

On conjugacy and perturbation of subalgebras

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Abstract. We study conjugacy orbits of certain types of subalgebras in tracial von Neumann algebras. We construct a highly indecomposable non-Gamma II_1 factor N such that every separable von Neumann subalgebra of N with Haagerup’s property admits a unique embedding up to unitary conjugation. Such a factor necessarily has to be non-separable, but we show that it can be taken to have density character $2^{\aleph_0}$. On the other hand, we are able to construct for any separable II_1 factor M_0 , a separable II_1 factor M containing M_0 such that every property (T) subfactor admits a unique embedding into M up to uniformly approximate unitary equivalence; that is, any pair of embeddings can be conjugated up to a small uniform 2-norm perturbation.

1. Statements of main results

In this paper, we make new contributions to the study of conjugacy and perturbation among subalgebras of tracial von Neumann algebras. These topics have been of interest in the area for many decades; see, for instance, [11, 13, 32, 37, 40, 41, 45].

Conjugating Haagerup subalgebras. Our first main result is the construction of the following exotic highly indecomposable non-separable non-Gamma II_1 factor.

Theorem A. *There exists a non-Gamma II_1 factor M such that every separable subalgebra of M with Haagerup’s property embeds in M uniquely up to unitary conjugacy.*

We first describe some key new ideas underlying our construction. Our II_1 factor is built via a concrete inductive construction consisting of iterated amalgamated free products and is tailored to satisfy the conjugacy condition above. One of the main challenges one typically faces in this situation is to maintain non-Gamma in the inductive limit. We adopt a strategy similar to [10]: begin the construction with a diffuse property (T) von Neumann algebra and then carefully apply the structure theorems of [28] at each stage of our construction, to get that the initial property (T) subalgebra is irreducible in the entire union. What allows us to carry this out is a new insight we make on the HNN

extension construction in von Neumann algebra theory [21, 48], which by design allows one to enlarge the algebra where one can conjugate isomorphic subalgebras. We design an a priori new construction involving amalgamated free products with free wreath products to achieve the end goal of an HNN extension and prove that it is indeed isomorphic to the HNN extension, via a universality argument. It is known that the HNN extension is isomorphic to a corner of an amalgamated free product (see [49, Proposition 3.1]). The contribution we make here is to show that the HNN extension is a natural unital subalgebra of an amalgamated free product, allowing us some ease in applying the non-intertwining criteria of [28]. Upon arranging a careful ordering of pairs of the subalgebras with Haagerup's property, we inductively apply our HNN construction to conjugate them. Note that the passage to non-separable is inevitable if we are to handle simultaneously all pairs of separable Haagerup subalgebras, as every abelian subalgebra has Haagerup's property. Moreover, our factor can be taken to have density character $2^{\aleph_0}$. Note also that we can choose our M to contain a copy of all separable Haagerup von Neumann algebras simply by beginning the iterative construction with the tensor product of all separable Haagerup von Neumann algebras. Another key ingredient in carrying out the entire procedure is a flexibility involving bypassing taking closures around certain limit ordinals (see also [20]).

Our construction can also be naturally modified to yield (see Corollary 3.12) a new approach to constructing existentially closed II_1 factor [17]. This is a model theoretic notion that has been studied in the literature recently [2, 9, 22, 30]. Note that existentially closed factors possess property Gamma (they are moreover McDuff [22]), hence will necessarily not be isomorphic to the construction in Theorem A. As such the Haagerup assumption cannot be removed from Theorem A, for this reason.

Theorem B. *There exists a II_1 factor (N, τ) of density character $2^{\aleph_0}$ that contains a unique copy of each separable tracial von Neumann algebra up to unitary conjugacy. Such a II_1 factor is necessarily existentially closed.*

Indecomposability and structural properties. We now describe the various indecomposability and structural properties of our factor M from Theorem A. These indicate that M cannot arise from more or less any standard von Neumann algebraic construction. The following results here are recorded in Proposition 3.17.

First we have that $M \not\cong \prod_{i \rightarrow \mathcal{U}} N_i$ where N_i are finite factors and $\mathcal{U}$ is a countably incomplete ultrafilter on any infinite set. This follows from [3, Corollary 3.8]. This result guarantees the existence inside any ultraproduct of II_1 factors of two embeddings of the free group factor $L(\mathbb{F}_2)$ that are not unitarily conjugate (in fact, the result shows the same for any Connes-embeddable non-amenable II_1 factor). Since $L(\mathbb{F}_2)$ has Haagerup's property, this contradicts the main feature of our construction.

Results of [36] (see also [42, 44]) prove the following indecomposability properties for M (in each case a pair of Haar unitaries that are not conjugate is identified): M does

not admit a diffuse regular subalgebra M_0 such that there is a Haar unitary $u \in \mathcal{U}(M)$ satisfying $\{u\}'' \perp M_0$. In particular, $M \not\cong N_1 \bar{\otimes} N_2$ where N_i are II_1 factors; $M \not\cong N \rtimes G$ where N is a diffuse tracial von Neumann algebra and G is an infinite group. Also, M does not admit Cartan subalgebras.

We also obtain the following using an elementary argument involving normal form decompositions: $M \not\cong N_1 *_B N_2$ where N_1 is diffuse and there exists $u_i \in \mathcal{U}(N_i)$ for $i = 1, 2$ such that $\mathbb{E}_B(u_i) = 0$; $M \not\cong \text{HNN}(N, N_0, \theta)$ where N is diffuse. Also, $M \not\cong L(G)$ where G is a discrete group that admits two elements $g, h \in G$ such that g^n is not conjugate to h^n for all $n \in \mathbb{N}$.

As a final remark, one can also take $h(M) \leq 0$ where h denotes the 1-bounded entropy [23, 33]. This result can be obtained by weaving into our inductive limit the 2-handle construction in [10], thereby ensuring sequential commutation which forces 1-bounded entropy to be non-positive (see also [34]). Moreover, by the arguments in [34], our construction can be shown to admit a unique sequential commutation orbit internally in the factor.

Conjugating property (T) subalgebras. In the setting of working with property (T) subalgebras, we are able to achieve a result similar to Theorem A, the significant difference being that the resulting construction can be made separable as opposed to the setting of conjugating Haagerup subalgebras in which case the construction is forced to be non-separable. However, the natural notion of equivalence to consider here is what we refer to as *uniformly approximate unitary equivalence*, as opposed to on the nose unitary equivalence, which is not likely possible to aim for in general in this setting in light of Remark 3.28. We define this concept as: $\pi_i : N \rightarrow M$, $i = 1, 2$ are uniformly approximate unitary equivalent embeddings if for all $\varepsilon > 0$ there exists a unitary $u \in \mathcal{U}(M)$ such that $\|\mathbb{E}_{\pi_2(N)}(u\pi_1(x)u^*) - u\pi_1(x)u^*\|_2 < \varepsilon$ for every $x \in (N)_1$.

Theorem C. *For any separable II_1 factor M_0 , there exists a separable II_1 factor M containing M_0 such that any diffuse property (T) von Neumann subalgebra $N \subset M$ admits a unique embedding into M up to uniform approximate unitarily equivalence.*

We actually prove a stronger statement than the above (see Theorem 3.27) which also further demonstrates the optimality of the result. The proof of Theorem C uses a previously studied metric between subalgebras (see [11–13, 45, 50]); we denote this metric d . When two subalgebras are close with respect to d , they have large corners which are isomorphic [45]; similarly, if one subalgebra is almost included in another, then there exists a genuine inclusion on large corners [13]. We also argue as in [37] that the set of property (T) subalgebras of a separable II_1 factor is separable with respect to d (see also [1, 14]). These notions are made precise and estimates are given in Section 3.4.

We now describe the construction of M in Theorem C. We proceed by induction as in Theorem A (see also [10]) to construct M from M_0 . Given II_1 factors $M_0 \subset M_1 \subset \dots \subset M_n$, we pick a dense set D_n of property (T) subalgebras and recursively apply our HNN extension to get that every pair of subalgebras $B_1, B_2 \in D_n$ which are isomorphic

on large corners is nearly unitarily conjugate in M_k for some $k \geq n$. We define M to be the inductive limit of the M_n . Given two isomorphic property (T) subalgebras N_i , $i = 1, 2$ of M , we use property (T) and [13] to embed large corners of each N_i into some M_n . Each N_i is d -close to some B_i in D_n , forcing B_1, B_2 to be isomorphic on large corners. By construction, the B_i are nearly unitarily equivalent in M so that the N_i are also nearly unitarily equivalent.

One consequence of uniformly approximate unitary equivalence is bi-intertwining [37, 41]: if $\pi_i : N \rightarrow M$, $i = 1, 2$ are two uniformly approximately unitarily equivalent embeddings, then it follows that $\pi_1(N) \prec_M \pi_2(N)$ and $\pi_2(N) \prec_M \pi_1(N)$. This follows from results of [13]; see also [37, 45]. This puts certain non-trivial restrictions on the structure of M ; for instance, this implies that our factor M from above cannot be isomorphic to $M_1 * M_2$ where each M_i contains a copy of the same diffuse property (T) subalgebra. These two embeddings of this subalgebra cannot bi-intertwine, contradicting this feature of our construction for M .

Two results of independent interest. We conclude the paper with two results of independent interest which arose from related considerations. The first is a genericity statement about elements generating factors inside tracial von Neumann algebras; see Proposition 3.30.

Proposition D. *Let $\{x_1, \dots, x_n\} \subset (M)_1$ be a finite set of elements of operator norm at most 1 in a II_1 factor M . Then for any $\varepsilon > 0$, there exists $\{y_1, \dots, y_n\} \subset (M)_1$, elements of operator norm at most 1, such that $\{y_1, \dots, y_n\}$ generates a subfactor of M and furthermore $\|y_i - x_i\|_2 < \varepsilon$ for all i .*

A delicate perturbation theorem in the above flavor has been obtained recently in [29], with applications to certain problems on trace spaces.

The second result is a sentence that perhaps could be used effectively to distinguish between non-Gamma factors up to elementary equivalence; see Proposition 3.32.

2. Preliminaries

2.1. Notation and background for II_1 factors

In a tracial von Neumann algebra (M, τ) , $\|\cdot\|_2$ is the norm defined by $\|x\|_2 = \sqrt{\tau(x^*x)}$. All inclusions of von Neumann algebras are unital unless otherwise specified. If $N \subset (M, \tau)$ is a von Neumann subalgebra, then there is a unique normal trace-preserving conditional expectation $E_N : M \rightarrow N$.

If $(M_\lambda)_{\lambda < \kappa} \subset N$ is an increasing chain of tracial von Neumann subalgebras with compatible traces of a von Neumann algebra N , then $M := \overline{\bigcup_{\lambda < \kappa} M_\lambda}^{\text{SOT}}$ is a tracial von Neumann subalgebra of N called the *inductive limit* of M_λ . The following is a basic fact (see, e.g., [35, Lemma 2.3]).

Lemma 2.1. *Let $(M, \tau) = \overline{\bigcup_{\lambda < \kappa} M_\lambda}^{\text{SOT}}$ be a tracial von Neumann algebra realized as an inductive limit where κ is an ordinal. Then $M'_1 \cap M = \overline{\bigcup_{\lambda < \kappa} M'_1 \cap M_\lambda}^{\text{SOT}}$.*

Two von Neumann subalgebras $N_1, N_2 \subset (M, \tau)$ are said to be *orthogonal* if $\tau(n_1 n_2) = \tau(n_1) \tau(n_2)$ for any choice of $n_i \in N_i$. In this case, we write $N_1 \perp N_2$. We say an element x is *orthogonal* to a subalgebra $N \subset (M, \tau)$ if $\tau(nx) = 0$ for all $n \in N$.

Lemma 2.2 ([36, Corollary 2.6]). *Let $B \subset (M, \tau)$ be a von Neumann subalgebra and $u \in \mathcal{U}(M)$ a unitary. If there is $B_0 \subset B$ a diffuse von Neumann subalgebra such that $u B_0 u^* \perp B$, then u is orthogonal to $\mathcal{N}(B)''$.*

In this paragraph, we follow [38, Section 3]. Let (M_i, τ_i) be tracial von Neumann algebras, $i = 1, 2$ with a common von Neumann subalgebra B . Let $E_i : M_i \rightarrow B$ denote the trace-preserving conditional expectation. The *amalgamated free product* $M_1 *_B M_2$ is a tracial von Neumann algebra which is SOT-densely spanned by elements of the form $x = b \in B$ and $x = x_{1,i_1} \cdots x_{k,i_k}$ where $x_{j,i_j} \in M_{i_j}$, $E_{i_j}(x_{j,i_j}) = 0$ (in other words, $x_{j,i_j} \in M_{i_j} \ominus B$), and $i_1 \neq \cdots \neq i_k$. We call such elements reduced words. If $x \notin B$ is a reduced word, then $E_B(x) = 0$, and in particular, $\tau(x) = 0$. We note that the subspaces B and $(M_{i_1} \ominus B) \cdots (M_{i_k} \ominus B)$ are all orthogonal for different tuples $(i_1, \dots, i_k)$ such that $i_1 \neq \cdots \neq i_k$. In the case $B = \mathbb{C}1$, we write $M_1 * M_2$ and call it the free product of M_1 with M_2 .

For a sequence $(M_n, \tau_n)_{n \in \mathbb{Z}}$ of tracial von Neumann algebras with a common von Neumann subalgebra B , we recall that the amalgamated free product is associative, so that $(M_1 *_B M_2) *_B M_3 \cong M_1 *_B (M_2 *_B M_3)$, so we write simply $M_1 *_B M_2 *_B M_3$. We define $*_B^{n \in \mathbb{Z}} M_n$ to be the inductive limit of the algebras $M_{-N} *_B M_{-N+1} *_B \cdots *_B M_N$ as $N \rightarrow \infty$. If all of the M_n , $n \in \mathbb{Z}$ are isomorphic to (M, τ) , then there is a natural permutation automorphism $\sigma \in \text{Aut}(*_B^{n \in \mathbb{Z}} M_n)$ which fixes B and takes M_n to M_{n+1} . The group this automorphism generates is isomorphic to $\mathbb{Z}$, and we define the *amalgamated free wreath product* $M \wr_B^* \mathbb{Z}$ as equal to the crossed product $*_B^{n \in \mathbb{Z}} M_n \rtimes_\sigma \mathbb{Z}$. We denote $M \wr^* \mathbb{Z} = M \wr_C^* \mathbb{Z}$. We note too that there is a natural isomorphism $M \wr_B^* \mathbb{Z} \cong M *_B (B \bar{\otimes} L\mathbb{Z})$ (see, e.g., [27, Section 3]).

Definition 2.3. A subalgebra B of tracial von Neumann algebra (M, τ) is said to be *relatively rigid* (or the inclusion has relative property (T)) if either of the following conditions hold:

- (1) For all $\varepsilon > 0$, there is $F \subset M$ finite and $\delta > 0$ such that for all unital, tracial, completely positive maps $\phi : M \rightarrow M$ satisfying $\max_{x \in F} \|\phi(x) - x\|_2 < \delta$, we have $\|\phi(b) - b\|_2 < \varepsilon$ for all $b \in (B)_1$ (the operator norm unit ball of B).
- (2) For all $\varepsilon > 0$, there is $F \subset M$ finite and $\delta > 0$ such that for all M - M bimodules $\mathcal{H}$ and tracial vectors $\xi \in \mathcal{H}$ satisfying $\max_{x \in F} \|x\xi - \xi x\| < \delta$, there is $\eta \in \mathcal{H}$ such that η is B -central (i.e., $\eta b = b\eta$ for all $b \in B$) and $\|\xi - \eta\| < \varepsilon$.

We say M is rigid (or has property (T)) if $M \subset M$ is relatively rigid.

We record a minor technical result which be used in the proof of Theorem C. The proof is nearly identical to the proof of Proposition 14.2.4 in [1].

Lemma 2.4. *Suppose (M, τ) is a tracial von Neumann algebra and $B \subset M$ is a subalgebra with property (T). Then for every $\varepsilon > 0$, there exist a finite set $F \subset B$ and a $\delta > 0$ such that whenever $\phi : M \rightarrow M$ is a tracial, unital, completely positive map such that $\max_{x \in F} \|\phi(x) - x\|_2 < \delta$, then $\|\phi(b) - b\|_2 < \varepsilon$ for all $b \in (B)_1$.*

Proof. Let $\varepsilon > 0$. Set $\varepsilon' = \frac{\varepsilon}{2}$. Since B has property (T), we can find $F \subset B$ finite and $\delta > 0$ such that for all B - B bimodules $\mathcal{H}$ and tracial vectors $\xi \in \mathcal{H}$ such that $\max_{x \in F} \|x\xi - \xi x\| < \delta$, there is a B -central vector η such that $\|\eta - \xi\| < \varepsilon'$.

Now let $\phi : M \rightarrow M$ be a tracial, unital, completely positive map such that $\max_{x \in F} \|\phi(x) - x\|_2 < \delta'$ where $2\delta' \max_{x \in F} \|x\|_2 = \delta^2$. Let $(\mathcal{H}, \xi)$ be the pointed M - M bimodule associated to ϕ . Then $\|x\xi - \xi x\|^2 \leq 2\|\phi(x) - x\|_2 \|x\|_2 < \delta^2$. Since $\mathcal{H}$ is an M - M bimodule, it is in particular a B - B bimodule, and so there is a B -central vector $\eta \in \mathcal{H}$ such that $\|\eta - \xi\| < \varepsilon'$. Then for $b \in (B)_1$, we have $\|\phi(b) - b\|_2 \leq \|b\xi - \xi b\| \leq \|b(\xi - \eta) - (\xi - \eta)b\| \leq 2\varepsilon' = \varepsilon$. ■

We omit the definitions of Haagerup's property, but it can be found in [1, Chapter 16]. We record the following facts about Haagerup's property and property (T).

A group Γ has property (T) if and only if $L\Gamma$ has property (T). In particular, $L(\text{SL}_3(\mathbb{Z}))$ has property (T). Furthermore, if M has property (T), then so does pMp for any projection $p \in M$ [1, Chapter 14]. Furthermore, a II_1 factor M has property (T) if and only if whenever N is a tracial von Neumann algebra such that $M \subset N$, we have $M' \cap N^{\mathcal{U}} = (M' \cap N)^{\mathcal{U}}$ for a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ [47]. The free group factors $L(\mathbb{F}_n)$ all have Haagerup's property [1, Chapter 16]. If M has Haagerup's property, then so does pMp , and a free product of II_1 factors with Haagerup's property again has Haagerup's property [5, Corollary 5]. A diffuse tracial von Neumann algebra with property (T) cannot embed into a tracial von Neumann algebra with Haagerup's property [1, Proposition 16.2.3] and [15].

Theorem 2.5 ([41]). *Let P, Q be von Neumann subalgebras of a tracial von Neumann algebra (M, τ) . Then the following are equivalent:*

- (a) *There is no net (u_i) of unitary elements in P such that for every $x, y \in M$, $\lim_i \|E_Q(x^*u_i y)\|_2 = 0$.*
- (b) *There exists an integer $n \geq 1$, a projection $q \in M_n(Q)$, a nonzero partial isometry $v \in M_{1,n}(M)$, and a normal unital homomorphism $\theta : P \rightarrow qM_n(Q)q$ such that $v^*v \leq q$ and $xv = v\theta(x)$ for all $x \in P$.*

When one of the above equivalent conditions is satisfied, we write $P \prec_M Q$. The following is well known and follows immediately from the definition above and [15].

Proposition 2.6. *If $A, B \subset (M, \tau)$ such that A is a von Neumann subalgebra with Haagerup's property and B is a diffuse von Neumann subalgebra with property (T), then $B \not\prec_M A$.*

Theorem 2.7 ([28, Theorem 1.1]). *Let $B \subset M_1, M_2$ be a common von Neumann subalgebra of two finite von Neumann algebras M_1 and M_2 . Let $M = M_1 *_B M_2$ and $Q \subset M_1$ a von Neumann subalgebra. If $Q \not\prec_M B$, then $Q' \cap M \subset M_1$.*

2.2. Ultraproducts

Let $\mathcal{U}$ be an ultrafilter on a set I and $(M_i, \tau_i)_{i \in I}$ a collection of tracial von Neumann algebras. We denote by $\prod_{i \rightarrow \mathcal{U}} M_i$ the *tracial ultraproduct*, that is, the quotient of $\{(x_i)_{i \in I} : \sup_{i \in I} \|x_i\| < \infty\}$ by the closed ideal $\mathcal{J} \subset \{(x_i)_{i \in I} : \sup_{i \in I} \|x_i\| < \infty\}$ consisting of $x = (x_i)_{i \in I}$ with $\lim_{i \rightarrow \mathcal{U}} \|x_i\|_2 = 0$. If all M_i are isomorphic for $i \in I$, then $M \cong M_i$ admits a natural diagonal inclusion $\iota : M \rightarrow \prod_{i \rightarrow \mathcal{U}} M_i$ given by $\iota(x) = (x)_{i \in I}$. For notational simplicity, we identify M with $\iota(M)$. In the case where all M_i are isomorphic to M , we write $M^{\mathcal{U}} := \prod_{i \rightarrow \mathcal{U}} M_i$. If (M, τ) is a II_1 factor, we say that M has *property Gamma* if $M' \cap M^{\mathcal{U}} \neq \mathbb{C}1$.

Proposition 2.8 ([1, Proposition 5.4.1]). *If (M_n, τ_n) is a sequence of tracial factors such that $\lim_n \dim M_n = \infty$ and $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$, then $\prod_{\mathcal{U}} M_n$ is a II_1 factor.*

The following is well known to experts. We include a proof for reader's convenience. Recall that a projection $p \in M$ is called *minimal* if $pMp = \mathbb{C}p$ and *central* if $px = xp$ for all $x \in M$.

Proposition 2.9. *Let (M_i, τ_i) be tracial von Neumann algebras. Then $\prod_{i \rightarrow \mathcal{U}} M_i$ is a factor if and only if there exists a sequence of either minimal projections in the center or zero projections $p_i \in M_i$ with $\tau_i(p_i) \rightarrow_{\mathcal{U}} 1$. Furthermore, it is a II_1 factor if and only if there exists a sequence of either minimal central projections or zero projections $p_i \in M_i$ with $\tau_i(p_i) \rightarrow_{\mathcal{U}} 1$ and that moreover $\dim(p_i M_i p_i) \rightarrow_{\mathcal{U}} \infty$.*

Proof. If there exists a sequence of either minimal central projections or zero projections $p_i \in M_i$ with $\tau_i(p_i) \rightarrow_{\mathcal{U}} 1$, consider $x = (x_i) = (x_i p_i) \in \prod_{i \rightarrow \mathcal{U}} M_i$ and follow the proof of Proposition 5.4.1 of [1], we have that $\prod_{i \rightarrow \mathcal{U}} M_i$ is a factor. The same argument proves the backward direction of the furthermore part.

Conversely, if $\prod_{i \rightarrow \mathcal{U}} M_i$ is a factor, then for any $p \in \mathcal{P}(\mathcal{Z}(\prod_{i \rightarrow \mathcal{U}} M_i))$, $p = 0$ or 1 . Then for all $p_i \in \mathcal{P}(\mathcal{Z}(M_i))$, $\tau_i(p_i)(1 - \tau_i(p_i)) \rightarrow_{\mathcal{U}} 0$. Let

$$I_0 = \left\{ i : \text{there exists a minimal projection } p_i \in \mathcal{P}(\mathcal{Z}(M_i)) \text{ with } \tau_i(p_i) > \frac{1}{3} \right\}.$$

Then for $i \notin I_0$, there exists $p_i \in \mathcal{P}(\mathcal{Z}(M_i))$ such that $\frac{1}{3} \leq \tau_i(p_i) < \frac{2}{3}$ and $\tau_i(p_i)(1 - \tau_i(p_i)) \geq \frac{2}{9}$. Therefore, $I_0 \in \mathcal{U}$. Take p_i to be a minimal projection in $\mathcal{Z}(M_i)$ with

$\tau_i(p_i) > \frac{1}{3}$ for $i \in I_0$ and $p_i = 0$ otherwise. Then by $\tau_i(p_i)(1 - \tau_i(p_i)) \rightarrow_{\mathcal{U}} 0$, for any small enough $\varepsilon > 0$, $I_\varepsilon := \{i : \tau_i(p_i) > 1 - \varepsilon\} \supseteq I_0 \cap \{i : \tau_i(p_i)(1 - \tau_i(p_i)) < \frac{\varepsilon}{3}\}$ belongs to $\mathcal{U}$. This means the sequence of either minimal central projections or zero projections $p_i \in M_i$ satisfies $\tau_i(p_i) \rightarrow_{\mathcal{U}} 1$.

Finally, for the forward direction of the furthermore part, assume to the contrary that any sequence of either minimal central projections or zero projections $p_i \in M_i$ with $\tau(p_i) \rightarrow 1$ satisfies $\dim(p_i M_i p_i) \not\rightarrow_{\mathcal{U}} \infty$. Since $\prod_{i \rightarrow \mathcal{U}} M_i$ is a factor, by what has been proved we may fix a sequence of either minimal central projections or zero projections $p_i \in M_i$ with $\tau(p_i) \rightarrow 1$. Then $\dim(p_i M_i p_i) \rightarrow_{\mathcal{U}} N$ for some natural number N , and therefore $J = \{i : \dim(p_i M_i p_i) = N\} \in \mathcal{U}$. But then $p_i M_i p_i$, being a factor, is necessarily $\mathbb{M}_{\sqrt{N}}(\mathbb{C})$. Thus, $\prod_{i \rightarrow \mathcal{U}} M_i = \prod_{i \rightarrow \mathcal{U}} p_i M_i p_i = \mathbb{M}_{\sqrt{N}}(\mathbb{C})$, a contradiction. ■

Definition 2.10. An ultrafilter $\mathcal{U}$ on a set S is called *countably incomplete* if there exists a sequence $(A_n)_{n \in \mathbb{N}}$ of sets in $\mathcal{U}$ such that $\bigcap_{n \geq 1} A_n = \emptyset$. Otherwise, $\mathcal{U}$ is called *countably complete*.

The following fact, for the case where the algebra involved is separable, is contained in [6, Lemma 2.3 (2)]. However, since we will apply the ultrapower construction to algebras with density characters up to the continuum, we need a slightly stronger result.

Lemma 2.11. *If (M, τ) is a tracial von Neumann algebra with density character below the first uncountable measurable cardinal (in particular, if M has density character less than or equal to the continuum) and $\mathcal{U}$ is countably complete, then the diagonal embedding $\Delta_M : M \rightarrow M^{\mathcal{U}}$ is a trace-preserving $*$ -isomorphism.*

Proof. By [8, Proposition 4.2.7], $\mathcal{U}$ is α -complete where α is an uncountable measurable cardinal; that is, if $\{A_i\}_{i \in I} \subset \mathcal{U}$ is a collection of sets with $|I| < \alpha$, then $\bigcap_i A_i \in \mathcal{U}$. Since M has density character below the first uncountable measurable cardinal, there is a $\|\cdot\|_2$ -dense subset $\{z_j\}_{j \in J}$ with $|J| < \alpha$. The proof then proceeds exactly the same as the proof of Lemma 2.3 (2) of [6].

For the in particular part, we note that by [8, Theorem 4.2.14], the first uncountable measurable cardinal α is inaccessible, so in particular $\beta < \alpha$ implies $2^\beta < \alpha$. Since α is uncountable, $\aleph_0 < \alpha$, so $2^{\aleph_0} < \alpha$. ■

Theorem 2.12 ([3, Corollary 3.8]). *Let $(M_n)_n$ be a sequence of II_1 factors. Suppose that N is a separable von Neumann algebra that embeds into $R^{\mathcal{U}}$ where $\mathcal{U}$ is an ultrafilter. Then N is amenable if and only if for any two embeddings $\pi, \rho : N \rightarrow M = \prod_{\mathcal{U}} M_n$ there is a unitary $u \in M$ such that $\pi(x) = u\rho(x)u^*$ for all $x \in N$.*

Definition 2.13. If $\mathcal{U}$ is an ultrafilter on S and $\mathcal{V}$ is an ultrafilter on T , then $\mathcal{U} \rtimes \mathcal{V}$ is defined by $X \in \mathcal{U} \rtimes \mathcal{V}$ if and only if $\{s : \{t : (s, t) \in X\} \in \mathcal{V}\} \in \mathcal{U}$. We see that $\mathcal{U} \rtimes \mathcal{V}$ is an ultrafilter on $S \times T$.

The following lemma appears as [7, Theorem 2.1].

Lemma 2.14. *If (M, τ) is a tracial von Neumann algebra and $\mathcal{U}, \mathcal{V}$ are ultrafilters on sets S, T , respectively, then $(M^{\mathcal{U}})^{\mathcal{V}} \cong M^{\mathcal{U} \times \mathcal{V}}$.*

2.3. Model theory

We say two tracial von Neumann algebras (M, τ_M) and (N, τ_N) are *elementarily equivalent* if there exists an ultrafilter $\mathcal{U}$ (possibly on an uncountable set) such that $M^{\mathcal{U}} \cong N^{\mathcal{U}}$. Note that by Lemma 2.11, we may assume $\mathcal{U}$ is countably incomplete when the algebras involved have density characters at most continuum. Let $\Delta_M : M \rightarrow M^{\mathcal{U}}$ denote the diagonal embedding. An embedding of tracial von Neumann algebras $\iota : M \rightarrow (N, \tau)$ is said to be an *elementary embedding* if there is an ultrafilter $\mathcal{U}$ and an isomorphism $\Phi : M^{\mathcal{U}} \rightarrow N^{\mathcal{U}}$ such that $\Phi \circ \Delta_M = \Delta_N \circ \iota$. When $M \subset N$ is an inclusion of tracial von Neumann algebras such that the inclusion map is an elementary embedding, we write $M \preceq N$ and say M is an *elementary substructure* of N .

The following can be found as [19, Theorem 2.3].

Theorem 2.15 (Downward Löwenheim–Skolem). *Let (M, τ) be a tracial von Neumann algebra. Let $X \subset M$ be a separable subset. Then there exists $M_0 \preceq M$ such that $X \subset M_0$ and M_0 is a separable tracial von Neumann algebra.*

Assuming the continuum hypothesis, all results in this paper which apply to ultrapowers of II_1 factors also apply to ultraproducts of matrices.

Proposition 2.16. *Assume the continuum hypothesis. Then every matrix ultraproduct $\mathcal{Q} = \prod_{\mathcal{U}} M_{k(n)}(\mathbb{C})$ such that $\lim_{\mathcal{U}} k(n) = \infty$ and $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$ is isomorphic to an ultrapower of a II_1 factor.*

Proof. By [18, Proposition 4.11], $\mathcal{Q}$ is countably saturated (i.e., $\aleph_1$ -saturated). Since the density character of $\mathcal{Q}$ is $2^{\aleph_0} = \aleph_1$, $\mathcal{Q}$ is saturated. By downward Löwenheim–Skolem, there is a separable tracial factor N such that N is elementarily equivalent to $\mathcal{Q}$. Again by [18, Proposition 4.11], $N^{\mathcal{U}}$ is countably saturated and therefore saturated. Since $N^{\mathcal{U}}$ and $\mathcal{Q}$ have the same density character and are both saturated, they are isomorphic by [18, Proposition 4.13]. It is clear that N is a II_1 factor since $\mathcal{Q}$ is not elementarily equivalent to a matrix algebra. ■

Definition 2.17. A tracial von Neumann algebra M is *existentially closed* (e.c.) in the class of tracial von Neumann algebras if for every tracial von Neumann algebra $N \supset M$, there is an ultrafilter $\mathcal{U}$ such that $M \subset N \subset M^{\mathcal{U}}$ where the inclusion of M is the diagonal embedding of M in $M^{\mathcal{U}}$.

We note that e.c. tracial von Neumann algebras are automatically II_1 factors [22].

Lemma 2.18. *A separable tracial von Neumann algebra M is e.c. if and only if for every separable tracial von Neumann algebra factor $N \supset M$, there is an ultrafilter $\mathcal{U}$ such that N embeds in $M^{\mathcal{U}}$ and the embedding commutes with the diagonal embedding of M in $M^{\mathcal{U}}$.*

Proof. It suffices to assume that for every separable tracial von Neumann algebra $N \supset M$, there is an ultrafilter $\mathcal{U}$ such that N embeds in $M^{\mathcal{U}}$ and the embedding commutes with the diagonal embedding of M in $M^{\mathcal{U}}$ and to prove that M is e.c., since the other direction is immediate.

Let $M \subset N$ be an inclusion of tracial von Neumann algebras. Then by downward Löwenheim–Skolem, there is an elementary separable substructure N_0 of N containing M . By hypothesis, there is an ultrafilter $\mathcal{U}$ and an inclusion $f : N_0 \hookrightarrow M^{\mathcal{U}}$ such that f restricts to the diagonal embedding $\Delta_{M, \mathcal{U}}$ on M . Since $N_0 \leq N$, there is an ultrafilter $\mathcal{V}$ and an inclusion $g : N \hookrightarrow N_0^{\mathcal{V}}$ such that g restricts to the diagonal embedding $\Delta_{N_0, \mathcal{V}}$ on N_0 . Lastly, since there is an inclusion $N_0 \subset M^{\mathcal{U}}$, there is also an induced inclusion $f^{\mathcal{V}} : N_0^{\mathcal{V}} \hookrightarrow (M^{\mathcal{U}})^{\mathcal{V}} = M^{\mathcal{U} \times \mathcal{V}}$. The composition of inclusions $M \subset N \subset N_0^{\mathcal{V}} \subset M^{\mathcal{U} \times \mathcal{V}}$ is then equal to the diagonal embedding $\Delta_{M, \mathcal{U} \times \mathcal{V}}$. In other words, the diagram below commutes:

$$\begin{array}{ccccc}
 M & \hookrightarrow & N_0 & \hookrightarrow & N \\
 & \searrow & \downarrow f & \searrow & \downarrow g \\
 & & M^{\mathcal{U}} & & N_0^{\mathcal{V}} \\
 & & & \searrow & \downarrow f^{\mathcal{V}} \\
 & & & & M^{\mathcal{U} \times \mathcal{V}}
 \end{array}$$

■

Lemma 2.19. *If M is a tracial von Neumann algebra such that all separable elementary substructures of M are e.c., then M is e.c.*

Proof. This follows from downward Löwenheim–Skolem and the fact that every inductive limit of a chain of e.c. tracial von Neumann algebras is e.c. [2, Fact 2.3.2 (4)]. ■

3. Proof of main results

3.1. Conjugating subalgebras

We follow the notation of [21, Section 3] (see also [48]). Let (M, τ) be a tracial von Neumann algebra, $A \subset M$ a von Neumann subalgebra, and $\theta : A \rightarrow M$ a trace-preserving $*$ -homomorphism. Then there is a tracial von Neumann algebra P which contains M and is generated by M and a single additional Haar unitary u such that $uxu^* = \theta(x)$ for all $x \in A$. The algebra P is called the HNN extension of M with respect to A and θ and denoted $\text{HNN}(M, A, \theta)$. Write $A_1 = A$ and $A_{-1} = \theta(A)$. An element

$x = x_0 u^{\varepsilon_1} \cdots u^{\varepsilon_n} x_n$ with $x_i \in M$ and $\varepsilon_i \in \{-1, 1\}$ is said to be reduced if $x_i \in M \ominus A_{\varepsilon_i}$ whenever $\varepsilon_i \neq \varepsilon_{i+1}$. By convention, elements of $M \ominus \mathbb{C}1$ are also considered reduced in P . The reduced words SOT-densely span P . Then P satisfies the following universal property.

Proposition 3.1 ([21, Proposition 3.2]). *Let $P = \text{HNN}(M, A, \theta)$ be an HNN extension. Assume that (Q, τ_Q) is any tracial von Neumann algebra, that $\pi : M \rightarrow Q$ is a trace-preserving embedding, and that $w \in Q$ is a unitary satisfying*

- $\pi(\theta(x)) = w\pi(x)w^*$ for all $x \in A$,
- for all reduced $x = x_0 u^{\varepsilon_1} \cdots u^{\varepsilon_n} x_n \in P$, we have $\tau_Q(\pi(x_0)w^{\varepsilon_1} \cdots w^{\varepsilon_n}\pi(x_n)) = 0$.

Then there exists a unique trace-preserving $$ -homomorphism $\tilde{\pi} : P \rightarrow Q$ extending π and satisfying $\tilde{\pi}(u) = w$.*

It is known that the HNN extension described above is isomorphic to a corner of an amalgamated free product (see [49, Proposition 3.1]). Our contribution is to show that the HNN extension is a natural unital subalgebra of an amalgamated free product, allowing us to easily apply intertwining results such as found in [28].

Definition 3.2. We proceed with our construction by letting (M, τ_M) be a tracial von Neumann algebra, $A \subseteq M$ be a subalgebra, and $\theta : A \rightarrow M$ be a trace-preserving $*$ -homomorphism.

Let $A \wr^* \mathbb{Z}$ be the free wreath product of A with $\mathbb{Z}$, that is, $A \wr^* \mathbb{Z} = (*^{n \in \mathbb{Z}} A_n) \rtimes \mathbb{Z}$ where A_n are copies of A and $\mathbb{Z}$ acts on $*^{n \in \mathbb{Z}} A_n$ by permuting A_n , that is, the conjugation by the generator of $\mathbb{Z}$ sends A_n to A_{n+1} . For notational purposes, let B_1 , B_2 , and B_3 be three copies of $A \wr^* \mathbb{Z}$, in which the copies of A are labeled A_n^1 , A_n^2 , and A_n^3 , respectively, and in which the generators of $\mathbb{Z}$ are labeled t , s , and r , respectively.

We now let $M_1 = M *_A B_1$ where the inclusion of A into B_1 sends A to A_0^1 . Then, let $M_2 = M_1 *_{\theta(A)} B_2$ where the inclusion of $\theta(A)$ into B_2 first sends $\theta(A)$ to A via θ^{-1} , then sends A to A_0^2 .

Now, we observe that in M_1 , A_1^1 is orthogonal to M , so in particular it is orthogonal to $\theta(A)$. We also have A_1^2 is orthogonal to A_0^2 . Hence, A_1^1 and A_1^2 are freely independent in M_2 , so $M_2 \supseteq A_1^1 \vee A_1^2 \cong A * A$. Thus, we may let $M_3 = M_2 *_{A_1^1 \vee A_1^2} B_3$ where A_1^1 is sent to A_0^3 and A_1^2 is sent to A_1^3 . Let $w = s^{-1}rt$. We easily see that $wxw^* = \theta(x)$ for all $x \in A$.

We define $\Phi(M, A, \theta) = \langle M, w \rangle'' \subset M_3$. To ease notation, we set $N = \Phi(M, A, \theta)$. For an element $x \in M_3$, we define $\tilde{x} = x - \tau(x)$.

Theorem 3.3. *Let (M, τ) be a tracial von Neumann algebra, $A \subset M$ a von Neumann subalgebra, and $\theta : A \rightarrow M$ an embedding. Then $\text{HNN}(M, A, \theta) \cong \Phi(M, A, \theta)$.*

Proof. The aim of the proof is to apply Proposition 3.1. There is clearly an embedding of M as a subalgebra into $\text{HNN}(M, A, \theta)$, and it is also immediately clear from the construction that $\theta(x) = wxw^*$ for all $x \in A$. Since $\text{HNN}(M, A, \theta)$ is generated by M and w ,

if we can show that all reduced words of the form $x_0 w^{\varepsilon_1} \cdots w^{\varepsilon_n} x_n$ have trace zero, then we will have proved the theorem.

We now make a series of four technical claims which will allow us to prove that the trace of all reduced words is equal to 0.

We note before proving the claims that $t \perp M$. Indeed, since t is a non-trivial group unitary in the crossed product $B_1 = *_n A_n^1 \rtimes \mathbb{Z}$, we have $\tau(at) = 0$ for all $a \in A_1^1$. That is, $t \perp A_1^1 = A$. But then t is a reduced word in $B_1 *_a M$ and therefore is orthogonal to M .

We note too that $s^{-1}b \perp A_0^2$ for any $b \in A_1^2$ for a similar reason. The unitary s^{-1} is a non-trivial group unitary in the crossed product $B_2 = *_n A_n^2 \rtimes \mathbb{Z}$, and so we have $\tau(s^{-1}c) = 0$ for all $c \in A_0^2 \vee A_1^2$. In particular, $\tau(s^{-1}bd) = 0$ for all $d \in A_0^2$.

We also remark that for any $x \in M$, $tx \perp \theta(A)$. Indeed, as $xa \in M$ for any $a \in \theta(A)$ and $t \perp M$, we have $\tau(txa) = \tau(t)\tau(xa) = 0$.

Claim 3.4. $txs^{-1} \perp A_1^1 \vee A_1^2$ for any $x \in M$.

Proof. We first note that, by construction, A_1^1 and A_1^2 are freely independent. Therefore, they are 2-norm densely spanned by elements of the form $a_1 b_1 \cdots b_{m-1} a_m$ where $a_1, a_m \in A_1^1$, $a_2, \dots, a_{m-1} \in A_1^1 \ominus \mathbb{C}$, and $b_1, \dots, b_{m-1} \in A_1^2 \ominus \mathbb{C}$.

To show that $txs^{-1} \perp A_1^1 \vee A_1^2$, it suffices to show $\tau(txs^{-1}a_1 b_1 \cdots b_{m-1} a_m) = 0$ for all $a_1, a_m \in A_1^1$, $a_2, \dots, a_{m-1} \in A_1^1 \ominus \mathbb{C}$, and $b_1, \dots, b_{m-1} \in A_1^2 \ominus \mathbb{C}$.

If $\tau(a_1) = 0$, then

$$txs^{-1}a_1 b_1 \cdots b_{m-1} a_m = (tx)s^{-1}a_1 b_1 \cdots b_{m-1} \overset{\circ}{a}_m + \tau(a_m)(tx)s^{-1}a_1 b_1 \cdots b_{m-1}$$

is a reduced word in $M_2 = M_1 *_\theta(A) B_2$ as $tx \perp \theta(A)$ and therefore has trace zero. In the case $\tau(a_1) \neq 0$, we similarly have

$$\begin{aligned} txs^{-1}a_1 b_1 \cdots b_{m-1} a_m &= (tx)s^{-1} \overset{\circ}{a}_1 b_1 \cdots b_{m-1} a_m + \tau(a_1)(tx)(s^{-1}b_1) \cdots b_{m-1} a_m \\ &= (tx)s^{-1} \overset{\circ}{a}_1 b_1 \cdots b_{m-1} a_m + \tau(a_1)(tx)(s^{-1}b_1) a_2 \cdots b_{m-1} \overset{\circ}{a}_m \\ &\quad + \tau(a_1)\tau(a_m)(tx)(s^{-1}b_1) a_2 \cdots b_{m-1}. \end{aligned}$$

The first term has trace zero, as we have already seen. The second and third terms are reduced words, as $s^{-1}b_1 \perp A_0^2$. ■

Claim 3.5. $sxt^{-1} \perp A_1^1 \vee A_1^2$ for any $x \in M$.

Proof. We first note that for $x \in M$ and $a \in A_1^1$, $xt^{-1}a \perp M$. Indeed, we note that $t^{-1} \perp M$ and $a = tbt^{-1}$ for some $b \in A_0^1 \subset M$ and so for any $y \in M$, $\tau(xt^{-1}ay) = \tau(xbt^{-1}y) = \tau(t^{-1})\tau(yxb) = 0$. In particular, we have that $xt^{-1}a \perp \theta(A)$.

Now, for $a_1, a_m \in A_1^1$, $a_2, \dots, a_{m-1} \in A_1^1 \ominus \mathbb{C}$, and $b_1, \dots, b_{m-1} \in A_1^2 \ominus \mathbb{C}$, we have that

$$sxt^{-1}a_1 b_1 \cdots b_{m-1} a_m = \tau(a_m)s(xt^{-1}a_1)b_1 \cdots b_{m-1} + s(xt^{-1}a_1)b_1 \cdots b_{m-1} \overset{\circ}{a}_m,$$

which is a sum of two terms which are reduced words in $M_2 = M_1 *_\theta(A) B_2$ and therefore has trace zero. ■

Claim 3.6. $txt^{-1} \perp A_1^1 \vee A_1^2$ for any $x \in M \ominus A$.

Proof. We first note that for $x \in M \ominus A$, $txt^{-1} \perp M$. Indeed, if $y \in M$, then $txt^{-1}y = txt^{-1}E_A(y) + txt^{-1}(y - E_A(y))$, a sum of two terms which are reduced words in $M_1 = M *_A B_1$. Therefore, $\tau(txt^{-1}y) = 0$.

We next observe that for $x \in M \ominus A$ and $a \in A_1^1$, $txt^{-1}a \perp \theta(A)$. We note that $a = tbt^{-1}$ for some $b \in A_0^1 = A$, and $bx \in M \ominus A$, so that $txt^{-1}a = txbt^{-1} \perp M \supset \theta(A)$.

Therefore, for $x \in M \ominus A$, $a_1, a_m \in A_1^1$, $a_2, \dots, a_{m-1} \in A_1^1 \ominus \mathbb{C}$, and $b_1, \dots, b_{m-1} \in A_1^2 \ominus \mathbb{C}$, we have that

$$txt^{-1}a_1b_1 \cdots b_{m-1}a_m = \tau(a_m)(txt^{-1}a_1)b_1 \cdots b_{m-1} + (txt^{-1}a_1)b_1 \cdots b_{m-1}a_m^\circ,$$

which is a sum of two terms which are reduced words in $M_2 = M_1 *_{\theta(A)} B_2$ and therefore has trace zero. ■

Claim 3.7. $sxs^{-1} \perp A_1^1 \vee A_1^2$ for any $x \in M \ominus \theta(A)$.

Proof. We first note that for $x \in M \ominus \theta(A)$, $sxs^{-1} \perp M$. Indeed, if $y \in M$, then $sxs^{-1}y = sxs^{-1}E_{\theta(A)}(y) + sxs^{-1}(y - E_{\theta(A)}(y))$, a sum of two terms which are reduced words in $M_2 = M_1 *_{\theta(A)} B_2$. Therefore, $\tau(sxs^{-1}y) = 0$.

We next observe that for $x \in M \ominus \theta(A)$ and $a \in A_1^2$, $sxs^{-1}a \perp \theta(A)$. We note that $a = sb s^{-1}$ for some $b \in A_0^2 = \theta(A)$, and $bx \in M \ominus \theta(A)$, so that $sxs^{-1}a = sxb s^{-1} \perp M \supset \theta(A)$.

Consequently, for $x \in M \ominus \theta(A)$, $a_1, a_m \in A_1^1$, $a_2, \dots, a_{m-1} \in A_1^1 \ominus \mathbb{C}$, and $b_1, \dots, b_{m-1} \in A_1^2 \ominus \mathbb{C}$, we have that

$$\begin{aligned} sxs^{-1}a_1b_1 \cdots b_{m-1}a_m &= (sxs^{-1})a_1^\circ b_1 \cdots b_{m-1}a_m^\circ \\ &\quad + \tau(a_1)(sxs^{-1}b_1) \cdots b_{m-1}a_m^\circ \\ &\quad + \tau(a_m)(sxs^{-1})a_1^\circ b_1 \cdots b_{m-1} \\ &\quad + \tau(a_1)\tau(a_m)(sxs^{-1}b_1)a_2 \cdots b_{m-1}, \end{aligned}$$

which is a sum of four terms which are reduced words in $M_2 = M_1 *_{\theta(A)} B_2$ and therefore has trace zero. ■

Now, let $x_0 u^{\varepsilon_1} \cdots u^{\varepsilon_n} x_n$ with $x_i \in M$, $\varepsilon_i \in \{-1, 1\}$, and $x_i \in M \ominus A_{\varepsilon_i}$ whenever $\varepsilon_i \neq \varepsilon_{i+1}$ be a reduced word in $P = \text{HNN}(M, A, \theta)$. We wish to show that $\tau_N(x_0 u^{\varepsilon_1} \cdots u^{\varepsilon_n} x_n) = 0$ where $N = \Phi(M, A, \theta)$. Write $w = s^{-1}rt$ and $w^{-1} = t^{-1}r^{-1}s$ as in the above construction. Then $x_0 u^{\varepsilon_1} \cdots u^{\varepsilon_n} x_n$ is a product of terms of the form $r, r^{-1}, s^{-1}xt, t^{-1}xs, tyt^{-1}$, and szs^{-1} where $x \in M$, $y \in M \ominus A$, and $z \in M \ominus \theta(A)$. Moreover, the product alternates between terms of the form $r^{\pm 1}$ and terms with an s or a t . But $B_3 \ni r, r^{-1} \perp A_1^1 \vee A_1^2$, while the other terms are in M_2 and orthogonal to $A_1^1 \vee A_1^2$. Therefore, the expression $x_0 u^{\varepsilon_1} \cdots u^{\varepsilon_n} x_n$ (possibly needing to conjugate by either s or t) is a reduced word in $M_3 = M_2 *_{A_1^1 \vee A_1^2} B_3$ and therefore has trace zero in N . ■

We record some more facts about $\text{HNN}(M, A, \theta)$ which follow from the above theorem and its proof.

Corollary 3.8. *In $\text{HNN}(M, A, \theta)$, $E_M(w^n) = 0$ for all $n \neq 0$ where w is as in the construction given in this section.*

Proof. We show that $\tau(w^n x) = 0$ for any $n > 0$ and $x \in M$. Indeed, we have,

$$\tau(w^n x) = \tau([s^{-1}rt]^n x) = \tau(r[(ts^{-1})r]^{n-1}txs^{-1}).$$

We observe that $r[(ts^{-1})r]^{n-1}txs^{-1}$ is a reduced word in $M_3 = M_2 *_{A_1^1 \vee A_1^2} B_3$, as $r \perp A_0^3 * A_1^3 = A_1^1 \vee A_1^2$, $ts^{-1} \perp A_1^1 \vee A_1^2$ per Claim 3.4, and $txs^{-1} \perp A_1^1 \vee A_1^2$ per Claim 3.4. ■

Corollary 3.9. *Let $N = \text{HNN}(M, A, \theta)$. Let $\tilde{A}$ be an isomorphic copy of A . Assume $Q \subset M$ is such that $Q \not\prec_{\tilde{M}_1} \tilde{A}$ for any tracial von Neumann algebra $\tilde{M}_1$ containing M and $\tilde{A}$ and $Q \not\prec_{\tilde{M}_2} \tilde{A} * \tilde{A}$ for any tracial von Neumann algebra $\tilde{M}_2$ containing M and $\tilde{A} * \tilde{A}$. Then $Q' \cap N \subset M$.*

Moreover, if M is a II_1 factor, $Q \subseteq M$ is an irreducible subfactor with property (T), and A has Haagerup's property, then $Q' \cap N = \mathbb{C}$.

Proof. The result follows from Theorem 2.7 since $Q' \cap N \subset Q' \cap M_3$ and

$$Q' \cap M_3 \subseteq Q' \cap M_2 \subseteq Q' \cap M_1 \subseteq Q' \cap M.$$

For the “moreover” part, since both A and $A_1^1 \vee A_1^2 \cong A * A$ have Haagerup's property, Proposition 2.6 says that every diffuse subalgebra Q with property (T) satisfies $Q \not\prec_{\tilde{M}_1} A$ and $Q \not\prec_{\tilde{M}_2} A * A$. Therefore, $Q' \cap N \subset Q' \cap M = \mathbb{C}1$. ■

3.2. Inductive constructions

Theorem 3.10. *For any diffuse tracial von Neumann algebra (M, τ_M) of density character at most $2^{\aleph_0}$, there exists a tracial von Neumann algebra (N, τ_N) containing M , of density character at most $2^{\aleph_0}$, such that whenever A is a separable tracial von Neumann algebra and $\pi_1, \pi_2 : A \rightarrow N$ are two embeddings, then there exists a unitary $u \in U(N)$ such that $u\pi_1(a)u^* = \pi_2(a)$ for all $a \in A$.*

Proof. Fix a bijection $\sigma = (\sigma_1, \sigma_2) : 2^{\aleph_0} \rightarrow 2^{\aleph_0} \times 2^{\aleph_0}$ such that $\sigma_1(\alpha) \leq \alpha$ for every $\alpha < 2^{\aleph_0}$. We construct an increasing sequence of algebras $(M_\alpha)_{\alpha < 2^{\aleph_0}}$, all of the density character at most $2^{\aleph_0}$, by induction. Let $M_0 = M$. Assume all M_λ with $\lambda < \alpha$ have been constructed. Let $\{(A^{\lambda, \kappa}, \pi_1^{\lambda, \kappa}, \pi_2^{\lambda, \kappa})\}_{\kappa < 2^{\aleph_0}}$ be an enumeration of all separable tracial von Neumann algebras $A^{\lambda, \kappa}$ and pairs of embeddings of $A^{\lambda, \kappa}$ into M_λ . If $\alpha = \lambda + 1$, we let $M_\alpha = \text{HNN}(M_\lambda, \pi_1^{\sigma(\lambda)}(A^{\sigma(\lambda)}), \pi_2^{\sigma(\lambda)} \circ (\pi_1^{\sigma(\lambda)})^{-1})$. If α is a limit ordinal, let $M_\alpha = \overline{\bigcup_{\lambda < \alpha} M_\lambda}$ where the closure is under the SOT topology on the GNS Hilbert space

associated with the trace on $\bigcup_{\lambda < \alpha} M_\lambda$. Finally, let $N = \bigcup_{\alpha < 2^{\aleph_0}} M_\alpha$. Since in a tracial von Neumann algebra, the SOT topology on the operator norm unit ball is induced by the 2-norm metric, and as the cofinality of $2^{\aleph_0}$ is larger than $\aleph_0$, we see that the operator norm unit ball of N is already SOT-closed, whence it is a tracial von Neumann algebra. For any separable A and embeddings $\pi_1, \pi_2 : A \rightarrow N$, again because the cofinality of $2^{\aleph_0}$ is larger than $\aleph_0$, the ranges of both embeddings must already be contained in M_α for some $\alpha < 2^{\aleph_0}$. Thus, $(A, \pi_1, \pi_2) = (A^{\alpha, \kappa}, \pi_1^{\alpha, \kappa}, \pi_2^{\alpha, \kappa})$ for some $\kappa < 2^{\aleph_0}$, so by construction, they are already unitarily conjugate in $M_{\sigma^{-1}(\alpha, \kappa)+1}$. ■

Corollary 3.11. *There is a tracial von Neumann algebra (N, τ) of density character $2^{\aleph_0}$ that contains a unique copy of each separable tracial von Neumann algebra up to unitary conjugacy.*

Proof. There are at most continuum many separable tracial von Neumann algebras since each is countably generated and is determined by the values of traces of all *-polynomials in the generators. Hence, there is a tracial von Neumann algebra M of density character $2^{\aleph_0}$ containing all separable tracial von Neumann algebras, for example, M being the tensor product of all separable tracial von Neumann algebras. Applying Theorem 3.10 inductively over all separable tracial von Neumann subalgebras proves the result. ■

Corollary 3.12. *Let N be as in Corollary 3.11. Then N is e.c.*

Proof. Let $N_0 \leq N$ be a separable elementary substructure. We note that by Theorem 2.15, for some ultrafilter $\mathcal{U}$, $N_0 \rightarrow N \rightarrow N_0^{\mathcal{U}}$ commutes with the diagonal embedding. Then if $M \supset N_0$ is a separable tracial von Neumann algebra, M embeds into N since N contains every separable tracial von Neumann algebra.

There are now two embeddings of N_0 into N : one that passes through M and the one from the beginning. By Corollary 3.11, these embeddings are unitarily conjugate. So we can conjugate M by a unitary u so that $N_0 \subset uMu^* \subset N \subset N_0^{\mathcal{U}}$ and all inclusions commute with the diagonal embedding. Therefore, N_0 is e.c.

That N is e.c. now follows from Lemma 2.19. ■

Lemma 3.13. *Any tracial (N, τ) with the property that it contains at least one non-amenable Connes-embeddable separable II_1 factor and that any two embeddings of this factor are unitarily conjugate is not isomorphic to an ultraproduct of II_1 factors.*

Proof. By Theorem 2.12, in an ultraproduct of II_1 factors, any non-amenable Connes-embeddable separable II_1 factor has at least two non-unitarily conjugate embeddings. ■

Definition 3.14 ([34, Definition 3.1]). Let (M, τ) be a diffuse tracial von Neumann algebra. Fix a countably incomplete ultrafilter $\mathcal{U}$. Define $\sim_M$ to be the equivalence relation defined on $\mathcal{H}(M)$ (Haar unitaries in M) in the following way: we say $u \sim_M v$ if there are $w_1, \dots, w_n \in \mathcal{H}(M^{\mathcal{U}})$ such that $[u, w_1] = [w_k, w_{k+1}] = [w_n, v] = 0$ for all $1 \leq k < n$.

Theorem 3.15 ([10, Theorem 4.2]). *Let (M, τ) be a diffuse tracial von Neumann algebra. Let $u_1, u_2 \in \mathcal{U}(M)$ be unitaries such that $\{u_1\}'' \perp \{u_2\}''$. Then there exists a II_1 factor $P = \Psi(M, u_1, u_2)$ containing M such that there exist Haar unitaries $v_1, v_2 \in \mathcal{H}(P)$ with the property that $[u_1, v_1] = [v_1, v_2] = [v_2, u_2] = 0$. Furthermore, if $Q \subset M$ is a von Neumann subalgebra such that $Q \not\prec_M \{u_i\}''$ for $i = 1$ and $i = 2$, then $Q' \cap P \subset M$. In particular, if $Q \subset M$ is a II_1 subfactor, then $Q' \cap P \subset M$.*

We are now ready to prove Theorem A, which follows immediately from the following theorem.

Theorem 3.16. *There exists a II_1 factor M that is not isomorphic to an ultraproduct of II_1 factors, is non-Gamma, and every separable subalgebra of M with Haagerup's property embeds in M uniquely up to unitary conjugacy. In particular, M is not e.c. The factor M can also be chosen to have non-positive 1-bounded entropy; that is, $h(M) \leq 0$.*

Proof. Let $M_0 = L(\mathrm{SL}_3(\mathbb{Z}))$. Similar to the proof of Theorem 3.10, we recursively define an increasing sequence of algebras $(M_\alpha)_{\alpha < 2^{\aleph_0}}$. Again, we fix a bijection $\sigma = (\sigma_1, \sigma_2) : 2^{\aleph_0} \rightarrow 2^{\aleph_0} \times 2^{\aleph_0}$ such that $\sigma_1(\alpha) \leq \alpha$ for every $\alpha < 2^{\aleph_0}$. Assume all M_λ with $\lambda < \alpha$ have been constructed. Let $\{Z^{\lambda, \kappa}\}_{\kappa < 2^{\aleph_0}}$ be an enumeration of all possible tuples of the form $Z^{\lambda, \kappa} = (A^{\lambda, \kappa}, \pi_1^{\lambda, \kappa}, \pi_2^{\lambda, \kappa})$ where $A^{\lambda, \kappa}$ is a separable tracial von Neumann algebra with Haagerup's property and $(\pi_1^{\lambda, \kappa}, \pi_2^{\lambda, \kappa})$ is a pair of embeddings of $A^{\lambda, \kappa}$ into M_λ , or of the form $Z^{\lambda, \kappa} = (u_1^{\lambda, \kappa}, u_2^{\lambda, \kappa})$ where $u_1, u_2 \in \mathcal{U}(M_\lambda)$ and $\{u_1\}'' \perp \{u_2\}''$.

If $\alpha = \lambda + 1$, we let

$$M_\alpha = \begin{cases} \mathrm{HNN}(M_\lambda, \pi_1^{\sigma(\lambda)}(A^{\sigma(\lambda)}), \pi_2^{\sigma(\lambda)} \circ (\pi_1^{\sigma(\lambda)})^{-1}), & Z^{\sigma(\lambda)} = (A^{\sigma(\lambda)}, \pi_1^{\sigma(\lambda)}, \pi_2^{\sigma(\lambda)}) \\ \Psi(M_\lambda, u_1^{\sigma(\lambda)}, u_2^{\sigma(\lambda)}), & Z^{\sigma(\kappa)} = (u_1^{\sigma(\lambda)}, u_2^{\sigma(\lambda)}) \end{cases}.$$

If α is a limit ordinal, let $M_\alpha = \overline{\bigcup_{\lambda < \alpha} M_\lambda}$. Finally, let $M = \bigcup_{\alpha < 2^{\aleph_0}} M_\alpha$. Again, it is a tracial von Neumann algebra.

We need to check that M is a non-Gamma II_1 factor. It is clear that any two embeddings of a separable subalgebra with Haagerup's property in M are unitarily conjugate. It is at this stage that Haagerup's property is crucial. By the moreover statement of Corollary 3.9, Theorem 3.15, and as M_0 has property (T), we have, using transfinite induction, that $M'_0 \cap M = M'_0 \cap M_0 = \mathbb{C}$. Since M_0 has property (T), we also have that $M'_0 \cap M^{\mathcal{U}} = (M'_0 \cap M)^{\mathcal{U}} = \mathbb{C}$. Therefore, $M' \cap M^{\mathcal{U}} = \mathbb{C}$ for any ultrafilter $\mathcal{U}$, so M is a non-Gamma II_1 factor. Applying Lemma 3.13 to $L(\mathbb{F}_2)$, a non-amenable Connes-embeddable separable II_1 factor with Haagerup's property shows that M is not isomorphic to an ultraproduct of II_1 factors.

We also need to check that $h(M) \leq 0$. By [34, Lemma 5.1], it suffices to check that $\sim_M$ has a unique orbit. Indeed, let $u_1, u_2 \in \mathcal{H}(M)$ be Haar unitaries. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. By [43], there is a diffuse separable abelian von Neumann algebra generated by a Haar unitary $v \in M^{\mathcal{U}}$ such that $\{u_1, u_2\}''$ and $\{v\}''$ are freely independent (see also [39]). By [26, Theorem 5.1], we can lift v to a sequence of unitaries $(v_n)_n$

in M such that $\{u_1\}'' \perp \{v_n\}''$ for each n . By construction, $\Psi(M_\lambda, u_1, v_n) \subset M$ for some ordinal $\lambda < 2^{\aleph_0}$ and therefore M contains Haar unitaries $w_{1,n}, w_{2,n}$ such that $[u_1, w_{1,n}] = [w_{1,n}, w_{2,n}] = [w_{2,n}, v_n] = 0$ for all n , as in Theorem 3.15. Therefore, $u_1 \sim_M u$. Similarly, $u_2 \sim_M u$, and so $u_1 \sim_M u_2$. Therefore, $\sim_M u$ has one orbit, implying by [34, Proposition 6.2] $\sim_M$ also only has one orbit. ■

3.3. Indecomposability results

Items (1) and (3) of the proposition below are results of [36, 42], and item (2) is quite an elementary observation, not appearing before in the literature. We include the proofs for the convenience of the readers. We also point out to the reader the note [24].

Proposition 3.17. *Let M be a II_1 factor such that all pairs of Haar unitaries are conjugate.*

- (1) *M does not contain a diffuse regular von Neumann subalgebra B and a Haar unitary u such that $\{u\}'' \perp B$. In particular, M is neither the tensor product of two II_1 factors nor a crossed product of a diffuse tracial von Neumann algebra by an infinite group; M does not have a Cartan subalgebra [36, 44].*
- (2) *M is not an amalgamated free product $M_1 *_B M_2$ with $B \subset M_i$, M_i diffuse, and where there exist unitaries $u_i \in M_i \ominus B$, $i = 1, 2$, moreover, M is not an HNN extension of a diffuse von Neumann algebra.*
- (3) *M does not have a separable MASA [42].*

Proof. (1) Suppose M has a diffuse tracial subalgebra B and a Haar unitary $u \in \mathcal{H}(M)$ such that $\{u\}'' \perp B$. Take $v \in \mathcal{H}(B)$. By the hypothesis, there is $w \in \mathcal{U}(M)$ such that $wvw^* = u$. Then $B_0 = \{v\}''$ is a diffuse subalgebra of B such that $wB_0w^* \perp B$. By Lemma 2.2, we have that $w \perp \mathcal{N}(B)''$. In particular, $\mathcal{N}(B)'' \neq M$, so B cannot be regular. In a tensor product $M_1 \otimes M_2$ of two II_1 factors, M_1 is regular and diffuse and any Haar unitary $u \in \mathcal{H}(M_2)$ satisfies $\{u\}'' \perp M_1$. In a crossed product of a diffuse algebra by an infinite group $B \rtimes \Gamma$, B is diffuse and regular and any Haar unitary $u \in L(\Gamma)$ again satisfies $\{u\}'' \perp B$. ($L(\Gamma)$ contains Haar unitaries since it is diffuse.)

We also include more elementary proofs for the cases of tensor products and crossed products. First suppose that M is a tensor product $M_1 \otimes M_2$ such that M_i are II_1 factors. Let u be a Haar unitary in M_1 . For pure tensors $x_1 \otimes x_2, y_1 \otimes y_2 \in M$, we see that $\lim_{n \rightarrow \infty} E_{1 \otimes M_2}((x_1 \otimes x_2)(u^n \otimes 1)(y_1 \otimes y_2)) = \tau(x_1 u^n y_1) x_2 y_2 \rightarrow 0$. Linearity shows the same holds for linear combinations of pure tensors. For general $x, y \in M$, and for $\varepsilon > 0$, take $x', y' \in M_1 \otimes M_2$ such that $\|x - x'\|_2 < \varepsilon$ and $\|y - y'\|_2 < \varepsilon$. By the Kaplansky density theorem, we can also choose $\|y'\|_\infty \leq \|y\|_\infty$. Then

$$\begin{aligned} \|E_{1 \otimes M_2}(xu^n y)\|_2 &\leq \|E_{1 \otimes M_2}(xu^n(y - y'))\|_2 + \|E_{1 \otimes M_2}((x - x')u^n y')\|_2 \\ &\quad + \|E_{1 \otimes M_2}(x'u^n y')\|_2 \end{aligned}$$

$$\begin{aligned} &\leq \|xu^n(y - y')\|_2 + \|(x - x')u^n y'\|_2 + \|E_{1 \otimes M_2}(x'u^n y')\|_2 \\ &\leq \|y - y'\|_2 \|xu^n\|_\infty + \|x - x'\|_2 \|u^n y'\|_\infty + \|E_{1 \otimes M_2}(x'u^n y')\|_2. \end{aligned}$$

This clearly goes to 0 as we choose better approximations x', y' and as $n \rightarrow \infty$. Applying the easy direction of Theorem 2.5 gives two non-conjugate Haar unitaries.

The proof for crossed products is very similar: if $M = B \rtimes_\sigma \Gamma$ where B is diffuse and Γ is infinite, then $L\Gamma$ is diffuse and there are Haar unitaries in both B and $L\Gamma$. Let u be a Haar unitary in B . Let bu_g, cu_h be elements in $B \cdot L\Gamma$ (such elements densely span $B \rtimes \Gamma$). Then $E_{L\Gamma}(bu_g u^n cu_h) = E_{L\Gamma}(b\sigma_g(u^n c)u_{gh}) = \tau(b\sigma_g(u^n c))\delta_{gh,e} = \tau(\sigma_{g^{-1}}(b)u^n c)\delta_{gh,e} \rightarrow 0$ as $n \rightarrow \infty$.

(2) In an amalgamated free product $M = M_1 *_B M_2$, take unitaries $u_i \in M_i \ominus B$. By the Kaplansky density theorem, it suffices to check that $\|E_{M_1}(x(u_1 u_2)^n y)\|_2 \rightarrow 0$ as $n \rightarrow \infty$ for any reduced words $x, y \in M$ since reduced words SOT-densely span M . Denote by X_i the set $M_i \ominus B$. We note that $BX_i B \subset X_i$ and $X_i X_i \subset B \oplus X_i$. Let $\underline{i} = (i_1, \dots, i_k)$ and $\underline{j} = (j_1, \dots, j_\ell)$ be possibly empty tuples of 1s and 2s such that $i_1 \neq \dots \neq i_k$ and $j_1 \neq \dots \neq j_\ell$. We take $x \in X_{i_1} \dots X_{i_k}$ and $y \in X_{j_1} \dots X_{j_\ell}$ with the convention that if $\underline{i}$ is empty, then $x \in B$, and similarly for y . Since $BX_i B \subset X_i$ and $X_i X_i \subset B \oplus X_i$, if n is greater than the number of occurrences of the number 2 in $\underline{i}$ and $\underline{j}$ combined, we will have that $x(u_1 u_2)^n y$ is a linear combination of words in $X_{m_1} \dots X_{m_r}$ where at least one of the m_s is equal to 2. Therefore, $E_{M_1}(x(u_1 u_2)^n y) = 0$ for all n sufficiently large. By the easy direction of the intertwining theorem, the unitary $u_1 u_2$ is not conjugate to any unitary in M_1 . Since M_1 is diffuse, this means M contains two non-conjugate unitaries.

To analyze the case of an HNN extension, we use the notation of Section 3.1. If $M = \text{HNN}(N, A, \theta)$, then $M = \langle N, w \rangle''$ where w is a Haar unitary and $\langle N, w \rangle'' \subset N_2 *_A *_A B$ for some diffuse algebras N_2, B such that $N \subset N_2$. Furthermore, we can write $w = s^{-1} r t$ where r, s, t are unitaries in $N_2 *_A *_A B$ such that $N_2 \ni t s^{-1} \perp A *_A A$ and $B \ni r \perp A *_A A$. By the previous paragraph, we have that $\|E_{N_2}(x(t s^{-1} r)^n y)\|_2 \rightarrow 0$ for all $x, y \in N_2 *_A *_A B$. In particular, $\|E_{N_2}(x t (s^{-1} r t)^{n-1} s^{-1} r y)\|_2 \rightarrow 0$ for all $x, y \in N_2 *_A *_A B$. Specializing to $N \subset N_2$, we have that $\|E_N(x w^n y)\|_2 \rightarrow 0$ for all $x, y \in N$. Since N is diffuse, it contains a Haar which must not be conjugate to w .

(3) If M had a separable MASA A , it would be isomorphic to $L\mathbb{Z}$. The canonical group unitaries u_1, u_2 would be conjugate, but $\{u_2\}'' = L(2\mathbb{Z})$ does not generate a MASA, a contradiction. ■

Remark 3.18. We also include the following auxiliary elementary observation: let Γ be a discrete group and if $g, h \in \Gamma$ are such that g^n is not conjugate to h^n for all nonzero integers n , then u_g is not conjugate to u_h in $L\Gamma$. Suppose towards a contradiction that u_g and u_h are conjugate in $L\Gamma$, then there is a unitary $u \in L\Gamma$ such that $u u_g = u_h u$. Write $u = \sum_{k \in \Gamma} \alpha_k u_k$. Then $\sum_k \alpha_k u_{kg} = \sum_k \alpha_k u_{hk}$. This implies that $\alpha_{h^{-1}k} = \alpha_{kg^{-1}}$ for all $k \in \Gamma$. Equivalently, we have $\alpha_{h^{-n}kg^n} = \alpha_k$ for all integers n and $k \in \Gamma$. But we know

that $h^{-n}kg^n \neq k$ for any n ; otherwise, g^n and h^n would be conjugate in Γ . Therefore, $\alpha_k = 0$, and so $u = 0$, a contradiction.

3.4. On conjugating property (T) subalgebras

We thank Adrian Ioana for suggesting we investigate conjugacy of property (T) subalgebras. We provide here a more or less optimal picture of the case of conjugation for property (T) subalgebras.

Let (M, τ) be a tracial von Neumann algebra. For a map $\phi : M \rightarrow M$, define

$$\|\phi\|_{\infty,2} = \sup\{\|\phi(x)\|_2 : x \in (M)_1\}$$

We briefly recall the notion of uniform distance between subalgebras of a tracial von Neumann algebra (M, τ) given by the metric

$$d(A, B) = \max\{\|(I - E_B)E_A\|_{\infty,2}, \|(I - E_A)E_B\|_{\infty,2}\}$$

(see, e.g., [13, 45, 50]). The metric d is equivalent to the metric $d_2(A, B) = \|E_A - E_B\|_{\infty,2}$. More precisely, $d(A, B) \leq d_2(A, B) \leq \sqrt{2d(A, B)}$ [45, Remark 6.6].

It is known that the set of relatively rigid subalgebras of M is closed in d [50, Proposition 3.2]. Here we prove the following, using the separability argument (for instance, see the proof of Theorem 4.5.1 in [37]).

Proposition 3.19. *Let (M, τ) be a separable tracial von Neumann algebra. Then the set of subalgebras of M with property (T) is separable with respect to d .*

Proof. Suppose for a contradiction that the set of subalgebras of M with property (T) is non-separable. Then there is $\varepsilon > 0$ such that M has property (T) subalgebras $(N_\alpha)_{\alpha \in I}$ such that I is uncountable and for each $\alpha \neq \beta \in I$, $d(N_\alpha, N_\beta) > \varepsilon$.

By property (T) of $N_\alpha \subset M$ for each $\alpha \in I$ and Lemma 2.4, there are finite sets $F_\alpha \subset N_\alpha$ and $\delta_\alpha > 0$ such that if $\phi : M \rightarrow M$ is a unital, tracial, completely positive map such that $\max_{x \in F_\alpha} \|\phi(x) - x\|_2 < \delta_\alpha$, then $\|\phi(y) - y\|_2 < \varepsilon$ for all $y \in (N_\alpha)_1$.

Since I is uncountable, there are $m, n \in \mathbb{N}$ such that $I_0 = \{\alpha \in I : \delta_\alpha \geq \frac{1}{n} \text{ and } |F_\alpha| = m\}$ is uncountable. Since M is separable, so is $L^2(M)^{\oplus m}$. Therefore, there exist $\alpha \neq \beta \in I_0$ such that, writing $F_\alpha = (x_1, \dots, x_m)$ and $F_\beta = (y_1, \dots, y_m)$, we have $\|x_i - y_i\|_2 < \frac{1}{2n}$ for each $i = 1, \dots, m$.

The map $E_{N_\alpha} : M \rightarrow M$ is unital, tracial, and completely positive and

$$\|E_{N_\alpha}(y_i) - y_i\|_2 \leq \|E_{N_\alpha}(y_i - x_i)\|_2 + \|y_i - x_i\|_2 < \frac{1}{n},$$

so by the relative rigidity of $N_\beta \subset N$, we get that $\|E_{N_\alpha}(x) - x\|_2 \leq \varepsilon$ for all $x \in (N_\beta)_1$. Similarly, we have that $\|E_{N_\beta}(x) - x\|_2 \leq \varepsilon$ for all $x \in (N_\alpha)_1$. Hence, $d(N_\alpha, N_\beta) \leq \varepsilon$, a contradiction. ■

The following is immediate from [45, Theorem 5.2] and the inequality $d_2(A, B) \leq \sqrt{2d(A, B)}$ (see also [50, Corollary 2.2]).

Proposition 3.20. *If $A, B \subset (M, \tau)$ are von Neumann subalgebras of a II_1 factor M and $d = d(A, B) < \frac{1}{9522}$, then A and B are stably isomorphic. More specifically, there are projections $p \in A$ and $q \in B$ such that $pAp \cong qBq$ and $\tau(p), \tau(q) \geq 1 - 2450d$.*

Compared to the statement of [45, Theorem 5.2], we note that we can drop the projections in A' and B' for the following reason: if $p' \in A'$, then $p'A$ is an SOT-closed two-sided ideal in A and is therefore isomorphic to zA for some $z \in Z(A)$. Similarly, for $p'pAp$ for any projection $p \in A$.

Using HNN extensions allows us to conjugate not just isomorphic subalgebras, but “almost” conjugate “almost” isomorphic subalgebras. Recall that if (M, τ_M) is a tracial von Neumann algebra and $p \in M$ is a projection, then pMp inherits the canonical trace $\tau_{pMp}(p xp) = \frac{\tau_M(p xp)}{\tau_M(p)}$.

Lemma 3.21. *If $A, B \subset (M, \tau_M)$ are von Neumann subalgebras of M and $p \in A, q \in B$ are projections such that $pAp \cong qBq$ via a trace-preserving isomorphism, then there is a tracial von Neumann algebra (N, τ_N) such that $M \subset N$, $\tau_N|_M = \tau_M$, and there is a unitary $u \in N$ such that $d(uAu^*, B) \leq 5\sqrt{1 - \min\{\tau(p), \tau(q)\}}$.*

Proof. Without loss of generality assume that $\tau(p) \geq \tau(q)$. Set $M_1 = M \bar{\otimes} L\mathbb{Z}$. Take a projection $p' \in L\mathbb{Z}$ such that $\tau(p') = \frac{\tau(q)}{\tau(p)}$. Then $\tilde{A} = p'pAp \oplus (1 - p')p\mathbb{C}$ and $\tilde{B} = qBq \oplus (1 - q)\mathbb{C}$ are isomorphic (via a trace-preserving isomorphism) subalgebras of M_1 . Let θ denote a trace-preserving embedding from $\tilde{A}$ into M_1 with image $\tilde{B}$. Now take $N = \text{HNN}(M_1, \tilde{A}, \theta)$. There is a unitary $u \in N$ such that $uxu^* = \theta(x)$ for all $x \in \tilde{A}$, and in particular $u\tilde{A}u^* = \tilde{B}$.

Since $d(A, \tilde{A}) = d(uAu^*, u\tilde{A}u^*) = d(uAu^*, \tilde{B})$, we have that $d(uAu^*, B) \leq d(A, \tilde{A}) + d(\tilde{B}, B)$. But for $x \in (A)_1$, $\|x - p'p xp\|_2 \leq \|1 - p'\|_2 + 2\|1 - p\|_2 \leq 3\sqrt{1 - \tau(p)}$. Thus, $d(A, \tilde{A}) \leq 3\sqrt{1 - \tau(p)}$. Similarly, $d(B, \tilde{B}) \leq 2\sqrt{1 - \tau(q)}$. ■

The following lemma is likely well known to experts, but we include a proof for completeness.

Lemma 3.22. *If (N, τ) is a tracial von Neumann algebra with projections p, q such that $\tau(p), \tau(q) > 1 - \delta$, then there are projections $p' \leq p$ and $q' \leq q$ such that $\tau(p') = \tau(q') \geq 1 - 2\delta$ and $p'Np' \cong q'Nq'$.*

Proof. By the comparison theorem for projections, there is a central projection $z \in Z(N)$ and partial isometries $v, w \in N$ such that $vv^* = pz$, $v^*v \leq qz$, $ww^* \leq p(1 - z)$, and $w^*w = q(1 - z)$. Take $p' = (v + w)(v + w)^*$ and $q' = (v + w)^*(v + w)$. ■

The following is due to Christensen; it can also be found in [37, Appendix].

Theorem 3.23 ([13, Theorem 4.6]). *Let $A, B \subset (M, \tau)$ with M type II_1 , A diffuse and B a subfactor. Suppose $\|E_B(x) - x\|_2 \leq \delta$ for all $x \in (A)_1$. If $0 < \delta < 10^{-6}$, then there are projections $e \in A$ and $f \in B$ and a unital homomorphism $\Phi : eAe \rightarrow fBf$ such that $\|1 - e\|_2 < 2\sqrt{\delta}$ and $\|\Phi(exe) - x\|_2 < 80\sqrt{\delta}$ for all $x \in (A)_1$.*

Definition 3.24. Let (M, τ) be a tracial von Neumann algebra. We say that two von Neumann subalgebras $A, B \subset N$ are *uniformly approximately unitarily equivalent* (u.a.u.e.) if for all $\varepsilon > 0$ there exists a unitary $u \in M$ such that $d(uAu^*, B) < \varepsilon$.

Remark 3.25. If $N_1, N_2 \subset N$ are u.a.u.e. type II_1 subfactors, it does not follow that $N_1 \cong N_2$; however, in light of Proposition 3.20 it does imply that there is a sequence $t_n \rightarrow 1$ such that $N_1^{t_n} \cong N_2$. Moreover, these stable isomorphisms will be implemented by partial isometries in N , and N_1, N_2 will be mutually s-intertwining via surjective *-homomorphisms. (More precisely, for all projections $p_i \in N'_i \cap M$, we have $p_i N_i \prec_M N_{3-i}$ for $i = 1, 2$, and the homomorphisms given by Theorem 2.5 can be taken to be surjective.)

We recall that there are separable II_1 factors with prescribed fundamental group; the first such examples were given in [41], with additional examples provided later on in [25, 28, 46]. If the fundamental groups $\mathcal{F}(N_1)$ and $\mathcal{F}(N_2)$ are discrete subgroups of $\mathbb{R}_+$, then N_1 and N_2 being u.a.u.e will imply $N_1 \cong N_2$. On the other hand, let (N, τ_N) be a II_1 factor with fundamental group $\mathbb{Q}^+$. Let M be an isomorphic copy of qNq where $\tau_N(q) = 2^{-1/2}$. Take positive numbers $r_n \in \mathbb{Q}\sqrt{2}$ which are increasing to 1. Then there are projections $p_n \in N$ such that $\tau_N(p_n) = r_n$. Since the fundamental group of N is $\mathbb{Q}^+$, $p_n N p_n \cong M$ for all n . By Lemma 3.21, we can inductively extend $N \bar{\otimes} M$ to $\tilde{N}$ to include unitaries u_n such that $d(u_n(p_n N p_n)u_n^*, M) \rightarrow 0$. But $d(p_n N p_n, N) \rightarrow 0$ too so that $d(u_n N u_n^*, M) \rightarrow 0$. So we have constructed an algebra $\tilde{N}$ in which N and M are u.a.u.e. even though they are not isomorphic.

We introduce one more definition before proving more general versions of Theorem C. We call a family $(M_\lambda)_{\lambda \in \Lambda}$ of tracial von Neumann algebras an *embedding universal inductive class* of separable tracial von Neumann algebras if (1) it is closed under taking inductive limits and (2) for every separable tracial von Neumann algebra N , there is an index $\lambda \in \Lambda$ such that N embeds into M_λ .

Theorem 3.26. *Let (M, τ) be a separable tracial von Neumann algebra. Then there is a separable tracial von Neumann algebra N containing M such that any two isomorphic diffuse property (T) von Neumann subalgebras of N are u.a.u.e. We may furthermore take N to belong to any embedding universal inductive class of separable tracial von Neumann algebras; for example, we may take N to be a II_1 factor or to have property Gamma.*

Proof. We take M_1 to equal M . By Proposition 3.19, enumerate a d -dense set $(B_{m,1})_{m=1}^\infty$ of property (T) subalgebras of M_1 . Fix a bijection $\sigma = (\sigma_1, \sigma_2, \sigma_3) : \mathbb{N} \rightarrow \mathbb{N}^3$ such that $\sigma_3(n) \leq n$. Define $\sigma_{13}(n) = (\sigma_1(n), \sigma_3(n))$ and similarly for σ_{23} .

Suppose inductively that $M_1, \dots, M_n$ have been constructed, along with d -dense sets of rigid subalgebras $(B_{m,k}) \subset M_k$ for $m \in \mathbb{N}$ and $k = 1, \dots, n$. If there exist projections $p_i \in B_{\sigma_{i3}(n)}$ such that $p_1 B_{\sigma_{13}(n)} p_1 \cong p_2 B_{\sigma_{23}(n)} p_2$ and $\tau(p_i) > 1 - \frac{1}{n}$ for $i = 1, 2$, using Proposition 3.21 define M_{n+1} to be a II_1 factor containing M_n and a unitary u such that $d(u B_{\sigma_{13}(n)} u^*, B_{\sigma_{23}(n)}) \leq 5 \frac{1}{\sqrt{n}}$. Otherwise, take $M_{n+1} = M_n$.

Take N to be the inductive limit of the M_n . Fix $\varepsilon > 0$. Let $N_1, N_2 \subset N$ be two isomorphic diffuse property (T) subalgebras. By property (T) and Lemma 2.4, there exist $F_i \subset N_i$ finite subsets and $\delta > 0$ such that if $\phi : N \rightarrow N$ is unital, tracial, and completely positive and $\|\phi(x) - x\|_2 < \delta$ for all $x \in F_i$, then $\|\phi(y) - y\|_2 < \varepsilon$ for all $y \in (N_i)_1$. In particular, for all $n \in \mathbb{N}$ sufficiently large, $\|E_{M_n}(x) - x\|_2 < \delta$ for all $x \in F_i$ and so $\|E_{M_n}(y) - y\|_2 < \varepsilon$ for all $y \in (N_i)_1$, $i = 1, 2$. We may in particular assume that $\frac{1}{\sqrt{n}} < 630\varepsilon^{1/4}$.

By Theorem 3.23, for ε sufficiently small, say $\varepsilon < \frac{1}{2024!}$, and for $i = 1, 2$ there are projections $e_i \in N_i$ and $f_i \in M_n$ and $*$ -homomorphisms $\Phi_i : e_i N_i e_i \rightarrow f_i M_n f_i$ such that $\|e_i - 1\|_2 < 2\sqrt{\varepsilon}$ and $\|\Phi_i(e_i x e_i) - x\|_2 \leq 80\sqrt{\varepsilon}$ for all $x \in (N_i)_1$.

Define $\widetilde{N}_i = \Phi_i(e_i N_i e_i) \oplus (1 - f_i)\mathbb{C}$. Then $d(N_i, \widetilde{N}_i) \leq 80\sqrt{\varepsilon}$. By the d -density of the $B_{m,n}$ in the rigid subalgebras of M_n , we can find $n \leq k \in \mathbb{N}$ such that $\sigma_3(k) = n$ and for $i = 1, 2$, $d(\widetilde{N}_i, B_{\sigma_{i3}(k)}) \leq \sqrt{\varepsilon}$. Now Proposition 3.20 implies that there are projections $p_i \in N_i$ and $q_i \in B_{\sigma_{i3}(k)}$ such that $p_i N_i p_i \cong q_i B_{\sigma_{i3}(k)} q_i$ and $\tau(p_i), \tau(q_i) \geq 1 - 198450\sqrt{\varepsilon}$. By Lemma 3.22, there are projections $s_i \leq p_i$ in N_i such that $s_1 N_1 s_1$ is isomorphic to $s_2 N_2 s_2$ and $\tau(s_i) \geq 1 - 396900\sqrt{\varepsilon}$ for $i = 1, 2$. In turn, this implies that there are projections $r_i \leq q_i$ in $B_{\sigma_{i3}(k)}$ such that $r_i B_{\sigma_{i3}(k)} r_i$ are isomorphic to r_i for $i = 1, 2$ and $\tau(r_i) \geq 1 - 396900\sqrt{\varepsilon}$. By construction, there is a unitary $u \in M_{k+1} \subset N$ such that $d(u B_{\sigma_{13}(k)} u^*, B_{\sigma_{23}(k)}) \leq \frac{5}{\sqrt{k}} \leq \frac{5}{\sqrt{n}} < 3150\varepsilon^{1/4}$.

Applying the triangle inequality repeatedly, we get that $d(u N_1 u^*, N_2) \leq 3312\varepsilon^{1/4}$. Since ε was chosen arbitrarily, we see that N_1 and N_2 are u.a.u.e. ■

The above proof actually shows the following stronger statement, which moreover provides a case for why the conclusion is optimal (see also Remark 3.28).

Theorem 3.27. *Let (M, τ) be a separable tracial von Neumann algebra. Then there is a separable II_1 factor N containing M such that any two diffuse isomorphic property (T) von Neumann subalgebras N_1 and N_2 of N satisfy that there exists a sequence of projections $p_{n,i} \rightarrow 1$ (where $p_{n,i} \in \mathcal{P}(N_i)$) such that $p_{n,1} N_1 p_{n,1} \cong p_{n,2} N_2 p_{n,2}$, and are u.a.u.e.*

Remark 3.28. We note that if M is a II_1 factor with property (T), then $M \bar{\otimes} R$ has full fundamental group, and therefore $M^t \subset (M \bar{\otimes} R)^t \cong M \bar{\otimes} R$ for all $t \in \mathbb{R}_+$. In particular, $M \bar{\otimes} R$ contains continuum many non-isomorphic, stably isomorphic property (T) subfactors since the fundamental group of M is countable. We contrast this remark with the fact that there are only countably many irreducible subfactors with (T) (see [37]). This

suggests that it is not likely possible to be able to exactly conjugate all isomorphic property (T) subalgebras inside of a separable II_1 factor, since our inductive limit arguments would require passing to the non-separable setting.

Miscellaneous results. The following is well known and is a folklore fact.

Lemma 3.29. *Let (M, τ_M) be a tracial von Neumann algebra. Then M is a factor if and only if for every $x \in M$ and $\varepsilon > 0$ there exists $y \in (M)_1$ such that $\|x - \tau(x)\|_2 < \|[x, y]\|_2 + \varepsilon$.*

The proof of the following proposition was suggested to us by D. Jekel and B. Hayes. We thank them profusely for allowing us to include it here.

Proposition 3.30. *Let $\{x_1, \dots, x_n\} \subset (M)_1$ be a finite set of elements of operator norm at most 1 in a II_1 factor M . Then for any $\varepsilon > 0$, there exists $\{y_1, \dots, y_n\} \subset (M)_1$, elements of operator norm at most 1, such that $\{y_1, \dots, y_n\}$ generates a subfactor of M and furthermore $\|y_i - x_i\|_2 < \varepsilon$ for all i .*

Proof. Let $A = C^*(T_1, \dots, T_n)$ be the universal C^* -algebra generated by n contractions $T_1, \dots, T_n$. For each positive integer $N > 0$ and for each $*$ -polynomial in n variables with rational coefficients p , we consider the following subset of $(M)_1^n$:

$$G_{N,p} = \left\{ (z_1, \dots, z_n) \in (M)_1^n : \exists \text{ } * \text{-polynomial in } n \text{ variables } q \text{ such that } \|q\|_A \leq 1 \right. \\ \left. \text{and } \|p(z_1, \dots, z_n) - \tau(p(z_1, \dots, z_n))\|_2 \right. \\ \left. < \|[p(z_1, \dots, z_n), q(z_1, \dots, z_n)]\|_2 + \frac{1}{N} \right\}$$

We observe that the above set is open in $(M)_1^n$ in the 2-norm topology. Indeed, for each fixed $*$ -polynomial q in n variables with $\|q\|_A \leq 1$, the set

$$G_{N,p,q} = \left\{ (z_1, \dots, z_n) \in (M)_1^n : \|p(z_1, \dots, z_n) - \tau(p(z_1, \dots, z_n))\|_2 \right. \\ \left. < \|[p(z_1, \dots, z_n), q(z_1, \dots, z_n)]\|_2 + \frac{1}{N} \right\}$$

is clearly open, and $G_{N,p}$ is the union of $G_{N,p,q}$ ranging over all such q . We now claim that $G_{N,p}$ is 2-norm dense in $(M)_1^n$ for each N and p . Indeed, it suffices to show, for any $(v_1, \dots, v_n) \in M^n$ with the operator norms of all v_i strictly smaller than 1, and any $\varepsilon > 0$, there exists $(w_1, \dots, w_n) \in G_{N,p}$ with $\|w_i - v_i\|_2 \leq \varepsilon$ for all i . Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. As $v_1, \dots, v_n \in M \subset M^{\mathcal{U}}$, by [43] there exists a family of free semicirculars $s_1, \dots, s_n \in M^{\mathcal{U}}$ free from $v_1, \dots, v_n$. By [16], for ε sufficiently small, $Q = W^*(v_1 + \varepsilon s_1, \dots, v_n + \varepsilon s_n)$ is a factor. Furthermore, again by taking ε sufficiently small, we may assume $\|v_i\| + \varepsilon \leq 1$. Write $z_i = v_i + \varepsilon s_i$.

Since Q is a factor, by the Kaplansky density theorem, [4, Proposition II.5.1.5], and Lemma 3.29, there exists a $*$ -polynomial q in n variables such that

$$\|p(z_1, \dots, z_n) - \tau(p(z_1, \dots, z_n))\|_2 < \| [p(z_1, \dots, z_n), q(z_1, \dots, z_n)] \|_2 + \frac{1}{N},$$

and furthermore $\|q(z_1, \dots, z_n)\|_\infty \leq \|q\|_A \leq 1$ (see also the argument in the proofs of Lemma 2.2 in [31] and Lemma 2.3 in [23]). Now, lift each s_i to a sequence of elements $(s_{ij})_{j \rightarrow \mathcal{U}}$ in M such that $\|s_{ij}\|_\infty \leq 1$ for all i and j . Write $z_{ij} = v_i + \varepsilon s_{ij}$. Then $\|z_{ij}\|_\infty \leq \|v_i\|_\infty + \varepsilon \leq 1$ and $\|z_{ij} - v_i\|_2 \leq \varepsilon$. Furthermore, $(z_{ij})_{j \rightarrow \mathcal{U}}$ is a lift of z_i . Thus,

$$\begin{aligned} & \lim_{j \rightarrow \mathcal{U}} (\|p(z_{1j}, \dots, z_{nj}) - \tau(p(z_{1j}, \dots, z_{nj}))\|_2 - \| [p(z_{1j}, \dots, z_{nj}), q(z_{1j}, \dots, z_{nj})] \|_2) \\ &= \|p(z_1, \dots, z_n) - \tau(p(z_1, \dots, z_n))\|_2 - \| [p(z_1, \dots, z_n), q(z_1, \dots, z_n)] \|_2 < \frac{1}{N}. \end{aligned}$$

So there exists $S \in \mathcal{U}$ such that whenever $j \in S$, we have

$$\|p(z_{1j}, \dots, z_{nj}) - \tau(p(z_{1j}, \dots, z_{nj}))\|_2 < \| [p(z_{1j}, \dots, z_{nj}), q(z_{1j}, \dots, z_{nj})] \|_2 + \frac{1}{N}.$$

Setting $w_i = z_{ij}$ for any $j \in S$ proves the claim.

Since we are restricting to $*$ -polynomials with rational coefficients p , there are only countably many such p . Hence, by Baire category theorem, the set

$$G = \bigcap_{\substack{N \in \mathbb{N}_+ \\ p \text{ } * \text{-polynomial with rational coefficients}}} G_{N,p}$$

is dense in $(M)_1^n$. In particular, there exists $(y_1, \dots, y_n) \in G$ such that $\|x_i - y_i\|_2 < \varepsilon$ for all i . We claim that $P = W^*(y_1, \dots, y_n)$ is a factor. Indeed, for any $x \in P$ and $\varepsilon > 0$, there exists a $*$ -polynomial p in n variables with rational coefficients such that $\|x - p(y_1, \dots, y_n)\|_2 < \frac{\varepsilon}{5}$. Let $N > 0$ be a positive integer such that $\frac{1}{N} \leq \frac{\varepsilon}{5}$. Then as $(y_1, \dots, y_n) \in G \subset G_{N,p}$, there exists a $*$ -polynomial q in n variables with $\|q\|_A \leq 1$ such that

$$\begin{aligned} \|x - \tau(x)\|_2 &\leq \|p(y_1, \dots, y_n) - \tau(p(y_1, \dots, y_n))\|_2 + \frac{2\varepsilon}{5} \\ &< \| [p(y_1, \dots, y_n), q(y_1, \dots, y_n)] \|_2 + \frac{1}{N} + \frac{2\varepsilon}{5} \\ &\leq \| [x, q(y_1, \dots, y_n)] \|_2 + \frac{2\|q(y_1, \dots, y_n)\|_\infty \varepsilon}{5} + \frac{3\varepsilon}{5}. \end{aligned}$$

Since $\|y_i\|_\infty \leq 1$ for all i , as $\|q\|_A \leq 1$, we have $\|q(y_1, \dots, y_n)\|_\infty \leq 1$, so the above yields that, for any $x \in P$ and $\varepsilon > 0$, there exists $y \in (M)_1$, namely $y = q(y_1, \dots, y_n)$, such that $\|x - \tau(x)\|_2 < \| [x, q(y_1, \dots, y_n)] \|_2 + \varepsilon$. Thus, P is a factor by Lemma 3.29. ■

Definition 3.31. Let (N, τ_N) , (M, τ_M) be separable II_1 factors, $\{x_i\}_{i \in I} \subset N$ be a countable set of elements of operator norm at most 1 that generates N , and $\{y_j\}_{j=1}^n \subset M$ be

a finite set of elements of operator norm at most 1 that generates M . Let $I_0 \subset I$ be a finite subset, $\varepsilon > 0$, and $m > 0$ be a positive integer. Then we say $(M, \tau_M, \{y_j\})$ is in the (I_0, ε, m) -neighborhood of $(N, \tau_N, \{x_i\})$, denoted by $(M, \{y_j\}) \in \Theta(N, I_0, \varepsilon, m)$, if there exists an injective map $\sigma : \{1, \dots, n\} \rightarrow I$ whose range contains I_0 such that

$$|\tau_M(p(y_1, \dots, y_n)) - \tau_N(p(x_{\sigma(1)}, \dots, x_{\sigma(n)}))| < \varepsilon$$

for all $*$ -monomials p in n variables of degree less than or equal to m . If we need to specify the correspondence between generators, that is, the injective map σ , we shall write $(M, \{y_j\}, \sigma) \in \Theta(N, I_0, \varepsilon, m)$.

If, for (M, τ_M) , there exists $\{y_j\}_{j=1}^n \subset M$, a finite set of elements of operator norm at most 1 that generates M such that $(M, \{y_j\}) \in \Theta(N, I_0, \varepsilon, n)$, we then say (M, τ_M) is in the (I_0, ε, m) -neighborhood of $(N, \tau_N, \{x_i\})$, denoted by $M \in \Theta(N, I_0, \varepsilon, n)$.

If I is finite and $I_0 = I$, we simply write $\Theta(N, \varepsilon, m)$ in place of $\Theta(N, I_0, \varepsilon, m)$.

Proposition 3.32. *Let N, M be II_1 factors, $\{x_i\}_{i \in I} \subset N$ be a countable set of elements of operator norm at most 1 that generates N , $\mathcal{U}$ be a countably incomplete ultrafilter on an index set J , then the following are equivalent:*

- (1) *For any countably incomplete ultrafilter $\mathcal{V}$ on any index set, for any embedding $\pi : N \rightarrow M^{\mathcal{V}}$, the relative commutant $\pi(N)' \cap M^{\mathcal{V}}$ contains a trace-zero unitary.*
- (2) *For any embedding $\pi : N \rightarrow M^{\mathcal{U}}$, the relative commutant $\pi(N)' \cap M^{\mathcal{U}}$ contains a trace-zero unitary.*
- (3) *For any $\varepsilon > 0$ and finite subset $I_0 \subset I$, there exists a finite subset $I' \subset I$ containing I_0 , $\delta > 0$, and a positive integer $n > 0$ such that whenever $(N_0, \{y_j\}, \sigma) \in \Theta(N, I', \delta, n)$ and $\pi : N_0 \rightarrow M$ is an embedding, then there exists a unitary $w \in M$ of trace zero such that $\|y_{\sigma^{-1}(i)}, w\|_2 \leq \varepsilon$ for all $i \in I_0$.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3) Assume to the contrary; that is, assume there exists $\varepsilon > 0$ and a finite subset $I_0 \subset I$ such that for any finite subset $I' \subset I$ containing I_0 , $\delta > 0$, and positive integer $n > 0$, there exists $(N_0, \{y_j\}, \sigma) \in \Theta(N, I_0, \delta, n)$ and $\pi : N_0 \rightarrow M$ an embedding such that for any unitary $w \in M$ of trace zero, we have $\|[\pi(y_{\sigma^{-1}(i)}), w]\|_2 > \varepsilon$ for some $i \in I_0$.

Thus, fix an increasing sequence of finite subsets $(I_k)_k \subset I$ containing I_0 and whose union is I , and a decreasing sequence $\delta_k \rightarrow 0$. Then for each k , there exists $(N_k, \{y_j\}_{j=1}^{m_k}, \sigma_k) \in \Theta(N, I_k, \delta_k, k)$ and $\pi_k : N_k \rightarrow M$ a trace-preserving embedding such that for any unitary $w \in M$ of trace zero, we have $\|[\pi_k(y_{\sigma_k^{-1}(i)}), w]\|_2 > \varepsilon$ for some $i \in I_0$. As $\mathcal{U}$ is countably incomplete, we may choose a decreasing sequence of sets $(J_k)_k \in \mathcal{U}$ whose intersection is empty. We then define, for each $j \in J$, a map $\phi_j : \{x_i\}_{i \in I} \rightarrow M$

$$\phi_j(x_i) = \begin{cases} \pi_k(y_{\sigma_k^{-1}(i)}), & \text{if } j \in J_k \setminus J_{k+1} \text{ and } i \in I_k, \\ 0, & \text{otherwise.} \end{cases}$$

Then the map $\phi : \{x_i\}_{i \in I} \rightarrow M^{\mathcal{U}}$ given by $\phi(x_i) = (\phi_j(x_i))_{j \rightarrow \mathcal{U}}$ is easily seen to preserve the law of $\{x_i\}_{i \in I} \subset N$ under τ_N . Since $\{x_i\}$ generates N , ϕ extends to an embedding $N \rightarrow M^{\mathcal{U}}$, which we shall still denote by ϕ .

By our assumptions, we may pick $w \in \phi(N)' \cap M^{\mathcal{U}}$ a trace-zero unitary. Lift it to $w = (w_j)_{j \rightarrow \mathcal{U}}$ where $w_j \in M$ are all trace-zero unitaries, so $\|[\phi_j(x_i), w_j]\|_2 \rightarrow 0$ as $j \rightarrow \mathcal{U}$ for any fixed $i \in I$. In particular, as $I_0 \subset I$ is finite, there exists $A \in \mathcal{U}$ such that $\|[\phi_j(x_i), w_j]\|_2 \leq \varepsilon$ whenever $i \in I_0$ and $j \in A$. Since $A \cap J_1 \in \mathcal{U}$, $A \cap J_1 \neq \emptyset$. Since $J_1 = \bigcup_{k=1}^{\infty} (J_k \setminus J_{k+1})$, $A \cap (J_k \setminus J_{k+1}) \neq \emptyset$ for some k . Fix $j \in A \cap (J_k \setminus J_{k+1})$. Then for any $i \in I_0 \subset I_k$, we have

$$\|[\pi_k(y_{\sigma_k^{-1}(i)}), w_j]\|_2 = \|[\phi_j(x_i), w_j]\|_2 \leq \varepsilon.$$

But by assumptions on N_k and π_k , we have $\|[\pi_k(y_{\sigma_k^{-1}(i)}), w_j]\|_2 > \varepsilon$ for some $i \in I_0$, a contradiction.

(3) $\Rightarrow$ (1) Let $\pi : N \rightarrow M^{\mathcal{V}}$ be an embedding. We may write $\pi(x_i) = (x_{ji})_{j \rightarrow \mathcal{V}}$ where x_{ji} are elements of M of operator norm at most 1. Fix an increasing sequence of finite subsets $I_k \subset I$ whose union is I . Then by assumptions there exist finite subsets $I'_k \subset I$ containing I_k , $\delta_k > 0$, and a positive integer $n_k > 0$ such that whenever $(N_0, \{y_j\}, \sigma) \in \Theta(N, I'_k, \delta_k, n_k)$ and $\pi : N_0 \rightarrow M$ is a trace-preserving embedding, then there exists a unitary $w \in M$ of trace zero such that $\|[\pi(y_{\sigma^{-1}(i)}), w]\|_2 \leq \frac{1}{k}$ for all $i \in I_k$.

Note that since the laws for $\{x_{ji}\}_{i \in I} \subset M$ converge to the law of $\{x_i\}_{i \in I}$ as $j \rightarrow \mathcal{V}$, for each k there exists $A_k \in \mathcal{V}$ such that, whenever $j \in A_k$, if we write $I'_k = \{i_{k,1}, \dots, i_{k,|I'_k|}\}$, then

$$|\tau_M(p(x_{ji_{k,1}}, \dots, x_{ji_{k,|I'_k|}})) - \tau_N(p(x_{i_{k,1}}, \dots, x_{i_{k,|I'_k|}}))| < \frac{\delta_k}{2}$$

for all $*$ -monomials p in $|I'_k|$ variables of degree less than or equal to n_k . By Proposition 3.30, for each $j \in A_k$, there exist $\{y_{jki}\}_{i \in I'_k} \subset M$ of operator norm at most 1 such that $\{y_{jki}\}_{i \in I'_k}$ generates a subfactor of M , and furthermore,

$$\|y_{jki} - x_{ji}\|_2 < \min\left\{\frac{\delta_k}{2n_k}, \frac{1}{k}\right\}$$

for all $i \in I'_k$. But then it is clear that

$$|\tau_M(p(y_{jki_{k,1}}, \dots, y_{jki_{k,|I'_k|}})) - \tau_N(p(x_{i_{k,1}}, \dots, x_{i_{k,|I'_k|}}))| < \delta_k$$

for all $*$ -monomials p in $|I'_k|$ variables of degree less than or equal to n_k . That is, if $N_{jk} = W^*(\{y_{jki}\}_{i \in I'_k}) \subset M$, then N_{jk} is a separable II_1 factor and $(N_{jk}, \{y_{jki}\}_{i \in I'_k}, \sigma_k) \in \Theta(N, I'_k, \delta_k, n_k)$ where σ_k is simply the inclusion map $I'_k \hookrightarrow I$. By assumption, there exists a trace-zero unitary $w_{jk} \in M$ such that $\|[y_{jki}, w_{jk}]\|_2 \leq \frac{1}{k}$ for all $i \in I_k$. But then, as $\|y_{jki} - x_{ji}\|_2 < \frac{1}{k}$ for all $i \in I'_k \supset I_k$, we have $\|[x_{ji}, w_{jk}]\|_2 \leq \frac{3}{k}$ for all $i \in I_k$.

As $\mathcal{V}$ is countably incomplete, we may choose a decreasing sequence of sets $(J_k)_k \in \mathcal{V}$ whose intersection is empty. Let $B_k = J_k \cap (\bigcap_{l=1}^k A_k)$. Then B_k is a sequence of sets

in $\mathcal{V}$ that decreases to $\emptyset$. We define, for each j ,

$$w_j = \begin{cases} w_{jk}, & \text{if } j \in B_k \setminus B_{k+1}, \\ 0, & \text{otherwise.} \end{cases}$$

One easily checks that $w = (w_j)_{j \rightarrow \mathcal{U}}$ is a trace-zero unitary that commutes with $\pi(x_i) = (x_{ji})_{j \rightarrow \mathcal{V}}$ for any $i \in I$; that is, $\pi(N)' \cap M^{\mathcal{V}}$ contains a trace-zero unitary. ■

We shall denote the equivalent conditions contained in the above proposition $\Gamma_N(M)$; that is, we shall write $\Gamma_N(M)$ when, for any countably incomplete ultrafilter $\mathcal{V}$ on any index set and for any embedding $\pi : N \rightarrow M^{\mathcal{V}}$, the relative commutant $\pi(N)' \cap M^{\mathcal{V}}$ contains a trace-zero unitary. By condition (2) in the proposition above, we see that $\Gamma_N(P)$ is preserved by changing P to Q which is elementarily equivalent to P when P and Q both have density characters at most continuum; that is, this leads to the following corollary.

Corollary 3.33. *If P and Q are elementarily equivalent II_1 factors which have density characters at most continuum and N is any fixed separable II_1 factor, then $\Gamma_N(P)$ if and only if $\Gamma_N(Q)$.*

One particular observation we make here is that, if P is non-Gamma with density character at most continuum, and $Q \equiv P$ with Q separable, then Q is non-Gamma as property Gamma is preserved by elementary equivalence. Then, because $P^{\mathcal{U}} \cong Q^{\mathcal{U}}$ for some countably incomplete $\mathcal{U}$ and the diagonal embedding $\Delta_Q : Q \rightarrow Q^{\mathcal{U}} \cong P^{\mathcal{U}}$ has trivial relative commutant, we have the following corollary.

Corollary 3.34. *If P and Q are elementarily equivalent non-Gamma II_1 factors with Q separable and P having density character at most continuum, then $\Gamma_Q(P)$ fails.*

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