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Noncommutative supports, local cohomology and spectral sequences

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Abstract. The purpose of this paper is to study local cohomology in the noncommutative algebraic geometry framework of Artin and Zhang. The noncommutative spaces are obtained by base change of a Grothendieck category that is locally noetherian or strongly locally noetherian. Using what we call elementary objects and their injective hulls, we develop a theory of supports and associated primes in these categories. We apply our theory to study a general functorial setup that requires certain conditions on the injective hulls of elementary objects and gives us spectral sequences for derived functors associated with local cohomology objects, as well as generalized local cohomology and also generalized Nagata ideal transforms.

1. Introduction

If k is a field and A is a commutative k-algebra, the category $\mathrm{Mod}-A$ of A-modules determines the geometry of the affine scheme $\mathrm{Spec}(A)$. If A is no longer commutative, the category $\mathrm{Mod}-A$ of right A-modules plays the role of a noncommutative affine scheme. Suppose that A is a commutative graded k-algebra and $\mathrm{Gr}(A)$ is its category of graded modules. Then, the geometry of the projective scheme $\mathrm{Proj}(A)$ can be understood by means of the category $\mathrm{QGr}(A)$ which is the quotient of $\mathrm{Gr}(A)$ over torsion modules, that is, modules which are obtained as filtered colimits of bounded graded modules (see [5]). Accordingly, the category $\mathrm{QGr}(A)$ plays the role of a noncommutative projective scheme in the theory of Artin and Zhang [5], where A is a noncommutative graded k-algebra. In general, one can approach noncommutative algebraic geometry by adapting notions from commutative rings and the usual theory of schemes to arbitrary Grothendieck categories.

One of the notions that is fundamental in algebraic geometry is that of base change. Let k be a field, and let A, R be k-algebras. Then, a module over the tensor product algebra $(A \otimes_k R)$ may be described as an ordinary A-module equipped with an additional R-module action. If we replace the category of A-modules by a general k-linear abelian category $\mathcal{S}$, we obtain the category $\mathcal{S}_R$ of "R-module objects in $\mathcal{S}$." These module objects were introduced by Popescu [21, p. 108] and may be treated as modules over a noncommutative base change of R by the abelian category $\mathcal{S}$. An R-module object in $\mathcal{S}$ consists

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of an object $\mathcal{M} \in \mathcal{S}$ along with a morphism of k-algebras

$$\rho_{\mathcal{M}} : R \longrightarrow \mathcal{S}(\mathcal{M}, \mathcal{M}).$$

In [6], Artin and Zhang systematically developed the properties of the abstract module category $\mathcal{S}_R$, where $\mathcal{S}$ is a Grothendieck category, and often a locally noetherian or strongly locally noetherian Grothendieck category. The theory developed by Artin and Zhang in [6] includes tensor products, internal hom, localization, versions of the Hilbert basis theorem, the Nakayama lemma, as well as the derived functors Tor and Ext. Several other fundamental constructions on $\mathcal{S}_R$, such as ind-objects, coherence and deformation theory, have been developed by Lowen and Van den Bergh [19]. In [7], we have also studied descent properties in $\mathcal{S}_R$.

We note that a number of properties of the category $\mathcal{S}$ extend to the module category $\mathcal{S}_R$, following the idea of noncommutative base change. For example, if $\mathcal{S}$ is locally finitely generated, so is $\mathcal{S}_R$. The version of the Hilbert basis theorem proved by Artin and Zhang in [6] shows that if $\mathcal{S}$ is locally noetherian and R is a finitely generated k-algebra, then $\mathcal{S}_R$ is also locally noetherian. Since Grothendieck categories abound in nature, the theory of the abstract module category $\mathcal{S}_R$ is very general and applies to a wide variety of situations. For instance, $\mathcal{S}$ could be the category of sheaves of abelian groups on a topological space, comodules over a coalgebra over a field, modules over a ringed space, quasi-coherent sheaves over a scheme or the category of comodules over a flat Hopf algebroid.

In this paper, we develop a local cohomology theory in the noncommutative algebraic geometry of Artin and Zhang [5, 6], by making use of abstract module categories. We let the k-linear Grothendieck category $\mathcal{S} = \mathcal{S}_k$ be strongly locally noetherian, whence it follows that $\mathcal{S}_R$ is locally noetherian for any noetherian commutative k-algebra R. We should note that our framework will give new results, even in the case we take $\mathcal{S}$ to be the category of modules over a noncommutative algebra A that is strongly locally noetherian, that is, $A \otimes_k R$ is right noetherian for every commutative noetherian k-algebra R (see [4]). If C is a coalgebra over a field, its category of comodules is locally noetherian (see [14, Section 2.4.8] and [23, Section V.4.3]). As with the theory in [6], we can take $\mathcal{S}$ to be the category $\mathrm{Gr}(A)$ of graded modules over a graded algebra A. The category $\mathrm{Gr}(A)$ is strongly locally noetherian whenever A is strongly noetherian in a graded sense (see [6, Section B5]). We can also take $\mathcal{S} = \mathrm{QGr}(A)$ for a noncommutative graded k-algebra A. These are the noncommutative projective schemes of Artin and Zhang [5]. If A is strongly noetherian in a graded sense (but not necessarily commutative), the category $\mathrm{QGr}(A)$ is strongly locally noetherian and $\mathrm{QGr}(A)_R = \mathrm{QGr}(A \otimes_k R)$ (see [6, Section B8]).

The aims of our paper are threefold. First, we build a theory of associated primes and supports in the abstract module category $\mathcal{S}_R$ using what we call R-elementary objects in $\mathcal{S}_R$. If $\mathcal{S}$ is the category of modules over a commutative k-algebra A, then an R-elementary object in $\mathcal{S}_R$ behaves somewhat like a quotient over a prime ideal. Accordingly, the injective hulls of these R-elementary objects in $\mathcal{S}_R$ play a key role in our theory.

Our second purpose is to develop local cohomology in $\mathcal{S}_R$. In fact, we work with a more general framework, extending the local cohomology with respect to a pair of ideals $I, J \subseteq R$ introduced by Takahashi, Yoshino and Yoshizawa [24]. One of the key aspects of the theory in [24] is that local cohomology is based on a nonclosed support $\mathbb{W}(I, J)$. In order to study the two-variable local cohomology objects $H^{\bullet}_{I,J}(\_\_)$ in $\mathcal{S}_R$, we will need the support theory, associated primes and R-elementary objects in $\mathcal{S}_R$. If $\mathcal{M} \in \mathcal{S}_R$ is a finitely generated object, we show that the set $\mathrm{Ass}(\mathcal{M})$ is finite. For an ideal $I \subseteq R$, the question of the finiteness of the set of associated primes of local cohomology objects is very interesting in commutative algebra and is also related to Faltings' local-global-principle for finiteness dimensions (see [16], and also [13, Section 9.6.2]). In fact, a number of questions related to the supports and associated primes of local cohomology have been studied extensively in the literature (see, for instance, [9–12, 17, 18, 22]). We give a condition for $\mathrm{Ass}(H^i_{I,J}(\mathcal{M})/\mathcal{N})$ to be finite, where $\mathcal{M} \in \mathcal{S}_R$ is finitely generated and $\mathcal{N} \subseteq H^i_{I,J}(\mathcal{M})$ is a finitely generated subobject. This generalizes a result of Brodmann and Faghani [12] to the two-variable local cohomology context, as well as to the abstract module category $\mathcal{S}_R$. In particular, by taking $\mathcal{N} = 0$, we have a condition for the collection of associated primes of $H^i_{I,J}(\mathcal{M})$ to be finite.

Finally, using properties of the injective hull of R-elementary objects, we apply our theory to describe a general functorial setup that is similar to the one recently introduced by Àlvarez Montaner, Boix and Zarzuela in [2] (see also [3]). By considering systems of functors on $\mathcal{S}_R$ satisfying certain conditions with respect to their values on injective hulls of R-elementary objects in $\mathcal{S}_R$, we obtain a general spectral sequence for their derived functors in this context. In particular, this gives us a spectral sequence for two-variable local cohomology objects in $\mathcal{S}_R$. In fact, it is striking that the nonclosed supports $\mathbb{W}(I, J)$ from [24] appear naturally in the theory developed in [2].

We conclude by showing two further applications of the spectral sequence in this formalism. The first is with generalized local cohomology functors given by derived functors of

$$\Gamma_{V_I} : \mathcal{S}_R \longrightarrow \mathcal{S}_R \quad \mathcal{M} \mapsto \varinjlim_{t \geq 1} \underline{\mathrm{Hom}}_R(V/I^t V, \mathcal{M}),$$

where V is a finitely generated R-module and $\underline{\mathrm{Hom}}(V, \_\_) : \mathcal{S}_R \to \mathcal{S}_R$ is the internal hom of an R-module V with respect to an object of $\mathcal{S}_R$. The second is the "generalized Nagata ideal transform" (see [2, Section 4] and [15]) that we extend to $\mathcal{S}_R$ as follows:

$$\Delta_{V_I} : \mathcal{S}_R \longrightarrow \mathcal{S}_R \quad \mathcal{M} \mapsto \varinjlim_{t \geq 1} \underline{\mathrm{Hom}}_R(I^t V, \mathcal{M}),$$

where V is again a finitely generated R-module. In each case, we obtain spectral sequences for their derived functors.

2. R-elementary objects, associated primes and injectives

Let k be a field. Throughout, we let $\mathcal{S} = \mathcal{S}_k$ be a strongly locally noetherian k-linear category. We let R be a commutative and noetherian k-algebra. Accordingly, the category $\mathcal{S}_R$ of R-module objects in $\mathcal{S}$ is locally noetherian. By [6, Section B3], there is a "tensor product" $\_\_ \otimes_R \_\_ : \mathcal{S}_R \times R-\mathrm{Mod} \to \mathcal{S}_R$ that is right exact in both variables. If V is a flat R-module, then the functor $\_\_ \otimes_R V : \mathcal{S}_R \to \mathcal{S}_R$ is exact. Since k is a field, the functor $\mathcal{M} \otimes_k \_\_ : k-\mathrm{Mod} \to \mathcal{S}$ is exact for any $\mathcal{M} \in \mathcal{S}$. Consequently, $(\mathcal{M} \otimes_k R) \otimes_R \_\_ : R-\mathrm{Mod} \to \mathcal{S}_R$ is exact for the k-algebra R (see [6, Section C1]). We will typically write $\otimes := \otimes_k$. We also note that an R-module object in $\mathcal{S}$ may be described by giving $\mathcal{M} \in \mathcal{S}$ and a "structure map" $\mathcal{M} \otimes R \to \mathcal{M}$ satisfying analogues of the usual associativity and unit conditions (see [6, Section B3.14]). For notions such as finitely generated objects, noetherian objects and locally noetherian categories that we will need throughout this paper, we refer the reader to [1].

We will now introduce some notation. Let $r_{\mathcal{M}}$ denote the endomorphism induced on $\mathcal{M} \in \mathcal{S}_R$ by $r \in R$. We set the annihilator $\mathrm{Ann}(\mathcal{M}) \subseteq R$ to be the ideal

$$\mathrm{Ann}(\mathcal{M}) := \{r \in R \mid r_{\mathcal{M}} = 0 : \mathcal{M} = \mathcal{M} \otimes_R R \xrightarrow{\mathcal{M} \otimes r} \mathcal{M} \otimes_R R = \mathcal{M}\}.$$

It is clear that if $\mathcal{M}' \subseteq \mathcal{M}$, then $\mathrm{Ann}(\mathcal{M}) \subseteq \mathrm{Ann}(\mathcal{M}')$. For any $\mathcal{M} \in \mathcal{S}_R$ and any element $a \in R$, we form the short exact sequence

$$0 \longrightarrow \mathrm{Ker}(a_{\mathcal{M}}) \longrightarrow \mathcal{M} \longrightarrow a\mathcal{M} = \mathrm{Im}(a_{\mathcal{M}}) \longrightarrow 0.$$

Since $\mathcal{S}_R$ is a Grothendieck category, an object $\mathcal{M} \in \mathcal{S}_R$ is said to be finitely generated (see, for instance, [23, Section V.3]) if for any directed family $\{\mathcal{M}_i\}_{i \in I}$ of subobjects of $\mathcal{M}$ in $\mathcal{S}_R$ such that $\sum_{i \in I} \mathcal{M}_i = \mathcal{M}$, there exists some $i_0 \in I$ such that $\mathcal{M}_{i_0} = \mathcal{M}$. For any $\mathcal{M} \in \mathcal{S}_R$, we denote by $\mathrm{fg}(\mathcal{M})$ the collection of finitely generated subobjects of $\mathcal{M}$ in $\mathcal{S}_R$. We also let $\mathrm{Pr}(\mathcal{M})$ denote the collection of finitely generated subobjects $0 \neq \mathcal{N} \subseteq \mathcal{M}$ in $\mathcal{S}_R$ such that $\mathrm{Ann}(\mathcal{N})$ is a prime ideal in R.

Definition 2.1. We will say that an object $0 \neq \mathcal{L} \in \mathcal{S}_R$ is R-elementary if it satisfies the following two conditions:

(a) $\mathcal{L}$ is finitely generated.

(b) There is a prime ideal $\mathfrak{p} \subseteq R$ such that for any non-zero subobject $\mathcal{L}' \subseteq \mathcal{L}$, we have $\mathrm{Ann}(\mathcal{L}') = \mathfrak{p}$.

For $\mathcal{M} \in \mathcal{S}_R$, we will say that a prime ideal $\mathfrak{p} \subseteq R$ is an associated prime of $\mathcal{M}$ if there exists an R-elementary object $\mathcal{L} \subseteq \mathcal{M}$ such that $\mathrm{Ann}(\mathcal{L}) = \mathfrak{p}$. We denote by $\mathrm{Ass}(\mathcal{M})$ the collection of associated primes of $\mathcal{M}$.

Our first purpose is to show that associated primes exist for every non-zero $\mathcal{M} \in \mathcal{S}_R$. For this, we first prove the following result.

Lemma 2.2. *Let* $0 \neq \mathcal{M} \in \mathcal{S}_R$*. Then, there exists a finitely generated subobject* $0 \neq \mathcal{N} \subseteq \mathcal{M}$ *in* $\mathcal{S}_R$ *such that* $\mathrm{Ann}(\mathcal{N})$ *is a prime ideal, that is,* $\mathrm{Pr}(\mathcal{M})$ *is non-empty.*

Proof. Because $\mathcal{S}_R$ is locally noetherian, the object $0 \neq \mathcal{M} \in \mathcal{S}_R$ must have non-zero finitely generated subobjects. Since R is noetherian, we can choose an ideal $I \subseteq R$ such that I is maximal among annihilators of finitely generated non-zero subobjects of $\mathcal{M}$. Suppose that $I = \mathrm{Ann}(\mathcal{N})$ for some $0 \neq \mathcal{N} \in \mathrm{fg}(\mathcal{M})$. We claim that I is prime. Otherwise, suppose we have $a, b \in R$ such that $a \notin I$, $b \notin I$ but $ab \in I$.

Since $ab \in I = \mathrm{Ann}(\mathcal{N})$, we have $a_{\mathcal{N}} \circ b_{\mathcal{N}} = 0$, which implies that $b_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}$ factors through the kernel $\mathrm{Ker}(a_{\mathcal{N}})$. Hence, $\pi^a_{\mathcal{N}} \circ b_{\mathcal{N}} = 0$, where $\pi^a_{\mathcal{N}}$ is the canonical epimorphism $\pi^a_{\mathcal{N}} : \mathcal{N} \to a\mathcal{N}$. Since $\pi^a_{\mathcal{N}}$ is a morphism of R-module objects, we obtain $b_{a\mathcal{N}} \circ \pi^a_{\mathcal{N}} = \pi^a_{\mathcal{N}} \circ b_{\mathcal{N}} = 0$. Since $\pi^a_{\mathcal{N}}$ is an epimorphism, it follows that $b_{a\mathcal{N}} = 0$, that is, $b \in \mathrm{Ann}(a\mathcal{N})$.

Since $\mathcal{S}_R$ is locally noetherian, we note that $a\mathcal{N} \subseteq \mathcal{N}$ is finitely generated. Since $a \notin I = \mathrm{Ann}(\mathcal{N})$, we know that $a\mathcal{N} \neq 0$. Clearly, $I = \mathrm{Ann}(\mathcal{N}) \subseteq \mathrm{Ann}(a\mathcal{N})$. Then, $I \subsetneq (b, I) \subseteq \mathrm{Ann}(a\mathcal{N})$, which contradicts the maximality of I. ■

Proposition 2.3. *Let* $0 \neq \mathcal{M} \in \mathcal{S}_R$*. Then,* $\mathrm{Ass}(\mathcal{M}) \neq \phi$*.*

Proof. Suppose that $\mathrm{Ass}(\mathcal{M}) = \phi$. By Lemma 2.2, we know that $\mathrm{Pr}(\mathcal{M}) \neq \phi$. As such, we may choose $0 \neq \mathcal{N}_0 \in \mathrm{fg}(\mathcal{M})$ such that $\mathfrak{p}_0 := \mathrm{Ann}(\mathcal{N}_0)$ is a prime ideal. Since $\mathfrak{p}_0$ cannot be an associated prime of $\mathcal{M}$, we can choose $0 \neq \mathcal{N}'_0 \subseteq \mathcal{N}_0$ such that $\mathfrak{p}_0 = \mathrm{Ann}(\mathcal{N}_0) \subsetneq \mathrm{Ann}(\mathcal{N}'_0)$. Again by Lemma 2.2, we know that $\mathrm{Pr}(\mathcal{N}'_0) \neq \phi$. Accordingly, we can choose $0 \neq \mathcal{N}_1 \subseteq \mathcal{N}'_0$ such that $\mathfrak{p}_1 := \mathrm{Ann}(\mathcal{N}_1)$ is a prime ideal. But then,

$$\mathfrak{p}_0 = \mathrm{Ann}(\mathcal{N}_0) \subsetneq \mathrm{Ann}(\mathcal{N}'_0) \subseteq \mathrm{Ann}(\mathcal{N}_1) = \mathfrak{p}_1.$$

By repeating the argument, we obtain an infinite increasing chain of prime ideals in R. This contradicts the fact that R is noetherian. ■

Proposition 2.4. *Suppose that* $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ *is a short exact sequence in* $\mathcal{S}_R$*. Then, we have* $\mathrm{Ass}(\mathcal{M}') \subseteq \mathrm{Ass}(\mathcal{M}) \subseteq \mathrm{Ass}(\mathcal{M}') \cup \mathrm{Ass}(\mathcal{M}'')$*.*

Proof. From Definition 2.1, it is clear that $\mathrm{Ass}(\mathcal{M}') \subseteq \mathrm{Ass}(\mathcal{M})$. We now consider $\mathfrak{p} \in \mathrm{Ass}(\mathcal{M})$ and an R-elementary object $\mathcal{L} \subseteq \mathcal{M}$ such that $\mathrm{Ann}(\mathcal{L}) = \mathfrak{p}$. We set $\mathcal{K} := \mathcal{M}' \cap \mathcal{L}$. Suppose that $\mathcal{K} = 0$. Then, we have

$$\mathcal{L} = \mathcal{L}/\mathcal{M}' \cap \mathcal{L} \cong (\mathcal{L} + \mathcal{M}')/\mathcal{M}' \subseteq \mathcal{M}/\mathcal{M}' = \mathcal{M}''.$$

Accordingly, $\mathfrak{p} \in \mathrm{Ass}(\mathcal{L}) \subseteq \mathrm{Ass}(\mathcal{M}'')$. Otherwise, suppose that $\mathcal{K} \neq 0$. Then, $0 \neq \mathcal{K} = \mathcal{M}' \cap \mathcal{L} \subseteq \mathcal{L}$ is also R-elementary with $\mathfrak{p} = \mathrm{Ann}(\mathcal{L}) = \mathrm{Ann}(\mathcal{K})$. By Definition 2.1, it is clear that $\mathfrak{p} \in \mathrm{Ass}(\mathcal{K}) \subseteq \mathrm{Ass}(\mathcal{M}')$. ■

Proposition 2.5. *Let $\mathcal{M} \in \mathcal{S}_R$ be finitely generated. Then, there exists a finite filtration*

$$0 = \mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \cdots \subseteq \mathcal{M}_p = \mathcal{M} \tag{2.1}$$

such that each successive quotient $\mathcal{M}_i/\mathcal{M}_{i-1}$ is R-elementary. Additionally, $\mathrm{Ass}(\mathcal{M}) \subseteq \bigcup_{i=1}^{p} \mathrm{Ass}(\mathcal{M}_i/\mathcal{M}_{i-1})$.

Proof. Let $\mathcal{M} \neq 0$. By Proposition 2.3, we can choose a subobject $0 \neq \mathcal{M}_1 \subseteq \mathcal{M}$ that is R-elementary. If $\mathcal{M} = \mathcal{M}_1$, we are done. Otherwise, we continue by applying Proposition 2.3 to $\mathcal{M}/\mathcal{M}_1$, which gives us a filtration as in (2.1). The filtration must be finite because $\mathcal{S}_R$ is a locally noetherian category and $\mathcal{M} \in \mathcal{S}_R$ is a finitely generated object. The last assertion follows by applying Proposition 2.4 repeatedly to the short exact sequences $0 \to \mathcal{M}_{i-1} \to \mathcal{M}_i \to \mathcal{M}_i/\mathcal{M}_{i-1} \to 0$. ■

Corollary 2.6. *Let $\mathcal{M} \in \mathcal{S}_R$ be finitely generated. Then, the set* $\mathrm{Ass}(\mathcal{M})$ *is finite.*

Proof. This is clear from Proposition 2.5. ■

Since $\mathcal{S}_R$ is a Grothendieck category, a subobject $\mathcal{N}$ of $\mathcal{M} \in \mathcal{S}_R$ is said to be essential if $\mathcal{N} \cap \mathcal{N}' \neq 0$ for every non-zero subobject $\mathcal{N}' \subseteq \mathcal{M}$ (see, for instance, [23, Section V.2]).

Corollary 2.7. *Let $\mathcal{M} \in \mathcal{S}_R$, and let $\mathcal{N} \subseteq \mathcal{M}$ be an essential subobject. Then,* $\mathrm{Ass}(\mathcal{N}) = \mathrm{Ass}(\mathcal{M})$.

Proof. It follows from Proposition 2.4 that $\mathrm{Ass}(\mathcal{N}) \subseteq \mathrm{Ass}(\mathcal{M})$. On the other hand, let $\mathfrak{p} \in \mathrm{Ass}(\mathcal{M})$ and $0 \neq \mathcal{L} \in \mathrm{fg}(\mathcal{M})$ be an R-elementary object such that $\mathrm{Ann}(\mathcal{L}) = \mathfrak{p}$. Since $\mathcal{N} \subseteq \mathcal{M}$ is essential, we know $\mathcal{N} \cap \mathcal{L} \neq 0$. It is now clear from Definition 2.1 that $\mathcal{N} \cap \mathcal{L}$ is R-elementary and $\mathrm{Ann}(\mathcal{N} \cap \mathcal{L}) = \mathfrak{p}$. ■

Since $\mathcal{S}_R$ is a Grothendieck category, every object $\mathcal{M} \in \mathcal{S}_R$ has an injective hull, which we will denote by $\mathcal{E}(\mathcal{M})$. We will now prove the main result of this section, which describes the injectives of $\mathcal{S}_R$ in terms of injective envelopes of R-elementary objects.

Theorem 2.8. *Let $\mathcal{S}$ be a strongly locally noetherian Grothendieck category, and let R be a commutative noetherian k-algebra. Then, every injective object $\mathcal{E}$ in $\mathcal{S}_R$ can be expressed as a direct sum of injective hulls of R-elementary objects.*

Proof. We choose a family $\{\mathcal{E}_i\}_{i \in I}$ of subobjects of $\mathcal{E}$ satisfying the following two conditions:

(1) Each $\mathcal{E}_i$ is isomorphic to the injective hull of an R-elementary object in $\mathcal{S}_R$.

(2) The sum $\sum_{i \in I} \mathcal{E}_i \subseteq \mathcal{E}$ is direct, that is, $(\mathcal{E}_i) \cap \left(\sum_{j \neq i} \mathcal{E}_j\right) = 0$ for each $i \in I$.

Since finite limits commute with filtered colimits in $\mathcal{S}_R$, condition (2) is equivalent to $0 = (\mathcal{E}_i) \cap \left(\sum_{j \in J} \mathcal{E}_j\right)$ for each $i \in I$ and any finite $J \subseteq I \backslash \{i\}$. Accordingly, by Zorn's lemma, we may choose a maximal such family $\{\mathcal{E}_i\}_{i \in I_0}$ inside $\mathcal{E}$.

We now set $\mathcal{E}' := \bigoplus_{i \in I_0} \mathcal{E}_i$. Since $\mathcal{S}_R$ is locally noetherian, the direct sum $\mathcal{E}'$ of injectives must be injective. Accordingly, we can split $\mathcal{E}$ as $\mathcal{E} = \mathcal{E}' \oplus \mathcal{E}''$. If $\mathcal{E}'' = 0$, the result is already proved. If $\mathcal{E}'' \neq 0$, it follows from Proposition 2.3 that there is an R-elementary object $\mathcal{L} \subseteq \mathcal{E}''$. Since $\mathcal{E}''$ is injective, the injective hull $\mathcal{E}(\mathcal{L}) \subseteq \mathcal{E}''$. This contradicts the maximality of the family $\{\mathcal{E}_i\}_{i \in I_0}$. ∎

If $T \subseteq R$ is a multiplicatively closed set, the localization $\mathcal{M}_T$ of an object $\mathcal{M} \in \mathcal{S}_R$ with respect to T is given by $\mathcal{M} \otimes_R R[T^{-1}]$ (see [6, Section B6]). The localization $\mathcal{M}_T$ may also be expressed as the filtered colimit of copies of $\mathcal{M}$ connected by morphisms $t_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ over all $t \in T$ (see [6, Section B6.1]). Accordingly, the kernel of the canonical map $\mathcal{M} \to \mathcal{M}_T$ is given by the filtered colimit (see [6, Proposition B6.2])

$$\operatorname{Ker}(\mathcal{M} \longrightarrow \mathcal{M}_T) = \bigcup_{t \in T} \operatorname{Ker}(t_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M}). \tag{2.2}$$

Moreover, since $R[T^{-1}]$ is a flat R-module, it follows from [6, Proposition C1.7] that localization is an exact functor on $\mathcal{S}_R$. If $\mathfrak{p}$ is a prime ideal, we denote by $\mathcal{M}_{\mathfrak{p}}$ the localization with respect to $R \backslash \mathfrak{p}$. We now have the following definition.

Definition 2.9. Let $\mathcal{M} \in \mathcal{S}_R$. Then, the support $\operatorname{Supp}(\mathcal{M})$ of $\mathcal{M}$ consists of all prime ideals $\mathfrak{p} \subseteq R$ such that $\mathcal{M}_{\mathfrak{p}} \neq 0$.

Proposition 2.10. *We now have the following result:*

(a) *Let* $\mathcal{M} \in \mathcal{S}_R$, *and let* $\mathfrak{p} \in \operatorname{Supp}(\mathcal{M})$. *Then,* $\mathfrak{p} \supseteq \operatorname{Ann}(\mathcal{M})$.

(b) *If* $\mathcal{M} \in \mathcal{S}_R$ *is finitely generated, then* $\operatorname{Supp}(\mathcal{M}) = \mathbb{V}(\operatorname{Ann}(\mathcal{M}))$, *the collection of prime ideals containing* $\operatorname{Ann}(\mathcal{M})$.

(c) *For a short exact sequence* $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ *in* $\mathcal{S}_R$, *we have* $\operatorname{Supp}(\mathcal{M}) = \operatorname{Supp}(\mathcal{M}') \cup \operatorname{Supp}(\mathcal{M}'')$.

Proof. (a) If $\mathfrak{p} \not\supseteq \operatorname{Ann}(\mathcal{M})$, we choose $t \in \operatorname{Ann}(\mathcal{M})$ such that $t \notin \mathfrak{p}$. Since t is a unit in $R_{\mathfrak{p}}$, it follows that $\mathcal{M} \otimes_R t = t_{\mathcal{M}} \otimes_R R_{\mathfrak{p}} = 0 : \mathcal{M}_{\mathfrak{p}} = \mathcal{M} \otimes_R R_{\mathfrak{p}} \to \mathcal{M} \otimes_R R_{\mathfrak{p}} = \mathcal{M}_{\mathfrak{p}}$ is an isomorphism. This gives $\mathcal{M}_{\mathfrak{p}} = 0$, and hence $\mathfrak{p} \notin \operatorname{Supp}(\mathcal{M})$. This proves (a).

Additionally, suppose that $\mathcal{M}$ is finitely generated and $\mathfrak{p} \in \mathbb{V}(\operatorname{Ann}(\mathcal{M}))$, but $\mathcal{M}_{\mathfrak{p}} = 0$. Since the colimit in (2.2) is filtered, this means that $\operatorname{Ker}(\mathcal{M} \to \mathcal{M}_{\mathfrak{p}}) = \mathcal{M} = \operatorname{Ker}(t_{\mathcal{M}})$ for some $t \notin \mathfrak{p}$. Then, $t \in \operatorname{Ann}(\mathcal{M})$ and $t \notin \mathfrak{p}$, which is a contradiction. This proves (b). The result of (c) follows directly from the fact that localization is exact. ∎

We conclude this section by establishing some properties of injective hulls in $\mathcal{S}_R$ that we will need throughout this paper. We begin with the following result.

Lemma 2.11. *Let* $\mathcal{M} \in \mathcal{S}_R$, *and let* $T \subseteq R$ *be a multiplicatively closed subset. Then, any subobject of* $\mathcal{M}_T \in \mathcal{S}_{R[T^{-1}]}$ *is of the form* $\mathcal{N}_T$ *for some subobject* $\mathcal{N} \subseteq \mathcal{M}$ *in* $\mathcal{S}_R$.

Proof. Let $\mathcal{K} \subseteq \mathcal{M}_T \in \mathcal{S}_{R[T^{-1}]}$ be a subobject. We set up the following two pullback diagrams

$$\begin{array}{ccccccc} \mathcal{N} & \longrightarrow & \mathcal{M} & & \mathcal{N} \otimes_R R[T^{-1}] & \longrightarrow & \mathcal{M} \otimes_R R[T^{-1}] = \mathcal{M}_T \\ \downarrow & & \downarrow & \Rightarrow & \downarrow & & \downarrow \\ \mathcal{K} & \longrightarrow & \mathcal{M}_T & & \mathcal{K} = \mathcal{K} \otimes_R R[T^{-1}] & \longrightarrow & \mathcal{M}_T \otimes_R R[T^{-1}] = \mathcal{M}_T, \end{array} \tag{2.3}$$

where the right-hand square in (2.3) follows by applying the exact functor $\_\_ \otimes_R R[T^{-1}]$ to the left-hand square. We note that $\mathcal{M}_T = \mathcal{M} \otimes_R R[T^{-1}] = \mathcal{M} \otimes_R R[T^{-1}] \otimes_R R[T^{-1}] = \mathcal{M}_T \otimes_R R[T^{-1}]$ and $\mathcal{K} \otimes_R R[T^{-1}] = \mathcal{K} \otimes_{R[T^{-1}]} R[T^{-1}] \otimes_R R[T^{-1}] = \mathcal{K}$. From (2.3), it is now clear that $\mathcal{N}$ is a subobject of $\mathcal{M}$ and that $\mathcal{K} = \mathcal{N} \otimes_R R[T^{-1}]$. ∎

Proposition 2.12. *Let $\mathcal{E} \in \mathcal{S}_R$ be an injective object, and let $T \subseteq R$ be a multiplicatively closed subset. Then, the localization $\mathcal{E}_T$ is an injective object in $\mathcal{S}_{R[T^{-1}]}$.*

Proof. Let $\{\mathcal{P}_i\}_{i \in I}$ be a set of noetherian generators for $\mathcal{S}_R$. Since $\mathcal{S}$ is strongly locally noetherian, it follows from [6, Proposition B5.1, Corollary B3.17] that $\{\mathcal{P}_i \otimes_R R[T^{-1}]\}_{i \in I}$ is a set of noetherian generators for $\mathcal{S}_{R[T^{-1}]}$. We choose one of these generators $\mathcal{P} \in \{\mathcal{P}_i\}_{i \in I}$ and consider a subobject $\mathcal{K} \subseteq \mathcal{P}_T$ in $\mathcal{S}_{R[T^{-1}]}$. Applying Lemma 2.11, we have a subobject $\mathcal{N} \hookrightarrow \mathcal{P}$ in $\mathcal{S}_R$ such that the localization $\mathcal{N}_T = \mathcal{K}$. We note that since $\mathcal{P} \in \mathcal{S}_R$ is noetherian, so is $\mathcal{N}$.

Since $\mathcal{E} \in \mathcal{S}_R$ is injective, it follows that we have a surjection $\mathcal{S}_R(\mathcal{P}, \mathcal{E}) \twoheadrightarrow \mathcal{S}_R(\mathcal{N}, \mathcal{E})$. As mentioned above, the localization $\mathcal{E}_T$ may be expressed as a filtered colimit of copies of $\mathcal{E}$. Since $\mathcal{N}$, $\mathcal{P}$ are finitely generated, we have induced surjections $\mathcal{S}_R(\mathcal{P}, \mathcal{E}_T) \twoheadrightarrow \mathcal{S}_R(\mathcal{N}, \mathcal{E}_T)$. In other words, we have a surjection $\mathcal{S}_{R[T^{-1}]}(\mathcal{P}_T, \mathcal{E}_T) = \mathcal{S}_{R[T^{-1}]}(\mathcal{P} \otimes_R R[T^{-1}], \mathcal{E}_T) \twoheadrightarrow \mathcal{S}_{R[T^{-1}]}(\mathcal{N} \otimes_R R[T^{-1}], \mathcal{E}_T) = \mathcal{S}_{R[T^{-1}]}(\mathcal{K}, \mathcal{E}_T)$. Applying Baer's criterion (see [23, Proposition V.2.9]) to the Grothendieck category $\mathcal{S}_{R[T^{-1}]}$, it follows that $\mathcal{E}_T$ is injective. ∎

We now need the following fact, which does not seem to have appeared before in the literature.

Lemma 2.13. *Let $\mathcal{D}$ be a locally noetherian category. Then, a filtered colimit of essential monomorphisms in $\mathcal{D}$ is an essential monomorphism.*

Proof. Let $\{\mathcal{B}_i \to \mathcal{A}_i\}_{i \in I}$ be a filtered system of essential monomorphisms. Then, $\mathcal{B} := \varinjlim_{i \in I} \mathcal{B}_i \to \mathcal{A} := \varinjlim_{i \in I} \mathcal{A}_i$ is a monomorphism, and we claim that $\mathcal{B} \subseteq \mathcal{A}$ is essential. For this, we consider a subobject $0 \neq \mathcal{F} \subseteq \mathcal{A}$ and suppose that $\mathcal{F} \cap \mathcal{B} = 0$. Since every object in $\mathcal{D}$ is the sum of its finitely generated subobjects, it is enough to consider the case where $\mathcal{F}$ is finitely generated.

Accordingly, we can choose $j \in I$ large enough such that the inclusion $\mathcal{F} \hookrightarrow \mathcal{A}$ factors through the canonical morphism $\mathcal{A}_j \to \mathcal{A}$. Then, $\mathcal{F} \to \mathcal{A}_j$ must be a monomorphism.

Since $\mathcal{B}_j \subseteq \mathcal{A}_j$ is essential, we have $\mathcal{F} \cap \mathcal{B}_j \neq 0$. Since $\mathcal{F} \cap \mathcal{B}_j \subseteq \mathcal{F} \cap \mathcal{B}$, it follows that $\mathcal{F} \cap \mathcal{B} \neq 0$, which is a contradiction. ∎

Proposition 2.14. *Let $T \subseteq R$ be a multiplicatively closed subset. Let $\mathcal{N} \in \mathcal{S}_R$, and let $\mathcal{E}(\mathcal{N})$ be its injective hull. Then, the localization $\mathcal{E}(\mathcal{N})_T$ is the injective hull of $\mathcal{N}_T$ in $\mathcal{S}_{R[T^{-1}]}$.*

Proof. We know that the canonical inclusion $\mathcal{N} \hookrightarrow \mathcal{E}(\mathcal{N})$ is essential. Since the localization is obtained by taking a filtered colimit, it follows from Lemma 2.13 that $\mathcal{N}_T \hookrightarrow \mathcal{E}(\mathcal{N})_T$ is essential. Since $\mathcal{E}(\mathcal{N})$ is injective in $\mathcal{S}_R$, we know from Proposition 2.12 that $\mathcal{E}(\mathcal{N})_T$ is injective in $\mathcal{S}_{R[T^{-1}]}$. The result is now clear. ∎

Proposition 2.15. *Let $\mathcal{S}$ be a strongly locally noetherian Grothendieck category, and let R be a commutative noetherian k-algebra. Let $\mathcal{N} \subseteq \mathcal{M}$ be an essential subobject in $\mathcal{S}_R$. Then,* $\mathrm{Supp}(\mathcal{N}) = \mathrm{Supp}(\mathcal{M})$.

Proof. According to Proposition 2.14, it is clear that $\mathrm{Supp}(\mathcal{N}) = \mathrm{Supp}(\mathcal{E}(\mathcal{N}))$. In general, if $\mathcal{N} \subseteq \mathcal{M}$ is essential, there is an inclusion $\mathcal{M} \hookrightarrow \mathcal{E}(\mathcal{N})$. Accordingly, $\mathrm{Supp}(\mathcal{N}) = \mathrm{Supp}(\mathcal{M}) = \mathrm{Supp}(\mathcal{E}(\mathcal{N}))$. ∎

3. Local cohomology objects for a pair of ideals

Let I, J be ideals in R. If $\mathcal{M} \in \mathcal{S}_R$ and $\mathfrak{G}$ is any family of subobjects of $\mathcal{M}$, we write $\bigoplus \mathfrak{G}$ (resp. $\sum \mathfrak{G}$) for the direct sum (resp. the sum in $\mathcal{M}$) of all the objects in $\mathfrak{G}$. We now define

$$\begin{aligned} |\Gamma_{I,J}(\mathcal{M})| &:= \{\mathcal{N} \in \mathrm{fg}(\mathcal{M}) \mid I^n \subseteq \mathrm{Ann}(\mathcal{N}) + J \text{ for } n \gg 1\}, \\ \Gamma_{I,J}(\mathcal{M}) &:= \sum |\Gamma_{I,J}(\mathcal{M})|. \end{aligned} \tag{3.1}$$

The definition in (3.1) is motivated by the notion of local cohomology for modules with respect to a pair of ideals introduced by Takahashi, Yoshino and Yoshizawa in [24]. We will say that $\mathcal{M} \in \mathcal{S}_R$ is (I, J)-torsion if $\Gamma_{I,J}(\mathcal{M}) = \mathcal{M}$. We will say that $\mathcal{M} \in \mathcal{S}_R$ is (I, J)-torsion free if $\Gamma_{I,J}(\mathcal{M}) = 0$. When $J = 0$, we will denote $\Gamma_{I,J}$ simply by Γ_I.

Lemma 3.1. *We start with the following result:*

(a) *For $\mathcal{M} \in \mathcal{S}_R$, the family $|\Gamma_{I,J}(\mathcal{M})|$ is closed under subobjects and finite sums. Further, any finitely generated $\mathcal{N} \subseteq \Gamma_{I,J}(\mathcal{M})$ lies in the family $|\Gamma_{I,J}(\mathcal{M})|$.*

(b) *$\Gamma_{I,J}$ is a left exact functor on $\mathcal{S}_R$.*

Proof. (a) It is clear that $|\Gamma_{I,J}(\mathcal{M})|$ is closed under subobjects. We consider $\mathcal{N}_1, \mathcal{N}_2 \in |\Gamma_{I,J}(\mathcal{M})|$, with $I^{n_1} \subseteq \mathrm{Ann}(\mathcal{N}_1) + J$ and $I^{n_2} \subseteq \mathrm{Ann}(\mathcal{N}_2) + J$. Then, we have

$$\begin{aligned} I^{n_1+n_2} &\subseteq (\mathrm{Ann}(\mathcal{N}_1) + J)(\mathrm{Ann}(\mathcal{N}_2) + J) \\ &\subseteq \mathrm{Ann}(\mathcal{N}_1) \cap \mathrm{Ann}(\mathcal{N}_2) + J \subseteq \mathrm{Ann}(\mathcal{N}_1 + \mathcal{N}_2) + J. \end{aligned}$$

Hence, $|\Gamma_{I,J}(\mathcal{M})|$ is closed under finite sums. Now if $\mathcal{N} \subseteq \Gamma_{I,J}(\mathcal{M})$ is finitely generated, it follows from the definition in (3.1) that $\mathcal{N}$ lies within a finite sum of objects in $|\Gamma_{I,J}(\mathcal{M})|$. Since $|\Gamma_{I,J}(\mathcal{M})|$ is closed under subobjects and finite sums, the result is now clear.

(b) Let $\phi : \mathcal{M}' \to \mathcal{M}$ be a morphism in $\mathcal{S}_R$. For any $\mathcal{N}' \subseteq \mathcal{M}'$, the quotient $\phi(\mathcal{N}') \subseteq \mathcal{M}$ satisfies $\mathrm{Ann}(\mathcal{N}') \subseteq \mathrm{Ann}(\phi(\mathcal{N}'))$. Accordingly, if $\mathcal{N}' \in |\Gamma_{I,J}(\mathcal{M}')|$, then $\phi(\mathcal{N}') \in |\Gamma_{I,J}(\mathcal{M})|$. This determines a morphism $\Gamma_{I,J}(\phi) : \Gamma_{I,J}(\mathcal{M}') \to \Gamma_{I,J}(\mathcal{M})$. In particular, if $\phi : \mathcal{M}' \to \mathcal{M}$ is a monomorphism, we have $|\Gamma_{I,J}(\mathcal{M}')| \subseteq |\Gamma_{I,J}(\mathcal{M})|$. Hence, $\Gamma_{I,J}(\mathcal{M}') \subseteq \Gamma_{I,J}(\mathcal{M})$. ∎

As defined in [24], we now consider for the pair of ideals (I, J) the set

$$\mathbb{W}(I, J) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid I^n \subseteq \mathfrak{p} + J \text{ for } n \gg 1\}.$$

When $J = 0$, we note that $\mathbb{W}(I, J) = \mathbb{V}(I)$, the set of prime ideals containing I.

Lemma 3.2. *Let $\mathcal{N} \in \mathcal{S}_R$ be finitely generated. Then,* $\mathrm{Supp}(\mathcal{N}) \subseteq \mathbb{W}(I, J)$ *if and only if* $I^n \subseteq \mathrm{Ann}(\mathcal{N}) + J$ *for $n \gg 1$.*

Proof. Since $\mathcal{N}$ is finitely generated, it follows from Proposition 2.10 that $\mathrm{Supp}(\mathcal{N}) = \mathbb{V}(\mathrm{Ann}(\mathcal{N}))$. First, we suppose that $I^n \subseteq \mathrm{Ann}(\mathcal{N}) + J$. Then, if $\mathfrak{p} \in \mathrm{Supp}(\mathcal{N})$, that is, $\mathfrak{p} \supseteq \mathrm{Ann}(\mathcal{N})$, we see that $I^n \subseteq J + \mathfrak{p}$, that is, $\mathfrak{p} \in \mathbb{W}(I, J)$.

Conversely, suppose that $\mathrm{Supp}(\mathcal{N}) \subseteq \mathbb{W}(I, J)$. If $\mathfrak{p}$ is a prime ideal containing $\mathrm{Ann}(\mathcal{N})$, then there is $n_{\mathfrak{p}} \geq 1$ such that $I^{n_{\mathfrak{p}}} \subseteq \mathfrak{p} + J$. Since R is noetherian, there are only finitely many primes minimal over $\mathrm{Ann}(\mathcal{N})$. It follows that we can find $n' \geq 1$ such that $I^{n'} \subseteq \mathrm{rad}(\mathrm{Ann}(\mathcal{N})) + J$. Again since R is noetherian, some power of $\mathrm{rad}(\mathrm{Ann}(\mathcal{N}))$ lies in $\mathrm{Ann}(\mathcal{N})$ and the result follows. ∎

It follows from Lemma 3.2 that $\Gamma_{I,J}(\mathcal{M})$ is the sum of all finitely generated subobjects $\mathcal{N} \in \mathrm{fg}(\mathcal{M})$ satisfying $\mathrm{Supp}(\mathcal{N}) \subseteq \mathbb{W}(I, J)$. We note that this implies $\mathrm{Supp}(\Gamma_{I,J}(\mathcal{M})) \subseteq \mathbb{W}(I, J)$. We will now describe the associated primes of $\Gamma_{I,J}(\mathcal{M})$.

Proposition 3.3. *For $\mathcal{M} \in \mathcal{S}_R$, we have* $\mathrm{Ass}(\mathcal{M}) \cap \mathbb{W}(I, J) = \mathrm{Ass}(\Gamma_{I,J}(\mathcal{M}))$.

Proof. Since $\Gamma_{I,J}(\mathcal{M}) \subseteq \mathcal{M}$, we have $\mathrm{Ass}(\Gamma_{I,J}(\mathcal{M})) \subseteq \mathrm{Ass}(\mathcal{M})$. If $\mathfrak{p} \in \mathrm{Ass}(\Gamma_{I,J}(\mathcal{M}))$, we consider an R-elementary object $\mathcal{L} \subseteq \Gamma_{I,J}(\mathcal{M})$ such that $\mathrm{Ann}(\mathcal{L}) = \mathfrak{p}$. By Lemmas 3.1 and 3.2, we see that $\mathfrak{p} \in \mathrm{Ass}(\mathcal{L}) \subseteq \mathrm{Supp}(\mathcal{L}) \subseteq \mathbb{W}(I, J)$. Hence, $\mathrm{Ass}(\Gamma_{I,J}(\mathcal{M})) \subseteq \mathrm{Ass}(\mathcal{M}) \cap \mathbb{W}(I, J)$.

Conversely, we take $\mathfrak{p} \in \mathrm{Ass}(\mathcal{M}) \cap \mathbb{W}(I, J)$ and consider an R-elementary object $\mathcal{L} \subseteq \mathcal{M}$ such that $\mathrm{Ann}(\mathcal{L}) = \mathfrak{p}$. Since $\mathfrak{p} \in \mathbb{W}(I, J)$, we can take $n \geq 1$ such that $I^n \subseteq \mathfrak{p} + J = \mathrm{Ann}(\mathcal{L}) + J$. Hence, $\mathcal{L} \in |\Gamma_{I,J}(\mathcal{M})|$ and therefore $\mathcal{L} \subseteq \Gamma_{I,J}(\mathcal{M})$. This gives $\mathfrak{p} \in \mathrm{Ass}(\Gamma_{I,J}(\mathcal{M}))$. ∎

If $\mathcal{A}$ is an abelian category, we recall (see, for instance, [8, Section 1.1]) that a torsion theory on $\mathcal{A}$ consists of a pair $(\mathcal{T}, \mathcal{F})$ of strict and full subcategories such that

$\mathcal{A}(X, Y) = 0$ for any $X \in \mathcal{T}$, $Y \in \mathcal{F}$ and any $Z \in \mathcal{A}$ fits into a short exact sequence $0 \to Z^{\mathcal{T}} \to Z \to Z^{\mathcal{F}} \to 0$ with $Z^{\mathcal{T}} \in \mathcal{T}$ and $Z^{\mathcal{F}} \in \mathcal{F}$. Additionally, $(\mathcal{T}, \mathcal{F})$ is said to be hereditary if the torsion class $\mathcal{T}$ is closed under subobjects.

Proposition 3.4. *For ideals I, J, the pair $(\mathcal{T}(\mathcal{S})_{I,J}, \mathcal{F}(\mathcal{S})_{I,J})$ of full subcategories of $\mathcal{S}_R$, where*

$$\mathrm{Ob}(\mathcal{T}(\mathcal{S})_{I,J}) := \{\mathcal{M} \in \mathcal{S}_R \mid \Gamma_{I,J}(\mathcal{M}) = \mathcal{M}\}$$
$$\mathrm{Ob}(\mathcal{F}(\mathcal{S})_{I,J}) := \{\mathcal{M} \in \mathcal{S}_R \mid \Gamma_{I,J}(\mathcal{M}) = 0\}$$

forms a hereditary torsion theory in $\mathcal{S}_R$.

Proof. We consider $\phi : \mathcal{M}' \to \mathcal{M}''$ in $\mathcal{S}_R$, where $\mathcal{M}' \in \mathcal{T}(\mathcal{S})_{I,J}$, $\mathcal{M}'' \in \mathcal{F}(\mathcal{S})_{I,J}$. We consider $\mathcal{N}' \in \mathrm{fg}(\mathcal{M}')$. Then, $\mathrm{Supp}(\phi(\mathcal{N}')) \subseteq \mathrm{Supp}(\mathcal{N}') \subseteq \mathbb{W}(I, J)$ and hence $\phi(\mathcal{M}') \subseteq \Gamma_{I,J}(\mathcal{M}'') = 0$. Now for any $\mathcal{M} \in \mathcal{S}_R$, we consider the short exact sequence

$$0 \longrightarrow \Gamma_{I,J}(\mathcal{M}) \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}/\Gamma_{I,J}(\mathcal{M}) \longrightarrow 0.$$

By Lemma 3.1, it is clear that $\Gamma_{I,J}(\mathcal{M}) \in \mathcal{T}(\mathcal{S})_{I,J}$. If $\mathcal{M}/\Gamma_{I,J}(\mathcal{M}) \notin \mathcal{F}(\mathcal{S})_{I,J}$, there is some $0 \neq \mathcal{N} \in \mathrm{fg}(\mathcal{M}/\Gamma_{I,J}(\mathcal{M}))$ such that $\mathrm{Supp}(\mathcal{N}) \subseteq \mathbb{W}(I, J)$. Then, we have a short exact sequence

$$0 \longrightarrow \Gamma_{I,J}(\mathcal{M}) \longrightarrow \mathcal{N}_0 \longrightarrow \mathcal{N} \longrightarrow 0,$$

where $\mathcal{N}_0$ is the preimage of $\mathcal{N}$ in $\mathcal{M}$. Then, $\mathrm{Supp}(\mathcal{N}_0) = \mathrm{Supp}(\Gamma_{I,J}(\mathcal{M})) \cup \mathrm{Supp}(\mathcal{N}) \subseteq \mathbb{W}(I, J)$ and hence $\mathcal{N}_0 \subseteq \Gamma_{I,J}(\mathcal{M})$, which implies that $\mathcal{N} = 0$. Hence, $(\mathcal{T}(\mathcal{S})_{I,J}, \mathcal{F}(\mathcal{S})_{I,J})$ is a torsion theory on $\mathcal{S}_R$. The fact that $\mathcal{T}(\mathcal{S})_{I,J}$ is closed under subobjects is also clear from Lemma 3.2. ∎

Definition 3.5. Let $\mathcal{M} \in \mathcal{S}_R$, and let $I, J \subseteq R$ be ideals. Then, for any $i \geq 0$, the i-th local cohomology object $H^i_{I,J}(\mathcal{M})$ of $\mathcal{M} \in \mathcal{S}_R$ with respect to (I, J) is given by the i-th right derived functor of $\Gamma_{I,J}$, that is, $H^i_{I,J}(\mathcal{M}) := \mathbb{R}^i\Gamma_{I,J}(\mathcal{M})$.

When $J = 0$, we set $H^i_I := H^i_{I,0}(\mathcal{M})$ and refer to $H^\bullet_I(\mathcal{M})$ as the local cohomology objects with respect to I.

We will now show that if $\mathcal{M} \in \mathcal{S}_R$ is (I, J)-torsion, then $H^i_{I,J}(\mathcal{M}) = 0$ for $i > 0$. This will be achieved by a sequence of steps. We will start by showing that the torsion theory $(\mathcal{T}(\mathcal{S})_{I,J}, \mathcal{F}(\mathcal{S})_{I,J})$ satisfies the property that the torsion part of an injective in $\mathcal{S}_R$ is still an injective.

Lemma 3.6. *Let $\mathcal{E} \in \mathcal{S}_R$ be an injective object. Then, for any ideal $I \subseteq R$, $\Gamma_I(\mathcal{E}) = \Gamma_{I,0}(\mathcal{E})$ is also an injective object in $\mathcal{S}_R$.*

Proof. We will show that for any finitely generated $\mathcal{M} \in \mathcal{S}_R$ and any subobject $\mathcal{N} \subseteq \mathcal{M}$, a morphism $\phi : \mathcal{N} \to \Gamma_I(\mathcal{E})$ extends to a morphism $\psi : \mathcal{M} \to \Gamma_I(\mathcal{E})$. Since $\mathcal{S}_R$ is locally noetherian, it has a set of finitely generated generators. Accordingly, it will follow from Baer's criterion [23, Proposition V.2.9] that $\Gamma_I(\mathcal{E})$ is injective.

Since $\phi(\mathcal{N}) \subseteq \Gamma_I(\mathcal{E}) \subseteq \mathcal{E}$ is finitely generated, we can choose $t \geq 1$ such that $I^t\phi(\mathcal{N}) = 0$. Since $\mathcal{E}$ is injective, the morphism $\mathcal{N} \xrightarrow{\phi} \Gamma_I(\mathcal{E}) \hookrightarrow \mathcal{E}$ extends to a morphism $\psi' : \mathcal{M} \to \mathcal{E}$. In particular, $\phi(\mathcal{N}) \subseteq \psi'(\mathcal{M})$. Since $\mathcal{M}$ is finitely generated, so is $\psi'(\mathcal{M})$. Applying the version of Artin–Rees lemma proved in [6, Proposition D1.8], we can find $c \geq 1$ such that for all $n \geq c$, we have

$$I^n\psi'(\mathcal{M}) \cap \phi(\mathcal{N}) = I^{n-c}(I^c\psi'(\mathcal{M}) \cap \phi(\mathcal{N})) \subseteq I^{n-c}\phi(\mathcal{N}). \tag{3.2}$$

Putting $n = t + c$, it follows from (3.2) that $I^{t+c}\psi'(\mathcal{M}) \cap \phi(\mathcal{N}) \subseteq I^t\phi(\mathcal{N}) = 0$. We now note that

$$\begin{aligned} &\phi(I^{t+c}\mathcal{M} \cap \mathcal{N}) \subseteq \phi(\mathcal{N}),\\ &\phi(I^{t+c}\mathcal{M} \cap \mathcal{N}) \subseteq \psi'(I^{t+c}\mathcal{M} \cap \mathcal{N}) \subseteq \psi'(I^{t+c}\mathcal{M}) \subseteq I^{t+c}\psi'(\mathcal{M}). \end{aligned} \tag{3.3}$$

Accordingly, it follows from (3.3) that $\phi(I^{t+c}\mathcal{M} \cap \mathcal{N}) \subseteq I^{t+c}\psi'(\mathcal{M}) \cap \phi(\mathcal{N}) = 0$. Now since

$$\begin{aligned} I^{t+c}\mathcal{M} \cap \mathcal{N} &= \mathrm{Ker}(I^{t+c}\mathcal{M} \oplus \mathcal{N} \longrightarrow \mathcal{M}),\\ I^{t+c}\mathcal{M} + \mathcal{N} &= \mathrm{Im}(I^{t+c}\mathcal{M} \oplus \mathcal{N} \longrightarrow \mathcal{M}), \end{aligned}$$

the morphism $0 \oplus \phi : I^{t+c}\mathcal{M} \oplus \mathcal{N} \to \Gamma_I(\mathcal{E})$ induces $\phi' : I^{t+c}\mathcal{M} + \mathcal{N} \to \Gamma_I(\mathcal{E})$ such that $\phi' \mid I^{t+c}\mathcal{M} = 0$ and $\phi' \mid \mathcal{N} = \phi$. Again since $\mathcal{E}$ is injective, there exists $\psi : \mathcal{M} \to \mathcal{E}$ extending $I^{t+c}\mathcal{M} + \mathcal{N} \xrightarrow{\phi'} \Gamma_I(\mathcal{E}) \hookrightarrow \mathcal{E}$. We now see that

$$I^{t+c}\psi(\mathcal{M}) = \psi(I^{t+c}\mathcal{M}) = \phi'(I^{t+c}\mathcal{M}) = 0$$

whence it follows that $\psi(\mathcal{M}) \subseteq \Gamma_I(\mathcal{E})$. This proves the result. ∎

The next step is to make use of the directed sets $\widetilde{\mathbb{W}}(I, J)$ defined in [24, Section 3] as follows: the elements of $\widetilde{\mathbb{W}}(I, J)$ are ideals $K \subseteq R$ satisfying $I^n \subseteq K + J$ for $n \gg 1$. For $K, K' \in \widetilde{\mathbb{W}}(I, J)$, we will say that $K \leq K'$ if $K \supseteq K'$. We observe that each $\widetilde{\mathbb{W}}(I, J)$ is filtered.

Lemma 3.7. *Let $\mathcal{M} \in \mathcal{S}_R$. Then, $\Gamma_{I,J}(\mathcal{M}) = \varinjlim_{K \in \widetilde{\mathbb{W}}(I,J)} \Gamma_K(\mathcal{M})$.*

Proof. We set $\mathcal{N} := \varinjlim_{K \in \widetilde{\mathbb{W}}(I,J)} \Gamma_K(\mathcal{M})$ and consider some $\mathcal{N}' \in \mathrm{fg}(\mathcal{N})$. Since $\widetilde{\mathbb{W}}(I, J)$ is filtered, we choose $K \in \widetilde{\mathbb{W}}(I, J)$ such that $\mathcal{N}' \subseteq \Gamma_K(\mathcal{M})$, that is, $K^m \subseteq \mathrm{Ann}(\mathcal{N}')$ for some $m \geq 1$. Since $K \in \widetilde{\mathbb{W}}(I, J)$, we may choose $n \geq 1$ such that $I^n \subseteq K + J$. Then, $I^{mn} \subseteq K^m + J \subseteq \mathrm{Ann}(\mathcal{N}') + J$ and hence $\mathcal{N}' \subseteq \Gamma_{I,J}(\mathcal{M})$. Conversely, we consider some $\mathcal{N}'' \in \mathrm{fg}(\Gamma_{I,J}(\mathcal{M}))$. Then, we have $I^l \subseteq \mathrm{Ann}(\mathcal{N}'') + J$ for some $l \geq 1$. Then, $\mathrm{Ann}(\mathcal{N}'') \in \widetilde{\mathbb{W}}(I, J)$, and we have $\mathcal{N}'' \subseteq \Gamma_{\mathrm{Ann}(\mathcal{N}'')}(\mathcal{M})$. Accordingly, we have $\mathrm{fg}(\Gamma_{I,J}(\mathcal{M})) = \mathrm{fg}(\mathcal{N})$ and hence $\Gamma_{I,J}(\mathcal{M}) = \mathcal{N}$ because $\mathcal{S}_R$ is locally noetherian. ∎

Proposition 3.8. *Let $\mathcal{E} \in \mathcal{S}_R$ be an injective object. Then, for any ideals $I, J \subseteq R$, $\Gamma_{I,J}(\mathcal{E})$ is also an injective object in $\mathcal{S}_R$.*

Proof. Since $\mathcal{E} \in \mathcal{S}_R$ is injective, it follows from Lemma 3.6 that $\Gamma_K(\mathcal{E})$ is injective for each $K \in \widetilde{\mathbb{W}}(I, J)$. We now consider some finitely generated $\mathcal{M} \in \mathcal{S}_R$, a subobject $\mathcal{N} \subseteq \mathcal{M}$ and a morphism $\phi : \mathcal{N} \to \Gamma_{I,J}(\mathcal{E})$. By Lemma 3.7, we know that $\Gamma_{I,J}(\mathcal{E}) = \varinjlim_{K \in \widetilde{\mathbb{W}}(I,J)} \Gamma_K(\mathcal{E})$. Since $\widetilde{\mathbb{W}}(I, J)$ is filtered and $\mathcal{N} \subseteq \mathcal{M}$ is finitely generated, we can find $K_0 \in \widetilde{\mathbb{W}}(I, J)$ such that ϕ factors through $\Gamma_{K_0}(\mathcal{E})$. Since $\Gamma_{K_0}(\mathcal{E})$ is injective, we now have $\psi' : \mathcal{M} \to \Gamma_{K_0}(\mathcal{E})$ extending $\mathcal{N} \to \Gamma_{K_0}(\mathcal{E})$. Composing with the canonical map $\Gamma_{K_0}(\mathcal{E}) \to \varinjlim_{K \in \widetilde{\mathbb{W}}(I,J)} \Gamma_K(\mathcal{M}) = \Gamma_{I,J}(\mathcal{E})$, we obtain an extension of $\phi : \mathcal{N} \to \Gamma_{I,J}(\mathcal{E})$ to a morphism $\psi : \mathcal{M} \to \Gamma_{I,J}(\mathcal{E})$. Since $\mathcal{S}_R$ is a locally noetherian category, it now follows from Baer's criterion (see [23, Proposition V.2.9]) that $\Gamma_{I,J}(\mathcal{E})$ is injective. ∎

Lemma 3.9. *Let $\mathcal{M} \in \mathcal{S}_R$ be an (I, J)-torsion object. Then, its injective hull $\mathcal{E}(\mathcal{M})$ is also (I, J)-torsion.*

Proof. Since $\mathcal{E}(\mathcal{M})$ is injective, Proposition 3.8 implies that $\Gamma_{I,J}(\mathcal{E}(\mathcal{M}))$ is also injective in $\mathcal{S}_R$. Accordingly, we have a direct sum decomposition $\mathcal{E}(\mathcal{M}) = \Gamma_{I,J}(\mathcal{E}(\mathcal{M})) \oplus \mathcal{N}$. We already know that $\mathcal{M} \in \mathcal{S}_R$ is an (I, J)-torsion object, and it is clear from the definition in (3.1) that $\mathcal{M}$ lies inside the torsion part of $\mathcal{E}(\mathcal{M})$. Then, $\mathcal{M} \cap \mathcal{N} = 0$. But $\mathcal{M}$ is essential in $\mathcal{E}(\mathcal{M})$, whence it follows that $\mathcal{N} = 0$, that is, $\mathcal{E}(\mathcal{M}) = \Gamma_{I,J}(\mathcal{E}(\mathcal{M}))$. ∎

Proposition 3.10. *Let $\mathcal{M} \in \mathcal{S}_R$ be an (I, J)-torsion object. Then, $H^i_{I,J}(\mathcal{M}) = 0$ for $i > 0$.*

Proof. By Lemma 3.9, we know that the injective hull of an (I, J)-torsion object is also (I, J)-torsion. By Proposition 3.4, we know that the torsion class $\mathcal{T}(\mathcal{S})_{I,J}$ is closed under both quotients and subobjects. Accordingly, we can obtain a resolution $\mathcal{M} \to \mathcal{E}^\bullet$ consisting of injective objects that are (I, J)-torsion. Therefore, applying the functor $\Gamma_{I,J}$ leaves this sequence unchanged. By definition, the cohomology of the resolution $\mathcal{E}^\bullet$ vanishes in positive degrees, and this proves the result. ∎

Corollary 3.11. *Let $\mathcal{M} \in \mathcal{S}_R$. Then,*

(a) $H^i_{I,J}(\mathcal{M}) \cong H^i_I(\mathcal{M}/\Gamma_{I,J}(\mathcal{M}))$ *for* $i > 0$.

(b) $H^i_{I,J}(\mathcal{M})$ *is* (I, J)*-torsion for any* $i \geq 0$.

Proof. By Proposition 3.10, we know that $H^i_I(\Gamma_{I,J}(\mathcal{M})) = 0$ for $i > 0$. The result of (a) is now clear from the long exact sequence of cohomologies obtained by applying $\Gamma_{I,J}(\_\_)$ to the short exact sequence $0 \to \Gamma_{I,J}(\mathcal{M}) \to \mathcal{M} \to \mathcal{M}/\Gamma_{I,J}(\mathcal{M}) \to 0$. To prove (b), we note that $H^\bullet_{I,J}(\mathcal{M})$ are, by definition, the homology objects of $\Gamma_{I,J}(\mathcal{E}^\bullet)$, where $\mathcal{M} \to \mathcal{E}^\bullet$ is an injective resolution. Since each $\Gamma_{I,J}(\mathcal{E}^\bullet)$ lies in the hereditary torsion class $\mathcal{T}(\mathcal{S})_{I,J}$, so does its subquotient $H^\bullet_{I,J}(\mathcal{M})$. ∎

4. Associated primes of local cohomology objects

In this section, we study finiteness conditions on local cohomology objects and their associated primes. We give a condition for the set of associated primes of the local cohomology object $H^\bullet_{I,J}(\_\_)$ with respect to an ideal pair (I, J) in an abstract module category to be finite. For $\mathcal{M} \in \mathcal{S}_R$, we say that $a \in R$ is a non-zero divisor on $\mathcal{M}$ if $a_\mathcal{M} : \mathcal{M} \to \mathcal{M}$ is a monomorphism.

Lemma 4.1. *Let $a \in R$ be a zero divisor on $\mathcal{M} \in \mathcal{S}_R$. Then, a lies in some associated prime of $\mathcal{M}$.*

Proof. By definition, we know that $\mathrm{Ker}(a_\mathcal{M}) \neq 0$. Using Proposition 2.3, we pick a prime ideal $\mathfrak{p} \in \mathrm{Ass}(\mathrm{Ker}(a_\mathcal{M})) \subseteq \mathrm{Ass}(\mathcal{M})$. Then, $\mathfrak{p} = \mathrm{Ann}(\mathcal{L})$ for some R-elementary object $\mathcal{L} \subseteq \mathrm{Ker}(a_\mathcal{M}) \subseteq \mathcal{M}$. Then, $a \in \mathrm{Ann}(\mathrm{Ker}(a_\mathcal{M})) \subseteq \mathrm{Ann}(\mathcal{L}) = \mathfrak{p}$. ∎

Lemma 4.2. *Let $I, J \subseteq R$ be ideals, and let $\mathcal{M} \in \mathcal{S}_R$ be a finitely generated object such that $\Gamma_{I,J}(\mathcal{M}) = 0$. Then, I contains a non-zero divisor on $\mathcal{M}$.*

Proof. Suppose that all elements of $I \subseteq R$ are zero divisors on $\mathcal{M}$. By Lemma 4.1, $I \subseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}(\mathcal{M})} \mathfrak{p}$. Since $\mathcal{M}$ is finitely generated, $\mathrm{Ass}(\mathcal{M})$ is finite by Corollary 2.6. By prime avoidance, there is some $\mathfrak{p}_0 \in \mathrm{Ass}(\mathcal{M})$ such that $I \subseteq \mathfrak{p}_0$. Hence, $\mathfrak{p}_0 \in \mathbb{W}(I, J)$.

Now since $\mathfrak{p}_0$ is an associated prime of $\mathcal{M}$, there is an R-elementary object $0 \neq \mathcal{L} \subseteq \mathcal{M}$ such that $\mathfrak{p}_0 = \mathrm{Ann}(\mathcal{L})$. Then, $\mathrm{Supp}(\mathcal{L}) = \mathbb{V}(\mathfrak{p}_0) \subseteq \mathbb{W}(I, J)$ and hence $\mathcal{L} \subseteq \Gamma_{I,J}(\mathcal{M}) = 0$, which is a contradiction. ∎

For any $\mathcal{M}$ in $\mathcal{S}_R$ and subobjects $\mathcal{M}', \mathcal{M}'' \subseteq \mathcal{M}$, we set

$$(\mathcal{M}'' : \mathcal{M}') := \{a \in R \mid \mathrm{Im}(a_{\mathcal{M}'}) \subseteq \mathcal{M}''\}.$$

It is clear that any such $(\mathcal{M}'' : \mathcal{M}')$ is an ideal in R.

Lemma 4.3. *Let $\mathcal{N} \subseteq \mathcal{M}$ in $\mathcal{S}_R$, and let $\mathcal{K} \subseteq \mathcal{M}/\mathcal{N}$ be a finitely generated subobject. Then, there exists a finitely generated subobject $\mathcal{K}' \subseteq \mathcal{M}$ such that*

(1) *The quotient map $\pi : \mathcal{M} \twoheadrightarrow \mathcal{M}/\mathcal{N}$ restricts to an epimorphism $\pi' : \mathcal{K}' \twoheadrightarrow \mathcal{K}$.*

(2) $\mathrm{Ann}(\mathcal{K}) = (\mathcal{N} : \mathcal{K}')$.

Proof. We put $\mathcal{K}'' := \pi^{-1}(\mathcal{K}) \subseteq \mathcal{M}$. Since finitely generated subobjects of $\mathcal{K}''$ form a filtered system and $\pi(\mathcal{K}'') = \mathcal{K}$, we can find $\mathcal{K}' \in \mathrm{fg}(\mathcal{K}'')$ such that $\pi(\mathcal{K}') = \mathcal{K}$. This proves (1). To prove (2), for any $a \in R$, we consider the commutative diagram

$$\begin{tikzcd}
\mathcal{K}' \arrow[r, "a_{\mathcal{K}'}"] \arrow[d, "\pi'"'] & \mathcal{K}' \arrow[r, "\iota'"] \arrow[d, "\pi'"'] & \mathcal{M} \arrow[d, "\pi"] \\
\mathcal{K} \arrow[r, "a_\mathcal{K}"] & \mathcal{K} \arrow[r, "\iota"] & \mathcal{M}/\mathcal{N}.
\end{tikzcd}$$

If $a \in \mathrm{Ann}(\mathcal{K})$, that is, $a_{\mathcal{K}} = 0$, we get $\pi \circ \iota' \circ a_{\mathcal{K}'} = \iota \circ a_{\mathcal{K}} \circ \pi' = 0$, which gives $\mathrm{Im}(a_{\mathcal{K}'}) \subseteq \mathrm{Ker}(\pi) = \mathcal{N}$, that is, $a \in (\mathcal{N} : \mathcal{K}')$. Conversely, if $a \in (\mathcal{N} : \mathcal{K}')$, we get $\iota \circ a_{\mathcal{K}} \circ \pi' = 0$. Since π' is an epimorphism and ι is a monomorphism, we get $a_{\mathcal{K}} = 0$ or $a \in \mathrm{Ann}(\mathcal{K})$. ∎

Proposition 4.4. *Let $\mathcal{M} \in \mathcal{S}_R$ be finitely generated, and let $I, J \subseteq R$ be ideals. Suppose that $i \geq 0$ is such that $H^j_{I,J}(\mathcal{M})$ is finitely generated for all $j < i$. Then, for any finitely generated $\mathcal{N} \subseteq H^i_{I,J}(\mathcal{M})$ such that $\mathcal{N} \supseteq JH^i_{I,J}(\mathcal{M})$, the collection $\mathrm{Ass}(H^i_{I,J}(\mathcal{M})/\mathcal{N})$ is finite.*

Proof. For $i = 0$, we know that $H^0_{I,J}(\mathcal{M}) = \Gamma_{I,J}(\mathcal{M}) \subseteq \mathcal{M}$ is finitely generated, and the result is clear. We will proceed by induction on i. By Corollary 3.11, we know that $H^j_{I,J}(\mathcal{M}) \cong H^j_{I,J}(\mathcal{M}/\Gamma_{I,J}(\mathcal{M}))$ for $j > 0$. Also, we know that $\Gamma_{I,J}(\mathcal{M}/\Gamma_{I,J}(\mathcal{M})) = 0$. Accordingly, we may suppose that $\mathcal{M} \in \mathcal{S}_R$ is finitely generated with $\Gamma_{I,J}(\mathcal{M}) = 0$, and it follows from Lemma 4.2 that we can find $a \in I$ which is a non-zero divisor on $\mathcal{M}$.

By Corollary 3.11, we know that $H^i_{I,J}(\mathcal{M})$ is (I, J)-torsion, that is, $\Gamma_{I,J}(H^i_{I,J}(\mathcal{M})) = H^i_{I,J}(\mathcal{M})$. Since $\mathcal{N} \subseteq H^i_{I,J}(\mathcal{M})$ is finitely generated, it follows from Lemma 3.1 that $I^t \subseteq \mathrm{Ann}(\mathcal{N}) + J$ for some $t > 0$. Accordingly, we have $a^t\mathcal{N} \subseteq J\mathcal{N}$. We now consider the short exact sequence

$$0 \longrightarrow \mathcal{M} \xrightarrow{a^t} \mathcal{M} \longrightarrow \mathcal{M}/a^t\mathcal{M} \longrightarrow 0. \tag{4.1}$$

Applying $\Gamma_{I,J}(\_\_)$ to (4.1), the long exact sequence of derived functors gives us

$$H^l_{I,J}(\mathcal{M}) \xrightarrow{a^t} H^l_{I,J}(\mathcal{M}) \longrightarrow H^l_{I,J}(\mathcal{M}/a^t\mathcal{M}) \longrightarrow H^{l+1}_{I,J}(\mathcal{M}) \xrightarrow{a^t} H^{l+1}_{I,J}(\mathcal{M}) \tag{4.2}$$

for every $l \geq 0$. Since $\mathcal{S}_R$ is locally noetherian, the collection of finitely generated objects in $\mathcal{S}_R$ is closed under extensions, subobjects and quotients (see [23, Section V.4.2]). Since $H^j_{I,J}(\mathcal{M})$ is finitely generated for all $j < i$, it follows from (4.2) that $H^l_{I,J}(\mathcal{M}/a^t\mathcal{M})$ is finitely generated for $l < i - 1$.

Since $a^t\mathcal{N} \subseteq J\mathcal{N}$, we also have the following diagram where the top row is exact and the vertical morphisms are epimorphisms

$$\begin{tikzcd}
H^{i-1}_{I,J}(\mathcal{M}) \arrow[r, "\iota"] & H^{i-1}_{I,J}(\mathcal{M}/a^t\mathcal{M}) \arrow[r, "\delta"] & H^i_{I,J}(\mathcal{M}) \arrow[r, "\mu:=a^t"] \arrow[d, "\pi"'] & H^i_{I,J}(\mathcal{M}) \arrow[d, "\pi'"] \\
 & & H^i_{I,J}(\mathcal{M})/\mathcal{N} \arrow[r, "\bar{\mu}:=\bar{a}^t"] & H^i_{I,J}(\mathcal{M})/J\mathcal{N}.
\end{tikzcd}$$

We now observe that $\mathrm{Ker}(\bar{\mu}) = \mu^{-1}(J\mathcal{N})/\mathcal{N}$. This gives us the short exact sequence

$$\begin{aligned}0 \longrightarrow H^{i-1}_{I,J}(\mathcal{M}/a^t\mathcal{M})/\delta^{-1}(\mathcal{N}) = (\mathrm{Im}(\delta) + \mathcal{N})/\mathcal{N} \longrightarrow \\ \mathrm{Ker}(\bar{\mu}) = \mu^{-1}(J\mathcal{N})/\mathcal{N} \longrightarrow \mu^{-1}(J\mathcal{N})/(\mathrm{Ker}(\mu) + \mathcal{N}) \longrightarrow 0,\end{aligned} \tag{4.3}$$

where we have used $\mathrm{Ker}(\mu) = \mathrm{Im}(\delta)$. Since $J\mathcal{N} \subseteq \mathcal{N}$ is finitely generated, so is its subobject $\mu^{-1}(J\mathcal{N})/\mathrm{Ker}(\mu)$. Then, the quotient $\mu^{-1}(J\mathcal{N})/(\mathrm{Ker}(\mu) + \mathcal{N})$ of $\mu^{-1}(J\mathcal{N})/\mathrm{Ker}(\mu)$ is also finitely generated.

On the other hand, we have the short exact sequence

$$0 \longrightarrow \mathrm{Im}(\iota) \cap \delta^{-1}(\mathcal{N}) = \mathrm{Ker}(\delta) \cap \delta^{-1}(\mathcal{N}) \longrightarrow \delta^{-1}(\mathcal{N}) \xrightarrow{\delta} \delta(\delta^{-1}(\mathcal{N})) \longrightarrow 0. \tag{4.4}$$

By assumption, $H^{i-1}_{I,J}(\mathcal{M})$ is finitely generated, and hence so is its subquotient $\mathrm{Im}(\iota) \cap \delta^{-1}(\mathcal{N})$. Also since $\mathcal{N}$ is finitely generated, so is $\delta(\delta^{-1}(\mathcal{N})) \subseteq \mathcal{N}$. Again since $\mathcal{S}_R$ is locally noetherian, it follows from (4.4) that $\delta^{-1}(\mathcal{N})$ is finitely generated. Additionally, we note that

$$\begin{gathered} \delta(JH^{i-1}_{I,J}(\mathcal{M}/a^t\mathcal{M})) \subseteq J\delta(H^{i-1}_{I,J}(\mathcal{M}/a^t\mathcal{M})) \subseteq JH^{i}_{I,J}(\mathcal{M}) \subseteq \mathcal{N} \\ \Downarrow \\ JH^{i-1}_{I,J}(\mathcal{M}/a^t\mathcal{M}) \subseteq \delta^{-1}(\mathcal{N}). \end{gathered}$$

We now put $\mathcal{K} := H^{i-1}_{I,J}(\mathcal{M}/a^t\mathcal{M})/\delta^{-1}(\mathcal{N})$ and $\mathcal{P} := \mathrm{Ker}(\overline{\mu})$. By the induction assumption, we see that $\mathrm{Ass}(\mathcal{K})$ is a finite set. Since $\mu^{-1}(J\mathcal{N})/(\mathrm{Ker}(\mu) + \mathcal{N})$ is finitely generated, from the short exact sequence (4.3), it now follows from Proposition 2.4 that $\mathrm{Ass}(\mathcal{P})$ is finite.

Since $\mathcal{N}$ is finitely generated, so is its image $\pi'(\mathcal{N})$. To prove the result, it now suffices to show that

$$\mathrm{Ass}(H^{i}_{I,J}(\mathcal{M})/\mathcal{N}) \subseteq \mathrm{Ass}(\mathcal{P}) \cup \mathrm{Ass}(\pi'(\mathcal{N})).$$

Accordingly, we choose some prime ideal $\mathfrak{p} \in \mathrm{Ass}(H^{i}_{I,J}(\mathcal{M})/\mathcal{N})\backslash\mathrm{Ass}(\mathcal{P})$. Then, there is an R-elementary object $\mathcal{L} \subseteq H^{i}_{I,J}(\mathcal{M})/\mathcal{N}$ such that $\mathfrak{p} = \mathrm{Ann}(\mathcal{L})$. Applying Lemma 4.3, we can choose a finitely generated subobject $\mathcal{L}' \subseteq H^{i}_{I,J}(\mathcal{M})$ such that $\pi : H^{i}_{I,J}(\mathcal{M}) \to H^{i}_{I,J}(\mathcal{M})/\mathcal{N}$ restricts to an epimorphism $\mathcal{L}' \twoheadrightarrow \mathcal{L}$ and $(\mathcal{N} : \mathcal{L}') = \mathrm{Ann}(\mathcal{L}) = \mathfrak{p}$. We now consider the short exact sequence

$$0 \longrightarrow \mathcal{P} \cap \mathcal{L} \longrightarrow \mathcal{L} \xrightarrow{\overline{\mu}|\mathcal{L} = \overline{a}^t} \overline{a}^t\mathcal{L} \longrightarrow 0.$$

By Proposition 2.4, we get $\mathrm{Ass}(\mathcal{L}) \subseteq \mathrm{Ass}(\mathcal{P} \cap \mathcal{L}) \cup \mathrm{Ass}(\overline{a}^t\mathcal{L}) \subseteq \mathrm{Ass}(\mathcal{P}) \cup \mathrm{Ass}(\overline{a}^t\mathcal{L})$. Since $\mathfrak{p} \in \mathrm{Ass}(\mathcal{L})$ does not lie in $\mathrm{Ass}(\mathcal{P})$, we obtain $\mathfrak{p} \in \mathrm{Ass}(\overline{a}^t\mathcal{L}) = \mathrm{Ass}(\overline{a}^t\pi(\mathcal{L}')) = \mathrm{Ass}(\pi'a^t\mathcal{L}')$.

We know that $a^t\mathcal{L}' \subseteq \mathcal{L}'$ is finitely generated. Again since $H^{i}_{I,J}(\mathcal{M})$ is (I, J)-torsion, it follows that $a^{t+s}\mathcal{L}' \subseteq J\mathcal{L}' \subseteq JH^{i}_{I,J}(\mathcal{M}) \subseteq \mathcal{N}$ for some $s \geq 1$. But since $(\mathcal{N} : \mathcal{L}') = \mathfrak{p}$, we get $a^{t+s} \in \mathfrak{p}$, that is, $a \in \mathfrak{p}$. Again since $\mathfrak{p} = (\mathcal{N} : \mathcal{L}')$, we obtain $a^t\mathcal{L}' \subseteq \mathcal{N}$ and hence $\pi'a^t\mathcal{L}' \subseteq \pi'(\mathcal{N})$. Then, $\mathfrak{p} \in \mathrm{Ass}(\pi'a^t\mathcal{L}') \subseteq \mathrm{Ass}(\pi'(\mathcal{N}))$. ■

Theorem 4.5. *Let $\mathcal{M} \in \mathcal{S}_R$ be finitely generated, and let $I, J \subseteq R$ be ideals. Suppose that $i \geq 0$ is such that $H^{j}_{I,J}(\mathcal{M})$ is finitely generated for all $j < i$. Suppose that $JH^{i}_{I,J}(\mathcal{M})$ is finitely generated. Then, for any finitely generated $\mathcal{N} \subseteq H^{i}_{I,J}(\mathcal{M})$, the collection $\mathrm{Ass}(H^{i}_{I,J}(\mathcal{M})/\mathcal{N})$ is finite.*

Proof. Since $\mathcal{N}$ and $JH^i_{I,J}(\mathcal{M})$ are both finitely generated, so is $\mathcal{N} + JH^i_{I,J}(\mathcal{M})$. By Proposition 4.4, it now follows that $\mathrm{Ass}(H^i_{I,J}(\mathcal{M})/(\mathcal{N} + JH^i_{I,J}(\mathcal{M})))$ is finite. We now consider the short exact sequence

$$0 \longrightarrow (\mathcal{N} + JH^i_{I,J}(\mathcal{M}))/\mathcal{N} \longrightarrow H^i_{I,J}(\mathcal{M})/\mathcal{N} \longrightarrow H^i_{I,J}(\mathcal{M})/(\mathcal{N} + JH^i_{I,J}(\mathcal{M})) \longrightarrow 0. \tag{4.5}$$

Since $(\mathcal{N} + JH^i_{I,J}(\mathcal{M}))/\mathcal{N}$ is finitely generated and $\mathrm{Ass}(H^i_{I,J}(\mathcal{M})/(\mathcal{N} + JH^i_{I,J}(\mathcal{M})))$ is finite, the result follows by applying Proposition 2.4 to the short exact sequence (4.5). ■

Corollary 4.6. *Let $\mathcal{M} \in \mathcal{S}_R$ be finitely generated, and let $I, J \subseteq R$ be ideals. Suppose that $i \geq 0$ is such that $H^j_{I,J}(\mathcal{M})$ is finitely generated for all $j < i$. Suppose that $JH^i_{I,J}(\mathcal{M})$ is finitely generated. Then, the collection $\mathrm{Ass}(H^i_{I,J}(\mathcal{M}))$ is finite.*

Proof. This follows directly from Theorem 4.5 by setting $\mathcal{N} = 0$. ■

We conclude this section with the following fact, which extends a result of Brodmann, Rotthaus and Sharp [11].

Proposition 4.7. *Let $\mathcal{M} \in \mathcal{S}_R$ be such that the collection χ of maximal elements in $\mathrm{Ass}(\mathcal{M})$ is finite. Let $I \subseteq R$ be an ideal. Suppose that for each $\mathfrak{p} \in \chi$, there is $n_{\mathfrak{p}} > 0$ such that $I^{n_{\mathfrak{p}}}\mathcal{M}_{\mathfrak{p}} = 0$. If $n := \max\{n_{\mathfrak{p}} \mid \mathfrak{p} \in \chi\}$, then $I^n\mathcal{M} = 0$.*

Proof. We consider some $\mathcal{N} \in \mathrm{fg}(\mathcal{M})$. It is clear that $(I^n\mathcal{N})_{\mathfrak{p}} \subseteq I^n\mathcal{M}_{\mathfrak{p}} = 0$. Since $\mathcal{S}_R$ is locally noetherian, we know that $I^n\mathcal{N} \subseteq \mathcal{N}$ is finitely generated. From (2.2), we see that for each $\mathfrak{p} \in \chi$, we have

$$(I^n\mathcal{N}) = \mathrm{Ker}((I^n\mathcal{N}) \longrightarrow (I^n\mathcal{N})_{\mathfrak{p}} = 0) = \bigcup_{t \in R\backslash\mathfrak{p}} \mathrm{Ker}((I^n\mathcal{N}) \xrightarrow{t} (I^n\mathcal{N})). \tag{4.6}$$

Since the union on the right-hand side of (4.6) is filtered, we can find $s_{\mathfrak{p}} \in R\backslash\mathfrak{p}$ such that $s_{\mathfrak{p}} \in \mathrm{Ann}(I^n\mathcal{N})$. Let $K \subseteq R$ be the ideal generated by the collection $\{s_{\mathfrak{p}}\}_{\mathfrak{p}\in\chi}$. It is clear that $K \subseteq \mathrm{Ann}(I^n\mathcal{N})$. Since K is not contained in any of the prime ideals in χ, and χ is finite, it follows from prime avoidance that we can choose an element $r \in K\backslash\bigcup_{\mathfrak{p}\in\chi}\mathfrak{p}$. From Lemma 4.1, it is clear that r must be a non-zero divisor on $\mathcal{M}$. But multiplication by r is zero on the subobject $I^n\mathcal{N} \subseteq \mathcal{M}$. Hence, $I^n\mathcal{N} = 0$. It follows that $I^n\mathcal{M} = \bigcup_{\mathcal{N}\in\mathrm{fg}(\mathcal{M})} I^n\mathcal{N} = 0$. ■

5. Spectral sequences for local cohomology type functors on $\mathcal{S}_R$

In this section, we will show that we can construct spectral sequences for "local cohomology type" functors on $\mathcal{S}_R$ by creating an axiomatic setup similar to Àlvarez Montaner, Boix and Zarzuela [2]. Our main tools will be the properties of injective hulls of R-elementary objects proved in Section 2. We will then exhibit three different situations where this axiomatic setup can be used to obtain spectral sequences.

Let P be a finite poset, and let $\mathrm{Fun}(P, \mathcal{S}_R)$ denote the category of systems of objects in $\mathcal{S}_R$ indexed over P. We suppose from now on that $\mathcal{S}$ has enough projectives. Then, it follows from Artin and Zhang [6, Lemma D3.2] that $\mathcal{S}_R$ has enough projectives. Then, we know that $\mathrm{Fun}(P, \mathcal{S}_R)$ has enough projectives and $\mathrm{Fun}(P^{\mathrm{op}}, \mathcal{S}_R)$ has enough injectives (see, for instance, [25, Section 2.3]). We also note that $\mathrm{Fun}(P, \mathcal{S}_R)$ and $\mathrm{Fun}(P^{\mathrm{op}}, \mathcal{S}_R)$ are Grothendieck categories.

For any functor $\mathfrak{G} \in \mathrm{Fun}(P^{\mathrm{op}}, \mathcal{S}_R)$, we consider the complex $(C^\bullet(\mathfrak{G}), \partial^\bullet)$ given by

$$C^k(\mathfrak{G}) := \prod_{p_0 < p_1 < \cdots < p_k} \mathfrak{G}(p_0) \qquad \partial^k = \sum_{j=0}^{k+1} (-1)^j \partial_j^k : C^k(\mathfrak{G}) \longrightarrow C^{k+1}(\mathfrak{G}),$$

where ∂_j^k is induced by deleting the j-th term in the sequence $\{p_0 < p_1 < \cdots < p_k < p_{k+1}\}$. Because P is a finite poset, it follows from the proof of [20, Lemma A.3.2] that this complex computes the derived functor of the inverse limit over P, that is, for any $\mathfrak{G} \in \mathrm{Fun}(P^{\mathrm{op}}, \mathcal{S}_R)$, we have $H^i(C^\bullet(\mathfrak{G})) = \mathbb{R}^i \lim_{p \in P} \mathfrak{G}(p)$.

Similarly, for any functor $\mathfrak{F} \in \mathrm{Fun}(P, \mathcal{S}_R)$, we consider the complex $(C_\bullet(\mathfrak{F}), \partial_\bullet)$ given by

$$C_k(\mathfrak{F}) := \bigoplus_{p_0 < p_1 < \cdots < p_k} \mathfrak{F}(p_0) \qquad \partial_k = \sum_{j=0}^{k+1} (-1)^j \partial_k^j : C_{k+1}(\mathfrak{F}) \longrightarrow C_k(\mathfrak{F}),$$

where ∂_k^j is induced by deleting the j-th term in the sequence $\{p_0 < p_1 < \cdots < p_k < p_{k+1}\}$. Then, it follows from [20, Section B.1.2] that this complex computes the derived functor of the direct limit over P, that is, for any $\mathfrak{F} \in \mathrm{Fun}(P, \mathcal{S}_R)$, we have $H_i(C_\bullet(\mathfrak{F})) = \mathbb{L}_i \operatorname{colim}_{p \in P} \mathfrak{F}(p)$.

From now onwards, we fix an ideal $I \subseteq R$. We suppose that I may be expressed as $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_k}$, where $I_{\alpha_1}, \ldots, I_{\alpha_k}$ are ideals of R. As in [2, Section 2], we let P denote the finite poset whose elements are all the possible different sums of the ideals $I_1, \ldots, I_n$, ordered by reverse inclusion. By $\widehat{P}$, we will mean the poset $\widehat{P} = P \cup \{1_{\widehat{P}}\}$ obtained by adding a final element $1_{\widehat{P}}$ to P (even if P already has a final element). The Alexandrov topology on P (see [2, Section 2.2.1]) is that whose open sets $U \subseteq P$ satisfy the property that if $p \in U$ and $p \leq q$, then $q \in U$. If P has the Alexandrov topology, we know (see [2, Lemma 2.9]) that the basic open set $[p, 1_{\widehat{P}}) = \{q \in P \mid q \geq p\}$ is contractible for each $p \in P$.

For any $p \in P$, we will denote by I_p the sum of ideals corresponding to $p \in P$. We now fix an ideal J and an additive functor $\Psi_{[*]} : \mathcal{S}_R \to \mathrm{Fun}(\widehat{P}, \mathcal{S}_R)$ that satisfies the following conditions analogous to [2, Setup 4.3].

(1) For each $p \in \widehat{P}$, the functor $\Psi_p := \Psi_{[*]}(\_\_)(p) : \mathcal{S}_R \to \mathcal{S}_R$ is left exact and preserves direct sums.

(2) Let $\mathcal{L} \in \mathcal{S}_R$ be an R-elementary object. Let $\mathcal{E}(\mathcal{L}) \in \mathcal{S}_R$ be the injective hull of $\mathcal{L}$, and let $\mathfrak{p} := \mathrm{Ann}(\mathcal{N})$. Then, for any maximal ideal $\mathfrak{m}$ in R, there are objects

$\mathcal{X}(\mathcal{L},\mathfrak{m})$ and $\mathcal{Y}(\mathcal{L},\mathfrak{m}) \in \mathcal{S}_R$ such that for $p \in \widehat{P}$, we have

$$\Psi_p(\mathcal{E}(\mathcal{L}))_{\mathfrak{m}} = \begin{cases} \mathcal{X}(\mathcal{L},\mathfrak{m}) & \text{if } \mathfrak{p} \in \mathbb{W}(I_p, J) \text{ and } \mathfrak{p} \subseteq \mathfrak{m}, \\ \mathcal{Y}(\mathcal{L},\mathfrak{m}) & \text{if } \mathfrak{p} \notin \mathbb{W}(I_p, J) \text{ and } \mathfrak{p} \subseteq \mathfrak{m}, \\ 0 & \text{otherwise.} \end{cases}$$

(3) For any $p \le q$, the natural transformation $\Psi_p \to \Psi_q$ satisfies, for any R-elementary object $\mathcal{L} \in \mathcal{S}_R$ and maximal ideal $\mathfrak{m}$,

$$\begin{aligned} &\Psi_p(\mathcal{E}(\mathcal{L}))_{\mathfrak{m}} \longrightarrow \Psi_q(\mathcal{E}(\mathcal{L}))_{\mathfrak{m}} \\ &\qquad = \begin{cases} \mathrm{id}_{\mathcal{X}(\mathcal{L},\mathfrak{m})} & \text{if } \Psi_p(\mathcal{E}(\mathcal{L}))_{\mathfrak{m}} = \mathcal{X}(\mathcal{L},\mathfrak{m}) = \Psi_q(\mathcal{E}(\mathcal{L}))_{\mathfrak{m}}, \\ \mathrm{id}_{\mathcal{Y}(\mathcal{L},\mathfrak{m})} & \text{if } \Psi_p(\mathcal{E}(\mathcal{L}))_{\mathfrak{m}} = \mathcal{Y}(\mathcal{L},\mathfrak{m}) = \Psi_q(\mathcal{E}(\mathcal{L}))_{\mathfrak{m}}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We also set $\Psi := \Psi_{1_{\widehat{P}}} : \mathcal{S}_R \to \mathcal{S}_R$. Moreover, for $i \ge 0$ and any $\mathcal{M} \in \mathcal{S}_R$, we let $\mathbb{R}^i\Psi_{[*]}(\mathcal{M})$ denote the direct system of derived functors $\{\mathbb{R}^i\Psi_p(\mathcal{M})\}_{p\in P}$.

Lemma 5.1. *Let $\mathcal{M} \in \mathcal{S}_R$ be such that the localization $\mathcal{M}_{\mathfrak{m}} = 0$ for every maximal ideal $\mathfrak{m} \subseteq R$. Then, $\mathcal{M} = 0$.*

Proof. We consider any $\mathcal{M}' \in \mathrm{fg}(\mathcal{M})$. Since localization is exact, we have inclusions $\mathcal{M}'_{\mathfrak{m}} \hookrightarrow \mathcal{M}_{\mathfrak{m}} = 0$ and hence $\mathcal{M}'_{\mathfrak{m}} = 0$ for each maximal ideal $\mathfrak{m}$. Since $\mathcal{M}'$ is finitely generated, it follows from Proposition 2.10 that $\mathrm{Supp}(\mathcal{M}') = \mathbb{V}(\mathrm{Ann}(\mathcal{M}'))$. Hence, we have $\mathbb{V}(\mathrm{Ann}(\mathcal{M}')) = \phi$, that is, $\mathcal{M}' = 0$. Since $\mathcal{S}_R$ is locally noetherian, $\mathcal{M}$ is the sum of all its finitely generated subobjects. Hence, $\mathcal{M} = 0$. ∎

Lemma 5.2. *Let $\mathcal{E} \in \mathcal{S}_R$ be an injective object. Then, the complex*

$$C_\bullet(\Psi_{[*]}(\mathcal{E})) \longrightarrow \Psi(\mathcal{E}) \longrightarrow 0$$

is exact.

Proof. Since localizations are exact, it follows from Lemma 5.1 that it suffices to show that $C_\bullet(\Psi_{[*]}(\mathcal{E}))_{\mathfrak{m}} \to \Psi(\mathcal{E})_{\mathfrak{m}} \to 0$ is exact for each maximal ideal $\mathfrak{m}$. Since $\mathcal{E} \in \mathcal{S}_R$ is injective, we know from Theorem 2.8 that $\mathcal{E}$ can be expressed as a direct sum of injective hulls of R-elementary objects. Since $\Psi_{[*]}$ and $C_\bullet$ preserve direct sums, it now suffices to check that $C_\bullet(\Psi_{[*]}(\mathcal{E}(\mathcal{L})))_{\mathfrak{m}} \to \Psi(\mathcal{E}(\mathcal{L}))_{\mathfrak{m}} \to 0$ is exact for each maximal ideal $\mathfrak{m}$, where $\mathcal{L} \in \mathcal{S}_R$ is an R-elementary object and $\mathcal{E}(\mathcal{L})$ its injective hull. We now set $\mathfrak{p} := \mathrm{Ann}(\mathcal{L})$. The rest of the proof now follows exactly as in [2, Lemma 4.4]. ∎

Theorem 5.3. *Let $\mathcal{S}$ be a strongly locally noetherian Grothendieck category, and let R be a commutative noetherian ring. Then, for any $\mathcal{M} \in \mathcal{S}_R$, we have a spectral sequence*

$$E_2^{-i,j} = \mathbb{L}_i\, \mathrm{colim}_{p\in P}\, \mathbb{R}^j\Psi_{[*]}(\mathcal{M}) \Longrightarrow \mathbb{R}^{j-i}\Psi(\mathcal{M}).$$

Proof. For $\mathcal{M} \in \mathcal{S}_R$, we consider an injective resolution $0 \to \mathcal{M} \to \mathcal{E}^\bullet$. We now consider the following bicomplex $D^{\bullet,\bullet}$ that is concentrated in the "second quadrant."

$$\begin{tikzcd}
& \uparrow & \uparrow & \\
\cdots \arrow[r] & C_1(\Psi_{[*]}(\mathcal{E}^1)) \arrow[r] \arrow[u] & C_0(\Psi_{[*]}(\mathcal{E}^1)) \arrow[r] \arrow[u] & 0 \\
\cdots \arrow[r] & C_1(\Psi_{[*]}(\mathcal{E}^0)) \arrow[r] \arrow[u] & C_0(\Psi_{[*]}(\mathcal{E}^0)) \arrow[r] \arrow[u] & 0
\end{tikzcd} \tag{5.1}$$

Here, we write $D^{-l,k} := C_l(\Psi_{[*]}(\mathcal{E}^k))$ to facilitate cohomological notation. As in [2, Theorem 4.6], we can now consider the two spectral sequences associated with the first and second filtrations of this bicomplex. Since the l-th column of this complex is given by $D^{-l,\bullet} = C_l(\Psi_{[*]}(\mathcal{E}^\bullet))$ and P is finite, the columns of this bicomplex vanish for $l \gg 0$ and hence both spectral sequences converge. Further, by Lemma 5.2, the rows of this bicomplex are exact up to the 0-th position, and the cohomologies at the 0-th position are given by $\Psi(\mathcal{E}^\bullet)$. It follows that the common abutment of these two spectral sequences is given by the cohomology groups of the complex

$$0 \longrightarrow \Psi(\mathcal{E}^0) \longrightarrow \Psi(\mathcal{E}^1) \longrightarrow \cdots . \tag{5.2}$$

Since $0 \to \mathcal{M} \to \mathcal{E}^\bullet$ is an injective resolution, it is clear that the cohomologies of (5.2) are given by $\mathbb{R}^\bullet\Psi(\mathcal{M})$. Taking the cohomology of the columns of (5.1), one obtains $E_1^{-i,j} = C_i(\mathbb{R}^j\Psi_{[*]}(\mathcal{M}))$. Now applying the cohomology of the rows of (5.1), we finally obtain the spectral sequence

$$E_2^{-i,j} = \mathbb{L}_i \operatorname{colim}_{p\in P} \mathbb{R}^j\Psi_{[*]}(\mathcal{M}) \Longrightarrow \mathbb{R}^{j-i}\Psi(\mathcal{M}).$$

This proves the result. ∎

In the following three subsections, we will now apply the formalism above to three separate contexts in order to obtain spectral sequences of derived functors. The first of these will be the local cohomology objects in $\mathcal{S}_R$ with respect to a pair of ideals (I, J).

5.1. Spectral sequences for $\Gamma_{I,J} : \mathcal{S}_R \to \mathcal{S}_R$

We continue with the ideal $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_k}$ and the partially ordered set P consisting of all possible sums of the ideals $\{I_{\alpha_1}, \ldots, I_{\alpha_k}\}$ as defined above. For an ideal $J \subseteq R$ and $p \in P$, we set $\Psi_p(\_\_) := \Gamma_{I_p,J}(\_\_)$ and $\Psi := \Gamma_{I,J}$.

Lemma 5.4. *Let $K, J \subseteq R$ be ideals. Then, the functor $\Gamma_{K,J}$ preserves direct sums.*

Proof. We consider a collection $\{\mathcal{M}_\beta\}_{\beta\in B}$ of objects in $\mathcal{S}_R$ and set $\mathcal{M} := \bigoplus_{\beta\in B} \mathcal{M}_\beta$. From the definition in (3.1), it is evident that $\Gamma_{K,J}(\mathcal{M}) \supseteq \bigoplus_{\beta\in B} \Gamma_{K,J}(\mathcal{M}_\beta)$. On the other hand, we consider $\mathcal{N} \in \mathrm{fg}(\mathcal{M})$ such that $K^n \subseteq \mathrm{Ann}(\mathcal{N}) + J$ for $n \gg 1$. Since $\mathcal{N}$ is finitely generated, we must have $\mathcal{N} \subseteq \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_r$ for some finite subcollection $\{\mathcal{M}_1, \dots, \mathcal{M}_r\}$ of objects from $\{\mathcal{M}_\beta\}_{\beta \in B}$. Let $\rho_l : \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_r \to \mathcal{M}_l$ denote the canonical projections for $1 \leq l \leq r$. We now note that

$$K^n \subseteq \mathrm{Ann}(\mathcal{N}) + J \subseteq \mathrm{Ann}(\rho_l(\mathcal{N})) + J \quad \text{for } n \gg 1 \quad \Rightarrow \quad \rho_l(\mathcal{N}) \subseteq \Gamma_{K,J}(\mathcal{M}_l)$$

for each $1 \leq l \leq r$. We now note that $\mathcal{N} \subseteq \rho_1(\mathcal{N}) \oplus \cdots \oplus \rho_r(\mathcal{N}) \subseteq \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_r$, which gives $\mathcal{N} \subseteq \bigoplus_{l=1}^r \Gamma_{K,J}(\mathcal{M}_l) \subseteq \bigoplus_{\beta \in B} \Gamma_{K,J}(\mathcal{M}_\beta)$. ∎

Lemma 5.5. *Let $\mathcal{L} \in \mathcal{S}_R$ be an R-elementary object, and let $\mathfrak{p} := \mathrm{Ann}(\mathcal{L})$. Let $K \subseteq R$ be an ideal. If $\mathfrak{p} \notin \mathbb{W}(K, J)$, then we have $\Gamma_{K,J}(\mathcal{E}(\mathcal{L})) = 0$.*

Proof. Since $\mathcal{L} \subseteq \mathcal{E}(\mathcal{L})$ is an essential subobject, we see that it follows from Corollary 2.7 that $\mathrm{Ass}(\mathcal{E}(\mathcal{L})) = \mathrm{Ass}(\mathcal{L}) = \{\mathfrak{p}\}$. We are given $\mathfrak{p} \notin \mathbb{W}(K, J)$. By Proposition 3.3, we now have $\mathrm{Ass}(\Gamma_{K,J}(\mathcal{E}(\mathcal{L}))) = \mathrm{Ass}(\mathcal{E}(\mathcal{L})) \cap \mathbb{W}(K, J) = \phi$. It follows that $\Gamma_{K,J}(\mathcal{E}(\mathcal{L})) = 0$. ∎

Lemma 5.6. *Let $\mathcal{L} \in \mathcal{S}_R$ be an R-elementary object, and let $\mathfrak{p} := \mathrm{Ann}(\mathcal{L})$. Let $K \subseteq R$ be an ideal.*

(a) *Suppose that $\mathfrak{p} \in \mathbb{W}(K, J)$. Then, we have $\Gamma_{K,J}(\mathcal{E}(\mathcal{L})) = \mathcal{E}(\mathcal{L})$.*

(b) *If $\mathfrak{p} \nsubseteq \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$, then $\mathcal{E}(\mathcal{L})_{\mathfrak{m}} = 0$.*

Proof. (a) Since $\mathfrak{p} = \mathrm{Ann}(\mathcal{L})$ and $\mathfrak{p} \in \mathbb{W}(K, J)$, we have $K^n \subseteq \mathrm{Ann}(\mathcal{L}) + J$ for $n \gg 1$. It follows from the definition in (3.1) that $\mathcal{L}$ is (K, J)-torsion. By Lemma 3.9, it follows that the injective hull $\mathcal{E}(\mathcal{L})$ is also (K, J)-torsion, that is, $\Gamma_{K,J}(\mathcal{E}(\mathcal{L})) = \mathcal{E}(\mathcal{L})$.

(b) Since $\mathcal{L}$ is finitely generated, we know that $\mathrm{Supp}(\mathcal{L}) = \mathbb{V}(\mathrm{Ann}(\mathcal{L})) = \mathbb{V}(\mathfrak{p})$. Accordingly, if $\mathfrak{p} \nsubseteq \mathfrak{m}$, then $\mathcal{L}_{\mathfrak{m}} = 0$. By Proposition 2.14, we know that $\mathcal{E}(\mathcal{L})_{\mathfrak{m}}$ is the injective hull of $\mathcal{L}_{\mathfrak{m}}$, and hence $\mathcal{E}(\mathcal{L})_{\mathfrak{m}} = 0$. ∎

Proposition 5.7. *For any $\mathcal{M} \in \mathcal{S}_R$, we have a spectral sequence*

$$E_2^{-i,j} = \mathbb{L}_i \, \mathrm{colim}_{\mathfrak{p} \in P} \, H^j_{[*],J}(\mathcal{M}) \Longrightarrow H^{j-i}_{I,J}(\mathcal{M}).$$

Proof. This follows directly from Lemmas 5.4, 5.5 and 5.6 and Theorem 5.3. ∎

5.2. Generalized local cohomology objects on $\mathcal{S}_R$

Let V be an R-module. By [6, Section B4], we know that $\otimes_R V : \mathcal{S}_R \to \mathcal{S}_R$ has a right adjoint $\underline{\mathrm{Hom}}_R(V, \_\_)$. The bifunctor $\underline{\mathrm{Hom}}_R(\_\_, \_\_) : (R-\mathrm{Mod})^{\mathrm{op}} \times \mathcal{S}_R \to \mathcal{S}_R$ is left exact in both variables and satisfies $\underline{\mathrm{Hom}}_R(R, \mathcal{M}) = \mathcal{M}$ for any $\mathcal{M} \in \mathcal{S}_R$. The right derived functors of $\underline{\mathrm{Hom}}_R(V, \_\_)$ are denoted by $\underline{\mathrm{Ext}}^\bullet_R(V, \_\_)$. If V, W are R-modules, it also follows from the adjunction that $\underline{\mathrm{Hom}}_R(V \otimes_R W, \_\_) \cong \underline{\mathrm{Hom}}_R(V, \underline{\mathrm{Hom}}_R(W, \_\_))$.

Given an R-module V and any ideal $K \subseteq R$, we denote by V_K the inverse system given by $V_K := \{V/K^tV\}_{t\geq 1}$. We now set

$$\Gamma_{V_K} : \mathcal{S}_R \longrightarrow \mathcal{S}_R \quad \mathcal{M} \mapsto \varinjlim_{t\geq 1} \underline{\mathrm{Hom}}_R(V/K^tV, \mathcal{M}). \tag{5.3}$$

It is immediate from (5.3) that Γ_{V_K} is left exact, and we denote by $H^\bullet_{V_K}(\_\_)$ the right derived functors of Γ_{V_K}.

Proposition 5.8. *Let V be an R-module, and let $K \subseteq R$ be an ideal. Then, for any $\mathcal{M} \in \mathcal{S}_R$, we have*

$$H^i_{V_K}(\mathcal{M}) = \varinjlim_{t\geq 1} \underline{\mathrm{Ext}}^i_R(V/K^tV, \mathcal{M}) \qquad i \geq 0.$$

Proof. For the sake of convenience, we set $F^i(\mathcal{M}) := \varinjlim_{t\geq 1} \underline{\mathrm{Ext}}^i_R(V/K^tV, \mathcal{M})$ for any $\mathcal{M} \in \mathcal{S}_R$. From the definition in (5.3), it follows that $H^0_{V_K}(\mathcal{M}) = F^0(\mathcal{M})$. The derived functors $\{H^\bullet_{V_K}(\_\_)\}$ give a family of cohomological δ-functors that is universal. Similarly, for each $t \geq 1$, the derived functors $\{\underline{\mathrm{Ext}}^\bullet_R(V/K^tV, \_\_)\}$ are universal δ-functors on $\mathcal{S}_R$. Since filtered colimits in $\mathcal{S}_R$ are also exact, we see that $\{F^\bullet\}$ is also a family of δ-functors. For any injective $\mathcal{E} \in \mathcal{S}_R$, the derived functor $\underline{\mathrm{Ext}}^i_R(V/K^tV, \mathcal{E})$ vanishes for $i > 0$, and hence so does $F^i(\mathcal{E})$. It follows that $\{F^\bullet\}$ is a universal δ-functor and since $F^0 = H^0_{V_K}$, we must have $F^i = H^i_{V_K}$ for every i. ∎

In order to understand the functor Γ_{V_K} better, we define, for any ideal $K \subseteq R$ and $\mathcal{M} \in \mathcal{S}_R$,

$$\gamma_K(\mathcal{M}) := \bigcap_{a\in K} \mathrm{Ker}(a_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M}) = \bigcap_{i=1}^{n} \mathrm{Ker}(a_{i,\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M}),$$

where $\{a_1, \dots, a_n\}$ is a set of generators for K.

Lemma 5.9. *Let $K \subseteq R$ be an ideal and $\mathcal{M} \in \mathcal{S}_R$. Then, $\gamma_K(\mathcal{M}) = \underline{\mathrm{Hom}}_R(R/K, \mathcal{M})$.*

Proof. If $\{a_1, \dots, a_n\}$ is a set of generators for K, we note that $R/K = (R/a_1R) \otimes_R \cdots \otimes_R (R/a_nR)$ and hence

$$\underline{\mathrm{Hom}}_R(R/K, \mathcal{M}) = \underline{\mathrm{Hom}}_R(R/a_1R, \underline{\mathrm{Hom}}_R(R/a_2R, \dots, \underline{\mathrm{Hom}}_R(R/a_nR, \mathcal{M}))).$$

Therefore, it suffices to check the result for a principal ideal $K = (a)$. In that case, $R \xrightarrow{a} R \to R/aR \to 0$ is a free presentation of R/aR, and it follows from [6, (B4.2)] that

$$0 \longrightarrow \underline{\mathrm{Hom}}_R(R/aR, \mathcal{M}) \longrightarrow \underline{\mathrm{Hom}}_R(R, \mathcal{M}) \cong \mathcal{M} \xrightarrow{\underline{\mathrm{Hom}}_R(R, a_{\mathcal{M}}) = a_{\mathcal{M}}} \underline{\mathrm{Hom}}_R(R, \mathcal{M}) \cong \mathcal{M}$$

is exact. The result is now clear. ∎

From Lemma 5.9 and the definition in (3.1), it now follows that for any ideal $K \subseteq R$, we have

$$\Gamma_K(\mathcal{M}) := \Gamma_{K,0}(\mathcal{M}) = \bigcup_{t\geq 1} \gamma_{K^t}(\mathcal{M}) = \varinjlim_{t\geq 1} \underline{\mathrm{Hom}}_R(R/K^t, \mathcal{M}). \tag{5.4}$$

For the rest of this subsection, we suppose that V is a finitely generated R-module. We note here the following fact.

Lemma 5.10. *Let V be a finitely generated R-module. Then, the functor $\underline{\mathrm{Hom}}_R(V, \_\_) : \mathcal{S}_R \to \mathcal{S}_R$ preserves filtered colimits.*

Proof. Let $\{\mathcal{M}_\beta\}_{\beta\in B}$ be a filtered system of objects in $\mathcal{S}_R$. We will show that

$$\begin{aligned}\mathcal{S}_R\Big(\mathcal{N} \otimes_R V, \varinjlim_{\beta\in B} \mathcal{M}_\beta\Big) &= \mathcal{S}_R\Big(\mathcal{N}, \underline{\mathrm{Hom}}_R\Big(V, \varinjlim_{\beta\in B} \mathcal{M}_\beta\Big)\Big) \\ &= \mathcal{S}_R\Big(\mathcal{N}, \varinjlim_{\beta\in B} \underline{\mathrm{Hom}}_R(V, \mathcal{M}_\beta)\Big)\end{aligned} \tag{5.5}$$

for any $\mathcal{N} \in \mathcal{S}_R$. Since $\mathcal{S}_R$ is locally noetherian, any object of $\mathcal{S}_R$ may be expressed as a filtered colimit of its finitely generated subobjects. Accordingly, it suffices to check (5.5) for $\mathcal{N} \in \mathcal{S}_R$ finitely generated. If $\mathcal{N}$ is finitely generated, we have

$$\mathcal{S}_R\Big(\mathcal{N}, \varinjlim_{\beta\in B} \underline{\mathrm{Hom}}_R(V, \mathcal{M}_\beta)\Big) = \varinjlim_{\beta\in B} \mathcal{S}_R(\mathcal{N}, \underline{\mathrm{Hom}}_R(V, \mathcal{M}_\beta)) = \varinjlim_{\beta\in B} \mathcal{S}_R(\mathcal{N} \otimes_R V, \mathcal{M}_\beta). \tag{5.6}$$

From (5.5) and (5.6), we see that it suffices to check

$$\mathcal{S}_R\Big(\mathcal{N} \otimes_R V, \varinjlim_{\beta\in B} \mathcal{M}_\beta\Big) = \varinjlim_{\beta\in B} \mathcal{S}_R(\mathcal{N} \otimes_R V, \mathcal{M}_\beta)$$

for $\mathcal{N} \in \mathcal{S}_R$ finitely generated. But we are given that V is a finitely generated R-module, that is, a quotient of R^n for some $n \geq 1$. Then, $\mathcal{N} \otimes_R V$ is a quotient of $\mathcal{N}^n$ and hence $\mathcal{N} \otimes_R V$ is finitely generated in $\mathcal{S}_R$. The result is now clear. ∎

Proposition 5.11. *Let V be a finitely generated R-module and $K \subseteq R$ an ideal. Then, for any $\mathcal{M} \in \mathcal{S}_R$, we have $\Gamma_{V_K}(\mathcal{M}) = \underline{\mathrm{Hom}}_R(V, \Gamma_K(\mathcal{M}))$. In particular, Γ_{V_K} preserves direct sums.*

Proof. Since V is finitely generated, it follows from (5.3), (5.4) and Lemma 5.10 that

$$\begin{aligned}\Gamma_{V_K}(\mathcal{M}) &= \varinjlim_{t\geq 1} \underline{\mathrm{Hom}}_R(V/K^t V, \mathcal{M}) = \varinjlim_{t\geq 1} \underline{\mathrm{Hom}}_R(V, \underline{\mathrm{Hom}}_R(R/K^t, \mathcal{M})) \\ &= \underline{\mathrm{Hom}}_R(V, \varinjlim_{t\geq 1} \underline{\mathrm{Hom}}_R(R/K^t, \mathcal{M})) \\ &= \underline{\mathrm{Hom}}_R(V, \Gamma_K(\mathcal{M})).\end{aligned}$$

The last statement is clear from Lemma 5.10 and from the fact that $\Gamma_K = \Gamma_{K,0}$ preserves direct sums. ∎

Lemma 5.12. *Let $R \to R'$ be an extension of k-algebras. For any R-module W and any $\mathcal{N} \in \mathcal{S}_{R'}$, we have an isomorphism*

$$\underline{\mathrm{Hom}}_{R'}(W \otimes_R R', \mathcal{N}) \cong \underline{\mathrm{Hom}}_R(W, \mathcal{N}).$$

Proof. We write W as the cokernel $R^{(Y)} \to R^{(X)} \to W \to 0$ of free modules. Then, $W \otimes_R R'$ is expressed as the cokernel $R'^{(Y)} \to R'^{(X)} \to W \otimes_R R' \to 0$. By definition (see [6, (B4.2)]), $\underline{\mathrm{Hom}}_R(W, \mathcal{N})$ is given as the kernel of the induced map $\mathcal{N}^X = \underline{\mathrm{Hom}}_R(R^{(X)}, \mathcal{N}) \to \underline{\mathrm{Hom}}_R(R^{(Y)}, \mathcal{N}) = \mathcal{N}^Y$. Similarly, $\underline{\mathrm{Hom}}_{R'}(W \otimes_R R', \mathcal{N})$ is also given by the kernel $\mathcal{N}^X = \underline{\mathrm{Hom}}_{R'}(R'^{(X)}, \mathcal{N}) \to \underline{\mathrm{Hom}}_{R'}(R'^{(Y)}, \mathcal{N}) = \mathcal{N}^Y$. ∎

Lemma 5.13. *Let $R \to R[T^{-1}]$ be the localization of R with respect to a multiplicatively closed subset T. Then, for any finitely generated R-module V and any $\mathcal{M} \in \mathcal{S}_R$, we have*

$$\underline{\mathrm{Hom}}_R(V, \mathcal{M})_T \cong \underline{\mathrm{Hom}}_{R[T^{-1}]}(V_T, \mathcal{M}_T).$$

Proof. Applying Lemma 5.12, we know that $\underline{\mathrm{Hom}}_{R[T^{-1}]}(V_T, \mathcal{M}_T) \cong \underline{\mathrm{Hom}}_R(V, \mathcal{M}_T)$. We have noted before that the localization $\mathcal{M}_T$ is given by a filtered colimit. Since V is a finitely generated R-module, we know from Lemma 5.10 that $\underline{\mathrm{Hom}}_R(V, \_\_)$ preserves filtered colimits. We now have $\underline{\mathrm{Hom}}_R(V, \mathcal{M}_T) \cong \underline{\mathrm{Hom}}_R(V, \mathcal{M})_T$. ∎

We now return to the ideal $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_k}$ and the partially ordered set P consisting of all possible sums of the ideals $\{I_{\alpha_1}, \ldots, I_{\alpha_k}\}$ as before. We fix $J = 0$ and a finitely generated R-module V. For $p \in P$, we set $\Psi_p(\_\_) := \Gamma_{V_{I_p}}(\_\_)$ and $\Psi := \Gamma_{V_I}$.

Lemma 5.14. *Let $\mathcal{L} \in \mathcal{S}_R$ be an R-elementary object, and let $\mathfrak{p} := \mathrm{Ann}(\mathcal{L})$. Let $K \subseteq R$ be an ideal.*

(a) *If $\mathfrak{p} \notin \mathbb{W}(K, 0) = \mathbb{V}(K)$, then we have $\Gamma_{V_K}(\mathcal{E}(\mathcal{L})) = 0$.*

(b) *Suppose that $\mathfrak{p} \in \mathbb{W}(K, 0) = \mathbb{V}(K)$. Then, we have $\Gamma_{V_K}(\mathcal{E}(\mathcal{L})) = \underline{\mathrm{Hom}}_R(V, \mathcal{E}(\mathcal{L}))$.*

Proof. (a) Since $\mathfrak{p} \notin \mathbb{W}(K, 0) = \mathbb{V}(K)$, we see that it follows from Lemma 5.5 that $\Gamma_K(\mathcal{E}(\mathcal{L})) = \Gamma_{K,0}(\mathcal{E}(\mathcal{L})) = 0$. Further, since V is finitely generated, we have by Proposition 5.11 that $\Gamma_{V_K}(\mathcal{E}(\mathcal{L})) = \underline{\mathrm{Hom}}_R(V, \Gamma_K(\mathcal{E}(\mathcal{L}))) = 0$.

(b) Since $\mathfrak{p} \in \mathbb{W}(K, 0) = \mathbb{V}(K)$, it follows from Lemma 5.6 that $\Gamma_K(\mathcal{E}(\mathcal{L})) = \Gamma_{K,0}(\mathcal{E}(\mathcal{L})) = \mathcal{E}(\mathcal{L})$. Since V is finitely generated, we have by Proposition 5.11 that $\Gamma_{V_K}(\mathcal{E}(\mathcal{L})) = \underline{\mathrm{Hom}}_R(V, \Gamma_K(\mathcal{E}(\mathcal{L}))) = \underline{\mathrm{Hom}}_R(V, \mathcal{E}(\mathcal{L}))$. ∎

Lemma 5.15. *Let $\mathcal{L} \in \mathcal{S}_R$ be an R-elementary object, and let $\mathfrak{p} := \mathrm{Ann}(\mathcal{L})$. Let $K \subseteq R$ be an ideal. If $\mathfrak{p} \nsubseteq \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$, then $\Gamma_{V_K}(\mathcal{E}(\mathcal{L}))_{\mathfrak{m}} = 0$.*

Proof. If $\mathfrak{p} \notin \mathbb{W}(K, 0) = \mathbb{V}(K)$, we already have $\Gamma_{V_K}(\mathcal{E}(\mathcal{L})) = 0$ by Lemma 5.14 (a). Otherwise, suppose $\mathfrak{p} \in \mathbb{W}(K, 0) = \mathbb{V}(K)$. Then, Lemma 5.14 (b) gives us $\Gamma_{V_K}(\mathcal{E}(\mathcal{L})) = \underline{\mathrm{Hom}}_R(V, \mathcal{E}(\mathcal{L}))$. Because V is finitely generated, applying Lemma 5.13 gives us

$$\Gamma_{V_K}(\mathcal{E}(\mathcal{L}))_{\mathfrak{m}} = \underline{\mathrm{Hom}}_{R_{\mathfrak{m}}}(V_{\mathfrak{m}}, \mathcal{E}(\mathcal{L})_{\mathfrak{m}})$$

for any maximal ideal $\mathfrak{m}$. Since $\mathfrak{p} \not\subseteq \mathfrak{m}$, we have by Lemma 5.6 that $\mathcal{E}(\mathcal{L})_{\mathfrak{m}} = 0$, which shows that $\Gamma_{V_K}(\mathcal{E}(\mathcal{L}))_{\mathfrak{m}} = 0$. ∎

Proposition 5.16. *Let V be a finitely generated R-module. Then, for any $\mathcal{M} \in \mathcal{S}_R$, we have a spectral sequence*

$$E_2^{-i,j} = \mathbb{L}_i \operatorname{colim}_{p \in P} H^j_{V_{I_p}}(\mathcal{M}) \Longrightarrow H^{j-i}_{V_I}(\mathcal{M}).$$

Proof. This follows directly from Lemmas 5.14 and 5.15 and Theorem 5.3. ∎

5.3. Generalized Nagata ideal transforms on $\mathcal{S}_R$

We let V be a finitely generated R-module and $K \subseteq R$ be an ideal. We define the generalized Nagata ideal transform Δ_{V_K} on $\mathcal{S}_R$ as follows:

$$\Delta_{V_K} : \mathcal{S}_R \longrightarrow \mathcal{S}_R \quad \mathcal{M} \mapsto \varinjlim_{t \geq 1} \underline{\operatorname{Hom}}_R(K^t V, \mathcal{M}).$$

It is immediate that the functor Δ_{V_K} is left exact. Since V is finitely generated, it is also clear from Lemma 5.10 that Δ_{V_K} preserves direct sums. We now need the following result.

Lemma 5.17. *Let $\mathcal{E} \in \mathcal{S}_R$ be an injective object. Then, the functor $\underline{\operatorname{Hom}}_R(\_\_, \mathcal{E}) : R - \mathrm{Mod} \to \mathcal{S}_R$ is exact.*

Proof. We already know that $\underline{\operatorname{Hom}}_R(\_\_, \mathcal{E})$ is left exact. Let $V' \hookrightarrow V$ be an inclusion of R-modules. We will show that $\underline{\operatorname{Hom}}_R(V, \mathcal{E}) \to \underline{\operatorname{Hom}}_R(V', \mathcal{E})$ is an epimorphism. Accordingly, we set $\mathcal{M} \in \mathcal{S}_R$ to be the cokernel $\underline{\operatorname{Hom}}_R(V, \mathcal{E}) \to \underline{\operatorname{Hom}}_R(V', \mathcal{E}) \to \mathcal{M} \to 0$.

In this section, we have assumed that $\mathcal{S}$ has enough projectives. We choose an epimorphism $\mathcal{P} \twoheadrightarrow \mathcal{M}$ in $\mathcal{S}$ with $\mathcal{P} \in \mathcal{S}$ projective. This induces an epimorphism $\mathcal{P} \otimes R \twoheadrightarrow \mathcal{M} \otimes R$ in $\mathcal{S}_R$. Also, we know that composing the structure map $\mathcal{M} \otimes R \to \mathcal{M}$ of $\mathcal{M} \in \mathcal{S}_R$ with the morphism $\mathcal{M} = \mathcal{M} \otimes k \to \mathcal{M} \otimes R$ induced by the unit gives the identity in $\mathcal{S}$. Accordingly, $\mathcal{M} \otimes R \to \mathcal{M}$ is an epimorphism in $\mathcal{S}_R$, since the underlying morphism in $\mathcal{S}$ is an epimorphism. We now have an epimorphism $\mathcal{P} \otimes R \twoheadrightarrow \mathcal{M} \otimes R \twoheadrightarrow \mathcal{M}$ in $\mathcal{S}_R$.

Since $\mathcal{P} \in \mathcal{S}$ is projective, the adjoint isomorphism $\mathcal{S}_R(\mathcal{P} \otimes R, \_\_) \cong \mathcal{S}_k(\mathcal{P}, \_\_)$ shows that $\mathcal{P} \otimes R \in \mathcal{S}_R$ is projective. Hence, the following sequence is exact:

$$\mathcal{S}_R(\mathcal{P} \otimes R, \underline{\operatorname{Hom}}_R(V, \mathcal{E})) \longrightarrow \mathcal{S}_R(\mathcal{P} \otimes R, \underline{\operatorname{Hom}}_R(V', \mathcal{E})) \longrightarrow \mathcal{S}_R(\mathcal{P} \otimes R, \mathcal{M}) \longrightarrow 0. \tag{5.7}$$

Since k is a field, the inclusion $V' \hookrightarrow V$ induces a monomorphism $(\mathcal{P} \otimes R) \otimes_R V' = \mathcal{P} \otimes V' \to \mathcal{P} \otimes V = (\mathcal{P} \otimes R) \otimes_R V$. Since $\mathcal{E} \in \mathcal{S}_R$ is injective, it now follows that $\mathcal{S}_R((\mathcal{P} \otimes R) \otimes_R V, \mathcal{E}) \to \mathcal{S}_R((\mathcal{P} \otimes R) \otimes_R V', \mathcal{E})$ is an epimorphism. Using (5.7), we now have $\mathcal{S}_R(\mathcal{P} \otimes R, \mathcal{M}) = 0$, which shows that $\mathcal{M} = 0$. ∎

Using Lemma 5.17, we see that if $\mathcal{E} \in \mathcal{S}_R$ is an injective object, then

$$0 \longrightarrow \underline{\operatorname{Hom}}_R(V/K^t V, \mathcal{E}) \longrightarrow \underline{\operatorname{Hom}}_R(V, \mathcal{E}) \longrightarrow \underline{\operatorname{Hom}}_R(K^t V, \mathcal{E}) \longrightarrow 0$$

is exact for any $t \geq 1$. Taking filtered colimits, we have a short exact sequence

$$\begin{aligned} 0 \longrightarrow \Gamma_{V_K}(\mathcal{E}) = \varinjlim_{t\geq 1} \underline{\mathrm{Hom}}_R(V/K^t V, \mathcal{E}) \longrightarrow \underline{\mathrm{Hom}}_R(V, \mathcal{E}) \longrightarrow \varinjlim_{t\geq 1} \underline{\mathrm{Hom}}_R(K^t V, \mathcal{E}) \\ = \Delta_{V_K}(\mathcal{E}) \longrightarrow 0. \end{aligned} \tag{5.8}$$

For our setup with $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_k}$ and $J = 0$, we now set $\Psi_p := \Delta_{V_{I_p}}(\_\_)$ for each $p \in P$ and $\Psi := \Delta_{V_I}$.

Lemma 5.18. *Let $\mathcal{L} \in \mathcal{S}_R$ be an R-elementary object, and let $\mathfrak{p} := \mathrm{Ann}(\mathcal{L})$. Let $K \subseteq R$ be an ideal.*

(a) *If $\mathfrak{p} \in \mathbb{W}(K, 0) = \mathbb{V}(K)$, then we have $\Delta_{V_K}(\mathcal{E}(\mathcal{L})) = 0$.*

(b) *If $\mathfrak{p} \notin \mathbb{W}(K, 0) = \mathbb{V}(K)$, we have $\Delta_{V_K}(\mathcal{E}(\mathcal{L})) = \underline{\mathrm{Hom}}_R(V, \mathcal{E}(\mathcal{L}))$.*

Proof. (a) If $\mathfrak{p} \in \mathbb{W}(K, 0) = \mathbb{V}(K)$, it follows from Lemma 5.14 (b) that

$$\Gamma_{V_K}(\mathcal{E}(\mathcal{L})) = \underline{\mathrm{Hom}}_R(V, \mathcal{E}(\mathcal{L}))$$

Accordingly, the short exact sequence in (5.8) gives us $\Delta_{V_K}(\mathcal{E}(\mathcal{L})) = 0$.

(b) If $\mathfrak{p} \notin \mathbb{W}(K, 0) = \mathbb{V}(K)$, it follows from Lemma 5.14 (a) that $\Gamma_{V_K}(\mathcal{E}(\mathcal{L})) = 0$. Accordingly, the short exact sequence in (5.8) gives us $\Delta_{V_K}(\mathcal{E}(\mathcal{L})) = \underline{\mathrm{Hom}}_R(V, \mathcal{E}(\mathcal{L}))$. This proves the result. ∎

Lemma 5.19. *Let $\mathcal{L} \in \mathcal{S}_R$ be an R-elementary object, and let $\mathfrak{p} := \mathrm{Ann}(\mathcal{L})$. Let $K \subseteq R$ be an ideal. If $\mathfrak{p} \not\subseteq \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$, then $\Delta_{V_K}(\mathcal{E}(\mathcal{L}))_{\mathfrak{m}} = 0$.*

Proof. If $\mathfrak{p} \in \mathbb{W}(K, 0)$, we already know from Lemma 5.18 (a) that $\Delta_{V_K}(\mathcal{E}(\mathcal{L})) = 0$. Otherwise, suppose that $\mathfrak{p} \notin \mathbb{W}(K, 0)$, so that it follows from Lemma 5.18 (b) that $\Delta_{V_K}(\mathcal{E}(\mathcal{L})) = \underline{\mathrm{Hom}}_R(V, \mathcal{E}(\mathcal{L}))$. Because V is finitely generated, applying Lemma 5.13 gives us $\Delta_{V_K}(\mathcal{E}(\mathcal{L}))_{\mathfrak{m}} = \underline{\mathrm{Hom}}_{R_{\mathfrak{m}}}(V_{\mathfrak{m}}, \mathcal{E}(\mathcal{L})_{\mathfrak{m}})$ for any maximal ideal $\mathfrak{m}$. Since $\mathfrak{p} \not\subseteq \mathfrak{m}$, we have by Lemma 5.6 (b) that $\mathcal{E}(\mathcal{L})_{\mathfrak{m}} = 0$, which shows that $\Delta_{V_K}(\mathcal{E}(\mathcal{L}))_{\mathfrak{m}} = 0$. ∎

Proposition 5.20. *Let V be a finitely generated R-module. For any $\mathcal{M} \in \mathcal{S}_R$, we have a spectral sequence*

$$E_2^{-i,j} = \mathbb{L}_i \operatorname{colim}_{p\in P} \mathbb{R}^j \Delta_{V_{I_p}}(\mathcal{M}) \Longrightarrow \mathbb{R}^{j-i} \Delta_{V_I}(\mathcal{M}).$$

Proof. This follows directly from Lemmas 5.18 and 5.19 and Theorem 5.3. ∎

References

[1] J. Adámek and J. Rosický, *Locally presentable and accessible categories*. London Math. Soc. Lecture Note Ser. 189, Cambridge University Press, Cambridge, 1994, 316 pp. Zbl 0795.18007 MR 1294136

[2] J. Àlvarez Montaner, A. F. Boix, and S. Zarzuela, On some local cohomology spectral sequences. *Int. Math. Res. Not. IMRN* **2020** (2020), no. 19, 6197–6293 Zbl 1486.13024 MR 4165477

[3] J. Àlvarez Montaner, R. García López, and S. Zarzuela Armengou, Local cohomology, arrangements of subspaces and monomial ideals. *Adv. Math.* **174** (2003), no. 1, 35–56 Zbl 1050.13009 MR 1959890

[4] M. Artin, L. W. Small, and J. J. Zhang, Generic flatness for strongly Noetherian algebras. *J. Algebra* **221** (1999), no. 2, 579–610 Zbl 0958.16024 MR 1728399

[5] M. Artin and J. J. Zhang, Noncommutative projective schemes. *Adv. Math.* **109** (1994), no. 2, 228–287 Zbl 0833.14002 MR 1304753

[6] M. Artin and J. J. Zhang, Abstract Hilbert schemes. *Algebr. Represent. Theory* **4** (2001), no. 4, 305–394 Zbl 1030.14003 MR 1863391

[7] A. Banerjee, An extension of the Beauville–Laszlo descent theorem. *Arch. Math. (Basel)* **120** (2023), no. 6, 595–604 Zbl 1527.13010 MR 4598542

[8] A. Beligiannis and I. Reiten, Homological and homotopical aspects of torsion theories. *Mem. Amer. Math. Soc.* **188** (2007), no. 883, 207 pp. Zbl 1124.18005 MR 2327478

[9] M. Brodmann, Asymptotic stability of $\mathrm{Ass}(M/I^n M)$. *Proc. Amer. Math. Soc.* **74** (1979), no. 1, 16–18 Zbl 0395.13008 MR 0521865

[10] M. Brodmann and M. Hellus, Cohomological patterns of coherent sheaves over projective schemes. *J. Pure Appl. Algebra* **172** (2002), no. 2–3, 165–182 Zbl 1011.13009 MR 1906872

[11] M. Brodmann, Ch. Rotthaus, and R. Y. Sharp, On annihilators and associated primes of local cohomology modules. *J. Pure Appl. Algebra* **153** (2000), no. 3, 197–227 Zbl 0968.13010 MR 1783166

[12] M. P. Brodmann and A. Lashgari Faghani, A finiteness result for associated primes of local cohomology modules. *Proc. Amer. Math. Soc.* **128** (2000), no. 10, 2851–2853 Zbl 0955.13007 MR 1664309

[13] M. P. Brodmann and R. Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications*. 2nd edn., Cambridge Stud. Adv. Math. 136, Cambridge University Press, Cambridge, 2013, 491 pp. Zbl 1263.13014 MR 3014449

[14] S. Dăscălescu, C. Năstăsescu, and Ş. Raianu, *Hopf algebras: an introduction*. Monographs and Textbooks in Pure Appl. Math. 235, Marcel Dekker, New York, 2001, 401 pp. Zbl 0962.16026 MR 1786197

[15] K. Divaani-Aazar and R. Sazeedeh, Cofiniteness of generalized local cohomology modules. *Colloq. Math.* **99** (2004), no. 2, 283–290 Zbl 1072.13011 MR 2079733

[16] G. Faltings, Der Endlichkeitssatz in der lokalen Kohomologie. *Math. Ann.* **255** (1981), no. 1, 45–56 Zbl 0451.13008 MR 0611272

[17] M. Hellus, On the set of associated primes of a local cohomology module. *J. Algebra* **237** (2001), no. 1, 406–419 Zbl 1027.13009 MR 1813886

[18] C. Huneke, D. Katz, and T. Marley, On the support of local cohomology. *J. Algebra* **322** (2009), no. 9, 3194–3211 Zbl 1186.13016 MR 2567416

[19] W. Lowen and M. Van den Bergh, Deformation theory of abelian categories. *Trans. Amer. Math. Soc.* **358** (2006), no. 12, 5441–5483 Zbl 1113.13009 MR 2238922

[20] A. Neeman, *Triangulated categories*. Ann. of Math. Stud. 148, Princeton University Press, Princeton, NJ, 2001, 449 pp. Zbl 0974.18008 MR 1812507

[21] N. Popescu, *Abelian categories with applications to rings and modules*. London Math. Soc. Monogr. 3, Academic Press, London-New York, 1973, 467 pp. Zbl 0271.18006 MR 0340375

[22] C. Rotthaus and L. M. Şega, Some properties of graded local cohomology modules. *J. Algebra* **283** (2005), no. 1, 232–247 Zbl 1105.13021 MR 2102081

[23] B. Stenström, *Rings of quotients: an introduction to methods of ring theory*. Grundlehren Math. Wiss. 217, Springer, New York-Heidelberg, 1975, 309 pp. Zbl 0296.16001 MR 0389953

[24] R. Takahashi, Y. Yoshino, and T. Yoshizawa, Local cohomology based on a nonclosed support defined by a pair of ideals. *J. Pure Appl. Algebra* **213** (2009), no. 4, 582–600 Zbl 1160.13013 MR 2483839

[25] C. A. Weibel, *An introduction to homological algebra*. Cambridge Stud. Adv. Math. 38, Cambridge University Press, Cambridge, 1994, 450 pp. Zbl 0797.18001 MR 1269324

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