dy' \right) iP_2 \right\|_{L^2} \\ &\lesssim \left\| \frac{1}{y} (\bar{w}w_1 - \bar{P}P_1) \right\|_{L^2} \|y^2 P_2\|_{L^\infty} \\ &\lesssim \left(\left\| \frac{1}{y \langle y \rangle} \varepsilon_1 \right\|_{L^2} + \|\varepsilon\|_{L^\infty} \left\| \frac{1}{y} \varepsilon_1 \right\|_{L^2} + \beta \left\| \frac{1}{y \langle y \rangle} \varepsilon \right\|_{L^2} \right) \cdot \beta \lesssim_M \mu X_3. \end{aligned}$$

This gives (4.62). Similarly, we have

$$\begin{aligned} \left\| \left(\int_0^y \text{Re}(\bar{w}w_1 - \bar{P}P_1)dy' \right) iP_2 \right\|_{-1} &\lesssim \left\| \frac{1}{y} (\bar{w}w_1 - \bar{P}P_1) \right\|_{L^2} \|y|P_2|_1\|_{L^\infty} \\ &\lesssim_M \mu^2 X_3, \end{aligned}$$

which is (4.64). This completes the proof of (4.62)–(4.64) for the term $(\int_0^y \text{Re}(\bar{w}w_1 - \bar{P}P_1)dy')iP_2$.

Nonlinear term $-(\int_y^\infty \operatorname{Re}(\bar{w}w_1)dy')i\varepsilon_2$. Estimate (4.62) easily follows from

$$\left\| \left(\int_y^\infty \operatorname{Re}(\bar{w}w_1)dy' \right) i\varepsilon_2 \right\|_{L^2} \lesssim \|\bar{w}w_1\|_{L^1} \|\varepsilon_2\|_{L^\infty} \lesssim_M \|w_1\|_{L^2} \|\varepsilon_2\|_{\dot{H}_{m+2}^1} \lesssim_M \mu X_3.$$

For (4.64), we decompose

$$\int_y^\infty \operatorname{Re}(\bar{w}w_1)dy' = \int_y^\infty \operatorname{Re}(Qw_1)dy' + \int_y^\infty \operatorname{Re}(\overline{(w-Q)}w_1)dy' =: V_{\text{loc}} + V_{\text{pert}}.$$

The term $-iV_{\text{loc}}\varepsilon_2$ is local; we use $Q \lesssim \langle y \rangle^{-3}$ to have

$$\begin{aligned} \|\langle y \rangle |iV_{\text{loc}}\varepsilon_2|_{-1}\|_{L^2} &\lesssim \left\| \langle y \rangle \int_y^\infty |Qw_1|dy' \cdot |\varepsilon_2|_{-1} \right\|_{L^2} + \|\langle y \rangle Qw_1\varepsilon_2\|_{L^2} \\ &\lesssim \|\langle y \rangle w_1\|_{L^2} \cdot \|\langle y \rangle^{-2}|\varepsilon_2|_{-1}\|_{L^2} + \|w_1\|_{L^2} \|\langle y \rangle^{-2}\varepsilon_2\|_{L^\infty} \\ &\lesssim_M \mu \|\langle y \rangle^{-2}|\varepsilon_2|_{-1}\|_{L^2} \lesssim_M \mu V_{7/2}. \end{aligned}$$

The term $-iV_{\text{pert}}\varepsilon_2$ is perturbative; we use $\|w - Q\|_{\dot{H}_m^1} + \|w_1\|_{L^2} \lesssim_M \mu$ to have

$$\begin{aligned} \|iV_{\text{pert}}\varepsilon_2|_{-1}\|_{L^2} &\lesssim \left\| \int_y^\infty |(w-Q)w_1|dy' \cdot |\varepsilon_2|_{-1} \right\|_{L^2} + \|(w-Q)w_1\varepsilon_2\|_{L^2} \\ &\lesssim \left\| \frac{1}{y}(w-Q)w_1 \right\|_{L^1} \|\varepsilon_2|_{-1}\|_{L^2} + \|(w-Q)w_1\|_{L^2} \|\varepsilon_2\|_{L^\infty} \\ &\lesssim \|w-Q\|_{\dot{H}_m^1} \|w_1\|_{L^2} \|\varepsilon_2\|_{\dot{H}_{m+2}^1} \lesssim_M \mu^2 X_3. \end{aligned}$$

This completes the proof of (4.62)–(4.64) for the term $-(\int_y^\infty \operatorname{Re}(\bar{w}w_1)dy')i\varepsilon_2$.

Nonlinear term $i(\bar{w}w_1^2 - \bar{P}P_1^2)$. We begin with the decomposition

$$\begin{aligned} \bar{w}w_1^2 - \bar{P}P_1^2 &= Q\varepsilon_1(2P_1 + \varepsilon_1) + \overline{(w-Q)}\varepsilon_1(2P_1 + \varepsilon_1) + \bar{\varepsilon}P_1^2 \\ &=: R_{\text{loc}} + R_{\text{pert},1} + R_{\text{pert},2}. \end{aligned}$$

First, R_{loc} is a local term; we use (2.42) and (4.8) to have

$$\|R_{\text{loc}}\|_{L^2} \lesssim \|\langle y \rangle^{-3}\varepsilon_1\|_{L^\infty} \|P_1 + 2\varepsilon_1\|_{L^2} \lesssim_M \mu \|\varepsilon_1\|_{\dot{H}_{m+1}^2} \lesssim_M \mu X_3,$$

and we use (2.42) and (4.26) to have

$$\begin{aligned} \|\langle y \rangle |R_{\text{loc}}|_{-1}\|_{L^2} &\lesssim \|\langle y \rangle^{-2}|\varepsilon_1|_{-1} \cdot (|P_1|_1 + |\varepsilon_1|)\|_{L^2} \lesssim_M \mu \|\langle y \rangle^{-2}|\varepsilon_1|_{-1}\|_{L^\infty} \\ &\lesssim_M \mu \|\varepsilon_1\|_{\dot{V}_{m+1}^{5/2}} \lesssim_M \mu V_{7/2}. \end{aligned}$$

Next, $R_{\text{pert},1}$ is a perturbative term; we use $\|w - Q\|_{\dot{H}_m^1} + \|P_1\|_{L^2 \cap y^{-1}L^\infty} \lesssim_M \mu$ to have

$$\begin{aligned} \|(w-Q)\varepsilon_1(P_1 + 2\varepsilon_1)\|_{L^2} &\lesssim \|w-Q\|_{L^\infty} \|\varepsilon_1\|_{L^\infty} \|P_1 + |\varepsilon_1|\|_{L^2} \\ &\lesssim_M \mu^2 \|\varepsilon_1\|_{\dot{H}_m^1} \lesssim_M \mu X_3 \end{aligned}$$

and (also using $\|\varepsilon_1\|_{L^\infty}^2 \lesssim \|\varepsilon_1\|_{L^2} \|\varepsilon_1\|_{\dot{H}_{m+1}^2}$)

$$\begin{aligned} \| |R_{\text{pert},1}|_{-1} \|_{L^2} &\lesssim \| |w - Q|_{-1} \|_{L^2} \left(\left\| \frac{1}{y} \varepsilon_1 \right\|_{L^\infty} \|y P_1\|_{L^\infty} + \|\varepsilon_1\|_{L^\infty}^2 \right) \\ &\quad + \|w - Q\|_{L^\infty} \|\varepsilon_1\|_{-1} \|_{L^\infty} (\|P_1\|_{L^2} + \|\varepsilon_1\|_{L^2}) \\ &\lesssim_M \mu^2 \|\varepsilon_1\|_{\dot{H}_{m+1}^2} \lesssim_M \mu^2 X_3. \end{aligned}$$

Finally, $R_{\text{pert},2}$ is a perturbative term; we use the estimate $|P_1^2|_1 \lesssim \beta^2 \langle y \rangle^{-2}$ to have for $\ell \in \{0, 1\}$,

$$\begin{aligned} \|\bar{\varepsilon} P_1^2|_{-\ell}\|_{L^2} &\lesssim \|\langle y \rangle^2 |P_1^2|_1\|_{L^\infty} \|\langle y \rangle^{-2} |\varepsilon|_{-\ell}\|_{L^2} \\ &\lesssim \begin{cases} \beta^2 \|\varepsilon\|_{\dot{H}_m^2} \lesssim_M \mu X_3 & \text{if } \ell = 0, \\ \beta^2 \|\varepsilon\|_{\dot{H}_m^3} \lesssim_M \mu^2 X_3 & \text{if } \ell = 1. \end{cases} \end{aligned}$$

This completes the proof of (4.62)–(4.64) for the term $i(\bar{w}w_1^2 - \bar{P}P_1^2)$. This completes the proof of Lemma 4.15. $\blacksquare$

5. Proof of the main theorem

In this section, we prove Theorem 1.4. Let $u(t)$ be an H_m^3 -solution to (CSS) that blows up at $T < +\infty$. Denote by $M = M[u]$ the L^2 -mass of u . Fix a small parameter $\alpha^* > 0$ such that the results in Section 4 are valid. By Remark 2.13, the solution $u(t)$ eventually belongs to the soliton tube $\mathcal{T}_{\alpha^*, M}$ (recall (4.5)). Note that $u(t)$ admits the decomposition as in Lemma 4.1.

Remark 5.1 (On implicit constants). In this section, the solution $u(t)$ is fixed and we allow the implicit constants to depend on u . In other words, we omit any dependences on $M[u]$, $E[u]$, $\|u(t_0)\|_{H_m^3}$, and so on, in the following inequalities. For example, the nonlinear coercivity estimate in Lemma 4.2 and the uniform bound in Corollary 4.17 can be stated as

$$\beta + \|\varepsilon\|_{\dot{H}_m^1} + \|\varepsilon_1\|_{L^2} \lesssim \mu \quad \text{and} \quad X_3 \lesssim \mu^3, \quad (5.1)$$

respectively.

5.1. Asymptotics of modulation parameters

The goal of this subsection is to derive sharp asymptotics of the modulation parameters. In particular, we show that the blow-up rate is pseudoconformal. We integrate the modulation equations. However, our b_s - and η_s -modulation estimates (4.42) are not sufficient,¹⁵ and we need to obtain more refined estimates. We achieve this by testing ε_1 -equation (4.40) against the almost nullspace elements $\chi_{\mu^{-1}} i y Q$ and $\chi_{\mu^{-1}} y Q$.

¹⁵Assuming $T = 0$, consider the regime $\lambda(t) \sim |t|^\nu$ and $b(t) \sim |t|^{2\nu-1}$ with $\nu \gg 1$.

Lemma 5.2 (Second modulation estimates). *Let*

$$\hat{b} := b - \frac{(\varepsilon_1, \chi_{1/\mu} 2iyQ)_r}{\|yQ\|_{L^2}^2} \quad \text{and} \quad \hat{\eta} := \eta - \frac{(\varepsilon_1, \chi_{1/\mu} 2yQ)_r}{\|yQ\|_{L^2}^2}. \quad (5.2)$$

Then, we have

$$|\hat{b} - b| + |\hat{\eta} - \eta| \lesssim \mu^2, \quad (5.3)$$

$$\left| \frac{\lambda_s}{\lambda} + \hat{b} \right| + |\gamma_s - (m+1)\hat{\eta}| \lesssim \mu^2, \quad (5.4)$$

$$|\hat{b}_s + \hat{b}^2 + \hat{\eta}^2| + |\hat{\eta}_s| \lesssim \beta^3 + \beta\mu^2 + \mu^4. \quad (5.5)$$

Proof. In the proof, we denote $R = \mu^{-1}$, $\psi_3 = -2iyQ$, $\psi_4 = -2yQ$.

Step 1. Proof of (5.3) and (5.4). Estimate (5.3) follows from (5.1) and (4.25),

$$(\varepsilon_1, \chi_R \psi_k)_r \lesssim \|y^{-2}\varepsilon_1\|_{L^2} \|\chi_R y^2 \psi_k\|_{L^2} \lesssim X_3 \cdot R \lesssim \mu^2.$$

Now (5.4) follows from (4.41), (5.3), and (4.45),

$$\begin{aligned} \left| \frac{\lambda_s}{\lambda} + \hat{b} \right| + |\gamma_s - (m+1)\hat{\eta}| &\lesssim |\mathbf{Mod}_1| + |b - \hat{b}| + |\widetilde{\mathbf{Mod}}_2| \\ &\quad + \left| \int_0^\infty \operatorname{Re}(\bar{w}w_1)dy + 2(m+1)\eta \right| + |\eta - \hat{\eta}| \lesssim \mu^2. \end{aligned}$$

Step 2. Proof of (5.5). First, we claim

$$|\partial_s(\varepsilon_1, \chi_R \psi_k)_r - (\widetilde{\mathbf{Mod}} \cdot \mathbf{v}_1, \chi_R \psi_k)_r| \lesssim (\beta + \mu^2)\mu^2 \quad \forall k \in \{3, 4\}. \quad (5.6)$$

To see this, we rewrite (4.47) as

$$\partial_s \varepsilon_1 = \widetilde{\mathbf{Mod}} \cdot \mathbf{v}_1 + \left(\frac{\lambda_s}{\lambda} \Lambda_{-1} - \gamma_s i \right) \varepsilon_1 - \check{F}_1$$

and test it against $\chi_R \psi_k$ to obtain

$$\begin{aligned} \partial_s(\varepsilon_1, \chi_R \psi_k)_r &= (\partial_s \varepsilon_1, \chi_R \psi_k)_r + (\varepsilon_1, \partial_s \chi_R \psi_k)_r \\ &= (\widetilde{\mathbf{Mod}} \cdot \mathbf{v}_1, \chi_R \psi_k)_r - (\check{F}_1, \chi_R \psi_k)_r \\ &\quad + \left(\varepsilon_1, \chi_R \left(-\frac{\lambda_s}{\lambda} y \partial_y \psi_k + \gamma_s i \psi_k \right) \right)_r + \left(\varepsilon_1, \left\{ \left(\partial_s - \frac{\lambda_s}{\lambda} y \partial_y \right) \chi_R \right\} \psi_k \right)_r. \end{aligned}$$

We keep the first term and estimate the remainders. By (4.51), we have

$$(\check{F}_1, \chi_R \psi_k)_r \lesssim \mu^4 + \beta^4 |\log \beta|^3 \lesssim \beta \mu^2 + \mu^4.$$

Next, we use (5.4) to have

$$\begin{aligned} \left(\varepsilon_1, \chi_R \left(-\frac{\lambda_s}{\lambda} y \partial_y \psi_k + \gamma_s i \psi_k \right) \right)_r &\lesssim \left(\left| \frac{\lambda_s}{\lambda} \right| + |\gamma_s| \right) \|y^{-2}\varepsilon_1\|_{L^2} \|y^2 \mathbf{1}_{(0,2R]} |\psi_k|_1\|_{L^2} \\ &\lesssim (\beta + \mu^2) \cdot \mu^3 \cdot \mu^{-1} \lesssim (\beta + \mu^2) \cdot \mu^2. \end{aligned}$$

Finally, since $\frac{\lambda_s}{\lambda} = \frac{\mu_s}{\mu}$, we have

$$\left(\varepsilon_1, \left\{ \left(\partial_s - \frac{\lambda_s}{\lambda} y \partial_y \right) \chi_R \right\} \psi_k \right)_r = 0.$$

The proof of claim (5.6) is completed.

Next, we claim that

$$\widetilde{\mathbf{Mod}}_j \cdot ((\mathbf{v}_1)_j, \chi_R \psi_k)_r \lesssim \mu^4 \quad \text{if } j \in \{1, 2\}, \quad (5.7)$$

$$\widetilde{\mathbf{Mod}}_j \cdot ((\mathbf{v}_1)_j, \chi_R \psi_k)_r \lesssim \beta \mu^2 \quad \text{if } j \in \{3, 4\} \text{ and } j \neq k, \quad (5.8)$$

$$\widetilde{\mathbf{Mod}}_3 \cdot ((\mathbf{v}_1)_3, \chi_R \psi_3)_r = -(b_s + b^2 + \eta^2) \|yQ\|_{L^2}^2 + O(\beta^3 + \beta \mu^2 + \mu^4), \quad (5.9)$$

$$\widetilde{\mathbf{Mod}}_4 \cdot ((\mathbf{v}_1)_4, \chi_R \psi_4)_r = -\eta_s \|yQ\|_{L^2}^2 + O(\beta^3 + \beta \mu^2 + \mu^4). \quad (5.10)$$

Indeed, (5.7) follows from (4.41) and $|P_1|_1 \lesssim \beta \langle y \rangle^{-2} + \beta^2$,

$$\widetilde{\mathbf{Mod}}_j \cdot ((\mathbf{v}_1)_j, \chi_R \psi_k)_r \lesssim X_3 \cdot \|\mathbf{1}_{(0,2R]} |P_1|_1 \cdot yQ\|_{L^1} \lesssim \mu^3 (\beta + \beta^2 |\log \mu|) \lesssim \mu^4.$$

We turn to the proof of (5.8)–(5.10). Let $j \in \{3, 4\}$. Using (3.33), we have

$$\begin{aligned} ((\mathbf{v}_1)_j, \chi_R \psi_k)_r &= \begin{cases} 0 & \text{if } j \neq k, \\ -(\chi_{B_1} yQ, \chi_R yQ)_r & \text{if } j = k \end{cases} + O(\|\mathbf{1}_{(0,2R]} \beta \cdot yQ\|_{L^1}) \\ &= \begin{cases} 0 & \text{if } j \neq k, \\ -\|yQ\|_{L^2}^2 + O(\mu^2) & \text{if } j = k \end{cases} + O(\beta |\log \mu|). \end{aligned}$$

Multiplying the above by $\widetilde{\mathbf{Mod}}_j$ and applying (4.42) with $V_{7/2} \leq X_3 \lesssim \mu^3$, we have

$$\begin{aligned} \widetilde{\mathbf{Mod}}_j \cdot ((\mathbf{v}_1)_j, \chi_R \psi_k)_r &= \begin{cases} 0 & \text{if } j \neq k, \\ -\widetilde{\mathbf{Mod}}_j \|yQ\|_{L^2}^2 + O(\mu^5) & \text{if } j = k \end{cases} + O(\beta \mu^3 |\log \mu|) \\ &= \begin{cases} O(\beta \mu^2) & \text{if } j \neq k, \\ -\widetilde{\mathbf{Mod}}_j \|yQ\|_{L^2}^2 + O(\beta \mu^2 + \mu^4) & \text{if } j = k. \end{cases} \end{aligned}$$

This completes the proof of (5.8). When $j = k$, further applying $|p_3^{(b)}| + |p_3^{(\eta)}| \lesssim \beta^3$ to the above completes the proof of (5.9)–(5.10).

We are now ready to finish the proof. Substituting claims (5.7)–(5.10) into (5.6) gives

$$\left| b_s + b^2 + \eta^2 + \frac{\partial_s(\varepsilon_1, \chi_R \psi_3)_r}{\|yQ\|_{L^2}^2} \right| + \left| \eta_s + \frac{\partial_s(\varepsilon_1, \chi_R \psi_4)_r}{\|yQ\|_{L^2}^2} \right| \lesssim \beta^3 + \beta \mu^2 + \mu^4.$$

By (5.2) and (5.3), this completes the proof of (5.5). This completes the proof of Lemma 5.2. $\blacksquare$

We now prove the main result of this subsection.

Proposition 5.3 (Asymptotics of modulation parameters). *There exist $\ell = \ell(u) \in (0, \infty)$ and $\gamma^* \in \mathbb{R}/2\pi\mathbb{Z}$ such that*

$$\lambda(t) = (\ell + o_{t \rightarrow T}(1)) \cdot (T - t), \quad (5.11)$$

$$\gamma(t) \rightarrow \gamma^*, \quad (5.12)$$

$$b(t) = (\ell^2 + o_{t \rightarrow T}(1)) \cdot (T - t), \quad (5.13)$$

$$\eta(t) \lesssim (T - t)^2. \quad (5.14)$$

In particular, the blow-up rate is pseudoconformal.

Proof. *Step 1. Preliminary controls.* Let $\hat{\beta} := (\hat{b}^2 + \hat{\eta}^2)^{1/2}$. We claim the following bound for all t close to T (cf. (1.13)):

$$\beta + \hat{\beta} + \lambda \lesssim T - t. \quad (5.15)$$

To see this, we write (4.43) as $\lambda_t \lesssim \frac{\mu}{\lambda} \lesssim 1$ and integrate it backwards from the blow-up time to get $\lambda \lesssim T - t$. The bounds for β and $\hat{\beta}$ then follow from (5.1) and (5.3).

Step 2. Nondegeneracy of $\frac{\hat{\beta}}{\mu}$. We claim that there exists $\ell > 0$ such that

$$\frac{\hat{\beta}}{\lambda} \rightarrow \ell > 0 \quad \text{as } t \rightarrow T. \quad (5.16)$$

To see this, we begin with the identity

$$\left(\frac{\hat{\beta}^2}{\lambda^2}\right)_s = -2\left(\frac{\hat{\beta}^2}{\lambda^2}\right)\left(\frac{\lambda_s}{\lambda} + \hat{b}\right) + \frac{2}{\lambda^2}\{\hat{b}(\hat{b}_s + \hat{b}^2 + \hat{\eta}^2) + \hat{\eta}\hat{\eta}_s\}.$$

Dividing both sides by $\frac{\hat{\beta}}{\lambda}$ and changing to the original time variable t , we have

$$\left(\frac{\hat{\beta}}{\lambda}\right)_t \lesssim \frac{\hat{\beta}}{\lambda} \cdot \frac{1}{\lambda^2} \left| \frac{\lambda_s}{\lambda} + \hat{b} \right| + \frac{1}{\lambda^3} (|\hat{b}_s + \hat{b}^2 + \hat{\eta}^2| + |\hat{\eta}_s|).$$

Substituting estimates (5.4) and (5.5) into the above gives

$$\left(\frac{\hat{\beta}}{\lambda}\right)_t \lesssim \frac{\hat{\beta}}{\lambda} \left(\frac{\mu^2}{\lambda^2} + \frac{\hat{\beta}^2}{\lambda^2} \right) + \frac{\mu^4}{\lambda^3} \lesssim \frac{\hat{\beta}}{\lambda} + \lambda.$$

Combining this with the modulation estimate (5.4), we have

$$\left(\frac{\hat{\beta}}{\lambda} + \lambda\right)_t \lesssim \left(\frac{\hat{\beta}}{\lambda} + \lambda\right) + \frac{1}{\lambda} (|\hat{b}| + \mu^2) \lesssim \frac{\hat{\beta}}{\lambda} + \lambda.$$

This inequality says that $\frac{\hat{\beta}}{\lambda} + \lambda$ cannot go to zero in finite time and there exists $\ell > 0$ such that

$$\frac{\hat{\beta}}{\lambda} + \lambda \rightarrow \ell.$$

Since $\lambda \rightarrow 0$ by the blow-up hypothesis, claim (5.16) follows.

Step 3. Proof of (5.13) and almost decreasing property of λ . In this step, we claim that

$$b(t) = (\ell^2 + o_{t \rightarrow T}(1)) \cdot (T - t), \quad (5.13)$$

$$\lambda(t_2) \lesssim \lambda(t_1) \quad \text{for all } t_1 \leq t_2 < T. \quad (5.17)$$

For the proof of (5.13), we use (5.5) to have

$$\left| \hat{b}_t + \frac{\hat{\beta}^2}{\lambda^2} \right| \lesssim \frac{\hat{\beta}^3 + \hat{\beta}\mu^2 + \mu^4}{\lambda^2} \lesssim \frac{\mu^3}{\lambda^2} \lesssim \lambda.$$

By (5.16), the above formula says

$$\hat{b}_t + \ell^2 = o_{t \rightarrow T}(1).$$

Integrating backwards in time using (5.15) and (5.3) gives

$$b(t) = \{b(t) - \hat{b}(t)\} - \int_t^T \hat{b}_t dt' = (\ell^2 + o_{t \rightarrow T}(1)) \cdot (T - t).$$

For the proof of (5.17), we use $\hat{b} > 0$ and (5.4) to have

$$\lambda_t \leq \lambda_t + \frac{\hat{b}}{\lambda} \leq C\lambda.$$

Applying Grönwall's inequality gives the almost decreasing property (5.17).

Step 4. Smallness of η and proof of (5.14). In this step, we claim that

$$|\eta| \lesssim \lambda(t) \cdot (T - t). \quad (5.18)$$

Note that (5.18) gives (5.14) in view of $\lambda(t) \lesssim T - t$. To show (5.18), we write (5.5) as $|\hat{\eta}_t| \lesssim \lambda$ and integrate it backwards in time using (5.15), (5.3), and (5.17):

$$\begin{aligned} \eta(t) &\leq |\eta(t) - \hat{\eta}(t)| + \int_t^T |\hat{\eta}_t| dt' \lesssim \lambda^2 + \int_t^T \lambda dt' \lesssim \lambda \cdot (T - t) + \lambda \int_t^T dt' \\ &\lesssim \lambda(t) \cdot (T - t). \end{aligned}$$

Step 5. Completion of the proof. It suffices to show (5.11) and (5.12). For the proof of (5.11), we combine (5.16), (5.18), and $b > 0$ to have

$$\frac{b}{\lambda} \rightarrow \ell.$$

Substituting b -asymptotics (5.13) into the above gives (5.11),

$$\lambda(t) = (\ell + o_{t \rightarrow T}(1)) \cdot (T - t).$$

For the proof of (5.12), we substitute (5.11) and (5.18) into (5.4) to have

$$|\gamma_t| \lesssim \frac{1}{\lambda^2}(|\eta| + \mu^2) \lesssim \frac{T - t}{\lambda} + 1 \lesssim 1.$$

Thus γ_t is integrable near the blow-up time and formula (5.12) follows. This completes the proof. ■

Remark 5.4. If one has modulation estimates of the form

$$\begin{aligned} \left| \frac{\lambda_s}{\lambda} + b \right| + |\gamma_s - (m+1)\eta| &\lesssim \|\varepsilon\|_{\text{loc}} + \beta^2, \\ |b_s + b^2 + \eta^2| + |\eta_s| &\lesssim \beta \|\varepsilon\|_{\text{loc}} + \|\varepsilon\|_{\text{loc}}^2 + \beta^3 \end{aligned}$$

with some $\|\varepsilon\|_{\text{loc}}$ satisfying

$$\int_{t^*}^T \frac{\|\varepsilon\|_{\text{loc}}^2}{\lambda^4} dt < +\infty, \quad (5.19)$$

then the above proof still works with simple modifications. Indeed, one has

$$\left(\frac{\beta}{\lambda} + \lambda \right)_t \lesssim \left(1 + \frac{\|\varepsilon\|_{\text{loc}}^2}{\lambda^4} \right) \left(\frac{\beta}{\lambda} + \lambda \right), \quad (5.20)$$

and hence (5.19) still guarantees $\frac{\beta}{\lambda} \rightarrow \ell > 0$ in view of Grönwall's inequality, which is the most important part of the proof. The previous proof is presented in the case where $b, \eta, \|\varepsilon\|_{\text{loc}}$ are replaced by $\hat{b}, \hat{\eta}, \mu^2$ and (5.19) is then obviously satisfied. This says that only the spacetime control of form (5.19) is important and the λ^7 -factor of the spacetime control (4.74) (note that $\mu^{-5}ds \sim \lambda^{-7}dt$) might be too strong. As the exponent 7 is obtained from the computation $7 = 2s + 1$ when $s = 3$, the spacetime control of form (5.19) might be available for any $H_m^{3/2}$ finite-time blow-up solutions. Thus Theorem 1.4 might be true for $H_m^{3/2}$ -solutions.

5.2. Asymptotic decomposition and structure of asymptotic profile

Completion of the proof of Theorem 1.4. By Proposition 5.3, it suffices to show the statements regarding the asymptotic profile. We note that, by Theorem 1.1, there exist parameters $\hat{\lambda}(t) \rightarrow 0$ and $\hat{\gamma}(t)$ (possibly not equal to $\lambda(t)$ and $\gamma(t)$) and the asymptotic profile z^* such that $u(t) - Q_{\hat{\lambda}(t), \hat{\gamma}(t)} \rightarrow z^*$ in L^2 . Therefore, it suffices to show the following:

- (1) $u(t) - Q_{\lambda(t), \gamma(t)} \rightarrow z^*$ in L^2 .
- (2) z^* satisfies the properties as in Theorem 1.4.

Step 1. Proof of (1). In this step, we show the first item (1), namely,

$$u(t) - Q_{\lambda(t), \gamma(t)} \rightarrow z^* \quad \text{in } L^2. \quad (5.21)$$

For this purpose, we claim the following.

Claim 1. If $u(t_n) - Q_{\lambda(t_n), \gamma(t_n)} \rightharpoonup \tilde{z}^*$ in H_m^1 for some $t_n \rightarrow T$ and $\tilde{z}^* \in H_m^1$, then $\tilde{z}^* = z^*$.

Assuming this claim, we finish the proof of (5.21). Note that $u(t) - Q_{\lambda(t), \gamma(t)}$ is H_m^1 -bounded:

$$\|u - Q_{\lambda, \gamma}\|_{H_m^1} \lesssim \|(P - Q) + \varepsilon\|_{L^2} + \lambda^{-1} \|(P - Q) + \varepsilon\|_{\dot{H}_m^1} \lesssim 1 + \frac{\mu}{\lambda} \lesssim 1.$$

Combining this fact with the further subsequence argument and Claim 1, we have $u(t) - Q_{\lambda(t), \gamma(t)} \rightharpoonup z^*$ in H_m^1 . By the Rellich–Kondrachov theorem, we have $u(t) - Q_{\lambda(t), \gamma(t)} \rightarrow z^*$ in L_{loc}^2 . On the other hand, since $\lambda(t), \hat{\lambda}(t) \rightarrow 0$ and $u(t) - Q_{\hat{\lambda}(t), \hat{\gamma}(t)} \rightarrow z^*$, we have $\mathbf{1}_{r \geq R} \{u(t) - Q_{\lambda(t), \gamma(t)}\} \rightarrow \mathbf{1}_{r \geq R} z^*$ for any $R > 0$. Therefore, we have (5.21).

It remains to show Claim 1. This again follows from the Rellich–Kondrachov theorem and $\lambda(t), \hat{\lambda}(t) \rightarrow 0$; for any $R > 1$, we have

$$\begin{aligned} \mathbf{1}_{[R^{-1}, R]} \tilde{z}^* &= \lim_{n \rightarrow \infty} \mathbf{1}_{[R^{-1}, R]} \{u(t_n) - Q_{\lambda(t_n), \gamma(t_n)}\} \\ &= \lim_{n \rightarrow \infty} \mathbf{1}_{[R^{-1}, R]} \{u(t_n) - Q_{\hat{\lambda}(t_n), \hat{\gamma}(t_n)}\} = \mathbf{1}_{[R^{-1}, R]} z^*, \end{aligned}$$

where the limits are taken in the L^2 -topology.

Step 2. Proof of (2). In this step, we show the second item of Theorem 1.4. Recall from (5.21) that the asymptotic profile z^* is the L^2 -limit of $u - Q^\# = [P - Q]^\# + \varepsilon^\#$. The limit of $[P - Q]^\#$ gives the singular part z_{sg}^* of z^* and the limit of $\varepsilon^\#$ gives the regular part z_{rg}^* . Thus it suffices to show the following two claims:

$$[P - Q]^\#(t, r) \rightarrow \begin{cases} -e^{i\gamma^*} \ell^2 \frac{\sqrt{2}}{8} \chi(\ell r) \cdot r \text{ in } L^2 & \text{if } m = 1, \\ e^{i\gamma^*} \ell^3 \frac{\sqrt{2}}{64} i \chi(\ell r) \cdot r^2 \text{ in } L^2 & \text{if } m = 2, \\ 0 \text{ in } L^2 & \text{if } m \geq 3, \end{cases} \quad (5.22)$$

$$\|\mathbf{1}_{r \geq \lambda(t)} |\varepsilon^\#(t)|^{-3}\|_{L^2} \lesssim 1. \quad (5.23)$$

Proof of (5.22). When $m = 1$, we show that only the $T_{2,0}^{(0)}$ -part¹⁶ of $P - Q$ contributes to the singular part of z^* . Indeed, using the pointwise bounds in Lemma 3.1 and the asymptotics of the modulation parameters in Proposition 5.3, we have

$$\begin{aligned} \|\chi_{B_1} T_1^{(0)}\|_{L^2} &\lesssim \|\mathbf{1}_{(0, 2B_1]} \beta \langle y \rangle^{-1}\|_{L^2} \lesssim \beta |\log \beta|^{1/2} \rightarrow 0, \\ \|\chi_{B_1} T_3^{(0)}\|_{L^2} &= 0, \\ \|\chi_{B_1} (T_{1,1}^{(0)} + T_{0,2}^{(0)})\|_{L^2} &\lesssim \|\mathbf{1}_{(0, 2B_1]} |\eta| \beta y\|_{L^2} \lesssim |\eta| \beta^{-1} \lesssim (T - t) \rightarrow 0. \end{aligned}$$

For the $T_{2,0}^{(0)}$ -part of P , using (3.20) and (1.9), we have

$$\left\| \chi_{B_1} \left(T_{2,0}^{(0)} + b^2 \frac{\sqrt{2}}{8} y \right) \right\|_{L^2} \lesssim \|\mathbf{1}_{(0, 2B_1]} b^2 \langle y \rangle^{-1} \log y\|_{L^2} \lesssim b^2 |\log \beta|^{3/2} \rightarrow 0$$

and using Proposition 5.3, we have

$$\begin{aligned} &\frac{e^{i\gamma(t)}}{\lambda(t)} \chi_{B_1} \left(\frac{r}{\lambda(t)} \right) \cdot \left(-(b(t))^2 \frac{\sqrt{2}}{8} \frac{r}{\lambda(t)} \right) \\ &= -e^{i\gamma(t)} \left(\frac{b(t)}{\lambda(t)} \right)^2 \frac{\sqrt{2}}{8} \cdot \chi \left(\frac{\beta(t)}{\lambda(t)} r \right) \cdot r \rightarrow -e^{i\gamma^*} \ell^2 \frac{\sqrt{2}}{8} \chi(\ell r) \cdot r \quad \text{in } L^2. \end{aligned}$$

¹⁶Recall notation (3.6).

When $m = 2$, we show that only the $T_{3,0}^{(0)}$ -part of $P - Q$ contributes to the singular part of z^* . Indeed, proceeding as in the previous paragraph but using the pointwise bounds for the $m = 2$ case, we have

$$\begin{aligned}\|\chi_{B_1} T_1^{(0)}\|_{L^2} &\lesssim \|\mathbf{1}_{(0,2B_1]} \beta \langle y \rangle^{-2}\|_{L^2} \lesssim \beta \rightarrow 0, \\ \|\chi_{B_1} T_2^{(0)}\|_{L^2} &\lesssim \|\mathbf{1}_{(0,2B_1]} \beta^2\|_{L^2} \lesssim \beta \rightarrow 0, \\ \|\chi_{B_1} (T_{2,1}^{(0)} + T_{1,2}^{(0)} + T_{0,3}^{(0)})\|_{L^2} &\lesssim \|\mathbf{1}_{(0,2B_1]} |\eta| \beta^2 y^2\|_{L^2} \lesssim |\eta| \beta^{-1} \lesssim (T - t) \rightarrow 0.\end{aligned}$$

For the $T_{3,0}^{(0)}$ -part of P , using (3.16) instead, we have

$$\left\| \chi_{B_1} \left(T_{3,0}^{(0)} - i b^3 \frac{\sqrt{2}}{64} y^2 \right) \right\|_{L^2} \lesssim \|\mathbf{1}_{(0,2B_1]} b^3 \langle y \rangle^{-4}\|_{L^2} \lesssim b^3 \rightarrow 0$$

and

$$\begin{aligned}& \frac{e^{i\gamma(t)}}{\lambda(t)} \chi_{B_1} \left(\frac{r}{\lambda(t)} \right) \cdot \left(i(b(t))^3 \frac{\sqrt{2}}{64} \left(\frac{r}{\lambda(t)} \right)^2 \right) \\ &= e^{i\gamma(t)} \left(\frac{b(t)}{\lambda(t)} \right)^3 \frac{\sqrt{2}}{64} i \cdot \chi \left(\frac{\beta(t)}{\lambda(t)} r \right) \cdot r^2 \rightarrow e^{i\gamma^*} \ell^3 \frac{\sqrt{2}}{64} i \chi(\ell r) \cdot r^2 \quad \text{in } L^2.\end{aligned}$$

When $m \geq 3$, there is no singular part of z^* :

$$\|P - Q\|_{L^2} \lesssim \|\mathbf{1}_{(0,2B_1]} (\beta \langle y \rangle^{-3} + \beta^2 \langle y \rangle^{-1})\|_{L^2} \lesssim \beta \rightarrow 0.$$

This completes the proof of (5.22).

Proof of (5.23). This easily follows from scaling

$$\|\mathbf{1}_{r \geq \lambda(t)} |\varepsilon^\sharp(t)|_{-3}\|_{L^2} = \lambda^{-3} \|\mathbf{1}_{y \geq 1} |\varepsilon(t)|_{-3}\|_{L^2} \lesssim \lambda^{-3} \|\varepsilon\|_{\dot{H}_m^3} \lesssim 1.$$

This completes the proof of Theorem 1.4. ■

Remark 5.5 (On the proof of Corollary 1.5). Corollary 1.5 is a standard consequence of Theorem 1.4 and pseudoconformal symmetry. Indeed, if an $H_m^{3,3}$ global solution u does not scatter, then $v := \mathcal{C}u$ becomes an H_m^3 finite-time blow-up solution. Thus v satisfies the properties as in Theorem 1.4. Applying the pseudoconformal transform in reverse, we see that u scatters to some Q_{ℓ, γ^*} , for some $\ell \in (0, \infty)$ and $\gamma^* \in \mathbb{R}/2\pi\mathbb{Z}$. We omit the details and refer the reader to [28, Section 5.2] or [29, Section 4].

Appendix A. Formal right inverses of linearized operators

In this appendix, we record explicit formulas of formal right inverses for the linear operators L_Q , A_Q , A_Q^* , H_Q , and $\tilde{H}_Q$ used in the profile construction (Section 3). As L_Q , A_Q , and H_Q have nontrivial kernels, their right inverses are not necessarily unique. As in [28], we fix the choice of formal right inverses, denoted by $^{(\text{out})}T^{-1}$ for a linear operator T , by

requiring the *outgoing property*; its Green's function $(^T)G(y, y')$ is determined by the properties

$$T(^T)G(y, y') = \delta_{y'}(y), \quad (^T)G(y, y') = 0 \quad \text{for } 0 < y < y'.$$

Roughly speaking, this means that if $f(y) = 0$ for $y \leq A$, then also $[^{(\text{out})}T^{-1}f](y) = 0$ for $y \leq A$.

We begin with the explicit formula for the formal right inverse of L_Q .

Lemma A.1 (Formal right inverse of L_Q). *Let $m \geq 0$. Let*

$$J_1(y) := \frac{1 - y^{2m+2}}{1 + y^{2m+2}}, \quad J_2(y) := \frac{1 - y^{2m+2}}{1 + y^{2m+2}} \log y + \frac{2}{m+1} \frac{1}{1 + y^{2m+2}}. \quad (\text{A.1})$$

- (Other expressions for J_k) We have

$$J_1 = \frac{m+1 + A_\theta[Q]}{m+1} = \frac{\Lambda Q}{(m+1)Q}, \quad J_2 = J_1 \log y + \frac{2m+2 + A_\theta[Q]}{(m+1)^2}. \quad (\text{A.2})$$

- (Asymptotics of J_1 and J_2 and their derivatives) For $y \geq 2$ and $k \in \mathbb{N}$,

$$|J_1 + 1|_k + (\log y)^{-1} |J_2 + \log y|_k \lesssim_k y^{-(2m+2)}. \quad (\text{A.3})$$

- (Formal right inverse of L_Q) The operator $^{(\text{out})}L_Q^{-1}$ defined by the formula

$$\begin{aligned} [^{(\text{out})}L_Q^{-1}f](y) := & Q(y) \left[J_1(y) \int_0^y [\partial_y J_2](y') \frac{\text{Re } f(y')}{Q(y')} y' dy' \right. \\ & - J_2(y) \int_0^y [\partial_y J_1](y') \frac{\text{Re } f(y')}{Q(y')} y' dy' \\ & \left. + \int_0^y \frac{i \text{Im } f(y')}{Q(y')} dy' \right] \end{aligned} \quad (\text{A.4})$$

is a formal right inverse of L_Q , i.e., for any function $f: (0, \infty) \rightarrow \mathbb{C}$ such that the integrals of (A.4) are finite, we have

$$L_Q(^{(\text{out})}L_Q^{-1}f) = f. \quad (\text{A.5})$$

- (Smooth extension) If f has a smooth (resp. analytic) $(m+1)$ -equivariant extension on $\mathbb{R}^2$, then $^{(\text{out})}L_Q^{-1}f$ has a smooth (resp. analytic) m -equivariant extension on $\mathbb{R}^2$.

Remark A.2. Formula (A.4) for the real part can be derived from the observation that

$$\left(\mathbf{D}_Q + \frac{1}{y}\right)L_Q = \left(\mathbf{D}_Q + \frac{1}{y}\right)\mathbf{D}_Q + Q^2$$

is a *local* second-order differential operator, the fact that ΛQ belongs to the kernel, and Wronskian relations of this operator. Note that this shares a similar spirit with the observation in Jackiw–Pi [16] that the Bogomol'nyi equation (1.8) can be linked to the Liouville equation, which is local.

Proof of Lemma A.1. We omit the obvious proof of (A.2)–(A.3).

We turn to the proof of (A.5). For the imaginary part of f , we have $L_Q = \mathbf{D}_Q$ and hence (A.5) is immediate from the relation $\mathbf{D}_Q Q = 0$. Henceforth, we assume f is real. Note that J_1 and J_2 solve

$$\partial_{yy} J_k + \frac{1}{y} \partial_y J_k + Q^2 J_k = 0 \quad \forall k \in \{1, 2\}, \quad (\text{A.6})$$

which easily follows from expressions (A.2) and the computation

$$\left(\partial_{yy} + \frac{1}{y} \partial_y\right) A_\theta[Q] = \frac{1}{y} \partial_y \left(-\frac{1}{2} y^2 Q^2\right) = -Q \Lambda Q.$$

We also note the Wronskian relation

$$J_1(\partial_y J_2) - (\partial_y J_1) J_2 = \frac{1}{y}. \quad (\text{A.7})$$

Letting $\tilde{f}(y) := \frac{f(y)}{Q(y)}$ and using $\mathbf{D}_Q Q = 0$, we have

$$\begin{aligned} L_Q^{(\text{out})} L_Q^{-1} f &= Q \left\{ \partial_y \left[J_1 \int_0^y (\partial_y J_2) \tilde{f} y' dy' - J_2 \int_0^y (\partial_y J_1) \tilde{f} y' dy' \right] \right. \\ &\quad \left. + \frac{1}{y} \int_0^y Q^2 \left[J_1 \int_0^{y'} (\partial_y J_2) \tilde{f} y'' dy'' - J_2 \int_0^{y'} (\partial_y J_1) \tilde{f} y'' dy'' \right] y' dy' \right\}. \end{aligned}$$

On the one hand, we use the Wronskian relation (A.7) to obtain

$$\begin{aligned} &\partial_y \left[J_1 \int_0^y (\partial_y J_2) \tilde{f} y' dy' - J_2 \int_0^y (\partial_y J_1) \tilde{f} y' dy' \right] \\ &= \tilde{f}(y') + \left[(\partial_y J_1) \int_0^y (\partial_y J_2) \tilde{f} y' dy' - (\partial_y J_2) \int_0^y (\partial_y J_1) \tilde{f} y' dy' \right]. \end{aligned}$$

On the other hand, using relation (A.6) in the form $y Q^2 J_k = -\partial_y (y \partial_y J_k)$ and integrating by parts, we obtain

$$\begin{aligned} &\frac{1}{y} \int_0^y Q^2 \left[J_1 \int_0^{y'} (\partial_y J_2) \tilde{f} y'' dy'' - J_2 \int_0^{y'} (\partial_y J_1) \tilde{f} y'' dy'' \right] y' dy' \\ &= - \left[(\partial_y J_1) \int_0^y (\partial_y J_2) \tilde{f} y' dy' - (\partial_y J_2) \int_0^y (\partial_y J_1) \tilde{f} y' dy' \right]. \end{aligned}$$

Summing up the previous two formulas completes the proof of (A.5).

We turn to the proof of the smoothness assertion. We recall a basic fact (see [27, Lemmas A.2 and A.3] for the proof of particularly similar statements) that a smooth function $f: (0, \infty) \rightarrow \mathbb{C}$ admits a smooth m -equivariant extension on $\mathbb{R}^2$ if and only if its even or odd extension on $\mathbb{R}$ (i.e., $\tilde{f}(x) = (-1)^m f(|x|)$) is also smooth and has vanishing Taylor coefficients at 0 up to $(|m| - 1)$ -th order. A similar statement holds for analytic extensions. With this fact and expression (A.4), the assertion is obvious for the imaginary part of f .

Henceforth, we show the assertion for the real part of f . We may assume f is real. Note that J_2 can be written in the form

$$J_2 = J_1 \log y + \tilde{J}_2,$$

where $\tilde{J}_2(y) = \frac{2}{m+1}(1 + y^{2m+2})^{-1}$. Since $\tilde{J}_2$ is analytic, its contribution to expression (A.4) is smooth. By a similar reason, the contribution of $\frac{J_1(y)}{y}$ of $\partial_y(J_1 \log y)$ is smooth. Thus it remains to show that (writing $\tilde{f}(y) := \frac{f(y)}{Q(y)}$ as before)

$$Q \left[J_1 \int_0^y (\partial_y J_1) \log y' \cdot \tilde{f} y' dy' - J_1 \log y \int_0^y (\partial_y J_1) \tilde{f} y' dy' \right] \quad (\text{A.8})$$

has a smooth expression, in particular at $y = 0$. This can be seen by the Fubini theorem,

$$-(\text{A.8}) = Q J_1 \int_0^y (\partial_y J_1) \left(\int_{y'}^y \frac{dy''}{y''} \right) \tilde{f} y' dy' = Q J_1 \int_0^y \left\{ \int_0^{y'} (\partial_y J_1) \tilde{f} y'' dy'' \right\} \frac{dy'}{y'}.$$

This completes the proof of the smoothness assertion. $\blacksquare$

It is easy to find formal right inverses of A_Q , A_Q^* , H_Q , and $\tilde{H}_Q$ using $A_Q(yQ) = 0$, $A_Q^*(\frac{1}{y^2Q}) = 0$, $H_Q = A_Q^* A_Q$, and $\tilde{H}_Q = A_Q A_Q^*$. In the following lemmas, we record explicit formulas of right inverses of these operators. We omit the proofs, which are clear from their explicit formulas. See also [28, Section 3.2].

Lemma A.3 (Formal inverse of A_Q). *Let $m \geq 0$.*

- The operator ${}^{(\text{out})}A_Q^{-1}$ defined by

$$[{}^{(\text{out})}A_Q^{-1}f](y) := yQ(y) \int_0^y \frac{f(y')}{y'Q(y')} dy' \quad (\text{A.9})$$

is a formal right inverse of A_Q .

- If the function f has a smooth (resp. analytic) $(m+2)$ -equivariant extension on $\mathbb{R}^2$, then ${}^{(\text{out})}A_Q^{-1}f$ has a smooth (resp. analytic) $(m+1)$ -equivariant extension.

Lemma A.4 (Formal inverse of A_Q^*). *Let $m \geq 0$.*

- The operator ${}^{(\text{out})}(A_Q^*)^{-1}$ defined by

$$[{}^{(\text{out})}(A_Q^*)^{-1}f](y) := -\frac{1}{y^2Q(y)} \int_0^y y'Q(y')f(y')y'dy' \quad (\text{A.10})$$

is a formal right inverse of A_Q .

- If in addition $f \perp yQ$ with respect to the $L^2(ydy)$ -inner product, then

$$[{}^{(\text{out})}(A_Q^*)^{-1}f](y) = \frac{1}{y^2Q(y)} \int_y^\infty y'Q(y')f(y')y'dy'. \quad (\text{A.11})$$

- If f has a smooth (resp. analytic) $(m+1)$ -equivariant extension on $\mathbb{R}^2$, then the function ${}^{(\text{out})}(A_Q^*)^{-1}f$ has a smooth (resp. analytic) $(m+2)$ -equivariant extension.

Lemma A.5 (Formal right inverse of H_Q). *Let $m \geq 0$. Let*

$$h_1(y) := yQ(y), \quad h_2(y) := yQ(y) \int_1^y \frac{1}{(y')^3 Q^2(y')} dy'.$$

- *The operator $^{(\text{out})}(H_Q)^{-1}$ defined by*

$$[^{(\text{out})}H_Q^{-1}f](y) := h_1(y) \int_0^y h_2(y') f(y') y' dy' - h_2(y) \int_0^y h_1(y') f(y') y' dy'$$

is a formal right inverse of H_Q .

- *If in addition $f \perp yQ$ with respect to the $L^2(ydy)$ -inner product, then*

$$[^{(\text{out})}H_Q^{-1}f](y) = h_1(y) \int_0^y h_2(y') f(y') y' dy' + h_2(y) \int_y^\infty h_1(y') f(y') y' dy'.$$

- *If the function f has a smooth (resp. analytic) $(m+1)$ -equivariant extension on $\mathbb{R}^2$, then $^{(\text{out})}H_Q^{-1}f$ has a smooth (resp. analytic) $(m+1)$ -equivariant extension.*

Lemma A.6 (Formal right inverses of $\tilde{H}_Q$). *Let $m \geq 0$. Let*

$$\begin{aligned} \tilde{h}_1(y) &:= \frac{1}{y^2 Q(y)} \int_0^y (y' Q(y'))^2 y' dy', \\ \tilde{h}_2(y) &:= \begin{cases} \frac{1}{y^2 Q(y)} \cdot \frac{1}{p} \int_y^\infty (y' Q(y'))^2 y' dy' & \text{if } m \geq 1, \\ \frac{1}{y^2 Q(y)} & \text{if } m = 0, \end{cases} \end{aligned}$$

where $p = \int_0^\infty (yQ(y))^2 y dy$.

- *The operators $^{(\text{out})}\tilde{H}_Q^{-1}$ and $^{(\text{in})}\tilde{H}_Q^{-1}$ defined by*

$$\begin{aligned} [^{(\text{out})}\tilde{H}_Q^{-1}f](y) &:= \tilde{h}_2(y) \int_0^y \tilde{h}_1(y') f(y') y' dy' - \tilde{h}_1(y) \int_0^y \tilde{h}_2(y') f(y') y' dy', \\ [^{(\text{in})}\tilde{H}_Q^{-1}f](y) &:= \tilde{h}_2(y) \int_0^y \tilde{h}_1(y') f(y') y' dy' + \tilde{h}_1(y) \int_y^\infty \tilde{h}_2(y') f(y') y' dy' \end{aligned}$$

are formal right inverses of $\tilde{H}_Q$. Note that the definition of $^{(\text{in})}\tilde{H}_Q^{-1}f$ requires a suitable integrability condition for f in the region $y \geq 1$.

- *If the function f has a smooth (resp. analytic) $(m+2)$ -equivariant extension on $\mathbb{R}^2$, then $^{(\text{out})}\tilde{H}_Q^{-1}f$ has a smooth (resp. analytic) $(m+2)$ -equivariant extension. The analogous statement holds for $^{(\text{in})}\tilde{H}_Q^{-1}f$ when it is well defined.*

Appendix B. Linear coercivity estimates

In this appendix, we give a sketch of the proof of Lemma 2.4. Estimates (2.32), (2.34), and (2.36) are already proved in [26, Appendix A]. The reader may fill the omitted details by following the proofs therein.

We collect some notations. Recall the Cauchy–Riemann operator $\partial_+ = \partial_y - \frac{m}{y}$ acting on m -equivariant functions (2.1). We denote $\dot{\mathcal{H}}_m^1 = \dot{H}_m^1$. We use the $\dot{\mathcal{H}}_{m,\leq}^k$ and $\dot{\mathcal{H}}_{m,\geq}^k$ -norms to denote the $\dot{\mathcal{H}}_m^k$ -norms restricted on $(0, \frac{1}{2}]$ and $[\frac{1}{2}, \infty)$, respectively. More precisely, one adds $\mathbf{1}_{(0,1/2]}$ or $\mathbf{1}_{[1/2,\infty)}$ to the norms appearing in the right-hand side of the definitions of $\dot{\mathcal{H}}_m^k$. Note that $\|f\|_{\dot{\mathcal{H}}_{m,\geq}^k} \sim \|\mathbf{1}_{[1/2,\infty)}|f|_{-k}\|_{L^2}$.

For the proof of Lemma 2.4, we first note that (2.30) is immediate from $\tilde{V}_Q \geq 1$. The remaining coercivity estimates for A_Q and L_Q follow from combining the corresponding subcoercivity estimates (Lemmas B.1 and B.2) below with a soft functional analytic argument; see, for instance, [26, Lemma A.6].

Lemma B.1 (Subcoercivity estimates for L_Q). *For any $k \in \{0, 1, 2\}$ and $f \in C_{c,m}^\infty$, we have*

$$\|L_Q f\|_{\dot{H}_{m+1}^k} + \|\mathbf{1}_{y \sim 1} f\|_{L^2} \sim \|f\|_{\dot{\mathcal{H}}_m^{k+1}}. \quad (\text{B.1})$$

Moreover, the kernel of $L_Q: H_{m,\text{loc}}^1 \rightarrow L_{\text{loc}}^2$ is $\text{span}_{\mathbb{R}}\{\Lambda Q, iQ\}$.

Proof. As mentioned above, here we only sketch the proof. The details for the $k \in \{0, 2\}$ cases can be found in [26, Appendix A].

Step 1. Estimates in the region $y \lesssim 1$. In the region $y \lesssim 1$, L_Q is well approximated by $\partial_y - \frac{m}{y} = \partial_+$, and we decompose

$$L_Q f = \partial_+ f - \frac{1}{y} A_\theta[Q]f - \frac{2}{y} A_\theta[Q, f]Q =: \partial_+ f + \tilde{L}_{Q,\leq} f. \quad (\text{B.2})$$

The following estimates hold for $k \in \{0, 1, 2\}$:

$$\|f\|_{\dot{\mathcal{H}}_{m,\leq}^{k+1}} \lesssim \|\mathbf{1}_{(0,1]}|\partial_+ f|_{-k}\|_{L^2} + \|\mathbf{1}_{[1/2,1]}f\|_{L^2}, \quad (\text{B.3})$$

$$\|\mathbf{1}_{(0,1]}\tilde{L}_{Q,\leq} f|_{-k}\|_{L^2} \lesssim \|\mathbf{1}_{(0,1]}y|f|_{-k}\|_{L^2}. \quad (\text{B.4})$$

The first estimate is a consequence of (logarithmic) Hardy's inequality and the operator identities $\partial_{yy} = (\partial_y + \frac{1}{y})^2 \partial_+$ for $m = 2$ and $\partial_{yy} = (\partial_y + \frac{1}{y}) \partial_+$ for $m = 1$. The second estimate can be proved using $|Q|_2 \lesssim y$ in the region $y \lesssim 1$.

Step 2. Estimates in the region $y \gtrsim 1$. In the region $y \gtrsim 1$, L_Q is well approximated by $\partial_y + \frac{m+2}{y}$, and we decompose

$$\begin{aligned} L_Q f &= \left(\partial_y + \frac{m+2}{y}\right)f - \frac{1}{y}(2m+2 + A_\theta[Q])f - \frac{2}{y}A_\theta[Q, f]Q \\ &=: \left(\partial_y + \frac{m+2}{y}\right)f + \tilde{L}_{Q,\geq} f. \end{aligned} \quad (\text{B.5})$$

The following estimates hold for $k \in \{0, 1, 2\}$:

$$\|f\|_{\dot{\mathcal{H}}_{m,\geq}^{k+1}} \lesssim \left\| \mathbf{1}_{[1/4,\infty)} \left| \left(\partial_y + \frac{m+2}{y}\right)f \right|_{-k} \right\|_{L^2} + \|\mathbf{1}_{[1/4,1/2]}f\|_{L^2}, \quad (\text{B.6})$$

$$\|\mathbf{1}_{[1/4,\infty)}\tilde{L}_{Q,\geq} f|_{-k}\|_{L^2} \lesssim \|\mathbf{1}_{[1/4,\infty)}y^{-3}|f|_{-k}\|_{L^2} + \|\mathbf{1}_{(0,1/4]}y^{1-k}f\|_{L^2}. \quad (\text{B.7})$$

The first estimate is a consequence of Hardy's inequality (using $m + 2 > 0$). The second estimate can be proved using (B.4) and $|Q|_2 \lesssim y^{-3}$ in the region $y \gtrsim 1$. Note that the second term of the right-hand side of (B.7) appears due to the nonlocal term $-\frac{2}{y} A_\theta[Q, f]Q$.

Step 3. Further treatment of localized lower order errors. We further estimate the localized lower order terms (i.e., (B.4) and (B.7)) as

$$\|(\mathbf{1}_{(0,1]y} + \mathbf{1}_{[1,\infty)}y^{-3}) \cdot |f|_{-k}\|_{L^2} \leq o_{r_0 \rightarrow 0}(\|f\|_{\dot{H}_m^{k+1}}) + O_{r_0}(\|\mathbf{1}_{[r_0, 1/r_0]}f\|_{L^2}). \quad (\text{B.8})$$

For the proof of (B.8), we first note that the contributions from the regions $y \leq 2r_0$ and $y \geq \frac{1}{2r_0}$ are bounded by $o_{r_0 \rightarrow 0}(\|f\|_{\dot{H}_m^{k+1}})$. The contribution from the region $2r_0 \leq y \leq \frac{1}{2r_0}$ can be estimated using interpolation, for example,

$$\begin{aligned} \|\mathbf{1}_{[2r_0, 1/(2r_0)]}|f|_{-2}\|_{L^2} &\lesssim_{r_0} \|\varphi f\|_{H_m^2} \lesssim_{r_0} \|\varphi f\|_{L^2}^{1/3} \|\varphi f\|_{H_m^3}^{2/3} \\ &\lesssim_{r_0} \|\mathbf{1}_{[r_0, 1/r_0]}f\|_{L^2}^{1/3} \|f\|_{\dot{H}_m^{k+1}}^{2/3}, \end{aligned}$$

where $\varphi = \varphi(y)$ is some smooth compactly supported radial function such that $\varphi \equiv 1$ on $[2r_0, \frac{1}{2r_0}]$ and $\varphi \equiv 0$ outside $[r_0, \frac{1}{r_0}]$. This gives (B.8).

Step 4. Proof of the subcoercivity estimate (B.1). The proof of the $(\lesssim)$ -direction follows from first applying decompositions (B.2) and (B.5), and then applying estimates (B.4) and (B.7),

$$\begin{aligned} \|L_Q f\|_{\dot{H}_{m+1}^k} + \|\mathbf{1}_{y \sim 1}f\|_{L^2} &\lesssim \|\mathbf{1}_{(0,1/2]}|\partial_+ f|_{-k}\|_{L^2} + \|\mathbf{1}_{(0,1/2]}y|f|_{-k}\|_{L^2} \\ &\quad + \|\mathbf{1}_{[1/2,\infty)}|f|_{-(k+1)}\|_{L^2} \lesssim \|f\|_{\dot{H}_m^{k+1}}. \end{aligned}$$

The proof of the $(\gtrsim)$ -direction follows from applying (B.3) and (B.6) first, then applying (B.2) and (B.5), (B.4) and (B.7), and finally (B.8),

$$\begin{aligned} \|f\|_{\dot{H}_m^{k+1}} &\lesssim \|\mathbf{1}_{(0,1]}|\partial_+ f|_{-k}\|_{L^2} + \left\| \mathbf{1}_{[1/4,\infty)} \left(\partial_y + \frac{m+2}{y} \right) f \right\|_{-k} \|_{L^2} + \|\mathbf{1}_{[1/4,1]}f\|_{L^2} \\ &\lesssim \|L_Q f|_{-k}\|_{L^2} + \|\mathbf{1}_{(0,1]}y|f|_{-k}\|_{L^2} + \|\mathbf{1}_{[1/4,\infty)}y^{-3}|f|_{-k}\|_{L^2} \\ &\lesssim \|L_Q f\|_{\dot{H}_{m+1}^k} + o_{r_0 \rightarrow 0}(\|f\|_{\dot{H}_m^{k+1}}) + O_{r_0}(\|\mathbf{1}_{[r_0, 1/r_0]}f\|_{L^2}). \end{aligned}$$

Choosing $r_0 > 0$ sufficiently small, the second term of the right-hand side can be absorbed into the left-hand side. This gives the $(\gtrsim)$ -inequality of (B.1).

Step 5. Kernel characterization. If $f \in H_{m,\text{loc}}^1$ belongs to the kernel of L_Q , then f is smooth by elliptic regularity; see for example [26, Lemma A.5]. Then, the result follows by the characterization of the smooth kernel of L_Q proved in [27, Lemma 3.3]. ■

Lemma B.2 (Subcoercivity estimates for A_Q). *For any $k \in \{0, 1\}$ and $f \in C_{c,m+1}^\infty$, we have*

$$\|A_Q f\|_{\dot{H}_{m+1}^k} + \|\mathbf{1}_{y \sim 1}f\|_{L^2} \sim \|f\|_{\dot{H}_{m+1}^{k+1}}, \quad \|A_Q f\|_{\dot{V}_{m+1}^{3/2}} + \|\mathbf{1}_{y \sim 1}f\|_{L^2} \sim \|f\|_{\dot{V}_{m+1}^{5/2}}.$$

Moreover, the kernel of $A_Q: H_{m+1,\text{loc}}^1 \rightarrow L_{\text{loc}}^2$ is $\text{span}_{\mathbb{C}}\{yQ\}$.

Proof. We proceed similarly as in the proof of Lemma B.1 and omit the details. In the region $y \lesssim 1$, we approximate A_Q by $\partial_y - \frac{m+1}{y} = \partial_+$ and decompose

$$A_Q f = \partial_+ f - \frac{1}{y} A_\theta[Q] f =: \partial_+ f + \tilde{A}_{Q,\leq} f.$$

We have the following estimates for $k \in \{0, 1\}$:

$$\begin{aligned} \|\mathbf{1}_{(0,1/2]} |f|_{-(k+1)}\|_{L^2} &\lesssim \|\mathbf{1}_{(0,1]} |\partial_+ f|_{-k}\|_{L^2} + \|\mathbf{1}_{[1/2,1]} f\|_{L^2}, \\ \|\mathbf{1}_{(0,1]} \tilde{A}_{Q,\leq} f|_{-k}\|_{L^2} &\lesssim \|\mathbf{1}_{(0,1]} y |f|_{-k}\|_{L^2}. \end{aligned}$$

On the other hand, in the region $y \gtrsim 1$, we approximate A_Q by $\partial_y + \frac{m+1}{y}$ and decompose

$$A_Q f = \partial_+ f - \frac{1}{y} (2m+2 + A_\theta[Q]) f =: \left(\partial_y + \frac{m+1}{y}\right) f + \tilde{A}_{Q,\geq} f.$$

We have the following estimates for $k \in \{0, 1\}$:

$$\begin{aligned} \|\mathbf{1}_{[1/2,\infty)} |f|_{-(k+1)}\|_{L^2} &\lesssim \left\| \mathbf{1}_{[1/4,\infty)} \left(\partial_y + \frac{m+1}{y}\right) f \right\|_{-k} \|_{L^2} + \|\mathbf{1}_{[1/4,1/2]} f\|_{L^2}, \\ \left\| \mathbf{1}_{[1/2,\infty)} \frac{1}{y^{3/2}} |f|_{-1} \right\|_{L^2} &\lesssim \left\| \mathbf{1}_{[1/4,\infty)} \frac{1}{y^{3/2}} \left(\partial_y + \frac{m+1}{y}\right) f \right\|_{L^2} + \|\mathbf{1}_{[1/4,1/2]} f\|_{L^2}, \\ \left\| \mathbf{1}_{[1/2,\infty)} \frac{1}{y} \partial_{yy} f \right\|_{L^2} &\lesssim \left\| \mathbf{1}_{[1/2,\infty)} \frac{1}{y} \partial_y \left(\partial_y + \frac{m+1}{y}\right) f \right\|_{L^2} + \left\| \mathbf{1}_{[1/2,\infty)} \frac{1}{y^2} |f|_{-1} \right\|_{L^2}, \\ \|\mathbf{1}_{[1/4,\infty)} \tilde{A}_{Q,\geq} f|_{-k}\|_{L^2} &\lesssim \|\mathbf{1}_{[1/4,\infty)} y^{-3} |f|_{-k}\|_{L^2}. \end{aligned}$$

Here, the first and second estimates are consequences of Hardy's inequality (using the fact that $m+1 > 0$). The second and third estimates are adapted to the $\dot{V}_{m+1}^{5/2}$ - and $\dot{V}_{m+2}^{3/2}$ -norms. Finally, we have for $k \in \{0, 1\}$

$$\|(\mathbf{1}_{(0,1]} y + \mathbf{1}_{[1,\infty)} y^{-3}) \cdot |f|_{-k}\|_{L^2} \leq o_{r_0 \rightarrow 0}(\|f\|_{\dot{H}_{m+1}^{k+1}}) + O_{r_0}(\|\mathbf{1}_{[r_0,1/r_0]} f\|_{L^2}),$$

and in fact we have a better estimate

$$\|(\mathbf{1}_{(0,1]} y + \mathbf{1}_{[1,\infty)} y^{-3}) \cdot |f|_{-1}\|_{L^2} \leq o_{r_0 \rightarrow 0}(\|f\|_{\dot{V}_{m+1}^{5/2}}) + O_{r_0}(\|\mathbf{1}_{[r_0,1/r_0]} f\|_{L^2}).$$

Proceeding as in Step 4 of the proof of Lemma B.1, we obtain the subcoercivity estimates. The kernel characterization of A_Q is in fact easier than that of L_Q because $A_Q(yQ) = 0$ (A_Q is $\mathbb{C}$ -linear) and we can use the uniqueness of ODE solutions. ■

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