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C^*-exponential rank of unital simple classifiable C^*-algebras

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Abstract. Let $\mathfrak{A}$ be the class of all unital simple separable amenable $\mathcal{Z}$-stable C^*-algebras satisfying UCT, which are classified by Elliott invariant. In this paper, we proved that any C^*-algebra in $\mathfrak{A}$ has C^*-exponential rank at most $1+\varepsilon$.

1. Introduction

Let A be a unital C^*-algebra and $U(A)$ the unitary group of A. We denote by $U_0(A)$ the connected component (in norm topology) of $U(A)$ containing the identity $\mathbf{1}_A$. It is well known that an element u in A belongs to $U_0(A)$ if and only if

$$u = \exp(2\pi i h_1)\exp(2\pi i h_2)\cdots\exp(2\pi i h_n)$$

for some natural number n and some self-adjoint elements $h_1, h_2, \ldots, h_n$ in A.

Based on the conclusion above, the conception of C^*-exponential rank, as defined by Phillips and Ringrose in 1991 (see [30] or Definition 2.24), can be considered. In the past around three decades, the C^*-exponential rank has been studied extensively. For instance, in [23], Lin proved that every C^*-algebra with real rank zero has C^*-exponential rank at most $1+\varepsilon$, Phillips in 1994 showed that the C^*-exponential rank of a C^*-algebra A can be arbitrarily large, that is, that for any n there are a unital C^*-algebra A and a unitary $u \in A$ which is a product of some number of exponentials but not of n or fewer (see [28]). Moreover, if A is a unital separable simple $\mathcal{Z}$-stable C^*-algebra which has rational tracial rank at most one, then A has the C^*-exponential rank at most $1+\varepsilon$ (see [22]). Further, the C^*-exponential rank plays an important role in the classification of C^*-algebras (see [8, 11–13, 20, 21] and so on).

In [16], a classification theorem is obtained for a class of unital simple separable amenable $\mathcal{Z}$-stable C^*-algebras satisfying UCT. This class of C^*-algebras exhausts all possible values of the Elliott invariant for unital stably finite simple separable amenable $\mathcal{Z}$-stable C^*-algebras. In this paper, we will prove that the C^*-exponential rank of a C^*-algebra in this class is at most $1+\varepsilon$.

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According to [15, Theorem 13.50], for any weakly unperforated Elliott invariant (see Definition 2.18 below) $((G, G_+, u), K, \triangle, r)$, there is a unital simple C^*-algebra $A = \varinjlim(A_n, \xi_{n,n+1})$ in the class $\mathcal{N}_0^{\mathcal{Z}}$ (see [15, Definition 13.6], relevant symbols are shown in [15, Definitions 2.9, 3.1 and 9.1]) such that

$$\mathrm{Ell}(A) \cong ((G, G_+, u), K, \triangle, r).$$

The above process of constructing a C^*-algebra A from a given weakly unperforated Elliott invariant is called *Elliott–GLN construction* in this paper. Our method of proof is done by modifying Elliott–GLN construction. Furthermore, combining the classification theorem of Elliott–Gong–Lin–Niu (see [9]) with Theorem 6.7 in this paper, we can obtain the conclusion that a unital simple separable amenable $\mathcal{Z}$-stable C^*-algebra satisfying UCT naturally has the C^*-exponential rank at most $1 + \varepsilon$. Note that a purely infinite simple C^*-algebra, automatically, has the C^*-exponential rank at most $1 + \varepsilon$ (see [27]).

2. Notation and preliminaries

This section includes a list of notations and definitions most of which are standard. We list them here for the reader's convenience.

Notation 2.1. Let q be a positive integer, we will use S_q to denote the collection of all permutations of $\{1, \ldots, q\}$.

Particularly, the permutation corresponding to the mapping $\{1, \ldots, q\} \ni j \mapsto (j+1) \bmod(q) \in \{1, \ldots, q\}$ will be denoted by $\varsigma \in S_q$.

Notation 2.2. Let n be a positive integer, we will use I_j^n to denote the arc

$$I_j^n := \left\{\exp(2\pi i t) : \frac{j-1}{n} \leqslant t < \frac{j}{n}\right\} \subset \mathbb{T}$$

for all $j = 1, \ldots, n$.

Definition 2.3. Let $\lambda_1, \ldots, \lambda_q$ be $q \geqslant 2$ elements in the unit circle $\mathbb{T}$. We say that $\{\lambda_j : j = 1, \ldots, q\}$ is *cyclically numbered* when there are real numbers $r_j \in \mathbb{R}$ such that $\exp(2\pi i r_j) = \lambda_j$ for all j, $r_1 \leqslant \cdots \leqslant r_q$ and $r_q - r_1 < 1$. When each r_j lies in $[0, 1)$, we say that $\{\lambda_j : j = 1, \ldots, q\}$ is *naturally numbered*.

Notation 2.4. Let L and K be two positive integers. We will use the symbol $\gamma = (c_{k,l}) : \mathbb{Z}^L \to \mathbb{Z}^K$ (or just $\gamma = (c_{k,l})$, If there is no ambiguity based on the context) to denote γ is a group homomorphism from $\mathbb{Z}^L$ to $\mathbb{Z}^K$, which is given by a $K \times L$ matrix $(c_{k,l})$ of integers, where $k = 1, \ldots, K$, $l = 1, \ldots, L$.

Notation 2.5. Let us denote by M_q the C^*-algebra formed by the $q \times q$ complex matrix, and write I_q for the unit element of M_q.

Notation 2.6. Let A be a unital C^*-algebra. We will write the spectrum of a as $\mathrm{sp}(a)$ for any $a \in A$.

Notation 2.7. Let A be a unital C^*-algebra. Denote by $A_{s.a.}$ the self-adjoint part of A.

Definition 2.8 ([31]). A pair (G, G_+) is called an *ordered Abelian group* if G is an Abelian group, G_+ is a subset of G with $G_+ + G_+ \subset G_+$, $G_+ \cap (-G_+) = \{0\}$ and $G_+ - G_+ = G$.

Define a relation $\leqslant$ on G by $x \leqslant y$ if $y - x$ belongs to G_+.

Particularly, the pair $(\mathbb{Z}, \mathbb{Z}_+)$, where $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$ is an ordered Abelian group.

Definition 2.9 ([31]). An element u in G_+ in an ordered Abelian group (G, G_+) is called an *order unit* if for any g in G there is a positive integer n with $-nu \leqslant g \leqslant nu$.

Definition 2.10 ([31]). A triple (G, G_+, u), where (G, G_+) is an ordered Abelian group and u is an order unit is called an *ordered Abelian group with an (distinguished) order unit*.

Particularly, the triple $(\mathbb{R}, \mathbb{R}_+, 1)$, where $\mathbb{R}_+ := [0, +\infty)$ is an ordered Abelian group with an order unit, and $(K_0(A), K_0(A)_+, [\mathbf{1}_A])$ is also, if A is a unital stably finite C^*-algebra.

Definition 2.11 ([31]). Let (G, G_+) and (H, H_+) be two ordered Abelian groups. A group homomorphism $\alpha : G \to H$ is said to be *positive* if $\alpha(G_+)$ is contained in H_+. Write $\alpha : (G, G_+) \to (H, H_+)$.

Assume now that (G, G_+, u) and (H, H_+, v) are two ordered Abelian groups with an order unit. A positive group homomorphism $\alpha : G \to H$ is said to be *order unit preserving* if $\alpha(u) = v$. Write $\alpha : (G, G_+, u) \to (H, H_+, v)$.

Notation 2.12 ([31]). Suppose that (G, G_+, u) is an ordered Abelian group with an order unit. A state on (G, G_+, u) is a group homomorphism $f : (G, G_+, u) \to (\mathbb{R}, \mathbb{R}_+, 1)$. The set of all states on (G, G_+, u) is denoted by $S(G)$.

Notation 2.13. Let A be a unital stably finite C^*-algebra with $T(A) \neq \emptyset$. Denote by $r_A : T(A) \to S(K_0(A))$ the map defined by $r_A(\tau)([p]) = \tau(p)$ for all projections $p \in \mathrm{M}_\infty(A)$ and for all $\tau \in T(A)$.

Notation 2.14. Let S be a compact convex set. Denote by $\mathrm{Aff}(S)$ the space of all real continuous affine functions on S, that is,

$$\begin{aligned}\mathrm{Aff}(S) := \big\{ f : S \to \mathbb{R} : f &\text{ is continuous and} \\ &f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda) f(y)\ \forall x, y \in S, \lambda \in [0,1] \big\}.\end{aligned}$$

In this paper, set

$$\mathrm{Aff}(S)_+ := \{ f \in \mathrm{Aff}(S) : f(s) > 0\ \forall s \in S \} \cup \{0\}.$$

It is easy to check that $(\mathrm{Aff}(S), \mathrm{Aff}(S)_+)$ is an ordered vector space with the strict (pointwise) order.

Notation 2.15. Let A be a unital C^*-algebra and let $T(A)$ denote the simplex of tracial states of A, a compact subset of A^*, the dual of A, with the weak* topology. Particularly, the tracial states space $T(A)$ is a compact convex set if A is a unital stably finite C^*-algebra with $T(A) \neq \emptyset$.

Notation 2.16. Let A and B be two C^*-algebras and $\varphi : A \to B$ be a $*$-homomorphism. Then, φ induces an affine map $T(\varphi) : T(B) \to T(A)$ defined by $T(\varphi)(\tau)(a) = \tau(\varphi(a))$ for all $\tau \in T(B)$ and $a \in A$. Denote by $\varphi^\sharp : \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(B))$ the contractive linear map defined by

$$\varphi^\sharp(f)(\tau) = f(T(\varphi)(\tau)) \quad \text{for all } f \in \mathrm{Aff}(T(A)) \text{ and } \tau \in T(B).$$

Definition 2.17. Let A be a unital C^*-algebra. The *Elliott invariant* of A, denoted by $\mathrm{Ell}(A)$ is the sextuple

$$\mathrm{Ell}(A) := ((K_0(A), K_0(A)_+, [\mathbf{1}_A]), K_1(A), T(A), r_A).$$

Definition 2.18. The sextuple $((G, G_+, u), K, \triangle, r)$ is called a *weakly unperforated Elliott invariant*, which means that (G, G_+, u) is a simple scaled ordered countable Abelian group with an order unit, K is a countable Abelian group, $\triangle$ is a metrizable Choquet simplex, and $r : \triangle \to S(G)$ is a surjective affine map such that for any $g \in G$

$$g \in G_+ - \{0\} \text{ if and only if } r(\tau)(g) > 0 \quad \forall \tau \in \triangle.$$

According to [15, Theorem 13.50], we know that for any weakly unperforated Elliott invariant $((G, G_+, u), K, \triangle, r)$, there is a unital simple C^*-algebra $A = \varinjlim(A_n, \xi_{n,n+1})$ in the class $\mathcal{N}_0^{\mathcal{Z}}$ (see [15, Definition 13.6]) such that

$$\mathrm{Ell}(A) \cong ((G, G_+, u), K, \triangle, r).$$

We write the symbol $\mathrm{Ell}(A) \cong ((G, G_+, u), K, \triangle, r)$, which means that there are an isomorphism κ_0 from $(K_0(A), K_0(A)_+, [\mathbf{1}_A])$ to (G, G_+, u), an isomorphism κ_1 from $K_1(A)$ to K and a continuous affine homeomorphism κ_T from $\triangle$ to $T(A)$ such that

$$r_A(\kappa_T(\tau))(x) = r(\tau)(\kappa_0(x))$$

for all $x \in K_0(A)$ and for all $\tau \in \triangle$.

Definition 2.19 ([11], [15, Notation 13.1]). Fix a set $\mathcal{X}$. We called a subset *with (finite) multiplicities* in $\mathcal{X}$ is a collection of elements in $\mathcal{X}$ which may be repeated finitely many times. Denote by $a^{\sim m}$ the subset with multiplicities $\{\underbrace{a, a, \ldots, a}_{m}\}$.

Let $A = \{a_1^{\sim m_1}, a_2^{\sim m_2}, \ldots, a_L^{\sim m_L}\}$ and $B = \{a_1^{\sim n_1}, a_2^{\sim n_2}, \ldots, a_L^{\sim n_L}\}$ (some of the m_l (or n_l) may be zero which means the element a_l does not appear in the set A (of B)). If

$m_l \leqslant n_l$ for all $l = 1, \ldots, L$, then we say that $A \subset B$. We define

$$\begin{aligned} A \cup B &:= \left\{a_1^{\sim \max\{m_1,n_1\}}, a_2^{\sim \max\{m_2,n_2\}}, \ldots, a_L^{\sim \max\{m_L,n_L\}}\right\}, \\ A \cap B &:= \left\{a_1^{\sim \min\{m_1,n_1\}}, a_2^{\sim \min\{m_2,n_2\}}, \ldots, a_L^{\sim \min\{m_L,n_L\}}\right\}, \\ A \backslash B &:= \left\{a_1^{\sim \max\{m_1-n_1,0\}}, a_2^{\sim \max\{m_2-n_2,0\}}, \ldots, a_L^{\sim \max\{m_L-n_L,0\}}\right\}, \\ A \sqcup B &:= \left\{a_1^{\sim (m_1+n_1)}, a_2^{\sim (m_2+n_2)}, \ldots, a_L^{\sim (m_L+n_L)}\right\}. \end{aligned}$$

When both A and B are ordinary sets, the symbols above yield the same results as the usual set operations of union, intersection, complement, while for the set of disjoint union operation is only good for the case $A \cap B = \emptyset$.

Notation 2.20. Let A be a finite set with multiplicities. We will use the symbol $\sharp(A)$ to denote the number of elements in A.

For example, if $A = \{a_1^{\sim m_1}, a_2^{\sim m_2}, \ldots, a_L^{\sim m_L}\}$, then $\sharp(A) = m_1 + \cdots + m_L$.

Notation 2.21. Let A be a unital sub-homogeneous C^*-algebra, that is, the maximal dimension of irreducible representations of A is finite. We will use $\mathrm{Sp}(A)$ to denote the set of equivalence classes of all irreducible representation of A and $\mathrm{RF}(A)$ to denote the set of equivalence classes of all (not necessarily irreducible) finite dimensional representations.

Particularly, the set $\mathrm{Sp}(A)$ will serve as a base set $\mathcal{X}$ when we talk about a finite set with multiplicities.

Notation 2.22. Let A and B be two C^*-algebras and let $\varphi : A \to B$ be a $*$-homomorphism. We will write $\mathrm{Sp}(\varphi)$ to denote the set

$$\mathrm{Sp}(\varphi) := \{x \in \mathrm{Sp}(A) : \ker \varphi \subset \ker x\}.$$

Moreover, when $B = \mathrm{M}_n$, let us use $\mathrm{SP}(\varphi)$ to denote the corresponding equivalence classes of φ in $\mathrm{RF}(A)$.

Therefore, if $\mathrm{SP}(\varphi) = \{x_1^{\sim m_1}, x_2^{\sim m_2}, \ldots, x_L^{\sim m_L}\}$, where $x_1, x_2, \ldots, x_L$ are all irreducible representations and $m_l > 0$ for all $l = 1, 2, \ldots, L$, then

$$\mathrm{Sp}(\varphi) = \{x_1, x_2, \ldots, x_L\} \subset \mathrm{Sp}(A).$$

So, $\mathrm{Sp}(\varphi)$ is an ordinary set which is a subset of $\mathrm{Sp}(A)$, while $\mathrm{SP}(\varphi)$ is a set with multiplicities, whose elements are also elements of $\mathrm{Sp}(A)$.

Note that if $\varphi, \psi : A \to \mathrm{M}_n$ are two $*$-homomorphisms, then φ and ψ are unitarily equivalent if and only if $\mathrm{SP}(\varphi) = \mathrm{SP}(\psi)$.

Notation 2.23. Let A and B be two C^*-algebras. If φ is a $*$-homomorphism from A to B, and $\pi \in \mathrm{RF}(B)$, then we will use $\varphi|_\pi$ for the composition $\pi \circ \varphi$.

In the last part of this section, we introduce the conception of C^*-exponential rank.

Definition 2.24 ([26,30]). Let A be a unital C^*-algebra and let u be an element of $U_0(A)$. We will define the C^*-exponential rank of u, written as $\mathrm{cer}(u)$, to be the *smallest* element of the set of symbols $\{1, 1+\varepsilon, 2, 2+\varepsilon, \ldots, \infty\}$ (with the obvious order) consistent with the following restrictions:

(i) $\operatorname{cer}(u) \leqslant n$ if u is the product $u = \prod_{j=1}^{n} \exp(2\pi i h_j)$ for some $h_1, \ldots, h_n \in A_{s.a.}$;

(ii) $\operatorname{cer}(u) \leqslant n + \varepsilon$ if u is a norm limit of products as in (i).

Moreover, define the C^*-exponential rank of A as follows:

$$\operatorname{cer}(A) := \sup\left\{\operatorname{cer}(u) : u \in U_0(A)\right\} \in \left\{1, 1+\varepsilon, 2, 2+\varepsilon, \ldots, \infty\right\}.$$

If A is non-unital, set $\operatorname{cer}(A) := \operatorname{cer}(A^{\sim})$.

3. Elliott–Thomsen algebras

In this section, we will introduce an important class of sub-homogeneous C^*-algebras, known as Elliott–Thomsen algebras. This class contains the familiar C^*-algebras, such as finite-dimensional C^*-algebras, $C([0,1])$, $C(\mathbb{T})$ and dimension drop C^*-algebras. This class of C^*-algebras is very important. Such as in [5], Elliott used it and some other building blocks with 2-dimensional spectra to realize any weakly unperforated simple ordered group with order unit as the K_0-group of a simple ASH-algebra. In addition, we introduce some simple properties for Elliott–Thomsen algebras, most of which have been proven or very easy.

Consistent with most literature, for convenience, we do not distinguish between the symbols $\mathrm{M}_q(C(X))$, $C(X, \mathrm{M}_q)$ and $C(X) \otimes \mathrm{M}_q$ for some compact Hausdorff space X.

Definition 3.1 ([5, 10]). Let F and E be two finite dimensional C^*-algebras and let $\beta_0, \beta_1 : F \to E$ be two unital $*$-homomorphisms. Set

$$\begin{aligned} A :&= C([0,1], E) \oplus_{\beta_0,\beta_1} F \\ &= \left\{(f; a) \in C([0,1], E) \oplus F : \beta_0(a) = f(0), \beta_1(a) = f(1)\right\}. \end{aligned}$$

Denote by $\mathcal{C}$ the class of all unital such C^*-algebras, and by $\mathcal{C}_0$ the sub-class of those C^*-algebras A in $\mathcal{C}$ such that $K_1(A) = \{0\}$.

Fact 3.2. Set $I := C_0((0,1), E)$. It is clearly that I is an (closed two-side) ideal in the Elliott–Thomsen algebra A and satisfies $A/I = F$. Consequently, the short exact sequence

$$0 \to I \hookrightarrow A \to F \to 0$$

can be obtained.

Let $F := \mathrm{M}_{[1]} \oplus \cdots \oplus \mathrm{M}_{[L]}$ and $E := \mathrm{M}_{\{1\}} \oplus \cdots \oplus \mathrm{M}_{\{K\}}$, where $[l]$ and $\{k\}$ are two positive integers depending on another two positive integers $1 \leqslant l \leqslant L$ and $1 \leqslant k \leqslant K$, respectively. Since A is of type I, (that is, any non-zero irreducible representation $\varphi : A \to B(\mathcal{H})$ satisfies $\mathcal{K}(\mathcal{H}) \subset \varphi(A)$ (see [24, p. 169])), the set $\operatorname{Sp}(A)$ (or the extreme points of

$T(A)$) has canonical one-to-one correspondence with the irreducible representations of A, which are given by

$$\mathrm{Sp}(A) := \bigsqcup_{k=1}^{K} (0,1)_k \cup \mathrm{Sp}(F),$$

where each $(0,1)_k$ is the open interval $(0,1)$ (We use the subscript k to indicate the k-th copy.) and $\mathrm{Sp}(F) = \{\theta_1, \ldots, \theta_L\}$. More precisely, the corresponding irreducible representation of an element $\theta_l \in \mathrm{Sp}(F) \subset \mathrm{Sp}(A)$ is the $*$-homomorphism

$$\begin{aligned}\Pi_{\theta_l} : C([0,1], E) \oplus_{\beta_0,\beta_1} F &\to \mathrm{M}_{[l]}\\ (f_1, \ldots, f_K; a_1, \ldots, a_L) &\mapsto a_l.\end{aligned}$$

If we use the symbol t_k to denote $t \in (0,1)_k \subset \mathrm{Sp}(A)$, then the corresponding irreducible representation of a given element $t_k \in \mathrm{Sp}(A)$ is the $*$-homomorphisms

$$\begin{aligned}\Pi_{t_k} : C([0,1], E) \oplus_{\beta_0,\beta_1} F &\to \mathrm{M}_{\{k\}}\\ (f_1, \ldots, f_K; a_1, \ldots, a_L) &\mapsto f_k(t).\end{aligned}$$

Suppose $K_0(\beta_0), K_0(\beta_1) : \mathbb{Z}^L \to \mathbb{Z}^K$ are given by matrices $(b^0_{k,l}), (b^1_{k,l}) \in \mathrm{M}_{K\times L}(\mathbb{Z}_+)$, respectively. Apparently,

$$\{k\} = \sum_{l=1}^{L} b^0_{k,l} \cdot [l] = \sum_{l=1}^{L} b^1_{k,l} \cdot [l], \quad k = 1, \ldots, K. \tag{3.1}$$

Further, we also have

$$\begin{aligned}\mathrm{Aff}(T(A)) = \Bigg\{ &(\mathfrak{R}_1, \ldots, \mathfrak{R}_K; r_1, \ldots, r_L) \in \bigoplus_{k=1}^{K} C([0,1], \mathbb{R}_k) \oplus \mathbb{R}^L :\\ &\mathfrak{R}_k(0) = \frac{1}{\{k\}} \sum_{l=1}^{L} b^0_{k,l}[l] \cdot r_l,\\ &\mathfrak{R}_k(1) = \frac{1}{\{k\}} \sum_{l=1}^{L} b^1_{k,l}[l] \cdot r_l,\ k = 1, \ldots, K \Bigg\},\end{aligned}$$

where $\mathbb{R}_k = \mathbb{R}$ for all $k = 1, \ldots, K$.

Proposition 3.3 ([15, Proposition 3.5] and [32, Lemma 2.1]). *Let $F := \mathrm{M}_{[1]} \oplus \cdots \oplus \mathrm{M}_{[L]}$, $E := \mathrm{M}_{\{1\}} \oplus \cdots \oplus \mathrm{M}_{\{K\}}$ and let C^*-algebra $A := C([0,1], E) \oplus_{\beta_0,\beta_1} F \in \mathcal{C}$. Then, $K_1(A) = \mathbb{Z}^K / \mathrm{Im}(K_0(\beta_1) - K_0(\beta_0))$ and $K_0(A) = \ker(K_0(\beta_1) - K_0(\beta_0))$ with positive cone $K_0(A) \cap \mathbb{Z}^L_+$ and scale*

$$\begin{pmatrix}[1]\\ \vdots \\ [L]\end{pmatrix} \in \mathbb{Z}^L.$$

Furthermore, if $K_1(A) = 0$, then $\mathbb{Z}^L / K_0(A) = K_1(C_0((0,1), E))$ is torsion free.

Remark 3.4. Let $U = (U_1, \ldots, U_K; \mathrm{I}_{[1]}, \ldots, \mathrm{I}_{[L]})$ be a unitary in $A = C([0,1], E) \oplus_{\beta_0,\beta_1} F$. Then,

$$[U] = [w_1, \ldots, w_K] \in K_1(A) = \mathbb{Z}^K / \operatorname{Im}(K_0(\beta_1) - K_0(\beta_0)),$$

where w_k is the winding number of $\det(U_k(t)) \in C([0,1], \mathbb{T})$ (note that $\det(U_k(0)) = \det(U_k(1)) = 1$) for any $k = 1, \ldots, K$.

Proposition 3.5 ([15, Proposition 3.6]). *For fixed finite dimensional C^*-algebras $F := \mathrm{M}_{[1]} \oplus \cdots \oplus \mathrm{M}_{[L]}$, $E := \mathrm{M}_{\{1\}} \oplus \cdots \oplus \mathrm{M}_{\{K\}}$, the C^*-algebras $C([0,1], E) \oplus_{\beta_0,\beta_1} F \in \mathcal{C}$ is completely determined (up to isomorphism) by the maps $K_0(\beta_0), K_0(\beta_1) : \mathbb{Z}^L \to \mathbb{Z}^K$.*

Remark 3.6. According to Proposition 3.5, without loss of generality, we can assume that

$$\begin{aligned} \beta_0 : \mathrm{M}_{[1]} \oplus \cdots \oplus \mathrm{M}_{[L]} &\to \mathrm{M}_{\{1\}} \oplus \cdots \oplus \mathrm{M}_{\{K\}} \\ (a_1, \ldots, a_L) &\mapsto (x_1, \ldots, x_K) \end{aligned}$$

and

$$\begin{aligned} \beta_1 : \mathrm{M}_{[1]} \oplus \cdots \oplus \mathrm{M}_{[L]} &\to \mathrm{M}_{\{1\}} \oplus \cdots \oplus \mathrm{M}_{\{K\}} \\ (a_1, \ldots, a_L) &\mapsto (y_1, \ldots, y_K), \end{aligned}$$

where

$$x_k = \operatorname{diag}(a_1 \otimes \mathrm{I}_{b^0_{k,1}}, \ldots, a_L \otimes \mathrm{I}_{b^0_{k,L}}), \quad y_k = \operatorname{diag}(a_1 \otimes \mathrm{I}_{b^1_{k,1}}, \ldots, a_L \otimes \mathrm{I}_{b^1_{k,L}})$$

for any $k = 1, \ldots, K$. Unless otherwise stated, we assume that β_0, β_1 have the *good form* as above in the remainder of this paper.

4. Some lemmas

In this section, we will prove some lemmas and collect some known results.

Lemma 4.1. *Let H, G be two self-adjoint elements in M_q, written as $\operatorname{sp}(H) = \{\lambda_1, \ldots, \lambda_q\}$ (see Notation 2.6 for the symbol $\operatorname{sp}(H)$) and $\operatorname{sp}(G) = \{\eta_1, \ldots, \eta_q\}$, respectively. If*

$$\exp(2\pi i H) = \exp(2\pi i G),$$

then there is a permutation $\tau \in S_q$ (see Notation 2.1) such that

$$\{\lambda_j - \eta_{\tau(j)} : j = 1, \ldots, q\} \subset \mathbb{Z}.$$

Moreover, if

$$\lambda_q - 1 < \lambda_1 \leqslant \cdots \leqslant \lambda_q$$

and

$$\eta_q - 1 < \eta_1 \leqslant \cdots \leqslant \eta_q,$$

(consequently, $\{\exp(2\pi i\lambda_1), \ldots, \exp(2\pi i\lambda_q)\}$ *and* $\{\exp(2\pi i\eta_1), \ldots, \exp(2\pi i\eta_q)\}$ *are both cyclically numbered (see Definition 2.3).) then there exist an integer* $k \in \{1, \ldots, q\}$ *such that*

$$\begin{aligned}\lambda_1 - \eta_{\varsigma^k(1)} &= \cdots = \lambda_{q-k} - \eta_{\varsigma^k(q-k)} \\ &= \lambda_{q-k+1} - \eta_{\varsigma^k(q-k+1)} - 1 = \cdots = \lambda_q - \eta_{\varsigma^k(q)} - 1 \in \mathbb{Z}\end{aligned}$$

(see Notation 2.1 for the symbol ς*).*

Proof. The proof is trivial. ∎

Lemma 4.2. *Let* $\{\lambda_1, \ldots, \lambda_q\}$ *and* $\{\eta_1, \ldots, \eta_q\}$ *be two finite subsets of the unit circle* $\mathbb{T}$*, which are both cyclically numbered. Then, for any* $\tau \in S_q$*, we can choose an integer* $k \in \{1, \ldots, q\}$ *such that*

$$\max\left\{|\lambda_j - \eta_{\varsigma^k(j)}| : j = 1, \ldots, q\right\} \leqslant \max\left\{|\lambda_j - \eta_{\tau(j)}| : j = 1, \ldots, q\right\}.$$

Proof. The proof is trivial. ∎

Lemma 4.3 (See the proof of [7, Lemma 3.27]). *Let* $\mathcal{Z} := \{z_1, \ldots, z_q\}$ *be a finite subset of the unit circle* $\mathbb{T}$ *such that* $z_j \in I_j^q$ *(see Notation 2.2) for any* $j = 1, \ldots, q$*. Suppose that* (x, y) *is a pair in* $\mathbb{T} \times \mathbb{T}$*, and write*

$$\mathcal{X} := \{\mu_1, \ldots, \mu_{q+1}\} = \{x\} \sqcup \mathcal{Z} \quad \text{and} \quad \mathcal{Y} := \{\nu_1, \ldots, \nu_{q+1}\} = \{y\} \sqcup \mathcal{Z}$$

(see Definition 2.19), then there exists a permutation $\tau \in S_{q+1}$ *such that*

$$\max\{|\mu_j - \nu_{\tau(j)}| : j = 1, \ldots, (q+1)\} < \frac{4\pi}{q},$$

measured along the circle.

Proof. The proof is trivial. ∎

Lemma 4.4. *Let* U *and* V *be two unitaries in* M_q*, and let their eigenvalues, counted with multiplicity, be* $\lambda_1, \ldots, \lambda_q$ *and* $\eta_1, \ldots, \eta_q$*, respectively. Then, there exists a permutation* $\tau \in S_q$ *such that*

$$\max\{|\lambda_j - \eta_{\tau(j)}| : j = 1, \ldots, q\} \leqslant \|U - V\|.$$

If both $\{\lambda_1, \ldots, \lambda_q\}$ *and* $\{\eta_1, \ldots, \eta_q\}$ *are cyclically numbered, then* τ *can be chosen in the form of* ς^k *for some integer* $k \in \{1, \ldots, q\}$*.*

Proof. The first part of the lemma has been demonstrated in the main theorem on [2, p. 71]. And the second part follows from Lemma 4.2. ∎

Lemma 4.5. *Let U be a unitary in $C([0,1],\mathrm{M}_q)$. Then, for any $0<\varepsilon<1$, there exists a unitary $\widetilde{U}$ in $C([0,1],\mathrm{M}_q)$ such that*

- (i) $\widetilde{U}=u\cdot\mathrm{diag}(\exp(2\pi i f_1),\ldots,\exp(2\pi i f_q))\cdot u^*$, *where* $f_1,\ldots,f_q\in C([0,1])_{s.a.}$ *and* $u\in U(C([0,1],\mathrm{M}_q))$,
- (ii) $\widetilde{U}(t)\in\mathrm{M}_q$ *has distinct eigenvalues for each* $t\in(0,1)$,
- (iii) $\widetilde{U}(0)=U(0),\widetilde{U}(1)=U(1)$,
- (iv) $\|\widetilde{U}-U\|<\varepsilon$,
- (v) *for any* $t\in[0,1]$, $\det(\widetilde{U}(t))=\det(U(t))$.

Proof. According to [15, Lemma 3.11] (see also [29, Lemma 1.5], and [4, Theorem 4]), there is a unitary $U'\in C([0,1],\mathrm{M}_q)$ such that

(1) $U'=u\cdot\mathrm{diag}(\exp(2\pi i g_1),\ldots,\exp(2\pi i g_q))\cdot u^*$, where $g_1,\ldots,g_q\in C([0,1])_{s.a.}$ and $u\in U(C([0,1],\mathrm{M}_q))$,

(2) $U'(t)\in\mathrm{M}_q$ has distinct eigenvalues for each $t\in(0,1)$,

(3) $U'(0)=U(0),U'(1)=U(1)$,

(4) $\|U'-U\|<\frac{\varepsilon}{4}$.

Because $\|U'U^*-1\|=\|U'-U\|<\frac{\varepsilon}{4}$, there is a self-adjoint element $H'\in C([0,1],\mathrm{M}_q)$ with $\|H'\|<\frac{\varepsilon}{16}$ such that $U'U^*=\exp(2\pi i H')$. As $U'(0)=U(0)$, $U'(1)=U(1)$, this implies $\exp(2\pi i H'(0))=\exp(2\pi i H'(1))=\mathrm{I}_q$. Combining with the condition $\|H'\|<\frac{\varepsilon}{16}$, it is clear that

$$H'(0)=H'(1)=0\in\mathrm{M}_q. \tag{4.1}$$

Define $F(t):=\det(\exp(2\pi i\frac{H'(t)}{q}))$, which is a continuous function in $C([0,1],\mathbb{T})$. On the one hand, it follows from (4.1) that $F(0)=F(1)=1$. On the other hand, we have

$$\det(U'(t))\cdot\det(U^*(t))=(F(t))^q\quad\forall t\in[0,1]. \tag{4.2}$$

Claim. $\|F-1\|<\frac{\pi}{8}\cdot\varepsilon$.

In fact, for a fixed $t_0\in[0,1]$, write $\mathrm{sp}(\frac{H'(t_0)}{q})=\{\eta_1,\ldots,\eta_q\}\subset\mathbb{R}$ and note that $\|H'\|<\frac{\varepsilon}{16}$, then $\{\eta_1,\ldots,\eta_q\}\subset(-\frac{\varepsilon}{16q},\frac{\varepsilon}{16q})$. Therefore,

$$\big|F(t_0)-1\big|=\left|\det\left(\exp\left(2\pi i\,\frac{H'(t_0)}{q}\right)\right)-1\right|=\left|\exp\left(2\pi i\sum_{j=1}^{q}\eta_j\right)-1\right|<\frac{\pi}{8}\cdot\varepsilon.$$

Combining the arbitrariness of t_0, the claim holds.

As $F\in C([0,1],\mathbb{T})$, there exists a function $h\in C([0,1])_{s.a.}$ with $\|h\|<\frac{\varepsilon}{16}$ such that $F=\exp(2\pi i h)$. Now, define $f_j=g_j-h$ for $j=1,\ldots,q$, and let $\widetilde{U}=\overline{F}\cdot U'$, where $\overline{F}$ is the conjugate element of F in C^*-algebra $C([0,1])$. Hence,

$$\widetilde{U}=\overline{F}\cdot U'=u\cdot\mathrm{diag}(\exp(2\pi i f_1),\ldots,\exp(2\pi i f_q))\cdot u^*,$$

which means that the condition (i) holds. Note that

$$\mathrm{sp}(U'(t)) = \{\exp(2\pi i g_1(t)), \dots, \exp(2\pi i g_q(t))\}$$

and

$$\mathrm{sp}(\widetilde{U}(t)) = \{\exp(2\pi i \cdot (g_1(t) - h(t))), \dots, \exp(2\pi i \cdot (g_q(t) - h(t)))\}.$$

Therefore, the condition (ii) follows from the fact that $U'(t) \in \mathrm{M}_q$ has distinct eigenvalues for each $t \in (0, 1)$. According to $F(0) = F(1) = 1$, the condition (iii) holds. Because

$$\|\widetilde{U} - U\| = \|\bar{F} \cdot U' - U\| \leqslant \|F - 1\| + \|U' - U\| < \varepsilon,$$

the condition (iv) holds. For the condition (v), we have

$$\begin{aligned}\det(\widetilde{U}(t)) &= \prod_{j=1}^{q} \exp(2\pi i \cdot (g_j(t) - h(t))) = \left(\prod_{j=1}^{q} \exp(2\pi i g_j(t))\right) \cdot (\exp(-2\pi i h(t)))^q \\ &= \det(U'(t)) \cdot \bar{F}^q(t) \overset{(4.2)}{=\!=} \det(U(t)),\end{aligned}$$

as desired. ∎

Remark 4.6. As [15, Remark 3.13], in Lemma 4.5, we may ensure that $f_1(0), \dots, f_q(0) \in [0, 1)$ and

$$f_q(t) - 1 < f_1(t) < \dots < f_q(t)$$

for any $t \in (0, 1)$, which means that $\{\exp(2\pi i f_1(0)), \dots, \exp(2\pi i f_q(0))\} \subset \mathbb{T}$ is cyclically numbered.

The following lemma is due to Nielsen and Thomsen.

Lemma 4.7 ([25, Lemma 2.2]). *Let $p_i : [0, 1] \to \mathbb{T}, i = 1, \dots, q$ be continuous functions such that the $p_i(t)$'s are mutually distinct for all $t \in [0, 1]$. Assume $\{p_i(0) : i = 1, \dots, q\}$ is naturally numbered and that $\{p_1(0), \dots, p_k(0)\} = \{p_1(1), \dots, p_q(1)\}$. Let $\tau \in S_k$ such that*

$$p_i(1) = p_{\tau(i)}(0), \quad i = 1, \dots, q.$$

Then,

(i) $\tau = \varsigma^k$ *for some* $k \in \{1, \dots, q\}$;

(ii) *the winding number of the loop* $[0, 1] \ni t \in \prod_{i=1}^{q} p_i(t) \in \mathbb{T}$ *lies in* $q\mathbb{Z} + k$.

Lemma 4.8. *Let U and V be two unitaries in $C([0, 1], \mathrm{M}_q)$ satisfying $U(0) = V(0)$, $U(1) = V(1)$, and for any $t \in [0, 1]$,*

$$\det(U(t)) = \det(V(t)). \tag{4.3}$$

Suppose that $\{z_1, z_2, \ldots, z_m\}$ *is a finite subset of the unit circle* $\mathbb{T}$ *and* n *is a positive integer. If for any* $j = 1, \ldots, n$,

$$\sharp(\{z_1, \ldots, z_m\} \cap I_j^n) > q, \tag{4.4}$$

(see Notation 2.20 for the symbol $\sharp(\cdot)$*), then there is a unitary* $W \in C([0,1], \mathrm{M}_{q+m})$ *such that* $W(0) = W(1) = \mathrm{I}_{q+m}$ *and*

$$\left\| \mathrm{diag}(U, z_1, \ldots, z_m) - W\,\mathrm{diag}(V, z_1, \ldots, z_m)W^* \right\| < \frac{23\pi}{n}.$$

Proof. The idea of this proof originates from [6, Theorem 4], and in addition, the reader can also refer to [25, Lemma 2.3]. Because, in the general case, the natural number n in this lemma is large enough, for easy calculation, we use the arc length as the distance between two points on the unit circle $\mathbb{T}$, which is not fundamentally different from the classical Euclidean distance.

Obviously, the inequality $m > nq$ holds according to the assumption (4.4). Without loss of generality, assume

$$\{z_{q(j-1)+1}, \ldots, z_{q(j-1)+q}\} \subset \{z_1, \ldots, z_m\} \cap I_j^n, \quad j = 1, \ldots, n. \tag{4.5}$$

It suffices to show that for some unitary $\widetilde{W} \in C([0,1], \mathrm{M}_{q+nq})$ such that $\widetilde{W}(0) = \widetilde{W}(1) = \mathrm{I}_{q+nq}$ and

$$\left\| \mathrm{diag}(U, z_1, \ldots, z_{nq}) - \widetilde{W} \cdot \mathrm{diag}(V, z_1, \ldots, z_{nq}) \cdot \widetilde{W}^* \right\| < \frac{23\pi}{n}. \tag{4.6}$$

Once the above is true, the lemma is justified by defining $W = \mathrm{diag}(\widetilde{W}, \mathrm{I}_{m-nq})$.

Now, we will begin to prove (4.6). Firstly, find a small enough $0 < 2\sigma < \frac{1}{2}$ such that for $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| < 2\sigma$, we have

$$\|U(t_1) - U(t_2)\| < \frac{\pi}{2n} \quad \text{and} \quad \|V(t_1) - V(t_2)\| < \frac{\pi}{2n}. \tag{4.7}$$

Next, choose two unitaries $\bar{U}$ and $\bar{\bar{U}}$ in M_q such that

(1) both $\bar{U}$ and $\bar{\bar{U}}$ have distinct eigenvalues, and

$$\mathrm{sp}(\bar{U}) \cap \{z_1, \ldots, z_{nq}\} = \mathrm{sp}(\bar{\bar{U}}) \cap \{z_1, \ldots, z_{nq}\} = \emptyset;$$

(2) for $\bar{U}$, there is a path $\bar{P}$ in $U(\mathrm{M}_q)$ satisfying $\bar{P}(0) = U(0) = V(0)$, $\bar{P}(1) = \bar{U}$ and

$$\left\| \bar{P}(t) - U(0) \right\| < \frac{\pi}{2n} \quad \forall t \in [0,1]; \tag{4.8}$$

(3) for $\bar{\bar{U}}$, there is a path $\bar{\bar{P}}$ in $U(\mathrm{M}_q)$ satisfying $\bar{\bar{P}}(0) = \bar{\bar{U}}$, $\bar{\bar{P}}(1) = U(1) = V(1)$ and

$$\left\| \bar{\bar{P}}(t) - U(1) \right\| < \frac{\pi}{2n} \quad \forall t \in [0,1]. \tag{4.9}$$

For σ, $\overline{P}$, $\overline{\overline{P}}$ above, we construct two unitaries $U', V' \in C([0,1], \mathrm{M}_q)$ as follows:

$$U'(t) = \begin{cases} \overline{P}(\frac{t}{\sigma}) & \text{if } 0 \leqslant t < \sigma, \\ \overline{P}(\frac{2\sigma - t}{\sigma}) & \text{if } \sigma \leqslant t < 2\sigma, \\ U(\frac{t-2\sigma}{1-4\sigma}) & \text{if } 2\sigma \leqslant t < 1-2\sigma, \\ \overline{\overline{P}}(\frac{1-\sigma-t}{\sigma}) & \text{if } 1-2\sigma \leqslant t < 1-\sigma, \\ \overline{\overline{P}}(\frac{t-1+\sigma}{\sigma}) & \text{if } 1-\sigma \leqslant t \leqslant 1, \end{cases}$$

$$\text{and} \quad V'(t) = \begin{cases} \overline{P}(\frac{t}{\sigma}) & \text{if } 0 \leqslant t < \sigma, \\ \overline{P}(\frac{2\sigma - t}{\sigma}) & \text{if } \sigma \leqslant t < 2\sigma, \\ V(\frac{t-2\sigma}{1-4\sigma}) & \text{if } 2\sigma \leqslant t < 1-2\sigma, \\ \overline{\overline{P}}(\frac{1-\sigma-t}{\sigma}) & \text{if } 1-2\sigma \leqslant t < 1-\sigma, \\ \overline{\overline{P}}(\frac{t-1+\sigma}{\sigma}) & \text{if } 1-\sigma \leqslant t \leqslant 1. \end{cases}$$

As a consequence,

$$\|U - U'\| < \frac{\pi}{n} \quad \text{and} \quad \|V - V'\| < \frac{\pi}{n} \tag{4.10}$$

and

$$U'(\sigma) = V'(\sigma) \quad \text{and} \quad U'(1-\sigma) = V'(1-\sigma). \tag{4.11}$$

Now, we divide into two cases for discussion.

Case 1. If $z_1, \ldots, z_{nq}$ are mutually distinct.

In this case, according to Lemma 4.5, for the restriction element $\mathrm{diag}(U', z_1, \ldots, z_{nq})|_{[\sigma, 1-\sigma]}$ there is a unitary $\widetilde{U} \in C([\sigma, 1-\sigma], \mathrm{M}_{q+nq})$ satisfying

(U-1) there exist a unitary $u \in C([\sigma, 1-\sigma], \mathrm{M}_{q+nq})$ and $(q+nq)$ continuous functions $h_1, \ldots, h_{q+nq} \in C([\sigma, 1-\sigma])_{s.a.}$ with $h_i(\sigma) \in [0,1)$ for $l = 1, \ldots, (q+nq)$ and $h_1 < \cdots < h_{q+nq}$ such that

$$\widetilde{U} = u \cdot \mathrm{diag}(\exp(2\pi i h_1), \ldots, \exp(2\pi i h_{q+nq})) \cdot u^*,$$

(U-2) $\widetilde{U}(t) \in \mathrm{M}_{q+nq}$ has distinct eigenvalues for each $t \in [\sigma, 1-\sigma]$,

(U-3) $\widetilde{U}(\sigma) = \mathrm{diag}(U'(\sigma), z_1, \ldots, z_{nq})$ and $\widetilde{U}(1-\sigma) = \mathrm{diag}(U'(1-\sigma), z_1, \ldots, z_{nq})$,

(U-4) $\|\widetilde{U} - \mathrm{diag}(U', z_1, \ldots, z_{nq})\| < \frac{\pi}{2n}$,

(U-5) for any $t \in [\sigma, 1-\sigma]$, $\det(\widetilde{U}(t)) = \det(\mathrm{diag}(U'(t), z_1, \ldots, z_{nq}))$.

Similar argument applies to the restriction element $\mathrm{diag}(V', z_1, \ldots, z_{nq})|_{[\sigma,1-\sigma]} \in C([\sigma, 1-\sigma], \mathrm{M}_{q+nq})$, hence, there is a unitary $\widetilde{V} \in C([\sigma, 1-\sigma], \mathrm{M}_{q+nq})$ satisfying

(V-1) there exist a unitary $v \in C([\sigma, 1-\sigma], \mathrm{M}_{q+nq})$ and $(q+nq)$ continuous functions $g_1, \ldots, g_{q+nq} \in C([\sigma, 1-\sigma])_{s.a.}$ with $g_l(\sigma) \in [0,1)$ for all $l = 1, \ldots, (q+nq)$ and $g_1 < \cdots < g_{q+nq}$ such that

$$\widetilde{V} = v \cdot \mathrm{diag}(\exp(2\pi i g_1), \ldots, \exp(2\pi i g_{q+nq})) \cdot v^*,$$

(V-2) $\widetilde{V}(t) \in \mathrm{M}_{q+nq}$ has distinct eigenvalues for each $t \in [\sigma, 1-\sigma]$,

(V-3) $\widetilde{V}(\sigma) = \mathrm{diag}(V'(\sigma), z_1, \ldots, z_{nq})$ and $\widetilde{V}(1-\sigma) = \mathrm{diag}(V'(1-\sigma), z_1, \ldots, z_{nq})$,

(V-4) $\|\widetilde{V} - \mathrm{diag}(V', z_1, \ldots, z_{nq})\| < \frac{\pi}{2n}$,

(V-5) for any $t \in [\sigma, 1-\sigma]$, $\det(\widetilde{V}(t)) = \det(\mathrm{diag}(V'(t), z_1, \ldots, z_{nq}))$.

On the one hand,

$$\widetilde{U}(\sigma) \overset{(4.11)}{=\!=} \widetilde{V}(\sigma) \quad \text{and} \quad \widetilde{U}(1-\sigma) \overset{(4.11)}{=\!=} \widetilde{V}(1-\sigma).$$

On the other hand, combining Lemma 4.4 with Remark 4.6, for any $t \in [\sigma, 1-\sigma]$, we have,

$$\mathrm{sp}(\widetilde{U}(t)) = \{\exp(2\pi i h_1(t)), \ldots, \exp(2\pi i h_{q+nq}(t))\}, \tag{4.12}$$

$$\mathrm{sp}(\widetilde{V}(t)) = \{\exp(2\pi i g_1(t)), \ldots, \exp(2\pi i g_{q+nq}(t))\}, \tag{4.13}$$

and assume that for an arbitrary $t_0 \in [\sigma, 1-\sigma]$,

$$\mathrm{sp}(\mathrm{diag}(U'(t_0), z_1, \ldots, z_{nq})) =: \{\lambda_1, \ldots, \lambda_{q+nq}\} \tag{4.14}$$

and

$$\mathrm{sp}(\mathrm{diag}(V'(t_0), z_1, \ldots, z_{nq})) =: \{\eta_1, \ldots, \eta_{q+nq}\}, \tag{4.15}$$

are all cyclically numbered. Now, noting that $h_l(\sigma) \in [0, 1)$ (see (U-1)) and $g_l(\sigma) \in [0, 1)$ (see (V-1)), we get

$$h_l(\sigma) = g_l(\sigma), \quad l = 1, \ldots, (q + nq). \tag{4.16}$$

Below, we will show that

$$h_l(1-\sigma) = g_l(1-\sigma), \quad l = 1, \ldots, (q + nq) \tag{4.17}$$

also hold.

On the one hand, it follows from the assumption (4.3) that

$$\det(\widetilde{U}(t)) \cdot \det(\widetilde{V}^*(t)) \overset{(4.12)}{\underset{(4.13)}{=\!=}} \exp\left(2\pi i \cdot \sum_{l=1}^{q+nq} (h_l(t) - g_l(t))\right) = 1 \quad \forall t \in [\sigma, 1-\sigma].$$

Consequently, for all $t \in [\sigma, 1-\sigma]$, $F(t) := \sum_{l=1}^{q+nq}(h_l(t) - g_l(t)) \in \mathbb{Z}$. Note that $\{h_l : l = 1, \ldots, (q+nq)\} \cup \{g_l : l = 1, \ldots, (q+nq)\} \subset C([\sigma, 1-\sigma])$, thus, the map $F(t) = \sum_{l=1}^{q+nq}(h_l(t) - g_l(t)) \in C([\sigma, 1-\sigma])$. So,

$$F(t) = F(\sigma) \overset{(4.16)}{=\!=} 0 \quad \forall t \in [\sigma, 1-\sigma].$$

Particularly,

$$\sum_{l=1}^{q+nq} (h_l(1-\sigma) - g_l(1-\sigma)) = 0. \tag{4.18}$$

On the other hand, by using Lemma 4.1, there exists an integer $k_0 \in \{1, \ldots, (q+nq)\}$ such that

$$\begin{aligned} & h_1(1-\sigma) - g_{\varsigma^{k_0}(1)}(1-\sigma) \\ & = \cdots = h_{(q+nq)-k_0}(1-\sigma) - g_{\varsigma^{k_0}((q+nq)-k_0)}(1-\sigma) \\ & = h_{(q+nq)-k_0+1}(1-\sigma) - g_{\varsigma^{k_0}((q+nq)-k_0+1)}(1-\sigma) - 1 \\ & = \cdots = h_{q+nq} - g_{\varsigma^{k_0}(q+nq)}(1-\sigma) - 1 \in \mathbb{Z}. \end{aligned}$$

Hence, it follows from (4.18) that $\varsigma(l) = l$ hold for all $l = 1, \ldots, (q+nq)$. That is, $h_l(1-\sigma) = g_l(1-\sigma)$ hold for all $l = 1, \ldots, (q+nq)$.

Now, for $t_0 \in [\sigma, 1-\sigma]$ above. By Lemma 4.4, (U-4) and (4.14), there exists an integer $\bar{k} \in \{1, \ldots, (q+nq)\}$ such that

$$\max\{|\lambda_l - \exp(2\pi i \cdot g_{\varsigma^{\bar{k}}(l)}(t_0))| : l = 1, \ldots, (q+nq)\} < \frac{\pi}{2n}. \tag{4.19}$$

Similarly, by Lemma 4.4, (U-4) and (4.15), there exists an integer $\hat{k} \in \{1, \ldots, (q+nq)\}$ such that

$$\max\{|\eta_l - \exp(2\pi i \cdot h_{\varsigma^{\hat{k}}(l)}(t_0))| : l = 1, \ldots, (q+nq)\} < \frac{\pi}{2n}. \tag{4.20}$$

Claim. There exists an integer $\tilde{k}$ such that

$$\max\{|\lambda_l - \eta_{\varsigma^{\tilde{k}}(l)}| : l = 1, \ldots, (q+nq)\} < \frac{4\pi}{n}. \tag{4.21}$$

According to Lemma 4.2, to prove the claim, it is sufficient to show that there exists a permutation $\tau \in S_{q+nq}$ such that

$$\max\{|\lambda_l - \eta_{\tau(l)}| : l = 1, \ldots, (q+nq)\} < \frac{4\pi}{n}.$$

Note that $\{z_1, \ldots, z_{nq}\} = \{\lambda_1, \ldots, \lambda_{q+nq}\} \cap \{\eta_1, \ldots, \eta_{q+nq}\}$. Write

$$\begin{aligned} \mathcal{X} &:= \{x_1, \ldots, x_q\} = \{\lambda_1, \ldots, \lambda_{q+nq}\} \backslash \{z_1, \ldots, z_{nq}\}, \\ \mathcal{Y} &:= \{y_1, \ldots, y_q\} = \{\eta_1, \ldots, \eta_{q+nq}\} \backslash \{z_1, \ldots, z_{nq}\} \end{aligned}$$

(see Definition 2.19) and note that $\sharp(\mathcal{X}) = \sharp(\mathcal{Y}) = q$. According to (4.5), for a fixed $l = 1, \ldots, q$, the set

$$\mathcal{Z}_l := \{z_l, z_{q+l}, \ldots, z_{(n-1)q+l}\}$$

satisfies $z_{(j-1)q+i} \in I_j^n$ for any $j = 1, \ldots, n$. Now, for the pair $(x_l, y_l) \in \mathcal{X} \times \mathcal{Y}$, write

$$\mathcal{X}_l := \{\mu_1, \ldots, \mu_{n+1}\} = \{x_l\} \sqcup \mathcal{Z}_l \quad \text{and} \quad \mathcal{Y}_l := \{\nu_1, \ldots, \nu_{n+1}\} = \{y_l\} \sqcup \mathcal{Z}_l.$$

By using Lemma 4.3, there exists a permutation $\tau_l \in S_{n+1}$ such that

$$\max\{|\mu_s - \nu_{\tau_l(s)}| : s = 1, \ldots, (n+1)\} < \frac{4\pi}{n}.$$

Note that

$$\mathcal{X} = \bigsqcup_{l=1}^{q} \mathcal{X}_l \quad \text{and} \quad \mathcal{Y} = \bigsqcup_{l=1}^{q} \mathcal{Y}_l.$$

Hence, there exists a permutation $\tau \in S_{q+nq}$ such that

$$\max\left\{\left|\lambda_l - \eta_{\tau(l)}\right| : l = 1, \ldots, q + nq\right\} < \frac{4\pi}{n},$$

which proves the claim.

Combining (4.19), (4.20) and (4.21), for $t_0 \in [\sigma, 1-\sigma]$ above there exists an integer $k \in \{1, \ldots, (q+nq)\}$ such that

$$\max\left\{\left|\exp(2\pi i g_l(t_0)) - \exp\big(2\pi i h_{\varsigma^k(l)}(t_0)\big)\right| : l = 1, \ldots, (q+nq)\right\} < \frac{5\pi}{n}. \tag{4.22}$$

Thus, we can define a path $p_l^{t_0} : [0,1] \to \mathbb{T}$ satisfying $p_l^{t_0}(0) = \exp(2\pi i g_l(t_0))$, $p_l^{t_0}(1) = \exp(2\pi i h_{\varsigma^k(l)}(t_0))$ and

$$|p_l^{t_0}(s) - p_l^{t_0}(0)| < \frac{5\pi}{n} \quad \forall s \in [0,1].$$

In addition, note that (4.16). Now, for each $l = 1, \ldots, (q+nq)$, set

$$p_l(s) = \begin{cases} \exp(2\pi i h_l(3(\sigma - t_0)s + t_0)) & \text{if } 0 \leqslant s < \frac{1}{3}, \\ \exp(2\pi i g_l(3(t_0 - \sigma)s + 2\sigma - t_0)) & \text{if } \frac{1}{3} \leqslant s < \frac{2}{3}, \\ p_l^{t_0}(3s-2) & \text{if } \frac{2}{3} \leqslant s \leqslant 1. \end{cases}$$

Consequently, a path P on the unit circle $\mathbb{T}$ can be obtained by defining

$$\begin{aligned} P : [0,1] &\to \mathbb{T}, \\ s &\mapsto \prod_{l=1}^{q+nq} p_l(s). \end{aligned}$$

Note that $P(0) = P(1) = \exp(2\pi i \sum_{l=1}^{q+nq} h_l(t_0))$. Let us denote by W the winding number of the loop P. According to the definition of P, first, we have

$$|W| \leqslant \frac{\frac{5\pi}{n} \cdot (q+nq)}{2\pi} = \frac{5q \cdot (1+n)}{2n}.$$

Since $\varsigma^k(l) = (l+k) \bmod (q+nq) \in \{1, \ldots, (q+nq)\}$, by using Lemma 4.7, we have $W \in (q+nq)\mathbb{Z} + k$, and so,

$$\min\{k, (q+nq) - k\} \leqslant |W|.$$

Thus,

$$\min\{k, (q+nq) - k\} \leqslant |W| \leqslant \frac{5q \cdot (1+n)}{2n}$$

holds. Hence, the open minor arc I with endpoint $\exp(2\pi i h_l(t_0))$ and $\exp(2\pi i h_{\varsigma^k(l)}(t_0))$ satisfies

$$\sharp(I \cap \{\exp(2\pi i h_1(t_0)), \ldots, \exp(2\pi i h_{q+nq}(t_0))\}) \leqslant \frac{5q \cdot (1+n)}{2n}.$$

Thus, by (4.4), we get

$$\sharp(I \cap \{I_j^n : j = 1, \ldots, n\}) \leqslant \frac{5(1+n)}{2n}.$$

It follows that

$$\begin{aligned} &\max\left\{\left|\exp\left(2\pi i h_{\varsigma^k(l)}(t_0)\right) - \exp(2\pi i h_l(t_0))\right| : l = 1, \ldots, (q+nq)\right\} \\ &\quad < \frac{2\pi}{n} \cdot \left(\frac{5(1+n)}{2n} + 2\right) < \frac{14\pi}{n}. \end{aligned}$$

Combining (4.22), we can show that

$$\max\left\{|\exp(2\pi i h_l(t_0)) - \exp(2\pi i g_l(t_0))| : l = 1, \ldots, (q+nq)\right\} < \frac{19\pi}{n}.$$

Consequently,

$$\left\|\operatorname{diag}(U'(t), z_1, \ldots, z_{nq}) - (uv^*)(t) \cdot \operatorname{diag}(V'(t), z_1, \ldots, z_{nq}) \cdot (vu^*)(t)\right\| < \frac{19\pi}{n}$$

for any $t \in [\sigma, 1-\sigma]$. Here, both u and v are elements in $U(C([\sigma, 1-\sigma], \mathrm{M}_{q+nq}))$. Recall that

$$\begin{aligned} &\operatorname{diag}(U'(\sigma), z_1, \ldots, z_{nq}) \\ &\quad \overset{\text{(U-3)}}{=\!=} \widetilde{U}(\sigma) \\ &\quad \overset{\text{(U-1)}}{=\!=} u(\sigma) \cdot \operatorname{diag}(\exp(2\pi i h_1(\sigma)), \ldots, \exp(2\pi i h_{q+nq}(\sigma))) \cdot u^*(\sigma), \\ &\operatorname{diag}(V'(\sigma), z_1, \ldots, z_{nq}) \\ &\quad \overset{\text{(V-3)}}{=\!=} \widetilde{V}(\sigma) \\ &\quad \overset{\text{(V-1)}}{=\!=} v(\sigma) \cdot \operatorname{diag}(\exp(2\pi i g_1(\sigma)), \ldots, \exp(2\pi i g_{q+nq}(\sigma))) \cdot v^*(\sigma) \end{aligned}$$

and

$$\begin{aligned} &\operatorname{diag}(U'(1-\sigma), z_1, \ldots, z_{nq}) \\ &\quad \overset{\text{(U-3)}}{=\!=} \widetilde{U}(1-\sigma) \\ &\quad \overset{\text{(U-1)}}{=\!=} u(1-\sigma) \cdot \operatorname{diag}(\exp(2\pi i h_1(1-\sigma)), \ldots, \exp(2\pi i h_{q+nq}(1-\sigma))) \cdot u^*(1-\sigma), \\ &\operatorname{diag}(V'(1-\sigma), z_1, \ldots, z_{nq}) \\ &\quad \overset{\text{(V-3)}}{=\!=} \widetilde{V}(1-\sigma) \\ &\quad \overset{\text{(V-1)}}{=\!=} v(1-\sigma) \cdot \operatorname{diag}(\exp(2\pi i g_1(1-\sigma)), \ldots, \exp(2\pi i g_{q+nq}(1-\sigma))) \cdot v^*(1-\sigma). \end{aligned}$$

Note that

$$\begin{aligned}&\mathrm{diag}(U'(\sigma), z_1, \ldots, z_{nq})\\ &\quad\overset{(4.11)}{=\!=\!=} \mathrm{diag}(V'(\sigma), z_1, \ldots, z_{nq}),\\ &\mathrm{diag}(\exp(2\pi i h_1(\sigma)), \ldots, \exp(2\pi i h_{q+nq}(\sigma)))\\ &\quad\overset{(4.16)}{=\!=\!=} \mathrm{diag}(\exp(2\pi i g_1(\sigma)), \ldots, \exp(2\pi i g_{q+nq}(\sigma)))\end{aligned}$$

and

$$\begin{aligned}&\mathrm{diag}(U'(1-\sigma), z_1, \ldots, z_{nq})\\ &\quad\overset{(4.11)}{=\!=\!=} \mathrm{diag}(V'(1-\sigma), z_1, \ldots, z_{nq}),\\ &\mathrm{diag}(\exp(2\pi h_1(1-\sigma)), \ldots, \exp(2\pi i h_{q+nq}(1-\sigma)))\\ &\quad\overset{(4.17)}{=\!=\!=} \mathrm{diag}(\exp(2\pi i g_1(1-\sigma)), \ldots, \exp(2\pi i g_{q+nq}(1-\sigma))).\end{aligned}$$

Thus,

$$\mathrm{diag}(U'(\sigma), z_1, \ldots, z_{nq}) = (uv^*)(\sigma) \cdot \mathrm{diag}(V'(\sigma), z_1, \ldots, z_{nq}) \cdot (vu^*)(\sigma)$$

and

$$\mathrm{diag}(U'(1-\sigma), z_1, \ldots, z_{nq}) = (uv^*)(1-\sigma) \cdot \mathrm{diag}(V'(1-\sigma), z_1, \ldots, z_{nq}) \cdot (vu^*)(1-\sigma).$$

By using [1, Proposition 4.4], there is a path $\bar{Q} : [0, \sigma] \to U(\mathrm{M}_{q+nq})$ with $\bar{Q}(0) = \mathrm{I}_{q+nq}$ and $\bar{Q}(\sigma) = (uv^*)(\sigma)$ such that

$$\mathrm{diag}(U'(\sigma), z_1, \ldots, z_{nq}) = \bar{Q}(t) \cdot \mathrm{diag}(V'(\sigma), z_1, \ldots, z_{nq}) \cdot \bar{Q}^*(t) \quad \forall t \in [0, \sigma].$$

For same reason, there is a path $\bar{\bar{Q}} : [1-\sigma, 1] \to U(\mathrm{M}_{q+nq})$ with $\bar{\bar{Q}}(1) = \mathrm{I}_{q+nq}$ and $\bar{\bar{Q}}(1-\sigma) = (uv^*)(1-\sigma)$ such that

$$\begin{aligned}&\mathrm{diag}(U'(1-\sigma), z_1, \ldots, z_{nq})\\ &\quad = \bar{\bar{Q}}(t) \cdot \mathrm{diag}(V'(1-\sigma), z_1, \ldots, z_{nq}) \cdot \bar{\bar{Q}}^*(t) \quad \forall t \in [1-\sigma, 1].\end{aligned}$$

Now, define

$$\widetilde{W}(t) = \begin{cases} \bar{Q}(t) & \text{if } 0 \leqslant t < \sigma, \\ (uv^*)(t) & \text{if } \sigma \leqslant t < 1-\sigma, \\ \bar{\bar{Q}}(t) & \text{if } 1-\sigma \leqslant t \leqslant 1. \end{cases}$$

Evidently, $\widetilde{W}(0) = \widetilde{W}(1) = \mathrm{I}_{q+nq}$. According to (4.8) and (4.9), it is easy to check

$$\left\| \mathrm{diag}(U', z_1, \ldots, z_{nq}) - \widetilde{W} \cdot \mathrm{diag}(V', z_1, \ldots, z_{nq}) \cdot \widetilde{W}^* \right\| < \frac{20\pi}{n}.$$

It follows from (4.10) that

$$\left\| \mathrm{diag}(U, z_1, \ldots, z_{nq}) - \widetilde{W} \cdot \mathrm{diag}(V, z_1, \ldots, z_{nq}) \cdot \widetilde{W}^* \right\| < \frac{21\pi}{n} < \frac{23\pi}{n}.$$

Case 2. If for some $j = 1, \ldots, n$, the set $\{z_{q(j-1)+1}, \ldots, z_{q(j-1)+q}\} = \{z_1, \ldots, z_{nq}\} \cap I_j^n$ satisfies that there exist two elements $z_{q(j-1)+l_1}, z_{q(j-1)+l_2} \in I_j^n$ such that $z_{q(j-1)+l_1} = z_{q(j-1)+l_2}$.

In this case, we can choose to replace $\{z_{q(j-1)+1}, \ldots, z_{q(j-1)+q}\}$ with $\{z'_{q(j-1)+1}, \ldots, z'_{q(j-1)+q}\} \subset I_j^n$, where $z'_{q(j-1)+1}, \ldots, z'_{q(j-1)+q}$ are mutually distinct. Thus, the problem now transforms into *Case 1*. That is, there is a unitary $\widetilde{W} \in C([0,1], \mathrm{M}_{q+nq})$ satisfying

$$\left\|\operatorname{diag}(U, z'_1, \ldots, z'_{nq}) - \widetilde{W} \cdot \operatorname{diag}(V, z'_1, \ldots, z'_{nq}) \cdot \widetilde{W}^*\right\| < \frac{21\pi}{n}.$$

Note that

$$\left\|\operatorname{diag}(z_1, \ldots, z_{q+nq}) - \operatorname{diag}(z'_1, \ldots, z'_{q+nq})\right\| < \frac{2\pi}{n}.$$

Thus,

$$\left\|\operatorname{diag}(U, z_1, \ldots, z_{nq}) - \widetilde{W} \cdot \operatorname{diag}(V, z_1, \ldots, z_{nq}) \cdot \widetilde{W}^*\right\| < \frac{23\pi}{n}.$$

Hence, (4.6) also holds in this case, as desired. ∎

Lemma 4.9. *Let F, E be two finite dimensional C^*-algebras and $A = C([0,1], E) \oplus_{\beta_0,\beta_1} F$ be a Elliott–Thomsen algebra. Suppose U is an element in $U_0(A)$, then there is a unitary V such that*

$$\det(V(t)) = \det(U(t)) \quad \forall t \in \operatorname{Sp}(A)$$

and $\operatorname{cer}(V) = 1$.

Proof. Let $F := \mathrm{M}_{[1]} \oplus \cdots \oplus \mathrm{M}_{[L]}$, $E := \mathrm{M}_{\{1\}} \oplus \cdots \oplus \mathrm{M}_{\{K\}}$. Suppose the two group homomorphisms

$$K_0(\beta_0) = (b^0_{k,l}), K_0(\beta_1) = (b^1_{k,l}) : K_0(F) = \mathbb{Z}^L \to K_0(E) = \mathbb{Z}^K$$

are induced by $\beta_0, \beta_1 : F \to E$, respectively, and set

$$(b_{k,l}) := (b^1_{k,l}) - (b^0_{k,l}) = K_0(\beta_1) - K_0(\beta_0).$$

Write $U = (U_1, \ldots, U_K; u_1, \ldots, u_L)$. It follows from Lemma 4.5, for each $k = 1, \ldots, K$ and a (fixed) small enough $\varepsilon > 0$ there is a unitary $\mathfrak{U}_k \in C([0,1], \mathrm{M}_{\{k\}})$ such that

(1) $\mathfrak{U}_k = \mathfrak{u}_k \cdot \operatorname{diag}(\exp(2\pi i f^k_1), \ldots, \exp(2\pi i f^k_{\{k\}})) \cdot \mathfrak{u}^*_k$ where $\mathfrak{u}_k \in U(C([0,1], \mathrm{M}_{\{k\}}))$ and $f^k_1, \ldots, f^k_{\{k\}} \in C([0,1])_{s.a.}$,

(2) $\mathfrak{U}_k(t) \in \mathrm{M}_{\{k\}}$ has distinct eigenvalues for each $t \in (0,1)$,

(3) $\mathfrak{U}_k(0) = U_k(0), \mathfrak{U}_k(1) = U_k(1)$,

(4) $\|\mathfrak{U}_k - U_k\| < \varepsilon$,

(5) for any $t \in [0,1]$, $\det(\mathfrak{U}_k(t)) = \det(U_k(t))$.

Let $\mathfrak{U} = (\mathfrak{U}_1, \ldots, \mathfrak{U}_K; u_1, \ldots, u_L)$. Note that for any $l = 1, \ldots, L$, the unitary $u_l \in \mathrm{M}_{[l]}$ have the form

$$u_l = \hat{u}_l \cdot \mathrm{diag}\big(\exp(2\pi i \lambda_1^l), \ldots, \exp\big(2\pi i \lambda_{[l]}^l\big)\big) \cdot \hat{u}_l^*,$$

where $\hat{u}_l \in U(\mathrm{M}_{[l]})$ and $\lambda_1^l, \ldots, \lambda_{[l]}^l \in [0, 1)$. Without loss of generality, for any $k = 1, \ldots, K$, assume

$$\big\{f_1^k(0), \ldots, f_{\{k\}}^k(0)\big\} = \big\{(\lambda_j^l)^{\sim b_{k,l}^0} : l = 1, \ldots, L;\ j = 1, \ldots, [l]\big\}.$$

And, set

$$\big\{z_1^k, \ldots, z_{\{k\}}^k\big\} := \big\{(\lambda_j^l)^{\sim b_{k,l}^1} : l = 1, \ldots, L;\ j = 1, \ldots, [l]\big\},$$

then there exist $\{k\}$ integers $m_1^k, \ldots, m_{\{k\}}^k$ such that

$$\big\{f_1^k(1), \ldots, f_{\{k\}}^k(1)\big\} = \big\{z_1^k + m_1^k, \ldots, z_{\{k\}}^k + m_{\{k\}}^k\big\}.$$

Now, write

$$\tilde{h}_l := \hat{u}_l \cdot \mathrm{diag}\big(-\lambda_1^l, \ldots, -\lambda_{[l]}^l\big) \cdot \hat{u}_l^*, \quad l = 1, \ldots, L,$$

and then, $\tilde{h} := (\tilde{h}_1, \ldots, \tilde{h}_L) \in F_{s.a.}$. Note that

$$\exp(2\pi i \tilde{h}_l) \cdot u_l = u_l \cdot \exp(2\pi i \tilde{h}_l) = \mathrm{I}_{[l]}, \quad l = 1, \ldots, L$$

and, for all $k = 1, \ldots, K$, both $\Pi_k(\beta_0(\tilde{h}))$ and $\Pi_k(\beta_1(\tilde{h}))$ are elements in $(\mathrm{M}_{\{k\}})_{s.a.}$ satisfying

$$\begin{aligned}
&\exp(2\pi i \cdot \Pi_k(\beta_0(\tilde{h}))) \cdot \mathfrak{U}_k(0) = \mathfrak{U}_k(0) \cdot \exp(2\pi i \cdot \Pi_k(\beta_0(\tilde{h}))) = \mathrm{I}_{\{k\}},\\
&\exp(2\pi i \cdot \Pi_k(\beta_1(\tilde{h}))) \cdot \mathfrak{U}_k(1) = \mathfrak{U}_k(1) \cdot \exp(2\pi i \cdot \Pi_k(\beta_1(\tilde{h}))) = \mathrm{I}_{\{k\}},
\end{aligned} \tag{4.23}$$

where Π_k is the quotient map from $E = \mathrm{M}_{\{1\}} \oplus \cdots \oplus \mathrm{M}_{\{K\}}$ to $\mathrm{M}_{\{k\}}$. Evidently, for an arbitrary $k = 1, \ldots, K$,

$$\begin{aligned}
\mathrm{sp}(\Pi_k(\beta_0(\tilde{h}))) &= \big\{(-\lambda_j^l)^{\sim b_{k,l}^0} : l = 1, \ldots, L;\ j = 1, \ldots, [l]\big\}(\subset \mathbb{R}),\\
\mathrm{sp}(\Pi_k(\beta_1(\tilde{h}))) &= \big\{(-\lambda_j^l)^{\sim b_{k,l}^1} : l = 1, \ldots, L;\ j = 1, \ldots, [l]\big\}(\subset \mathbb{R}).
\end{aligned}$$

For convenience, we renumber the elements in both $\mathrm{sp}(\Pi_k(\beta_0(\tilde{h})))$ and $\mathrm{sp}(\Pi_k(\beta_1(\tilde{h})))$, written as $\{\theta_1^k, \ldots, \theta_{\{k\}}^k\} \subset \mathbb{R}$ and $\{\vartheta_1^k, \ldots, \vartheta_{\{k\}}^k\} \subset \mathbb{R}$, respectively. Now, we can arbitrarily choose $\{k\}$ continuous real-valued functions $\tilde{g}_1^k, \ldots, \tilde{g}_{\{k\}}^k$ such that

$$\big\{\tilde{g}_1^k(0), \ldots, \tilde{g}_{\{k\}}^k(0)\big\} = \big\{\theta_1^k, \ldots, \theta_{\{k\}}^k\big\} \quad \text{and} \quad \big\{\tilde{g}_1^k(1), \ldots, \tilde{g}_{\{k\}}^k(1)\big\} = \big\{\vartheta_1^k, \ldots, \vartheta_{\{k\}}^k\big\}.$$

Set

$$\widetilde{H}_k := \mathfrak{u}_k \cdot \mathrm{diag}\big(\tilde{g}_1^k, \ldots, \tilde{g}_{\{k\}}^k\big) \cdot \mathfrak{u}_k^*, \quad k = 1, \ldots, K$$

and then, $\widetilde{H} := (\widetilde{H}_1, \dots, \widetilde{H}_K) \in C([0,1], E)_{s.a.}$. Obviously, $(\widetilde{H}; \tilde{h}) \in A_{s.a.}$ and

$$[U] = [\mathfrak{U}] = [\mathfrak{U} \cdot \exp(2\pi i(\widetilde{H}; \tilde{h}))] = 0 \in K_1(A) = \mathbb{Z}^K / \operatorname{Im}(K_0(\beta_1) - K_0(\beta_0)). \quad (4.24)$$

Note that for each $k = 1, \dots, K$, we have

$$\det(\mathfrak{U}_k(t) \exp(2\pi i \widetilde{H}_k(t))) = \exp\left(2\pi i \sum_{s=1}^{\{k\}} (f_s^k(t) + \tilde{g}_s^k(t))\right).$$

Thus, the function $\det(\mathfrak{U}_k(\cdot) \exp(2\pi i \widetilde{H}_k(\cdot))) \in C([0,1], \mathbb{T})$ and

$$\det(\mathfrak{U}_k(0) \exp(2\pi i \widetilde{H}_k(0))) \overset{(4.23)}{=\!=} \det(\mathfrak{U}_k(1) \exp(2\pi i \widetilde{H}_k(1))).$$

We can calculate that the winding number of $\det(\mathfrak{U}_k(\cdot) \exp(2\pi i \widetilde{H}_k(\cdot)))$ is

$$\begin{aligned}
\nu_k &:= w(\det(\mathfrak{U}_k(t) \exp(2\pi i \widetilde{H}_k(t)))) \\
&= \sum_{s=1}^{\{k\}} (f_s^k(1) + \tilde{g}_s^k(1)) - \sum_{s=1}^{\{k\}} (f_s^k(0) + \tilde{g}_s^k(0)) \\
&= \sum_{s=1}^{\{k\}} (f_s^k(1) - f_s^k(0)) + \sum_{s=1}^{\{k\}} (\tilde{g}_s^k(1) - \tilde{g}_s^k(0)) \\
&= \sum_{s=1}^{\{k\}} (z_s^k + m_s^k) - \sum_{l=1}^{L} \sum_{j=1}^{[l]} b_{k,l}^0 \lambda_j^l + \sum_{s=1}^{\{k\}} \vartheta_s^k - \sum_{s=1}^{\{k\}} \theta_s^k \\
&= \sum_{l=1}^{L} \sum_{j=1}^{[l]} b_{k,l}^1 \lambda_j^l - \sum_{l=1}^{L} \sum_{j=1}^{[l]} b_{k,l}^0 \lambda_j^l + \sum_{s=1}^{\{k\}} m_s^k + \sum_{l=1}^{L} \sum_{j=1}^{[l]} b_{k,l}^1 (-\lambda_j^l) \\
&\quad - \sum_{l=1}^{L} \sum_{j=1}^{[l]} b_{k,l}^0 (-\lambda_j^l) \\
&= \sum_{s=1}^{\{k\}} m_i^k.
\end{aligned}$$

Hence, by using Remark 3.4 and (4.24), we get

$$(\nu_1, \dots, \nu_K) \in \operatorname{Im}(K_0(\beta_1) - K_0(\beta_0)) (\subset \mathbb{Z}^K). \quad (4.25)$$

That is, there exists an element $(\mu_1, \dots, \mu_L) \in \mathbb{Z}^L$ such that

$$\sum_{l=1}^{L} \mu_l \cdot b_{k,l} = \nu_k, \quad k = 1, \dots, K. \quad (4.26)$$

Now, for each $l = 1, \dots, L$, defining

$$\eta_j^l := \begin{cases} \lambda_1^l + \mu_l & \text{if } j = 1, \\ \lambda_j^l & \text{if } j = 2, \dots, [l]. \end{cases} \quad (4.27)$$

We see at once the matrix $h_l := \hat{u}_l \cdot \mathrm{diag}(\eta_1^l, \ldots, \eta_{[l]}^l) \cdot \hat{u}_l^*$ is an element in $(\mathrm{M}_{[l]})_{s.a.}$ with $\exp(2\pi i h_l) = u_l$, and hence write $h := (h_1, \ldots, h_L) \in F_{s.a.}$. Then, for each $k = 1, \ldots, K$, both $\Pi_k(\beta_0(h))$ and $\Pi_k(\beta_1(h))$ are elements in $(\mathrm{M}_{\{k\}})_{s.a.}$ satisfying $\exp(2\pi i \cdot \Pi_k(\beta_0(h))) = \mathfrak{U}_k(0)$ and $\exp(2\pi i \cdot \Pi_k(\beta_1(h))) = \mathfrak{U}_k(1)$, where Π_k is the quotient map from $E = \mathrm{M}_{\{1\}} \oplus \cdots \oplus \mathrm{M}_{\{K\}}$ to $\mathrm{M}_{\{k\}}$. Evidently,

$$\mathrm{sp}(\Pi_k(\beta_0(h))) = \{(\eta_j^l)^{\sim b_{k,l}^0} : l = 1, \ldots, L;\ j = 1, \ldots, [l]\},$$
$$\mathrm{sp}(\Pi_k(\beta_1(h))) = \{(\eta_j^l)^{\sim b_{k,l}^1} : l = 1, \ldots, L;\ j = 1, \ldots, [l]\}.$$

For convenience, we renumber the elements in $\mathrm{sp}(\Pi_k(\beta_0(h)))$and $\mathrm{sp}(\Pi_k(\beta_1(h)))$, written as $\{x_1^k, \ldots, x_{\{k\}}^k\}$ and $\{y_1^k, \ldots, y_{\{k\}}^k\}$, respectively. It follows from Lemma 4.1 that we can assume

$$\left\{f_s^k(0) - x_s^k, f_s^k(1) - y_s^k : k = 1, \ldots, K;\ s = 1, \ldots, \{k\}\right\} \subset \mathbb{Z}, \tag{4.28}$$

and hence, $\sum_{s=1}^{\{k\}}(f_s^k(0) - x_s^k)$ and $\sum_{s=1}^{\{k\}}(f_s^k(1) - y_s^k)$ are two integers.

Claim. We have
$$\sum_{s=1}^{\{k\}}(f_s^k(1) - y_s^k) = \sum_{s=1}^{\{k\}}(f_s^k(0) - x_s^k) \in \mathbb{Z}.$$

Note that

$$\sum_{s=1}^{\{k\}} x_s^k = \sum_{l=1}^{L}\sum_{j=1}^{[l]} \eta_j^l b_{k,l}^0 \overset{(4.27)}{=\!=} \sum_{l=1}^{L}\sum_{j=1}^{[l]} \lambda_j^l b_{k,l}^0 + \sum_{l=1}^{L} \mu_l b_{k,l}^0,$$
$$\sum_{s=1}^{\{k\}} y_s^k = \sum_{l=1}^{L}\sum_{j=1}^{[l]} \eta_j^l b_{k,l}^1 \overset{(4.27)}{=\!=} \sum_{l=1}^{L}\sum_{j=1}^{[l]} \lambda_j^l b_{k,l}^1 + \sum_{l=1}^{L} \mu_l b_{k,l}^1,$$

thus,

$$\sum_{s=1}^{\{k\}}(y_s^k - x_s^k) = \sum_{l=1}^{L}\sum_{j=1}^{[l]} \lambda_j^l \cdot b_{k,l} + \sum_{l=1}^{L} \mu_l \cdot b_{k,l} \overset{(4.26)}{=\!=} \sum_{s=1}^{\{k\}}(f_s^k(1) - f_s^k(0)).$$

That is,
$$\sum_{s=1}^{\{k\}}(f_s^k(1) - y_s^k) = \sum_{s=1}^{\{k\}}(f_s^k(0) - x_s^k) \in \mathbb{Z}, \tag{4.29}$$

which proves the claim.

For each $k = 1, \ldots, K$, set $a_k := \sum_{s=1}^{\{k\}}(f_s^k(1) - y_s^k) = \sum_{s=1}^{\{k\}}(f_s^k(0) - x_s^k) \in \mathbb{Z}$. For any $s = 1, \ldots, (\{k\} - 1)$, we can arbitrarily choose the real-valued function g_s^k on the closed interval $[0, 1]$ only satisfying

$$g_s^k(0) = x_s^k \quad \text{and} \quad g_s^k(1) = y_s^k.$$

For the case $s = \{k\}$, let

$$g_{\{k\}}^k(t) := \sum_{i=1}^{\{k\}} f_s^k(t) - \sum_{s=1}^{\{k\}-1} g_s^k(t) - a_k.$$

For the $\{k\}$ functions $g_1^k, \dots, g_{\{k\}}^k \in C([0,1])_{s.a.}$, it is evident that

(1) $g_{\{k\}}^k(0) = x_{\{k\}}^k$ and $g_{\{k\}}^k(1) = y_{\{k\}}^k$,

(2) $\sum_{s=1}^{\{k\}} g_s^k(t) = \sum_{s=1}^{\{k\}} f_s^k(t) - a_k \; \forall t \in [0,1]$.

Now, we can define $G_k := \mathfrak{u}_k \cdot \mathrm{diag}(g_1^k, \dots, g_{\{k\}}^k) \cdot \mathfrak{u}_k^*$ for any $k = 1, \dots, K$. It follows from 1 that $G := (G_1, \dots, G_K; h_1, \dots, h_L) \in A_{s.a.}$ and $\det(\exp(2\pi i G(t))) = \det(\mathfrak{U}(t)) = \det(U(t))$, for any $t \in \mathrm{Sp}(A)$, which is clear from 2. Therefore, set $V = \exp(2\pi i G)$, as desired. ∎

Remark 4.10. If we write $V = (V_1, \dots, V_K; v_1, \dots, v_L)$, then $v_l = u_l$ for all $l = 1, \dots, L$ and $V_k(0) = U_k(0)$, $V_k(1) = U_k(1)$ for all $k = 1, \dots, K$, which are clear from the process of the proof.

5. Elliott–GLN construction

It has been showed in [15, Theorem 13.50] for any weakly unperforated Elliott invariant $((G, G_+, u), K, \triangle, r)$, there is a unital simple C^*-algebra $A = \varinjlim(A_n, \xi_{n,n+1})$ in the class $\mathcal{N}_0^{\mathcal{Z}}$ such that

$$\mathrm{Ell}(A) \cong ((G, G_+, u), K, \triangle, r).$$

We will modify the construction—mainly we will modify the definition of $*$-homomorphism $\xi_{n,n+1}$, now we will call it $\ell_{n,n+1}$. Therefore, it is necessary for us to recall the part concerning the treatment of groups in the Elliott–GLN construction, as well as to introduce the construction of C^*-algebras A_n.

Now, fix a weakly unperforated Elliott invariant $((G, G_+, u), K, \triangle, r)$. It is important to point out that we do *not* make the two assumptions that G is torsion free or K is a trivial group.

Fact 5.1. Below, we will present the construction on the level of groups in Elliott–GLN construction.

Step 1. [15, 13.9]. Let $G^1 \subset \mathrm{Aff}(\triangle)$ be a countable dense subgroup with at least three $\mathbb{Q}$-linearly independent elements and let $H = G \oplus G^1$. Define

$$H_+ := \{(g, f) \in G \oplus G^1 : r(\tau)(g) + f(\tau) > 0 \; \forall \tau \in \triangle\} \cup \{(0,0)\}.$$

And, the order unit $u \in G$ when regarded as $(u, 0) \in G \oplus G^1 = H$ is also an order unit of H_+. Note that $\mathrm{Tor}(H) = \mathrm{Tor}(G)$, and G is a relatively divisible subgroup of H, that is, if $g \in G, h \in H$ and $m \in \mathbb{Z} \backslash \{0\}$ such that $g = mh$, then there is $g' \in G$ such that $g = mg'$.

Step 2. It was shown that $H/\operatorname{Tor}(H)$ is a simple dimension group [15, 13.9]. Hence, we can obtain the commutative diagram of groups [15, Lemma 13.24]:

$$\begin{tikzcd}
D_1 \arrow[r, hook, "\gamma_{1,2}|_{D_1}"] \arrow[d, hook] & D_2 \arrow[r, hook] \arrow[d, hook] & \cdots \arrow[r, hook] & \operatorname{Tor}(H) \arrow[d, hook] \\
H_1 \arrow[r, "\gamma_{1,2}"] \arrow[d] & H_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & H \arrow[d] \\
H_1' \arrow[r, "\gamma_{1,2}'"] & H_2' \arrow[r] & \cdots \arrow[r] & H/\operatorname{Tor}(H),
\end{tikzcd}$$

where $H_n = D_n \oplus H_n'$ is a finitely generated Abelian group with $D_n = \operatorname{Tor}(H_n)$. Write $H_n' = \mathbb{Z}^{p_n} = H_n/\operatorname{Tor}(H_n)(p_n \geqslant 3)$ and define

$$(H_n)_+ := ((\mathbb{Z}_+\backslash\{0\}) \oplus \operatorname{Tor}(H_n) \cup (0,0)) \oplus \mathbb{Z}_+^{p_n-1},$$

where $\mathbb{Z}_+$ is the set of natural numbers $\{0, 1, 2, \dots\}$, then $\gamma_{n,n+1}$ is a positive group homomorphism. Moreover, we can guarantee that H_n has an order unit $u_n \in (H_n)_+$ such that $\gamma_{n,\infty}(u_n) = (u, 0)$ for all positive integers n. That is, $\gamma_{n,n+1} : (H_n, (H_n)_+, u_n) \to (H_{n+1}, (H_{n+1})_+, u_{n+1})$ (see Definition 2.11).

Step 3. From Step 2, we can obtain the commutative diagram of groups [15, 13.26]:

$$\begin{tikzcd}
G_1 \arrow[r, "\gamma_{1,2}|_{G_1}"] \arrow[d, hook] & G_2 \arrow[r] \arrow[d, hook] & \cdots \arrow[r] & G \arrow[d, hook] \\
H_1 \arrow[r, "\gamma_{1,2}"] \arrow[d] & H_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & H \arrow[d] \\
H_1/G_1 \arrow[r, "\tilde{\gamma}_{1,2}"] & H_2/G_2 \arrow[r] & \cdots \arrow[r] & H/G,
\end{tikzcd}$$

where $G_n := H_n \cap \gamma_{n,\infty}^{-1}(G)$ is a relatively divisible subgroup of H_n and $(G_n)_+ := (H_n)_+ \cap G_n$.

Note that $u \in G \subset H$, thus both H_n and G_n share the same order unit u_n. Because G_n is a relatively divisible subgroup of H_n, hence, H_n/G_n is a finitely generated free Abelian group, written as $H_n/G_n := \mathbb{Z}^{l_n}$. Obviously, the conclusion that $\operatorname{Tor}(G_n) = \operatorname{Tor}(H_n) = D_n \subset G_n$ holds, so, the quotient map $H_n \to H_n/G_n$ induces the map

$$\pi_n : H_n/\operatorname{Tor}(H_n) = \mathbb{Z}^{p_n} \to H_n/G_n = \mathbb{Z}^{l_n}.$$

Step 4. (see the end of [15, 13.26]) The group K is written as the inductive limit of finitely generated Abelian groups, as follows:

$$K_1 \xrightarrow{\chi_{1,2}} K_2 \xrightarrow{\chi_{2,3}} \cdots \to K_n \xrightarrow{\chi_{n,n+1}} \cdots \to K.$$

To avoid ambiguity, we replace the subscript "n" with the subscript "$\star$".

Fact 5.2. Let $((H_\star, (H_\star)_+, u_\star, G_\star), K_\star)$ be a 5-tuple of groups satisfying

(i) $H_\star$ is a finitely generated Abelian group, written as $H_\star = \mathrm{Tor}(H_\star) \oplus \mathbb{Z}^{p_\star} = \oplus_{e=1}^{p_\star} H_\star^e$, where $p_\star$ is a positive integer greater than or equal to 3 depending on the subscript "$\star$", and

$$H_\star^e = \begin{cases} \mathbb{Z} \oplus \mathrm{Tor}(H_\star) & \text{if } e = 1, \\ \mathbb{Z} & \text{if } e = 2, \dots, p_\star; \end{cases}$$

(ii) $G_\star$ is a proper relatively divisible subgroup of subgroup of $H_\star$ with $\mathrm{Tor}(H_\star) \subset G_\star$. Consequently, $\mathrm{Tor}(G_\star) = \mathrm{Tor}(H_\star)$, and $H_\star / G_\star$ is a finitely generated free Abelian group, written as $\mathbb{Z}^{l_\star}$;

(iii) set $(H_\star)_+ := ((\mathbb{Z}_+ \backslash \{0\}) \oplus \mathrm{Tor}(H_\star) \cup (0,0)) \oplus \mathbb{Z}_+^{p_\star - 1}$, and define $(G_\star)_+ := G_\star \cap (H_\star)_+$;

(iv) both $H_\star$ and $G_\star$ share the same order unit $u_\star \in (H_\star)_+ \backslash \{0\}$;

(v) $K_\star$ is a finitely generated Abelian group.

Now, we will construct a separable C^*-algebra $A_\star$ such that

$$((K_0(A_\star), K_0(A_\star)_+, [\mathbf{1}_{A_\star}]), K_1(A_\star)) \cong ((G_\star, (G_\star)_+, u_\star), K_\star).$$

Fact 5.3. *Step 1*. Write

$$u_\star = (([\star, 1], \tau_\star), [\star, 2], \dots, [\star, p_*]) \in H_\star = \bigoplus_{e=1}^{p_\star} H_\star^e,$$

where $\tau_\star$ is an element in $\mathrm{Tor}(H_\star) \subset H_\star^1$, and for each $e = 1, \dots, p_\star$, the element $[\star, e]$ is a positive integer depending on the indices "$\star$" and "e". Thus, we can obtain a finite dimensional C^*-algebra

$$F_\star := \mathrm{M}_{[\star,1]} \oplus \cdots \oplus \mathrm{M}_{[\star, p_\star]}.$$

Step 2. Because $\mathrm{Tor}(H_\star) \subset G_\star$, $H_\star \to H_\star / G_\star$ induces a quotient $\pi_\star : H_\star / \mathrm{Tor}(H_\star) =: \mathbb{Z}^{p_\star} \to H_\star / G_\star =: \mathbb{Z}^{l_\star}$. Write $\pi_\star = (b_{j,e}^\star)$ (see Notation 2.4).

Step 3. Now, choose two $l_\star \times p_\star$ matrices $(b_{j,e}^{\star,0})$, $(b_{j,e}^{\star,1})$ with entries in positive integers satisfying $(b_{j,e}^{\star,1}) - (b_{j,e}^{\star,0}) = (b_{j,e}^\star)$. Note $u_\star \in G_\star$. Thus, $\pi_\star([\star, 1], [\star, 2], \dots, [\star, p_*]) = 0$. Consequently, $\sum_{e=1}^{p_\star} b_{j,e}^{\star,0} \cdot [\star, e] = \sum_{e=1}^{p_\star} b_{j,e}^{\star,1} \cdot [\star, e]$ for any $j = 1, \dots, l_\star$. Set

$$\{\star, j\} := \sum_{e=1}^{p_\star} b_{j,e}^{\star,0} \cdot [\star, e] = \sum_{e=1}^{p_\star} b_{j,e}^{\star,1} \cdot [\star, e], \quad j = 1, \dots, l_\star,$$

and define a finite dimensional C^*-algebra

$$E_\star := \mathrm{M}_{\{\star,1\}} \oplus \cdots \oplus \mathrm{M}_{\{\star, l_\star\}}.$$

Step 4. Furthermore, we can choose two unital $*$-homomorphisms $\beta_{\star,0}, \beta_{\star,1} : F_\star \to E_\star$ such that $K_0(\beta_{\star,0}) = (b_{j,e}^{\star,0})$ and $K_0(\beta_{\star,1}) = (b_{j,e}^{\star,1})$, respectively.

Notice: We have freedom to choose a sufficiently large positive integer M, and replace $b_{j,e}^{\star,0}, b_{j,e}^{\star,1}$ by $b_{j,e}^{\star,0} + M, b_{j,e}^{\star,1} + M$, respectively. Note that the number

$$(b_{j,e}^{\star,1} + M) - (b_{j,e}^{\star,0} + M) = (b_{j,e}^{\star,1}) - (b_{j,e}^{\star,0}) = (b_{j,e}^{\star}).$$

That is, we can guarantee that

$$\min\{b_{j,e}^{\star,0}, b_{j,e}^{\star,1} : j = 1, \dots, l_\star; e = 1, \dots, p_\star\}$$

is large enough. In addition, both $\beta_{\star,0}$ and $\beta_{\star,1}$ could have good forms (see Remark 3.6).

Step 5. Define

$$A_{C,\star} := C([0,1], E_\star) \oplus_{\beta_{\star,0},\beta_{\star,1}} F_\star \in \mathcal{C}_0.$$

Step 6. Fix a topological space $X_\star$ with the base point $\hat{\mathrm{x}}_\star$, which is a connected finite CW complex with dimension three and the cohomology groups

$$H^2(X_\star) = \mathrm{Tor}(H_\star), \quad H^3(X_\star) = \mathrm{Tor}(K_\star) \quad \text{and} \quad H^1(X_\star) \oplus H^3(X_\star) = K_\star.$$

(For the concrete form of $X_\star$, readers can refer to [15, 13.27].)

Step 7. Fix a projection $P_\star$ in $\mathrm{M}_\infty(C(X_\star))$ with $\mathrm{rank}(P_\star) = [\star, 1]$ and $ch_1(P_\star) = \tau_\star \in H^2(X_\star) = \mathrm{Tor}(H_\star)$. It is natural to obtain a C^*-algebra $P_\star \mathrm{M}_\infty(C(X_\star))P_\star$. And define a C^*-algebra

$$\begin{aligned}&(P_\star \mathrm{M}_\infty(C(X_\star))P_\star)|_{\hat{\mathrm{x}}_\star}\\ &\quad := \{a \in P_\star(\hat{\mathrm{x}}_\star)\mathrm{M}_\infty P_\star(\hat{\mathrm{x}}_\star) : \exists \mathfrak{a} \in P_\star \mathrm{M}_\infty(C(X_\star))P_\star \text{ such that } \mathfrak{a}(\hat{\mathrm{x}}_\star) = a\}.\end{aligned}$$

Because $\mathrm{rank}(P_\star) = [\star, 1]$, $(P_\star \mathrm{M}_\infty(C(X_\star))P_\star)|_{\hat{\mathrm{x}}_\star} \cong \mathrm{M}_{[\star,1]}$. Fix a $*$-isomorphism $\mathrm{j}_\star$ from $(P_\star \mathrm{M}_\infty(C(X_\star))P_\star)|_{\hat{\mathrm{x}}_\star}$ to $\mathrm{M}_{[\star,1]}$.

Step 8. Define

$$A_{X,\star}^e := \begin{cases} P_\star \mathrm{M}_\infty(C(X_\star))P_\star & \text{if } e = 1,\\ \mathrm{M}_{[\star,e]} & \text{if } e = 2, \dots, p_\star \end{cases}$$

and $A_{X,\star} := A_{X,\star}^1 \oplus A_{X,\star}^2 \oplus \cdots \oplus A_{X,\star}^{p_\star}$.

Step 9. Define

$$\begin{aligned}A_\star := \{&(\bar{f}; \mathfrak{a}, a_2, \dots, a_{p_\star}) \in C([0,1], E_\star) \oplus A_{X,\star}^1 \oplus A_{X,\star}^2 \oplus \cdots \oplus A_{X,\star}^{p_\star} :\\ &\bar{f}(0) = \beta_{\star,0}((\mathrm{j}_\star(\mathfrak{a}(\hat{\mathrm{x}}_\star)), a_2, \dots, a_{p_\star})), \bar{f}(1) = \beta_{\star,1}((\mathrm{j}_\star(\mathfrak{a}(\hat{\mathrm{x}}_\star)), a_2, \dots, a_{p_\star}))\}.\end{aligned}$$

Fact 5.4. In addition, for later convenience, we will introduce more symbols and facts.

(1) Let us assume that

$$
\begin{aligned}
I_\star &:= \big\{(\bar{f};\mathfrak{a},a_2,\ldots,a_{p_\star}) \in A_\star : \mathfrak{a}=0 \text{ and } a_e=0,\ e=2,\ldots,p_\star\big\},\\
J_\star &:= \big\{(\bar{f};\mathfrak{a},a_2,\ldots,a_{p_\star}) \in A_\star : \bar{f}=0, \mathfrak{a}(\hat{\mathrm{x}}_\star)=0 \text{ and } a_e=0,\ e=2,\ldots,p_\star\big\},\\
I_\star + J_\star &:= \big\{(\bar{f};\mathfrak{a},a_2,\ldots,a_{p_\star}) \in A_\star : \mathfrak{a}(\hat{\mathrm{x}}_\star)=0 \text{ and } a_e=0,\ e=2,\ldots,p_\star\big\}.
\end{aligned}
$$

We see at once that $I_\star$, $J_\star$ and $I_\star + J_\star$ are (closed two-side) ideals of $A_\star$, and

$$
\begin{aligned}
A_{C,\star} &= A_\star/J_\star = C([0,1],E_\star) \oplus_{\beta_{\star,0},\beta_{\star,1}} F_\star \in \mathcal{C}_0,\\
A_{X,\star} &= A_\star/I_\star = \bigoplus_{e=1}^{p_\star} A^e_{X,\star},\\
F_\star &= A_\star/(I_\star + J_\star).
\end{aligned}
$$

Denote by $\Pi_{J,\star} : A_\star \to A_{C,\star}$, $\Pi_{I,\star} : A_\star \to A_{X,\star}$ and $\Pi^q_\star : A_\star \to F_\star$ the corresponding quotient maps.

Note that

$$
\begin{aligned}
&((K_0(A_{X,\star}), K_0(A_{X,\star})_+, [\mathbf{1}_{A_{X,\star}}]), K_1(A_{X,\star}))\\
&\quad = ((H_\star, (H_\star)_+, u_\star), K_\star),\\
&\big((K_0(A^1_{X,\star}), K_0(A^1_{X,\star})_+, [\mathbf{1}_{A^1_{X,\star}}]), K_1(A^1_{X,\star})\big)\\
&\quad = ((\mathbb{Z} \oplus \mathrm{Tor}(H_\star), (\mathbb{Z}_+\backslash\{0\}) \oplus \mathrm{Tor}(H_\star) \cup \{(0,0)\}, ([\star,1],\tau_\star)), K_\star).
\end{aligned}
$$

(2) Set

$$
\begin{aligned}
\overline{I}_\star &= \{(\bar{f};a) \in A_{\mathrm{C},\star} : a = 0\};\\
\overline{J}_\star &= \{(\mathfrak{a},a_2,\ldots,a_{p_\star}) \in A_{X,\star} : \mathfrak{a}(\hat{\mathrm{x}}_\star)=0 \text{ and } a_e=0,\ e=2,\ldots,p_\star\}.
\end{aligned}
$$

Note that $\overline{I}_\star$ and $\overline{J}_\star$ are ideal for $A_{C,\star}$ and $A_{X,\star}$, respectively, and satisfy $A_{C,\star}/\overline{I}_\star = A_{X,\star}/\overline{J}_\star = F_\star$. Denote by $\Pi^q_{C,\star} : A_{C,\star} \to F_\star$, $\Pi^q_{X,\star} : A_{X,\star} \to F_\star$ the corresponding quotient maps.

(3) The quotient maps involved above satisfy the following commutative diagram:

$$
\begin{tikzcd}
A_\star \arrow[r, "\Pi_{J,\star}"] \arrow[d, "\Pi_{I,\star}"'] \arrow[dr, "\Pi^q_\star"] & A_{C,\star} \arrow[d, "\Pi^q_{C,\star}"] \\
A_{X,\star} \arrow[r, "\Pi^q_{X,\star}"'] & F_\star.
\end{tikzcd}
\tag{5.1}
$$

(4) It is clear that

$$\mathrm{Sp}(I_\star) = \bigsqcup_{j=1}^{l_\star} (0,1)_{\star,j}, \quad \mathrm{Sp}(J_\star) = X_\star \backslash \{\hat{\mathrm{x}}_\star\},$$

$$\mathrm{Sp}(A_{C,\star}) = \mathrm{Sp}(I_\star) \sqcup \mathrm{Sp}(F_\star), \quad \mathrm{Sp}(A_{X,\star}) = \mathrm{Sp}(J_\star) \sqcup \mathrm{Sp}(F_\star),$$

$$\mathrm{Sp}(F_\star) = \{\theta_{\star,1}, \dots, \theta_{\star,p_\star}\} = \mathrm{Sp}(A_{X,\star}) \cap \mathrm{Sp}(A_{C,\star}),$$

$$\mathrm{Sp}(A_\star) = \mathrm{Sp}(I_\star) \sqcup \mathrm{Sp}(A_{X,\star}) = \mathrm{Sp}(J_\star) \sqcup \mathrm{Sp}(A_{C,\star}) = \mathrm{Sp}(A_{X,\star}) \cup \mathrm{Sp}(A_{C,\star}).$$

We can also see that $\mathrm{Sp}(A_{X,\star}) = X_\star \cup \mathrm{Sp}(F_\star)$, with $\theta_{\star,1} \in \mathrm{Sp}(F_\star)$ identified with $\hat{\mathrm{x}}_\star \in X_\star = \mathrm{Sp}(A^1_{X,\star})$.

(5) In general, $0_{\star,j}, 1_{\star,j} \notin \mathrm{Sp}(A_{C,\star})$, but belong to the set $\mathrm{RF}(A_{C,\star})$ (see Notation 2.22), that is, both $0_{\star,j}$ and $1_{\star,j}$ correspond to a finite-dimensional representation of $A_{C,\star}$ (also, of $A_\star$), respectively. Precisely,

$$0_{\star,j} = \left\{\theta_{\star,1}^{\sim b_{j,1}^{\star,0}}, \dots, \theta_{\star,p_\star}^{\sim b_{j,p_\star}^{\star,0}}\right\} \quad \text{and} \quad 1_{\star,j} = \left\{\theta_{\star,1}^{\sim b_{j,1}^{\star,1}}, \dots, \theta_{\star,p_\star}^{\sim b_{j,p_\star}^{\star,1}}\right\} \tag{5.2}$$

for any $j = 1, \dots, l_\star$.

(6) Note that

$$\mathrm{Aff}(T(F_\star)) = \mathbb{R}^{p_\star} \quad \text{and} \quad \mathrm{Aff}(T(E_\star)) = \mathbb{R}^{l_\star}.$$

As $K_0(\beta_{\star,0}) = (b_{j,e}^{\star,0})$ and $K_0(\beta_{\star,1}) = (b_{j,e}^{\star,1})$, hence,

$$\beta^\sharp_{\star,0} : \mathbb{R}^{p_\star} \to \mathbb{R}^{l_\star},$$
$$\mathrm{w}_\star = (\mathrm{w}_{\star,1}, \dots, \mathrm{w}_{\star,p_\star}) \mapsto (\mathrm{z}^0_{\star,1}, \dots, \mathrm{z}^0_{\star,l_\star})$$

and

$$\beta^\sharp_{\star,1} : \mathbb{R}^{p_\star} \to \mathbb{R}^{l_\star},$$
$$\mathrm{w}_\star = (\mathrm{w}_{\star,1}, \dots, \mathrm{w}_{\star,p_\star}) \mapsto (\mathrm{z}^1_{\star,1}, \dots, \mathrm{z}^1_{\star,l_\star}),$$

where for any $j = 1, \dots, l_\star$,

$$\mathrm{z}^0_{\star,j} = \frac{1}{\{\star, j\}} \sum_{e=1}^{p_\star} b_{j,e}^{\star,0}[\star, e] \cdot \mathrm{w}_{\star,e} \quad \text{and} \quad \mathrm{z}^1_{\star,j} = \frac{1}{\{\star, j\}} \sum_{e=1}^{p_\star} b_{j,e}^{\star,1}[\star, e] \cdot \mathrm{w}_{\star,e}.$$

(7) For any $j = 1, \dots, l_\star$, let $\Pi_{\star,j}$ be the quotient map from $E_\star = \mathrm{M}_{\{\star,1\}} \oplus \cdots \oplus \mathrm{M}_{\{\star,l_\star\}}$ to $\mathrm{M}_{\{\star,j\}}$. Evidently, the induced map $\Pi^\sharp_{\star,j}$ is just the j-th coordinate projection onto $\mathbb{R}^{l_\star}$.

(8) Obviously,

$$\mathrm{Aff}(T(A_{X,\star})) = C(X_\star, \mathbb{R}) \oplus \mathbb{R}^{p_\star - 1}$$

and

$$\begin{aligned}\mathrm{Aff}(T(A_\star)) = \Bigg\{ &(G_{\star,1}, \dots, G_{\star,l_\star}; \mathrm{g}_\star, g_{\star,2}, \dots, g_{\star,p_\star}) \\ &\in \bigoplus_{j=1}^{l_\star} C([0,1], \mathbb{R}_j) \oplus C(X_\star, \mathbb{R}) \oplus \mathbb{R}^{p_\star - 1} : \\ &G_{\star,j}(0) = \Pi^\sharp_{\star,j}(\beta^\sharp_{\star,0}(g_\star)), G_{\star,j}(1) = \Pi^\sharp_{\star,j}(\beta^\sharp_{\star,1}(g_\star)), j = 1, \dots, l_\star \Bigg\},\end{aligned}$$

by writing $g_{\star,1} := \mathrm{g}_\star(\hat{\mathrm{x}}_1) \in \mathbb{R}$ and $g_\star := (g_{\star,1}, \dots, g_{\star,p_\star}) \in \mathbb{R}^{p_\star}$, and where $\mathbb{R}_j = \mathbb{R}$ for all $j = 1, \dots, l_\star$.

6. Main theorem

Lemma 6.1. *Let X_1 be a connected finite CW complex with dimension at most three, with the base point $\hat{\mathrm{x}}_1$, such that the cohomology groups $H^2(X_1)$ and $H^3(X_1)$ are finite, and let $\{x_1, x_2, \dots\} \subset X_1 \backslash \{\hat{\mathrm{x}}_1\}$ be a countable dense subset in X_1. Suppose that P_1 is a projection in $\mathrm{M}_\infty(C(X_1))$, then for any finite set $\mathfrak{F} \subset U_0(P_1 \mathrm{M}_\infty(C(X_1)) P_1)$, $\varepsilon > 0$ there exists a natural number N such that we have the following.*

If X_2 is a connected finite CW complex with dimension at most three, with the base point $\hat{\mathrm{x}}_2$, such that the cohomology groups $H^2(X_2)$ and $H^3(X_2)$ are finite, and for any $\alpha \in KK(C(X_1), C(X_2))_+$ with $\mathrm{rank}(\alpha([\mathbf{1}_{C(X_1)}])) \geqslant N$, then there exist a projection $Q \in \mathrm{M}_\infty(C(X_2))$ with $\alpha([P_1]) = [Q]$ and a unital $*$-homomorphism $\psi : P_1 \mathrm{M}_\infty(C(X_1)) P_1 \to Q \mathrm{M}_\infty(C(X_2)) Q$ satisfying

(i) $KK(\psi) = \alpha$,

(ii) *for any $u \in \mathfrak{F}$, there exists a self-adjoint element $\mathfrak{h} \in Q \mathrm{M}_\infty(C(X_2)) Q$ such that*

$$\|\psi(u) - \exp(2\pi i \mathfrak{h})\| < \varepsilon.$$

Proof. It is obvious that for some positive integer l such that $u \in U_0(P_1 \mathrm{M}_l(C(X_1)) P_1)$ for any $u \in \mathfrak{F}$. Note that the following inductive sequence:

$$\mathrm{M}_l(C(X_1)) \xrightarrow{\phi_{1,2}} \mathrm{M}_{2!\cdot l}(C(X_1)) \to \cdots \mathrm{M}_{n!\cdot l}(C(X_1)) \xrightarrow{\phi_{n,n+1}} \mathrm{M}_{(n+1)!\cdot l}(C(X_1)) \cdots \to A,$$

where

$$\begin{aligned}\phi_{n,n+1} : \mathrm{M}_{n!\cdot l}(C(X_1)) &\to \mathrm{M}_{(n+1)!\cdot l}(C(X_1)) \\ f &\mapsto \mathrm{diag}(f(x), \underbrace{f(x_n), \dots, f(x_n)}_{n}).\end{aligned}$$

Let $A = \varinjlim(\mathrm{M}_{n!\cdot l}(C(X_1)), \phi_{n,n+1})$, according to [17] (or [18]), then A is real rank zero, hence, $\mathrm{cer}(A) \leqslant 1 + \varepsilon$ (see [23]). It means that for each $u \in \mathfrak{F}$ there are a positive integer

$n_0(u)$ and a self-adjoint element $h \in \phi_{1,n_0(u)}(P_1)\mathrm{M}_{n_0(u)!\cdot l}(C(X_1))\phi_{1,n_0(u)}(P_1)$ such that $\phi_{1,n_0(u)}(u) \in \phi_{1,n_0(u)}(P_1)\mathrm{M}_{n_0(u)!\cdot l}(C(X_1))\phi_{1,n_0(u)}(P_1)$, $\|\phi_{1,n_0(u)}(u) - \exp(2\pi i h)\| < \varepsilon$.

Now, set $n_0 = \max\{n_0(u) : u \in \mathfrak{F}\}$ and let $N = n_0! \cdot (13l)$.

(1) It follows from $\alpha \in KK(C(X_1), C(X_2))_+$ and $\mathrm{rank}(\alpha([\mathbf{1}_{C(X_1)}])) \geqslant N$ that there exists a $*$-homomorphism $\xi : C(X_1) \to \mathrm{M}_{13}(C(X_2))$ with $KK(\xi|_{C_0(X_1)}) = \alpha|_{C_0(X_1)}$. Now, we can consider $*$-homomorphisms $\xi \otimes \mathrm{id}_{n_0!\cdot l} : \mathrm{M}_{n_0!\cdot l}(C(X_1)) \to \mathrm{M}_N(C(X_2))$ and $(\xi \otimes \mathrm{id}_{n_0!\cdot l}) \circ \phi_{1,n_0} : \mathrm{M}_l(C(X_1)) \to \mathrm{M}_N(C(X_2))$, and obtain the conclusion $KK((\xi \otimes \mathrm{id}_{n_0!\cdot l}) \circ \phi_{1,n_0}) = \alpha$.

(2) Because $\alpha([\mathbf{1}_{C(X_1)}]) \in K_0(C(X_1))$, if write $N_0 = \mathrm{rank}(\alpha([\mathbf{1}_{C(X_1)}])) \geqslant N \geqslant n_0! \cdot 13$, then there exists a projection $Q_0 \in \mathrm{M}_\infty(C(X_2))$ with $\mathrm{rank}(Q_0) = N_0$. It is evident that we can guarantee that Q_0 has the form $Q_1 \oplus \mathbf{1}_{n_0!\cdot 13}$ from [19, Theorem 1.2 of Chapter 9]. Now, we can define a $*$-homomorphism

$$\begin{aligned} \eta : C(X_1) &\to Q_1\mathrm{M}_\infty(C(X_2))Q_1 \\ f &\mapsto f(\hat{\mathrm{x}}_1)Q_1. \end{aligned}$$

Set

$$Q = \mathrm{diag}(\mathbf{1}_N, \underbrace{Q_1, \ldots, Q_1}_{l}),$$

which is a projection in $\mathrm{M}_\infty(C(X_2))$. Let $\mathrm{rank}(Q_1) = N_1 := N_0 - n_0! \cdot 13$, then $\mathrm{rank}(Q) = \overline{N} = N + l \cdot N_1 = l \cdot N_0 \geqslant N$. If we consider the $*$-homomorphism $\Lambda := ((\xi \otimes \mathrm{id}_{n_0!\cdot l}) \circ \phi_{1,n_0}) \oplus (\eta \otimes \mathrm{id}_l) : \mathrm{M}_l(C(X_1)) \to Q\mathrm{M}_\infty(C(X_2))Q$ and let $\psi := \Lambda|_{P_1\mathrm{M}_l(C(X_1))P_1}$, combining the definition of η, it is easy to check that ψ satisfies the conditions appearing in this lemma, as desired. ∎

Notation 6.2. For convenience, we write

$$\mathrm{SP}(\psi|_{\hat{\mathrm{x}}_2}) = \left\{\hat{\mathrm{x}}_1^{\sim\hat{c}}, x_1^{\sim\varpi_1}, \ldots, x_M^{\sim\varpi_M}\right\}. \tag{6.1}$$

(See Notation 2.22 for the symbol $\mathrm{SP}(\psi|_{\hat{\mathrm{x}}_2})$, which is a set with multiplicities.)

Remark 6.3. According to [23,26], in fact, we can guarantee that the self-adjoint element $h \in \phi_{1,n_0}(P)\mathrm{M}_{n_0!\cdot l}(C(X_1))\phi_{1,n_0}(P)$ with finite spectrum. Hence, for the self-adjoint element $\mathfrak{h} \in Q\mathrm{M}_\infty(C(X_2))Q$, too. Accurately speaking, by [19, Theorem 1.2 of Chapter 9], $\mathfrak{h}$ can have the form

$$\mathfrak{h} = \lambda_1 \mathrm{q}_1 + \cdots + \lambda_{\overline{N}} \mathrm{q}_{\overline{N}},$$

where $\overline{N} = \mathrm{rank}(Q)$, $\lambda_1, \ldots, \lambda_{\overline{N}} \in [0, 1)$ and $\mathrm{q}_1, \ldots, \mathrm{q}_{\overline{N}}$ are mutually orthogonal rank one projections with $\mathrm{q}_1 + \cdots + \mathrm{q}_{\overline{N}} = Q$.

Now, we will begin to formally explore how to obtain a C^*-algebra A_2 and a $*$-homomorphism from A_1 to A_2 by a given C^*-algebra A_1 with the form in Fact 5.3.

Fact 6.4. Given two 5-tuples

$$((H_1,(H_1)_+,u_1,G_1),K_1) \quad \text{and} \quad ((H_2,(H_2)_+,u_2,G_2),K_2)$$

with the conditions in Fact 5.2 (with the index "$\star$" replaced by the indices "1" and "2", respectively. In the remainder of the article, in most cases, we will omit similar statement). On the one hand, recall that there exist a finite dimensional C^*-algebra $E_1 = \mathrm{M}_{\{1,1\}} \oplus \cdots \mathrm{M}_{\{1,l_1\}}$ and a C^*-algebra with the following form:

$$A_1 := \Big\{(\bar{f};\mathfrak{a},a_2,\ldots,a_{p_1}) \in C([0,1],E_1) \oplus P_1\mathrm{M}_\infty(C(X_1))P_1 \oplus \bigoplus_{e=2}^{p_1}\mathrm{M}_{[1,e]} : \\ \bar{f}(0) = \beta_{1,0}((\mathrm{j}_1(\mathfrak{a}(\hat{\mathrm{x}}_1)),a_2,\ldots,a_{p_1})),\ \bar{f}(1) = \beta_{1,1}((\mathrm{j}_1(\mathfrak{a}(\hat{\mathrm{x}}_1)),a_2,\ldots,a_{p_1}))\Big\} \tag{6.2}$$

such that

$$\mathrm{Ell}(A_1) \cong ((G_1,(G_1)_+,u_1),K_1).$$

Now, fix the C^*-algebra A_1 and set $K_0(\beta_{1,0}) = (b_{j,e}^{1,0})$, $K_0(\beta_{1,1}) = (b_{j,e}^{1,1})$. Hence,

$$\mathrm{Aff}(T(F_1)) = \mathbb{R}^{p_1} \quad \text{and} \quad \mathrm{Aff}(T(A_{X,1})) = C(X_1,\mathbb{R}) \oplus \mathbb{R}^{p_1-1}, \tag{6.3}$$

and

$$\mathrm{Aff}(T(A_1)) = \Big\{(G_{1,1},\ldots,G_{1,l_1};\mathrm{g}_1,g_{1,2},\ldots,g_{1,p_1}) \\ \in \bigoplus_{j=1}^{l_1} C([0,1],\mathbb{R}_j) \oplus C(X_1,\mathbb{R}) \oplus \mathbb{R}^{p_1-1} : \\ G_{1,j}(0) = \Pi_{1,j}^{\sharp}(\beta_{1,0}^{\sharp}(g_1)),\ G_{1,j}(1) = \Pi_{1,j}^{\sharp}(\beta_{1,1}^{\sharp}(g_1)),\ j = 1,\ldots,l_1\Big\}, \tag{6.4}$$

by writing $g_{1,1} := \mathrm{g}_1(\hat{\mathrm{x}}_1) \in \mathbb{R}$ and $g_1 := (g_{1,1},\ldots,g_{1,p_1}) \in \mathbb{R}^{p_1}$, and where $\mathbb{R}_j = \mathbb{R}$ for all $j = 1,\ldots,l_1$ (see (8) in Fact 5.4).

For convenience, let

$$\mathcal{B} := \max\{b_{j,e}^{1,0}, b_{j,e}^{1,1} : j = 1,\ldots,l_1, e = 1,\ldots,p_1\},$$
$$\mathcal{M} := \max\{[1,1],\ldots,[1,p_1]\}.$$

On the other hand, let $\gamma : (H_1,(H_1)_+,u_1) \to (H_2,(H_2)_+,u_2)$ be a group homomorphism with $\gamma(G_1) \subset G_2$ and χ be a group homomorphism from K_1 to K_2. Note that $\gamma(\mathrm{Tor}(H_1)) \subset \mathrm{Tor}(H_2)$. So, the map $\gamma : H_1 \to H_2$ induces a positive homomorphism

$$\gamma' = (c_{k,e}) : \mathbb{Z}^{p_1} := H_1/\mathrm{Tor}(H_1) \to \mathbb{Z}^{p_2} := H_2/\mathrm{Tor}(H_2).$$

Using (ii) in Fact 5.2 and the following commutative condition $\gamma(G_1) \subset G_2$, we naturally obtain a group homomorphism

$$\widetilde{\gamma} = (d_{s,j}) : \mathbb{Z}^{l_1} := H_1/G_1 \to \mathbb{Z}^{l_2} := H_2/G_2$$

and the commutative diagram

$$\begin{tikzcd} \mathbb{Z}^{p_1} \arrow[r, "\gamma'=(c_{k,e})"] \arrow[d, "\pi_1=(b^1_{j,e})"'] & \mathbb{Z}^{p_2} \arrow[d, "\pi_2=(b^2_{s,k})"] \\ \mathbb{Z}^{l_1} \arrow[r, "\widetilde{\gamma}=(d_{s,j})"'] & \mathbb{Z}^{l_2} \end{tikzcd} \tag{6.5}$$

Set

$$\mathfrak{C}_{1,2} := \min\{c_{k,e} : e = 1, \dots, p_1,\ k = 1, \dots, p_2\},$$
$$\mathfrak{B}_2 := \min\{b^{2,0}_{s,k}, b^{2,1}_{s,k} : k = 1, \dots, p_2,\ s = 1, \dots, l_2\}$$

and

$$\mathcal{D} := \max\{2, |d_{s,j}| : s = 1, \dots, l_2,\ j = 1, \dots, l_1\}.$$

Below, we will show that when $\mathfrak{C}_{1,2}$ and $\mathfrak{B}_2$ is sufficiently large, the construction can go through under our requirements.

Lemma 6.5. *Suppose γ and χ be given as Fact 6.4 and let A_1 be the given C^*-algebra (see (6.2)). If $\mathfrak{F}$ is an arbitrary non-empty finite subset of $U_0(A_1)$, then for any $\varepsilon > 0$ and $0 < \delta < 1$ there exist a positive integer T with $T > \frac{1}{\delta}$ and a finite set $\mathcal{Y} := \{y_1, \dots, y_M\} \subset X_1 \backslash \{\hat{x}_1\}$, which is δ-dense in X_1, satisfying the following property.*

If

$$\mathfrak{C}_{1,2} > \frac{2}{\varepsilon} \cdot \max\{13, N, c_0 + c'_{1,1}, c'_{1,2}, \dots, c'_{1,p_1}\}, \tag{6.6}$$

$$\mathfrak{B}_2 > \max\left\{L_1, \dots, L_{l_2}, \frac{2}{\varepsilon} \cdot \frac{\mathcal{B}\mathcal{D}\mathcal{M} p_1 l_1}{p_2}\right\}, \tag{6.7}$$

where

$$\varpi := \varpi_1 + \cdots + \varpi_M, \tag{6.8}$$

$$c_0 := \hat{c} + \varpi = \hat{c} + \varpi_1 + \cdots + \varpi_M, \tag{6.9}$$

$$c'_{1,e} := (T-1) \cdot \sum_{j=1}^{l_1} b^{1,0}_{j,e}, \quad e = 1, \dots, p_1, \tag{6.10}$$

$$L_s := \mathcal{B}\mathcal{D} l_1 \cdot \sum_{e=1}^{p_1} [1, e] + |b^2_{s,1}| \cdot \left(\varpi \cdot [1,1] + (T-1) \cdot \sum_{j=1}^{l_1} \{1, j\}\right), \quad s = 1, \dots, l_2, \tag{6.11}$$

and N, $\hat{c}$ *and* $\varpi_1, \ldots, \varpi_M$ *(see* (6.1)*), which are all positive integers, are obtained by using Lemma* 6.1*, then there are a* C^**-algebra* A_2 *with the form in Fact* 5.3 *and a* $*$*-homomorphism* $\ell : A_1 \to A_2$ *such that*

(i) $K_0(\ell) = \gamma|_{G_1}$, $K_1(\ell) = \chi$;

(ii) $\mathcal{Y} \cup \{(\frac{t}{T})_{1,j} : j = 1, \ldots, l_1,\ t = 1, \ldots, (T-1)\} \subset \mathrm{SP}(\ell|_{\theta_{2,1}})$;
(Note that $\theta_{2,1} = \hat{\mathrm{x}}_2 \in \mathrm{Sp}(A_2)$*.)*

(iii) $\mathrm{Sp}(F_1) = \{\theta_{1,1}, \ldots, \theta_{1,p_1}\} \subset \mathrm{SP}(\ell|_x)$ *holds for any* $x \in \mathrm{Sp}(A_2)$;

(iv) *let* Θ *be a* $*$*-homomorphism from* F_1 *to* F_2 *with* $K_0(\Theta) = \gamma' = (c_{k,e})$ *(see* (6.5)*), and* $\Pi_1^{q\sharp}$ *and* $\Pi_2^{q\sharp}$ *the affine maps induced by* $\Pi_1^q : A_1 \to F_1$ *and* $\Pi_2^q : A_2 \to F_2$*, respectively (See Notation* 2.16 *and* (5.1)*). Then,*

$$\left\|\Theta^\sharp(\Pi_1^{q\sharp}(\mathfrak{G}_1)) - \Pi_2^{q\sharp}(\ell^\sharp(\mathfrak{G}_1))\right\| < \varepsilon,$$

for all $\mathfrak{G}_1 \in \mathrm{Aff}(T(A_1))$ *(see* (6.4)*) with* $\|\mathfrak{G}_1\| \leqslant 1$;

(v) *there exist two contractive linear and order preserving maps* Γ_1 *from* $\mathrm{Aff}(T(F_1))$ *to* $\mathrm{Aff}(T(A_1))$ *and* Γ_2 *from* $\mathrm{Aff}(T(F_2))$ *to* $\mathrm{Aff}(T(A_2))$ *such that*

$$\left\|\Gamma_2(\Theta^\sharp(\mathrm{w}_1)) - \ell^\sharp(\Gamma_1(\mathrm{w}_1))\right\| < \varepsilon$$

for all $\mathrm{w}_1 = (\mathrm{w}_{1,1}, \ldots, \mathrm{w}_{1,p_1}) \in \mathrm{Aff}(F_1) = \mathbb{R}^{p_1}$ *with* $\|\mathrm{w}_1\| \leqslant 1$;

(vi) *for any* $U \in \mathfrak{F}$*, there is a self-adjoint element* $H \in A_2$ *such that* $\|\ell(U) - \exp(2\pi i H)\| < \varepsilon$.

(See Notation 2.23 *for the symbols both* $\ell|_{\theta_{2,1}}$ *and* $\ell|_x$*, and see also Notation* 2.22 *for the symbol* $\mathrm{SP}(\ell|_x)$*.)*

Proof. Set

$$\mathfrak{F}^1 := \{\Pi^1_{X,1}(U) : U \in \mathfrak{F}\},$$

where $\Pi^1_{X,1}$ is given by the composition of $\Pi_{I,1} : A_1 \to A_{X,1}$ (see (5.1)) and the quotient map $\Pi^1_X : A_{X,1} = \oplus_{e=1}^{p_1} A^e_{X,1} \to A^1_{X,1} = P_1 \mathrm{M}_\infty(C(X_1)) P_1$. Without loss of generality, we assume for any $U \in \mathfrak{F} \subset U_0(A_1)$, $\mathrm{sp}(U) = \mathbb{T}$. Thus, for an arbitrary $U = (U_1, \ldots, U_{l_1}; \mathfrak{u}, u_2, \ldots, u_{p_1}) \in \mathfrak{F}$ (with $\mathfrak{u} \in \mathfrak{F}^1$), we have

$$\left(\bigcup_{j=1}^{l_1} \mathrm{sp}(U_j)\right) \cup \mathrm{sp}(\mathfrak{u}) = \mathbb{T}. \tag{6.12}$$

Now, fix a sufficiently large positive integer n with

$$n > \frac{46\pi}{\varepsilon}. \tag{6.13}$$

For the integer n, we can define a set $\mathcal{R}_U := \{r \in \{1, \ldots, n\} : I_r^n \cap \mathrm{sp}(\mathfrak{u}) \neq \emptyset\} (\neq \emptyset)$. Thus, for any $r \in \mathcal{R}_U$ there exists $y_r \in X_1 \backslash \{\hat{\mathrm{x}}_1\}$ such that $I_r^n \cap \mathrm{sp}(\mathfrak{u}(y_r)) \neq \emptyset$. Now, fix one y_r

for each $r \in \mathcal{R}_U$ and define $\mathcal{Y}_1' := \bigcup_{U \in \mathfrak{F}} \{y_r \in X_1 \backslash \{\hat{x}_1\} : r \in \mathcal{R}_U\}$, which is a finite subset with multiplicities in $X_1 \backslash \{\hat{x}_1\}$. If $\mathcal{Y}_1'$ is already δ-dense in X_1, then let $\mathcal{Y}_1$ be an ordinary set corresponding to $\mathcal{Y}_1'$. Otherwise, add a finite number of elements in $X_1 \backslash \{\hat{x}_1\}$ to $\mathcal{Y}_1'$ such that the new $\mathcal{Y}_1'$ is δ-dense in X_1. Further, Let $\mathcal{Y}_1$ be an ordinary set corresponding to the new $\mathcal{Y}_1'$. Combining the assumption (6.12), it is clear that there is a positive integer T with $\frac{1}{T} < \delta$ such that for any $U = (U_1, \ldots, U_{l_1}; \mathfrak{u}, u_2, \ldots, u_{p_1}) \in \mathfrak{F} \subset U_0(A_1)$ and for any $r = 1, \ldots, n$, one of the following properties holds:

(a′) $I_r^n \cap \mathrm{sp}(U_i(\frac{t}{T})) \neq \emptyset$ for some pair $(t, j) \in \{1, \ldots, (T-1)\} \times \{1, \ldots, l_1\}$;

(b′) $I_r^n \cap \mathrm{sp}(\mathfrak{u}(y)) \neq \emptyset$ for some element $y \in \mathcal{Y}_1$.

Step 1. Construct the C^*-algebra A_2.

Recall γ is a positive group homomorphism from $H_1 = \oplus_{e=1}^{p_1} H_1^e$ to $H_2 = \oplus_{k=1}^{p_2} H_2^k$ and $H_1^1 = \mathbb{Z} \oplus \mathrm{Tor}(H_1)$ and $H_2^1 = \mathbb{Z} \oplus \mathrm{Tor}(H_2)$. Let

$$c_{1,1} : \mathbb{Z} \xrightarrow{\mathbf{i}_{1,1}^1} H_1^1 \xrightarrow{\mathbf{i}_{1,1}} H_1 \xrightarrow{\gamma} H_2 \xrightarrow{\bar{\pi}_{2,1}} H_2^1 \xrightarrow{\bar{\pi}_{2,1}^1} \mathbb{Z}$$

$$\mathbf{q}_{2,1} : \mathbb{Z} \xrightarrow{\mathbf{i}_{1,1}^1} H_1^1 \xrightarrow{\mathbf{i}_{1,1}} H_1 \xrightarrow{\gamma} H_2 \xrightarrow{\bar{\pi}_{2,1}} H_2^1 \xrightarrow{\bar{\pi}_{2,2}^1} \mathrm{Tor}(H_2)$$

$$\mathbf{q}_{2,2} : \mathrm{Tor}(H_1) \xrightarrow{\mathbf{i}_{1,2}^1} H_1^1 \xrightarrow{\mathbf{i}_{1,1}} H_1 \xrightarrow{\gamma} H_2 \xrightarrow{\bar{\pi}_{2,1}} H_2^1 \xrightarrow{\bar{\pi}_{2,2}^1} \mathrm{Tor}(H_2),$$

where $\mathbf{i}_{1,1}$, $\mathbf{i}_{1,1}^1$ and $\mathbf{i}_{1,2}^1$ are natural embedding, $\bar{\pi}_{2,1}$, $\bar{\pi}_{2,1}^1$ and $\bar{\pi}_{2,2}^1$ are quotient maps. Now, we get a group homomorphism, writing $\begin{pmatrix} c_{1,1} & 0 \\ \mathbf{q}_{2,1} & \mathbf{q}_{2,2} \end{pmatrix}$, from $H_1^1 = \mathbb{Z} \oplus \mathrm{Tor}(H_1)$ to $H_2^1 = \mathbb{Z} \oplus \mathrm{Tor}(H_2)$. Applying Lemma 6.1 to $\mathfrak{F}^1$ in place of $\mathfrak{F}$ and noting $\mathfrak{C}_{1,2} > N$, thus we can choose a topological space X_2 with the base point $\hat{x}_2$ and with the conditions in Lemma 6.1 such that

(1) $H^2(X_2) = \mathrm{Tor}(H_2)$, $H^3(X_2) = \mathrm{Tor}(K_2)$ and $H^1(X_2) \oplus H^3(X_2) = K_2$;

(2) there exist a projection $Q_1 \in \mathrm{M}_\infty(C(X_2))$ and a unital $*$-homomorphism ψ_1 from $P_1 \mathrm{M}_\infty(C(X_1)) P_1$ to $Q_1 \mathrm{M}_\infty(C(X_2)) Q_1$ satisfying

 (2-a) $K_0(\psi_1) = \begin{pmatrix} c_0 & 0 \\ \mathbf{q}_{2,1} & \mathbf{q}_{2,2} \end{pmatrix}$, $K_1(\psi_1) = \chi$;

 (2-b) for any $\mathfrak{u} \in \mathfrak{F}^1$, $\|\psi_1(\mathfrak{u}) - \exp(2\pi i \mathfrak{h}_1)\| < \frac{\varepsilon}{4}$ for some self-adjoint element $\mathfrak{h}_1 \in Q_1 \mathrm{M}_\infty(C(X_2)) Q_1$.

Let $\mathrm{SP}(\psi_1|_{\hat{x}_2}) = \{\hat{x}_1^{\sim \hat{c}'}, \bar{y}_1^{\sim \varpi_1'}, \ldots, \bar{y}_K^{\sim \varpi_K'}\}$, (see Notation 2.22), where $\{\bar{y}_1, \ldots, \bar{y}_K\} \subset X_1 \backslash \{\hat{x}_1\}$. Then, set

$$\mathcal{Y} := \mathcal{Y}_1 \cup \{\bar{y}_1, \ldots, \bar{y}_K\} = \{y_1, \ldots, y_M\} (\subset X_1 \backslash \{\hat{x}_1\}),$$

which is δ-dense in X_1. Obviously, for the above positive integer T and the set $\mathcal{Y}$, we still have for any $U = (U_1, \ldots, U_{l_1}; \mathfrak{u}, u_2, \ldots, u_{p_1}) \in \mathfrak{F} \subset U_0(A_1)$ and for any I_r^n, $r = 1, \ldots, n$, one of the following properties holds:

(a) $I_r^n \cap \mathrm{sp}(U_j(\frac{t}{T})) \neq \emptyset$ for some pair $(t, j) \in \{1, \ldots, (T-1)\} \times \{1, \ldots, l_1\}$,

(b) $I_r^n \cap \mathrm{sp}(\mathfrak{u}(y)) \neq \emptyset$ for some element $y \in \mathcal{Y}$.

Now, applying Lemma 6.1 to $\mathfrak{F}^1$ in place of $\mathfrak{F}$, we can choose another projection $Q \in \mathrm{M}_\infty(C(X_2))$ and a unital $*$-homomorphism

$$\psi : P_1\mathrm{M}_\infty(C(X_1))P_1 \to Q\mathrm{M}_\infty(C(X_2))Q \tag{6.14}$$

satisfying

(1) $\mathrm{SP}(\psi|_{\hat{x}_2}) = \{\hat{\mathrm{x}}_1^{\sim\hat{c}}, y_1^{\sim\varpi_1}, \ldots, y_M^{\sim\varpi_M}\}$ where all $\hat{c}, \varpi_1, \ldots, \varpi_M$ are positive integers;

(2) $K_0(\psi) = \begin{pmatrix} c_0 & 0 \\ \mathbf{q}_{2,1} & \mathbf{q}_{2,2} \end{pmatrix}$ and $K_1(\psi) = \chi$;

(3) for any $\mathfrak{u} \in \mathfrak{F}^1$, there is a self-adjoint element $\mathfrak{h} \in Q\mathrm{M}_\infty(C(X_2))Q$ such that

$$\|\psi(\mathfrak{u}) - \exp(2\pi i\mathfrak{h})\| < \frac{\varepsilon}{4}. \tag{6.15}$$

Now, recall (6.8) and (6.9), that is,

$$\varpi = \varpi_1 + \cdots + \varpi_M \quad \text{and} \quad c_0 = \hat{c} + \varpi = \hat{c} + \varpi_1 + \cdots + \varpi_M.$$

Consequently,

$$\mathrm{rank}(Q) = c_0 \cdot [1, 1], \tag{6.16}$$

and

$$\begin{aligned} \psi^\# : C(X_1, \mathbb{R}) &\to C(X_2, \mathbb{R}) \\ \mathrm{g}_1 &\mapsto \mathrm{g}_2, \end{aligned} \tag{6.17}$$

where

$$\mathrm{g}_2(\hat{x}_2) = \frac{1}{c_0}\left(\hat{c} \cdot \mathrm{g}_1(\hat{\mathrm{x}}_1) + \sum_{m=1}^{M} \varpi_m \cdot \mathrm{g}_1(y_m)\right).$$

It follows from the path-connectivity of the topological space X_1 that there exists a path $y_m(\omega)$ in X_1 such that $y_m(0) = \hat{\mathrm{x}}_1$, $y_m(1) = y_m$ for any $y_m \in \mathcal{Y}$. Hence, for any $\omega \in [0, 1]$ and $y_m \in \mathcal{Y}$, there is an $*$-isomorphism $\varrho_{m,\omega}$ from $P_1(y_m(\omega))\mathrm{M}_\infty(C(y_m(\omega)))$ $P_1(y_m(\omega))$ to $\mathrm{M}_{[1,1]}$. Moreover, we can guarantee that the $*$-isomorphism $\varrho_{m,\omega}$ satisfies

(1) $\varrho_{m,0} = \mathrm{j}_1$;

(2) the $*$-isomorphism $\varrho_{m,\omega}$ could be chosen to be continuously depending on ω by using the fact that the interval [0, 1] is contractible.

Consequently, for any $\mathfrak{a} \in P_1\mathrm{M}_\infty(C(X_1))P_1$, the function $[0, 1] \ni \omega \mapsto \varrho_{m,\omega}(\mathfrak{a}(y_m(\omega))) \in \mathrm{M}_{[1,1]}$ is a path in $\mathrm{M}_{[1,1]}$ with $\varrho_{m,0}(\mathfrak{a}(y_m(0))) = a_1$ and $\varrho_{m,1}(\mathfrak{a}(y_m(1))) = \varrho_{m,1}(\mathfrak{a}(y_m))$, where $a_1 := \mathrm{j}_1(\mathfrak{a}(\hat{\mathrm{x}}_1))$. That is, for any $y_m \in \mathcal{Y}$, we have

$$\mathrm{j}_1(\mathfrak{a}(\hat{\mathrm{x}}_1)) =: a_1 \sim_h \varrho_{m,1}(\mathfrak{a}(y_m)).$$

Define a unital $*$-homomorphism:

$$\begin{aligned} \Omega : P_1\mathrm{M}_\infty(C(X_1))P_1 &\to \mathrm{M}_{\varpi\cdot[1,1]}(\subset \mathrm{M}_{[2,1]}) \\ \mathfrak{a} &\mapsto \mathrm{diag}(\varrho_{1,1}(\mathfrak{a}(y_1)) \otimes \mathrm{I}_{\varpi_1}, \ldots, \varrho_{M,1}(\mathfrak{a}(y_M)) \otimes \mathrm{I}_{\varpi_M}). \end{aligned}$$

Consequently,

$$a_1 \otimes \mathrm{I}_{\varpi} \sim_h \Omega(\mathfrak{a}). \tag{6.18}$$

and

$$\begin{aligned} \Omega^\sharp : C([0,1],\mathbb{R}) &\to \mathbb{R} \\ \mathrm{g}_1 &\mapsto \frac{1}{\varpi}\sum_{m=1}^{M} \varpi_m \cdot \mathrm{g}_1(y_m). \end{aligned} \tag{6.19}$$

As $c_{1,1} > \mathfrak{C}_{1,2} > \frac{2}{\varepsilon} \cdot (c_0 + c'_{1,1})$ (see (6.6)), set

$$c''_{1,1} := c_{1,1} - (c_0 + c'_{1,1}) > 0 \quad \text{and} \quad c_1 := c'_{1,1} + c''_{1,1}. \tag{6.20}$$

Consequently,

$$c_{1,1} = c_0 + c_1 = \hat{c} + \varpi + c'_{1,1} + c''_{1,1}. \tag{6.21}$$

Now, let $P_2 := \mathrm{diag}(Q, \mathrm{I}_{c_1 \cdot [1,1]}, \mathrm{I}_{c_{1,2} \cdot [1,2]}, \ldots, \mathrm{I}_{c_{1,p_1} \cdot [1,p_1]})$, which is also a projection in $\mathrm{M}_\infty(C(X_2))$ and set $A^1_{X,2} := P_2 \mathrm{M}_\infty(C(X_2)) P_2$. Evidently,

$$[2,1] := \mathrm{rank}(P_2) = c_0 \cdot [1,1] + c_1 \cdot [1,1] + \sum_{e=2}^{p_1} c_{1,e} \cdot [1,e] = \sum_{e=1}^{p_1} c_{1,e} \cdot [1,e]. \tag{6.22}$$

For convenience, define a unital $*$-homomorphism

$$\begin{aligned} \overline{\phi} : A_{X,1} = \bigoplus_{e=1}^{p_1} A^e_{X,1} &\to A^1_{X,2} = P_2 \mathrm{M}_\infty(C(X_2)) P_2 \\ (\mathfrak{a}, a_2, \ldots, a_{p_1}) &\mapsto \mathrm{diag}(\psi(\mathfrak{a}), a_1 \otimes \mathrm{I}_{c_1}, a_2 \otimes \mathrm{I}_{c_{1,2}}, \ldots, a_{p_1} \otimes \mathrm{I}_{c_{1,p_1}}), \end{aligned}$$

where $a_1 = \mathrm{j}_1(\mathfrak{a}(\hat{\mathrm{x}}_1))$. Without loss of generality, we can fix an identification j_2 of $P_2 \mathrm{M}_\infty(C(X_2) P_2)|_{\hat{\mathrm{x}}_2}$ with $\mathrm{M}_{[2,1]}$ and assume that for any $\mathfrak{a} \in A^1_{X,1}$,

$$\psi(\mathfrak{a})(\hat{\mathrm{x}}_2) = \mathrm{diag}(a_1 \otimes \mathrm{I}_{\hat{c}}, \Omega(\mathfrak{a})). \tag{6.23}$$

Further, set

$$[2,k] := \sum_{e=1}^{p_1} c_{k,e} \cdot [1,e], \quad k = 2, \ldots, p_2 \tag{6.24}$$

and $A^k_{X,2} := \mathrm{M}_{[2,k]}$. Note that

$$A_{X,1} = \bigoplus_{e=1}^{p_1} A^e_{X,1} = P_1 \mathrm{M}_\infty(C(X_1)) P_1 \oplus \left(\bigoplus_{e=2}^{p_1} \mathrm{M}_{[1,e]} \right)$$

and set

$$\begin{aligned} A_{X,2} &:= \bigoplus_{k=1}^{p_2} A^k_{X,2} = P_2 \mathrm{M}_\infty(C(X_2)) P_2 \oplus \left(\bigoplus_{k=2}^{p_2} \mathrm{M}_{[2,k]} \right), \\ F_2 &:= \mathrm{M}_{[2,1]} \oplus \cdots \oplus \mathrm{M}_{[2,p_2]}. \end{aligned}$$

Then, according to the information above and recall $c_1 = c'_{1,1} + c''_{1,1}$ (see (6.20)), we can define a unital $*$-homomorphism:

$$\phi : A_{X,1} = \bigoplus_{e=1}^{p_1} A_{X,1}^e \to A_{X,2} = \bigoplus_{k=1}^{p_2} A_{X,1}^k$$
$$(\mathfrak{a}, a_2, \ldots, a_{p_1}) \mapsto (\mathfrak{b}, b_2, \ldots, b_{p_2}),$$

by

$$\mathfrak{b} = \overline{\phi}(\mathfrak{a}, a_2, \ldots, a_{p_1}) = \operatorname{diag}(\psi(\mathfrak{a}), a_1 \otimes \mathrm{I}_{c_1}, a_2 \otimes \mathrm{I}_{c_{1,2}}, \ldots, a_{p_1} \otimes \mathrm{I}_{c_{1,p_1}}), \tag{6.25}$$
$$b_k = \operatorname{diag}(a_1 \otimes \mathrm{I}_{c_{k,1}}, a_2 \otimes \mathrm{I}_{c_{k,2}}, \ldots, a_{p_1} \otimes \mathrm{I}_{c_{k,p_1}}), \quad k = 2, \ldots, p_2, \tag{6.26}$$

where $a_1 = \mathrm{j}_1(\mathfrak{a}(\hat{\mathrm{x}}_1))$. Note that $K_0(\phi) = \gamma$ and $K_1(\phi) = \chi$. And by (6.23), set

$$b_1 := \mathrm{j}_2(\mathfrak{b}(\hat{\mathrm{x}}_2)) = \operatorname{diag}\big(a_1 \otimes \mathrm{I}_{\hat{c}}, \Omega(\mathfrak{a}), a_1 \otimes \mathrm{I}_{c_1}, a_2 \otimes \mathrm{I}_{c_{1,2}}, \ldots, a_{p_1} \otimes \mathrm{I}_{c_{1,p_1}}\big) \in \mathrm{M}_{[2,1]}. \tag{6.27}$$

Below, we continue to construct A_2. Because the left side of the following commutative diagram (see (6.5))

$$\begin{tikzcd}
\mathbb{Z}^{p_1} \arrow[r, "\gamma'=(c_{k,e})"] \arrow[d, "\pi_1=(b^1_{j,e})"'] & \mathbb{Z}^{p_2} \arrow[d, "\pi_2=(b^2_{s,k})"] \\
\mathbb{Z}^{l_1} \arrow[r, "\widetilde{\gamma}=(d_{s,j})"'] & \mathbb{Z}^{l_2}
\end{tikzcd}$$

is given, that is, the group homomorphism $\pi_2 = (b^2_{s,k}) : \mathbb{Z}^{p_2} \to \mathbb{Z}^{l_2}$ is fixed. Now, choose two $l_2 \times p_2$ matrices $(b^{2,0}_{s,k})$ and $(b^{2,1}_{s,k})$ with entries in positive integers satisfying

$$(b^{2,1}_{s,k}) - (b^{2,0}_{s,k}) = (b^2_{s,k}) \tag{6.28}$$

and the condition (6.7). Note that the order unit $u_1 = (([1,1], \tau_1), [1,2], \ldots, [1,p_1])$ is an element in G_1, so $\pi_2(\gamma'([1,1], \ldots, [1,p_1])) = 0$. According to (6.22) and (6.24), define

$$\begin{aligned}
\{2,s\} :&= \sum_{k=1}^{p_2} b^{2,0}_{s,k} \cdot [2,k] = \sum_{e=1}^{p_1} \sum_{k=1}^{p_2} b^{2,0}_{s,k} \cdot c_{k,e}[1,e] \\
&= \sum_{k=1}^{p_2} b^{2,1}_{s,k} \cdot [2,k] = \sum_{e=1}^{p_1} \sum_{k=1}^{p_2} b^{2,1}_{s,k} \cdot c_{k,e}[1,e], \quad s = 1, \ldots, l_2
\end{aligned} \tag{6.29}$$

and

$$E_2 := \mathrm{M}_{\{2,1\}} \oplus \cdots \oplus \mathrm{M}_{\{2,l_2\}}. \tag{6.30}$$

Further, by Remark 3.6, we can choose two unital $*$-homomorphisms with good form $\beta_{2,0}, \beta_{2,1} : F_2 \to E_2$ such that $K_0(\beta_{2,0}) = (b^{2,0}_{s,k})$ and $K_0(\beta_{2,1}) = (b^{2,1}_{s,k})$. Now, define

$$A_2 := \Big\{ (\bar{g}; \mathfrak{b}, b_2, \ldots, b_{p_2}) \in C([0,1], E_2) \oplus \bigoplus_{k=1}^{p_2} A^k_{X,2} : \\ \bar{g}(0) = \beta_{2,0}((\mathrm{j}_2(\mathfrak{b}(\hat{\mathrm{x}}_2)), b_2, \ldots, b_{p_2})), \bar{g}(1) = \beta_{2,1}((\mathrm{j}_2(\mathfrak{b}(\hat{\mathrm{x}}_2)), b_2, \ldots, b_{p_2})) \Big\}.$$

We see at once that A_2 has the form in Fact 5.3 and

$$\mathrm{Aff}(T(F_2)) = \mathbb{R}^{p_2}, \quad \mathrm{Aff}(T(A_{X,2})) = C(X_2, \mathbb{R}) \oplus \mathbb{R}^{p_2-1},$$

and

$$\mathrm{Aff}(T(A_2)) = \Big\{ (G_{2,1}, \ldots, G_{2,l_2}; \mathrm{g}_2, g_{2,2}, \ldots, g_{2,p_2}) \\ \in \bigoplus_{s=1}^{l_2} C([0,1], \mathbb{R}_s) \oplus C(X_2, \mathbb{R}) \oplus \mathbb{R}^{p_2-1} : \\ G_{2,s}(0) = \Pi^\sharp_{2,s}(\beta^\sharp_{2,0}(g_2)), G_{2,s}(1) = \Pi^\sharp_{2,s}(\beta^\sharp_{2,1}(g_2)),\ s = 1, \ldots, l_2 \Big\},$$

by writing $g_{2,1} := \mathrm{g}_2(\hat{\mathrm{x}}_2) \in \mathbb{R}$ and $g_2 = (g_{2,1}, \ldots, g_{2,p_2}) \in \mathbb{R}^{p_2}$, and where $\mathbb{R}_s = \mathbb{R}$ for all $s = 1, \ldots, l_2$ (see (8) in Fact 5.4).

Step 2. Construct the $*$-homomorphism $\ell : A_1 \to A_2$.

To construct the map $\ell : A_1 \to A_2$, note that for any $(f_1, \ldots, f_{l_1}; \mathfrak{a}, a_2, \ldots, a_{p_1}) \in A_1$ and the element $[0,1] \ni \omega \mapsto f_j(\frac{t}{T} \cdot \omega) \in \mathrm{M}_{\{1,j\}}$ can be viewed as a path in $\mathrm{M}_{\{1,j\}}$ connecting $f_j(0)$ and $f_j(\frac{t}{T})$ for any pair $(j,t) \in \{1, \ldots, l_1\} \times \{1, \ldots, (T-1)\}$. Furthermore, by using

$$0_{1,j} = \left\{ \theta_{1,1}^{\sim b^{1,0}_{j,1}}, \ldots, \theta_{1,p_1}^{\sim b^{1,0}_{j,p_1}} \right\} \quad \text{(see (5.2))},$$

and setting $a_1 = \mathrm{j}_1(\mathfrak{a}(\hat{\mathrm{x}}_1))$, we get

$$f_j(0) = \mathrm{diag}\big(a_1 \otimes \mathrm{I}_{b^{1,0}_{j,1}}, \ldots, a_{p_1} \otimes \mathrm{I}_{b^{1,0}_{j,p_1}}\big) \sim_h f_j\left(\frac{t}{T}\right).$$

Noting that $c'_{1,e} = (T-1) \cdot \sum_{j=1}^{l_1} b^{1,0}_{j,e}$ (see (6.10)) and according to the condition (6.6), we get

$$c''_{1,e} := c_{1,e} - c'_{1,e} > 0, \quad e = 2, \ldots, p_1. \tag{6.31}$$

Note that $\{1,j\} = \sum_{e=1}^{p_1} b^{1,0}_{j,e} \cdot [1,e]$ for any $j = 1, \ldots, l_1$, so

$$\mathbf{c} := (T-1) \cdot \sum_{j=1}^{l_1} \{1,j\} = \sum_{e=1}^{p_1} c'_{1,e} \cdot [1,e]. \tag{6.32}$$

Further, define a unital $*$-homomorphism

$$\begin{aligned}\Xi : C([0,1], E_1) = \bigoplus_{j=1}^{l_1} C([0,1], \mathrm{M}_{\{1,j\}}) &\to \mathrm{M}_{\mathbf{c}} \\ \bar{f} = (f_1, \ldots, f_{l_1}) &\mapsto \\ &\operatorname{diag}\left(f_1\left(\frac{1}{T}\right), \ldots, f_1\left(\frac{T-1}{T}\right), \ldots, f_{l_1}\left(\frac{1}{T}\right), \ldots, f_{l_1}\left(\frac{T-1}{T}\right)\right).\end{aligned} \tag{6.33}$$

Consequently,

$$\begin{aligned}&\operatorname{diag}\left(a_1 \otimes \mathrm{I}_{c'_{1,1}}, \ldots, a_{p_1} \otimes \mathrm{I}_{c'_{1,p_1}}\right) \\ &\sim_u \operatorname{diag}(\underbrace{f_1(0), \ldots, f_1(0)}_{T-1}, \ldots, \underbrace{f_{l_1}(0), \ldots, f_{l_1}(0)}_{T-1}) \sim_h \Xi(\bar{f})\end{aligned} \tag{6.34}$$

and

$$\begin{aligned}\Xi^\sharp : \oplus_{j=1}^{l_1} C([0,1], \mathbb{R}_j) &\to \mathbb{R} \\ \bar{G}_1 = (G_{1,1}, \ldots, G_{1,l_1}) &\mapsto \frac{1}{\mathbf{c}} \cdot \sum_{j=1}^{l_1} \sum_{t=1}^{T-1} \{1, j\} \cdot G_{1,j}\left(\frac{t}{T}\right),\end{aligned} \tag{6.35}$$

where $\mathbb{R}_j = \mathbb{R}$ for all $j = 1, \ldots, l_1$.

Now, by using (6.34) and noting that $c_1 = c'_{1,1} + c''_{1,1}$ (see (6.20)), we want to modify

$$\mathfrak{b} = \operatorname{diag}\left(\psi(\mathfrak{a}), a_1 \otimes \mathrm{I}_{c_1}, a_2 \otimes \mathrm{I}_{c_{1,2}}, \ldots, a_{p_1} \otimes \mathrm{I}_{c_{1,p_1}}\right) \quad \text{(see (6.25))}$$

to

$$\mathfrak{b}' = \operatorname{diag}\left(\psi(\mathfrak{a}), \Xi(\bar{f}), a_1 \otimes \mathrm{I}_{c''_{1,1}}, \ldots, a_{p_1} \otimes \mathrm{I}_{c''_{1,p_1}}\right).$$

And note that $\psi(\mathfrak{a}) \in Q\mathrm{M}_\infty(C(X_2))Q$ (see (6.14)) and $\operatorname{rank}(Q) = c_0 \cdot [1,1]$ (see (6.16)), thus there exists a unitary $\mathfrak{p}$ in $C([0,1], P_2\mathrm{M}_\infty(C(X_2))P_2)$ such that

(1) $\mathfrak{p} = \operatorname{diag}(Q, \mathfrak{w})$ for some $\mathfrak{w} \in C([0,1], \mathrm{M}_{c_1 \cdot [1,1] + \sum_{e=2}^{p_1} c_{1,e} \cdot [1,e]})$;

(2) $\mathfrak{p}(0) = \mathfrak{b}'$ and $\mathfrak{p}(1) = \mathfrak{b}$ (see (6.31)).

Hence, we can define a unital $*$-homomorphism φ from A_1 to $A_{X,2}$ as follows:

$$\begin{aligned}\varphi : A_1 &\to A_{X,2} \\ f = (f_1, \ldots, f_{l_1}; \mathfrak{a}, a_2, \ldots, a_{p_1}) &\mapsto (\mathfrak{b}', b_2, \ldots, b_{p_2}),\end{aligned}$$

by

$$\mathfrak{b}' = \operatorname{diag}\left(\psi(\mathfrak{a}), \Xi(\bar{f}), a_1 \otimes \mathrm{I}_{c''_{1,1}}, \ldots, a_{p_1} \otimes \mathrm{I}_{c''_{1,p_1}}\right), \tag{6.36}$$

$$b_k = \operatorname{diag}\left(a_1 \otimes \mathrm{I}_{c_{k,1}}, \ldots, a_{p_1} \otimes \mathrm{I}_{c_{k,p_1}}\right), \quad k = 2, \ldots, p_2, \tag{6.37}$$

where $\bar{f} = (f_1, \ldots, f_{l_1}) \in C([0,1], E_1) = \oplus_{j=1}^{l_1} C([0,1], \mathrm{M}_{\{1,j\}})$ and $a_1 = \mathrm{j}_1(\mathfrak{a}(\hat{\mathrm{x}}_1))$. (Note that, here, $b_2, \ldots, b_{p_2}$ are the same as (6.26), respectively.) As a consequence,

$$b_1' := \mathrm{j}_2(\mathfrak{b}'(\hat{\mathrm{x}}_2)) \overset{(6.23)}{=\!=\!=} \operatorname{diag}\big(a_1 \otimes \mathrm{I}_{\hat{c}}, \Omega(\mathfrak{a}), \Xi(\bar{f}), a_1 \otimes \mathrm{I}_{c''_{1,1}}, \ldots, a_{p_1} \otimes \mathrm{I}_{c''_{1,p_1}}\big). \quad (6.38)$$

Further, it is clear that $K_0(\varphi) = \gamma|_{G_1}$, $K_1(\varphi) = \chi$ and

$$\begin{aligned} \varphi^\sharp : \mathrm{Aff}(T(A_1)) &\to \mathrm{Aff}(T(A_{X,2})) = C(X_2, \mathbb{R}) \oplus \mathbb{R}^{p_2-1} \\ \mathfrak{G}_1 = (\bar{G}_1; \mathrm{g}_1, g_{1,2}, \ldots, g_{1,p_1}) &\mapsto (\mathrm{g}_2, g_{2,1}, \ldots, g_{2,p_2}), \end{aligned} \quad (6.39)$$

where

$$\mathrm{g}_2(x) \overset{(6.17)}{\underset{(6.35)}{=\!=\!=}} \frac{1}{[2,1]}\Big(c_0[1,1]\cdot\psi^\sharp(\mathrm{g}_1)(x) + \mathbf{c}\cdot\Xi^\sharp(\bar{G}_1) + \sum_{e=1}^{p_1} c''_{1,e}[1,e]\cdot g_{1,e}\Big) \quad \forall x \in X_2,$$

$$\mathrm{g}_2(\hat{\mathrm{x}}_2) \overset{(6.19)}{\underset{(6.38)}{=\!=\!=}} \frac{1}{[2,1]}\cdot\Big([1,1](\hat{c}\cdot g_{1,1} + \varpi\cdot\Omega^\sharp(\mathrm{g}_1)) + \mathbf{c}\cdot\Xi^\sharp(\bar{G}_1) + \sum_{e=1}^{p_1} c''_{1,e}[1,e]\cdot g_{1,e}\Big),$$

$$g_{2,k} \overset{(6.37)}{=\!=\!=} \frac{1}{[2,k]}\sum_{e=1}^{p_1} c_{k,e}[1,e]\cdot g_{1,j}, \quad k = 2, \ldots, p_2.$$

Now, to define ℓ, we will connect

$$\beta_{2,0}(b_1', b_2, \ldots, b_{p_2}) \quad \text{and} \quad \beta_{2,1}(b_1', b_2, \ldots, b_{p_2})$$

in $E_2 = \oplus_{s=1}^{l_2}\mathrm{M}_{\{2,s\}}$ (see (6.30)). Evidently, it is enough for each $s = 1, \ldots, l_2$ to connect $\Pi_{2,s}(\beta_{2,0}(b_1', b_2, \ldots, b_{p_2}))$ and $\Pi_{2,s}(\beta_{2,1}(b_1', b_2, \ldots, b_{p_2}))$ in $\mathrm{M}_{\{2,s\}}$, where $\Pi_{2,s}$ is the quotient map from E_2 to $\mathrm{M}_{\{2,s\}}$. This is equivalent to construct the $*$-homomorphism $\Pi_{2,s}\circ\ell : A_1 \to C([0,1], \mathrm{M}_{\{2,s\}})$. Note that

$$\begin{aligned} &\Pi_{2,s}(\beta_{2,0}(b_1', b_2, \ldots, b_{p_2})) \\ &\quad \sim_u \operatorname{diag}\big(\Omega(\mathfrak{a}) \otimes \mathrm{I}_{b^{2,0}_{s,1}}, \Xi(\bar{f}) \otimes \mathrm{I}_{b^{2,0}_{s,1}}, a_1 \otimes \mathrm{I}_{\zeta^0_{s,1}}, \ldots, a_{p_1} \otimes \mathrm{I}_{\zeta^0_{s,p_1}}\big) \end{aligned} \quad (6.40)$$

and

$$\begin{aligned} &\Pi_{2,s}(\beta_{2,1}(b_1', b_2, \ldots, b_{p_2})) \\ &\quad \sim_u \operatorname{diag}\big(\Omega(\mathrm{a}) \otimes \mathrm{I}_{b^{2,1}_{s,1}}, \Xi(\bar{f}) \otimes \mathrm{I}_{b^{2,1}_{s,1}}, a_1 \otimes \mathrm{I}_{\zeta^1_{s,1}}, \ldots, a_{p_1} \otimes \mathrm{I}_{\zeta^1_{s,p_1}}\big), \end{aligned} \quad (6.41)$$

where

$$\zeta^0_{s,e} = \begin{cases} b^{2,0}_{s,1}\cdot(\hat{c} + c''_{1,1}) + \sum_{k=2}^{p_2} b^{2,0}_{s,k}\cdot c_{k,1} & \text{if } e = 1, \\ b^{2,0}_{s,1}\cdot c''_{1,e} + \sum_{k=2}^{p_2} b^{2,0}_{s,k}\cdot c_{k,e} & \text{if } e = 2, \ldots, p_1 \end{cases} \quad (6.42)$$

and

$$\zeta^1_{s,e} = \begin{cases} b^{2,1}_{s,1}\cdot(\hat{c} + c''_{1,1}) + \sum_{k=2}^{p_2} b^{2,1}_{s,k}\cdot c_{k,1} & \text{if } e = 1, \\ b^{2,1}_{s,1}\cdot c''_{1,e} + \sum_{k=2}^{p_2} b^{2,1}_{s,k}\cdot c_{k,e} & \text{if } e = 2, \ldots, p_1. \end{cases}$$

Thus,

$$\zeta_{s,e}^1-\zeta_{s,e}^0 \xlongequal{(6.28)} \begin{cases} b_{s,1}^2\cdot(\hat{c}+c_{1,1}'')+\sum_{k=2}^{p_2} b_{s,k}^2\cdot c_{k,1} & \text{if } e=1,\\ b_{s,1}^2\cdot c_{1,e}''+\sum_{k=2}^{p_2} b_{s,k}^2\cdot c_{k,e} & \text{if } e=2,\dots,p_1. \end{cases} \tag{6.43}$$

Fix an $s=1,\dots,l_2$. Now, for each $j=1,\dots,l_1$, let

$$d'_{s,j}=\begin{cases} |d_{s,j}| & \text{if } d_{s,j}\neq 0,\\ 2 & \text{if } d_{s,j}=0. \end{cases}$$

According to the proof of [15, Lemma 13.15], define a unital $*$-homomorphism

$$\heartsuit_s: C([0,1],E_1)=\bigoplus_{j=1}^{l_1} C([0,1],\mathrm{M}_{\{1,j\}})\to C\Big([0,1],\mathrm{M}_{\sum_{j=1}^{l_1} d'_{s,j}\cdot\{1,j\}}\Big)$$
$$\bar{f}=(f_1,\dots,f_{l_1})\mapsto \operatorname{diag}(F_{s,1},\dots,F_{s,l_1}),$$

for any $j=1,\dots,l_1$, where

$$F_{s,j}(\omega)=\begin{cases} f_j(\omega)\otimes \mathrm{I}_{|d_{s,j}|} & \text{if } d_{s,j}>0,\\ f_j(1-\omega)\otimes \mathrm{I}_{|d_{s,j}|} & \text{if } d_{s,j}<0,\\ \operatorname{diag}(f_j(\omega),f_j(1-\omega)) & \text{if } d_{s,j}=0. \end{cases}$$

Obviously, $\heartsuit_s(\bar{f})(0)$ and $\heartsuit_s(\bar{f})(1)$ only depend on $\bar{f}(0)$ and $\bar{f}(1)$. Of course, we also have

$$\begin{aligned} \heartsuit_s^\sharp: &\bigoplus_{j=1}^{l_1} C([0,1],\mathbb{R}_j)\to C([0,1],\mathbb{R})\\ \bar{G}_1=(G_{1,1},\dots,G_{1,l_1})\mapsto &\frac{1}{\sum_{j=1}^{l_1} d'_{s,j}\cdot\{1,j\}}\\ &\cdot\Bigg(\sum_{\{j:d_{s,j}>0\}}|d_{s,j}|\{1,j\}\cdot G_{1,j}(\omega)\\ &\quad+\sum_{\{j:d_{s,j}<0\}}|d_{s,j}|\{1,j\}\cdot G_{1,j}(1-\omega)\\ &\quad+\sum_{\{j:d_{s,j}=0\}}\{1,j\}\cdot(G_{1,j}(\omega)+G_{1,j}(1-\omega))\Bigg), \end{aligned}\tag{6.44}$$

where $\mathbb{R}_j=\mathbb{R}$ for all $j=1,\dots,l_1$.

Note that for any $(\bar{f};a_1,\dots,a_{p_1})\in A_{\mathrm{C},1}=C([0,1],E_1)\oplus_{\beta_{1,0};\beta_{1,1}}F_1$, we have

$$(\heartsuit_s(\bar{f}))(0)\sim_u \operatorname{diag}\big(a_1\otimes\mathrm{I}_{\mu^0_{s,1}},\dots,a_{p_1}\otimes\mathrm{I}_{\mu^0_{s,p_1}}\big)$$

and

$$(\heartsuit_s(\bar{f}))(1)\sim_u \operatorname{diag}\big(a_1\otimes\mathrm{I}_{\mu^1_{s,1}},\dots,a_{p_1}\otimes\mathrm{I}_{\mu^1_{s,p_1}}\big),$$

where, for any $e = 1, \ldots, p_1$,

$$\mu^0_{s,e} = \sum_{\{j: d_{s,j}>0\}} |d_{s,j}| b^{1,0}_{j,e} + \sum_{\{j: d_{s,j}<0\}} |d_{s,j}| b^{1,1}_{j,e} + \sum_{\{j: d_{s,j}=0\}} \left(b^{1,0}_{j,e} + b^{1,1}_{j,e}\right),$$
$$\mu^1_{s,e} = \sum_{\{j: d_{s,j}>0\}} |d_{s,j}| b^{1,1}_{j,e} + \sum_{\{j: d_{s,j}<0\}} |d_{s,j}| b^{1,0}_{j,e} + \sum_{\{j: d_{s,j}=0\}} \left(b^{1,0}_{j,e} + b^{1,1}_{j,e}\right).$$

Consequently,

$$\sum_{j=1}^{l_1} d'_{s,j} \cdot \{1, j\} = \sum_{e=1}^{p_1} \mu^0_{s,e} \cdot [1, e] = \sum_{e=1}^{p_1} \mu^1_{s,e} \cdot [1, e]. \tag{6.45}$$

Noting that the diagram (6.5) is commutative, for all $e = 1, \ldots, p_1$, we get the following equalities:

$$\mu^1_{s,e} - \mu^0_{s,e} = \sum_{k=1}^{p_2} d_{s,j} \cdot b^1_{j,e} = \sum_{k=1}^{p_2} b^2_{s,k} \cdot c_{k,e} = \sum_{k=1}^{p_2} b^{2,1}_{s,k} \cdot c_{k,e} - \sum_{k=1}^{p_2} b^{2,0}_{s,k} \cdot c_{k,e} \tag{6.46}$$

hold. On the one hand, by the assumption (6.7), we get

$$\max\left\{\mu^0_{s,e}, \mu^1_{s,e} : s = 1, \ldots, l_2; e = 1, \ldots, p_1\right\} < \mathscr{B}\mathscr{D} l_1 < \frac{\mathfrak{B} \cdot p_2 \varepsilon}{2}. \tag{6.47}$$

On the other hand, for each $s = 1, \ldots, l_2$,

$$\{2, s\} = \sum_{k=1}^{p_2} b^{2,0}_{s,k} \cdot [2, k] > \mathfrak{B}_2 \cdot \sum_{k=1}^{p_2} [2, k] > \mathfrak{B}_2 p_2. \tag{6.48}$$

Thus, combining (6.45), (6.47) with (6.48),

$$\sum_{j=1}^{l_1} d'_{s,j} \cdot \{1, j\} < \mathscr{B}\mathscr{D}\mathscr{M} p_1 l_1 < \mathfrak{B}_2 p_2 < \{2, s\}.$$

This means that the C^*-algebra $C([0, 1], \mathrm{M}_{\sum_{j=1}^{l_1} d'_{s,j} \cdot \{1, j\}})$ can be regarded as a corner of the C^*-algebra $C([0, 1], \mathrm{M}_{\{2,s\}})$. Now, for each $e = 1, \ldots, p_1$, let

$$\nu^0_{s,e} := \zeta^0_{s,e} - \mu^0_{s,e} \quad \text{and} \quad \nu^1_{s,e} := \zeta^1_{s,e} - \mu^1_{s,e}. \tag{6.49}$$

Consequently, combining (6.47) again with (6.48),

$$\min\left\{\nu^0_{s,e}, \nu^1_{s,e} : e = 1, \ldots, p_1\right\} > \mathfrak{B}_2 p_2 - \mathscr{B}\mathscr{D} l_1 > 0.$$

Thus, we can rewrite (6.40) and (6.41), respectively, as follows:

$$\begin{aligned} &\Pi_{2,s}(\beta_{2,0}(b'_1, b_2, \ldots, b_{p_2})) \\ &\quad \sim_u \operatorname{diag}\left((\heartsuit_s(\bar{f}))(0), \Omega(\mathfrak{a}) \otimes \mathrm{I}_{b^{2,0}_{s,1}}, \Xi(\bar{f}) \otimes \mathrm{I}_{b^{2,0}_{s,1}}, a_1 \otimes \mathrm{I}_{\nu^0_{s,1}}, \ldots, a_{p_1} \otimes \mathrm{I}_{\nu^0_{s,p_1}}\right) \end{aligned} \tag{6.50}$$

and

$$\begin{aligned}&\Pi_{2,s}(\beta_{2,1}(b_1', b_2, \ldots, b_{p_2}))\\&\quad\sim_u \operatorname{diag}\big((\heartsuit_s(\bar{f}))(1), \Omega(\mathfrak{a}) \otimes \mathrm{I}_{b_{s,1}^{2,1}}, \Xi(\bar{f}) \otimes \mathrm{I}_{b_{s,1}^{2,1}}, a_1 \otimes \mathrm{I}_{\nu_{s,1}^1}, \ldots, a_{p_1} \otimes \mathrm{I}_{\nu_{s,p_1}^1}\big).\end{aligned}\tag{6.51}$$

By using (6.20), (6.43) and (6.46), we get

$$\nu_{s,e}^0 - \nu_{s,e}^1 = \begin{cases} \big(b_{s,1}^{2,1} - b_{s,1}^{2,0}\big) \cdot (\varpi + c_{1,1}') & \text{if } e = 1,\\ \big(b_{s,1}^{2,1} - b_{s,1}^{2,0}\big) \cdot c_{1,e}' & \text{if } e = 2, \ldots, p_1.\end{cases}$$

Now, for an arbitrary $s = 1, \ldots, l_2$, the construction of $\Pi_{2,s} \circ \ell$ will be divided into two cases.

Case 1. $b_{s,1}^{2,1} \geqslant b_{s,1}^{2,0}$. That is, $b_{s,1}^2 = b_{s,1}^{2,1} - b_{s,1}^{2,0} \geqslant 0$.

In this case, obviously, $\nu_{s,e}^0 \geqslant \nu_{s,e}^1$ holds for each $e = 1, \ldots, p_1$. And according to (6.18) and (6.34), for any $f = (f_1, \ldots, f_{l_1}; \mathfrak{a}, a_2, \ldots, a_{p_1}) \in A_1$, and set $a_1 := \mathrm{j}_1(\mathfrak{a}(\hat{\mathrm{x}}_1))$, $\bar{f} := (f_1, \ldots, f_{l_1}) \in C([0,1], E_1)$, then we have

$$\begin{aligned}&\operatorname{diag}(\Omega(\mathfrak{a}), \Xi(\bar{f})) \otimes \mathrm{I}_{b_{s,1}^{2,1} - b_{s,1}^{2,0}}\\&\quad\sim_h \operatorname{diag}(\underbrace{a_1, \ldots, a_1}_{\varpi}, \underbrace{f_1(0), \ldots, f_1(0)}_{(T-1)}, \ldots, \underbrace{f_{l_1}(0), \ldots, f_{l_1}(0)}_{T-1}) \otimes \mathrm{I}_{b_{s,1}^{2,1} - b_{s,1}^{2,0}}\\&\quad\sim_u \operatorname{diag}\big(a_1 \otimes \mathrm{I}_{\varpi + c_{1,1}'}, a_2 \otimes \mathrm{I}_{c_{1,2}'}, \ldots, a_{p_1} \otimes \mathrm{I}_{c_{1,p_1}'}\big) \otimes \mathrm{I}_{b_{s,1}^{2,1} - b_{s,1}^{2,0}}.\end{aligned}\tag{6.52}$$

Note that

$$\begin{aligned}&\sum_{j=1}^{l_1} d_{s,1}' \cdot \{1, j\} + \varpi[1,1] \cdot b_{s,1}^{2,0} + \mathbf{c} \cdot b_{s,1}^{2,0} + \sum_{e=1}^{p_1} \nu_{s,e}^0 \cdot [1, e]\\
&\overset{(6.45)}{\underset{(6.42)}{=\!=}} \left(b_{s,1}^{2,0} \cdot (\hat{c} + c_{1,1}'') + \sum_{k=2}^{p_2} b_{s,k}^{2,0} \cdot c_{k,1}\right) \cdot [1,1] + \varpi[1,1] \cdot b_{s,1}^{2,0} + \mathbf{c} \cdot b_{s,1}^{2,0}\\
&\quad + \sum_{e=2}^{p_1} \left(b_{s,1}^{2,0} \cdot c_{1,e}'' + \sum_{k=2}^{p_2} b_{s,k}^{2,0} \cdot c_{k,e}\right) \cdot [1, e]\\
&\overset{(6.32)}{=\!=} (\hat{c} + \varpi + c_{1,1}' + c_{1,1}'')[1,1] \cdot b_{s,1}^{2,0} + \sum_{e=2}^{p_1} c_{1,e}'[1,e] \cdot b_{s,1}^{2,0} + \sum_{k=2}^{p_2} c_{k,1}[1,1] \cdot b_{s,k}^{2,0}\\
&\quad + \sum_{e=2}^{p_1} c_{1,e}''[1,e] \cdot b_{s,1}^{2,0} + \sum_{e=2}^{p_1} \sum_{k=2}^{p_2} c_{k,e}[1,e] \cdot b_{s,k}^{2,0}\\
&\overset{(6.21)}{\underset{(6.31)}{=\!=}} \sum_{e=1}^{p_1} \sum_{k=1}^{p_2} b_{s,k}^{2,0} \cdot c_{k,e}[1,e] \overset{(6.29)}{=\!=} \{2, s\}\end{aligned}$$

and

$$\sum_{e=1}^{l_1} d'_{s,1}\cdot\{1,e\}+\varpi[1,1]\cdot b_{s,1}^{2,1}+\mathbf{c}\cdot b_{s,1}^{2,1}+\sum_{e=1}^{p_1}\nu_{s,1}^1\cdot[1,e]=\{2,s\}.$$

For convenience, for an arbitrary $s=1,\ldots,l_2$, define

$$\diamondsuit_{s,0}:=\mathrm{diag}(\underbrace{a_1,\ldots,a_1}_{\varpi},\underbrace{f_1(0),\ldots,f_1(0)}_{T-1},\ldots,\underbrace{f_{l_1}(0),\ldots,f_{l_1}(0)}_{T-1})\otimes \mathrm{I}_{b_{s,1}^{2,1}-b_{s,1}^{2,0}},$$

$$\begin{aligned}\diamondsuit_{s,1}&:=\mathrm{diag}(\Omega(\mathfrak{a}),\Xi(\bar{f}))\otimes \mathrm{I}_{b_{s,1}^{2,1}-b_{s,1}^{2,0}}\\&=\mathrm{diag}\Big(\varrho_{1,1}(\mathfrak{a}(y_1))\otimes \mathrm{I}_{\varpi_1},\ldots,\varrho_{M,1}(\mathfrak{a}(y_M))\otimes \mathrm{I}_{\varpi_M},\\&\qquad f_1\Big(\frac{1}{T}\Big),\ldots,f_1\Big(\frac{T-1}{T}\Big),\ldots,f_{l_1}\Big(\frac{1}{T}\Big),\ldots,f_{l_1}\Big(\frac{T-1}{T}\Big)\Big)\otimes \mathrm{I}_{b_{s,1}^{2,1}-b_{s,1}^{2,0}},\end{aligned}$$

$$\beth_s:=\mathrm{diag}(\Omega(\mathfrak{a}),\Xi(\bar{f}))\otimes \mathrm{I}_{b_{s,1}^{2,0}},$$

$$\daleth_s:=\mathrm{diag}\big(a_1\otimes \mathrm{I}_{\nu_{s,1}^1},\ldots,a_{p_1}\otimes \mathrm{I}_{\nu_{s,p_1}^1}\big)\quad\text{(see (6.49))}.$$

Hence, we can rewrite again (6.40) and (6.41) in the following form:

$$\Pi_{2,s}(\beta_{2,0}(b'_1,b_2,b_3,\ldots,b_{p_2}))=\mathcal{U}_{s,0}\cdot\mathrm{diag}((\heartsuit_s(\bar{f}))(0),\diamondsuit_{s,0},\beth_s,\daleth_s)\cdot\mathcal{U}_{s,0}^*$$

and

$$\Pi_{2,s}(\beta_{2,1}(b'_1,b_2,b_3,\ldots,b_{p_2}))=\mathcal{U}_{s,1}\cdot\mathrm{diag}((\heartsuit_s(\bar{f}))(1),\diamondsuit_{s,1},\beth_s,\daleth_s)\cdot\mathcal{U}_{s,1}^*,$$

respectively, where $\mathcal{U}_{s,0},\mathcal{U}_{s,1}\in U(\mathrm{M}_{\{2,s\}})$. Recalling that

$$\mathbf{c}=(T-1)\cdot\sum_{j=1}^{l_1}\{1,j\}$$

(see (6.32)), we can define a $*$-homomorphism

$$\diamondsuit_s:A_1\to C([0,1],\mathrm{M}_{\varpi\cdot[1,1]+\mathbf{c}})\otimes \mathrm{M}_{b_{s,1}^{2,1}-b_{s,1}^{2,0}}$$

$$\begin{aligned}(f_1,\ldots,f_{l_1};\mathfrak{a},a_2,\ldots,a_{p_1})\mapsto \mathrm{diag}\Big(&\varrho_{1,\omega}(\mathfrak{a}(y_1(\omega)))\\&\otimes \mathrm{I}_{\varpi_1},\ldots,\varrho_{M,\omega}(\mathfrak{a}(y_M(\omega)))\otimes \mathrm{I}_{\varpi_M},\\&f_1\Big(\frac{1}{T}\cdot\omega\Big),\ldots,f_1\Big(\frac{T-1}{T}\cdot\omega\Big),\ldots,\\&f_{l_1}\Big(\frac{1}{T}\cdot\omega\Big),\ldots,f_{l_1}\Big(\frac{T-1}{T}\cdot\omega\Big)\Big)\otimes \mathrm{I}_{b_{s,1}^{2,1}-b_{s,1}^{2,0}}.\end{aligned}$$

Obviously, $(\lozenge_s(f))(0) = \lozenge_{s,0}$, $(\lozenge_s(f))(1) = \lozenge_{s,1}$ and

$$
\begin{aligned}
\lozenge_s^{\sharp} : \mathrm{Aff}(T(A_1)) &\to \mathbb{R} \\
\mathfrak{G}_1 = (\bar{G}_1; \mathfrak{g}_1, g_{1,2}, \dots, g_{1,p_1}) &\mapsto \frac{1}{\varpi \cdot [1,1] + \mathbf{c}} \\
&\quad \cdot \Bigg([1,1] \sum_{m=1}^{M} \varpi_m \cdot \mathfrak{g}_1(y_m(\omega)) \\
&\quad + \sum_{j=1}^{l_1} \sum_{t=1}^{T-1} \{1, j\} \cdot G_{1,j}\left(\frac{t}{T} \cdot \omega\right)\Bigg),
\end{aligned}
\tag{6.53}
$$

where

$$\bar{G}_1 = (G_{1,1}, \dots, G_{1,l_1}) \in \mathrm{Aff}(T(C([0,1], E_1))).$$

Now, for an arbitrary $s = 1, \dots, l_2$, fix a unitary $\mathcal{U}_s \in C([0,1], \mathrm{M}_{\{2,s\}})$ such that $\mathcal{U}_s(0) = \mathcal{U}_{s,0}$ and $\mathcal{U}_s(1) = \mathcal{U}_{s,1}$, and define

$$
\begin{aligned}
\Pi_{2,s} \circ \ell : A_1 &\to C([0,1], \mathrm{M}_{\{2,s\}}) \\
(f_1, \dots, f_{l_1}; \alpha, a_2, \dots, a_{p_1}) &\mapsto \mathcal{U}_s \cdot \mathrm{diag}((\heartsuit_s(\bar{f}))(\omega), (\lozenge_s(f))(\omega), \beth_s, \daleth_s) \cdot \mathcal{U}_s^*.
\end{aligned}
$$

Case 2. $b_{s,1}^{2,1} < b_{s,1}^{2,0}$. That is, $b_{s,1}^2 = b_{s,1}^{2,1} - b_{s,1}^{2,0} < 0$.

In this case, $\nu_{s,e}^0 < \nu_{s,e}^1$ holds for each $e = 1, \dots, p_1$. Similar to *Case 1*, for any $f = (f_1, \dots, f_{l_1}; \alpha, a_2, \dots, a_{p_1}) \in A_1$, set $a_1 := \mathrm{j}_1(\alpha(\hat{x}_1))$, $\bar{f} := (f_1, \dots, f_{l_1}) \in C([0,1], E_1)$ and define

$$
\begin{aligned}
\tilde{\lozenge}_{s,0} &= \mathrm{diag}\Bigg(\varrho_{1,1}(\alpha(y_1)) \otimes \mathrm{I}_{\varpi_1}, \dots, \varrho_{M,1}(\alpha(y_M)) \otimes \mathrm{I}_{\varpi_M}, \\
&\qquad f_1\left(\frac{1}{T}\right), \dots, f_1\left(\frac{T-1}{T}\right), \dots, f_{l_1}\left(\frac{1}{T}\right), \dots, f_{l_1}\left(\frac{T-1}{T}\right)\Bigg) \otimes \mathrm{I}_{b_{s,1}^{2,0} - b_{s,1}^{2,1}}, \\
\tilde{\lozenge}_{s,1} &= \mathrm{diag}(\underbrace{a_1, \dots, a_1}_{\varpi}, \underbrace{f_1(0), \dots, f_1(0)}_{T-1}, \dots, \underbrace{f_{l_1}(0), \dots, f_{l_1}(0)}_{T-1}) \otimes \mathrm{I}_{b_{s,1}^{2,0} - b_{s,1}^{2,1}}, \\
\tilde{\beth}_s &= \mathrm{diag}(\Omega(\alpha), \Xi(\bar{f})) \otimes \mathrm{I}_{b_{s,1}^{2,1}}, \\
\tilde{\daleth}_s &= \mathrm{diag}\big(a_1 \otimes \mathrm{I}_{\nu_{s,1}^0}, \dots, a_{p_1} \otimes \mathrm{I}_{\nu_{s,p_1}^0}\big) \quad \text{(see (6.49))}.
\end{aligned}
$$

Then, we rewrite (6.40) and (6.41) as follows:

$$\Pi_{2,s}(\beta_{2,0}(b_1', b_2, b_3, \dots, b_{p_2})) = \tilde{\mathcal{U}}_{s,0} \cdot \mathrm{diag}\big((\heartsuit_s(\bar{f}))(0), \tilde{\lozenge}_{s,0}, \tilde{\beth}_s, \tilde{\daleth}_s\big) \cdot \tilde{\mathcal{U}}_{s,0}^*$$

and

$$\Pi_{2,s}(\beta_{2,1}(b_1', b_2, b_3, \dots, b_{p_2})) = \tilde{\mathcal{U}}_{s,0} \cdot \mathrm{diag}\big((\heartsuit_s(\bar{f}))(1), \tilde{\lozenge}_{s,1}, \tilde{\beth}_s, \tilde{\daleth}_s\big) \cdot \tilde{\mathcal{U}}_{s,1}^*,$$

respectively, where $\widetilde{\mathcal{U}}_{s,0}, \widetilde{\mathcal{U}}_{s,1} \in U(\mathrm{M}_{\{2,s\}})$. Recalling that $\mathbf{c} = (T-1)\cdot\sum_{j=1}^{l_1}\{1,j\}$ (see (6.32)), we can define another $*$-homomorphism

$$\begin{aligned}
&\widetilde{\Diamond}_s : A_1 \to C([0,1], \mathrm{M}_{\varpi\cdot[1,1]+\mathbf{c}}) \otimes \mathrm{M}_{b_{s,1}^{2,0}-b_{s,1}^{2,1}}\\
&(f_1,\ldots,f_{l_1};\mathfrak{a},a_2,\ldots,a_{p_1})\\
&\quad\mapsto \mathrm{diag}\Big(\varrho_{1,1-\omega}(\mathfrak{a}(y_1(1-\omega)))\otimes \mathrm{I}_{\varpi_1},\ldots,\\
&\qquad\qquad \varrho_{M,1-\omega}(\mathfrak{a}(y_M(1-\omega)))\otimes \mathrm{I}_{\varpi_M},\\
&\qquad\qquad f_1\left(\frac{1}{T}\cdot(1-\omega)\right),\ldots,f_1\left(\frac{T-1}{T}\cdot(1-\omega)\right),\ldots,\\
&\qquad\qquad f_{l_1}\left(\frac{1}{T}\cdot(1-\omega)\right),\ldots,f_{l_1}\left(\frac{T-1}{T}\cdot(1-\omega)\right)\Big)\otimes \mathrm{I}_{b_{s,1}^{2,0}-b_{s,1}^{2,1}}.
\end{aligned}$$

Obviously, $(\widetilde{\Diamond}_s(f))(0) = \widetilde{\Diamond}_{s,0}$, $(\widetilde{\Diamond}_s(f))(1) = \widetilde{\Diamond}_{s,1}$ and

$$\begin{aligned}
&\widetilde{\Diamond}_s^\sharp : \mathrm{Aff}(T(A_1)) \to \mathbb{R}\\
&\mathfrak{G}_1 = (\bar{G}_1; \mathrm{g}_1, g_{1,2},\ldots,g_{1,p_1}) \mapsto \frac{1}{\varpi\cdot[1,1]+\mathbf{c}}\\
&\qquad\cdot\Bigg([1,1]\sum_{m=1}^{M}\varpi_m\cdot \mathrm{g}_1(y_m(1-\omega))\\
&\qquad + \sum_{j=1}^{l_1}\sum_{t=1}^{T-1}\{1,j\}\cdot G_{1,j}\left(\frac{t}{T}\cdot(1-\omega)\right)\Bigg),
\end{aligned}\tag{6.54}$$

where $\bar{G}_1 = (G_{1,1},\ldots,G_{1,l_1}) \in \mathrm{Aff}(T(C([0,1],E_1)))$. Now, for an arbitrary $s=1,\ldots,l_2$, fix a unitary $\widetilde{\mathcal{U}}_s \in C([0,1],\mathrm{M}_{\{2,s\}})$ such that $\widetilde{\mathcal{U}}_s(0) = \widetilde{\mathcal{U}}_{s,0}$ and $\widetilde{\mathcal{U}}_s(1) = \widetilde{\mathcal{U}}_{s,1}$, and define

$$\begin{aligned}
&\Pi_{2,s}\circ\ell : A_1 \to C([0,1],\mathrm{M}_{\{2,s\}})\\
&(f_1,\ldots,f_{l_1};\mathfrak{a},a_2,\ldots,a_{p_1}) \mapsto \widetilde{\mathcal{U}}_s\cdot \mathrm{diag}((\heartsuit_s(\bar{f}))(\omega),(\widetilde{\Diamond}_s(f))(\omega),\beth_s,\daleth_s)\cdot\widetilde{\mathcal{U}}_s^*.
\end{aligned}$$

On the one hand, recall that

$$\begin{aligned}
&\varphi : A_1 \to A_{X,2}\\
&(f_1,\ldots,f_{l_1};\mathfrak{a},a_2,\ldots,a_{p_1}) \mapsto (\mathfrak{b}',b_2,\ldots,b_{p_2}).
\end{aligned}$$

On the other hand, for each $s=1,\ldots,l_2$, let $\mathcal{V}_s \in C([0,1],\mathrm{M}_{\{2,s\}})$ is a unitary satisfying

$$\mathcal{V}_s = \begin{cases} \mathcal{U}_s & \text{if } b_{s,1}^{2,1} \geqslant b_{s,1}^{2,0},\\ \widetilde{\mathcal{U}}_s & \text{if } b_{s,1}^{2,1} < b_{s,1}^{2,0}.\end{cases}$$

Now, we can define the $*$-homomorphism as follows:

$$\begin{aligned}
&\ell : A_1 \to A_2\\
&f = (f_1,\ldots,f_{l_1};\mathfrak{a},a_2,\ldots,a_{p_1}) \mapsto (\hbar_1,\ldots,\hbar_{l_2};\mathfrak{b}',b_2,\ldots,b_{p_2}),
\end{aligned}\tag{6.55}$$

where, for any $s = 1, \ldots, l_2$,

$$\hbar_s(\omega) = \begin{cases} \mathcal{V}_s(\omega) \cdot \operatorname{diag}\big((\heartsuit_s(\bar{f}))(\omega), (\Diamond_s(f))(\omega), \beth_s, \daleth_s\big) \cdot \mathcal{V}_s^*(\omega) & \text{if } b_{s,1}^{2,1} \geqslant b_{s,1}^{2,0}, \\ \mathcal{V}_s(\omega) \cdot \operatorname{diag}\big((\heartsuit_s(\bar{f}))(\omega), (\widetilde{\Diamond}_s(f))(\omega), \widetilde{\beth}_s, \widetilde{\daleth}_s\big) \cdot \mathcal{V}_s^*(\omega) & \text{if } b_{s,1}^{2,1} < b_{s,1}^{2,0}, \end{cases} \tag{6.56}$$

and $\bar{f} = (f_1, \ldots, f_{l_1}) \in C([0,1], E_1)$.

It is easy to verify that the conditions (i), (ii) and (iii) hold for the map ℓ.

Step 3. Show that the map ℓ satisfies the condition (iv).

Step 3-1. Compute the function $\ell^\sharp : \operatorname{Aff}(T(A_1)) \to \operatorname{Aff}(T(A_2))$.

For an arbitrary $\mathfrak{G}_1 = (G_{1,1}, \ldots, G_{1,l_1}; \mathfrak{g}_1, g_{1,2}, \ldots, g_{1,p_1}) \in \operatorname{Aff}(T(A_1))$, set $\bar{G}_1 := (G_{1,1}, \ldots, G_{1,l_1}) \in \operatorname{Aff}(T(C([0,1], E_1)))$ and $g_{1,1} := \mathfrak{g}_1(\hat{\mathfrak{x}}_1) \in \mathbb{R}$. For each $s = 1, \ldots, l_2$, if $b_{s,1}^{2,1} \geqslant b_{s,1}^{2,0}$, then by (6.53) and (6.56), define

$$\begin{aligned} G_{2,s} : [0,1] &\to \mathbb{R} \\ \omega &\mapsto \frac{1}{\{2,s\}} \Bigg(\Bigg(\sum_{j=1}^{l_1} d'_{s,j}\{1,j\} \Bigg) \cdot \heartsuit_s^\sharp(\bar{G}_1)(\omega) \\ &\qquad + \big(b_{s,1}^{2,1} - b_{s,1}^{2,0}\big)(\varpi \cdot [1,1] + \mathbf{c}) \cdot \Diamond_s^\sharp(\mathfrak{G}_1)(\omega) \\ &\qquad + b_{s,1}^{2,0} \cdot ([1,1]\varpi \cdot \Omega^\sharp(\mathfrak{g}_1) + \mathbf{c} \cdot \Xi^\sharp(\bar{G}_1)) \\ &\qquad + \sum_{e=1}^{p_1} \nu_{s,e}^1 [1,e] \cdot g_{1,e} \Bigg), \end{aligned} \tag{6.57}$$

and if $b_{s,1}^{2,1} < b_{s,1}^{2,0}$, then by (6.54) and (6.56), define

$$\begin{aligned} G_{2,s} : [0,1] &\to \mathbb{R} \\ \omega &\mapsto \frac{1}{\{2,s\}} \Bigg(\Bigg(\sum_{j=1}^{l_1} d'_{s,j}\{1,j\} \Bigg) \cdot \heartsuit_s^\sharp(\bar{G}_1)(\omega) \\ &\qquad + \big(b_{s,1}^{2,0} - b_{s,1}^{2,1}\big)(\varpi \cdot [1,1] + \mathbf{c}) \cdot \widetilde{\Diamond}_s^\sharp(\mathfrak{G}_1)(\omega) \\ &\qquad + b_{s,1}^{2,1} \cdot ([1,1]\varpi \cdot \Omega^\sharp(\mathfrak{g}_1) + \mathbf{c} \cdot \Xi^\sharp(\bar{G}_1)) \\ &\qquad + \sum_{e=1}^{p_1} \nu_{s,e}^0 [1,e] \cdot g_{1,e} \Bigg). \end{aligned} \tag{6.58}$$

And, for any $s = 1, \ldots, l_2$, by using (6.50) and (6.51), we get

$$G_{2,s}(0) = \Pi_{2,s}^\sharp\big(\beta_{2,0}^\sharp\big(\Pi_2^{q\sharp}(\ell^\sharp(\mathfrak{G}_1))\big)\big) \quad \text{and} \quad G_{2,s}(1) = \Pi_{2,s}^\sharp\big(\beta_{2,1}^\sharp\big(\Pi_2^{q\sharp}(\ell^\sharp(\mathfrak{G}_1))\big)\big). \tag{6.59}$$

Thus, from (6.57), (6.58), (6.59) and the definition of ℓ (see (6.55)), we have

$$\begin{aligned} \ell^\sharp : \operatorname{Aff}(T(A_1)) &\to \operatorname{Aff}(T(A_2)) \\ \mathfrak{G}_1 = (G_{1,1}, \ldots, G_{1,l_1}; \mathfrak{g}_1, g_{1,2}, \ldots, g_{1,p_1}) &\mapsto (G_{2,1}, \ldots, G_{2,l_2}; \mathfrak{g}_2, g_{2,2}, \ldots, g_{2,p_2}), \end{aligned}$$

where $\mathrm{g}_2, g_{2,2}, \ldots, g_{2,p_2}$ are the same as (6.39), respectively. That is,

$$\mathrm{g}_2(x) = \frac{1}{[2,1]}\Big(c_0[1,1]\cdot\psi^\sharp(\mathrm{g}_1)(x) + \mathbf{c}\cdot\Xi^\sharp(\overline{G}_1) + \sum_{e=1}^{p_1} c''_{1,e}[1,e]\cdot g_{1,e}\Big) \quad \forall x\in X_2, \tag{6.60}$$

$$\mathrm{g}_2(\hat{\mathrm{x}}_2) = \frac{1}{[2,1]}\cdot\Big([1,1](\hat{c}\cdot g_{1,1} + \varpi\cdot\Omega^\sharp(\mathrm{g}_1)) + \mathbf{c}\cdot\Xi^\sharp(\overline{G}_1) + \sum_{e=1}^{p_1} c''_{1,e}[1,e]\cdot g_{1,e}\Big), \tag{6.61}$$

$$g_{2,k} = \frac{1}{[2,k]}\sum_{e=1}^{p_1} c_{k,e}[1,e]\cdot g_{1,e}, \quad k=2,\ldots,p_2, \tag{6.62}$$

where $\overline{G}_1 := (G_{1,1},\ldots,G_{1,l_1}) \in \mathrm{Aff}(T(C([0,1],E_1)))$ and $g_{1,1} := \mathrm{g}_1(\hat{\mathrm{x}}_1)\in\mathbb{R}$.

Step 3-2. Complete the proof of *Step 3*.

Recall that Θ is a $*$-homomorphism from F_1 to F_2 with $K_0(\Theta)=\gamma'=(c_{k,j})$ (see (6.5)). Thus,

$$\begin{aligned}\Theta^\sharp : \mathbb{R}^{p_1} = \mathrm{Aff}(T(F_1)) &\to \mathbb{R}^{p_2} = \mathrm{Aff}(T(F_2)),\\ \mathrm{w}_1 = (\mathrm{w}_{1,1},\ldots,\mathrm{w}_{1,p_1}) &\mapsto (\mathrm{w}_{2,1},\ldots,\mathrm{w}_{2,p_2}),\end{aligned} \tag{6.63}$$

where

$$\mathrm{w}_{2,k} = \frac{1}{[2,k]}\cdot\sum_{e=1}^{p_1} c_{k,e}[1,e]\cdot\mathrm{w}_{1,e}, \quad k=1,\ldots,p_2. \tag{6.64}$$

Of course, we also have

$$\begin{aligned}\Pi_1^{q\sharp} &: \mathrm{Aff}(T(A_1)) \to \mathrm{Aff}(T(F_1))\\ (G_{1,1},\ldots,G_{1,l_1};\mathrm{g}_1,g_{1,2},\ldots,g_{1,p_1}) &\mapsto (\mathrm{g}_1(\hat{\mathrm{x}}_1),g_{1,2},\ldots,g_{1,p_1})\end{aligned}$$

and

$$\begin{aligned}\Pi_2^{q\sharp} &: \mathrm{Aff}(T(A_2)) \to \mathrm{Aff}(T(F_2))\\ (G_{2,1},\ldots,G_{2,l_2};\mathrm{g}_2,g_{2,2},\ldots,g_{2,p_1}) &\mapsto (\mathrm{g}_2(\hat{\mathrm{x}}_2),g_{2,2},\ldots,g_{2,p_1}).\end{aligned}$$

Now, for an arbitrary $\mathfrak{G}_1 = (G_{1,1},\ldots,G_{1,l_1};\mathrm{g}_1,g_{1,2},\ldots,g_{1,p_1}) \in \mathrm{Aff}(T(A_1))$ with $\|\mathfrak{G}_1\|\leqslant 1$, set $\overline{G}_1 = (G_{1,1},\ldots,G_{1,l_1}) \in \mathrm{Aff}(T(C([0,1],E_1)))$ and $g_{1,1} := \mathrm{g}_1(\hat{\mathrm{x}}_1)\in\mathbb{R}$. Apparently,

$$\begin{aligned}\Theta^\sharp(\Pi_1^{q\sharp}(\mathfrak{G}_1)) &= \Theta^\sharp(g_{1,1},\ldots,g_{1,p_1}) = (\dot{\mathrm{w}}_{2,1},\ldots,\dot{\mathrm{w}}_{2,p_2}),\\ \Pi_2^{q\sharp}(\ell^\sharp(\mathfrak{G}_1)) &= \Pi_2^{q\sharp}(\overline{G}_2;\mathrm{g}_2,g_{2,2},\ldots,g_{2,p_2}) = (\ddot{\mathrm{w}}_{2,1},\ldots,\ddot{\mathrm{w}}_{2,p_2}),\end{aligned}$$

where $\overline{G}_2 := (G_{2,1}, \ldots, G_{2,l_2}) \in \mathrm{Aff}(T(C([0,1], E_2)))$, and for any $k = 1, \ldots, p_2$,

$$\dot{\mathrm{w}}_{2,k} = \frac{1}{[2,k]} \sum_{e=1}^{p_1} c_{k,e}[1,e] \cdot g_{1,e},$$

$$\ddot{\mathrm{w}}_{2,k} = \begin{cases} \mathrm{g}_2(\hat{\mathrm{x}}_2) & \text{if } k = 1, \\ g_{2,k} & \text{if } k = 2, \ldots, p_2. \end{cases}$$

According to (6.62), for all $k = 2, \ldots, p_2$,

$$\dot{\mathrm{w}}_{2,k} = \ddot{\mathrm{w}}_{2,k}.$$

Note that $\varphi : A \to B$ is a $*$-homomorphism, then $\varphi^\sharp : \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(B))$ is, induced by φ, contractive (see Notation 2.16). Thus, by using (6.6) and (6.22) and noting that $\mathfrak{G}_1 = (G_{1,1}, \ldots, G_{1,l_1}; \mathrm{g}_1, g_{1,2}, \ldots, g_{1,p_1}) \in \mathrm{Aff}(T(A_1))$ with $\|\mathfrak{G}_1\| \leqslant 1$, we have

$$\begin{aligned}
&\left\|\Theta^\sharp(\Pi_1^{q\sharp}(\mathfrak{G}_1)) - \Pi_2^{q\sharp}(\ell^\sharp(\mathfrak{G}_1))\right\| \\
&\quad = |\dot{\mathrm{w}}_{2,1} - \ddot{\mathrm{w}}_{2,1}| \\
&\quad \overset{(6.61)}{=\!=} \frac{1}{[2,1]} \Bigg| \sum_{e=1}^{p_1} c_{1,e}[1,e] \cdot g_{1,e} \\
&\qquad\qquad - \left([1,1](\hat{c} \cdot g_{1,1} + \varpi \cdot \Omega^\sharp(\mathrm{g}_1)) + \mathbf{c} \cdot \Xi^\sharp(\overline{G}_1) + \sum_{e=1}^{p_1} c''_{1,e}[1,e] \cdot g_{1,e} \right) \Bigg| \\
&\quad \underset{(6.31)}{\overset{(6.21)}{=\!=}} \frac{1}{[2,1]} \Bigg| \varpi[1,1] \cdot (g_{1,1} - \Omega^\sharp(g_1)) + c'_{1,1}[1,1] \cdot (g_{1,1} - \Xi^\sharp(\overline{G}_1)) \\
&\qquad\qquad + \sum_{e=2}^{p_1} c'_{1,e}[1,e] \cdot g_{1,e} \Bigg| \\
&\quad < \frac{1}{[2,1]} \left(2(\varpi + c'_{1,1})[1,1] + \sum_{e=2}^{p_1} c'_{1,e}[1,e] \right) < \varepsilon,
\end{aligned}$$

as desired.

Step 4. Show that the map ℓ satisfies the condition (v). (The idea of this proof originates from [15, 13.18].)

Step 4-1. Construct the two maps Γ_1 and Γ_2.

Define

$$\Gamma_1 : \mathrm{Aff}(T(F_1)) \to \mathrm{Aff}(T(A_1))$$
$$\mathrm{w}_1 = (\mathrm{w}_{1,1}, \ldots, \mathrm{w}_{1,p_1}) \mapsto (G_{1,1}, \ldots, G_{1,l_1}; \mathrm{g}_1, g_{1,2}, \ldots, g_{1,p_1}),$$

where

$$g_1(x) = w_{1,1} \quad \forall x \in X_1, \tag{6.65}$$
$$g_{1,e} = w_{1,e}, \quad e = 2, \dots, p_1, \tag{6.66}$$
$$G_{1,j}(\omega) = (1-\omega)\cdot \Pi^{\sharp}_{1,j}(\beta^{\sharp}_{1,0}(w_1)) + \omega \cdot \Pi^{\sharp}_{1,j}(\beta^{\sharp}_{1,1}(w_1)) \quad \forall \omega \in [0,1], \tag{6.67}$$

and $\Pi^{\sharp}_{1,j}$ is induced by the quotient map $\Pi_{1,j} : E_1 \to M_{\{1,j\}}$ for any $j = 1, \dots, l_1$.

Similarly, define

$$\begin{aligned} \Gamma_2 : \mathrm{Aff}(T(F_2)) &\to \mathrm{Aff}(T(A_2)) \\ w_2 = (w_{2,1}, \dots, w_{2,p_2}) &\mapsto (G_{2,1}, \dots, G_{2,l_2}; g_2, g_{2,2}, \dots, g_{2,p_2}), \end{aligned} \tag{6.68}$$

where

$$g_2(x) = w_{2,1} \quad \forall x \in X_2, \tag{6.69}$$
$$g_{2,k} = w_{2,k}, \quad k = 2, \dots, p_2, \tag{6.70}$$
$$G_{2,s}(\omega) = (1-\omega)\cdot \Pi^{\sharp}_{2,s}(\beta^{\sharp}_{2,0}(w_2)) + \omega \cdot \Pi^{\sharp}_{2,s}(\beta^{\sharp}_{2,1}(w_2)) \quad \forall \omega \in [0,1], \tag{6.71}$$

and $\Pi^{\sharp}_{2,s}$ is induced by the quotient map $\Pi_{2,s} : E_2 \to M_{\{2,s\}}$ for any $s = 1, \dots, l_2$.

Step 4-2. Complete the proof of *Step 4*.

Recall that if S is a compact convex set, then define

$$\mathrm{Aff}(S)_+ := \{f \in \mathrm{Aff}(S) : f(s) > 0 \ \forall s \in S\} \cup \{0\} \quad \text{(see Notation 2.14)}.$$

It is easy to check that Γ_1 and Γ_2 are both contractive linear and order preserving maps. Thus, for an arbitrary $w_1 = (w_{1,1}, \dots, w_{1,p_1}) \in \mathrm{Aff}(F_1) = \mathbb{R}^{p_1}$ with $\|w_1\| \leqslant 1$, we have

$$\begin{aligned} \Gamma_2(\Theta^{\sharp}(w_1)) &\xlongequal{(6.63)} \Gamma_2(w_{2,1}, \dots, w_{2,p_2}) \\ &\xlongequal{(6.68)} (\dot{G}_{2,1}, \dots, \dot{G}_{2,l_2}; \dot{g}_2, \dot{g}_{2,2}, \dots, \dot{g}_{2,p_2}), \end{aligned}$$

where

$$\dot{g}_2(x) \xlongequal[(6.69)]{(6.64)} \frac{1}{[2,1]} \cdot \sum_{e=1}^{p_1} c_{1,e}[1,e]\cdot w_{1,e} \quad \forall x \in X_2, \tag{6.72}$$
$$\dot{g}_{2,k} \xlongequal[(6.70)]{(6.64)} \frac{1}{[2,k]} \cdot \sum_{e=1}^{p_1} c_{k,e}[1,e]\cdot w_{1,e}, \quad k = 2, \dots, p_2, \tag{6.73}$$
$$\dot{G}_{2,s}(\omega) \xlongequal{(6.71)} (1-\omega)\cdot \Pi^{\sharp}_{2,s}(\beta^{\sharp}_{2,0}(\Theta^{\sharp}(w_1))) + \omega \cdot \Pi^{\sharp}_{2,s}(\beta^{\sharp}_{2,1}(\Theta^{\sharp}(w_1))), \quad \omega \in [0,1]. \tag{6.74}$$

Of course, we also have

$$\Gamma_1(w_1) = (G'_{1,1}, \dots, G'_{1,l_1}; g'_1, g'_{1,2}, \dots, g'_{1,p_1}) \in \mathrm{Aff}(T(A_1)),$$

where

$$\mathrm{g}_1'(x) \xlongequal{(6.65)} \mathrm{w}_{1,1} \quad \forall x \in X_1, \tag{6.75}$$

$$g_{1,e}' \xlongequal{(6.66)} \mathrm{w}_{1,e}, \quad e = 2, \ldots, p_1, \tag{6.76}$$

$$G_{1,j}'(\omega) \xlongequal{(6.67)} (1-\omega) \cdot \Pi_{1,j}^\sharp(\beta_{1,0}^\sharp(\mathrm{w}_1)) + \omega \cdot \Pi_{1,j}^\sharp(\beta_{1,1}^\sharp(\mathrm{w}_1)) \quad \forall \omega \in [0,1]. \tag{6.77}$$

Further, by setting $\mathfrak{G}_1' := \Gamma_1(\mathrm{w}_1) = (G_{1,1}', \ldots, G_{1,l_1}'; \mathrm{g}_1', g_{1,2}', \ldots, g_{1,p_1}') \in \mathrm{Aff}(T(A_1))$ and $\bar{G}_1' := (G_{1,1}', \ldots, G_{1,l_1}') \in \mathrm{Aff}(T(C([0,1], E_1)))$, we get

$$\begin{aligned} \ell^\sharp(\Gamma_1(\mathrm{w}_1)) = \ell^\sharp(\mathfrak{G}_1') &= \ell^\sharp(G_{1,1}', \ldots, G_{1,l_1}'; \mathrm{g}_1', g_{1,2}', \ldots, g_{1,p_1}') \\ &= (\ddot{G}_{2,1}, \ldots, \ddot{G}_{2,l_2}; \ddot{\mathrm{g}}_2, \ddot{g}_{2,2}, \ldots, \ddot{g}_{2,p_2}), \end{aligned}$$

where

$$\begin{aligned} \ddot{\mathrm{g}}_2(x) \xlongequal{(6.60)} \frac{1}{[2,1]} \Big(& c_0[1,1] \cdot \mathrm{w}_{1,1} + \mathbf{c} \cdot \Xi^\sharp(\bar{G}_1') \\ & + \sum_{e=1}^{p_1} c_{1,e}''[1,e] \cdot \mathrm{w}_{1,e} \Big) \quad \forall x \in X_2, \end{aligned} \tag{6.78}$$

$$\ddot{g}_{2,k} \xlongequal{(6.62)} \frac{1}{[2,k]} \sum_{e=1}^{p_1} c_{k,e}[1,e] \cdot \mathrm{w}_{1,e}, \quad k = 2, \ldots, p_2 \tag{6.79}$$

and for each $s = 1, \ldots, l_2$, if $b_{s,1}^{2,1} \geqslant b_{s,1}^{2,0}$, then, for any $\omega \in [0,1]$,

$$\begin{aligned} \ddot{G}_{2,s}(\omega) \xlongequal{(6.57)} \frac{1}{\{2,s\}} \Bigg(& \Big(\sum_{j=1}^{l_1} d_{s,j}'\{1,j\} \Big) \cdot \heartsuit_s^\sharp(\bar{G}_1')(\omega) \\ & + \big(b_{s,1}^{2,1} - b_{s,1}^{2,0}\big)(\varpi \cdot [1,1] + \mathbf{c}) \cdot \Diamond_s^\sharp(\mathfrak{G}_1')(\omega) \\ & + b_{s,1}^{2,0} \cdot ([1,1]\varpi \cdot \Omega^\sharp(\mathrm{g}_1) + \mathbf{c} \cdot \Xi^\sharp(\bar{G}_1')) \\ & + \sum_{e=1}^{p_1} \nu_{s,e}^1[1,e] \cdot \mathrm{w}_{1,e} \Bigg) \end{aligned} \tag{6.80}$$

and if $b_{s,1}^{2,1} < b_{s,1}^{2,0}$, then, for any $\omega \in [0,1]$,

$$\begin{aligned} \ddot{G}_{2,s}(\omega) \xlongequal{(6.58)} \frac{1}{\{2,s\}} \Bigg(& \Big(\sum_{j=1}^{l_1} d_{s,j}'\{1,j\} \Big) \cdot \heartsuit_s^\sharp(\bar{G}_1')(\omega) \\ & + \big(b_{s,1}^{2,0} - b_{s,1}^{2,1}\big)(\varpi \cdot [1,1] + \mathbf{c}) \cdot \tilde{\Diamond}_s^\sharp(\mathfrak{G}_1')(\omega) \\ & + b_{s,1}^{2,1} \cdot ([1,1]\varpi \cdot \Omega^\sharp(\mathrm{g}_1) + \mathbf{c} \cdot \Xi^\sharp(\bar{G}_1')) \\ & + \sum_{e=1}^{p_1} \nu_{s,e}^0[1,e] \cdot g_{1,e} \Bigg). \end{aligned} \tag{6.81}$$

Below, we will show that

$$
\begin{aligned}
&\left\|\Gamma_2(\Theta^\sharp(\mathrm{w}_1)) - \ell^\sharp(\Gamma_1(\mathrm{w}_1))\right\| \\
&< \left\|(\dot{G}_{2,1}, \dots, \dot{G}_{2,l_2}; \dot{g}_2, \dot{g}_{2,2}, \dots, \dot{g}_{2,p_2}) - (\ddot{G}_{2,1}, \dots, \ddot{G}_{2,l_2}; \ddot{g}_2, \ddot{g}_{2,2}, \dots, \ddot{g}_{2,p_2})\right\| < \varepsilon.
\end{aligned}
$$

Firstly,

$$\dot{g}_{2,k} \overset{(6.73)}{\underset{(6.79)}{=\!=}} \ddot{g}_{2,k}, \quad k = 2, \dots, p_2.$$

Secondly, according to (6.72), (6.78) and $\mathrm{w}_1 = (\mathrm{w}_{1,1}, \dots, \mathrm{w}_{1,p_1}) \in \mathrm{Aff}(F_1) = \mathbb{R}^{p_1}$ with $\|\mathrm{w}_1\| \leqslant 1$, and recalling that $\mathbf{c} = \sum_{e=1}^{p_1} c'_{1,e} \cdot [1, e]$ (see (6.32)), then for any $x \in X_2$

$$
\begin{aligned}
|\dot{g}(x) - \ddot{g}(x)| &= \frac{1}{[2,1]}\Bigg|\sum_{e=1}^{p_1} c_{1,e}[1,e] \cdot \mathrm{w}_{1,e} \\
&\quad - \left(c_0[1,1] \cdot \mathrm{w}_{1,1} + \mathbf{c} \cdot \Xi^\sharp(\bar{G}'_1) + \sum_{e=1}^{p_1} c''_{1,e}[1,e] \cdot \mathrm{w}_{1,e}\right)\Bigg| \\
&\overset{(6.31)}{\underset{(6.21)}{=\!=}} \frac{1}{[2,1]}\left|\sum_{e=1}^{p_1} c'_{1,e}[1,e] \cdot (\mathrm{w}_{1,e} - \Xi^\sharp(\bar{G}'_1))\right| < \frac{2}{[2,1]} \sum_{e=1}^{p_1} c'_{1,e}[1,e] < \varepsilon.
\end{aligned}
$$

Finally, for each $s = 1, \dots, l_2$, if $b^{2,1}_{s,1} \geqslant b^{2,0}_{s,1}$, then, for any $\omega \in [0, 1]$,

$$
\begin{aligned}
&\left|\Pi^\sharp_{2,s}(\beta^\sharp_{2,0}(\Theta^\sharp(\mathrm{w}_1))) - \ddot{G}_{2,s}(\omega)\right| \\
&\overset{(6.74)}{\underset{(6.80)}{=\!=}} \frac{1}{\{2,s\}}\Bigg|\sum_{k=1}^{p_2}\sum_{e=1}^{p_1} b^{2,0}_{s,k} c_{k,e}[1,e] \cdot \mathrm{w}_{1,e} - \Bigg(\left(\left(\sum_{j=1}^{l_1} d'_{s,j}\{1,j\}\right) \cdot \heartsuit^\sharp_s(\bar{G}'_1)(\omega)\right. \\
&\quad + \left(b^{2,1}_{s,1} - b^{2,0}_{s,1}\right)(\varpi \cdot [1,1] + \mathbf{c}) \cdot \diamondsuit^\sharp_s(\mathfrak{G}'_1)(\omega) \\
&\quad + b^{2,0}_{s,1} \cdot ([1,1]\varpi \cdot \Omega^\sharp(\mathrm{g}_1) + \mathbf{c} \cdot \Xi^\sharp(\bar{G}'_1)) + \sum_{e=1}^{p_1} \nu^1_{s,e}[1,e] \cdot \mathrm{w}_{1,j}\Bigg)\Bigg| \\
&\overset{(6.45)}{\underset{(6.32)}{=\!=}} \frac{1}{\{2,s\}}\Bigg|\sum_{e=1}^{p_1} \mu^1_{s,e}[1,e](\mathrm{w}_{1,e} - \heartsuit^\sharp_s(\bar{G}'_1)(\omega)) \\
&\quad + \left(b^{2,1}_{s,1} - b^{2,0}_{s,1}\right)\varpi[1,1](\mathrm{w}_{1,1} - \diamondsuit^\sharp_s(\mathfrak{G}'_1)(\omega)) \\
&\quad + \left(b^{2,1}_{s,1} - b^{2,0}_{s,1}\right)\sum_{e=1}^{p_1} c'_{1,e}[1,e](\mathrm{w}_{1,e} - \diamondsuit^\sharp_s(\mathfrak{G}'_1)(\omega)) \\
&\quad + b^{2,0}_{s,1}[1,1]\varpi(\mathrm{w}_{1,1} - \Omega^\sharp(\mathrm{g}_1)) + b^{2,0}_{s,1}\sum_{e=1}^{p_1} c'_{1,e}[1,e](\mathrm{w}_{1,e} - \Xi^\sharp(\bar{G}'_1))\Bigg| \\
&< \frac{2}{\{2,s\}}\left(\sum_{e=1}^{p_1} \mu^1_{s,e}[1,e] + b^{2,1}_{s,1}\varpi[1,1] + b^{2,1}_{s,1}\sum_{e=1}^{p_1} c'_{1,e}[1,e]\right) \\
&< \frac{2}{\{2,s\}}\left(\sum_{e=1}^{p_1} \frac{\mathfrak{B}_2 \cdot p_2\varepsilon}{2}[1,e] + b^{2,1}_{s,1}\varpi[1,1] + b^{2,1}_{s,1}\sum_{e=1}^{p_1} c'_{1,e}[1,e]\right) \quad \text{(see (6.47))} \\
&< \varepsilon.
\end{aligned}
$$

Similarly, if $b_{s,1}^{2,1} < b_{s,1}^{2,0}$, then

$$\left|\Pi_{2,s}^{\sharp}\big(\beta_{2,0}^{\sharp}(\Theta^{\sharp}(\mathrm{w}_1))\big) - \ddot{G}_{2,s}(\omega)\right| < \varepsilon \quad \forall \omega \in [0,1].$$

For the same reason,

$$\left|\Pi_{2,s}^{\sharp}\big(\beta_{2,1}^{\sharp}(\Theta^{\sharp}(\mathrm{w}_1))\big) - \ddot{G}_{2,s}(\omega)\right| < \varepsilon \quad \forall s = 1,\dots,l_2,\ \omega \in [0,1].$$

Thus, for any $s = 1,\dots,l_2$, $\omega \in [0,1]$, we have

$$\begin{aligned}
&\left|\dot{G}_{2,s}(\omega) - \ddot{G}_{2,s}(\omega)\right| \\
&= \frac{1}{\{2,s\}}\left|(1-\omega)\cdot \Pi_{2,s}^{\sharp}\big(\beta_{2,0}^{\sharp}(\Theta^{\sharp}(\mathrm{w}_1))\big) + \omega\cdot\Pi_{2,s}^{\sharp}\big(\beta_{2,1}^{\sharp}(\Theta^{\sharp}(\mathrm{w}_1))\big) - \ddot{G}_{2,s}(\omega)\right| \\
&< \frac{(1-\omega)\cdot|\Pi_{2,s}^{\sharp}(\beta_{2,0}^{\sharp}(\Theta^{\sharp}(\mathrm{w}_1))) - \ddot{G}_{2,s}(\omega)| + \omega\cdot|\Pi_{2,s}^{\sharp}(\beta_{2,1}^{\sharp}(\Theta^{\sharp}(\mathrm{w}_1))) - \ddot{G}_{2,s}(\omega)|}{\{2,s\}} < \varepsilon,
\end{aligned}$$

as desired.

Step 5. Show that the map ℓ satisfies the condition (vi).

For an arbitrary element $U = (U_1,\dots,U_{l_1};\mathfrak{u},u_2,\dots,u_{p_1}) \in \mathfrak{F} \subset U_0(A_1)$, set $u_1 := \mathrm{j}_1(\mathfrak{u}(\hat{\mathrm{x}}_1))$ and $\bar{U} := (U_1,\dots,U_{l_1}) \in C([0,1],E_1)$. So, the element $\Pi_{J,1}(U) = (U_1,\dots,U_{l_1}; u_1,\dots,u_{p_1}) = (\bar{U}; u_1,\dots,u_{p_1}) \in U_0(A_{C,1})$. By using Lemma 4.9, there is a unitary $V' \in A_{\mathrm{C},1}$, written $V' = (V_1,\dots,V_{l_1}; u_1,\dots,u_{p_1})$, such that $\mathrm{cer}(V') = 1$ and

$$\det(V'(x)) = \det((\Pi_{J,1}(U))(x)) \quad \forall x \in \mathrm{SP}(A_{C,1}).$$

Write $V = (V_1,\dots,V_{l_1};\mathfrak{u},u_2,\dots,u_{p_1})$ which is a unitary in A_1 and let $\bar{V} := (V_1,\dots,V_{l_1}) \in C([0,1],E_1)$. Set

$$\ell(U) = \ell(U_1,\dots,U_{l_1};\mathfrak{u},u_2,\dots,u_{p_1}) = (\mathbb{X}_1,\dots,\mathbb{X}_{l_2};\mathfrak{x},x_2,\dots,x_{p_2}), \tag{6.82}$$

$$\ell(V) = \ell(V_1,\dots,V_{l_1};\mathfrak{u},u_2,\dots,u_{p_1}) = (\mathbb{Y}_1,\dots,\mathbb{Y}_{l_2};\mathfrak{z},x_2,\dots,x_{p_2}), \tag{6.83}$$

where $\mathfrak{x},\mathfrak{z} \in U(A_{X,2}^1) = U(P_2\mathrm{M}_\infty(C(X_2))P_2)$. According to the definition of ℓ (see (6.55)), we get

$$\mathbb{X}_s(\omega) = \mathcal{V}_s(\omega)\cdot\mathrm{diag}((\heartsuit_s(\bar{U}))(\omega), \mathbb{X}_s^u(\omega))\cdot\mathcal{V}_s^*(\omega),$$
$$\mathbb{Y}_s(\omega) = \mathcal{V}_s(\omega)\cdot\mathrm{diag}((\heartsuit_s(\bar{V}))(\omega), \mathbb{Y}_s^v(\omega))\cdot\mathcal{V}_s^*(\omega),$$

where

$$\mathbb{X}_s^u(\omega) = \begin{cases} \mathrm{diag}((\Diamond_s(U))(\omega), \beth_s, \daleth_s) & \text{if } b_{s,1}^{2,1} \geqslant b_{s,1}^{2,0}, \\ \mathrm{diag}((\widetilde{\Diamond}_s(U))(\omega), \widetilde{\beth}_s, \widetilde{\daleth}_s) & \text{if } b_{s,1}^{2,1} < b_{s,1}^{2,0}, \end{cases}$$

and

$$\mathbb{Y}_s^v(\omega) = \begin{cases} \mathrm{diag}((\Diamond_s(V))(\omega), \beth_s, \daleth_s) & \text{if } b_{s,1}^{2,1} \geqslant b_{s,1}^{2,0}, \\ \mathrm{diag}((\widetilde{\Diamond}_s(V))(\omega), \widetilde{\beth}_s, \widetilde{\daleth}_s) & \text{if } b_{s,1}^{2,1} < b_{s,1}^{2,0}. \end{cases}$$

For all $s = 1,\dots,l_2$, let

$$Z_s(\omega) := \mathcal{V}_s(\omega)\cdot\mathrm{diag}((\heartsuit_s(\bar{V}))(\omega), \mathbb{X}_s^u(\omega))\cdot\mathcal{V}_s^*(\omega) \quad \forall\omega\in[0,1]. \tag{6.84}$$

Step 5-1. We have the following claim:

$$Z = (Z_1, \dots, Z_{l_2}; \mathfrak{x}, x_2, \dots, x_{p_2}) \in A_2 \tag{6.85}$$

and there exists a self-adjoint element $\widehat{H} = (H_1, \dots, H_{l_2}; \mathfrak{h}, h_2, \dots, h_{p_2}) \in A_2$ such that

$$\left\| Z - \exp(2\pi i \widehat{H}) \right\| < \frac{\varepsilon}{2}.$$

For the first part of the claim, by setting $x_1 = \mathrm{j}_2(\mathfrak{x}(\hat{\mathrm{x}}_2))$, it is sufficient to show that for any $s = 1, \dots, l_2$,

$$Z_s(0) = \Pi_{2,s}(\beta_{2,0}(x_1, \dots, x_{p_2})) \quad \text{and} \quad Z_s(1) = \Pi_{2,s}(\beta_{2,1}(x_1, \dots, x_{p_2})),$$

where $\Pi_{2,s}$ is the quotient map from E_2 to $\mathrm{M}_{\{2,s\}}$. As $\ell(U)$ is an element in A_2, the equality

$$\mathbb{X}_s(0) = \mathcal{V}_s(0) \cdot \mathrm{diag}((\heartsuit_s(\bar{U}))(0), \mathbb{X}_s^u(0)) \cdot \mathcal{V}_s^*(0) = \Pi_{2,s}(\beta_{2,0}(x_1, \dots, x_{p_2}))$$

holds. And note that

$$Z_s(0) = \mathcal{V}_s(0) \cdot \mathrm{diag}((\heartsuit_s(\bar{V}))(0), \mathbb{X}_s^u(0)) \cdot \mathcal{V}_s^*(0),$$

and $\heartsuit_s(\bar{V})(0)$ and $\heartsuit_s(\bar{V})(1)$ only depends $\bar{V}(0)$ and $\bar{V}(1)$. It follows from Remark 4.10 that $(\heartsuit_s(\bar{U}))(0) = (\heartsuit_s(\bar{V}))(0)$. Hence, we have

$$Z_s(0) = \mathbb{X}_s(0) = \Pi_{s,2}(\beta_{2,0}(x_1, \dots, x_{p_2})), \quad s = 1, \dots, l_2.$$

For the same reason,

$$Z_s(1) = \mathbb{X}_s(1) = \Pi_{s,2}(\beta_{2,1}(x_1, \dots, x_{p_2})), \quad s = 1, \dots, l_2.$$

Thus, the conclusion $Z \in A_2$ is proved.

We begin to prove the second part of the claim, that is, there is a self-adjoint element $\widehat{H} \in A_2$ such that $\|Z - \exp(2\pi i \widehat{H})\| < \frac{\varepsilon}{2}$. From $\mathrm{cer}(V') = 1$, there exists a self-adjoint element $H' = (H_1', \dots, H_{l_1}'; h_1', \dots, h_{p_1}') \in A_{\mathrm{C},1}$ such that

$$\begin{aligned} V' &= (V_1, \dots, V_{l_1}; u_1, \dots, u_{p_1}) \\ &= \exp(2\pi i (H_1', \dots, H_{l_1}'; h_1', \dots, h_{p_1}')) = \exp(2\pi i H'). \end{aligned} \tag{6.86}$$

Particularly,

$$u_e = \exp(2\pi i h_e'), \quad e = 1, \dots, p_1. \tag{6.87}$$

Recall that $[0, 1] \ni \omega \mapsto \varrho_{m,\omega}(\mathfrak{u}(y_m(\omega)))$ is an element in $U(C([0, 1], \mathrm{M}_{[1,1]}))$ with $\varrho_{m,0}(\mathfrak{u}(y_m(0))) = u_1$ and $\varrho_{m,1}(\mathfrak{u}(y_m(1))) = \varrho_{m,1}(\mathfrak{u}(y_m))$ for each $y_m \in \mathcal{Y}$. Thus, there exists a self-adjoint element $h_m'' \in C([0, 1], \mathrm{M}_{[1,1]})$ such that $h_m''(0) = h_1'$, $\varrho_{m,1}(\mathfrak{u}(y_m)) = \exp(2\pi i h_m''(1))$ and

$$\left\| \varrho_{m,\omega}(\mathfrak{u}(y_m(\omega))) - \exp(2\pi i h_m'') \right\| < \frac{\varepsilon}{4}.$$

Recall that $\mathbf{c} = (T-1)\cdot\sum_{j=1}^{l_1}\{1, j\}$ (see (6.32)). Choose a self-adjoint element

$$\operatorname{diag}(H_{1,1}(\omega), \ldots, H_{1,(T-1)}(\omega), \ldots, H_{l_1,1}(\omega), \ldots, H_{l_1,(T-1)}(\omega)) \in C([0,1], \mathrm{M}_{\mathbf{c}})$$

such that for any pair $(j,t) \in \{1,\ldots,l_1\} \times \{1,\ldots,(T-1)\}$,

$$\exp(2\pi i H_{j,t}(0)) = U_j(0) \quad \text{and} \quad \left\| \exp(2\pi i H_{j,t}(\omega)) - U_j\left(\frac{t}{T}\cdot\omega\right)\right\| < \frac{\varepsilon}{4}. \tag{6.88}$$

Set

$$\mho := \operatorname{diag}(h_1''(1)\otimes \mathrm{I}_{\varpi_1}, \ldots, h_M''(1)\otimes \mathrm{I}_{\varpi_M})$$

and

$$\eth := \operatorname{diag}(H_{1,1}(1), \ldots, H_{1,(T-1)}(1), \ldots, H_{l_1,1}(1), \ldots, H_{l_1,(T-1)}(1)),$$

then

$$\operatorname{diag}(\exp(2\pi i\mho), \exp(2\pi i\eth)) = \operatorname{diag}(\Omega(\alpha), \Xi(\bar{U})). \tag{6.89}$$

Now, according to (6.37) and (6.38), define a self-adjoint element $h = (h_1, \ldots, h_{p_2})$ in $F_2 = \oplus_{k=1}^{p_2}\mathrm{M}_{[2,k]}$ with

$$h_k = \begin{cases} \operatorname{diag}\big(h_1'\otimes \mathrm{I}_{\hat{c}}, \mho, \eth, h_1'\otimes \mathrm{I}_{c''_{1,1}}, \ldots, h_{p_1}'\otimes \mathrm{I}_{c''_{1,p_1}}\big) & \text{if } k=1,\\ \operatorname{diag}\big(h_1'\otimes \mathrm{I}_{c_{k,1}}, \ldots, h_{p_1}'\otimes \mathrm{I}_{c_{k,p_1}}\big) & \text{if } k = 2,\ldots,p_2. \end{cases} \tag{6.90}$$

Once again, using the same means as above to connect $\Pi_{2,s}(\beta_{2,0}(h))$ and $\Pi_{2,s}(\beta_{2,1}(h))$.

For each $s = 1, \ldots, l_2$, if $b_{s,1}^{2,1} \geqslant b_{s,1}^{2,0}$, then define

$$H_s(\omega) := \mathcal{V}_s(\omega)\cdot \operatorname{diag}((\heartsuit_s(\bar{H}'))(\omega), \Diamond_s^h(\omega), \beth_s^h, \daleth_s^h)\cdot \mathcal{V}_s^*(\omega) \in C([0,1], \mathrm{M}_{\{2,s\}}),$$

where

$$\begin{aligned} \Diamond_s^h : [0,1] &\to \mathrm{M}_{\varpi\cdot[1,1]+\mathbf{c}}\otimes \mathrm{M}_{b_{s,1}^{2,1}-b_{s,1}^{2,0}} \\ \omega &\mapsto \operatorname{diag}(h_1''(\omega)\otimes \mathrm{I}_{\varpi_1}, \ldots, h_M''(\omega)\otimes \mathrm{I}_{\varpi_M},\\ &\qquad H_{1,1}(\omega), \ldots, H_{1,(T-1)}(\omega), \ldots,\\ &\qquad H_{l_1,1}(\omega), \ldots, H_{l_1,(T-1)}(\omega))\otimes \mathrm{I}_{b_{s,1}^{2,1}-b_{s,1}^{2,0}} \end{aligned}$$

and $\beth_s^h = \operatorname{diag}(\mho, \eth)\otimes \mathrm{I}_{b_{s,1}^{2,0}}$, $\daleth_s^h = \operatorname{diag}(h_1'\otimes \mathrm{I}_{\nu_{s,1}^1}, \ldots, h_{p_1}'\otimes \mathrm{I}_{\nu_{s,p_1}^1})$.

If $b_{s,1}^{2,1} < b_{s,1}^{2,0}$, then define

$$H_s(\omega) := \mathcal{V}_s(\omega)\cdot \operatorname{diag}((\heartsuit_s(\bar{H}'))(\omega), \widetilde{\Diamond}_s^h(\omega), \widetilde{\beth}_s^h, \widetilde{\daleth}_s^h)\cdot \mathcal{V}_s^*(\omega) \in C([0,1], \mathrm{M}_{\{2,s\}}),$$

where

$$\begin{aligned} \widetilde{\Diamond}_s^h : [0,1] &\to \mathrm{M}_{\varpi\cdot[1,1]+\mathbf{c}}\otimes \mathrm{M}_{b_{s,1}^{2,0}-b_{s,1}^{2,1}} \\ \omega &\mapsto \operatorname{diag}(h_1''(1-\omega)\otimes \mathrm{I}_{\varpi_1}, \ldots, h_M''(1-\omega)\otimes \mathrm{I}_{\varpi_M},\\ &\qquad H_{1,1}(1-\omega), \ldots, H_{1,(T-1)}(1-\omega), \ldots,\\ &\qquad H_{l_1,1}(1-\omega), \ldots, H_{l_1,(T-1)}(1-\omega))\otimes \mathrm{I}_{b_{s,1}^{2,0}-b_{s,1}^{2,1}} \end{aligned}$$

and $\widetilde{\beth}_s^h = \operatorname{diag}(\mho, \eth)\otimes \mathrm{I}_{b_{s,1}^{2,1}}$, $\widetilde{\daleth}_s^h = \operatorname{diag}(h_1'\otimes \mathrm{I}_{\nu_{s,1}^0}, \ldots, h_{p_1}'\otimes \mathrm{I}_{\nu_{s,p_1}^0})$.

By means above, we obtain a self-adjoint element $(H_1, \dots, H_{l_2}; h_1, \dots, h_{p_2})$ in $A_{C,2}$. Below, we will show

$$\big\|(Z_1, \dots, Z_{l_2}; x_1, \dots, x_{p_2}) - \exp(2\pi i(H_1, \dots, H_{l_2}; h_1, \dots, h_{p_2}))\big\| < \frac{\varepsilon}{4}. \tag{6.91}$$

First, we have

$$x_k \overset{(6.87)}{=\!=\!=} \exp(2\pi i h_k), \quad k = 2, \dots, p_2. \tag{6.92}$$

Next, by writing $u_1 = \mathrm{j}_1(\mathfrak{u}(\hat{\mathrm{x}}_1))$, we get

$$x_1 := \mathrm{j}_2(\mathfrak{x}(\hat{\mathrm{x}}_2)) \overset{(6.38)}{=\!=\!=} \mathrm{diag}\big(u_1 \otimes \mathrm{I}_{\hat{c}}, \Omega(\mathfrak{u}), \Xi(\bar{U}), u_1 \otimes \mathrm{I}_{c''_{1,1}}, \dots, u_{p_1} \otimes \mathrm{I}_{c''_{1,p_1}}\big).$$

Recalling that $h_1 = \mathrm{diag}(h'_1 \otimes \mathrm{I}_{\hat{c}}, \mho, \eth, h'_1 \otimes \mathrm{I}_{c''_{1,1}}, \dots, h'_{p_1} \otimes \mathrm{I}_{c''_{1,p_1}})$ (see (6.90)), we have

$$x_1 \underset{(6.89)}{\overset{(6.87)}{=\!=\!=}} \exp(2\pi i h_1). \tag{6.93}$$

Further, recall that, for any $s = 1, \dots, l_2$,

$$Z_s(\omega) = \mathcal{V}_s(\omega) \cdot \mathrm{diag}((\heartsuit_s(\bar{V}))(\omega), \mathbb{X}^u_s(\omega)) \cdot \mathcal{V}^*_s(\omega) \quad \forall \omega \in [0, 1] \quad (\text{see } (6.84))$$

and

$$\mathbb{X}^u_s(\omega) = \begin{cases} \mathrm{diag}((\Diamond_s(U))(\omega), \beth_s, \daleth_s) & \text{if } b^{2,1}_{s,1} \geqslant b^{2,0}_{s,1}, \\ \mathrm{diag}((\widetilde{\Diamond}_s(U))(\omega), \widetilde{\beth}_s, \widetilde{\daleth}_s) & \text{if } b^{2,1}_{s,1} < b^{2,0}_{s,1}. \end{cases}$$

From (6.88), we get

$$\|Z_s(\omega) - \exp(2\pi i H_s)\| < \frac{\varepsilon}{4}, \quad s = 1, \dots, l_2.$$

Hence, we have completed the proof of (6.91).

Below, let's proceed to prove this claim. Recall that,

$$\|\psi(\mathfrak{u}) - \exp(2\pi i \mathfrak{h})\| < \frac{\varepsilon}{4} \quad (\text{see } (6.15)),$$

for some self-adjoint element $\mathfrak{h} \in Q\mathrm{M}_\infty(C(X_2))Q$. And note that

$$\begin{aligned} \Pi^1_{X,2}(\Pi_{I,2}(\ell(U))) &\overset{(6.82)}{=\!=\!=} \mathfrak{x} \\ &\overset{(6.36)}{=\!=\!=} \mathrm{diag}\big(\psi(\mathfrak{u}), \Xi(\bar{U}), u_1 \otimes \mathrm{I}_{c''_{1,1}}, \dots, u_{p_1} \otimes \mathrm{I}_{c''_{1,p_1}}\big) \in U(A^1_{X,2}), \end{aligned}$$

where $\Pi^1_{X,2}$ is the quotient map from $A_{X,2}$ to $A^1_{X,2} = P_2\mathrm{M}_\infty(X_2)P_2$, $\bar{U} = (U_1, \dots, U_{l_1}) \in C([0, 1], E_1)$. Let us assume that $\bar{\mathfrak{h}} = \mathrm{diag}(\mathfrak{h}, \eth, h'_1 \otimes \mathrm{I}_{c''_{1,1}}, \dots, h'_{p_1} \otimes \mathrm{I}_{c''_{1,p_1}}) \in (A^1_{X,2})_{s.a.} = (P_2\mathrm{M}_\infty(X_2)P_2)_{s.a.}$. But $h_1 = \mathrm{j}_2(\bar{\mathfrak{h}}(\hat{\mathrm{x}}_2))$ may be *not* true. Below, we will modify $\bar{\mathfrak{h}}$ to $\mathfrak{h} \in (A^1_{X,2})_{s.a.}$ such that $h_1 = \mathrm{j}_2(\mathfrak{h}(\hat{\mathrm{x}}_2))$. Note that

$$\big\|\Pi^1_{X,2}(\Pi_{I,2}(\ell(U))) - \exp(2\pi i \bar{\mathfrak{h}})\big\| = \big\|\mathfrak{x} - \exp(2\pi i \bar{\mathfrak{h}})\big\| < \frac{\varepsilon}{4}. \tag{6.94}$$

Further, recall $\mathrm{j}_2(\mathfrak{x}(\hat{\mathrm{x}}_2)) = x_1 = \exp(2\pi i h_1)$ (see (6.93)). Thus,

$$\left\|\exp(2\pi i h_1) - \exp(2\pi i \cdot \mathrm{j}_2(\bar{\mathfrak{h}}(\hat{\mathrm{x}}_2)))\right\| < \frac{\varepsilon}{4}. \tag{6.95}$$

According to Remark 6.3 and the construction of P_2 (with $\operatorname{rank}(P_2) = [2,1]$), the element $\bar{\mathfrak{h}} \in P_2\mathrm{M}_\infty(C(X_2))P_2$ has the form $\bar{\mathfrak{h}} = \overline{\lambda}_1\mathrm{q}_1 + \cdots + \overline{\lambda}_{[2,1]}\mathrm{q}_{[2,1]}$, where $\overline{\lambda}_1, \ldots, \overline{\lambda}_{[2,1]} \in [0,1)$, $\mathrm{q}_1, \ldots, \mathrm{q}_{[2,1]}$ are mutually orthogonal rank one projections with $\mathrm{q}_1 + \cdots + \mathrm{q}_{[2,1]} = P_2$. As $h_1 \in \mathrm{M}_{[2,1]}$, write $\operatorname{sp}(h_1) = \{\lambda_1, \ldots, \lambda_{[2,1]}\}$. Combining (6.95) with Lemma 4.1, there exist integers $n_1, \ldots, n_{[2,1]}$ such that $|\overline{\lambda}_i - (\lambda_i + n_i)| < \frac{\varepsilon}{4}$ for any $i = 1, \ldots, [2,1]$. Write

$$\mathfrak{h} = \lambda_1\mathrm{q}_1 + \cdots + \lambda_{[2,1]}\mathrm{q}_{[2,1]}.$$

Thus, $\mathrm{j}_2(\mathfrak{h}(\hat{\mathrm{x}}_2)) = h_1$. Now, define $\widehat{H} = (H_1, \ldots, H_{l_2}; \mathfrak{h}, h_2, \ldots, h_{p_2})$. It is easy to check that $\widehat{H} \in (A_2)_{s.a.}$ and

$$\left\|Z - \exp(2\pi i \widehat{H})\right\| < \frac{\varepsilon}{2},$$

which completes the proof of the claim.

Step 5-2. Complete the proof of *Step 5*.

Recall that $\ell(U) = (\mathbb{X}_1, \ldots, \mathbb{X}_{l_2}; \mathfrak{x}, x_2, \ldots, x_{p_2})$ (see (6.82)) and $Z = (Z_1, \ldots, Z_{l_2}; \mathfrak{x}, x_2, \ldots, x_{p_2})$ (see (6.85)), where

$$\begin{aligned} \mathbb{X}_s(\omega) &= \mathcal{V}_s(\omega) \cdot \operatorname{diag}((\heartsuit_s(\bar{U}))(\omega), \mathbb{X}_s^u(\omega)) \cdot \mathcal{V}_s^*(\omega), \\ Z_s(\omega) &= \mathcal{V}_s(\omega) \cdot \operatorname{diag}((\heartsuit_s(\bar{V}))(\omega), \mathbb{X}_s^u(\omega)) \cdot \mathcal{V}_s^*(\omega) \end{aligned}$$

and

$$\mathbb{X}_s^u(\omega) = \begin{cases} \operatorname{diag}((\diamondsuit_s(U))(\omega), \beth_s, \daleth_s) & \text{if } b_{s,1}^{2,1} \geqslant b_{s,1}^{2,0}, \\ \operatorname{diag}((\widetilde{\diamondsuit}_s(U))(\omega), \widetilde{\beth}_s, \widetilde{\daleth}_s) & \text{if } b_{s,1}^{2,1} < b_{s,1}^{2,0}. \end{cases}$$

An easy computation shows that the elements

$$\operatorname{diag}\big((\heartsuit_s(\bar{U}))(\omega), (\diamondsuit_s(U))(\omega)\big) \quad \text{and} \quad \operatorname{diag}\big((\heartsuit_s(\bar{U}))(\omega), (\widetilde{\diamondsuit}_s(U))(\omega)\big)$$

are exactly both in $C([0,1], \mathrm{M}_{L_s'})$, where

$$\begin{aligned} L_s' &= \sum_{e=1}^{p_1} \mu_{s,e}^0[1,e] + |b_{s,1}^2| \cdot \left(\varpi[1,1] + (T-1)\sum_{j=1}^{l_1}\{1,j\}\right) \\ &< \mathcal{B}\mathcal{D}l_1 \sum_{e=1}^{p_1}[1,e] + |b_{s,1}^2| \cdot \left(\varpi[1,1] + (T-1)\sum_{j=1}^{l_1}\{1,j\}\right) = L_s. \end{aligned}$$

Thus, because the condition (6.7) holds, which implies $\mathfrak{B}_2 > \max\{L_s : s = 1, \ldots, l_2\}$, for any $s = 1, \ldots, l_2$, either

$$\sharp(I_r^n \cap \operatorname{sp}(\operatorname{diag}(\beth_s, \daleth_s))) > L_s$$

or

$$\sharp(I_r^n \cap \mathrm{sp}(\mathrm{diag}(\widetilde{\beth}_s, \widetilde{\daleth}_s))) > L_s.$$

By using Lemma 4.8, (6.13) for each $s = 1, \ldots, l_2$, there is a unitary $W_s \in C([0,1], \mathrm{M}_{\{2,s\}})$ such that

$$\|\mathbb{X}_s - W_s Z_s W_s^*\| < \frac{23\pi}{n} < \frac{\varepsilon}{2}$$

and $W_s(0) = W_s(1) = \mathrm{I}_{\{2,s\}}$.

Set $W = (W_1, \ldots, W_{l_2}; P_2, \mathrm{I}_{[2,2]}, \ldots, \mathrm{I}_{[2,p_2]})$, which is an element in $U(A_2)$, and let $H = W\widehat{H}W^*$. So, we get

$$\|\ell(U) - \exp(2\pi i H)\| < \varepsilon.$$

Thus, the $*$-homomorphism ℓ satisfies the condition (vi), as desired. ∎

Fact 6.6. Let us assume that

$$((H_n, (H_n)_+, u_n, G_n), K_n) \quad \text{and} \quad ((H_{n+1}, (H_{n+1})_+, u_{n+1}, G_{n+1}), K_{n+1})$$

be two 5-tuples with the conditions in Fact 5.2. And the homomorphisms both $\gamma_{n,n+1}$ and $\chi_{n,n+1}$ satisfy the conditions in Fact 6.4, respectively. In this time, the commutative diagram (6.5) becomes the following form:

$$\begin{array}{ccc} \mathbb{Z}^{p_n} & \xrightarrow{\gamma'_{n,n+1}=(c_{k,e}^{n,n+1})} & \mathbb{Z}^{p_{n+1}} \\ {\scriptstyle \pi_n=(b_{j,e}^n)}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi_{n+1}=(b_{s,k}^{n+1})} \\ \mathbb{Z}^{l_n} & \xrightarrow[\widetilde{\gamma}_{n,n+1}=(d_{s,j}^{n,n+1})]{} & \mathbb{Z}^{l_{n+1}} \end{array} . \tag{6.96}$$

Now, set

$$\mathfrak{C}_{n,n+1} := \min\{c_{k,e}^{n,n+1} : e = 1, \ldots, p_n;\ k = 1, \ldots, p_{n+1}\}$$

and

$$\mathfrak{B}_{n+1} := \min\{b_{s,k}^{n+1,0}, b_{s,k}^{n,1} : k = 1, \ldots, p_{n+1};\ s = 1, \ldots, l_{n+1}\}.$$

It is a simple matter to guarantee that $\mathfrak{C}_{n,n+1}$ is larger enough passing to a subsequence. Combining Fact 5.1 and Lemma 6.5, we can obtain the following main theorem in this paper.

Theorem 6.7. *For a given weakly unperforated Elliott invariant* $((G, G_+, u), K, \triangle, r)$*, then there exists a simple* C^**-algebra* $A = \varinjlim(A_n, \ell_{n,n+1})$ *such that*

$$\mathrm{cer}(A) \leqslant 1 + \varepsilon$$

and $\mathrm{Ell}(A) \cong ((G, G_+, u), K, \triangle, r)$.

Proof. We will prove it by induction.

(I) Through the 5-tuple $((H_1,(H_1)_+,u_1,G_1),K_1)$, we fix a C^*-algebra A_1 with the form in Fact 5.3. Because A_1 is separable, there are finite sets $\mathfrak{F}_{1,k}, k=1,2,\dots,$ such that $\mathfrak{F}_{1,1}\subset\mathfrak{F}_{1,2}\subset\cdots\subset U_0(A_1)$, and the countable set $\bigcup_{k=1}^{\infty}\mathfrak{F}_{1,k}$ is dense in $U_0(A_1)$.

(II) Set $\mathfrak{F}_1:=\mathfrak{F}_{1,1}$, which is finite set contained $U_0(A_1)$, and let $\varepsilon_1=\delta_1=\frac{1}{2^1}$. From Lemma 6.5, there exist a 5-tuple, without loss of generality, written

$$((H_2,(H_2)_+,u_2,G_2),K_2)$$

and the diagram (6.96) satisfying the following property.

If $\mathfrak{C}_{1,2},\mathfrak{B}_2$ are large enough, then there exist a C^*-algebra A_2 with the form in Fact 5.3 and a $*$-homomorphism $\ell_{1,2}:A_1\to A_2$ such that $K_0(\ell_{1,2})=\gamma_{1,2}|_{G_1}$, $K_1(\ell_{1,2})=\chi_{1,2}$, and for any $U\in\mathfrak{F}_1$ there is a self-adjoint element $H\in A_2$ such that $\|\ell_{1,2}(U)-\exp(2\pi iH)\|<\frac{1}{2^1}$.

(III) Now, assume that we have obtained the diagram below

$$A_1\xrightarrow{\ell_{1,2}}A_2\xrightarrow{\ell_{2,3}}\cdots\xrightarrow{\ell_{n-1,n}}A_n.$$

we will obtain A_{n+1} and $\ell_{n,n+1}$ by induction. Because A_n is separable, there are finite sets $\mathfrak{F}_{n,k}, k=1,2,\dots,$ such that $\mathfrak{F}_{n,1}\subset\mathfrak{F}_{n,2}\subset\cdots\subset U_0(A_n)$, and the countable set $\bigcup_{k=1}^{\infty}\mathfrak{F}_{n,k}$ is dense in $U_0(A_n)$.

Set $\mathfrak{F}_n=\bigcup_{i=1}^{n}\ell_{i,n}(\mathfrak{F}_{i,(n+1)-i})$, and let $\varepsilon_n=\delta_n=\frac{1}{2^n}$. From Lemma 6.5, there exist a 5-tuple, without loss of generality, written $((H_{n+1},(H_{n+1})_+,u_{n+1},G_{n+1}),K_{n+1})$ and the diagram (6.96) satisfying the following property.

If $\mathfrak{C}_{n,n+1},\mathfrak{B}_{n+1}$ are large enough, then there are a C^*-algebra A_{n+1} with the form in Fact 5.3 and a $*$-homomorphism $\ell_{n,n+1}:A_n\to A_{n+1}$ such that $K_0(\ell_{n,n+1})=\gamma_{n,n+1}|_{G_n}$, $K_1(\ell_{n,n+1})=\chi_{n,n+1}$, and for any $U\in\mathfrak{F}_n$ there is a self-adjoint element $H\in A_n$ such that $\|\ell_{n,n+1}(U)-\exp(2\pi iH)\|<\frac{1}{2^n}$ hold.

At the same time, by using (iv) and (v) in Lemma 6.5, we also obtain the following approximate intertwining diagram:

$$\begin{tikzcd}
\mathrm{Aff}(T(A_1)) \arrow[r, "\ell_{1,2}^{\sharp}"] \arrow[d, bend right, "\Pi_1^{q\sharp}"'] & \mathrm{Aff}(T(A_2)) \arrow[r] \arrow[d, bend right, "\Pi_2^{q\sharp}"'] & \cdots \arrow[r] & \mathrm{Aff}(T(A_n)) \arrow[r, "\ell_{n,n+1}^{\sharp}"] \arrow[d, bend right, "\Pi_n^{q\sharp}"'] & \mathrm{Aff}(T(A_{n+1})) \arrow[r] \arrow[d, bend right, "\Pi_{n+1}^{q\sharp}"'] & \cdots \\
\mathrm{Aff}(T(F_1)) \arrow[r, "\Theta_{1,2}^{\sharp}"'] \arrow[u, bend right, "\Gamma_1"'] & \mathrm{Aff}(T(F_2)) \arrow[r] \arrow[u, bend right, "\Gamma_2"'] & \cdots \arrow[r] & \mathrm{Aff}(T(F_n)) \arrow[r, "\Theta_{n,n+1}^{\sharp}"'] \arrow[u, bend right, "\Gamma_n"'] & \mathrm{Aff}(T(F_{n+1})) \arrow[r] \arrow[u, bend right, "\Gamma_{n+1}"'] & \cdots
\end{tikzcd}$$

Thus, by using [15, Remark 13.10 and Proposition 13.5], we obtain a C^*-algebra $A=\varinjlim(A_n,\ell_{n,n+1})$, which satisfies

$$\mathrm{cer}(A)\leqslant 1+\varepsilon$$

and $((K_0(A), K_0(A)_+, [\mathbf{1}_A]), K_1(A), T(A), r_A) \cong ((G, G_+, u), K, \triangle, r)$, as desired. ■

Corollary 6.8. *For a given weakly unperforated Elliott invariant* $((G, G_+, u), K, \triangle, r)$*, then there exists a* C^**-algebra* $A = \varinjlim(A_n, \ell_{n,n+1}) \in \mathcal{N}_0^{\mathcal{Z}}$ *such that*

$$\mathrm{cer}(A) \leqslant 1 + \varepsilon$$

and $\mathrm{Ell}(A) \cong ((G, G_+, u), K, \triangle, r)$.

Proof. It follows from [15, Theorem 13.50] and Theorem 6.7. ■

Theorem 6.9. *Let* A *be a unital separable simple amenable* $\mathcal{Z}$*-stable* C^**-algebras which satisfy the UCT. Then,* $\mathrm{cer}(A) \leqslant 1 + \varepsilon$.

Proof. By [14], A is either stably finite or purely infinite. If A is purely infinite, then by [27], $\mathrm{cer}(A) \leqslant 1 + \varepsilon$ (see also [23, 34]). If A is stably finite, by [3, 33], A satisfies the condition $T_{qd}(A) = T(A) \neq \emptyset$. By [9, Theorem 4.10], $gTR(A) \leqslant 1$. Now, the theorem follows from Corollary 6.8 and [16, Corollary 29.9]. ■

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