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From analytic monads to ∞-operads through Lawvere theories

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Abstract. We show that a variant of Lurie's ∞-operads, where the category of finite pointed sets is replaced by that of spans of finite sets, can be identified as precisely the Lawvere theories of the analytic monads previously studied by Gepner, Kock, and the author. This gives an equivalence between ∞-operads (or more precisely a "flagged" or "pinned" version thereof) and analytic monads, with an ∞-operad $\mathcal{O}$ corresponding to the monad for $\mathcal{O}$-algebras in spaces. In particular, the ∞-operad $\mathcal{O}$ is completely determined by this monad. To prove this we extend the equivalence between ∞-categorical Lawvere theories and algebraic monads, due to Gepner, Groth, and Nikolaus, to the many-object (or many-sorted) case.

Contents

1. Introduction

The theory of ∞-*operads* gives a powerful framework for working with homotopy-coherent algebraic structures. Our goal in this paper is to give a direct comparison between Lurie's approach to ∞-operads [26] and the *analytic monads* introduced in [13], by identifying the Lawvere theories of analytic monads as ∞-operads in a slight variant of Lurie's definition. In particular, we will show that an ∞-operad $\mathcal{O}$ is completely determined by the monad for free $\mathcal{O}$-algebras in spaces.

Before we explain these results in more detail, it is helpful to first recall the relation between operads in Set and monads; for simplicity, we focus on the case of one-object

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operads[1] in Set. One common description of such operads is that they are associative monoids in symmetric sequences with respect to the composition product. Here a *symmetric sequence* in Set consists of a sequence of sets $\mathcal{O}(n)$, $n = 0, 1, \ldots,$ where $\mathcal{O}(n)$ has an action of the symmetric group Σ_n; this data can also be encoded as a functor $\mathbb{F}^{\simeq} \to \mathsf{Set}$, where $\mathbb{F}^{\simeq}$ is the groupoid of finite sets and bijections. The *composition product* of two symmetric sequences $\mathcal{O}$ and $\mathcal{P}$ is then given by the formula

$$(\mathcal{O} \circ \mathcal{P})(n) \cong \coprod_{k=0}^{\infty} \mathcal{O}(k) \times_{\Sigma_k} \Big(\coprod_{i_1+\cdots+i_k=n} \big(\mathcal{P}(i_1) \times \cdots \times \mathcal{P}(i_k)\big) \times_{\Sigma_{i_1} \times \cdots \times \Sigma_{i_k}} \Sigma_n \Big),$$

analogous to the formula for composition of power series.

Joyal [21] extended the analogy with power series by proving that under left Kan extension along the inclusion $\mathbb{F}^{\simeq} \to \mathsf{Set}$, symmetric sequences in Set (or *species* in Joyal's terminology) are identified with a certain class of *analytic* endofunctors of Set. Not all natural transformations among such functors come from morphisms of symmetric sequences, but we get an equivalence of categories if we consider a certain class of "weakly cartesian" transformations. Moreover, under this equivalence the composition product is identified with composition of endofunctors, which means that one-object operads can be identified with associative monoids in analytic functors under composition, i.e., with the class of *analytic* monads on Set.[2] Moreover, the analytic monad that corresponds to an operad $\mathcal{O}$ can be identified with the monad for free $\mathcal{O}$-algebras in Set.

Analytic monads in the ∞-categorical setting were introduced by Gepner, Kock, and the author in [13]. Here the appropriate[3] notion of analytic functor admits a considerably simpler characterization than in the classical 1-categorical context: if $\mathcal{S}$ is the ∞-category of spaces or ∞-groupoids, then a functor $\mathcal{S}_{/I} \to \mathcal{S}_{/J}$ is *analytic* precisely when it preserves sifted colimits and weakly contractible limits. Moreover, the appropriate morphisms between these are precisely the *cartesian* natural transformations, i.e., those whose naturality squares are all cartesian. An analytic monad is then a monad on a slice $\mathcal{S}_{/I}$ whose underlying endofunctor is analytic, and whose multiplication and unit transformations are cartesian.

In [13], we defined an ∞-category of analytic monads on slices $\mathcal{S}_{/I}$ where the base space I can vary, and related these analytic monads to ∞-operads by proving that this ∞-category is equivalent to that of *dendroidal Segal spaces* due to Cisinski and Moerdijk [9]. These are known to be equivalent to other models for ∞-operads by the comparison results of Cisinski–Moerdijk [9, 10], Heuts–Hinich–Moerdijk [19], Barwick [4], Chu, Heuts, and the author [8], and most recently Hinich and Moerdijk [20] (who directly compare Lurie's ∞-operads to complete dendroidal Segal spaces).

[1] As opposed to (colored) operads with more general sets of objects/colours.

[2] A similar description applies to operads with an arbitrary set of objects, we have only restricted to the one-object case for simplicity.

[3] For example, appropriate in the sense of corresponding to symmetric sequences and their generalizations, cf. [13, Section 3.2].

In particular, it follows from these comparisons that analytic monads are equivalent to Lurie's model for ∞-operads [26], which is by far the best developed approach. However, the resulting connection between the two is rather indirect, since it passes through at least one additional model for ∞-operads. This is rather unsatisfying, since it seems clear what a direct equivalence ought to do in one direction: If $\mathcal{O}$ is an ∞-operad, then we can explicitly describe the monad for free $\mathcal{O}$-algebras in $\mathcal{S}$, and this is an analytic monad.

The main goal of the present paper is therefore to prove a direct comparison between Lurie's ∞-operads and analytic monads (i.e., without passing through dendroidal Segal spaces and the other equivalences cited above), which assigns to an ∞-operad $\mathcal{O}$ the monad for $\mathcal{O}$-algebras in $\mathcal{S}$. To state this precisely we first need to explain a slight wrinkle, however: we proved in [13] that analytic monads correspond to dendroidal Segal spaces, but ∞-operads should instead correspond to the full subcategory of *complete* dendroidal Segal spaces. To correct for this, we can (following [1]) instead consider what we will called *pinned ∞-operads*: pairs $(\mathcal{O}, p)$ where $\mathcal{O}$ is an ∞-operad and $p\colon X \twoheadrightarrow \mathcal{O}^{\simeq}_{\langle 1\rangle}$ is an essentially surjective morphism from an ∞-groupoid X to the ∞-groupoid of objects of the ∞-operad $\mathcal{O}$. The composite

$$\mathsf{Alg}_{\mathcal{O}}(\mathcal{S}) \to \mathsf{Fun}(\mathcal{O}^{\simeq}_{\langle 1\rangle}, \mathcal{S}) \xrightarrow{p^*} \mathsf{Fun}(X, \mathcal{S}) \tag{1.1}$$

is then also a monadic right adjoint corresponding to an analytic monad. We can now state our main result as follows.

Theorem A. *Let* $\mathsf{POpd}(\mathbb{F}_*)$ *denote the ∞-category of pinned ∞-operads and* AnMnd *that of analytic monads. Then the functor*

$$\mathsf{POpd}(\mathbb{F}_*) \to \mathsf{AnMnd},$$

taking a pinned ∞-operad $(\mathcal{O}, p)$ to the monad (1.1), is an equivalence.

To prove this, we also want an explicit functor in the other direction, extracting a pinned ∞-operad from an analytic monad. To this end, we will study *Lawvere theories* of analytic monads. Let us say that a monad T on $\mathsf{Fun}(X, \mathcal{S})$ for some ∞-groupoid X is *algebraic* if the functor T preserves sifted colimits; then the *Lawvere theory* $(\mathcal{L}(T), p)$ of T is given by taking $\mathcal{L}(T)^{\mathrm{op}}$ to be the full subcategory of $\mathsf{Alg}(T)$ spanned by the free algebras on finite coproducts of representables in $\mathsf{Fun}(X, \mathcal{S})$, together with the functor $p\colon X \to \mathcal{L}(T)$ taking a point of X to the free algebra on the presheaf represented by it. More generally, a Lawvere theory can be defined as an ∞-category $\mathcal{L}$ with finite products, together with a functor $p\colon X \to \mathcal{L}$ such that every object of $\mathcal{L}$ is a finite product of objects in the image of p. Generalizing a result of Gepner–Groth–Nikolaus [12] (for the case $X \simeq *$), we prove an ∞-categorical version of the monad–theory correspondence for multi-sorted Lawvere theories.

Theorem B. *Let* AlgMnd *be the ∞-category of algebraic monads and* LwvTh *that of Lawvere theories. Taking Lawvere theories of algebraic monads gives a functor*

$$\mathfrak{Th}\colon \mathsf{AlgMnd} \to \mathsf{LwvTh}.$$

This functor is an equivalence, with inverse the functor

$$\mathfrak{Mod}\colon \mathsf{LwvTh} \to \mathsf{AlgMnd}$$

that assigns to a Lawvere theory $(\mathcal{L}, p)$ *its* ∞*-category* $\mathrm{Mod}_{\mathcal{L}}(\mathcal{S})$ *of* models *in* $\mathcal{S}$*, meaning functors* $\mathcal{L} \to \mathcal{S}$ *that preserve finite products, together with the functor to* $\mathrm{Fun}(X, \mathcal{S})$ *given by restriction along* p *(which is a monadic right adjoint corresponding to an algebraic monad).*

We can regard analytic monads as forming a (non-full) subcategory of algebraic monads, and therefore Theorem B identifies AnMnd with a (non-full) subcategory of LwvTh. For ordinary categories (and one-sorted Lawvere theories) this subcategory was identified by Szawiel–Zawadowski [29], but this is not quite the relevant identification for us: The ∞-category AnMnd has a terminal object Sym, which is the monad for commutative monoids, and it turns out that AnMnd is a *full* subcategory of algebraic monads over Sym. The Lawvere theory of Sym can be identified with the pair $\mathrm{Span}(\mathbb{F})^{\natural} := (\mathrm{Span}(\mathbb{F}), \{\mathbf{1}\})$ consisting of the $(2,1)$-category $\mathrm{Span}(\mathbb{F})$ of spans of finite sets, together with the inclusion of the one-element set $\mathbf{1}$. From Theorem B we then get an identification between AnMnd and a full subcategory $\mathsf{LwvTh}^{\mathrm{an}}_{/\mathrm{Span}(\mathbb{F})^{\natural}}$ of the ∞-category $\mathsf{LwvTh}_{/\mathrm{Span}(\mathbb{F})^{\natural}}$ of Lawvere theories over $\mathrm{Span}(\mathbb{F})^{\natural}$. This ∞-category turns out to have a very nice alternative description.

The $(2,1)$-category $\mathrm{Span}(\mathbb{F})$ has many of the same features as $\mathbb{F}_*$ (which can be regarded as the subcategory containing only those spans where the backwards map is injective), so that we can modify Lurie's definition of ∞-operads over $\mathbb{F}_*$ to get a notion of $\mathrm{Span}(\mathbb{F})$-$\infty$*-operads* as certain ∞-categories over $\mathrm{Span}(\mathbb{F})$. A key difference is that $\mathrm{Span}(\mathbb{F})$-$\infty$-operads turn out to be ∞-categories with finite products, so that we can regard pinned $\mathrm{Span}(\mathbb{F})$-$\infty$-operads as Lawvere theories. This gives in fact a fully faithful functor from the ∞-category $\mathrm{POpd}(\mathrm{Span}(\mathbb{F}))$ of pinned $\mathrm{Span}(\mathbb{F})$-$\infty$-operads to $\mathsf{LwvTh}_{/\mathrm{Span}(\mathbb{F})^{\natural}}$, and just as for pinned ∞-operads over $\mathbb{F}_*$, the corresponding monads are analytic. We thus have an inclusion

$$\mathrm{POpd}\big(\mathrm{Span}(\mathbb{F})\big) \hookrightarrow \mathsf{LwvTh}^{\mathrm{an}}_{/\mathrm{Span}(\mathbb{F})^{\natural}}. \tag{1.2}$$

Conversely, we prove the following.

Theorem C. *The Lawvere theory of any analytic monad is a pinned* $\mathrm{Span}(\mathbb{F})$*-*$\infty$*-operad.*

The functor (1.2) is thus essentially surjective, and combining this with Theorem B we have equivalences

$$\mathrm{POpd}\big(\mathrm{Span}(\mathbb{F})\big) \simeq \mathsf{LwvTh}^{\mathrm{an}}_{/\mathrm{Span}(\mathbb{F})^{\natural}} \simeq \mathsf{AnMnd}.$$

To deduce Theorem A from this we only need to know that the two version of ∞-operads are equivalent. This has already been shown as part of joint work with Barkan and Steinebrunner [2], using the symmetric monoidal envelopes over $\mathbb{F}_*$ and $\mathrm{Span}(\mathbb{F})$.

Specifically, [2, Corollaries 5.1.13 and 5.3.17] imply that pulling back along the inclusion $\mathbb{F}_* \hookrightarrow \mathsf{Span}(\mathbb{F})$ gives an equivalence between (pinned) ∞-operads over $\mathsf{Span}(\mathbb{F})$ and $\mathbb{F}_*$, and this is compatible with ∞-categories of algebras.

Our last main result is that when we combine Theorem A with the equivalence between analytic monads and dendroidal Segal spaces proven in [13], we get the expected relation between complete dendroidal Segal spaces and ∞-operads.

Theorem D. *Under the composite equivalence*

$$\mathsf{POpd}(\mathbb{F}_*) \simeq \mathsf{AnMnd} \simeq \mathsf{Seg}_{\mathbf{\Omega}^{\mathrm{op}}}(\mathcal{S})$$

the full subcategory $\mathsf{CSeg}_{\mathbf{\Omega}^{\mathrm{op}}}(\mathcal{S})$ *of complete dendroidal Segal spaces corresponds to the* ∞*-category* $\mathsf{Opd}(\mathbb{F}_*)$ *of* ∞*-operads (identified as those pinned* ∞*-operads where the morphism of spaces is an equivalence).*

Overview. In Section 2 we first introduce (pinned) ∞-operads over $\mathbb{F}_*$ and $\mathsf{Span}(\mathbb{F})$, and recall the comparison between them. We also introduce Lawvere theories and their models, and show that pinned $\mathsf{Span}(\mathbb{F})$-$\infty$-operads are examples of Lawvere theories, and their models are precisely algebras (or monoids) in the ∞-category of spaces.

In Section 3, we introduce algebraic monads and their associated Lawvere theories, before proving Theorem B.

We then study analytic monads and their Lawvere theories in Section 4. Here we also show that pinned $\mathsf{Span}(\mathbb{F})$-$\infty$-operads give analytic monads, and prove Theorem C.

Finally, in Section 5 our goal is to prove Theorem D; this turns out to require an understanding of how the equivalences between pinned ∞-operads, analytic monads, and dendroidal Segal spaces restrict to equivalences between full subcategories of pinned ∞-categories, linear monads, and Segal spaces on $\mathbf{\Delta}$.

Notation. Much of the notation used is hopefully fairly standard, but we mention a couple of points here for the reader's convenience:

- If T is a monad on an ∞-category $\mathcal{C}$, we will generally denote the corresponding monadic adjunction by
$$F_T : \mathcal{C} \rightleftarrows \mathsf{Alg}(T) : U_T.$$
- If $\mathcal{C}$ is a locally small ∞-category, we denote the Yoneda embedding for $\mathcal{C}$ by $\mathsf{y}_{\mathcal{C}}\colon \mathcal{C} \to \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$. If the ∞-category $\mathcal{C}$ is clear from context, we will also just write y for $\mathsf{y}_{\mathcal{C}}$.
- We write $\mathsf{P}(\mathcal{C})$ for the ∞-category $\mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ of presheaves on $\mathcal{C}$. We write P^* for the contravariant presheaves functor and $\mathsf{P}_!$ for the covariant version (obtained from P^* by taking left adjoints). We will use without comment that the Yoneda embedding gives a natural transformation $\mathsf{y}\colon \mathrm{id} \to \mathsf{P}_!$, cf. [17, Section 8].
- In ∞-categories with an inert–active factorization system, we will often indicate inert morphisms as $X \rightarrowtail Y$ and active morphisms as $X \rightsquigarrow Y$.

2. ∞-operads and Lawvere theories

Our main concern in this section is ∞-operads over $\mathsf{Span}(\mathbb{F})$ and their relation to Lurie's ∞-operads and to Lawvere theories. In Section 2.1 we introduce (pinned) ∞-operads over both $\mathsf{Span}(\mathbb{F})$ and $\mathbb{F}_*$ and recall the equivalence between them, while in Section 2.2 we define Lawvere theories and show that pinned $\mathsf{Span}(\mathbb{F})$-$\infty$-operads are examples of these. Finally, in the brief Section 2.3 we show that if $\mathcal{O}$ is a $\mathsf{Span}(\mathbb{F})$-$\infty$-operad, then $\mathcal{O}$-algebras or $\mathcal{O}$-monoids in an ∞-category with finite products are the same thing as *models* of $\mathcal{O}$ when it is viewed as a Lawvere theory.

2.1. ∞-operads over $\mathbb{F}_*$ and $\mathsf{Span}(\mathbb{F})$

In this subsection we review the notions of ∞-operads over $\mathbb{F}_*$ and $\mathsf{Span}(\mathbb{F})$, and the comparison between them proved in [2]. We start by recalling Lurie's definition of ∞-operads over $\mathbb{F}_*$ from [26].

Notation 2.1.1. Let $\mathbb{F}$ denote the ordinary category of finite sets and $\mathbb{F}_*$ that of finite pointed sets. We write $S_+ := (S \amalg \{*\}, *)$ for the pointed set obtained by adding a disjoint base point $*$ to a set S; all objects of $\mathbb{F}_*$ are of this form. A morphism $\phi\colon S_+ \to T_+$ is called *inert* if $|\phi^{-1}(t)| = 1$ for $t \neq *$, or in other words if ϕ restricts to an isomorphism $S \setminus \phi^{-1}(*) \xrightarrow{\sim} T$, and *active* if $\phi^{-1}(*) = \{*\}$, i.e., if ϕ restricts to a morphism of sets $S \to T$; we write $\mathbb{F}_*^{\mathrm{int}}$ and $\mathbb{F}_*^{\mathrm{act}}$ for the wide subcategories of $\mathbb{F}_*$ containing the inert and active morphisms, respectively. Then $(\mathbb{F}_*, \mathbb{F}_*^{\mathrm{int}}, \mathbb{F}_*^{\mathrm{act}})$ is a factorization system: for every morphism ϕ of $\mathbb{F}_*$, the groupoid of factorizations of ϕ as an inert morphism followed by an active morphism is contractible. For $s \in S$ we write $\rho_s\colon S_+ \to \mathbf{1}_+$ for the inert map that sends all elements of S except s to the base point.

Definition 2.1.2. An ∞-*operad* is a functor of ∞-categories $p\colon \mathcal{O} \to \mathbb{F}_*$ such that

(1) $\mathcal{O}$ has all p-cocartesian lifts of inert morphisms in $\mathbb{F}_*$.

(2) Given $X, Y \in \mathcal{O}$ over $S_+, T_+ \in \mathbb{F}_*$ and cocartesian lifts $Y \to Y_t$ over ρ_t, the commutative square

$$\begin{tikzcd} \mathsf{Map}_{\mathcal{O}}(X, Y) \arrow[r] \arrow[d] & \prod_{t \in T} \mathsf{Map}_{\mathcal{O}}(X, Y_t) \arrow[d] \\ \mathsf{Map}_{\mathbb{F}_*}(S_+, T_+) \arrow[r] & \prod_{t \in T} \mathsf{Map}_{\mathbb{F}_*}(S_+, \mathbf{1}_+) \end{tikzcd}$$

is cartesian.

(3) For every $S_+ \in \mathbb{F}_*$, the functor

$$\mathcal{O}_{S_+} \to \prod_{s \in S} \mathcal{O}_{\mathbf{1}_+}$$

given by cocartesian transport along the maps ρ_s ($s \in S$) is an equivalence.

The definition of ∞-operad does not use anything special about $\mathbb{F}_*$ except for the inert-active factorization system and the object $\mathbf{1}_+$—as spelled out in [7, Section 9] it can be straightforwardly extended to any ∞-category equipped with a factorization system and a collection of special "elementary" objects. Here we are interested in the variant where $\mathbb{F}_*$ is replaced by the $(2,1)$-category of *spans* of finite sets.

Definition 2.1.3. We write $\mathsf{Span}(\mathbb{F})$ for the ∞-category (in fact a $(2,1)$-category) of spans of finite sets, defined as in [14] or [3]. Informally, this has finite sets as objects with the morphisms given by *spans*, that is diagrams of the form

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ S & & T, \end{tikzcd}$$

with composition given by taking pullbacks. More formally, $\mathsf{Span}(\mathbb{F})$ can be defined as the complete Segal space given by

$$\mathsf{Map}\big([n], \mathsf{Span}(\mathbb{F})\big) \simeq \mathsf{Map}_{\mathrm{cart}}\big(\mathsf{Tw}\big([n]\big), \mathbb{F}\big);$$

here $\mathsf{Tw}([n])$ is the *twisted arrow category* of $[n]$, which can be succinctly described as the partially ordered set of pairs (i, j), $0 \le i \le j \le n$, with

$$(i, j) \le (i', j') \iff i \le i' \le j' \le j,$$

and $\mathsf{Map}_{\mathrm{cart}}(\mathsf{Tw}([n]), \mathbb{F})$ denotes the subspace of $\mathsf{Map}(\mathsf{Tw}([n]), \mathbb{F})$ spanned by functors $F \colon \mathsf{Tw}([n]) \to \mathbb{F}$ such that F takes all squares of the form

$$\begin{tikzcd} (i, j) \arrow[r] \arrow[d] & (i', j) \arrow[d] \\ (i, j') \arrow[r] & (i', j') \end{tikzcd}$$

to cartesian squares in $\mathbb{F}$.

Definition 2.1.4. We say a morphism $S \xleftarrow{f} X \xrightarrow{g} T$ in $\mathsf{Span}(\mathbb{F})$ is *active* if f is an isomorphism, and *inert* if g is an isomorphism. The inert and active morphisms are closed under composition, and if we write $\mathsf{Span}(\mathbb{F})^{\mathrm{int}}$ and $\mathsf{Span}(\mathbb{F})^{\mathrm{act}}$ for the wide subcategories of $\mathsf{Span}(\mathbb{F})$ containing only these morphisms, then it is easy to see (cf. [17, Proposition 2.15]) that we have equivalences

$$\mathsf{Span}(\mathbb{F})^{\mathrm{int}} \simeq \mathbb{F}^{\mathrm{op}}, \quad \mathsf{Span}(\mathbb{F})^{\mathrm{act}} \simeq \mathbb{F}.$$

For $s \in S$, let ρ'_s denote the (inert) span $S \hookleftarrow \{s\} \xrightarrow{\sim} \mathbf{1}$.

Observation 2.1.5. The inert and active morphisms form a factorization system on the ∞-category $\mathsf{Span}(\mathbb{F})$; indeed, the analogous claim holds for any ∞-category of spans by [17, Proposition 4.9].

We can then consider the following variant of Definition 2.1.2.

Definition 2.1.6. A $\mathsf{Span}(\mathbb{F})$-$\infty$-*operad* is a functor of ∞-categories $p\colon \mathcal{O} \to \mathsf{Span}(\mathbb{F})$ such that

(1) $\mathcal{O}$ has all p-cocartesian lifts of inert morphisms in $\mathsf{Span}(\mathbb{F})$.
(2) Given $X, Y \in \mathcal{O}$ over $S, T \in \mathbb{F}_*$ and cocartesian lifts $Y \to Y_t$ over ρ'_t, the commutative square

$$\begin{tikzcd} \mathsf{Map}_{\mathcal{O}}(X,Y) \arrow[r] \arrow[d] & \prod_{t\in T} \mathsf{Map}_{\mathcal{O}}(X, Y_t) \arrow[d] \\ \mathsf{Map}_{\mathsf{Span}(\mathbb{F})}(S,T) \arrow[r] & \prod_{t\in T} \mathsf{Map}_{\mathsf{Span}(\mathbb{F})}(S, \mathbf{1}) \end{tikzcd}$$

is cartesian.
(3) For every $S \in \mathsf{Span}(\mathbb{F})$, the functor

$$\mathcal{O}_S \to \prod_{s\in S} \mathcal{O}_{\mathbf{1}}$$

given by cocartesian transport along the maps ρ'_s ($s \in S$) is an equivalence.

Observation 2.1.7. Condition (3) is slightly redundant: Given (1) and (2), we have for all $X, X' \in \mathcal{O}_{\mathbf{n}}$ equivalences

$$\mathsf{Map}_{\mathcal{O}_{\mathbf{n}}}(X', X) \xrightarrow{\sim} \prod_{i=1}^{n} \mathsf{Map}_{\mathcal{O}}(X', X_i)_{\rho'_i} \xleftarrow{\sim} \prod_{i=1}^{n} \mathsf{Map}_{\mathcal{O}_{\mathbf{1}}}(X'_i, X_i),$$

where the first map is an equivalence of fibres from one of the cartesian squares in (2) and the second is an equivalence by the universal property of the cocartesian maps $X' \to X'_i$. This shows that the functor $\mathcal{O}_{\mathbf{n}} \to \prod_{i=1}^{n} \mathcal{O}_{\mathbf{1}}$ in (3) is fully faithful. To conclude that it is an equivalence it therefore suffices to additionally assume that these functors are essentially surjective.

Definition 2.1.8. For $\mathfrak{F} = \mathbb{F}_*, \mathsf{Span}(\mathbb{F})$ and $p\colon \mathcal{O} \to \mathfrak{F}$ an $\mathfrak{F}$-∞-operad, we say that a morphism $\phi\colon X \to Y$ in $\mathcal{O}$ is *inert* if it is cocartesian and its image in $\mathfrak{F}$ is inert, and *active* if its image in $\mathfrak{F}$ is active. The inert and active morphisms then form a factorization system on $\mathcal{O}$ by [26, Proposition 2.1.2.5].

Definition 2.1.9. For $\mathfrak{F} = \mathbb{F}_*, \mathsf{Span}(\mathbb{F})$, a *morphism of* $\mathfrak{F}$-∞-*operads* is a commutative triangle

$$\begin{tikzcd} \mathcal{O} \arrow[rr] \arrow[dr] & & \mathcal{P} \arrow[dl] \\ & \mathfrak{F}, & \end{tikzcd}$$

where the two diagonal maps are $\mathfrak{F}$-∞-operads and the horizontal map preserves inert morphisms. We define $\mathsf{Opd}(\mathfrak{F})$ to be the subcategory of $\mathsf{Cat}_{\infty/\mathfrak{F}}$ spanned by the $\mathfrak{F}$-∞-operads and the morphisms thereof. For $\mathfrak{F}$-∞-operads $\mathcal{O}$ and $\mathcal{P}$, we also write $\mathsf{Alg}_{\mathcal{O}}(\mathcal{P})$ for the full subcategory of $\mathsf{Fun}_{/\mathfrak{F}}(\mathcal{O}, \mathcal{P})$ spanned by the morphisms of $\mathfrak{F}$-∞-operads.

Observation 2.1.10. We can think of $\mathbb{F}_*$ as the wide subcategory of $\mathsf{Span}(\mathbb{F})$ that contains only those morphisms

$$\begin{tikzcd} & X \arrow[dl, hook'] \arrow[dr, "f"] & \\ S & & T \end{tikzcd}$$

where the backwards map is injective: we identify this with the map of pointed sets $S_+ \to T_+$ given by

$$s \mapsto \begin{cases} *, & s \notin X, \\ f(s), & s \in X. \end{cases}$$

We thus have an inclusion $\mathrm{i}\colon \mathbb{F}_* \to \mathsf{Span}(\mathbb{F})$, and the factorization system on $\mathbb{F}_*$ is obtained by restricting that on $\mathsf{Span}(\mathbb{F})$ along i.

Theorem 2.1.11 ([2, Corollaries 5.1.13 and 5.3.17]). *Pullback along* i *induces an equivalence*

$$\mathrm{i}^*\colon \mathsf{Opd}\big(\mathsf{Span}(\mathbb{F})\big) \xrightarrow{\sim} \mathsf{Opd}(\mathbb{F}_*).$$

Moreover, for any $\mathcal{O}, \mathcal{P} \in \mathsf{Opd}(\mathsf{Span}(\mathbb{F}))$ *we also get a natural equivalence*

$$\mathsf{Alg}_{\mathcal{O}}(\mathcal{P}) \xrightarrow{\sim} \mathsf{Alg}_{\mathrm{i}^*\mathcal{O}}(\mathrm{i}^*\mathcal{P})$$

between ∞*-categories of algebras.* ■

Definition 2.1.12. For $\mathfrak{F}$ either $\mathsf{Span}(\mathbb{F})$ or $\mathbb{F}_*$ (which we think of as a subcategory of $\mathsf{Span}(\mathbb{F})$), we say that a *pinned* $\mathfrak{F}$-∞-operad is an $\mathfrak{F}$-∞-operad $\mathcal{O}$ together with a morphism $p\colon X \to \mathcal{O}_1^{\simeq}$ that is essentially surjective (i.e., surjective on π_0); we refer to the morphism p as a *pinning* (of $\mathcal{O}$). We then define the ∞-category $\mathsf{POpd}(\mathfrak{F})$ as the pullback

$$\begin{tikzcd} \mathsf{POpd}(\mathfrak{F}) \arrow[r] \arrow[d] & \mathsf{Opd}(\mathfrak{F}) \arrow[d, "(-)_1^{\simeq}"] \\ \mathsf{Fun}\big([1], \mathcal{S}\big)_{\mathrm{es}} \arrow[r, "\mathrm{ev}_1"] & \mathcal{S}, \end{tikzcd}$$

where $\mathsf{Fun}([1], \mathcal{S})_{\mathrm{es}}$ is the full subcategory of $\mathsf{Fun}([1], \mathcal{S})$ spanned by the essentially surjective morphisms.

Notation 2.1.13. If $\mathcal{O}$ is an $\mathfrak{F}$-∞-operad, we write $\mathcal{O}^{\natural}$ for the pinned $\mathfrak{F}$-∞-operad $(\mathcal{O}, \mathcal{O}_1^{\simeq} \xrightarrow{=} \mathcal{O}_1^{\simeq})$. (It is easy to see that this gives a functor $(-)^{\natural}\colon \mathsf{Opd}(\mathfrak{F}) \to \mathsf{POpd}(\mathfrak{F})$, which is right adjoint to the functor that forgets the pinning.)

Remark 2.1.14. The term *pinned* is taken from [5]. The analogous notion for (∞, n)-categories has previously been studied by Ayala and Francis [1] under the name *flagged*

(∞, n)*-categories*. Their terminology is inspired by that of "flags" in vector spaces, which makes sense for general n where one considers a sequence of (∞, k)-categories for $k = 0, 1, \ldots, n$. Since we only consider the case $n = 1$, however, there is not much of a flag to speak of, which is why we have chosen to use different terminology.

Base change along i is compatible with the pullback squares defining pinned ∞-operads, giving the following.

Corollary 2.1.15. *Pullback along* i *induces an equivalence*

$$\mathrm{i}^*\colon \mathsf{POpd}(\mathsf{Span}(\mathbb{F})) \xrightarrow{\sim} \mathsf{POpd}(\mathbb{F}_*)$$

between ∞-categories of pinned ∞-operads. ∎

2.2. $\mathsf{Span}(\mathbb{F})$-$\infty$-operads and Lawvere theories

Our goal in this subsection is to show that we can think of $\mathsf{Span}(\mathbb{F})$-$\infty$-operads as *Lawvere theories* in the following sense.

Definition 2.2.1. A *Lawvere theory* $(\mathcal{L}, p)$ is an ∞-category $\mathcal{L}$ that has finite products, together with a functor $p\colon X \to \mathcal{L}$ from an ∞-groupoid X, such that every object in $\mathcal{L}$ can be written as a product of objects in the image of p.

Observation 2.2.2. If we freely adjoin finite coproducts to an ∞-groupoid X, we get the ∞-category $\mathbb{F}_{/X} := \mathbb{F} \times_{\mathcal{S}} \mathcal{S}_{/X}$ of finite sets with a map to X. If $\mathcal{L}$ is an ∞-category with finite products, a morphism $p\colon X \to \mathcal{L}$ therefore uniquely extends to a product-preserving functor $\bar{p}\colon \mathbb{F}_{/X}^{\mathrm{op}} \to \mathcal{L}$, and the requirement for $(\mathcal{L}, p)$ to be a Lawvere theory can be formulated as: $\bar{p}$ is essentially surjective.

Remark 2.2.3. Our definition of Lawvere theory is an ∞-categorical version of the classical notion of a (finitary) multi-sorted Lawvere theory. The one-sorted version was first introduced in Lawvere's thesis [23], and has been studied for ∞-categories by Cranch [11], Gepner–Groth–Nikolaus [12] and Berman [6]. There is a very substantial literature on 1-categorical Lawvere theories and generalizations thereof, which we will not attempt to survey; in the ∞-categorical setting, general notions of algebraic theories have recently been developed by Henry–Meadows [18] and Kositsyn [22].

Definition 2.2.4. A *morphism of Lawvere theories* from $(\mathcal{L}, p\colon X \to \mathcal{L})$ to $(\mathcal{L}', p'\colon X' \to \mathcal{L}')$ is a product-preserving functor $F\colon \mathcal{L} \to \mathcal{L}'$ together with a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ \mathcal{L} & \xrightarrow{F} & \mathcal{L}'. \end{array}$$

We write LwvTh for the ∞-category of Lawvere theories, defined as a subcategory of $\mathsf{Fun}([1], \mathsf{Cat}_\infty)$.

Lemma 2.2.5. *The disjoint union of finite sets gives cartesian products in* $\mathsf{Span}(\mathbb{F})$*. In particular,* $\mathsf{Span}(\mathbb{F})^\natural := (\mathsf{Span}(\mathbb{F}), \{\mathbf{1}\} \hookrightarrow \mathsf{Span}(\mathbb{F}))$ *is a Lawvere theory.*

Proof. For $S, T \in \mathbb{F}$, pullback along the inclusions $S, T \hookrightarrow S \amalg T$ gives an equivalence

$$\mathbb{F}_{/S \amalg T} \xrightarrow{\sim} \mathbb{F}_{/S} \times \mathbb{F}_{/T}.$$

Since we have $\mathsf{Map}_{\mathsf{Span}(\mathbb{F})}(S', S) \simeq \mathbb{F}^{\simeq}_{/S' \times S}$, we get an equivalence

$$\mathsf{Map}_{\mathsf{Span}(\mathbb{F})}(S', S \amalg T) \xrightarrow{\sim} \mathsf{Map}_{\mathsf{Span}(\mathbb{F})}(S', S) \times \mathsf{Map}_{\mathsf{Span}(\mathbb{F})}(S', T)$$

given by composition with the maps

$$S \xleftarrow{=} S \hookrightarrow S \amalg T \quad \text{and} \quad T \xleftarrow{=} T \hookrightarrow S \amalg T.$$

These maps therefore exhibit $S \amalg T$ as the product of S and T in $\mathsf{Span}(\mathbb{F})$. ∎

Proposition 2.2.6. *A functor* $\pi\colon \mathcal{O} \to \mathsf{Span}(\mathbb{F})$ *is a* $\mathsf{Span}(\mathbb{F})$*-*$\infty$*-operad if and only if the following conditions hold:*

(1′) $\mathcal{O}$ *has* π*-cocartesian morphisms over inert morphisms in* $\mathsf{Span}(\mathbb{F})$*.*

(2′) *For* $X \in \mathcal{O}$ *over* $\mathbf{n}$ *in* $\mathsf{Span}(\mathbb{F})$*, the* π*-cocartesian morphisms* $X \to X_i$ *over* $\rho'_i\colon \mathbf{n} \rightarrowtail \mathbf{1}$ *exhibit* X *as the product* $\prod_{i=1}^n X_i$ *in* $\mathcal{O}$*.*

(3′) *The functor* $\mathcal{O}_{\mathbf{n}} \to \prod_{i=1}^n \mathcal{O}_{\mathbf{1}}$*, given by cocartesian transport over the maps* ρ'_i*, is essentially surjective.*

Proof. We check that conditions (1)–(3) in Definition 2.1.6 correspond to the given conditions (1′)–(3′). Here (1′) is identical to (1). For (2), consider the commutative square

$$\begin{array}{ccc} \mathsf{Map}_{\mathcal{O}}(X', X) & \longrightarrow & \prod_{i=1}^n \mathsf{Map}_{\mathcal{O}}(X', X_i) \\ \downarrow & & \downarrow \\ \mathsf{Map}_{\mathsf{Span}(\mathbb{F})}(\mathbf{n}', \mathbf{n}) & \xrightarrow{\sim} & \prod_{i=1}^n \mathsf{Map}_{\mathsf{Span}(\mathbb{F})}(\mathbf{n}', \mathbf{1}), \end{array}$$

where X and X' are objects of $\mathcal{O}$ over $\mathbf{n}$ and $\mathbf{n}'$, respectively, and $X \to X_i$ is cocartesian over $\rho'_i\colon \mathbf{n} \rightarrowtail \mathbf{1}$. Here the bottom horizontal map is an equivalence, since the maps ρ'_i exhibit $\mathbf{n}$ as the product of n copies of $\mathbf{1}$ in $\mathsf{Span}(\mathbb{F})$ by Lemma 2.2.5. This means the square is cartesian if and only if the top horizontal morphism is an equivalence, which is true for all X' if and only if the cocartesian morphisms $X \to X_i$ exhibit X as the product $\prod_{i=1}^n X_i$ in $\mathcal{O}$. Thus (2) is equivalent to (2′). By Observation 2.1.7 we know that we can replace (3) by the additional assumption that the map

$$\mathcal{O}_{\mathbf{n}} \to \prod_{i=1}^n \mathcal{O}_{\mathbf{1}}$$

is essentially surjective, which is precisely (3′). ∎

Lemma 2.2.7. *Suppose $\pi\colon \mathcal{O} \to \mathsf{Span}(\mathbb{F})$ is a $\mathsf{Span}(\mathbb{F})$-$\infty$-operad. Given objects $X_i \in \mathcal{O}_{S_i}$ for $i = 1, \ldots, n$, then a collection of morphisms $p_i\colon X \to X_i$ in $\mathcal{O}$ exhibit X as the product $\prod_i X_i$ if and only if*

- (i) *$\pi(p_i)$ exhibit $\pi(X)$ as the product of S_i in $\mathsf{Span}(\mathbb{F})$ (so that they are the inert morphisms corresponding to the inclusions of S_i in $\coprod_i S_i$),*
- (ii) *p_i is cocartesian for each i.*

In particular, $\mathcal{O}$ has finite products, π preserves these, and $(\mathcal{O}, \mathcal{O}_{\mathbf{1}}^{\simeq})$ is a Lawvere theory.

Proof. Since products are unique up to equivalence if they exist, it suffices to show there exists a unique object satisfying the conditions, and this is a product.

Let η_j denote the inert morphism $S := \coprod_i S_i \rightarrowtail S_j$ in $\mathsf{Span}(\mathbb{F})$ corresponding to the inclusion of S_j in the coproduct. Then cocartesian transport along the maps η_j fits in a commutative square

$$\begin{tikzcd} \mathcal{O}_S \arrow[r, "(\eta_{i,!})_i"] \arrow[d, "\sim"] & \prod_i \mathcal{O}_{S_i} \arrow[d, "\sim"] \\ \prod_S \mathcal{O}_{\mathbf{1}} \arrow[r, "\sim"] & \prod_i \left(\prod_{S_i} \mathcal{O}_{\mathbf{1}}\right), \end{tikzcd}$$

where the vertical morphisms are equivalences by condition (3) in Definition 2.1.6. Then the top horizontal functor is also an equivalence, which implies there exists a unique object $X \in \mathcal{O}_S$ with cocartesian morphisms $X \rightarrowtail X_i$ over η_i. It remains to show that these exhibit X as a product.

For this, observe that if $X_i \rightarrowtail X_{i,s}$ denotes the cocartesian morphisms over $\rho'_s\colon S_i \rightarrowtail \mathbf{1}$, then for any $Y \in \mathcal{O}$ we have a commutative square

$$\begin{tikzcd} \mathsf{Map}_{\mathcal{O}}(Y, X) \arrow[r] \arrow[d, "\sim"] & \prod_i \mathsf{Map}_{\mathcal{O}}(Y, X_i) \arrow[d, "\sim"] \\ \prod_{(i,s)\in S} \mathsf{Map}_{\mathcal{O}}(Y, X_{i,s}) \arrow[r, "\sim"] & \prod_i \left(\prod_{s\in S_i} \mathsf{Map}_{\mathcal{O}}(Y, X_{i,s})\right), \end{tikzcd}$$

where the vertical maps are equivalences by Proposition 2.2.6. Hence the top horizontal map is also an equivalence, which shows that X is a product, as required. It only remains to observe that $(\mathcal{O}, \mathcal{O}_{\mathbf{1}}^{\simeq})$ is a Lawvere theory since every object is a product of objects in $\mathcal{O}_{\mathbf{1}}$ by Proposition 2.2.6. ∎

Observation 2.2.8. More generally, any pinned $\mathsf{Span}(\mathbb{F})$-$\infty$-operad $(\mathcal{O}, q\colon X \twoheadrightarrow \mathcal{O}_{\mathbf{1}}^{\simeq})$ is a Lawvere theory.

Lemma 2.2.9. *Let $\pi\colon \mathcal{O} \to \mathsf{Span}(\mathbb{F})$ be a $\mathsf{Span}(\mathbb{F})$-$\infty$-operad. A morphism $\phi\colon X \to X'$ in $\mathcal{O}$ is inert if and only if the composite $X \to X' \rightarrowtail X'_i$ is inert for all $i \in \pi(X')$, where $X' \to X'_i$ is cocartesian over ρ'_i.*

Proof. Since inert morphisms are closed under composition, the "only if" direction is immediate. In the "if" direction, we first check that a morphism

$$S \xleftarrow{f} T \xrightarrow{g} S'$$

in $\mathsf{Span}(\mathbb{F})$ is inert if and only if its composite with ρ'_i is inert for each $i \in S'$. This composite is the span

$$S \leftarrow T_i \rightarrow \mathbf{1},$$

where T_i is the fibre of g at i. This is inert for all i if and only if each of the fibres T_i has a single element, which is equivalent to g being an isomorphism, i.e., to the original span being inert, as required.

Thus if ϕ satisfies the condition then $\pi(\phi)$ is inert in $\mathsf{Span}(\mathbb{F})$. If $X \rightarrowtail X'' \rightsquigarrow X'$ is the inert–active factorization of ϕ, then it follows that $X'' \rightsquigarrow X'$ lies over $\mathrm{id}_{\pi(X')}$, and since we have a factorization system the map ϕ is inert if and only if this active map is an equivalence. Now observe that the inert-active factorization of the composite $X \to X'_i$ yields a commutative diagram

$$\begin{tikzcd}
X \arrow[r, tail] \arrow[dr, tail] & X'' \arrow[r, squiggly] \arrow[d, tail] & X' \arrow[d, tail] \\
& X''_i \arrow[r, squiggly] & X'_i,
\end{tikzcd}$$

where by assumption the map $X \to X'_i$ is inert, and so $X''_i \rightsquigarrow X'_i$ is an equivalence. But the equivalence of ∞-categories $\mathcal{O}_{\pi(X')} \simeq (\mathcal{O}_{\mathbf{1}})^{\times|\pi(X')|}$ shows that $X'' \rightsquigarrow X'$ is an equivalence if and only if $X''_i \rightsquigarrow X'_i$ is one for each i. ■

Proposition 2.2.10. *Given a commutative triangle*

$$\begin{tikzcd}
\mathcal{O} \arrow[rr, "f"] \arrow[dr, "\pi"'] & & \mathcal{O}' \arrow[dl, "\pi'"] \\
& \mathsf{Span}(\mathbb{F}) &
\end{tikzcd}$$

where $\mathcal{O}$ and $\mathcal{O}'$ are $\mathsf{Span}(\mathbb{F})$-$\infty$-operads, the functor f preserves inert morphisms if and only if it preserves finite products.

Proof. Let us first suppose that f preserves finite products. To show that f preserves inert morphisms, we start by observing that f preserves inert morphisms over ρ'_i: Consider $X \in \mathcal{O}_{\mathbf{n}}$, with inert morphisms $\xi_i \colon X \rightarrowtail X_i$ over ρ'_i. Then by assumption the morphisms $f(\xi_i)\colon f(X) \to f(X_i)$ exhibit $f(X)$ as the product of the $f(X_i)$'s, and so are indeed inert by Lemma 2.2.7. Now consider an arbitrary inert morphism $X \rightarrowtail X'$ in $\mathcal{O}$. To show that $f(X) \to f(X')$ is inert, it suffices by Lemma 2.2.9 to show that $f(X) \to f(X') \rightarrowtail f(X')_i$ is inert for each i, where $f(X') \to f(X')_i$ is cocartesian over ρ'_i. But we know $f(X') \rightarrowtail f(X')_i$ is the image of the inert morphism $X' \rightarrowtail X'_i$, and hence $f(X) \to f(X')_i$ is indeed inert, since it is the image of the inert composite $X \rightarrowtail X' \rightarrowtail X'_i$.

Conversely, suppose we know that f preserves inert morphisms. Then Lemma 2.2.7 implies that f preserves products, since these are characterized in terms of inert morphisms. ■

Putting the preceding results together, we have shown the following.

Corollary 2.2.11. *If we view both* $\mathsf{POpd}(\mathsf{Span}(\mathbb{F}))$ *and* $\mathsf{LwvTh}_{/\mathsf{Span}(\mathbb{F})^\natural}$ *as subcategories of the pullback of the functors*

$$\mathsf{Fun}([1], \mathcal{S}) \xrightarrow{\mathrm{ev}_1} \mathcal{S} \xleftarrow{(-)^{\simeq}_{\mathbf{1}}} \mathsf{Cat}_{\infty/\mathsf{Span}(\mathbb{F})},$$

then we have a full subcategory inclusion $\mathsf{POpd}(\mathsf{Span}(\mathbb{F})) \hookrightarrow \mathsf{LwvTh}_{/\mathsf{Span}(\mathbb{F})^\natural}$. ■

2.3. Algebras, monoids and models

If $\mathcal{O}$ is a $\mathsf{Span}(\mathbb{F})$-$\infty$-operad and $\mathcal{C}$ is an ∞-category with finite products, our goal in this subsection is to identify three structures in $\mathcal{C}$ parametrized by $\mathcal{O}$: $\mathcal{O}$-algebras in the symmetric monoidal ∞-category given by the cartesian product on $\mathcal{C}$, $\mathcal{O}$-monoids in $\mathcal{C}$, and models for $\mathcal{O}$ as a Lawvere theory in $\mathcal{C}$.

We begin by introducing the definitions of the last two structures.

Definition 2.3.1. If $\mathcal{L}$ and $\mathcal{C}$ are ∞-categories with finite products (such as the underlying ∞-category of a Lawvere theory), then a *model* of $\mathcal{L}$ in $\mathcal{C}$ is a functor $\mathcal{L} \to \mathcal{C}$ that preserves finite products. We write $\mathsf{Mod}_{\mathcal{L}}(\mathcal{C})$ for the full subcategory of $\mathsf{Fun}(\mathcal{L}, \mathcal{C})$ spanned by the models.

Definition 2.3.2. If $\mathcal{O}$ is a $\mathsf{Span}(\mathbb{F})$-$\infty$-operad and $\mathcal{C}$ is an ∞-category with finite products, then an $\mathcal{O}$*-monoid* in $\mathcal{C}$ is a functor $M \colon \mathcal{O} \to \mathcal{C}$ such that for every $X \in \mathcal{O}$ over $\mathbf{n} \in \mathsf{Span}(\mathbb{F})$, the morphism

$$M(X) \to \prod_{i=1}^{n} M(X_i),$$

induced by the inert morphisms $X \rightarrowtail X_i$ over ρ_i', is an equivalence. We write $\mathsf{Mon}_{\mathcal{O}}(\mathcal{C})$ for the full subcategory of $\mathsf{Fun}(\mathcal{O}, \mathcal{C})$ spanned by the $\mathcal{O}$-monoids.

Proposition 2.3.3. *If* $\mathcal{O}$ *is a* $\mathsf{Span}(\mathbb{F})$*-*$\infty$*-operad and* $\mathcal{C}$ *is an* ∞*-category, then a functor* $M \colon \mathcal{O} \to \mathcal{C}$ *is an* $\mathcal{O}$*-monoid if and only if* M *preserves finite products. In particular,* $\mathsf{Mon}_{\mathcal{O}}(\mathcal{C}) \simeq \mathsf{Mod}_{\mathcal{O}}(\mathcal{C})$ *as full subcategories of* $\mathsf{Fun}(\mathcal{O}, \mathcal{C})$.

Proof. It follows from (2′) in Proposition 2.2.6 that if M is an $\mathcal{O}$-model, then M is an $\mathcal{O}$-monoid. To prove the converse, we again use the description of products in $\mathcal{O}$ from Lemma 2.2.7: Suppose the (necessarily inert) morphisms $X \rightarrowtail X_i$ for $i = 1, \ldots, n$ exhibit X in $\mathcal{O}$ over S as a product of the objects X_i, which lie over S_i. If $X_i \rightarrowtail X_{i,s}$ denotes the cocartesian morphisms over $\rho_s' \colon S_i \rightarrowtail \mathbf{1}$, then we have a commutative square

$$\begin{tikzcd}
M(X) \arrow[r] \arrow[d, "\sim"] & \prod_i M(X_i) \arrow[d, "\sim"] \\
\prod_{(i,s)\in S} M(X_{i,s}) \arrow[r, "\sim"] & \prod_i \left(\prod_{s\in S_i} M(X_{i,s})\right).
\end{tikzcd}$$

It follows that the top horizontal morphism is also an equivalence, so that M preserves this product. ∎

Proposition 2.3.4. *For every ∞-category $\mathcal{C}$ with finite products, there exists a $\mathsf{Span}(\mathbb{F})$-$\infty$-operad $\mathcal{C}^{\times} \to \mathsf{Span}(\mathbb{F})$ (which in fact is a cocartesian fibration) together with a functor $\mathcal{C}^{\times} \to \mathcal{C}$ (which preserves products) such that composition with this functor induces an equivalence*

$$\mathsf{Alg}_{\mathcal{O}}(\mathcal{C}^{\times}) \xrightarrow{\sim} \mathsf{Mon}_{\mathcal{O}}(\mathcal{C})$$

for every $\mathsf{Span}(\mathbb{F})$-$\infty$-operad $\mathcal{O}$.

Proof. This follows by the same proof as for the version with $\mathbb{F}_*$ in place of $\mathsf{Span}(\mathbb{F})$, proved in [26, Section 2.4.1], and we do not repeat it here. ∎

3. Lawvere theories and algebraic monads

It was first shown by Linton [24] that one-sorted Lawvere theories are equivalent to finitary monads on the category of sets, and the ∞-categorical analogue of this correspondence has been proved by Gepner–Groth–Nikolaus [12]. In this section we will extend this result to our multi-sorted notion of Lawvere theories, proving Theorem B. We start by introducing the relevant notion of *algebraic* monad and showing that any Lawvere theory gives rise to one of these in Section 3.1. In Section 3.2 we then define the Lawvere theory associated to an algebraic monad and prove that its models recover the monad we started with.

3.1. Algebraic monads from Lawvere theories

Our goal in this subsection is to show that models for a Lawvere theory in the ∞-category of spaces always gives an algebraic monad, in the following sense.

Definition 3.1.1. An *algebraic monad* is a monad T on $\mathsf{Fun}(X, \mathcal{S}) \simeq \mathcal{S}_{/X}$ for some $X \in \mathcal{S}$ such that the underlying endofunctor of T preserves sifted colimits.

Warning 3.1.2. In the 1-categorical context, Lawvere theories are related to *finitary* monads, i.e., monads that preserve *filtered* colimits. This is not the case in our ∞-categorical setting, where the monads are required to preserve *sifted* colimits (which in $\mathcal{S}_{/X}$ boils down to filtered colimits and simplicial colimits). The fundamental reason for this difference is that Set is obtained from the category $\mathbb{F}$ of finite sets by freely adding filtered colimits, but $\mathcal{S}$ is obtained from $\mathbb{F}$ by freely adding sifted colimits. Here the relevance of $\mathbb{F}$ is that in both cases we want Lawvere theories to be defined in terms of finite products rather than more general limits.

We note the following general result about monads and colimits, taken from Henry and Meadows [18].

Proposition 3.1.3. *Let T be a monad on an ∞-category $\mathcal{C}$, which has all $\mathcal{I}$-shaped colimits for some ∞-category $\mathcal{I}$. Then the following are equivalent:*

(1) *The endofunctor $T\colon \mathcal{C} \to \mathcal{C}$ preserves $\mathcal{I}$-shaped colimits.*

(2) *The ∞-category $\mathsf{Alg}(T)$ has $\mathcal{I}$-shaped colimits and the functor $U_T\colon \mathsf{Alg}(T) \to \mathcal{C}$ preserves these.*

Proof. Since $T \simeq U_T F_T$ where F_T is a left adjoint, the implication from (2) to (1) is immediate. The non-trivial direction is [18, Lemma 6.6], which in turn is an application of [26, Corollary 4.2.3.5]. ∎

Corollary 3.1.4. *A monad T on $\mathcal{S}_{/X}$ is algebraic if and only if for the corresponding monadic adjunction*

$$F_T\colon \mathcal{S}_{/X} \rightleftarrows \mathsf{Alg}(T) : U_T,$$

the ∞-category $\mathsf{Alg}(T)$ has sifted colimits and the right adjoint U_T preserves these. ∎

Observation 3.1.5. If T is an algebraic monad on $\mathsf{Fun}(X, \mathcal{S})$, then $\mathsf{Alg}(T)$ is locally small. Indeed, if T is any monad on a locally small ∞-category $\mathcal{C}$ then $\mathsf{Alg}(T)$ is locally small: any object $A \in \mathsf{Alg}(T)$ can be written as the simplicial colimit of a free resolution, which means that $\mathsf{Map}_{\mathsf{Alg}(T)}(A, B)$ is a cosimplicial limit of mapping spaces in $\mathcal{C}$ and so is small.

Proposition 3.1.6. *Let $(\mathcal{L}, p\colon X \twoheadrightarrow \mathcal{L})$ be a Lawvere theory. Then:*

(i) *The ∞-category $\mathsf{Mod}_{\mathcal{L}}(\mathcal{S})$ is presentable, and the inclusion*

$$\mathsf{Mod}_{\mathcal{L}}(\mathcal{S}) \hookrightarrow \mathsf{Fun}(\mathcal{L}, \mathcal{S})$$

has a left adjoint.

(ii) *The functor $p^*\colon \mathsf{Mod}_{\mathcal{L}}(\mathcal{S}) \to \mathsf{Fun}(X, \mathcal{S})$ is a monadic right adjoint.*

(iii) *The corresponding monad $T_{\mathcal{L}}$ is algebraic.*

(iv) *The left adjoint L of p^* satisfies*

$$L\mathsf{y}_X x \simeq \mathsf{y}_{\mathcal{L}^{\mathrm{op}}}(px).$$

Proof. The ∞-category $\mathsf{Mod}_{\mathcal{L}}(\mathcal{S})$ can be defined as the full subcategory of $\mathsf{Fun}(\mathcal{L}, \mathcal{S})$ spanned by the objects that are local with respect to the set of maps of the form

$$\coprod_{i=1}^{n} \mathsf{y}_{\mathcal{L}^{\mathrm{op}}}(px_i) \to \mathsf{y}_{\mathcal{L}^{\mathrm{op}}}\left(\prod_{i=1}^{n} px_i\right)$$

for $x_1, \dots, x_n$ in X. Thus $\mathsf{Mod}_{\mathcal{L}}(\mathcal{S})$ is an accessible localization of $\mathsf{Fun}(\mathcal{L}, \mathcal{S})$, which means the inclusion has a left adjoint L_{prod} and $\mathsf{Mod}_{\mathcal{L}}(\mathcal{S})$ is presentable. This proves (i).

If $p_!\colon \mathsf{Fun}(X, \mathcal{S}) \to \mathsf{Fun}(\mathcal{L}, \mathcal{S})$ denotes left Kan extension along p, then the composite $L_{\mathrm{prod}} p_!$ is a left adjoint to p^*. To prove that this is the monadic adjunction for an algebraic monad, it suffices by the ∞-categorical monadicity theorem, [26, Theorem 4.7.3.5], to prove that p^* is conservative and preserves sifted colimits.

To see that p^* is conservative, observe that if $\eta\colon F \to G$ is a natural transformation between product-preserving functors $\mathcal{L} \to \mathcal{S}$, then since every object of $\mathcal{L}$ is a product of objects in the image of p, the component η_A must be an equivalence for every $A \in \mathcal{L}$ if it is an equivalence for objects in the image of p, i.e., if $p^*\eta$ is an equivalence.

If we view p^* as a functor $\mathsf{Fun}(\mathcal{L}, \mathcal{S}) \to \mathsf{Fun}(X, \mathcal{S})$, it preserves all colimits, so to show p^* preserves sifted colimits it suffices to check that $\mathsf{Mod}_{\mathcal{L}}(\mathcal{S})$ is closed under sifted colimits in $\mathsf{Fun}(\mathcal{L}, \mathcal{S})$. In other words, given a sifted diagram $\phi\colon \mathcal{I} \to \mathsf{Fun}(\mathcal{L}, \mathcal{S})$ where each $\phi(i)$ lies in $\mathsf{Mod}_{\mathcal{L}}(\mathcal{S})$, so does the colimit of ϕ. This holds because colimits in $\mathsf{Fun}(\mathcal{L}, \mathcal{S})$ are computed pointwise, and for a product $A \times B$ in $\mathcal{L}$ we have

$$\operatorname{colim}_{i\in\mathcal{I}} \phi_i(A \times B) \xrightarrow{\sim} \operatorname{colim}_{i\in\mathcal{I}} \phi_i(A) \times \phi_i(B) \xrightarrow{\sim} \big(\operatorname{colim}_{i\in\mathcal{I}} \phi_i(A)\big) \times \big(\operatorname{colim}_{i\in\mathcal{I}} \phi_i(B)\big),$$

where the second equivalence holds since sifted colimits by definition commute with finite products in $\mathcal{S}$. This proves (ii) and (iii).

It remains to prove (iv). We saw that the left adjoint of p^* is $L_{\mathrm{prod}} p_!$, and we know $p_! \mathsf{y}_X x \simeq \mathsf{y}_{\mathcal{L}^{\mathrm{op}}} p(x)$, so this amounts to checking that $\mathsf{y}_{\mathcal{L}^{\mathrm{op}}} p(x)$ already lies in the full subcategory $\mathsf{Mod}_{\mathcal{L}}(\mathcal{S})$. In other words, we must show that the functor

$$\mathsf{Map}_{\mathcal{L}}(p(x), -)\colon \mathcal{L} \to \mathcal{S}$$

preserves finite products, which is obvious. ∎

Remark 3.1.7. The appropriate notion of morphism between algebraic monads for us is that of *lax* morphisms of monads. If we work with $(\infty, 2)$-categories, we can view a monad on an ∞-category as a functor of $(\infty, 2)$-categories from the universal monad 2-category to CAT_∞, the $(\infty, 2)$-category of ∞-categories. A lax morphism of monads is then a lax natural transformation $S \to T$ between such functors. However, if we instead encode monads by the corresponding monadic right adjoints, we can avoid working with $(\infty, 2)$-categories: as was shown in [15], a lax morphism of monads $S \to T$ is equivalent to a commutative square

$$\begin{tikzcd} \mathsf{Alg}(T) \arrow[r, "F^*"] \arrow[d, "U_T"] & \mathsf{Alg}(T) \arrow[d, "U_S"] \\ \mathsf{Fun}(X, \mathcal{S}) \arrow[r, "f^*"] & \mathsf{Fun}(Y, \mathcal{S}), \end{tikzcd} \tag{3.1}$$

where in the case of algebraic monads we want f^* to be given by composition with a morphism $f\colon Y \to X$ of ∞-groupoids. We can thus encode the ∞-category of algebraic monads and lax morphisms thereof in terms of such squares, which explains the following definition.

Definition 3.1.8. We define the ∞-category AlgMnd of algebraic monads by taking its opposite $\mathsf{AlgMnd}^{\mathrm{op}}$ to be the full subcategory of the pullback of $\mathrm{ev}_1\colon \mathsf{Fun}([1], \widehat{\mathsf{Cat}}_\infty) \to \widehat{\mathsf{Cat}}_\infty$ along $\mathsf{Fun}(-, \mathcal{S})\colon \mathcal{S}^{\mathrm{op}} \to \widehat{\mathsf{Cat}}_\infty$ whose objects are the monadic right adjoints corresponding to algebraic monads. Equivalently, by Corollary 3.1.4 these are the right adjoints that are conservative and preserve sifted colimits.

Observation 3.1.9. Given a morphism of algebraic monads as in (3.1), the functor F^* automatically preserves limits and sifted colimits: since U_S is conservative and preserves limits and sifted colimits, it suffices to show that $U_S F^*$ preserves these, but this is by assumption equivalent to $f^* U_T$, which preserves them since both U_T and f^* do so.

Observation 3.1.10. Given a morphism of Lawvere theories

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"] & X' \arrow[d, "p'"] \\ \mathcal{L} \arrow[r, "F"] & \mathcal{L}', \end{tikzcd}$$

composition with F preserves product-preserving functors, and so restricts to a functor

$$F^* \colon \mathsf{Mod}_{\mathcal{L}'}(\mathcal{S}) \to \mathsf{Mod}_{\mathcal{L}}(\mathcal{S}).$$

This fits in a commutative square

$$\begin{tikzcd} \mathsf{Mod}_{\mathcal{L}'}(\mathcal{S}) \arrow[r, "F^*"] \arrow[d, "p'^*"] & \mathsf{Mod}_{\mathcal{L}}(\mathcal{S}) \arrow[d, "p^*"] \\ \mathsf{Fun}(X', \mathcal{S}) \arrow[r, "f^*"] & \mathsf{Fun}(X, \mathcal{S}), \end{tikzcd}$$

which is a morphism of algebraic monads $T_{\mathcal{L}} \to T_{\mathcal{L}'}$ by Proposition 3.1.6. We thus have a functor $\mathfrak{Mod} \colon \mathsf{LwvTh} \to \mathsf{AlgMnd}$ that takes a Lawvere theory to its ∞-category of models in $\mathcal{S}$, viewed as an algebraic monad via the given map from an ∞-groupoid.

3.2. Lawvere theories from algebraic monads

Our next goal is to prove that *every* algebraic monad arises from a Lawvere theory via the following construction:

Definition 3.2.1. If T is an algebraic monad on $\mathsf{Fun}(X, \mathcal{S})$, then the *Lawvere theory* $\mathfrak{Th}(T)$ *of* T is the pair $(\mathcal{L}(T), p_T)$ where $\mathcal{L}(T)^{\mathrm{op}}$ is the full subcategory of $\mathsf{Alg}(T)$ spanned by the objects of the form $\coprod_{i=1}^n F_T \mathsf{y}_X x_i$ for x_i in X, together with the map $p_T \colon X \to \mathcal{L}(T)$ obtained by restricting F_T.

Notation 3.2.2. It will be convenient to write $(x_1, \ldots, x_n)$ for the coproduct $\coprod_{i=1}^n \mathsf{y}_X(x_i)$ in $\mathsf{Fun}(X, \mathcal{S})$, and $F_T(x_1, \ldots, x_n)$ for its image $F_T(\coprod_{i=1}^n \mathsf{y}_X(x_i)) \simeq \coprod_{i=1}^n F_T \mathsf{y}_X x_i$ in $\mathcal{L}(T)^{\mathrm{op}}$.

Lemma 3.2.3. *If $(\mathcal{L}, p)$ is a Lawvere theory, then the Yoneda embedding for $\mathcal{L}^{\mathrm{op}}$ factors through a fully faithful functor*

$$\mathcal{L}^{\mathrm{op}} \hookrightarrow \mathsf{Mod}_{\mathcal{L}}(\mathcal{S})$$

whose image is precisely $\mathcal{L}(T_{\mathcal{L}})$. This gives an equivalence of Lawvere theories

$$(\mathcal{L}, p) \simeq \mathfrak{Th}(T_{\mathcal{L}}).$$

Proof. First observe that for $X \in \mathcal{L}$, the copresheaf

$$\mathsf{y}_{\mathcal{L}^{\mathrm{op}}} X \simeq \mathsf{Map}_{\mathcal{L}}(X, -)$$

clearly preserves products, so that $\mathsf{y}_{\mathcal{L}^{\mathrm{op}}}$ factors through the full subcategory $\mathsf{Mod}_{\mathcal{L}}(\mathcal{S})$ of $\mathsf{Fun}(\mathcal{L}, \mathcal{S})$.

If L denotes the left adjoint to $p^*\colon \mathsf{Mod}_{\mathcal{L}}(\mathcal{S}) \to \mathsf{Fun}(X, \mathcal{S})$, then we saw in Proposition 3.1.6 that we have

$$L\mathsf{y}_X(x) \simeq \mathsf{y}_{\mathcal{L}^{\mathrm{op}}}(px). \tag{3.2}$$

Then $L(\coprod_{i=1}^n \mathsf{y}_X(x_i)) \simeq \coprod_{i=1}^n \mathsf{y}_{\mathcal{L}^{\mathrm{op}}}(px)$, giving natural equivalences

$$\begin{aligned}
\mathsf{Map}_{\mathsf{Mod}_{\mathcal{L}}(\mathcal{S})}\Big(L\Big(\coprod_{i=1}^n \mathsf{y}_X(x_i)\Big), \Phi\Big) &\simeq \mathsf{Map}_{\mathsf{Fun}(\mathcal{L},\mathcal{S})}\Big(\coprod_{i=1}^n \mathsf{y}_{\mathcal{L}^{\mathrm{op}}}(px_i), \Phi\Big) \\
&\simeq \prod_{i=1}^n \Phi(px_i) \simeq \Phi\Big(\prod_{i=1}^n px_i\Big) \\
&\simeq \mathsf{Map}_{\mathsf{Mod}_{\mathcal{L}}(\mathcal{S})}\Big(\mathsf{y}_{\mathcal{L}^{\mathrm{op}}}\Big(\prod_{i=1}^n px_i\Big), \Phi\Big)
\end{aligned}$$

for any $\Phi \in \mathsf{Mod}_{\mathcal{L}}(\mathcal{S})$. From the Yoneda Lemma we then have an equivalence

$$L\Big(\coprod_{i=1}^n \mathsf{y}_X(x_i)\Big) \simeq \mathsf{y}_{\mathcal{L}^{\mathrm{op}}}\Big(\prod_{i=1}^n px_i\Big).$$

By definition, these are precisely the objects that lie in the full subcategory $\mathcal{L}(T_{\mathcal{L}})^{\mathrm{op}}$. On the other hand, by assumption every object in $\mathcal{L}$ is a product of objects in the image of p, so these are also precisely the objects in the image of $\mathcal{L}^{\mathrm{op}}$. The image of $\mathcal{L}^{\mathrm{op}}$ is thus $\mathcal{L}(T_{\mathcal{L}})^{\mathrm{op}}$, as required.

The equivalence (3.2) can be rephrased as giving a commutative square

$$\begin{tikzcd}
X \arrow[r, hook, "\mathsf{y}_X"] \arrow[d, "p^{\mathrm{op}}"'] & \mathsf{Fun}(X, \mathcal{S}) \arrow[d, "L"] \\
\mathcal{L}^{\mathrm{op}} \arrow[r, hook, "\mathsf{y}_{\mathcal{L}^{\mathrm{op}}}"] & \mathsf{Mod}_{\mathcal{L}}(\mathcal{S}).
\end{tikzcd}$$

This restricts to a commutative triangle

$$\begin{tikzcd}
& X \arrow[dl, "p^{\mathrm{op}}"'] \arrow[dr, "p_T^{\mathrm{op}}"] & \\
\mathcal{L}^{\mathrm{op}} \arrow[rr, "\sim"] & & \mathcal{L}(T_{\mathcal{L}})^{\mathrm{op}},
\end{tikzcd}$$

which gives the required equivalence of Lawvere theories. ∎

Proposition 3.2.4. *Let T be an algebraic monad on X. Then the fully faithful inclusion $\mathcal{L}(T)^{\mathrm{op}} \hookrightarrow \mathsf{Alg}(T)$ induces a commutative triangle*

$$\begin{tikzcd} \mathsf{Alg}(T) \arrow[rr, "\nu"] \arrow[dr, "U_T"'] & & \mathsf{Mod}_{\mathcal{L}(T)}(\mathcal{S}) \arrow[dl, "p^*"] \\ & \mathsf{Fun}(X, \mathcal{S}), & \end{tikzcd}$$

where ν arises from the Yoneda embedding of $\mathsf{Alg}(T)$ restricted to $\mathcal{L}(T)^{\mathrm{op}}$ and p is the induced map $X \to \mathcal{L}(T)$. Moreover, the functor ν is an equivalence.

Proof. By Observation 3.1.5 the ∞-category $\mathsf{Alg}(T)$ is locally small, so we have a Yoneda embedding $\mathsf{Alg}(T) \hookrightarrow \mathsf{Fun}(\mathsf{Alg}(T)^{\mathrm{op}}, \mathcal{S})$. Composing this with the inclusion $\mathcal{L}(T)^{\mathrm{op}} \hookrightarrow \mathsf{Alg}(T)$, we get a restricted Yoneda functor

$$\nu \colon \mathsf{Alg}(T) \to \mathsf{Fun}\big(\mathcal{L}(T), \mathcal{S}\big).$$

This functor takes $A \in \mathsf{Alg}(T)$ to the copresheaf

$$F_T(x_1, \ldots, x_n) \in \mathcal{L}(T) \mapsto \mathsf{Map}_{\mathsf{Alg}(T)}\big(F_T(x_1, \ldots, x_n), A\big) \simeq \prod_{i=1}^{n} U_T A(x_i).$$

This takes finite products in $\mathcal{L}(T)$ to products in $\mathcal{S}$, since the former correspond to coproducts in $\mathsf{Alg}(T)$, and so factors through the full subcategory $\mathsf{Mod}_{\mathcal{L}(T)}(\mathcal{S})$. Note also that we have a natural equivalence

$$\nu F_T(x_1, \ldots, x_n) \simeq \mathsf{y}_{\mathcal{L}(T)^{\mathrm{op}}} F_T(x_1, \ldots, x_n), \tag{3.3}$$

since by Lemma 3.2.3 we know that $\mathsf{y}_{\mathcal{L}(T)^{\mathrm{op}}}\Lambda$ lies in $\mathsf{Mod}_{\mathcal{L}(T)}(\mathcal{S})$ for any object $\Lambda \simeq F_T(y_1, \ldots, y_m)$ in $\mathcal{L}(T)$, and so we have natural equivalences

$$\begin{aligned} &\mathsf{Map}_{\mathsf{Mod}_{\mathcal{L}(T)}(\mathcal{S})}\big(\mathsf{y}_{\mathcal{L}(T)^{\mathrm{op}}}\Lambda, \nu F_T(x_1, \ldots, x_n)\big) \\ &\quad \simeq \big(\nu F_T(x_1, \ldots, x_n)\big)(\Lambda) \\ &\quad \simeq \mathsf{Map}_{\mathsf{Alg}(T)}\big(\Lambda, F_T(x_1, \ldots, x_n)\big) \\ &\quad \simeq \mathsf{Map}_{\mathcal{L}(T)}\big(\Lambda, F_T(x_1, \ldots, x_n)\big) \\ &\quad \simeq \mathsf{Map}_{\mathsf{Mod}_{\mathcal{L}(T)}(\mathcal{S})}\big(\mathsf{y}_{\mathcal{L}(T)^{\mathrm{op}}}\Lambda, \mathsf{y}_{\mathcal{L}(T)^{\mathrm{op}}} F_T(x_1, \ldots, x_n)\big). \end{aligned}$$

Now for $A \in \mathsf{Alg}(T)$ we have natural equivalences

$$\begin{aligned} \mathsf{Map}_{\mathsf{Fun}(X,\mathcal{S})}(\mathsf{y}_X x, p^*\nu A) &\simeq (\nu A)(px) \simeq \mathsf{Map}_{\mathsf{Alg}(T)}(F_T x, A) \\ &\simeq \mathsf{Map}_{\mathsf{Fun}(X,\mathcal{S})}(\mathsf{y}_X x, U_T A), \end{aligned} \tag{3.4}$$

and so by the Yoneda lemma there is a natural equivalence $p^*\nu \simeq U_T$. This means that we have the required commutative triangle

$$\begin{tikzcd} \mathsf{Alg}(T) \arrow[rr, "\nu"] \arrow[dr, "U_T"'] & & \mathsf{Mod}_{\mathcal{L}(T)}(\mathcal{S}) \arrow[dl, "p^*"] \\ & \mathsf{Fun}(X, \mathcal{S}). & \end{tikzcd}$$

Here both the diagonal functors are monadic right adjoints by Proposition 3.1.6, so to prove that the horizontal functor is an equivalence it suffices by [26, Corollary 4.7.3.16] to check that the induced transformation

$$L \to \nu F_T$$

is an equivalence, where L denotes the left adjoint $\mathsf{Fun}(X, \mathcal{S}) \to \mathsf{Mod}_{\mathcal{L}(T)}(\mathcal{S})$ to p^*. Since p^* is conservative, this is equivalent to the transformation of endofunctors

$$p^* L \to p^* \nu F_T \simeq T$$

being an equivalence. Note that here p^* preserves sifted colimits, since sifted colimits in $\mathsf{Mod}_{\mathcal{L}(T)}(\mathcal{S})$ are computed in $\mathsf{Fun}(\mathcal{L}(T), \mathcal{S})$, while T preserves sifted colimits since it is algebraic. Moreover, every object of $\mathsf{Fun}(X, \mathcal{S})$ is a sifted colimit of objects of the form $(x_1, \dots, x_n) := \coprod_{i=1}^{n} \mathsf{y}_X(x_i)$ for $x_i \in X$, so it suffices to check that we have an equivalence for such objects.

We first consider the case where $n = 1$. Then for $M \in \mathsf{Mod}_{\mathcal{L}(T)}(\mathcal{S})$ we have natural equivalences

$$\begin{aligned}\mathsf{Map}_{\mathsf{Mod}_{\mathcal{L}(T)}(\mathcal{S})}(Lx, M) &\simeq \mathsf{Map}_{\mathsf{Fun}(X,\mathcal{S})}(x, p^* M) \simeq p^* M(x) \simeq M(F_T x) \\ &\simeq \mathsf{Map}_{\mathsf{Mod}_{\mathcal{L}(T)}(\mathcal{S})}(\mathsf{y}_{\mathcal{L}(T)^{\mathrm{op}}} F_T x, M).\end{aligned}$$

Thus by the Yoneda Lemma we have an equivalence $Lx \xrightarrow{\sim} \mathsf{y}_{\mathcal{L}(T)^{\mathrm{op}}} F_T x$. Since the Yoneda embedding is fully faithful, this equivalence is the point of the mapping space $\mathsf{Map}_{\mathsf{Mod}_{\mathcal{L}(T)}(\mathcal{S})}(Lx, \mathsf{y}_{\mathcal{L}(T)^{\mathrm{op}}} F_T x)$ that corresponds to $\mathrm{id}_{F_T x}$ under the equivalence

$$\begin{aligned}\mathsf{Map}_{\mathsf{Mod}_{\mathcal{L}(T)}(\mathcal{S})}(Lx, \mathsf{y}_{\mathcal{L}(T)^{\mathrm{op}}} F_T x) &\simeq \mathsf{Map}_{\mathsf{Fun}(X,\mathcal{S})}(x, p^* \mathsf{y}_{\mathcal{L}(T)^{\mathrm{op}}} F_T x) \\ &\simeq p^* \mathsf{y}_{\mathcal{L}(T)^{\mathrm{op}}}(F_T x)(x) \simeq (\mathsf{y}_{\mathcal{L}(T)^{\mathrm{op}}} F_T x)(F_T x) \\ &\simeq \mathsf{Map}_{\mathsf{Alg}(T)}(F_T x, F_T x).\end{aligned} \tag{3.5}$$

On the other hand, the canonical map $Lx \to \nu F_T x$ corresponds under the equivalences

$$\mathsf{Map}_{\mathsf{Mod}_{\mathcal{L}(T)}(\mathcal{S})}(Lx, \nu F_T x) \simeq \mathsf{Map}_{\mathsf{Fun}(X,\mathcal{S})}(x, p^* \nu F_T x) \simeq \mathsf{Map}_{\mathsf{Fun}(X,\mathcal{S})}(x, U_T F_T x)$$

to the unit map $x \to U_T F_T x$. For the second equivalence here we used the equivalence $U_T \simeq p^* \nu$, and using (3.4) we can decompose this as

$$\begin{aligned}\mathsf{Map}_{\mathsf{Fun}(X,\mathcal{S})}(x, p^* \nu F_T x) &\simeq (p^* \nu F_T x)(x) \\ &\simeq \mathsf{Map}_{\mathsf{Alg}(T)}(F_T x, F_T x) \\ &\simeq \mathsf{Map}_{\mathsf{Fun}(X,\mathcal{S})}(x, U_T F_T x),\end{aligned}$$

where the unit map corresponds to the identity of $F_T x$ under the last adjunction equivalence.

In other words, we have shown that the canonical map $Lx \to \nu F_T x$ is the point that corresponds to $\mathrm{id}_{F_T x}$ under the equivalence

$$\begin{aligned}\mathsf{Map}_{\mathsf{Mod}_{\mathcal{L}(T)}(\mathcal{S})}(Lx, \nu F_T x) &\simeq \mathsf{Map}_{\mathsf{Fun}(X,\mathcal{S})}(x, p^*\nu F_T x) \\ &\simeq (p^*\nu F_T x)(x) \\ &\simeq \mathsf{Map}_{\mathsf{Alg}(T)}(F_T x, F_T x).\end{aligned}$$

It now only remains to observe that when we identify $\nu F_T x$ with $\mathsf{y} F_T x$ this equivalence is precisely that of (3.5), which is clear from the definitions. We conclude that the composite $Lx \to \nu F_T x \simeq \mathsf{y} F_T x$ is precisely the map we showed was an equivalence, and so the canonical map $Lx \to \nu F_T x$ is indeed an equivalence.

Now we consider the general case, where we have a commutative square

$$\begin{tikzcd} \coprod_{i=1}^n Lx_i \arrow[r, "\sim"] \arrow[d, "\sim"] & L(x_1, \ldots, x_n) \arrow[d] \\ \coprod_{i=1}^n \nu F_T x_i \arrow[r] & \nu F_T(x_1, \ldots, x_n) \end{tikzcd}$$

in which the top horizontal map is an equivalence since L preserves coproducts and the left vertical map is an equivalence by what we just proved. To show the right vertical map is an equivalence it is then sufficient to show that the bottom horizontal map is one. Mapping this into an object $M \in \mathsf{Mod}_{\mathcal{L}(T)}(\mathcal{S})$ we get using the identification (3.3) a commutative diagram

$$\begin{tikzcd} \mathsf{Map}\big(\nu F_T(x_1, \ldots, x_n), M\big) \arrow[r, "\sim"] \arrow[d] & M\big(F_T(x_1, \ldots, x_n)\big) \arrow[d] \\ \mathsf{Map}\big(\coprod_{i=1}^n \nu F_T(x_i), M\big) \arrow[r, "\sim"] & \prod_{i=1}^n M(F_T x_i). \end{tikzcd}$$

Here the right vertical map is an equivalence since M is an $\mathcal{L}(T)$-model, hence by the Yoneda lemma the canonical map

$$L(x_1, \ldots, x_n) \to \nu F_T(x_1, \ldots, x_n)$$

is an equivalence in $\mathsf{Mod}_{\mathcal{L}(T)}(\mathcal{S})$, as required. ■

Remark 3.2.5. If T is an algebraic monad, we have seen that $\mathsf{Alg}(T)$ is equivalent to $\mathsf{Mod}_{\mathcal{L}(T)}(\mathcal{S})$. In the terminology of [25], this identifies $\mathsf{Alg}(T)$ with the *nonabelian derived ∞-category* of $\mathcal{L}(T)^{\mathrm{op}}$, which by [25, Proposition 5.5.8.15] is the ∞-category $\mathcal{P}_\Sigma(\mathcal{L}(T)^{\mathrm{op}})$ obtained by freely adjoining sifted colimits to $\mathcal{L}(T)^{\mathrm{op}}$.

Corollary 3.2.6. *The preceding description of ∞-categories of algebras for algebraic monads implies the following:*

(i) *If T is an algebraic monad on $\mathsf{Fun}(X, \mathcal{S})$, then $\mathsf{Alg}(T)$ is a presentable ∞-category.*

(ii) *For any morphism of algebraic monads as in* (3.1)*, the functor* $F^*\colon \mathsf{Alg}(T) \to \mathsf{Alg}(S)$ *has a left adjoint* $F_!$.

(iii) *The left adjoint* $F_!$ *restricts to a functor* $\mathcal{L}(S)^{\mathrm{op}} \to \mathcal{L}(T)^{\mathrm{op}}$*, which gives a morphism of Lawvere theories*

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "p_S"'] & X \arrow[d, "p_T"] \\ \mathcal{L}(S) \arrow[r, "F"] & \mathcal{L}(T). \end{tikzcd}$$

Proof. Part (i) is immediate from Proposition 3.2.4: We know that $\mathsf{Alg}(T)$ is equivalent to $\mathsf{Mod}_{\mathcal{L}(T)}(\mathcal{S})$, which is presentable by Proposition 3.1.6 (i). The functor

$$F^*\colon \mathsf{Alg}(T) \to \mathsf{Alg}(S)$$

preserves limits and sifted colimits by Observation 3.1.9; it is thus a limit-preserving accessible functor between presentable ∞-categories, and therefore admits a left adjoint $F_!$ by [25, Corollary 5.5.2.9]. This proves (ii), and passing to left adjoints in the square (3.1) we get a commutative square

$$\begin{tikzcd} \mathsf{Fun}(Y, \mathcal{S}) \arrow[r, "f_!"] \arrow[d, "F_S"] & \mathsf{Fun}(X, \mathcal{S}) \arrow[d, "F_T"] \\ \mathsf{Alg}(S) \arrow[r, "F_!"] & \mathsf{Alg}(T). \end{tikzcd}$$

Thus we have

$$F_! F_S\left(\coprod_{i=1}^{n} \mathrm{y}_Y(y_i)\right) \simeq F_T f_!\left(\coprod_{i=1}^{n} \mathrm{y}_Y(y_i)\right) \simeq F_T\left(\coprod_{i=1}^{n} \mathrm{y}_X f(y_i)\right),$$

which shows that $F_!$ takes the full subcategory $\mathcal{L}(S)^{\mathrm{op}}$ into $\mathcal{L}(T)^{\mathrm{op}}$, and also that it gives the required commutative square. Moreover, $F_!$ preserves coproducts, being a left adjoint, and so the induced functor $F\colon \mathcal{L}(S) \to \mathcal{L}(T)$ preserves products, since these correspond to coproducts of algebras. ∎

Observation 3.2.7. From Corollary 3.2.6 we see that the assignment of the Lawvere theory $\mathfrak{Th}(T)$ to an algebraic monad T extends to a functor $\mathfrak{Th}\colon \mathsf{AlgMnd} \to \mathsf{LwvTh}$. Moreover, this functor is inverse to the functor $\mathfrak{Mod}$ from Observation 3.1.10: for a Lawvere theory $(\mathcal{L}, q)$ we have a natural equivalence

$$\mathfrak{Th}\big(\mathfrak{Mod}(\mathcal{L}, q)\big) \simeq (\mathcal{L}, q)$$

by Lemma 3.2.3, while for an algebraic monad T we have a natural equivalence

$$\mathfrak{Mod}\big(\mathfrak{Th}(T)\big) \simeq T$$

by Proposition 3.2.4.

We have thus proved the following.

Theorem 3.2.8. *The functors*

$$\mathfrak{Mod} : \mathsf{LwvTh} \rightleftarrows \mathsf{AlgMnd} : \mathfrak{Th}$$

are inverse equivalences of ∞-categories.

Remark 3.2.9. There are many versions of more general "monad/theory correspondences" for ordinary categories in the literature. Versions of these for "monads with arities" have been generalized to the ∞-categorical setting by Henry–Meadows [18] and Kositsyn [22], though they do not explicitly discuss how the situation simplifies in the special case of Lawvere theories.

4. Analytic monads and ∞-operads

We saw in Corollary 2.2.11 that pinned $\mathsf{Span}(\mathbb{F})$-$\infty$-operads give a full subcategory of Lawvere theories over $\mathsf{Span}(\mathbb{F})^{\natural}$. In particular, we obtain an algebraic monad from every pinned $\mathsf{Span}(\mathbb{F})$-$\infty$-operad, and our overall goal in this section is to identify this full subcategory with that coming from a special class of algebraic monads, namely the *analytic* ones. We start by introducing analytic monads in Section 4.1, and then show that any pinned $\mathsf{Span}(\mathbb{F})$-$\infty$-operad gives such an analytic monad in Section 4.2. After this we are ready to study the Lawvere theories of analytic monads in Section 4.3, where we in particular prove Theorem C (which together with our previous work completes the proof of Theorem A).

4.1. Analytic and algebraic monads

In this first subsection we will recall the definition of analytic monads, as introduced in [13], and give a useful description of them, as a full subcategory of a certain slice of algebraic monads.

Definition 4.1.1. A functor $F\colon \mathcal{S}_{/X} \to \mathcal{S}_{/Y}$ is *analytic* if F preserves sifted colimits and weakly contractible limits, that is limits indexed by ∞-categories with contractible classifying space.

Remark 4.1.2. By [13, Theorem 2.2.3 and Proposition 3.1.9] any analytic functor $\mathcal{S}_{/X} \to \mathcal{S}_{/Y}$ is (uniquely) of the form $t_! p_* s^*$ for some diagram of spaces

$$X \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} Y,$$

where the fibres of p are finite sets. This is the *polynomial diagram* corresponding to the functor.

Definition 4.1.3. A natural transformation $\alpha\colon F \to G$ of functors $\mathcal{C} \to \mathcal{D}$ is *cartesian* if for every morphism $\phi\colon x \to y$ in $\mathcal{C}$, the naturality square

$$\begin{tikzcd} Fx \arrow[r, "\alpha_x"] \arrow[d, "F\phi"] & Gx \arrow[d, "G\phi"] \\ Fy \arrow[r, "\alpha_y"] & Gy \end{tikzcd}$$

is cartesian. A *cartesian monad* is a monad whose unit and multiplication transformations are cartesian, and an *analytic monad* is a cartesian monad on an ∞-category of the form $\mathcal{S}_{/X}$ whose underlying endofunctor is analytic.

Remark 4.1.4. An analytic monad can also be defined as a monad in an $(\infty, 2)$-category whose objects are the ∞-categories $\mathrm{Fun}(X, \mathcal{S})$ for $X \in \mathcal{S}$, with analytic functors as morphisms and cartesian natural transformations as 2-morphisms. The appropriate notion of morphisms between analytic monads for us is then certain lax morphisms of monads in this $(\infty, 2)$-category. These can also be described in terms of monadic adjunctions as follows.

Definition 4.1.5. Suppose T is an analytic monad on $\mathrm{Fun}(X, \mathcal{S})$ and S is an analytic monad on $\mathrm{Fun}(Y, \mathcal{S})$. A *morphism of analytic monads* from S to T is a commutative square of ∞-categories

$$\begin{tikzcd} \mathrm{Alg}(T) \arrow[r, "F^*"] \arrow[d, "U_T"] & \mathrm{Alg}(S) \arrow[d, "U_S"] \\ \mathrm{Fun}(X, \mathcal{S}) \arrow[r, "f^*"] & \mathrm{Fun}(Y, \mathcal{S}), \end{tikzcd} \tag{4.1}$$

where the bottom horizontal functor comes from a map of spaces $f\colon Y \to X$, and the induced transformation

$$Sf^* \to f^*T$$

is cartesian. (Since U_S and U_T detect pullbacks, we can equivalently ask for the mate transformation

$$F_S f^* \to F^* F_T \tag{4.2}$$

to be cartesian.)

Definition 4.1.6. We define AnMnd to be the subcategory of AlgMnd whose objects are the analytic monads and whose morphisms are those whose mate transformations (4.2) are cartesian.

The first step towards a more useful description of the ∞-category AnMnd is to identify its terminal object.

Proposition 4.1.7. AnMnd *has a terminal object* Sym*, given by the unique analytic monad structure on the terminal analytic endofunctor on* $\mathcal{S}$*. The latter is the endofunctor*

$$X \mapsto \coprod_{n=0}^{\infty} X^{\times n}_{h\Sigma_n},$$

given by the diagram

$$* \leftarrow \mathbb{F}_*^{\simeq} \to \mathbb{F}^{\simeq} \to *, \tag{4.3}$$

where the map $\mathbb{F}_*^{\simeq} \to \mathbb{F}^{\simeq}$ is given by forgetting the base point; this is the classifying map for morphisms of spaces with finite discrete fibres.[4]

Proof. Restricting [13, Proposition 4.4.2] to analytic monads, we get in particular that the forgetful functor $\mathsf{AnMnd} \to \mathcal{S}$ is a cartesian fibration, which lives in a commutative triangle

$$\begin{tikzcd} \mathsf{AnMnd} \arrow[rr, "\mathfrak{U}"] \arrow[dr] & & \mathsf{AnEnd} \arrow[dl] \\ & \mathcal{S}, & \end{tikzcd}$$

where AnEnd is the ∞-category of analytic endofunctors. Here the functor $\mathsf{AnEnd} \to \mathcal{S}$ is a cartesian and cocartesian fibration, and the forgetful functor $\mathfrak{U}\colon \mathsf{AnMnd} \to \mathsf{AnEnd}$ preserves cartesian morphisms. To show that AnMnd has a terminal object, it suffices to prove that the fibre $\mathsf{AnMnd}(*)$ has a terminal object Sym, and for every morphism $q\colon X \to *$ in $\mathcal{S}$, its cartesian transport $q^*\mathrm{Sym}$ is terminal in $\mathsf{AnMnd}(X)$.

The fibre $\mathsf{AnMnd}(X)$ is the ∞-category of associative algebras in the ∞-category $\mathsf{AnEnd}(X)$ of analytic endofunctors of $\mathsf{Fun}(X, \mathcal{S})$ under composition, and the forgetful functor to $\mathsf{AnEnd}(X)$ detects limits. Moreover, we have a commutative square

$$\begin{tikzcd} \mathsf{AnMnd}(*) \arrow[r, "q^*"] \arrow[d, "\mathfrak{U}_*"] & \mathsf{AnMnd}(X) \arrow[d, "\mathfrak{U}_X"] \\ \mathsf{AnEnd}(*) \arrow[r, "q^*"] & \mathsf{AnEnd}(*) \end{tikzcd}$$

where the lower horizontal functor preserves limits, being a right adjoint, and the vertical functors detect limits. Thus if Sym is a terminal object in $\mathsf{AnMnd}(*)$, then $q^*\mathrm{Sym}$ is terminal in $\mathsf{AnMnd}(X)$ if and only if $\mathfrak{U}_X q^*\mathrm{Sym} \simeq q^*\mathfrak{U}_*\mathrm{Sym}$ is terminal in $\mathsf{AnEnd}(X)$, which is true since q^* and $\mathfrak{U}_*$ preserve limits. Moreover, by [26, Corollary 3.2.2.5] the ∞-category $\mathsf{AnMnd}(*)$ has a terminal object if $\mathsf{AnEnd}(*)$ has one, and this is given by the unique associative algebra structure on this terminal analytic endofunctor. To complete the proof we now observe that [13, Corollary 3.1.13] implies that (4.3) gives the terminal object of $\mathsf{AnEnd}(*)$, as required. ∎

Since AnMnd has a terminal object, the inclusion $\mathsf{AnMnd} \to \mathsf{AlgMnd}$ factors through $\mathsf{AlgMnd}_{/\mathrm{Sym}}$. We now have the following result.

Lemma 4.1.8. *The functor $\mathsf{AnMnd} \to \mathsf{AlgMnd}_{/\mathrm{Sym}}$ is fully faithful.*

[4] Note that the groupoids $\mathbb{F}_*^{\simeq}$ and $\mathbb{F}^{\simeq}$ are both equivalent to $\coprod_{n=0}^{\infty} B\Sigma_n$, and this classifying map can be described as the disjoint union of the maps $B\Sigma_n \to B\Sigma_{n+1}$ induced by the inclusion of Σ_n in Σ_{n+1} as the subgroup of automorphisms that fix one element in the set $\mathbf{n+1}$.

Proof. Suppose we have analytic monads T and S over spaces X and Y, respectively. Then a morphism from S to T in $\mathsf{AlgMnd}_{/\mathrm{Sym}}$ gives a morphism $f\colon Y \to X$, a natural transformation $\alpha\colon Sf^* \to f^*T$, and a commutative triangle

$$\begin{tikzcd} Sp^* \arrow[rr, "\alpha q^*"] \arrow[dr] & & f^*Tq^* \arrow[dl] \\ & p^*\mathrm{Sym}, & \end{tikzcd}$$

where q and p denote the maps $X \to *$, $Y \to *$, respectively. Here the two diagonal transformations are cartesian, since by assumption the two maps to Sym come from AnMnd, and our goal is to prove that α is then cartesian.

The pasting lemma for pullback squares implies that given a commutative triangle of natural transformations

$$\begin{tikzcd} F \arrow[rr, "\phi"] \arrow[dr, "\psi"'] & & G \arrow[dl, "\chi"] \\ & H & \end{tikzcd}$$

where χ is cartesian, the transformation ϕ is cartesian if and only if ψ is cartesian. From the triangle above we can thus conclude that αq^* is cartesian. We can then make a commutative square

$$\begin{tikzcd} Sf^* \arrow[r, "\alpha"] \arrow[d] & f^*T \arrow[d] \\ Sp^*q_! \arrow[r, "\alpha q^* q_!"] & f^*Tq^*q_! \end{tikzcd}$$

using the counit transformation $\mathrm{id} \to q^*q_!$, which is cartesian by [13, Lemma 2.1.5]. Then the vertical maps in the square are cartesian transformations, as is the bottom horizontal map. Using the pasting lemma again, it follows that the top horizontal transformation α is also cartesian. ∎

Observation 4.1.9. From Theorem 3.2.8 it now follows that the functor

$$\mathsf{AnMnd} \hookrightarrow \mathsf{AlgMnd}_{/\mathrm{Sym}} \xrightarrow{\mathfrak{Th}} \mathsf{LwvTh}_{/\mathfrak{Th}(\mathrm{Sym})}$$

identifies AnMnd with a full subcategory $\mathsf{LwvTh}^{\mathrm{an}}_{/\mathfrak{Th}(\mathrm{Sym})}$ of $\mathsf{LwvTh}_{/\mathfrak{Th}(\mathrm{Sym})}$. Over the next few sections we will identify this full subcategory with that of pinned $\mathsf{Span}(\mathbb{F})$-$\infty$-operads.

Remark 4.1.10. The ∞-category AnMnd can also be seen as a (non-full) subcategory of AlgMnd, which means that it also corresponds to some subcategory of LwvTh. It would be interesting to know if this has an explicit description. In the 1-categorical setting, Szawiel and Zawadowski give such a description of the Lawvere theories of one-sorted analytic monads in [29].

4.2. Analytic monads from $\mathsf{Span}(\mathbb{F})$-$\infty$-operads

In this subsection we will describe the monad corresponding to a pinned $\mathsf{Span}(\mathbb{F})$-$\infty$-operad, and show that this is always analytic. Using this, we will then identify the ∞-category $\mathsf{POpd}(\mathsf{Span}(\mathbb{F}))$ with a full subcategory of AnMnd.

Notation 4.2.1. Suppose $\mathcal{O}$ is a $\mathsf{Span}(\mathbb{F})$-$\infty$-operad. We write $U_{\mathcal{O}}$ for the restriction functor

$$\mathsf{Mon}_{\mathcal{O}}(\mathcal{S}) \hookrightarrow \mathsf{Fun}(\mathcal{O}, \mathcal{S}) \to \mathsf{Fun}(\mathcal{O}_{\mathbf{1}}^{\simeq}, \mathcal{S})$$

and $F_{\mathcal{O}}$ for its left adjoint. Note that $(\mathcal{O}, \mathcal{O}_{\mathbf{1}}^{\simeq})$ is a Lawvere theory by Lemma 2.2.7, and by Proposition 2.3.3 the ∞-category $\mathsf{Mon}_{\mathcal{O}}(\mathcal{S})$ is the ∞-category of models for this Lawvere theory. Thus $U_{\mathcal{O}}$ is the monadic right adjoint for an algebraic monad by Proposition 3.1.6; we write $T_{\mathcal{O}}$ for this monad.

Proposition 4.2.2. *Suppose $\mathcal{O}$ is an $\mathsf{Span}(\mathbb{F})$-$\infty$-operad. Then $T_{\mathcal{O}}$ is an analytic monad, and its underlying endofunctor is given on $F \in \mathsf{Fun}(\mathcal{O}_{\mathbf{1}}^{\simeq}, \mathcal{S})$ by the formula*

$$(T_{\mathcal{O}} F)(x) \simeq \operatorname{colim}_{y \rightsquigarrow x \in \mathsf{Act}_{\mathcal{O}}(x)} \prod_{y \rightarrowtail y_i} F(y_i),$$

where $\mathsf{Act}_{\mathcal{O}}(x) := (\mathcal{O}^{\mathrm{act}}_{/x})^{\simeq}$ denotes the ∞-groupoid of active maps to x in $\mathcal{O}$ and the limit is over the set of inert maps $y \rightarrowtail y_i$ over ρ_i'. Moreover, any morphism of $\mathsf{Span}(\mathbb{F})$-$\infty$-operads induces a morphism of analytic monads.

Proof. In this proof we use some terminology from [7]. The ∞-category $\mathcal{O}$ equipped with its inert-active factorization system and the objects over $\mathbf{1}$ is an *algebraic pattern*, and $\mathsf{Mon}_{\mathcal{O}}(\mathcal{S})$ is the ∞-category of Segal $\mathcal{O}$-spaces for this pattern.

The algebraic pattern $\mathsf{Span}(\mathbb{F})$ is *extendable* by [2, Example 3.3.25]. It therefore follows from [7, Corollary 9.17] (or [2, Lemma 4.1.15]) that any $\mathsf{Span}(\mathbb{F})$-$\infty$-operad is also extendable; the formula for $T_{\mathcal{O}}$ is then a special case of [7, Corollary 8.12].

It then follows from [7, Proposition 10.6] that the monad $T_{\mathcal{O}}$ is cartesian and preserves weakly contractible limits. Since we already know that $T_{\mathcal{O}}$ is algebraic, i.e., preserves sifted colimits, this shows that $T_{\mathcal{O}}$ is an analytic monad. A morphism of $\mathsf{Span}(\mathbb{F})$-$\infty$-operads then induces a morphism of analytic monads by [7, Proposition 10.10]. ∎

Corollary 4.2.3. *Suppose $(\mathcal{O}, q\colon X \twoheadrightarrow \mathcal{O}_{\mathbf{1}}^{\simeq})$ is a pinned $\mathsf{Span}(\mathbb{F})$-$\infty$-operad. Then the composite*

$$\mathsf{Mon}_{\mathcal{O}}(\mathcal{S}) \xrightarrow{U_{\mathcal{O}}} \mathsf{Fun}(\mathcal{O}_{\mathbf{1}}^{\simeq}, \mathcal{S}) \xrightarrow{q^*} \mathsf{Fun}(X, \mathcal{S})$$

is the monadic right adjoint for an analytic monad $T_{(\mathcal{O},q)}$. Moreover, any morphism of pinned $\mathsf{Span}(\mathbb{F})$-$\infty$-operads induces a morphism of analytic monads.

Proof. The endofunctor $T_{(\mathcal{O},q)}$ is the composite $q^* T_{\mathcal{O}} q_!$. Here the functor q^* preserves all limits and colimits, and $q_!$ preserves all colimits (being a left adjoint) as well as weakly contractible limits (by [13, Lemma 2.2.10]), hence $T_{(\mathcal{O},q)}$ is an analytic functor since $T_{\mathcal{O}}$

is one. Moreover, the multiplication and unit transformations for $T_{(\mathcal{O},q)}$ are given by the compositions

$$q^*T_{\mathcal{O}}q_!q^*T_{\mathcal{O}}q_! \to q^*T_{\mathcal{O}}T_{\mathcal{O}}q_! \to q^*T_{\mathcal{O}}q_!, \quad \mathrm{id} \to q^*q_! \to q^*T_{\mathcal{O}}q_!$$

of the corresponding transformations for $T_{\mathcal{O}}$ and the counit and unit of the adjunction $q_! \dashv q^*$. The latter are cartesian transformations by [13, Lemma 2.1.5], hence so are these compositions, since we know $T_{\mathcal{O}}$ is a cartesian monad. Thus $T_{(\mathcal{O},q)}$ is indeed an analytic monad. Similarly, the natural transformation associated to a morphism of pinned $\mathsf{Span}(\mathbb{F})$-$\infty$-operads is built from that coming from the underlying morphism of $\mathsf{Span}(\mathbb{F})$-$\infty$-operads and cartesian (co)unit transformations, so that we get a morphism of analytic monads. ■

The monad $T_{(\mathcal{O},q)}$ for a pinned $\mathsf{Span}(\mathbb{F})$-$\infty$-operad $(\mathcal{O},q)$ is by definition the monad corresponding to the Lawvere theory $(\mathcal{O},q)$. We next observe that the monad corresponding to the terminal (pinned) $\mathsf{Span}(\mathbb{F})$-$\infty$-operad is the terminal analytic monad.

Lemma 4.2.4. *The monad $T_{\mathsf{Span}(\mathbb{F})^\natural}$ corresponding to the Lawvere theory $\mathsf{Span}(\mathbb{F})^\natural$ is* Sym. *Equivalently, the Lawvere theory $\mathfrak{T}\mathfrak{h}(\mathrm{Sym})$ is $\mathsf{Span}(\mathbb{F})^\natural$.*

Proof. We know from Proposition 4.1.7 that Sym is the unique analytic monad structure on its underlying endofunctor on $\mathcal{S}$, which is

$$X \mapsto \coprod_{n=0}^{\infty} X^{\times n}_{h\Sigma_n}.$$

It thus suffices to show that this is also the underlying endofunctor of the analytic monad $T_{\mathsf{Span}(\mathbb{F})}$. This follows from the formula in Proposition 4.2.2 since we have $\mathsf{Act}_{\mathsf{Span}(\mathbb{F})}(\mathbf{1}) \simeq \coprod_{n=0}^{\infty} B\Sigma_n$, giving

$$T_{\mathsf{Span}(\mathbb{F})}(X) \simeq \operatorname{colim}_{\mathbf{n}\rightsquigarrow\mathbf{1}\in\mathsf{Act}_{\mathsf{Span}(\mathbb{F})}(\mathbf{1})} X^{\times n} \simeq \coprod_{n=0}^{\infty} X^{\times n}_{h\Sigma_n}. \qquad ■$$

We have thus shown that both $\mathsf{POpd}(\mathsf{Span}(\mathbb{F}))$ and AnMnd give full subcategories of $\mathsf{LwvTh}_{/\mathsf{Span}(\mathbb{F})^\natural}$, and that the former subcategory is contained in the latter.

Corollary 4.2.5. *$\mathsf{POpd}(\mathsf{Span}(\mathbb{F}))$ is a full subcategory of $\mathsf{LwvTh}^{\mathrm{an}}_{/\mathsf{Span}(\mathbb{F})^\natural} \simeq \mathsf{AnMnd}$.* ■

4.3. Lawvere theories of analytic monads

In the previous subsection we showed that we have a fully faithful functor

$$\mathsf{POpd}(\mathsf{Span}(\mathbb{F})) \hookrightarrow \mathsf{LwvTh}^{\mathrm{an}}_{/\mathsf{Span}(\mathbb{F})^\natural}.$$

Our goal in this subsection is to prove that this is also essentially surjective. In other words, we want to show that the Lawvere theory of *any* analytic monad is a pinned $\mathsf{Span}(\mathbb{F})$-$\infty$-operad. To show this, we will first describe an inert–active factorization system on such

Lawvere theories. This arises from a factorization system on the entire Kleisli ∞-category, which was constructed in [7]. We begin by recalling this, which requires first introducing some terminology:

Definition 4.3.1. A functor $F: \mathcal{C} \to \mathcal{D}$ is a *local right adjoint* if for every object $C \in \mathcal{C}$ the induced functor on slices $F_{/C}: \mathcal{C}_{/C} \to \mathcal{D}_{/FC}$ is a right adjoint.

Observation 4.3.2. By [13, Theorem 2.2.3], a functor $F: \mathsf{Fun}(X, \mathcal{S}) \to \mathsf{Fun}(Y, \mathcal{S})$ is a local right adjoint if and only if it is accessible and preserves weakly contractible limits. In particular, every analytic functor is a local right adjoint.

Observation 4.3.3. For an analytic functor $F: \mathsf{Fun}(X, \mathcal{S}) \to \mathsf{Fun}(Y, \mathcal{S})$ we can describe the left adjoint L_* of $F_{/*}$ quite explicitly: Recall that an analytic functor F is of the form $t_! p_* s^*$ for a unique bispan of spaces

$$X \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} Y$$

where p has finite discrete fibres. Here $t: B \to Y$ corresponds to $F(*)$ under the straightening equivalence $\mathsf{Fun}(Y, \mathcal{S}) \simeq \mathcal{S}_{/Y}$, since $t_!$ is given on slices by composition with t and $p_* s^*$ preserves the terminal object. Thus $F_{/*}$ can be identified with the functor

$$p_* s^*: \mathsf{Fun}(X, \mathcal{S}) \to \mathsf{Fun}(B, \mathcal{S}),$$

with the left adjoint L_* being $s_! p^*$. In particular, a map $\mathsf{y}_Y(y) \to F(*)$ corresponds to a point $b \in B$ over $y \in Y$, and $L_* \mathsf{y}_Y(y)$ corresponds to $s_! p^* b$, which is the composite $E_b \hookrightarrow E \xrightarrow{s} X$. Since E_b is a finite set, we have

$$L_*\big(\mathsf{y}_Y(y) \to F(*)\big) \simeq \coprod_{e \in E_b} \mathsf{y}\big(s(e)\big). \tag{4.4}$$

Definition 4.3.4. For any monad T on some ∞-category $\mathcal{C}$, we write $\mathcal{K}(T)$ for the *Kleisli ∞-category* of T, defined as the full subcategory of $\mathsf{Alg}(T)$ containing the free algebras $F_T C$ for $C \in \mathcal{C}$.

Definition 4.3.5. Suppose T is an analytic monad on $\mathsf{Fun}(X, \mathcal{S})$, and consider a morphism $\phi: F_T I \to F_T J$ in $\mathcal{K}(T)$. Under the adjunction $F_T \dashv U_T$ this corresponds to a morphism $\phi': I \to TJ$ in $\mathsf{Fun}(X, \mathcal{S})$, which we can regard as a morphism in $\mathsf{Fun}(X, \mathcal{S})_{/T*}$ via the image $TJ \to T*$ under T of the unique map from J to the terminal object. Then ϕ' corresponds to a morphism $\phi'': L_* I \to J$ in $\mathsf{Fun}(X, \mathcal{S})$, where L_* is the left adjoint to $T_{/*}$. The unit of the adjunction $L_* \dashv T_{/*}$ then gives a factorization of ϕ' as

$$J \to T L_* J \xrightarrow{T\phi''} TI,$$

which is adjoint to a factorization of ϕ as

$$F_T J \to F_T L_* J \xrightarrow{F_T \phi''} F_T I.$$

We call this the *canonical factorization* of ϕ, and say that ϕ is *inert* if the map $F_T J \to F_T L_* J$ is an equivalence, and *active* if $F_T \phi''$ is an equivalence.

Theorem 4.3.6. *If T is an analytic monad, then the active and inert morphisms form a factorization system on $\mathcal{K}(T)$, and the canonical factorization is an active–inert factorization. Moreover, for any morphism of analytic monads (4.1), the functor $F_!\colon \mathcal{K}(S) \to \mathcal{K}(T)$ preserves active and inert morphisms.*

Proof. That active and inert morphisms form a factorization system is a special case of [7, Theorem 12.1], while their preservation by morphisms of analytic monads follows from the same argument as in the proof of [7, Lemma 11.18]. ∎

Corollary 4.3.7. *Let T be an analytic monad on $\mathsf{Fun}(X, \mathcal{S})$. The inert–active factorization system on $\mathcal{K}(T)^{\mathrm{op}}$ restricts to the Lawvere theory $\mathcal{L}(T)$. Moreover, for any morphism of analytic monads (4.1), the functor $F\colon \mathcal{L}(S) \to \mathcal{L}(T)$ preserves inert and active morphisms.*

Proof. Given a morphism $F_T(\coprod_{i=1}^n \mathsf{y}_X(x_i)) \to F_T(\coprod_{j=1}^m \mathsf{y}_X(y_j))$, its canonical factorization is

$$F_T\Big(\coprod_{i=1}^n \mathsf{y}_X(x_i)\Big) \to F_T L_*\Big(\coprod_{i=1}^n \mathsf{y}_X(x_i)\Big) \to F_T\Big(\coprod_{j=1}^m \mathsf{y}_X(y_j)\Big)$$

where $L_*(\coprod_{i=1}^n \mathsf{y}_X(x_i))$ is defined with respect to the composite

$$\coprod_{i=1}^n \mathsf{y}_X(x_i) \to T\Big(\coprod_{j=1}^m \mathsf{y}_X(y_j)\Big) \to T(*)$$

of the adjoint map and T applied to the unique map $\coprod_{j=1}^m \mathsf{y}_X(y_j) \to *$. It thus suffices to check that $L_*(\coprod_{i=1}^n \mathsf{y}_X(x_i))$ is again a finite coproduct of representables for any map $\coprod_{i=1}^n \mathsf{y}_X(x_i) \to T(*)$. This follows from (4.4) in the case $n = 1$ and so in general since we have $L_*(\coprod_{i=1}^n \mathsf{y}_X(x_i)) \simeq \coprod_{i=1}^n L_*\mathsf{y}_X(x_i)$ as L_* is a left adjoint. The preservation of active and inert morphisms then follows from Theorem 4.3.6. ∎

Proposition 4.3.8. *Consider a morphism of analytic monads (4.1), and let $F_!$ be the left adjoint to F^* (which exists by Corollary 3.2.6). Then:*

(i) *Every inert morphism in $\mathcal{K}(S)$ is $F_!$-cartesian.*

(ii) *Given $F_S I \in \mathcal{K}(S)$ and a morphism $\psi\colon J \to f_! I$ in $\mathsf{Fun}(X, \mathcal{S})$, there exists an $F_!$-cartesian morphism over $F_T\psi$ in $\mathcal{K}(T)$, given by $F_S\phi$ where ϕ is determined by the pullback square*

$$\begin{tikzcd} K \arrow[r, "\phi"] \arrow[d] & I \arrow[d] \\ f^*J \arrow[r, "f^*\psi"] & f^* f_! I \end{tikzcd}$$

in $\mathsf{Fun}(Y, \mathcal{S})$.

(iii) *A morphism in $\mathcal{K}(S)$ is inert if and only if it is cartesian over an inert morphism in $\mathcal{K}(T)$.*

Proof. Every inert morphism in $\mathcal{K}(S)$ is a composite of a free morphism and an equivalence. Since equivalences are always cartesian, to prove (i) it suffices to show that free morphisms in $\mathcal{K}(S)$ are cartesian. Given a morphism $\phi\colon I \to J$ in $\mathsf{Fun}(Y, \mathcal{S})$, we must show that for every object $K \in \mathsf{Fun}(Y, \mathcal{S})$ the commutative square

$$\begin{tikzcd} \mathsf{Map}_{\mathsf{Alg}(S)}(F_S K, F_S I) \arrow[r, "(F_S\phi)_*"] \arrow[d] & \mathsf{Map}_{\mathsf{Alg}(S)}(F_S K, F_S J) \arrow[d] \\ \mathsf{Map}_{\mathsf{Alg}(T)}(F_! F_S K, F_! F_S I) \arrow[r, "(F_! F_S\phi)_*"] & \mathsf{Map}_{\mathsf{Alg}(T)}(F_! F_S K, F_! F_S J) \end{tikzcd}$$

is cartesian. Using the natural equivalence

$$F_! F_S \simeq F_T f_!$$

and the various adjunctions involved, we identify this with the commutative square

$$\begin{tikzcd} \mathsf{Map}_{\mathsf{Fun}(Y,\mathcal{S})}(K, SI) \arrow[r, "(S\phi)_*"] \arrow[d] & \mathsf{Map}_{\mathsf{Fun}(Y,\mathcal{S})}(K, SJ) \arrow[d] \\ \mathsf{Map}_{\mathsf{Fun}(Y,\mathcal{S})}(K, f^*Tf_!I) \arrow[r, "(f^*Tf_!\phi)_*"] & \mathsf{Map}_{\mathsf{Fun}(Y,\mathcal{S})}(K, f^*Tf_!J). \end{tikzcd}$$

This is cartesian for all K if and only if the commutative square

$$\begin{tikzcd} SI \arrow[r, "S\phi"] \arrow[d] & SJ \arrow[d] \\ f^*Tf_!I \arrow[r, "f^*Tf_!\phi"] & f^*Tf_!J \end{tikzcd}$$

is a cartesian square in $\mathsf{Fun}(Y, \mathcal{S})$. Now this square is indeed cartesian since it is a naturality square for the natural transformation $S \to Sf^*f_! \to f^*Tf_!$ which is cartesian since all the functors involved preserve pullbacks and the unit transformation $\mathrm{id} \to f^*f_!$ and the transformation $Sf^* \to f^*T$ are both cartesian. This proves (i).

To prove (ii), first define $\phi\colon K \to I$ by the pullback square

$$\begin{tikzcd} K \arrow[r, "\phi"] \arrow[d] & I \arrow[d] \\ f^*J \arrow[r, "f^*\psi"] & f^*f_!I. \end{tikzcd}$$

Then consider the commutative diagram

$$\begin{tikzcd} f_!K \arrow[r, "f_!\phi"] \arrow[d] & f_!I \arrow[d] \arrow[dd, bend left, equal] \\ f_!f^*J \arrow[r, "f_!f^*\psi"] \arrow[d] & f_!f^*f_!I \arrow[d] \\ J \arrow[r, "\psi"] & f_!I \end{tikzcd}$$

in $\mathsf{Fun}(X, \mathcal{S})$. Here the bottom square is cartesian since the counit $f_! f^* \to \mathrm{id}$ is cartesian, and the top square is cartesian since $f_!$ preserves pullbacks. The composite square is then also cartesian, and since the right vertical composite is the identity, it follows that the map $f_! K \to J$ is an equivalence. The composite square thus gives an equivalence between $f_!\phi$ and ψ. The free morphism $F_S\phi$ is then cartesian over $F_T f_!\phi \simeq F_T\psi$ by part (i), which proves (ii).

To prove (iii), it remains to prove that every cartesian morphism in $\mathcal{K}(S)$ that lies over an inert morphism in $\mathcal{K}(T)$ is inert. Since inert morphisms are composites of free morphisms and equivalences, we may assume the morphism in $\mathcal{K}(T)$ is free. Then the construction in (ii) gives a cartesian morphism with the same image in $\mathcal{K}(T)$ that is manifestly inert. Since cartesian morphisms are unique when they exist, this completes the proof. ■

Restricting this to Lawvere theories, we have the following.

Corollary 4.3.9. *Consider a morphism of analytic monads* (4.1)*, and let* $F\colon \mathcal{L}(S) \to \mathcal{L}(T)$ *be the induced morphism of Lawvere theories. Then:*

(i) *$\mathcal{L}(S)$ has F-cocartesian morphisms over inert morphisms in $\mathcal{L}(T)$, and these are precisely the inert morphisms in $\mathcal{L}(S)$.*

(ii) *A morphism in $\mathcal{L}(S)$ is active if and only if it lies over an active morphism in $\mathcal{L}(T)$.*

Proof. By Proposition 4.3.8 every inert morphism in $\mathcal{K}(S)$ is cartesian over $\mathcal{K}(T)$. Restricting to the full subcategories $\mathcal{L}(S)^{\mathrm{op}}$ and $\mathcal{L}(T)^{\mathrm{op}}$, we get after taking op that every inert morphism in $\mathcal{L}(S)$ is cocartesian over $\mathcal{L}(T)$. To show that $\mathcal{L}(S)$ has cocartesian morphisms over inert maps in $\mathcal{L}(T)$, it is enough to check that given $\Xi := \coprod_{i=1}^{n} \mathsf{y}_Y(y_i)$ and a morphism

$$\psi\colon \coprod_{j=1}^{m} \mathsf{y}_X(x_j) \to f_!\Xi \simeq \coprod_{i=1}^{n} \mathsf{y}_X f(y_i)$$

in $\mathsf{Fun}(X, \mathcal{S})$, the cartesian morphism over ψ from Proposition 4.3.8 (ii) lies in the full subcategory $\mathcal{L}(S)^{\mathrm{op}}$. First observe that the map ψ amounts to specifying a morphism $h\colon \mathbf{m} \to \mathbf{n}$ and equivalences $x_j \simeq f(y_{h(j)})$ in X. In the cartesian square

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \coprod_{i=1}^{n} \mathsf{y}(y_i) \arrow[d] \\ f^* \coprod_{j=1}^{m} \mathsf{y}_X(x_j) \arrow[r] & f^* \coprod_{i=1}^{n} \mathsf{y}_X f(y_i), \end{tikzcd}$$

that induces the cartesian morphism in $\mathcal{K}(S)$, the object K can therefore be identified with $\coprod_{j=1}^{m} \mathsf{y}(y_{h(j)})$, using that coproducts are disjoint and colimits are universal in $\mathsf{Fun}(Y, \mathcal{S})$. To prove (i) it remains to show that *every* morphism in $\mathcal{L}(S)$ that is cocartesian over an inert map in $\mathcal{L}(T)$ is inert; this is true since cocartesian morphisms are unique when they exist and the construction in Proposition 4.3.8 (ii) gives a cocartesian morphism that *is* inert.

Since $\mathcal{L}(S)$ has cocartesian morphisms over inert maps in $\mathcal{L}(T)$, we can lift the inert–active factorization system on $\mathcal{L}(T)$ to a factorization system on $\mathcal{L}(S)$, whereby a morphism factors as a cocartesian morphism over an inert map in $\mathcal{L}(T)$ followed by a morphism that maps to an active map in $\mathcal{L}(T)$. By (i), the first class is precisely that of inert morphisms in $\mathcal{L}(S)$. But one class in a factorization system is completely determined by the other by [25, Proposition 5.2.8.11], so this means that the active morphisms in $\mathcal{L}(S)$ must be exactly those that lie over active morphisms in $\mathcal{L}(T)$. ■

Lemma 4.3.10. *Under the identification* $\mathsf{Span}(\mathbb{F}) \simeq \mathcal{L}(\mathrm{Sym})$*, the inert–active factorization system on* $\mathsf{Span}(\mathbb{F})$ *(as in Definition 2.1.4) corresponds to that from Corollary 4.3.7.*

Proof. We will show that both $\mathsf{Span}(\mathbb{F})$ and $\mathcal{L}(\mathrm{Sym})$ can be identified with the canonical pattern $\mathcal{W}(\mathrm{Sym})$ associated to the monad Sym in [7, Section 13]. There $\mathcal{W}(\mathrm{Sym})^{\mathrm{op}}$ is defined as the full subcategory of $\mathsf{Alg}(\mathrm{Sym}) \simeq \mathsf{Mon}_{\mathsf{Span}(\mathbb{F})}(\mathcal{S})$ of free algebras on objects on the full subcategory $\mathcal{U}(\mathrm{Sym})^{\mathrm{op}} \subseteq \mathcal{S}$ consisting of objects of the form $L_*(p)$ for morphisms $p \colon * \to \mathrm{Sym}(*)$. By (4.4) these objects are precisely the finite coproducts of the point, i.e., the finite sets, so that $\mathcal{W}(\mathrm{Sym})^{\mathrm{op}}$ is precisely $\mathcal{L}(\mathrm{Sym})^{\mathrm{op}}$. The factorization system on both is obtained by restricting that on $\mathcal{K}(\mathrm{Sym})$, and both have the free algebra on a point as their unique elementary object.

It remains to show that the canonical pattern $\mathcal{W}(\mathrm{Sym})$ is in fact $\mathsf{Span}(\mathbb{F})$. For this we apply [7, Corollary 14.18]. As already observed, the pattern $\mathsf{Span}(\mathbb{F})$ is *extendable* by [2, Example 3.3.25], and by inspection every object of $\mathsf{Span}(\mathbb{F})$ admits a (unique) active morphism to the elementary object $\mathbf{1}$. The only condition left to check is then that the pattern $\mathsf{Span}(\mathbb{F})$ is *saturated*, meaning that for every object $S \in \mathsf{Span}(\mathbb{F})$, the functor $\mathsf{Map}_{\mathsf{Span}(\mathbb{F})}(S, -) \colon \mathsf{Span}(\mathbb{F}) \to \mathcal{S}$ is an $\mathsf{Span}(\mathbb{F})$-monoid. This amounts to the functor taking disjoint unions of sets to products, or in other words that the disjoint union is the cartesian product in $\mathsf{Span}(\mathbb{F})$, which we saw in Lemma 2.2.5. ■

Proposition 4.3.11. *Let T be an analytic monad on* $\mathsf{Fun}(X, \mathcal{S})$*. Then the morphism of Lawvere theories* $\mathcal{L}(T) \to \mathsf{Span}(\mathbb{F})$*, corresponding to the unique morphism of analytic monads*

$$\begin{array}{ccc} \mathsf{Alg}(\mathrm{Sym}) & \xrightarrow{Q^*} & \mathsf{Alg}(T) \\ {\scriptstyle U_{\mathrm{Sym}}}\downarrow & & \downarrow{\scriptstyle U_T} \\ \mathcal{S} & \xrightarrow{q^*} & \mathsf{Fun}(X, \mathcal{S}) \end{array}$$

to Sym*, is a* $\mathsf{Span}(\mathbb{F})$*-$\infty$-operad. Moreover, the functor $p \colon X \to \mathcal{L}(T)$ makes $\mathfrak{Th}(T) = (\mathcal{L}(T), p)$ a pinned* $\mathsf{Span}(\mathbb{F})$*-$\infty$-operad.*

Proof. We check the conditions (1′)–(3′) in Proposition 2.2.6. Condition (1′) follows from Corollary 4.3.9 and the identification of the inert morphisms in the ∞-category $\mathsf{Span}(\mathbb{F})$ from Lemma 4.3.10. The pullback square

$$\begin{array}{ccc} F_T x_i & \longrightarrow & F_T(x_1, \ldots, x_n) \\ \downarrow & & \downarrow \\ F_T q^*\{i\} & \longrightarrow & F_T q^* \mathbf{n} \end{array}$$

in $\mathcal{K}(T)$ shows, by the description of the cartesian morphisms in Proposition 4.3.8, that the inclusion of the factor $F_T x_i$ in the coproduct $F_T(x_1, \ldots, x_n) \simeq \coprod_{i=1}^n F_T x_i$ is cartesian over the inclusion $\{i\} \hookrightarrow \mathbf{n}$. Over in $\mathcal{L}(T)$, this means that the cocartesian morphisms $F_T(x_1, \ldots, x_n) \to F_T(x_i)$ over ρ_i exhibit $F_T(x_1, \ldots, x_n)$ as the product $\prod_{i=1}^n F_T x_i$, which is precisely condition (2′). Moreover, given objects $F_T x_i$ for $i = 1, \ldots, n$, the same argument shows that the object $F_T(x_1, \ldots, x_n)$ in $\mathcal{L}(T)_{\mathbf{n}}$ is sent to $(F_T(x_1), \ldots, F_T(x_n))$ in $\prod_{i=1}^n \mathcal{L}(T)_{\mathbf{1}}$ under cocartesian transport along the maps ρ_i. This gives condition (3′), which completes the proof that $\mathcal{L}(T)$ is a $\mathsf{Span}(\mathbb{F})$-$\infty$-operad.

We have a commutative square

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d] & \mathcal{L}(T) \arrow[d] \\ \{\mathbf{1}\} \arrow[r] & \mathsf{Span}(\mathbb{F}), \end{tikzcd}$$

which shows that p factors through a morphism $X \to \mathcal{L}(T)_{\mathbf{1}}^{\simeq}$. It remains to show that this is essentially surjective. Every object of $\mathcal{L}(T)$ is a product of objects in the image of p, but since the functor to $\mathsf{Span}(\mathbb{F})$ preserves products, these products cannot lie over $\mathbf{1}$ if they have more than a single factor. Thus the only objects of $\mathcal{L}(T)$ that map to $\mathbf{1}$ are those in the image of p, as required. ∎

Proposition 4.3.11 shows that the Lawvere theory of any analytic monad is a pinned $\mathsf{Span}(\mathbb{F})$-$\infty$-operad. We already saw in Corollary 4.2.5 that $\mathsf{POpd}(\mathsf{Span}(\mathbb{F}))$ was a full subcategory of $\mathsf{LwvTh}^{\mathrm{an}}_{/\mathsf{Span}(\mathbb{F})^\natural}$, so this means this subcategory is actually the entire ∞-category $\mathsf{LwvTh}^{\mathrm{an}}_{/\mathsf{Span}(\mathbb{F})^\natural}$. Combining this with Theorem 3.2.8 and Observation 4.1.9, we get the following.

Corollary 4.3.12. *We have mutually inverse equivalences*

$$\mathfrak{Mod} : \mathsf{POpd}(\mathsf{Span}(\mathbb{F})) \rightleftarrows \mathsf{AnMnd} : \mathfrak{Th}$$

induced by those of Theorem 3.2.8.

5. ∞-operads and complete dendroidal Segal spaces

Our work so far gives an equivalence

$$\mathsf{POpd}\big(\mathsf{Span}(\mathbb{F})\big) \simeq \mathsf{AnMnd}.$$

On the other hand, in [13] we showed that AnMnd is equivalent to $\mathsf{Seg}_{\mathbf{\Omega}^{\mathrm{op}}}(\mathcal{S})$, the ∞-category of *dendroidal Segal spaces*. Our goal in this section is to prove Theorem D, i.e., to show that the resulting composite equivalence between $\mathsf{POpd}(\mathsf{Span}(\mathbb{F}))$ and $\mathsf{Seg}_{\mathbf{\Omega}^{\mathrm{op}}}(\mathcal{S})$ restricts to an equivalence between $\mathsf{Opd}(\mathsf{Span}(\mathbb{F}))$, which we identify as the full subcategory of $\mathsf{POpd}(\mathsf{Span}(\mathbb{F}))$ spanned by those pinned $\mathsf{Span}(\mathbb{F})$-$\infty$-operads $(\mathcal{O}, \alpha\colon X \twoheadrightarrow \mathcal{O}_{\mathbf{1}}^{\simeq})$

where the map α is an equivalence, and the full subcategory of *complete* dendroidal Segal spaces, meaning those dendroidal Segal spaces whose underlying Segal space is complete in the sense of Rezk [27].

We can regard (pinned) ∞-categories as (pinned) ∞-operads with only unary operations, and similarly we can regard Segal spaces over $\mathbf{\Delta}$ as a full subcategory of dendroidal Segal spaces (whose values at all non-unary corollas are $\emptyset$). Most of this section is concerned with showing that our equivalence and that of [13] restricts to an equivalence between these full subcategories, thus producing the following commutative diagram:

$$\begin{tikzcd} \mathsf{POpd}(\mathsf{Span}(\mathbb{F})) \arrow[r, "\sim", "4.3.12"'] & \mathsf{AnMnd} & \mathsf{Seg}_{\mathbf{\Omega}^{\mathrm{op}}}(\mathcal{S}) \arrow[l, "\sim"', "{[13]}"] \\ \mathsf{PCat}_\infty \arrow[u, hook] \arrow[r, "\sim", "5.1.15"'] & \mathsf{LinMnd} \arrow[u, hook] & \mathsf{Seg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{S}) \arrow[u, hook] \arrow[l, "\sim"', "5.2.6"] \end{tikzcd} \tag{5.1}$$

Here:

- PCat_∞ is the ∞-category of *pinned ∞-categories*, meaning ∞-categories $\mathcal{C}$ equipped with essentially surjective maps of ∞-groupoids $X \twoheadrightarrow \mathcal{C}^{\simeq}$.
- The fully faithful inclusion $\mathsf{PCat}_\infty \hookrightarrow \mathsf{POpd}(\mathsf{Span}(\mathbb{F}))$ is left adjoint to the functor given by

$$(\mathcal{O}, p\colon X \twoheadrightarrow \mathcal{O}_{\mathbf{1}}^{\simeq}) \mapsto (\mathcal{O}_{\mathbf{1}}, p).$$

- LinMnd is the full subcategory of AnMnd containing the *linear monads* (which preserve all colimits).
- The fully faithful inclusion $\mathsf{Seg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{S}) \hookrightarrow \mathsf{Seg}_{\mathbf{\Omega}^{\mathrm{op}}}(\mathcal{S})$ is given by left Kan extension along the inclusion $\mathbf{\Delta}^{\mathrm{op}} \hookrightarrow \mathbf{\Omega}^{\mathrm{op}}$, and so is left adjoint to the underlying Segal space functor.

We begin by constructing the left-hand square in Section 5.1, which amounts to understanding the restriction of the equivalence $\mathsf{POpd}(\mathsf{Span}(\mathbb{F})) \simeq \mathsf{AnMnd}$ to the full subcategory of pinned ∞-operads with only unary operations, which we identify with pinned ∞-categories. We then construct the right-hand square in Section 5.2; this requires proving that the equivalence between analytic monads and dendroidal Segal spaces from [13] restricts to an equivalence between linear monads and Segal spaces on $\mathbf{\Delta}^{\mathrm{op}}$. Finally, in Section 5.3, we show that the composite equivalence between pinned ∞-categories and Segal spaces restricts to one between ∞-categories and complete Segal spaces, and then extend this to the operadic setting.

5.1. Linear monads and ∞-categories

Any ∞-category can be regarded as an ∞-operad with only unary operations. In this subsection we will first describe this embedding concretely in terms of $\mathsf{Span}(\mathbb{F})$-$\infty$-operads, and then identify the full subcategory of AnMnd that corresponds to (pinned) ∞-categories when we restrict the equivalence of Corollary 4.3.12.

Definition 5.1.1. For $\mathcal{C} \in \mathsf{Cat}_\infty$, we define $\mathbb{F}^{\mathrm{op}}_{\mathcal{C}} \to \mathbb{F}^{\mathrm{op}}$ as the cocartesian fibration corresponding to the functor

$$j_*\mathcal{C}\colon \mathbb{F}^{\mathrm{op}} \to \mathsf{Cat}_\infty$$

obtained by right Kan extension of $\mathcal{C}$ along the inclusion $j\colon \{\mathbf{1}\} \hookrightarrow \mathbb{F}^{\mathrm{op}}$. The functor $j_*\mathcal{C}$ is then given by

$$j_*\mathcal{C}(\mathbf{n}) \simeq \mathsf{Fun}(\mathbf{n}, \mathcal{C}) \simeq \mathcal{C}^{\times n}.$$

Lemma 5.1.2. *The composite* $\mathbb{F}^{\mathrm{op}}_{\mathcal{C}} \to \mathbb{F}^{\mathrm{op}} \to \mathsf{Span}(\mathbb{F})$ *is a* $\mathsf{Span}(\mathbb{F})$*-$\infty$-operad, and* $\mathbb{F}^{\mathrm{op}}_{(-)}$ *gives a functor* $\mathsf{Cat}_\infty \to \mathsf{Opd}(\mathsf{Span}(\mathbb{F}))$.

Proof. We can regard $\mathbb{F}^{\mathrm{op}}_{(-)} \to \mathsf{Span}(\mathbb{F})$ as the composite functor

$$\mathsf{Cat}_\infty \xrightarrow{j_*} \mathsf{Fun}(\mathbb{F}^{\mathrm{op}}, \mathsf{Cat}_\infty) \simeq \mathsf{Cat}^{\mathrm{coc}}_{\infty/\mathbb{F}^{\mathrm{op}}} \to \mathsf{Cat}_{\infty/\mathsf{Span}(\mathbb{F})},$$

where the last functor is given by composition with the inclusion $i\colon \mathbb{F}^{\mathrm{op}} \to \mathsf{Span}(\mathbb{F})$ of the inert (i.e., backwards) morphisms. Here i clearly admits cocartesian lifts of inert morphisms in $\mathsf{Span}(\mathbb{F})$, hence so does the composite $\mathbb{F}^{\mathrm{op}}_{\mathcal{C}} \to \mathsf{Span}(\mathbb{F})$ (given by cocartesian lifts of the morphisms in $\mathbb{F}^{\mathrm{op}}$), and the functor $\mathbb{F}^{\mathrm{op}}_{\mathcal{C}} \to \mathbb{F}^{\mathrm{op}}_{\mathcal{D}}$ induced by any functor $\mathcal{C} \to \mathcal{D}$ in Cat_∞ preserves these, as it preserves cocartesian morphisms over $\mathbb{F}^{\mathrm{op}}$.

It remains to check that $\mathbb{F}^{\mathrm{op}}_{\mathcal{C}}$ is an ∞-operad. By definition we have the Segal condition $(\mathbb{F}^{\mathrm{op}}_{\mathcal{C}})_{\mathbf{n}} \simeq \prod_{i=1}^{n} \mathcal{C}$. For any objects $\mathbf{n}$, $\mathbf{m}$ in $\mathbb{F}$ we have a pullback square

$$\begin{tikzcd}
\mathsf{Map}_{\mathbb{F}^{\mathrm{op}}}(\mathbf{n}, \mathbf{m}) \arrow[r, "\sim"] \arrow[d] & \prod_{i=1}^{m} \mathsf{Map}_{\mathbb{F}^{\mathrm{op}}}(\mathbf{n}, \mathbf{1}) \arrow[d] \\
\mathsf{Map}_{\mathsf{Span}(\mathbb{F})}(\mathbf{n}, \mathbf{m}) \arrow[r, "\sim"] & \prod_{i=1}^{m} \mathsf{Map}_{\mathsf{Span}(\mathbb{F})}(\mathbf{n}, \mathbf{1}),
\end{tikzcd}$$

so it suffices to show that for $\phi\colon \mathbf{n} \to \mathcal{C}$, $\psi\colon \mathbf{m} \to \mathcal{C}$ we have a pullback square

$$\begin{tikzcd}
\mathsf{Map}_{\mathbb{F}^{\mathrm{op}}_{\mathcal{C}}}(\phi, \psi) \arrow[r, "\sim"] \arrow[d] & \prod_{i=1}^{m} \mathsf{Map}_{\mathbb{F}^{\mathrm{op}}_{\mathcal{C}}}\big(\phi, \psi(i)\big) \arrow[d] \\
\mathsf{Map}_{\mathbb{F}^{\mathrm{op}}}(\mathbf{n}, \mathbf{m}) \arrow[r, "\sim"] & \prod_{i=1}^{m} \mathsf{Map}_{\mathbb{F}^{\mathrm{op}}}(\mathbf{n}, \mathbf{1}).
\end{tikzcd}$$

To see this, observe that for $f\colon \mathbf{m} \to \mathbf{n}$, the map on fibres over f can be identified with

$$\mathsf{Map}_{\mathsf{Fun}(\mathbf{m},\mathcal{C})}(\phi \circ f, \psi) \xrightarrow{\sim} \prod_{i=1}^{m} \mathsf{Map}_{\mathcal{C}}\big(\phi\big(f(i)\big), \psi(i)\big).$$ ■

Lemma 5.1.3. *If* $\mathcal{O}$ *is a* $\mathsf{Span}(\mathbb{F})$*-$\infty$-operad, then there is a natural equivalence*

$$\mathcal{O} \times_{\mathsf{Span}(\mathbb{F})} \mathbb{F}^{\mathrm{op}} \xrightarrow{\sim} \mathbb{F}^{\mathrm{op}}_{\mathcal{O}_1}.$$

Proof. The projection $\mathcal{O} \times_{\mathsf{Span}(\mathbb{F})} \mathbb{F}^{\mathrm{op}} \to \mathbb{F}^{\mathrm{op}}$ is a cocartesian fibration, corresponding to a functor $\mathbb{F}^{\mathrm{op}} \to \mathsf{Cat}_\infty$. Hence the unit of the adjunction $j^* \dashv j_*$ corresponds to a morphism $\mathcal{O} \times_{\mathsf{Span}(\mathbb{F})} \mathbb{F}^{\mathrm{op}} \to \mathbb{F}^{\mathrm{op}}_{\mathcal{O}_{\mathbf{1}}}$ of cocartesian fibrations over $\mathbb{F}^{\mathrm{op}}$. To show that this is an equivalence it suffices to show that it is an equivalence on the fibre over every $\mathbf{n} \in \mathbb{F}^{\mathrm{op}}$. To see this we observe that since the functor preserves cocartesian morphisms we have a commutative square

$$\begin{tikzcd} \mathcal{O}_{\mathbf{n}} \arrow[r] \arrow[d, "\sim"] & (\mathbb{F}^{\mathrm{op}}_{\mathcal{O}_{\mathbf{1}}})_{\mathbf{n}} \arrow[d, "\sim"] \\ \prod_{i=1}^{n} \mathcal{O}_{\mathbf{1}} \arrow[r, "\sim"] & \prod_{i=1}^{n} (\mathbb{F}^{\mathrm{op}}_{\mathcal{O}_{\mathbf{1}}})_{\mathbf{1}}, \end{tikzcd}$$

where the vertical maps are equivalences since both sides satisfy the Segal condition, and the bottom horizontal map is an equivalence since by construction the two fibres over $\mathbf{1}$ are the same. ∎

Proposition 5.1.4. *The functor $\mathbb{F}^{\mathrm{op}}_{(-)}$ is fully faithful, and is left adjoint to the functor*

$$(-)_{\mathbf{1}} : \mathsf{Opd}(\mathsf{Span}(\mathbb{F})) \to \mathsf{Cat}_\infty$$

that extracts the underlying ∞-category of unary operations in a $\mathsf{Span}(\mathbb{F})$-$\infty$-operad.

Proof. Using Lemma 5.1.3, for $\mathcal{C} \in \mathsf{Cat}_\infty$, $\mathcal{O} \in \mathsf{Opd}(\mathsf{Span}(\mathbb{F}))$ we have natural equivalences

$$\begin{aligned} \mathsf{Map}_{\mathsf{Opd}(\mathsf{Span}(\mathbb{F}))}(\mathbb{F}^{\mathrm{op}}_{\mathcal{C}}, \mathcal{O}) &\simeq \mathsf{Map}_{\mathsf{Cat}^{\mathrm{coc}}_{\infty/\mathbb{F}^{\mathrm{op}}}}(\mathbb{F}^{\mathrm{op}}_{\mathcal{C}}, \mathcal{O} \times_{\mathsf{Span}(\mathbb{F})} \mathbb{F}^{\mathrm{op}}) \\ &\simeq \mathsf{Map}_{\mathsf{Cat}^{\mathrm{coc}}_{\infty/\mathbb{F}^{\mathrm{op}}}}(\mathbb{F}^{\mathrm{op}}_{\mathcal{C}}, \mathbb{F}^{\mathrm{op}}_{\mathcal{O}_{\mathbf{1}}}) \\ &\simeq \mathsf{Map}_{\mathsf{Fun}(\mathbb{F}^{\mathrm{op}}, \mathsf{Cat}_\infty)}(j_*\mathcal{C}, j_*\mathcal{O}_{\mathbf{1}}) \\ &\simeq \mathsf{Map}_{\mathsf{Cat}_\infty}(j^* j_*\mathcal{C}, \mathcal{O}_{\mathbf{1}}) \\ &\simeq \mathsf{Map}_{\mathsf{Cat}_\infty}(\mathcal{C}, \mathcal{O}_{\mathbf{1}}). \end{aligned}$$

This shows that $\mathbb{F}^{\mathrm{op}}_{(-)}$ is left adjoint to $(-)_{\mathbf{1}}$. Moreover, the unit map $\mathcal{C} \to (\mathbb{F}^{\mathrm{op}}_{\mathcal{C}})_{\mathbf{1}}$ is an equivalence, which implies that $\mathbb{F}^{\mathrm{op}}_{(-)}$ is fully faithful. ∎

Definition 5.1.5. We define PCat_∞ as the pullback $\mathsf{Cat}_\infty \times_{\mathcal{S}} \mathsf{Fun}([1], \mathcal{S})_{\mathrm{es}}$ (given by the functors $(-)^{\simeq}$ and ev_1), where $\mathsf{Fun}([1], \mathcal{S})_{\mathrm{es}}$ is the full subcategory of $\mathsf{Fun}([1], \mathcal{S})$ spanned by the essentially surjective maps.

Notation 5.1.6. If $\mathcal{C}$ is an ∞-category, we denote the pinned ∞-category

$$(\mathcal{C}, \mathcal{C}^{\simeq} \xrightarrow{=} \mathcal{C}^{\simeq})$$

by $\mathcal{C}^{\natural}$.

Observation 5.1.7. If $\mathcal{C}$ is an ∞-category and $(\mathcal{D}, q : X \twoheadrightarrow \mathcal{D}^{\simeq})$ is a pinned ∞-category, we have

$$\mathsf{Map}_{\mathsf{PCat}_\infty}(\mathcal{C}^{\natural}, (\mathcal{D}, q)) \simeq \mathsf{Map}_{\mathsf{Cat}_\infty}(\mathcal{C}, \mathcal{D}) \times_{\mathsf{Map}_{\mathcal{S}}(\mathcal{C}^{\simeq}, \mathcal{D}^{\simeq})} \mathsf{Map}_{\mathcal{S}}(\mathcal{C}^{\simeq}, X).$$

Lemma 5.1.8. *The adjunction* $\mathbb{F}^{\mathrm{op}}_{(-)} \dashv (-)_{\mathbf{1}}$ *extends to an adjunction*

$$\mathbb{F}^{\mathrm{op}}_{(-)} : \mathsf{PCat}_\infty \rightleftarrows \mathsf{POpd}\big(\mathsf{Span}(\mathbb{F})\big) : (-)_{\mathbf{1}}$$

where the left adjoint $\mathbb{F}^{\mathrm{op}}_{(-)}$ *is still fully faithful.*

Proof. Immediate from the observation that both adjoints and the (co)unit transformations are compatible with the functors $(-)^{\simeq}\colon \mathsf{Cat}_\infty \to \mathcal{S}$ and $(-)^{\simeq}_{\mathbf{1}}\colon \mathsf{Opd}(\mathsf{Span}(\mathbb{F})) \to \mathcal{S}$. ∎

Proposition 5.1.9. *The ∞-category $\mathbb{F}^{\mathrm{op}}_{\mathcal{C}}$ has finite products. If $\mathcal{D}$ is an ∞-category with finite products, then a functor $\mathbb{F}^{\mathrm{op}}_{\mathcal{C}} \to \mathcal{D}$ preserves products if and only if it is right Kan extended along the inclusion $\mathcal{C} \hookrightarrow \mathbb{F}^{\mathrm{op}}_{\mathcal{C}}$. Restriction along this inclusion therefore gives an equivalence*

$$\mathsf{Mon}_{\mathbb{F}^{\mathrm{op}}_{\mathcal{C}}}(\mathcal{D}) \xrightarrow{\sim} \mathsf{Fun}(\mathcal{C}, \mathcal{D}).$$

Proof. Since $\mathbb{F}^{\mathrm{op}}_{\mathcal{C}}$ is a $\mathsf{Span}(\mathbb{F})$-$\infty$-operad, we know it has finite products by Proposition 2.2.6, and by Proposition 2.3.3 a functor $F\colon \mathbb{F}^{\mathrm{op}}_{\mathcal{C}} \to \mathcal{D}$ preserves these if and only if it is an $\mathbb{F}^{\mathrm{op}}_{\mathcal{C}}$-monoid, i.e., for every object $X = (x_1, \dots, x_n)$ the canonical map

$$F(X) \to \prod_{i=1}^{n} F(x_i) \tag{5.2}$$

is an equivalence.

The ∞-category

$$\mathcal{C}_{X/} := \mathcal{C} \times_{\mathbb{F}^{\mathrm{op}}_{\mathcal{C}}} (\mathbb{F}^{\mathrm{op}}_{\mathcal{C}})_{X/}$$

contains the discrete set S_X of cocartesian maps $X \to x_i$ as a subcategory. We claim the inclusion $S_X \to \mathcal{C}_{X/}$ is coinitial. To see this, first observe that the target of the projection

$$\mathcal{C}_{X/} \to \{\mathbf{1}\} \times_{\mathbb{F}^{\mathrm{op}}} \mathbb{F}^{\mathrm{op}}_{\mathbf{n}/}$$

is the discrete set of morphisms $\mathbf{1} \to \mathbf{n}$ in $\mathbb{F}$, i.e., the set $\mathbf{n}$. The fibre over an element $i \in \mathbf{n}$ identifies (via the cocartesian morphism $X \to x_i$) with $\mathcal{C}_{x_i/}$, where x_i is of course initial. From this the criterion of [25, Theorem 4.1.3.1] immediately implies that S_X is coinitial.

It follows that pointwise right Kan extensions along the inclusion

$$\mathcal{C} \hookrightarrow \mathbb{F}^{\mathrm{op}}_{\mathcal{C}}$$

exist provided the target has finite products, and that a functor $F\colon \mathbb{F}^{\mathrm{op}}_{\mathcal{C}} \to \mathcal{D}$ is a right Kan extension of its restriction to $\mathcal{C}$ precisely when the maps (5.2) are equivalences. ∎

The composite functor

$$\mathsf{PCat}_\infty \to \mathsf{POpd}\big(\mathsf{Span}(\mathbb{F})\big) \xrightarrow{\sim} \mathsf{AnMnd}$$

thus takes the pair $(\mathcal{C}, p\colon X \twoheadrightarrow \mathcal{C}^{\simeq})$ to the monadic right adjoint

$$\mathsf{Mon}_{\mathbb{F}^{\mathrm{op}}_{\mathcal{C}}}(\mathcal{S}) \simeq \mathsf{Fun}(\mathcal{C}, \mathcal{S}) \xrightarrow{p^*} \mathsf{Fun}(X, \mathcal{S}).$$

Our next goal is to identify the image of PCat_∞ as precisely the full subcategory of *linear* monads inside AnMnd.

Definition 5.1.10. A *linear monad* on $\mathsf{Fun}(X, \mathcal{S})$ is an analytic monad such that the underlying endofunctor preserves small colimits.

Observation 5.1.11. Since presheaves on an ∞-category are its free cocompletion, we can identify colimit-preserving functors $\mathsf{Fun}(X, \mathcal{S}) \to \mathsf{Fun}(Y, \mathcal{S})$ for $X, Y \in \mathcal{S}$ with

$$\mathsf{Fun}\big(X, \mathsf{Fun}(Y, \mathcal{S})\big) \simeq \mathsf{Fun}(X \times Y, \mathcal{S}) \simeq \mathcal{S}_{/X \times Y},$$

so that a colimit-preserving functor is identified with a *span*

$$\begin{tikzcd} & E \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y, \end{tikzcd}$$

with this span corresponding to the functor $g_! f^*$. In terms of polynomial diagrams, this means a colimit-preserving functor corresponds to a diagram

$$X \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} Y,$$

where p is an *equivalence*.

As a special case of Proposition 3.1.3, we have the following.

Corollary 5.1.12. *A monad T on $\mathsf{Fun}(X, \mathcal{S})$ is linear if and only if for the corresponding monadic adjunction $F_T \dashv U_T$, the ∞-category $\mathsf{Alg}(T)$ has small colimits and the right adjoint $U_T\colon \mathsf{Alg}(T) \to \mathsf{Fun}(X, \mathcal{S})$ preserves these.* ∎

Proposition 5.1.13. *Suppose T is a linear monad on $\mathsf{Fun}(X, \mathcal{S})$, and let $\mathcal{C}^{\mathrm{op}}$ be the full subcategory of $\mathsf{Alg}(T)$ spanned by the objects $F_T \mathsf{y}_X(x)$ for $x \in X$. Then the restricted Yoneda embedding $\mathsf{Alg}(T) \to \mathsf{Fun}(\mathcal{C}, \mathcal{S})$ is an equivalence.*

Proof. We apply [13, Lemma 3.3.11], for which it suffices to check that the objects of $\mathcal{C}^{\mathrm{op}}$ are completely compact and jointly conservative. We have a natural equivalence

$$\mathsf{Map}_{\mathsf{Alg}(T)}\big(F_T \mathsf{y}_X(x), A\big) \simeq \mathsf{Map}_{\mathsf{Fun}(X,\mathcal{S})}\big(\mathsf{y}_X(x), U_T A\big) \simeq (U_T A)(x),$$

from which we see that $F_T \mathsf{y}_X(x)$ is completely compact since $\mathsf{y}_X(x)$ is completely compact in $\mathsf{Fun}(X, \mathcal{S})$ and U_T preserves small colimits by Corollary 5.1.12. Moreover, the objects of $\mathcal{C}^{\mathrm{op}}$ are jointly conservative since U_T is conservative and equivalences in $\mathsf{Fun}(X, \mathcal{S})$ are detected by evaluation at the points of X. ∎

Corollary 5.1.14. *An analytic monad T is linear if and only if it is in the image of* PCat_∞. ∎

Corollary 5.1.15. *We have a commutative square*

$$\begin{tikzcd} \mathsf{POpd}(\mathsf{Span}(\mathbb{F})) \arrow[r, "\sim"] & \mathsf{AnMnd} \\ \mathsf{PCat}_\infty \arrow[u, hook, "\mathbb{F}^{\mathrm{op}}_{(-)}"] \arrow[r, "\sim"] & \mathsf{LinMnd}, \arrow[u, hook] \end{tikzcd}$$

where the bottom horizontal functor takes a pinned ∞-category $(\mathcal{C}, p\colon X \twoheadrightarrow \mathcal{C}^{\simeq})$ to the monadic right adjoint $\mathsf{Fun}(\mathcal{C}, \mathcal{S}) \xrightarrow{p^*} \mathsf{Fun}(X, \mathcal{S})$. ■

5.2. Linear monads and Segal spaces

Our goal in this section is to show that the equivalence between analytic monads and dendroidal Segal spaces from [13] restricts to an equivalence between linear monads and Segal spaces.

In order to do this, we will briefly revisit each step in the comparison and discuss how it restricts to the linear case; we refer the reader to [13] for full details.

Firstly, by giving an alternative description of the ∞-category AnMnd we constructed (in [13, Section 5.1]) a forgetful functor $U_{\mathrm{an}}\colon \mathsf{AnMnd} \to \mathsf{AnEnd}$ where AnEnd is an ∞-category of *analytic endofunctors* on ∞-categories of the form $\mathcal{S}_{/X}$; an object of AnEnd is a space X together with an analytic endofunctor $T\colon \mathcal{S}_{/X} \to \mathcal{S}_{/X}$, and a morphism $(X,T) \to (Y,S)$ is a morphism of spaces $f\colon X \to Y$ together with a cartesian transformation $Tf^* \to f^*S$.

Definition 5.2.1. We define LinEnd as the full subcategory of AnEnd containing the linear (i.e., colimit-preserving) endofunctors. By construction we then have a pullback square

$$\begin{tikzcd} \mathsf{LinMnd} \arrow[r, hook] \arrow[d, "U_{\mathrm{lin}}"] & \mathsf{AnMnd} \arrow[d, "U_{\mathrm{an}}"] \\ \mathsf{LinEnd} \arrow[r, hook] & \mathsf{AnEnd}. \end{tikzcd}$$

Next, we defined the category $\boldsymbol{\Omega}^{\mathrm{int}}$ of trees and inert morphisms (or subtree inclusions) as a full subcategory of AnEnd. Here we regard a tree T as the endofunctor corresponding to the polynomial diagram

$$E \xleftarrow{s} V_* \xrightarrow{p} V \xrightarrow{t} E,$$

where E is the set of edges of T, V is the set of vertices, and V_* is the set of vertices with a marked incoming edge. The map s is the projection $(v,e) \mapsto e$ and p is the projection $(v,e) \mapsto v$, while t takes a vertex v to its unique outgoing edge. Here the fibre of p at v is the set of incoming edges of v, so the tree T is linear in the sense that it has only unary vertices if and only if p is an equivalence, which we saw in Observation 5.1.11 is equivalent to the corresponding functor being linear. It is easy to see that the full subcategory of $\boldsymbol{\Omega}^{\mathrm{int}}$ on the linear trees is $\boldsymbol{\Delta}^{\mathrm{int}}$, so we get a pullback square

$$\begin{tikzcd} \boldsymbol{\Delta}^{\mathrm{int}} \arrow[r, hook] \arrow[d, hook] & \mathsf{LinEnd} \arrow[d, hook] \\ \boldsymbol{\Omega}^{\mathrm{int}} \arrow[r, hook] & \mathsf{AnEnd}. \end{tikzcd}$$

Let us write $\mathbf{\Omega}^{\mathrm{el}}$ for the full subcategory of $\mathbf{\Omega}^{\mathrm{int}}$ on the edge η and the corollas C_n (with C_n having n leaves for $n = 0, 1, \ldots$). Then we define $\mathbf{\Delta}^{\mathrm{el}}$ as the intersection of $\mathbf{\Omega}^{\mathrm{el}}$ with $\mathbf{\Delta}^{\mathrm{int}}$, so that $\mathbf{\Delta}^{\mathrm{el}}$ has objects $\eta = [0]$ and $C_1 = [1]$.

In [13, Section 3.3] we showed that the restricted Yoneda embedding $\mathsf{AnEnd} \to \mathcal{P}(\mathbf{\Omega}^{\mathrm{el}})$ is an equivalence, while $\mathsf{AnEnd} \to \mathcal{P}(\mathbf{\Omega}^{\mathrm{int}})$ is fully faithful with image the functors that satisfy the dendroidal Segal condition, or equivalently are right Kan extended from $\mathbf{\Omega}^{\mathrm{el}}$. To deduce an equivalent statement for linear endofunctors we first prove the following characterization.

Proposition 5.2.2. *The following are equivalent for an object $F \in \mathsf{AnEnd}$.*

(i) *F is linear.*

(ii) $\mathsf{Map}_{\mathsf{AnEnd}}(C_n, F) \simeq \emptyset$ *for $n \neq 1$.*

(iii) $\mathsf{Map}_{\mathsf{AnEnd}}(T, F) \simeq \emptyset$ *for every tree $T \in \mathbf{\Omega}^{\mathrm{int}}$ which is not linear.*

(iv) $\mathsf{Map}_{\mathsf{AnEnd}}(-, F)|_{\mathbf{\Omega}^{\mathrm{int}}} \in \mathcal{P}(\mathbf{\Omega}^{\mathrm{int}})$ *is left Kan extended from $\mathbf{\Delta}^{\mathrm{int,op}}$.*

(v) $\mathsf{Map}_{\mathsf{AnEnd}}(-, F)|_{\mathbf{\Omega}^{\mathrm{el}}} \in \mathcal{P}(\mathbf{\Omega}^{\mathrm{el}})$ *is left Kan extended from $\mathbf{\Delta}^{\mathrm{el,op}}$.*

Proof. Suppose F corresponds to the polynomial diagram

$$X \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} X,$$

where p has finite discrete fibres. The map p is then classified by a map $\pi \colon B \to \mathbb{F}^{\simeq}$, and by [13, Lemma 3.3.6] the space $\mathsf{Map}_{\mathsf{AnEnd}}(C_n, F)$ is equivalent to the fibre of π at an n-element set. This fibre is empty if and only if no fibres of p are n-element sets, so that condition (ii) is equivalent to all fibres of p having size 1, i.e., to p being an equivalence, which by Observation 5.1.11 corresponds to F being linear.

Condition (ii) is a special case of (iii), but it also implies (iii) since every tree is a colimit in AnEnd of its edges and corollas by [13, Lemma 3.3.5], and so $\mathsf{Map}_{\mathsf{AnEnd}}(T, F)$ must be empty whenever T contains a non-unary vertex.

To see that (ii) and (iii) are equivalent to (iv) and (v), respectively, it suffices to note that there are no maps in $\mathbf{\Omega}^{\mathrm{int}}$ from a non-linear tree to a linear one. ∎

Corollary 5.2.3. *We have commutative squares*

$$\begin{tikzcd} \mathsf{LinEnd} \arrow[r, "\sim"] \arrow[d, hook] & \mathsf{P}(\mathbf{\Delta}^{\mathrm{el}}) \arrow[d, hook] \\ \mathsf{AnEnd} \arrow[r, "\sim"] & \mathsf{P}(\mathbf{\Omega}^{\mathrm{el}}), \end{tikzcd} \qquad \begin{tikzcd} \mathsf{LinEnd} \arrow[r, "\sim"] \arrow[d, hook] & \mathsf{Seg}_{\mathbf{\Delta}^{\mathrm{int}}}(\mathcal{S}) \arrow[d, hook] \\ \mathsf{AnEnd} \arrow[r, "\sim"] & \mathsf{Seg}_{\mathbf{\Omega}^{\mathrm{int}}}(\mathcal{S}), \end{tikzcd}$$

where in both the right-hand vertical arrow is a left Kan extension.

Proof. Let $\lambda, \delta^{\mathrm{el}}, \omega^{\mathrm{el}}, L^{\mathrm{el}}$ denote the inclusions $\mathsf{LinEnd} \hookrightarrow \mathsf{AnEnd}$, $\mathbf{\Delta}^{\mathrm{el}} \hookrightarrow \mathsf{LinEnd}$, $\mathbf{\Omega}^{\mathrm{el}} \hookrightarrow \mathsf{AnEnd}$, $\mathbf{\Delta}^{\mathrm{el}} \hookrightarrow \mathbf{\Omega}^{\mathrm{el}}$, respectively. From the commutative square

$$\begin{tikzcd} \mathbf{\Delta}^{\mathrm{el}} \arrow[r, hook, "\delta^{\mathrm{el}}"] \arrow[d, hook, "L^{\mathrm{el}}"] & \mathsf{LinEnd} \arrow[d, hook, "\lambda"] \\ \mathbf{\Omega}^{\mathrm{el}} \arrow[r, hook, "\omega^{\mathrm{el}}"] & \mathsf{AnEnd} \end{tikzcd}$$

we obtain a commutative diagram

$$\begin{tikzcd} \mathsf{LinEnd} \arrow[r, hook, "y_{\mathsf{LinEnd}}"] \arrow[d, hook, "\lambda"] & \mathsf{P}(\mathsf{LinEnd}) \arrow[r, "\delta^{\mathrm{el},*}"] \arrow[d, "\lambda_!"] & \mathsf{P}(\mathbf{\Delta}^{\mathrm{el}}) \arrow[d, hook, "L_!^{\mathrm{el}}"] \arrow[dl, Rightarrow] \\ \mathsf{AnEnd} \arrow[r, hook, "y_{\mathsf{AnEnd}}"] & \mathsf{P}(\mathsf{AnEnd}) \arrow[r, "\omega^{\mathrm{el},*}"] & \mathsf{P}(\mathbf{\Omega}^{\mathrm{el}}), \end{tikzcd}$$

where the left square comes from the Yoneda embedding and the right square is the mate of the square of restriction functors above. To obtain the first commutative square it then suffices to show that the composite lax square actually commutes, i.e., that for a linear endofunctor F the map $L_!^{\mathrm{el}}\delta^{\mathrm{el},*}y_{\mathsf{LinEnd}}F \to \omega^{\mathrm{el},*}y_{\mathsf{AnEnd}}\lambda F$ is an equivalence. We can check this by evaluating at every object $T \in \mathbf{\Omega}^{\mathrm{el}}$, where we get the canonical map

$$\operatorname{colim}_{X\in(\mathbf{\Delta}^{\mathrm{el}})^{\mathrm{op}}_{/T}} \mathsf{Map}\big(\delta^{\mathrm{el}}(X), F\big) \to \mathsf{Map}\big(\omega^{\mathrm{el}}(T), \lambda F\big);$$

if T lies in $\mathbf{\Delta}^{\mathrm{el}}$ then this is an equivalence since $(\mathbf{\Delta}^{\mathrm{el}})^{\mathrm{op}}_{/T}$ has a terminal object, and if T does not lie in $\mathbf{\Delta}^{\mathrm{el}}$ it is an equivalence because $(\mathbf{\Delta}^{\mathrm{el}})^{\mathrm{op}}_{/T}$ is empty and the right-hand side is $\emptyset$ because F is linear. We thus have a commutative square

$$\begin{tikzcd} \mathsf{LinEnd} \arrow[r] \arrow[d, hook] & \mathsf{P}(\mathbf{\Delta}^{\mathrm{el}}) \arrow[d, hook, "L_!^{\mathrm{el}}"] \\ \mathsf{AnEnd} \arrow[r, "\sim"] & \mathsf{P}(\mathbf{\Omega}^{\mathrm{el}}), \end{tikzcd}$$

and to obtain the first square it remains only to show that the top horizontal morphism is an equivalence. To see this, we first observe that this functor is automatically fully faithful (since essentially surjective and fully faithful functors are a factorization system on Cat_∞). It then suffices to observe that the image of LinEnd in $\mathsf{P}(\mathbf{\Omega}^{\mathrm{el}})$ consists precisely of the functors left Kan extended from $\mathbf{\Delta}^{\mathrm{el}}$ by Proposition 5.2.2. We omit the argument for the second square, since it is essentially the same. ∎

In [13, Section 5.2], we proved that U_{an} has a left adjoint $F_{\mathrm{an}}\colon \mathsf{AnEnd} \to \mathsf{AnMnd}$, taking an analytic endofunctor to the free monad on it. We can define $\mathbf{\Omega}$ as the full subcategory of AnMnd containing the free monads on the objects of $\mathbf{\Omega}^{\mathrm{int}}$; $\mathbf{\Delta} \subseteq \mathbf{\Omega}$ is then the full subcategory of AnMnd containing the free monads on linear trees.

Lemma 5.2.4. *F_{an} restricts to a functor $F_{\mathrm{lin}}\colon \mathsf{LinEnd} \to \mathsf{LinMnd}$, left adjoint to U_{lin}.*

Proof. If P is a linear endofunctor, then we must show that the free analytic monad $F_{\mathrm{an}}P$ is a linear monad. This amounts to checking that the underlying endofunctor of $F_{\mathrm{an}}P$ is linear. This follows from the explicit description of this endofunctor in [13, Theorem 5.2.4], since the space of maps from any non-linear tree to P is empty if P is a linear endofunctor. ∎

Proposition 5.2.5. *The following are equivalent for an analytic monad $M \in \mathsf{AnMnd}$.*

(i) *M is linear.*

(ii) $\mathsf{Map}_{\mathsf{AnMnd}}(C_n, M) \simeq \emptyset$ *for* $n \neq 1$.

(iii) $\mathsf{Map}_{\mathsf{AnMnd}}(T, M) \simeq \emptyset$ *for every* $T \in \mathbf{\Omega}$ *which is not linear.*

(iv) $\mathsf{Map}_{\mathsf{AnMnd}}(-, M)|_{\mathbf{\Omega}^{\mathrm{op}}} \in \mathsf{P}(\mathbf{\Omega})$ *is left Kan extended from* $\mathbf{\Delta}^{\mathrm{op}}$.

Proof. By definition, M is linear precisely if $U_{\mathrm{an}}M$ is linear; since the elements of $\mathbf{\Omega}$ are free, the equivalence of the first three conditions follows immediately from Proposition 5.2.2. The last two conditions are then equivalent because a presheaf on $\mathbf{\Omega}$ is left Kan extended from its restriction to $\mathbf{\Delta}^{\mathrm{op}}$ precisely when its value at all non-linear trees is $\emptyset$, because there are no maps from a non-linear tree to a linear one in $\mathbf{\Omega}$. ∎

In [13, Section 5.3] we proved as our main theorem that the restricted Yoneda embedding $\mathsf{AnMnd} \to \mathsf{P}(\mathbf{\Omega})$ fits in a pullback square

$$\begin{tikzcd}
\mathsf{AnMnd} \arrow[r, hook] \arrow[d, "U_{\mathrm{an}}"] & \mathsf{P}(\mathbf{\Omega}) \arrow[d] \\
\mathsf{AnEnd} \arrow[r, hook] & \mathsf{P}(\mathbf{\Omega}^{\mathrm{int}}),
\end{tikzcd}$$

and so identifies AnMnd with the full subcategory $\mathsf{Seg}_{\mathbf{\Omega}^{\mathrm{op}}}(\mathcal{S}) \subseteq \mathsf{P}(\mathbf{\Omega})$. Together with Proposition 5.2.5 this gives the following analogue of Corollary 5.2.3 (with essentially the same proof).

Corollary 5.2.6. *There is a commutative square*

$$\begin{tikzcd}
\mathsf{LinMnd} \arrow[r, "\sim"] \arrow[d, hook] & \mathsf{Seg}_{\mathbf{\Delta}}(\mathcal{S}) \arrow[d, hook] \\
\mathsf{AnMnd} \arrow[r, "\sim"] & \mathsf{Seg}_{\mathbf{\Omega}}(\mathcal{S}),
\end{tikzcd}$$

where the right-hand vertical arrow is a left Kan extension. ∎

Our final goal in this section is to combine this square and that from Corollary 5.2.3 into a commutative cube:

Proposition 5.2.7. *There is a commutative diagram*

$$\begin{tikzcd}
\mathsf{LinMnd} \arrow[rr, "\sim"] \arrow[dr, hook] \arrow[dd] & & \mathsf{Seg}_{\mathbf{\Delta}}(\mathcal{S}) \arrow[dr, hook] \arrow[dd] & \\
& \mathsf{AnMnd} \arrow[rr, "\sim"] \arrow[dd] & & \mathsf{Seg}_{\mathbf{\Omega}}(\mathcal{S}) \arrow[dd] \\
\mathsf{LinEnd} \arrow[rr, "\sim"] \arrow[dr, hook] & & \mathsf{Seg}_{\mathbf{\Delta}^{\mathrm{int}}}(\mathcal{S}) \arrow[dr, hook] & \\
& \mathsf{AnEnd} \arrow[rr, "\sim"] & & \mathsf{Seg}_{\mathbf{\Omega}^{\mathrm{int}}}(\mathcal{S})
\end{tikzcd}$$

where all the squares are cartesian.

Proof. We have a commutative cube

$$\begin{tikzcd}
\mathbf{\Delta}^{\mathrm{int}} \arrow[rr, hook] \arrow[dr, hook] \arrow[dd] & & \mathsf{LinEnd} \arrow[dr, hook] \arrow[dd, "F_{\mathrm{lin}}"] & \\
& \mathbf{\Omega}^{\mathrm{int}} \arrow[rr, hook] \arrow[dd] & & \mathsf{AnEnd} \arrow[dd, "F_{\mathrm{an}}"] \\
\mathbf{\Delta} \arrow[rr, hook] \arrow[dr, hook] & & \mathsf{LinMnd} \arrow[dr, hook] & \\
& \mathbf{\Omega} \arrow[rr, hook] & & \mathsf{AnMnd}.
\end{tikzcd}$$

Taking presheaves and mates we get (since mates in Cat_∞ are functorial, as was proved in [16]) the following diagram, where all except the front and back squares are a priori lax:

$$\begin{tikzcd}
\mathsf{P}(\mathsf{LinMnd}) \arrow[rr] \arrow[dr, hook] \arrow[dd, "F^*_{\mathrm{lin}}"] & & \mathsf{P}(\mathbf{\Delta}) \arrow[dr, hook] \arrow[dd] & \\
& \mathsf{P}(\mathsf{AnMnd}) \arrow[rr] \arrow[dd, "F^*_{\mathrm{an}}"] & & \mathsf{P}(\mathbf{\Omega}) \arrow[dd] \\
\mathsf{P}(\mathsf{LinEnd}) \arrow[rr] \arrow[dr, hook] & & \mathsf{P}(\mathbf{\Delta}^{\mathrm{int}}) \arrow[dr, hook] & \\
& \mathsf{P}(\mathsf{AnEnd}) \arrow[rr] & & \mathsf{P}(\mathbf{\Omega}^{\mathrm{int}}).
\end{tikzcd}$$

The functor P^* preserves adjunctions, so we can identify F^*_{an} as the left adjoint of U^*_{an}. Moreover, P^* preserves mate squares, so that the mate in the vertical direction of the square

$$\begin{tikzcd}
\mathsf{P}(\mathsf{AnMnd}) \arrow[r] \arrow[d, "F^*_{\mathrm{an}}"] & \mathsf{P}(\mathsf{LinMnd}) \arrow[d, "F^*_{\mathrm{lin}}"] \\
\mathsf{P}(\mathsf{AnEnd}) \arrow[r] & \mathsf{P}(\mathsf{LinEnd})
\end{tikzcd}$$

is the commutative square

$$\begin{tikzcd}
\mathsf{P}(\mathsf{AnEnd}) \arrow[r] \arrow[d, "U^*_{\mathrm{an}}"] & \mathsf{P}(\mathsf{LinEnd}) \arrow[d, "U^*_{\mathrm{lin}}"] \\
\mathsf{P}(\mathsf{AnMnd}) \arrow[r] & \mathsf{P}(\mathsf{LinMnd})
\end{tikzcd}$$

obtained by applying P^* to the square

$$\begin{tikzcd}
\mathsf{LinMnd} \arrow[r, hook] \arrow[d, "U_{\mathrm{lin}}"] & \mathsf{AnMnd} \arrow[d, "U_{\mathrm{an}}"] \\
\mathsf{LinEnd} \arrow[r, hook] & \mathsf{AnEnd}.
\end{tikzcd}$$

The left-hand square in our cube above is therefore obtained by taking mates in both vertical and horizontal directions for this last square, and so it is precisely the corresponding commutative square of left adjoints (and so is in particular not actually lax). We can therefore compose our cube with the following commutative cube arising from the naturality of the Yoneda embedding:

$$\begin{tikzcd}
\mathsf{LinMnd} \arrow[rr, hook] \arrow[dr, hook] \arrow[dd, "U_{\mathrm{lin}}"] & & \mathsf{P}(\mathsf{LinMnd}) \arrow[dr, hook] \arrow[dd, "(U_{\mathrm{lin}})_!" near start] & \\
& \mathsf{AnMnd} \arrow[rr, hook] \arrow[dd, "U_{\mathrm{an}}" near start] & & \mathsf{P}(\mathsf{AnMnd}) \arrow[dd, "(U_{\mathrm{an}})_!"] \\
\mathsf{LinEnd} \arrow[rr, hook] \arrow[dr, hook] & & \mathsf{P}(\mathsf{LinEnd}) \arrow[dr, hook] & \\
& \mathsf{AnEnd} \arrow[rr, hook] & & \mathsf{P}(\mathsf{AnEnd}).
\end{tikzcd}$$

This produces a diagram of the form

$$\begin{tikzcd}
\mathsf{LinMnd} \arrow[rr] \arrow[dr, hook] \arrow[dd] & & \mathsf{P}(\boldsymbol{\Delta}) \arrow[dr, hook] \arrow[dd] & \\
& \mathsf{AnMnd} \arrow[rr] \arrow[dd] & & \mathsf{P}(\boldsymbol{\Omega}) \arrow[dd] \\
\mathsf{LinEnd} \arrow[rr] \arrow[dr, hook] & & \mathsf{P}(\boldsymbol{\Delta}^{\mathrm{int}}) \arrow[dr, hook] & \\
& \mathsf{AnEnd} \arrow[rr] & & \mathsf{P}(\boldsymbol{\Omega}^{\mathrm{int}})
\end{tikzcd}$$

where the top, bottom and right-hand squares are potentially lax. To complete the proof it remains to show that these squares actually commute. For the top and bottom this follows from Corollary 5.2.6 and Corollary 5.2.3, respectively, while for the right-hand square it is immediate (since both left Kan extensions amount to extending the functors by $\emptyset$ on non-linear trees). ∎

This completes the construction of the diagram (5.1).

Remark 5.2.8. Combining our work so far in this section, we have in particular an equivalence $\mathsf{PCat}_\infty \simeq \mathsf{Seg}_{\boldsymbol{\Delta}^{\mathrm{op}}}(\mathcal{S})$ between pinned ∞-categories and (not necessarily complete) Segal spaces in the sense of Rezk [27]. Such an equivalence has previously been proved by Ayala and Francis [1] (as the case $n = 1$ of a more general equivalence between *flagged* (∞, n)*-categories* and Rezk's Segal Θ_n-spaces [28]).

5.3. Completeness

Our goal in this final section is to understand the composite equivalence

$$\mathsf{PCat}_\infty \xrightarrow{\sim} \mathsf{LinMnd} \xleftarrow{\sim} \mathsf{Seg}_{\boldsymbol{\Delta}^{\mathrm{op}}}(\mathcal{S}),$$

and to show that it restricts to the standard equivalence between ∞-categories and complete Segal spaces. Using this, we can then complete the proof that $\mathsf{Span}(\mathbb{F})$-$\infty$-operads are equivalent to complete dendroidal Segal spaces.

Lemma 5.3.1. *The composite functor*

$$\mathsf{PCat}_\infty \xrightarrow{\sim} \mathsf{LinMnd} \xrightarrow{U_{\mathrm{lin}}} \mathsf{LinEnd}$$

takes a pinned ∞*-category* $(\mathcal{C}, p\colon X \twoheadrightarrow \mathcal{C}^{\simeq})$*, to the span*

$$S(\mathcal{C}, p) := X \leftarrow \mathsf{Map}\big([1], \mathcal{C}\big) \times_{(\mathcal{C}^{\simeq}\times\mathcal{C}^{\simeq})} (X \times X) \to X,$$

where the pullback is along $(\mathrm{ev}_0, \mathrm{ev}_1)\colon \mathsf{Map}([1], \mathcal{C}) \to \mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq}$ *and two copies of* p.

Proof. We first identify the functor $X \times X \to \mathcal{S}$ corresponding to the endofunctor

$$\mathsf{Fun}(X, \mathcal{S}) \xrightarrow{p_!} \mathsf{Fun}(\mathcal{C}, \mathcal{S}) \xrightarrow{p^*} \mathsf{Fun}(X, \mathcal{S}).$$

This is the functor

$$\begin{aligned}(x, x') \mapsto \mathsf{Map}_{\mathsf{Fun}(X,\mathcal{S})}\big(\mathsf{y}_X(x), p^* p_! \mathsf{y}_X(x')\big) &\simeq \mathsf{Map}_{\mathsf{Fun}(\mathcal{C},\mathcal{S})}\big(p_!\mathsf{y}_X(x), p_!\mathsf{y}_X(x')\big)\\ &\simeq \mathsf{Map}_{\mathcal{C}}\big(p(x), p(x')\big).\end{aligned}$$

To find the corresponding span we now observe that $\mathsf{Map}([1], \mathcal{C}) \to \mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq}$ is the fibration for the mapping space functor restricted to the underlying ∞-groupoid of $\mathcal{C}$ (since $\mathsf{Map}([1], \mathcal{C})$ is the underlying ∞-groupoid of the twisted arrow ∞-category), and composition of functors corresponds to pulling back fibrations. ■

Proposition 5.3.2. *Under the composite equivalence* $\varepsilon\colon \mathsf{Seg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{S}) \xrightarrow{\sim} \mathsf{LinMnd} \xrightarrow{\sim} \mathsf{PCat}_\infty$*, the full subcategory* $\mathbf{\Delta}$ *in* $\mathsf{Seg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{S})$ *corresponds to the standard embedding of* $\mathbf{\Delta}$ *in* $\mathsf{Cat}_\infty \subseteq \mathsf{PCat}_\infty$.

Proof. Since the resulting functor $\mathbf{\Delta} \to \mathsf{PCat}_\infty$ is fully faithful, it almost suffices to check that it does the right thing on objects. Moreover, since we have the colimit decomposition $[n] \simeq [1] \amalg_{[0]} \cdots \amalg_{[0]} [1]$ in $\mathsf{Seg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{S})$, and hence for the image in PCat_∞, it is enough to look at the objects [0] and [1].

By construction, [0] and [1] correspond to the free linear monads on the spans

$$\begin{aligned}\eta &= \mathbf{1} \leftarrow \emptyset \to \mathbf{1},\\ C_1 &= \mathbf{2} \xleftarrow{1\mapsto 1} \mathbf{1} \xrightarrow{1\mapsto 2} \mathbf{2}.\end{aligned}$$

By [13, Lemma 3.3.6] we then have for any span $T = X \leftarrow Y \to X$ that

$$\begin{aligned}\mathsf{Map}_{\mathsf{LinEnd}}(\eta, T) &\simeq \mathsf{Map}_{\mathcal{S}}(\mathbf{1}, X) \simeq X,\\ \mathsf{Map}_{\mathsf{LinEnd}}(C_1, T) &\simeq Y.\end{aligned}$$

Thus using Observation 5.1.7 we have

$$\begin{aligned}\mathsf{Map}\big([0], \varepsilon^{-1}(\mathcal{C}, p)\big) &\simeq \mathsf{Map}_{\mathsf{LinEnd}}\big(\eta, S(\mathcal{C}, p)\big) \simeq X\\ &\simeq \mathsf{Map}_{\mathsf{PCat}_\infty}\big([0]^\natural, (\mathcal{C}, p)\big),\end{aligned}$$

$$\mathsf{Map}\big([1], \varepsilon^{-1}(\mathcal{C}, p)\big) \simeq \mathsf{Map}_{\mathsf{LinEnd}}\big(C_1, S(\mathcal{C}, p)\big) \simeq \mathsf{Map}\big([1], \mathcal{C}\big) \times_{\mathcal{C}^{\simeq}\times\mathcal{C}^{\simeq}} X \times X$$
$$\simeq \mathsf{Map}_{\mathsf{PCat}_\infty}\big([1]^{\natural}, (\mathcal{C}, p)\big),$$

as required. This pins down the functor $\mathbf{\Delta} \to \mathsf{PCat}_\infty$ up to automorphisms of $\mathbf{\Delta}$, of which the only non-trivial one is the order-reversing functor op: $\mathbf{\Delta} \xrightarrow{\sim} \mathbf{\Delta}$. To see that our functor is not reversing the order we need only observe that the two inclusions of [0] in [1] indeed occur in the expected order. ∎

Observation 5.3.3. It follows that the Segal space corresponding to a pinned ∞-category $(\mathcal{C}, p\colon X \twoheadrightarrow \mathcal{C}^{\simeq})$ under the equivalence ε is given by

$$[n] \mapsto \mathsf{Map}_{\mathsf{PCat}_\infty}\big([n]^{\natural}, (\mathcal{C}, p)\big) \simeq \mathsf{Map}_{\mathsf{Cat}_\infty}\big([n], \mathcal{C}\big) \times_{(\mathcal{C}^{\simeq})^{\times n}} X^{\times n}.$$

Proposition 5.3.4. *An object of* PCat_∞ *lies in the full subcategory* Cat_∞ *if and only if it is local with respect to the map*

$$E := \big([0], \mathbf{2} \twoheadrightarrow *\big) \to [0]^{\natural}.$$

Proof. For $(\mathcal{C}, p\colon X \twoheadrightarrow \mathcal{C}^{\simeq})$, we have a natural equivalence

$$\mathsf{Map}_{\mathsf{PCat}_\infty}\big(E, (\mathcal{C}, p)\big) \simeq X \times_{\mathcal{C}^{\simeq}} X,$$

under which the map $X \simeq \mathsf{Map}([0]^{\natural}, (\mathcal{C}, p)) \to \mathsf{Map}(E, (\mathcal{C}, p))$ corresponds to the diagonal. But the diagonal $X \to X \times_{\mathcal{C}^{\simeq}} X$ is an equivalence if and only if the map p is a monomorphism by [25, Lemma 5.5.6.15]. Since p is surjective by assumption, if it is a monomorphism then it is necessarily an equivalence, which completes the proof. ∎

Corollary 5.3.5. *Under the equivalence* $\mathsf{Seg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{S}) \xrightarrow{\sim} \mathsf{LinMnd} \xrightarrow{\sim} \mathsf{PCat}_\infty$, *the full subcategory* $\mathsf{Cat}_\infty \subseteq \mathsf{PCat}_\infty$ *is identified with the full subcategory* $\mathsf{CSeg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{S})$ *of* complete *Segal spaces.*

Proof. From the formula of Observation 5.3.3, the object $E := ([0], \mathbf{2} \twoheadrightarrow *)$ corresponds to the simplicial set

$$\mathsf{Map}\big([n], [0]\big) \times_{*^{\times n}} \mathbf{2}^{\times n} \simeq \mathbf{2}^{\times n}.$$

This is precisely the nerve of the contractible groupoid with two objects, and by definition $\mathsf{CSeg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{S})$ is the full subcategory of $\mathsf{Seg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{S})$ of objects that are local with respect to the map from this to the terminal object. ∎

Corollary 5.3.6. *Under the equivalence*

$$\mathsf{Seg}_{\mathbf{\Omega}^{\mathrm{op}}}(\mathcal{S}) \simeq \mathsf{AnMnd} \simeq \mathsf{POpd}\big(\mathsf{Span}(\mathbb{F})\big),$$

the full subcategory

$$\mathsf{Opd}\big(\mathsf{Span}(\mathbb{F})\big) \subseteq \mathsf{POpd}\big(\mathsf{Span}(\mathbb{F})\big)$$

is identified with the full subcategory $\mathsf{CSeg}_{\mathbf{\Omega}^{\mathrm{op}}}(\mathcal{S})$ *of* complete *dendroidal Segal spaces.*

Proof. We have constructed a commutative diagram

$$\begin{tikzcd} \mathsf{Seg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{S}) \arrow[r, "\sim"] \arrow[d, hook] & \mathsf{LinMnd} \arrow[d, hook] & \mathsf{PCat}_\infty \arrow[l, "\sim"'] \arrow[d, hook] \\ \mathsf{Seg}_{\mathbf{\Omega}^{\mathrm{op}}}(\mathcal{S}) \arrow[r, "\sim"] & \mathsf{AnMnd} & \mathsf{POpd}(\mathsf{Span}(\mathbb{F})). \arrow[l, "\sim"'] \end{tikzcd}$$

Here the left and right vertical arrows have left adjoints, given respectively by restriction along the inclusion $\mathbf{\Delta} \hookrightarrow \mathbf{\Omega}$ and by extracting the fibre over $\mathbf{1}$. We therefore also have a commutative square

$$\begin{tikzcd} \mathsf{Seg}_{\mathbf{\Omega}^{\mathrm{op}}}(\mathcal{S}) \arrow[r, "\sim"] \arrow[d] & \mathsf{POpd}(\mathsf{Span}(\mathbb{F})) \arrow[d] \\ \mathsf{Seg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{S}) \arrow[r, "\sim"] & \mathsf{PCat}_\infty \end{tikzcd}$$

with these left adjoints. In this picture we can define $\mathsf{CSeg}_{\mathbf{\Omega}^{\mathrm{op}}}(\mathcal{S})$ as the full subcategory of $\mathsf{Seg}_{\mathbf{\Omega}^{\mathrm{op}}}(\mathcal{S})$ comprising those objects whose image in $\mathsf{Seg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{S})$ lies in $\mathsf{CSeg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{S})$. From Corollary 5.3.5 it follows that under our equivalence this full subcategory is identified with the full subcategory of $\mathsf{POpd}(\mathsf{Span}(\mathbb{F}))$ that contains the objects whose image in PCat_∞ lies in Cat_∞, which is precisely $\mathsf{Opd}(\mathsf{Span}(\mathbb{F}))$. ∎

Remark 5.3.7. We can also explicitly identify the full subcategory of AnMnd that corresponds to $\mathsf{Opd}(\mathsf{Span}(\mathbb{F}))$ and $\mathsf{CSeg}_{\mathbf{\Omega}^{\mathrm{op}}}(\mathcal{S})$ under our equivalences: this contains the analytic monads T on $\mathsf{Fun}(X, \mathcal{S})$ that are *complete* in the sense that the induced essentially surjective map $X \to \mathcal{L}(T)^{\simeq}_{\mathbf{1}}$ is an equivalence. (Here $\mathcal{L}(T)^{\simeq}_{\mathbf{1}}$ is the same object as $\mathcal{U}(T)$ in the notation of [7, Section 15], so this definition of complete analytic monads agrees with the notion of completeness defined there.)

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