Comment. Math. Helv. 79 (2004) 840–841
0010-2571/04/040840-2
DOI 10.1007/s00014-004-0817-x

Commentarii Mathematici Helvetici

Errata

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In the article "Cohomology of toric bundles" by P. Sankaran and V. Uma published in Volume 78/3 (2003), pp. 540–554 in the journal Commentarii Mathematici Helvetici were errors. The corrections are as follows:

We correct here the errors in our paper [6] which we found recently much to our embarrassment. The notations of [6] will be in force unless otherwise stated.

1. In Lemma 2.2(i) it was asserted that the elements $z_u \in \mathcal{I} := \mathcal{I}_S$ for every $u \in M$ the proof of which was left out as an "easy exercise". Upon re-examining our proof we realized that it is not valid without further hypotheses! We circumvent the problem by modifying the definition of $\mathcal{I}$ as follows so that Lemma 2.2(i) is redundant:

Assume that $r_i, 1 \leq i \leq n$, are invertible elements in the centre of S. Let $\mathcal{I}$ be the (two-sided) ideal of the polynomial algebra $S[x_1, \cdots, x_d]$ generated by the following two types of elements:

$$x_{j_1} \cdots x_{j_k}, \qquad 1 \leq j_p \leq d, \tag{i}$$

whenever $v_{j_1}, \cdots, v_{j_k}$ do *not* span a cone of Δ; for each $u := \sum_{1\leq i\leq n} a_i u_i \in M$, the element

$$z_u := \prod_{j,\langle u,v_j\rangle>0} (1-x_j)^{\langle u,v_j\rangle} - r_u \prod_{j,\langle u,v_j\rangle<0} (1-x_j)^{-\langle u,v_j\rangle} \tag{ii$'$}$$

where $r_u = \prod_{1\leq i\leq n} r_i^{a_i}$. Define $\mathcal{R}(S,\Delta) := S[x_1, \cdots, x_d]/\mathcal{I}$.

With this definition of $\mathcal{I}$, Lemma 2.2(i) is a tautology. Remaining parts of Lemma 2.2 (the proofs of which used part (i)) are now valid as given in [6].

Lemma 2.2 was used in Proposition 4.3(iii). But with the corrected definition of $\mathcal{R}$, it continues to hold because in 4.3(ii), we established the stronger condition $\prod [L_j]^{\langle u,v_j\rangle} = 1$. This (together with 4.3(i)) ensures that $x_j \mapsto (1-[L_j]^\vee)$, $1 \leq j \leq d$, does yield a well-defined ring homomorphism $\mathcal{R} \longrightarrow K(X)$ (where $r_i = 1$, $1 \leq i \leq n$).

Thanks to equations (7) and (8), p. 552 of [6], the proof of Theorem 1.2(iv) is valid verbatim with this modified definition of $\mathcal{R}$.

2. In Theorem 1.2 (ii), we need, besides the new definition of $\mathcal{R}$, that B be Hausdorff.

Proof of Theorem 1.2 (ii). We now give a proof that $K^*(E(X))$ is a free $K^*(B)$-module of rank m, the number of n-dimensional cones in Δ where $X = X(\Delta)$. With notations as in §4, [6] the restriction of $[\mathcal{L}(\tau_i)]$, $1 \le i \le m$, to the fibre X forms a $\mathbb{Z}$-basis for $K^*(X)$. Since B is compact Hausdorff, it is locally compact and normal. Therefore B can be covered by finitely many *compact* subsets $W_1, \cdots, W_k$ such that the bundle $\pi|W_r$ is trivial for $1 \le r \le k$. Let Y be a closed subspace of W_r. Now using the Künneth theorem for K-theory, which is also valid for general compact spaces (cf. [2]), we see that $K^*(\pi^{-1}(Y))$ is a free $K^*(Y)$-module with basis $[\mathcal{L}(\tau_i)|\pi^{-1}(Y)]$, $1 \le i \le m$. Applying Theorem 1.3, Ch. IV, [4], we conclude that $K^*(E(X))$ is a free $K(B)$-module with basis $[\mathcal{L}(\tau_i)]$, $1 \le i \le m$. In view of equations (7) and (8), p. 552, [6], setting $r_i = \pi^*(\xi_i^\vee)$, one has a well-defined homomorphism $\mathcal{R}(K(B), \Delta) \longrightarrow K(E(X))$ of $K(B)$ algebras defined by $x_j \mapsto (1 - \mathcal{L}_j)$. Rest of the proof is exactly as given in p. 552, [6]. □

3. It was asserted after the proof of Lemma 4.2, [6], that flag varieties G/B where G is semi simple and B a Borel subgroup and smooth Schubert varieties in G/B satisfy the hypotheses of Lemma 4.2. In fact it turns out that $H^*(G/B;\mathbb{Z})$ is not generated by $H^2(G/B;\mathbb{Z})$ in general. This is related to the presence of torsion in the integral cohomology of the classifying space BG. (See §4 of [3].) However $H^*(SL(n,\mathbb{C})/B;\mathbb{Z})$ is generated as an algebra by $H^2(SL(n,\mathbb{C})/B;\mathbb{Z})$. More importantly, the *conclusion* of Lemma 4.2 is valid for any G/B. This follows from the surjectivity of the "α-construction" established by Atiyah–Hirzebruch (Theorem 5.8, [1]) and Pittie [5].

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