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Cohomologies, deformations and extensions of pre-LieDer pairs

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Abstract. In this paper, we use the higher derived bracket to give the controlling algebra of pre-LieDer pairs. We give the cohomology of pre-LieDer pairs by using the twist L_∞-algebra of this controlling algebra. In particular, we define the cohomology of regular pre-LieDer pairs. We study infinitesimal deformations of pre-LieDer pairs, which are characterized by the second cohomology group of pre-LieDer pairs. We also define the cohomology of regular pre-LieDer pairs with coefficients in an arbitrary representation and using the second cohomology group to classify abelian extensions of regular pre-LieDer pairs.

1. Introduction

Derivations play an important role in the study of various algebraic structures. They can be used to construct higher derived brackets and homotopy Lie algebras [36], deformation formulas [10] and differential Galois theory [30]. They are also important in control theory [1,2] and functional analysis [6,7]. Moreover, they also play a fundamental role in the study of gauge theories in quantum field theory via the BV formalism [5]. In [16,29], the authors studied the operad of associative algebras with derivation. Recently, in [33], the authors studied the cohomology, extensions and deformations of Lie algebras with derivations. Using similar ideas, the author studied associative algebras with derivations [13] and Leibniz algebras with derivations [12].

The notion of a pre-Lie algebra (Its other names are left-symmetric algebras, quasi-associative algebras, Vinberg algebras and so on) has been introduced independently by M. Gerstenhaber in the deformation theory of rings and algebras [19]. Pre-Lie algebra arose from the study of affine manifolds and affine structures on Lie groups [24] and homogeneous convex cones [34]. Its defining identity is weaker than associativity. This algebraic structure describes some properties of cochains space in Hochschild cohomology of an associative algebra, rooted trees and vector fields on affine spaces. Moreover, it is playing an increasing role in algebra, geometry and physics due to their applications in nonassociative algebras, combinatorics, numerical analysis and quantum field theory; see also in [3, 4, 8, 11]. There is a close relationship between pre-Lie algebras and Lie

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algebras: a pre-Lie algebra $(\mathfrak{g}, \cdot)$ gives rise to a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_C)$ via the commutator bracket, which is called the subadjacent Lie algebra and denoted by $\mathfrak{g}^C$. Furthermore, the map $L : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, defined by $L_x y = x \cdot y$ for all $x, y \in \mathfrak{g}$, gives rise to a representation of the subadjacent Lie algebra $\mathfrak{g}^C$ on $\mathfrak{g}$.

The purpose of the paper is to study the cohomology of a pre-LieDer pair and its applications. The motivation for such a study is due to the study of relative difference Lie algebras and relative Rota–Baxter Lie algebras [21–23, 27]. We use the higher derived bracket to give the controlling algebra of pre-LieDer pairs. Using the Maurer–Cartan element of this controlling algebra, we construct a twist L_∞-algebra. We give the cohomology of pre-LieDer pairs by using this twist L_∞-algebra. Compared to Lie algebras, associative algebras and Leibniz algebras, this is a different way to constructing cohomologies with derivations. Using this cohomology, we study infinitesimal deformations of pre-LieDer pairs. We also define the cohomology of regular pre-LieDer pairs with coefficients in an arbitrary representation and study abelian extensions of regular pre-LieDer pairs.

The paper is organized as follows. In Section 2, first, we give the notion of pre-LieDer pairs. Then, by a given representation $(V; \rho, \mu)$ of a pre-Lie algebra $(\mathfrak{g}, \pi)$, we recall the graded Lie algebra whose Maurer–Cartan elements are $\pi + \rho + \mu$. Finally, we use this Maurer–Cartan element to give the cohomology of pre-Lie algebras with representations. In Section 3, first, we recall the notion of L_∞-algebras and higher derived brackets. Then, we use higher derived brackets to construct an L_∞-algebra. Given a pre-LieDer pair, we can obtain the Maurer–Cartan element of this L_∞-algebra. Finally, using this Maurer–Cartan element, we construct a twist L_∞-algebra, which controls deformations of pre-LieDer pairs. In Section 4, first, using the twist L_∞-algebra constructed from the last section, we give the cohomology of pre-LieDer pairs. Then, we study infinitesimal deformations of pre-LieDer pairs by using this cohomology. We show that equivalent infinitesimal deformations are in the same second cohomology group. Finally, we use this cohomology to give the cohomology of regular pre-LieDer pairs. In Section 5, first, we introduce the notion of representations of regular pre-LieDer pairs. Then, we give cohomologies of regular pre-LieDer pairs with coefficients in an arbitrary representation. Finally, we deal with abelian extensions of regular pre-LieDer pairs. We show that the second cohomology group classifies abelian extensions of regular pre-LieDer pairs.

In this paper, we work over an algebraically closed field $\mathbb{K}$ of characteristic 0, and all the vector spaces are over $\mathbb{K}$.

The following are cohomologies that appeared in this paper. We made a summary.

(1) The cohomology of the cochain complex $\left(\bigoplus_{n=1}^{+\infty} C^n(\mathfrak{g}; V), \mathrm{d}\right)$ is called the cohomology of the pre-Lie algebra $(\mathfrak{g}, \cdot)$ with coefficients in the representation $(V; \rho, \mu)$.

(2) The cohomology of the cochain complex $\left(\bigoplus_{n=1}^{+\infty} C^n(\mathfrak{g}, \rho, \mu), \partial\right)$ is called the cohomology of the pre-Lie algebra $(\mathfrak{g}, \cdot)$ with the representation $(V; \rho, \mu)$.

(3) The cohomology of the cochain complex $\big(\bigoplus_{n=1}^{+\infty} C^n(\mathfrak{g}, \rho, \mu, D), \mathcal{D}\big)$ is called the cohomology of the pre-LieDer pair $((\mathfrak{g}, \cdot), D, \rho, \mu)$.

(4) The cohomology of the cochain complex $\big(\bigoplus_{n=1}^{+\infty} C^n(\mathfrak{g}, D), \bar{\mathcal{D}}\big)$ is called the cohomology of the regular pre-LieDer pair $((\mathfrak{g}, \cdot), D)$.

(5) The cohomology of the cochain complex $\big(\bigoplus_{n=1}^{+\infty} C^n(\mathfrak{g}, D; V, K, \widetilde{\rho}, \widetilde{\mu}), \mathcal{D}_{(\widetilde{\rho}, \widetilde{\mu})}\big)$ is called the cohomology of the regular pre-LieDer pair $((\mathfrak{g}, \cdot), D)$ with coefficients in the representation $(V, K, \widetilde{\rho}, \widetilde{\mu})$.

2. Cohomologies of pre-Lie algebras with representations

In this section, first, we define a pre-LieDer pair, which consists of a pre-Lie algebra $(\mathfrak{g}, \pi)$ and a derivation D. Then, we recall the graded Lie algebra whose Maurer–Cartan element is $\pi + \rho + \mu$, where $(V; \rho, \mu)$ is a representation of the pre-Lie algebra $(\mathfrak{g}, \pi)$. Finally, we use this Maurer–Cartan element to give the cohomology of the pre-Lie algebra $(\mathfrak{g}, \pi)$ with the representation $(V; \rho, \mu)$.

Definition 2.1 ([8]). A *pre-Lie algebra* $(\mathfrak{g}, \cdot)$ is a vector space $\mathfrak{g}$ equipped with a bilinear product $\cdot : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ such that for all $x, y, z \in \mathfrak{g}$, the following equality is satisfied:

$$(x \cdot y) \cdot z - x \cdot (y \cdot z) = (y \cdot x) \cdot z - y \cdot (x \cdot z).$$

Let $(\mathfrak{g}, \cdot)$ be a pre-Lie algebra. The commutator $[x, y]_C = x \cdot y - y \cdot x$ gives a Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_C)$, which is denoted by $\mathfrak{g}^C$ and called the *sub-adjacent Lie algebra* of $(\mathfrak{g}, \cdot)$.

Definition 2.2 ([3]). A *representation* of a pre-Lie algebra $(\mathfrak{g}, \cdot)$ on a vector space V consists of a pair (ρ, μ), where $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of the sub-adjacent Lie algebra $\mathfrak{g}^C$ on V, and $\mu : \mathfrak{g} \to \mathfrak{gl}(V)$ is a linear map such that for all $x, y \in \mathfrak{g}$:

$$\mu(y) \circ \mu(x) - \mu(x \cdot y) = \mu(y) \circ \rho(x) - \rho(x) \circ \mu(y).$$

We denote a representation of a pre-Lie algebra $(\mathfrak{g}, \cdot)$ by $(V; \rho, \mu)$. Furthermore, let $L, R : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ be linear maps, where $L_x y = x \cdot y$, $R_x y = y \cdot x$. Then $(\mathfrak{g}; L, R)$ is also a representation, which is called the regular representation.

Definition 2.3. Let $(V; \rho, \mu)$ be a representation of a pre-Lie algebra $(\mathfrak{g}, \cdot)$ and $D : \mathfrak{g} \to V$ a linear map such that for all $x, y \in \mathfrak{g}$,

$$D(x \cdot y) = \rho(x)D(y) + \mu(y)D(x).$$

Then D is a *derivation* of the pre-Lie algebra $(\mathfrak{g}, \cdot)$ with respect to the representation $(V; \rho, \mu)$. A pre-Lie algebra $(\mathfrak{g}, \cdot)$ with D is called a *pre-LieDer pair*, which is denoted by $((\mathfrak{g}, \cdot), D, \rho, \mu)$. In particular, let $(\mathfrak{g}; L, R)$ be a regular representation, then $((\mathfrak{g}, \cdot), D, L, R)$ is called a regular pre-LieDer pair and simply denoted by $((\mathfrak{g}, \cdot), D)$.

Example 1. Let $\{e_1, e_2, e_3\}$ be a basis of 3-dimensional vector space $\mathfrak{g}$. Define a product on $\mathfrak{g}$ by

$$\begin{aligned} &e_1 \cdot e_1 = e_1, \quad && e_1 \cdot e_2 = e_3, \quad && e_1 \cdot e_3 = 0, \\ &e_2 \cdot e_1 = 0, && e_2 \cdot e_2 = 0, && e_2 \cdot e_3 = 0, \\ &e_3 \cdot e_1 = 0, && e_3 \cdot e_2 = 0, && e_3 \cdot e_3 = 0. \end{aligned}$$

Then $(\mathfrak{g}, \cdot)$ is a pre-Lie algebra. Let $D = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right)$. More precisely, $D(e_1) = 0$, $D(e_2) = e_2 + e_3$, $D(e_3) = e_3$. Then $((\mathfrak{g}, \cdot), D)$ is a regular pre-LieDer pair.

Definition 2.4. Let $((\mathfrak{g}, \cdot), D, \rho, \mu)$ and $((\mathfrak{g}', \cdot'), D', \rho', \mu')$ be two pre-LieDer pairs. A *homomorphism* $(f_\mathfrak{g}, f_V)$ from $((\mathfrak{g}, \cdot), D, \rho, \mu)$ to $((\mathfrak{g}', \cdot'), D', \rho', \mu')$ consists of a pre-Lie algebra homomorphism $f_\mathfrak{g} : (\mathfrak{g}, \cdot) \to (\mathfrak{g}', \cdot')$ and a linear map $f_V : V \to V'$ such that for all $x \in \mathfrak{g}$,

$$\begin{aligned} f_V \circ \rho(x) &= \rho'(f_\mathfrak{g}(x)) \circ f_V, \\ f_V \circ \mu(x) &= \mu'(f_\mathfrak{g}(x)) \circ f_V, \\ f_V \circ D &= D' \circ f_\mathfrak{g}. \end{aligned}$$

Let $(V; \rho, \mu)$ be a representation of a pre-Lie algebra $(\mathfrak{g}, \cdot)$. The set of n-cochains is given by

$$C^n(\mathfrak{g}; V) = \mathrm{Hom}(\wedge^{n-1}\mathfrak{g} \otimes \mathfrak{g}, V) \quad \forall n \geq 1.$$

For all $f \in C^n(\mathfrak{g}; V)$, $x_1, \ldots, x_{n+1} \in \mathfrak{g}$, define the coboundary operator $\mathrm{d} : C^n(\mathfrak{g}; V) \to C^{n+1}(\mathfrak{g}; V)$ by

$$\begin{aligned} (\mathrm{d}f)(x_1, \ldots, x_{n+1}) = &\sum_{i=1}^{n} (-1)^{i+1} \rho(x_i) f(x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^{i+1} \mu(x_{n+1}) f(x_1, \ldots, \widehat{x_i}, \ldots, x_n, x_i) \\ &- \sum_{i=1}^{n} (-1)^{i+1} f(x_1, \ldots, \widehat{x_i}, \ldots, x_n, x_i \cdot x_{n+1}) \\ &+ \sum_{1 \leq i < j \leq n} (-1)^{i+j} f([x_i, x_j]_C, x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_{n+1}). \end{aligned}$$

Theorem 2.5 ([17]). *With the above notation,* $\left(\bigoplus_{n=1}^{+\infty} C^n(\mathfrak{g}; V), \mathrm{d}\right)$ *is a chain complex.*

Definition 2.6 ([17]). The cohomology of the cochain complex $\left(\bigoplus_{n=1}^{+\infty} C^n(\mathfrak{g}; V), \mathrm{d}\right)$ is called the cohomology of the pre-Lie algebra $(\mathfrak{g}, \cdot)$ with coefficients in the representation $(V; \rho, \mu)$. The corresponding cohomology group is denoted by $H^n(\mathfrak{g}; V)$.

Remark 2.7. We use $\mathrm{d}_{\mathrm{reg}}$ to denote the coboundary operator of $(\mathfrak{g}, \cdot)$ with the coefficients in the regular representation.

A permutation $\sigma \in \mathbb{S}_n$ is called an $(i, n-i)$-unshuffle if $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(n)$. If $i = 0$ and $i = n$, we assume $\sigma = \mathrm{Id}$. The set of all $(i, n-i)$-unshuffles will be denoted by $\mathbb{S}_{(i,n-i)}$. The notion of an $(i_1, \ldots, i_k)$-unshuffle and the set $\mathbb{S}_{(i_1,\ldots,i_k)}$ are defined similarly.

Let $\mathfrak{g}$ be a vector space. It was shown in [9, 31, 38] that the graded vector space $C^*(\mathfrak{g};\mathfrak{g}) = \bigoplus_{n=1}^{+\infty} C^n(\mathfrak{g};\mathfrak{g}) = \bigoplus_{n=1}^{+\infty} \mathrm{Hom}(\wedge^{n-1}\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$ equipped with the Matsushima–Nijenhuis bracket

$$[P, Q]^{MN} = P \circ Q - (-1)^{pq} Q \circ P \quad \forall P \in C^{p+1}(\mathfrak{g};\mathfrak{g}),\ Q \in C^{q+1}(\mathfrak{g};\mathfrak{g})$$

is a graded Lie algebra, where $P \circ Q \in C^{p+q+1}(\mathfrak{g};\mathfrak{g})$ is defined by

$$\begin{aligned} P \circ Q(x_1, \ldots, x_{p+q+1}) &= \sum_{\sigma \in \mathbb{S}(q,1,p-1)} \mathrm{sgn}(\sigma) P(Q(x_{\sigma(1)}, \ldots, x_{\sigma(q)}, x_{\sigma(q+1)}), \\ &\qquad x_{\sigma(q+2)}, \ldots, x_{\sigma(p+q)}, x_{p+q+1}) \\ &\quad + (-1)^{pq} \sum_{\sigma \in \mathbb{S}(p,q)} \mathrm{sgn}(\sigma) P(x_{\sigma(1)}, \ldots, x_{\sigma(p)}, \\ &\qquad Q(x_{\sigma(p+1)}, \ldots, x_{\sigma(p+q)}, x_{p+q+1})). \end{aligned}$$

In particular, $\pi \in \mathrm{Hom}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$ defines a pre-Lie algebra if and only if $[\pi, \pi]^{MN} = 0$. If π is a pre-Lie algebra structure, then $\mathrm{d}_\pi := [\pi, \cdot]^{MN}$ is a graded derivation of the graded Lie algebra $(C^*(\mathfrak{g};\mathfrak{g}), [\cdot,\cdot]^{MN})$ satisfying $\mathrm{d}_\pi \circ \mathrm{d}_\pi = 0$, so that $(C^*(\mathfrak{g};\mathfrak{g}), [\cdot,\cdot]^{MN}, \mathrm{d}_\pi)$ becomes a differential graded Lie algebra.

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be vector spaces, and let elements of $\mathfrak{g}_1$ be denoted by x, y, x_i and elements of $\mathfrak{g}_2$ be denoted by u, v, v_i. Let $c : \wedge^{n-1}\mathfrak{g}_1 \otimes \mathfrak{g}_1 \to \mathfrak{g}_2$ be a linear map. We can construct a linear map $\widehat{c} \in C^n(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2)$ by

$$\widehat{c}(x_1 + v_1, \ldots, x_n + v_n) := c(x_1, \ldots, x_n).$$

In general, for a given linear map $f : \wedge^{k-1}\mathfrak{g}_1 \otimes \wedge^{l}\mathfrak{g}_2 \otimes \mathfrak{g}_1 \to \mathfrak{g}_j$ for $j \in \{1, 2\}$, we define a linear map $\widehat{f} \in C^{k+l}(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2)$ by

$$\begin{aligned} \widehat{f}(x_1 + v_1, \ldots, x_{k+l} + v_{k+l}) &= \sum_{\sigma \in \mathbb{S}(k-1,l)} \mathrm{sgn}(\sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(k-l)}, v_{\sigma(k)}, \ldots, v_{\sigma(k+l-1)}, x_{k+l}). \end{aligned}$$

Similarly, for $f : \wedge^{k}\mathfrak{g}_1 \otimes \wedge^{l-1}\mathfrak{g}_2 \otimes \mathfrak{g}_2 \to \mathfrak{g}_j$ for $j \in \{1, 2\}$, we define a linear map $\widehat{f} \in C^{k+l}(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2)$ by

$$\begin{aligned} \widehat{f}(x_1 + v_1, \ldots, x_{k+l} + v_{k+l}) &= \sum_{\sigma \in \mathbb{S}(k,l-1)} \mathrm{sgn}(\sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(k)}, v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l-1)}, v_{k+l}). \end{aligned}$$

We call the linear map $\widehat{f}$ a *horizontal lift* of f or simply a lift. We define $\mathscr{G}^{k,l} = \wedge^{k-1}\mathfrak{g}_1 \otimes \wedge^l \mathfrak{g}_2 \otimes \mathfrak{g}_1 + \wedge^k \mathfrak{g}_1 \otimes \wedge^{l-1}\mathfrak{g}_2 \otimes \mathfrak{g}_2$. The vector space $\wedge^{n-1}(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \otimes (\mathfrak{g}_1 \oplus \mathfrak{g}_2)$ is isomorphic to the direct sum of $\mathscr{G}^{k,l}, k + l = n$.

Definition 2.8 ([37]). A linear map $f \in \mathrm{Hom}(\wedge^{n-1}(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \otimes (\mathfrak{g}_1 \oplus \mathfrak{g}_2), \mathfrak{g}_1 \oplus \mathfrak{g}_2)$ has a *bidegree* $k \mid l$ if the following four conditions hold:

(1) $k + l + 1 = n$.

(2) If X is an element in $\mathscr{G}^{k+1,l}$, then $f(X) \in \mathfrak{g}_1$.

(3) If X is an element in $\mathscr{G}^{k,l+1}$, then $f(X) \in \mathfrak{g}_2$.

(4) All the other cases, $f(X) = 0$.

We denote a linear map f with bidegree $k \mid l$ by $\|f\| = k \mid l$.

We call a linear map f *homogeneous* if f has a bidegree. We denote the set of homogeneous linear maps of bidegree $k \mid l$ by $C^{k|l}(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2)$. We have $k + l \geq 0, k, l \geq -1$ because $n \geq 1$ and $k + 1, l + 1 \geq 0$.

In our later study, the subspaces $C^{k|0}(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2)$ and $C^{l|-1}(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2)$ will be frequently used. By the above lift, we have the following isomorphisms:

$$\begin{aligned} C^{k|0}(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2) \cong\ & \mathrm{Hom}(\wedge^k \mathfrak{g}_1 \otimes \mathfrak{g}_1, \mathfrak{g}_1) \oplus \mathrm{Hom}(\wedge^k \mathfrak{g}_1 \otimes \mathfrak{g}_2, \mathfrak{g}_2) \\ & \oplus \mathrm{Hom}(\wedge^{k-1} \mathfrak{g}_1 \otimes \mathfrak{g}_2 \otimes \mathfrak{g}_1, \mathfrak{g}_2); \\ C^{l|-1}(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2) \cong\ & \mathrm{Hom}(\wedge^{l-1} \mathfrak{g}_1 \otimes \mathfrak{g}_1, \mathfrak{g}_2). \end{aligned}$$

Lemma 2.9 ([37]). *If $\|f\| = k_f \mid l_f$ and $\|g\| = k_g \mid l_g$, then $[f, g]^{MN}$ has the bidegree $k_f + k_g \mid l_f + l_g$.*

Lemma 2.10 ([37]). *If $\|f\| = -1 \mid k$ (resp. $k \mid -1$) and $\|g\| = -1 \mid l$ (resp. $l \mid -1$), then $[f, g]^{MN} = 0$.*

By Lemmas 2.9 and 2.10, we obtain that $\left(\bigoplus_{k=0}^{+\infty} C^{k|0}(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2), [\cdot, \cdot]^{MN}\right)$ is a graded Lie sub-algebra of the graded Lie algebra $(C^*(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2), [\cdot, \cdot]^{MN})$ and $\left(\bigoplus_{l=0}^{+\infty} C^{l|-1}(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2), [\cdot, \cdot]^{MN}\right)$ is an abelian graded Lie sub-algebra of the graded Lie algebra $(C^*(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2), [\cdot, \cdot]^{MN})$.

For all $\pi \in \mathrm{Hom}(\mathfrak{g} \otimes \mathfrak{g}, V)$, $\rho \in \mathrm{Hom}(\mathfrak{g} \otimes V, V)$ and $\mu \in \mathrm{Hom}(V \otimes \mathfrak{g}, V)$, we have the following.

Proposition 2.11 ([28]). *Let $\mathfrak{g}$ and V be two vector spaces. Then $\pi + \rho + \mu \in C^{1|0}(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V)$ is a Maurer–Cartan element of the graded Lie algebra $\left(\bigoplus_{k=0}^{+\infty} C^{k|0}(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V), [\cdot, \cdot]^{MN}\right)$ if and only if $(V; \rho, \mu)$ is a representation of the pre-Lie algebra $(\mathfrak{g}, \pi)$.*

Let $(V; \rho, \mu)$ be a representation of a pre-Lie algebra $(\mathfrak{g}, \pi)$. By Proposition 2.11, $\pi + \rho + \mu$ is a Maurer–Cartan element of $\left(\bigoplus_{k=0}^{+\infty} C^{k|0}(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V), [\cdot, \cdot]^{MN}\right)$. It

follows from the graded Jacobi identity that $\mathrm{d}_{\pi+\rho+\mu} := [\pi+\rho+\mu, \cdot]^{MN}$ is a graded derivation of the graded Lie algebra $\big(\bigoplus_{k=0}^{+\infty} C^{k|0}(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V), [\cdot,\cdot]^{MN}\big)$ satisfying $\mathrm{d}^2_{\pi+\rho+\mu} = 0$. Thus, we have the following.

Theorem 2.12. *Let $(\mathfrak{g}, \pi)$ be a pre-Lie algebra and $(V; \rho, \mu)$ be a representation of $(\mathfrak{g}, \pi)$. Then $\big(\bigoplus_{k=0}^{+\infty} C^{k|0}(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V), [\cdot,\cdot]^{MN}, \mathrm{d}_{\pi+\rho+\mu}\big)$ is a differential graded Lie algebra.*

Furthermore, $(V; \rho+\rho', \mu+\mu')$ is a representation of the pre-Lie algebra $(\mathfrak{g}, \pi+\pi')$ for $\pi' \in \mathrm{Hom}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$, $\rho' \in \mathrm{Hom}(\mathfrak{g} \otimes V, V)$ and $\mu' \in \mathrm{Hom}(V \otimes \mathfrak{g}, \mathfrak{g})$ if and only if $\pi' + \rho' + \mu'$ is a Maurer–Cartan element of the differential graded Lie algebra $\big(\bigoplus_{k=0}^{+\infty} C^{k|0}(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V), [\cdot,\cdot]^{MN}, \mathrm{d}_{\pi+\rho+\mu}\big)$.

Let $(V; \rho, \mu)$ be a representation of the pre-Lie algebra $(\mathfrak{g}, \pi)$. Define the set of 0-cochains $C^0(\mathfrak{g}, \rho, \mu)$ to be 0. For $n \geq 1$, we define the set of n-cochains $C^n(\mathfrak{g}, \rho, \mu)$ by

$$\begin{aligned} C^n(\mathfrak{g}, \rho, \mu) := {} & \mathrm{Hom}(\wedge^{n-1}\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g}) \oplus \mathrm{Hom}(\wedge^{n-1}\mathfrak{g} \otimes V, V) \\ & \oplus \mathrm{Hom}(\wedge^{n-2}\mathfrak{g} \otimes V \otimes \mathfrak{g}, V). \end{aligned}$$

Define the coboundary operator $\partial : C^n(\mathfrak{g}, \rho, \mu) \to C^{n+1}(\mathfrak{g}, \rho, \mu)$ by

$$\partial f := (-1)^{n-1} \mathrm{d}_{\pi+\rho+\mu} f = (-1)^{n-1} [\pi+\rho+\mu, f]^{MN}.$$

Since $\mathrm{d}^2_{\pi+\rho+\mu} = 0$, we obtain that $\partial^2 = 0$. Thus, $\big(\bigoplus_{n=0}^{+\infty} C^n(\mathfrak{g}, \rho, \mu), \partial\big)$ is a cochain complex.

Definition 2.13 ([28]). The cohomology of the cochain complex $\big(\bigoplus_{n=1}^{+\infty} C^n(\mathfrak{g}, \rho, \mu), \partial\big)$ is called the cohomology of the pre-Lie algebra $(\mathfrak{g}, \cdot)$ with the representation $(V; \rho, \mu)$. The corresponding cohomology group is denoted by $H^n(\mathfrak{g}, \rho, \mu)$.

For all $f = (f_\mathfrak{g}, f_\rho, f_\mu) \in C^n(\mathfrak{g}, \rho, \mu)$, where $f_\mathfrak{g} \in \mathrm{Hom}(\wedge^{n-1}\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$, $f_\rho \in \mathrm{Hom}(\wedge^{n-1}\mathfrak{g} \otimes V, V)$ and $f_\mu \in \mathrm{Hom}(\wedge^{n-2}\mathfrak{g} \otimes V \otimes \mathfrak{g}, V)$. Then we have

$$\partial f = ((\partial f)_\mathfrak{g}, (\partial f)_\rho, (\partial f)_\mu).$$

By straightforward computation, we obtain that

$$(\partial f)_\mathfrak{g}(x_1, \ldots, x_{n+1}) = \mathrm{d}_{\mathrm{reg}} f_\mathfrak{g}(x_1, \ldots, x_{n+1}). \tag{2.1}$$

$$\begin{aligned} & (\partial f)_\rho(x_1, \ldots, x_n, u) \\ & \quad = \sum_{i=1}^{n} (-1)^{i+1} \rho(f_\mathfrak{g}(x_1, \ldots, \widehat{x_i}, \ldots, x_n, x_i))u \\ & \qquad + \sum_{i=1}^{n} (-1)^{i+1} \rho(x_i) f_\rho(x_1, \ldots, \widehat{x_i}, \ldots, x_n, u) \end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=1}^{n} (-1)^{i+1} f_\rho(x_1, \ldots, \widehat{x_i}, \ldots, x_n, \rho(x_i)u) \\
&+ \sum_{1 \le i < j \le n} (-1)^{i+j} f_\rho([x_i, x_j]_C, x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_n, u).
\end{aligned}
$$

$$
\begin{aligned}
&(\partial f)_\mu(x_1, \ldots, x_{n-1}, u, x_n) \\
&\quad = (-1)^{n-1}(\mu(f_{\mathfrak{g}}(x_1, \ldots, x_n))u + \mu(x_n) f_\rho(x_1, \ldots, x_{n-1}, u) \\
&\qquad\qquad - f_\rho(x_1, \ldots, x_{n-1}, \mu(x_n)u)) \\
&\qquad - \sum_{i=1}^{n-1} (-1)^{i+1} f_\mu(x_1, \ldots, \widehat{x_i}, \ldots, x_{n-1}, u, x_i \cdot x_n) \\
&\qquad + \sum_{i=1}^{n-1} (-1)^{i+1} \rho(x_i) f_\mu(x_1, \ldots, \widehat{x_i}, \ldots, x_{n-1}, u, x_n) \\
&\qquad - \sum_{i=1}^{n-1} (-1)^{i+1} f_\mu(x_1, \ldots, \widehat{x_i}, \ldots, x_{n-1}, \rho(x_i)u, x_n) \\
&\qquad + \sum_{i=1}^{n-1} (-1)^{i+1} \mu(x_n) f_\mu(x_1, \ldots, \widehat{x_i}, \ldots, x_{n-1}, u, x_i) \\
&\qquad + \sum_{i=1}^{n-1} (-1)^{i+1} f_\mu(x_1, \ldots, \widehat{x_i}, \ldots, x_{n-1}, \mu(x_i)u, x_n) \\
&\qquad + \sum_{1 \le i < j \le n-1} (-1)^{i+j} f_\mu([x_i, x_j]_C, x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_{n-1}, u, x_n).
\end{aligned}
$$

3. Maurer–Cartan characterizations and deformations of pre-LieDer pairs

In this section, first, we recall L_∞-algebras and higher derived brackets. Then, we use higher derived brackets to construct an L_∞-algebra. Given a pre-LieDer pair, we can obtain the Maurer–Cartan element of this L_∞-algebra. Finally, we construct a twist L_∞-algebra by this Maurer–Cartan element, which controls deformations of pre-LieDer pairs.

3.1. L_∞-algebras and higher derived brackets

Let $\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}^k$ be a $\mathbb{Z}$-graded vector space. The desuspension operator s^{-1} changes the grading of $\mathfrak{g}$ according to the rule $(s^{-1}\mathfrak{g})^i := \mathfrak{g}^{i+1}$. The degree -1 map $s^{-1} : \mathfrak{g} \to s^{-1}\mathfrak{g}$ is defined by sending $x \in \mathfrak{g}$ to its copy $s^{-1}x \in s^{-1}\mathfrak{g}$.

The notion of an L_∞-algebra was introduced by Stasheff in [32]. See [25,26] for more details.

Definition 3.1. An *L_∞-algebra* is a $\mathbb{Z}$-graded vector space $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}^k$ equipped with a collection ($k \geq 1$) of linear maps $l_k : \otimes^k \mathfrak{g} \to \mathfrak{g}$ of degree 1 with the property that, for any homogeneous elements $x_1, \ldots, x_n \in \mathfrak{g}$, we have

(1) *(graded symmetry)* for every $\sigma \in S_n$,

$$l_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \varepsilon(\sigma) l_n(x_1, \ldots, x_n),$$

(2) *(generalized Jacobi identity)* for all $n \geq 1$,

$$\sum_{i=1}^{n} \sum_{\sigma\in S(i,n-i)} \varepsilon(\sigma) l_{n-i+1}(l_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0,$$

where $\varepsilon(\sigma) = \varepsilon(\sigma; x_1, \ldots, x_n)$ is the Koszul sign for a permutation $\sigma \in S_n$ and $x_i \in \mathfrak{g}$.

We denote an L_∞-algebra by $(\mathfrak{g}, \{l_k\}_{k=1}^{+\infty})$.

Definition 3.2. A *Maurer–Cartan element* of an L_∞-algebra $(\mathfrak{g}, \{l_k\}_{k=1}^{+\infty})$ is an element $\alpha \in \mathfrak{g}^0$ satisfying

$$\sum_{k=1}^{+\infty} \frac{1}{k!} l_k(\alpha, \ldots, \alpha) = 0. \tag{3.1}$$

Remark 3.3. In general, the Maurer–Cartan equation (3.1) makes sense when the L_∞-algebra is a filtered L_∞-algebra [14]. In the following, the L_∞-algebra under consideration satisfies $l_k = 0$ for k sufficiently big, so the Maurer–Cartan equation makes sense.

Let α be a Maurer–Cartan element of an L_∞-algebra $(\mathfrak{g}, \{l_k\}_{k=1}^{+\infty})$. Define $l_k^\alpha : \otimes^k \mathfrak{g} \to \mathfrak{g} (k \geq 1)$ by

$$l_k^\alpha(x_1, \ldots, x_k) = \sum_{n=0}^{+\infty} \frac{1}{n!} l_{k+n}(\underbrace{\alpha, \ldots, \alpha}_{n}, x_1, \ldots, x_k).$$

Theorem 3.4 ([15, 20]). *With the above notation, $(\mathfrak{g}, \{l_k^\alpha\}_{k=1}^{+\infty})$ is an L_∞-algebra which is called the twisted L_∞-algebra by α.*

Theorem 3.5 ([20]). *Let $(\mathfrak{g}, \{l_k^\alpha\}_{k=1}^{+\infty})$ be a twist L_∞-algebra by α. Then $\alpha + \alpha'$ is a Maurer–Cartan element of the L_∞-algebra $(\mathfrak{g}, \{l_k\}_{k=1}^{+\infty})$ if and only if α' is a Maurer–Cartan element of the twist L_∞-algebra $(\mathfrak{g}, \{l_k^\alpha\}_{k=1}^{+\infty})$.*

Definition 3.6 ([35]). A *V-data* consists of a quadruple $(L, \mathfrak{h}, P, \Delta)$, where

(1) $(L, [\cdot, \cdot])$ is a graded Lie algebra,

(2) $\mathfrak{h}$ is an abelian graded Lie subalgebra of $(L, [\cdot, \cdot])$,

(3) $P : L \to L$ is a projection, that is $P \circ P = P$, whose image is $\mathfrak{h}$ and its kernel is a graded Lie subalgebra of $(L, [\cdot,\cdot])$,

(4) Δ is an element in $\ker(P)^1$ such that $[\Delta, \Delta] = 0$.

Theorem 3.7 ([18, 35]). *Let $(L, \mathfrak{h}, P, \Delta)$ be a V-data. Then the graded vector space $s^{-1}L \oplus \mathfrak{h}$ is an L_∞-algebra, where*

$$\begin{aligned} l_1(s^{-1}f, \theta) &= (-s^{-1}[\Delta, f], P(f + [\Delta, \theta])), \\ l_2(s^{-1}f, s^{-1}g) &= (-1)^{|f|}s^{-1}[f, g], \\ l_k(s^{-1}f, \theta_1, \dots, \theta_{k-1}) &= P([\cdots[[f, \theta_1], \theta_2], \dots, \theta_{k-1}]), \quad k \geq 2, \\ l_k(\theta_1, \dots, \theta_{k-1}, \theta_k) &= P([\cdots[[\Delta, \theta_1], \theta_2], \dots, \theta_k]), \quad k \geq 2. \end{aligned}$$

Here $\theta, \theta_1, \dots, \theta_k$ are homogeneous elements of $\mathfrak{h}$ and f, g are homogeneous elements of L. All the other L_∞-algebra products that are not obtained from the ones written above by permutations of arguments will vanish.

Moreover, if L' is a graded Lie subalgebra of L that satisfies $[\Delta, L'] \subset L'$, then $s^{-1}L' \oplus \mathfrak{h}$ is an L_∞-subalgebra of the above L_∞-algebra $(s^{-1}L \oplus \mathfrak{h}, \{l_k\}_{k=1}^{+\infty})$.

3.2. The L_∞-algebra that controls deformations of pre-LieDer pairs

Proposition 3.8. *We have a V-data $(L, \mathfrak{h}, P, \Delta)$ as follows:*

(1) *Let $\mathfrak{g}$ and V be vector spaces. The graded Lie algebra $(L, [\cdot,\cdot])$ is given by $\big(\bigoplus_{n=1}^{+\infty} \mathrm{Hom}(\wedge^{n-1}(\mathfrak{g} \oplus V) \otimes (\mathfrak{g} \oplus V), \mathfrak{g} \oplus V), [\cdot,\cdot]^{MN}\big)$.*

(2) *The abelian graded Lie subalgebra $\mathfrak{h}$ is given by*

$$\mathfrak{h} = \bigoplus_{n=0}^{+\infty} C^{n|-1}(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V) = \bigoplus_{n=0}^{+\infty} \mathrm{Hom}(\wedge^{n-1}\mathfrak{g} \otimes \mathfrak{g}, V).$$

(3) *$P : L \to L$ is a projection onto the subspace $\mathfrak{h}$.*

(4) *$\Delta = 0$.*

Consequently, we obtain an L_∞-algebra $(s^{-1}L \oplus \mathfrak{h}, \{l_k\}_{k=1}^{+\infty})$, where l_i are given by

$$\begin{aligned} l_1(s^{-1}f, \theta) &= P(f), \\ l_2(s^{-1}f, s^{-1}g) &= (-1)^{|f|}s^{-1}[f, g]^{MN}, \\ l_k(s^{-1}f, \theta_1, \dots, \theta_{k-1}) &= P([\cdots[[f, \theta_1]^{MN}, \theta_2]^{MN}, \dots, \theta_{k-1}]^{MN}), \quad k \geq 2 \end{aligned}$$

for homogeneous elements $\theta, \theta_1, \dots, \theta_{k-1} \in \mathfrak{h}$, $f, g \in L$ and all the other possible combinations vanish.

Proof. Obviously, $\mathfrak{h}$ is an abelian graded Lie subalgebra of $(L, [\cdot,\cdot])$, and $\mathrm{Ker}(P)$ is a graded Lie subalgebra of $(L, [\cdot,\cdot])$. Thus, $(L, \mathfrak{h}, P, \Delta)$ is a V-data. By Theorem 3.7, we obtain an L_∞-algebra $(s^{-1}L \oplus \mathfrak{h}, \{l_k\}_{k=1}^{+\infty})$. ∎

We set L' by

$$\begin{aligned} L' &= \bigoplus_{n=0}^{+\infty} C^{n|0}(\mathfrak{g} \oplus V, \mathfrak{g} \oplus V) \\ &= \bigoplus_{n=0}^{+\infty} (\mathrm{Hom}(\wedge^n \mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g}) \oplus \mathrm{Hom}(\wedge^n \mathfrak{g} \otimes V, V) \oplus \mathrm{Hom}(\wedge^{n-1} \mathfrak{g} \otimes V \otimes \mathfrak{g}, V)). \end{aligned}$$

Now, we give the main result in this subsection.

Theorem 3.9. *With the above notation, $(s^{-1}L' \oplus \mathfrak{h}, \{l_k\}_{k=1}^{+\infty})$ is an L_∞-algebra, where l_k are given by*

$$\begin{aligned} l_2(s^{-1}f, s^{-1}g) &= (-1)^{|f|} s^{-1}[f, g]^{MN}, \\ l_2(s^{-1}f, \theta) &= P([f, \theta]^{MN}), \\ l_k(s^{-1}f, \theta_1, \ldots, \theta_{k-1}) &= 0, \quad k \geq 3 \end{aligned}$$

for homogeneous elements $\theta, \theta_1, \ldots, \theta_{k-1} \in \mathfrak{h}$, $f, g \in L'$ and all the other possible combinations vanish.

Moreover, for all $\pi \in \mathrm{Hom}(\mathfrak{g} \otimes \mathfrak{g}, V)$, $\rho \in \mathrm{Hom}(\mathfrak{g} \otimes V, V)$, $\mu \in \mathrm{Hom}(V \otimes \mathfrak{g}, V)$ and $D \in \mathrm{Hom}(\mathfrak{g}, V)$, $(s^{-1}(\pi + \rho + \mu), D)$ is a Maurer–Cartan element of the L_∞-algebra $(s^{-1}L' \oplus \mathfrak{h}, \{l_k\}_{k=1}^{+\infty})$ if and only if $((\mathfrak{g}, \pi), D, \rho, \mu)$ is a pre-LieDer pair.

Proof. Obviously, L' is a graded Lie subalgebra. Since $\Delta = 0$, we have $[\Delta, L'] = 0$. Thus, $(s^{-1}L' \oplus \mathfrak{h}, \{l_k\}_{k=1}^{+\infty})$ is an L_∞-algebra. It is straightforward to deduce that

$$[\pi + \rho + \mu, D]^{MN} \in \mathrm{Hom}(\mathfrak{g} \otimes \mathfrak{g}, V), \quad [[\pi + \rho + \mu, D]^{MN}, D]^{MN} = 0.$$

Then, we have

$$\begin{aligned} &\frac{1}{2} l_2((s^{-1}(\pi + \rho + \mu), D), (s^{-1}(\pi + \rho + \mu), D)) \\ &\quad = \frac{1}{2} l_2(s^{-1}(\pi + \rho + \mu), s^{-1}(\pi + \rho + \mu)) + l_2(s^{-1}(\pi + \rho + \mu), D) \\ &\quad = \Big(-\frac{1}{2} s^{-1}[\pi + \rho + \mu, \pi + \rho + \mu]^{MN}, P[\pi + \rho + \mu, D]^{MN}\Big) \\ &\quad = \Big(-\frac{1}{2} s^{-1}[\pi + \rho + \mu, \pi + \rho + \mu]^{MN}, [\pi + \rho + \mu, D]^{MN}\Big). \end{aligned}$$

Thus, $(s^{-1}(\pi + \rho + \mu), D)$ is a Maurer–Cartan element of the L_∞-algebra $(s^{-1}L' \oplus \mathfrak{h}, \{l_k\}_{k=1}^{+\infty})$ if and only if

$$[\pi + \rho + \mu, \pi + \rho + \mu]^{MN} = 0, \quad [\pi + \rho + \mu, D]^{MN} = 0.$$

By Proposition 2.11, $[\pi+\rho+\mu, \pi+\rho+\mu]^{MN}=0$ if and only if $(V;\rho,\mu)$ is a representation of the pre-Lie algebra $(\mathfrak{g},\pi)$. For all $x, y\in\mathfrak{g}$, we have

$$\begin{aligned}0&=[\pi+\rho+\mu, D]^{MN}(x,y)\\&=\rho(x,D(y))+\mu(D(x),y)-D(\pi(x,y))\\&=\rho(x)D(y)+\mu(y)D(x)-D(x\cdot y),\end{aligned}$$

which implies that $[\pi+\rho+\mu, D]^{MN}=0$ if and only if D is a derivation of the pre-Lie algebra $(\mathfrak{g},\pi)$ with respect to the representation $(V;\rho,\mu)$. Thus, $(s^{-1}(\pi+\rho+\mu), D)$ is a Maurer–Cartan element of the L_∞-algebra $(s^{-1}L'\oplus\mathfrak{h}, \{l_k\}_{k=1}^{+\infty})$ if and only if $((\mathfrak{g},\pi), D, \rho, \mu)$ is a pre-LieDer pair. This finishes the proof. ∎

Theorem 3.10. *With the above notation, let $((\mathfrak{g},\pi), D, \rho, \mu)$ be a pre-LieDer pair. Then $(s^{-1}L'\oplus\mathfrak{h}, \{l_k^{(s^{-1}(\pi+\rho+\mu),D)}\}_{k=1}^{+\infty})$ is an L_∞-algebra, where $l_k^{(s^{-1}(\pi+\rho+\mu),D)}$ are given by*

$$\begin{aligned}l_1^{(s^{-1}(\pi+\rho+\mu),D)}(s^{-1}f,\theta)&=l_2((s^{-1}(\pi+\rho+\mu),D),(s^{-1}f,\theta)),\\l_2^{(s^{-1}(\pi+\rho+\mu),D)}((s^{-1}f_1,\theta_1),(s^{-1}f_2,\theta_2))&=l_2((s^{-1}f_1,\theta_1),(s^{-1}f_2,\theta_2)),\\l_k^{(s^{-1}(\pi+\rho+\mu),D)}((s^{-1}f_1,\theta_1),\ldots,(s^{-1}f_k\theta_k))&=0,\quad k\geq 3.\end{aligned}$$

Furthermore, for all $\pi'\in\mathrm{Hom}(\mathfrak{g}\otimes\mathfrak{g},V)$, $\rho'\in\mathrm{Hom}(\mathfrak{g}\otimes V,V)$, $\mu'\in\mathrm{Hom}(V\otimes\mathfrak{g},V)$ and $D'\in\mathrm{Hom}(\mathfrak{g},V)$, $((\mathfrak{g},\pi+\pi'), D+D', \rho+\rho', \mu+\mu')$ is a pre-LieDer pair if and only if $(s^{-1}(\pi'+\rho'+\mu'), D')$ is a Maurer–Cartan element of the twist L_∞-algebra $(s^{-1}L'\oplus\mathfrak{h}, \{l_k^{(s^{-1}(\pi+\rho+\mu),D)}\}_{k=1}^{+\infty})$.

Proof. By Theorem 3.9, $((\mathfrak{g},\pi+\pi'), D+D', \rho+\rho', \mu+\mu')$ is a pre-LieDer pair if and only if $(s^{-1}(\pi+\pi'+\rho+\rho'+\mu+\mu'), D+D')$ is a Maurer–Cartan element of the L_∞-algebra $(s^{-1}L'\oplus\mathfrak{h}, \{l_k\}_{k=1}^{+\infty})$. By Theorem 3.5, $(s^{-1}(\pi+\pi'+\rho+\rho'+\mu+\mu'), D+D')$ is a Maurer–Cartan element of the L_∞-algebra $(s^{-1}L'\oplus\mathfrak{h}, \{l_k\}_{k=1}^{+\infty})$ if and only if $(s^{-1}(\pi'+\rho'+\mu'), D')$ is a Maurer–Cartan element of the twist L_∞-algebra $(s^{-1}L'\oplus\mathfrak{h}, \{l_k^{(s^{-1}(\pi+\rho+\mu),D)}\}_{k=1}^{+\infty})$. ∎

4. Cohomologies and infinitesimal deformations of pre-LieDer pairs

In this section, first, we give the cohomology of pre-LieDer pairs. Then, we study infinitesimal deformations of pre-LieDer pairs by using this cohomology. We show that equivalent infinitesimal deformations are in the same second cohomology group. Finally, we use this cohomology to give the cohomology of regular pre-LieDer pairs.

In this section, We will also denote the pre-Lie multiplication "$\cdot$" by π.

4.1. Cohomologies of pre-LieDer pairs

Let $((\mathfrak{g},\cdot), D, \rho, \mu)$ be a pre-LieDer pair. Define the set of 0-cochains $C^0(\mathfrak{g},\rho,\mu,D)$ to be 0 and define the set of 1-cochains $C^1(\mathfrak{g},\rho,\mu,D)$ to be $\mathrm{Hom}(\mathfrak{g},\mathfrak{g})\oplus\mathrm{Hom}(V,V)$. For $n\geq 2$, define the set of n-cochains $C^n(\mathfrak{g},\rho,\mu,D)$ by

$$\begin{aligned}C^n(\mathfrak{g},\rho,\mu,D) &:= C^n(\mathfrak{g},\rho,\mu)\oplus C^{n-1}(\mathfrak{g};V)\\ &= \mathrm{Hom}(\wedge^{n-1}\mathfrak{g}\otimes\mathfrak{g},\mathfrak{g})\oplus\mathrm{Hom}(\wedge^{n-1}\mathfrak{g}\otimes V,V)\\ &\quad\oplus\mathrm{Hom}(\wedge^{n-2}\mathfrak{g}\otimes V\otimes\mathfrak{g},V)\oplus\mathrm{Hom}(\wedge^{n-2}\mathfrak{g}\otimes\mathfrak{g},V).\end{aligned}$$

Define the coboundary operator $\mathcal{D}: C^n(\mathfrak{g},\rho,\mu,D)\to C^{n+1}(\mathfrak{g},\rho,\mu,D)$ by

$$\mathcal{D}(f,\theta)=(-1)^{n-2}l_1^{(s^{-1}(\pi+\rho+\mu),D)}(s^{-1}f,\theta),\tag{4.1}$$

where $f\in\mathrm{Hom}(\wedge^{n-1}\mathfrak{g}\otimes\mathfrak{g},\mathfrak{g})\oplus\mathrm{Hom}(\wedge^{n-1}\mathfrak{g}\otimes V,V)\oplus\mathrm{Hom}(\wedge^{n-2}\mathfrak{g}\otimes V\otimes\mathfrak{g},V)$ and $\theta\in\mathrm{Hom}(\wedge^{n-2}\mathfrak{g}\otimes\mathfrak{g},V)$.

Theorem 4.1. *$\big(\bigoplus_{n=1}^{+\infty}C^n(\mathfrak{g},\rho,\mu,D),\mathcal{D}\big)$ is a cochain complex, that is, $\mathcal{D}\circ\mathcal{D}=0$.*

Proof. By Theorem 3.10, we obtain that $(s^{-1}L'\oplus\mathfrak{h},\{l_k^{(s^{-1}(\pi+\rho+\mu),D)}\}_{k=1}^{+\infty})$ is an L_∞-algebra. Thus, we have $l_1^{(s^{-1}(\pi+\rho+\mu),D)}\circ l_1^{(s^{-1}(\pi+\rho+\mu),D)}=0$, which implies that $\mathcal{D}\circ\mathcal{D}=0$. ∎

Definition 4.2. The cohomology of the cochain complex $\big(\bigoplus_{n=1}^{+\infty}C^n(\mathfrak{g},\rho,\mu,D),\mathcal{D}\big)$ is called the cohomology of the pre-LieDer pair $((\mathfrak{g},\cdot),D,\rho,\mu)$. The corresponding cohomology group is denoted by $H^n(\mathfrak{g},\rho,\mu,D)$.

In the sequel, by straightforward computation, we give the explicit formula of the operator $\mathcal{D}$. For all $f\in\mathrm{Hom}(\wedge^{n-1}\mathfrak{g}\otimes\mathfrak{g},\mathfrak{g})\oplus\mathrm{Hom}(\wedge^{n-1}\mathfrak{g}\otimes V,V)\oplus\mathrm{Hom}(\wedge^{n-2}\mathfrak{g}\otimes V\otimes\mathfrak{g},V)$ and $\theta\in\mathrm{Hom}(\wedge^{n-2}\mathfrak{g}\otimes\mathfrak{g},V)$, we have

$$\|[\pi+\rho+\mu,f]^{MN}\|=n\,|\,0,\quad \|[\pi+\rho+\mu,\theta]^{MN}\|=n\,|\,-1,\quad \|[f,D]^{MN}\|=n\,|\,-1.$$

Thus, by (4.1), we have

$$\begin{aligned}\mathcal{D}(f,\theta)&=(-1)^{n-2}l_1^{(s^{-1}(\pi+\rho+\mu),D)}(s^{-1}f,\theta)\\ &=(-1)^{n-2}l_2((s^{-1}(\pi+\rho+\mu),D),(s^{-1}f,\theta))\\ &=(-1)^{n-2}(l_2(s^{-1}(\pi+\rho+\mu),s^{-1}f)+l_2(s^{-1}(\pi+\rho+\mu),\theta)\\ &\qquad+l_2(D,s^{-1}f))\\ &=(-1)^{n-2}(-s^{-1}[\pi+\rho+\mu,f]^{MN},P[\pi+\rho+\mu,\theta]^{MN}+P[f,D]^{MN})\\ &=(-1)^{n-2}(-s^{-1}[\pi+\rho+\mu,f]^{MN},[\pi+\rho+\mu,\theta]^{MN}+[f,D]^{MN})\\ &=(\partial f,\mathrm{d}\theta+\delta f).\end{aligned}$$

Lemma 4.3. *With the above notation, for all* $f = (f_\mathfrak{g}, f_\rho, f_\mu) \in \mathrm{Hom}(\wedge^{n-1}\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g}) \oplus \mathrm{Hom}(\wedge^{n-1}\mathfrak{g} \otimes V, V) \oplus \mathrm{Hom}(\wedge^{n-2}\mathfrak{g} \otimes V \otimes \mathfrak{g}, V)$, $\delta : \mathrm{Hom}(\wedge^{n-1}\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g}) \oplus \mathrm{Hom}(\wedge^{n-1}\mathfrak{g} \otimes V, V) \oplus \mathrm{Hom}(\wedge^{n-2}\mathfrak{g} \otimes V \otimes \mathfrak{g}, V) \to \mathrm{Hom}(\wedge^{n-1}\mathfrak{g} \otimes \mathfrak{g}, V)$ *is given by*

$$\begin{aligned}
&(\delta f)(x_1, \ldots, x_{n-1}, x_n) \\
&\quad = \sum_{i=1}^{n-1} (-1)^{i+1} f_\mu(x_1, \ldots, \widehat{x_i}, \ldots, x_{n-1}, D(x_i), x_n) \\
&\qquad + (-1)^{n-2} \cdot (f_\rho(x_1, \ldots, x_{n-1}, D(x_n)) - D(f_\mathfrak{g}(x_1, \ldots, x_n))).
\end{aligned}$$

Proof. For all $x_1, \ldots, x_n \in \mathfrak{g}$, we have

$$\begin{aligned}
&(\delta f)(x_1, \ldots, x_n) \\
&\quad = (-1)^{n-2}[f, D]^{MN}(x_1, \ldots, x_n) \\
&\quad = (-1)^{n-2}\Bigg(\sum_{i=1}^{n-1} (-1)^{i+1} f_\mu(D(x_i), x_1, \ldots, \widehat{x_i}, \ldots, x_n) \\
&\qquad\qquad + f_\rho(x_1, \ldots, x_{n-1}, D(x_n)) - D(f_\mathfrak{g}(x_1, \ldots, x_n)) \Bigg) \\
&\quad = \sum_{i=1}^{n-1} (-1)^{i+1} f_\mu(x_1, \ldots, \widehat{x_i}, \ldots, x_{n-1}, D(x_i), x_n) \\
&\qquad + (-1)^{n-2}(f_\rho(x_1, \ldots, x_{n-1}, D(x_n)) - D(f_\mathfrak{g}(x_1, \ldots, x_n))).
\end{aligned}$$

The proof is finished. ∎

For all $f = (f_\mathfrak{g}, f_\rho, f_\mu) \in \mathrm{Hom}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g}) \oplus \mathrm{Hom}(\mathfrak{g} \otimes V, V) \oplus \mathrm{Hom}(V \otimes \mathfrak{g}, V)$, $\theta \in \mathrm{Hom}(\mathfrak{g}, V)$, $x_1, x_2, x_3 \in \mathfrak{g}$, and $u \in V$, we have

$$\begin{aligned}
(\partial f)_\mathfrak{g}(x_1, x_2, x_3) &= x_1 \cdot f_\mathfrak{g}(x_2, x_3) + x_2 \cdot f_\mathfrak{g}(x_1, x_3) + f_\mathfrak{g}(x_2, x_1) \cdot x_3 \\
&\quad - f_\mathfrak{g}(x_1, x_2) \cdot x_3 - f_\mathfrak{g}(x_2, x_1 \cdot x_3) + f_\mathfrak{g}(x_1, x_2 \cdot x_3) \\
&\quad - f_\mathfrak{g}(x_1 \cdot x_2 - x_2 \cdot x_1, x_3). \\
(\partial f)_\rho(x_1, x_2, u) &= \rho(f_\mathfrak{g}(x_2, x_1))u - \rho(f_\mathfrak{g}(x_1, x_2))u + \rho(x_1) f_\rho(x_2, u) \\
&\quad - \rho(x_2) f_\rho(x_1, u) - f_\rho(x_2, \rho(x_1)u) + f_\rho(x_1, \rho(x_2)u) \\
&\quad - f_\rho(x_1 \cdot x_2 - x_2 \cdot x_1, u). \\
(\partial f)_\mu(x_1, u, x_2) &= -(\mu(f_\mathfrak{g}(x_1, x_2))u + \mu(x_2) f_\rho(x_1, u) - f_\rho(x_1, \mu(x_2)u)) \\
&\quad - f_\mu(u, x_1 \cdot x_2) + \rho(x_1) f_\mu(u, x_2) - f_\mu(\rho(x_1)u, x_2) \\
&\quad + \mu(x_2) f_\mu(u, x_1) + f_\mu(\mu(x_1)u, x_2). \\
(\mathrm{d}\theta + \delta f)(x_1, x_2) &= \rho(x_1) f(x_2) + \mu(x_2) f(x_1) - f(x_1 \cdot x_2) + f_\mu(D(x_1), x_2) \\
&\quad + f_\rho(x_1, D(x_2)) - D(f_\mathfrak{g}(x_1, x_2)).
\end{aligned}$$

By $\partial f = ((\partial f)_{\mathfrak{g}}, (\partial f)_{\rho}, (\partial f)_{\mu})$ and $\mathcal{D}(f,\theta) = (\partial f, \mathrm{d}\theta + \delta f)$. We obtain the 2-coboundary operator of the pre-LieDer pair $((\mathfrak{g},\cdot), D, \rho, \mu)$.

The following diagram well explains the above operators.

$$\begin{array}{ccccc} C^n(\mathfrak{g},\rho,\mu) & \xrightarrow{\partial} & C^{n+1}(\mathfrak{g},\rho,\mu) & \xrightarrow{\partial} & C^{n+2}(\mathfrak{g},\rho,\mu) \\ \oplus & \overset{\delta}{\searrow} & \oplus & \overset{\delta}{\searrow} & \oplus \\ \mathcal{C}^{n-1}(\mathfrak{g};V) & \xrightarrow[\mathrm{d}]{} & \mathcal{C}^{n}(\mathfrak{g};V) & \xrightarrow[\mathrm{d}]{} & \mathcal{C}^{n+1}(\mathfrak{g};V) \end{array}$$

At the end of this subsection, we give the relation between various cohomologies groups.

Theorem 4.4. *There is a short exact sequence of the cochain complexes:*

$$0 \longrightarrow \left(\bigoplus_{n=0}^{+\infty} C^n(\mathfrak{g};V), \mathrm{d}\right) \xrightarrow{\iota} \left(\bigoplus_{n=0}^{+\infty} C^n(\mathfrak{g},\rho,\mu,D), \mathcal{D}\right) \xrightarrow{p} \left(\bigoplus_{n=0}^{+\infty} C^n(\mathfrak{g},\rho,\mu), \partial\right) \longrightarrow 0,$$

where $\iota(\theta) = (0,\theta)$ and $p(f,\theta) = f$ for all $f \in C^n(\mathfrak{g},\rho,\mu)$ and $\theta \in C^{n-1}(\mathfrak{g};V)$.

Consequently, there is a long exact sequence of the cohomology groups:

$$\cdots \longrightarrow H^n(\mathfrak{g};V) \xrightarrow{H^n(\iota)} H^n(\mathfrak{g},\rho,\mu,D) \xrightarrow{H^n(p)} H^n(\mathfrak{g},\rho,\mu) \xrightarrow{c^n} H^{n+1}(\mathfrak{g};V) \longrightarrow \cdots,$$

where the connecting map c^n is defined by $c^n([\alpha]) = [\delta(\alpha)]$, for all $[\alpha] \in H^n(\mathfrak{g},\pi,\rho,\mu)$.

Proof. By the explicit formula of the coboundary operator $\mathcal{D}$, we have the short exact sequence of chain complexes which induces a long exact sequence of cohomology groups. ∎

4.2. Infinitesimal deformations of pre-LieDer pairs

Definition 4.5. Let $((\mathfrak{g},\pi), D, \rho, \mu)$ be a pre-LieDer pair. If for all $t \in \mathbb{K}$, $((\mathfrak{g}, \pi_t = \pi + t\omega), D_t = D + t\widehat{D}, \rho_t = \rho + t\sigma, \mu_t = \mu + t\tau)$ is a pre-LieDer pair, where $\omega : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}, \sigma : \mathfrak{g}\otimes V \to \mathfrak{g}, \tau : V\otimes\mathfrak{g} \to \mathfrak{g}$ and $\widehat{D} : \mathfrak{g} \to V$ are linear maps. We say that $(\omega,\sigma,\tau,\widehat{D})$ generates an *infinitesimal deformation* of the pre-LieDer pair $((\mathfrak{g},\pi), D, \rho, \mu)$.

It is direct to check that $(\omega,\sigma,\tau,\widehat{D})$ generates an infinitesimal deformation of the pre-LieDer pair $((\mathfrak{g},\pi), D, \rho, \mu)$ if and only if for all $x, y \in \mathfrak{g}$, the following equalities are satisfied:

$$[\pi + \rho + \mu, \omega + \sigma + \tau]^{MN} = 0, \tag{4.2}$$

$$[\omega + \sigma + \tau, \omega + \sigma + \tau]^{MN} = 0, \tag{4.3}$$

$$[\pi + \rho + \mu, \widehat{D}]^{MN} + [\omega + \sigma + \tau, D]^{MN} = 0, \tag{4.4}$$

$$[\omega + \sigma + \tau, \widehat{D}]^{MN} = 0. \tag{4.5}$$

Obviously, $(\omega,\sigma,\tau)\in C^2(\mathfrak{g},\pi,\rho,\mu)$, (4.2) means that (ω,σ,τ) is a 2-cocycle of the pre-Lie algebra $(\mathfrak{g},\pi)$ with the representation $(V;\rho,\mu)$, that is, $\partial(\omega,\sigma,\tau)=0$. Equation (4.3) means that $(V;\sigma,\tau)$ is a representation of the pre-Lie algebra $(\mathfrak{g},\omega)$. Equation (4.4) means that $\mathrm{d}\widehat{D}+\delta(\omega,\sigma,\tau)=0$. (4.5) means that $((\mathfrak{g},\omega),\widehat{D},\sigma,\tau)$ is a pre-LieDer pair. By (4.2) and (4.4), we have

$$\mathcal{D}(\omega,\sigma,\tau,\widehat{D})=(\partial(\omega,\sigma,\tau),\mathrm{d}\widehat{D}+\delta(\omega,\sigma,\tau))=0.$$

Theorem 4.6. *With the above notation,* $(\omega,\sigma,\tau,\widehat{D})$ *is a* 2*-cocycle of the pre-LieDer pair* $((\mathfrak{g},D),\rho,\mu)$.

Definition 4.7. Let $(\pi_t'=\pi+t\omega',\rho_t'=\rho+t\sigma',\mu_t'=\mu+t\tau',D_t'=D+t\widehat{D}')$ and $(\pi_t=\pi+t\omega,\rho_t=\rho+t\sigma,\mu_t=\mu+t\tau,D_t=D+t\widehat{D})$ be two infinitesimal deformations of the pre-LieDer pair $((\mathfrak{g},\pi),D,\rho,\mu)$. We call them *equivalent* if there exist $N\in\mathfrak{gl}(\mathfrak{g})$ and $S\in\mathfrak{gl}(V)$ such that $(\mathrm{Id}_{\mathfrak{g}}+tN,\mathrm{Id}_V+tS)$ is a homomorphism from the pre-LieDer pair $((\mathfrak{g},\pi_t'),D_t',\rho_t',\mu_t')$ to the pre-LieDer pair $((\mathfrak{g},\pi_t),D_t,\rho_t,\mu_t)$, that is, for all $x,y\in\mathfrak{g}$, the following equalities hold:

$$\begin{aligned}
(\mathrm{Id}_{\mathfrak{g}}+tN)(x\cdot_t' y)&=(\mathrm{Id}_{\mathfrak{g}}+tN)(x)\cdot_t(\mathrm{Id}_{\mathfrak{g}}+tN)(y),\\
(\mathrm{Id}_V+tS)\circ\rho_t'(x)&=\rho_t((\mathrm{Id}_{\mathfrak{g}}+tN)(x))\circ(\mathrm{Id}_V+tS),\\
(\mathrm{Id}_V+tS)\circ\mu_t'(x)&=\mu_t((\mathrm{Id}_{\mathfrak{g}}+tN)(x))\circ(\mathrm{Id}_V+tS),\\
(\mathrm{Id}_V+tS)\circ D_t'&=D_t\circ(\mathrm{Id}_{\mathfrak{g}}+tN).
\end{aligned}$$

An infinitesimal deformation is said to be *trivial* if it is equivalent to (π,ρ,μ,D).

By direct calculations, $(\pi_t',\rho_t',\mu_t',D_t')$ and (π_t,ρ_t,μ_t,D_t) are infinitesimal equivalent deformations if and only if for all $x,y\in\mathfrak{g},u\in V$, the following equalities hold:

$$\begin{aligned}
\omega'(x,y)-\omega(x,y)&=N(x)\cdot y+x\cdot N(y)-N(x\cdot y),&(4.6)\\
N(\omega'(x,y))&=N(x)\cdot N(y)+\omega(x,N(y))+\omega(N(x),y),\\
\omega(N(x),N(y))&=0,\\
\sigma'(x)u-\sigma(x)u&=\rho(x)S(u)+\rho(N(x))u-S(\rho(x)u),&(4.7)\\
S(\sigma'(x)u)&=\rho(N(x))S(u)+\sigma(x)S(u)+\sigma(N(x))u,\\
\sigma(N(x))S(u)&=0,\\
\tau'(x)u-\tau(x)u&=\mu(x)S(u)+\mu(N(x))u-S(\mu(x)u),&(4.8)\\
S(\tau'(x)u)&=\mu(N(x))S(u)+\tau(x)S(u)+\tau(N(x))u,\\
\tau(N(x))S(u)&=0,\\
\widehat{D}'(x)-\widehat{D}(x)&=D(N(x))-S(D(x)),&(4.9)\\
S(\widehat{D}'(x))&=\widehat{D}(N(x)).
\end{aligned}$$

We summarize the above discussion into the following conclusion.

Theorem 4.8. *Let $((\mathfrak{g},\pi), D, \rho, \mu)$ be a pre-LieDer pair. If two infinitesimal deformations $(\pi_t' = \pi + t\omega', \rho_t' = \rho + t\sigma', \mu_t' = \mu + t\tau', D_t' = D + t\widehat{D}')$ and $(\pi_t = \pi + t\omega, \rho_t = \rho + t\sigma, \mu_t = \mu + t\tau, D_t = D + t\widehat{D})$ are equivalent, then $(\omega', \sigma', \tau', \widehat{D}')$ and $(\omega, \sigma, \tau, \widehat{D})$ are in the same cohomology class of $H^2(\mathfrak{g}, \rho, \mu, D)$.*

Proof. By Theorem 4.6, we obtain that $(\omega', \sigma', \tau', \widehat{D}'), (\omega, \sigma, \tau, \widehat{D}) \in Z^2(\mathfrak{g}, \rho, \mu, D)$. Obviously, $(N, S) \in C^1(\mathfrak{g}, \rho, \mu, D)$. For all $x, y \in \mathfrak{g}, u, v \in V$, by (4.6), (4.7) and (4.8), we have

$$\begin{aligned}
&((\omega', \sigma', \tau') - (\omega, \sigma, \tau))((x, u), (y, v))\\
&\quad = (N(x)\cdot y + x\cdot N(y) - N(x\cdot y), \rho(x)S(v) + \rho(N(x))v - S(\rho(x)v),\\
&\mu(y)S(u) + \mu(N(y))u - S(\mu(y)u))\\
&\quad = \partial(N, S)((x, u), (y, v)),
\end{aligned}$$

which implies that $(\omega', \sigma', \tau') - (\omega, \sigma, \tau) = \partial(N, S)$. For all $x \in \mathfrak{g}$, by (4.9), we have

$$(\widehat{D}' - \widehat{D})(x) = D(N(x)) - S(D(x)) = \delta(N, S)(x),$$

which implies that $\widehat{D}' - \widehat{D} = \delta(N, S)$. Thus, we have

$$(\omega', \sigma', \tau', \widehat{D}') - (\omega, \sigma, \tau, \widehat{D}) = (\partial(N, S), \delta(N, S)) = \mathcal{D}(N, S).$$

This finishes the proof. ∎

4.3. Cohomologies of regular pre-LieDer pairs

Let $((\mathfrak{g}, \cdot), D)$ be a regular pre-LieDer pair. Define the set of 0-cochains $C^0(\mathfrak{g}, D)$ to be 0, and define the set of 1-cochains $C^1(\mathfrak{g}, D)$ to be $\mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$. For $n \geq 2$, define the set of n-cochains $C^n(\mathfrak{g}, D)$ by

$$C^n(\mathfrak{g}, D) := \mathrm{Hom}(\wedge^{n-1}\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g}) \oplus \mathrm{Hom}(\wedge^{n-2}\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g}).$$

For all $f \in \mathrm{Hom}(\wedge^{n-1}\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$ and $\theta \in \mathrm{Hom}(\wedge^{n-2}\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$. Define the embedding $i : C^n(\mathfrak{g}, D) \to C^n(\mathfrak{g}, L, R, D)$ by

$$i(f, \theta) = (f, f, f, \theta).$$

Denote by $\mathrm{Im}^n(i) = i(C^n(\mathfrak{g}, D))$. Then, we have the following.

Proposition 4.9. *With the above notation, $\big(\bigoplus_{n=0}^{+\infty} \mathrm{Im}^n(i), \mathcal{D}\big)$ is a subcomplex of the cochain complex $\big(\bigoplus_{n=0}^{+\infty} C^n(\mathfrak{g}, L, R, D), \mathcal{D}\big)$.*

Proof. For all $f \in \mathrm{Hom}(\wedge^{n-1}\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g}), \theta \in \mathrm{Hom}(\wedge^{n-2}\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g}), (f, f, f, \theta) \in \mathrm{Im}^n(i)$, we have

$$\begin{aligned}
\mathcal{D}(f, f, f, \theta) &= (\partial(f, f, f), \mathrm{d}\theta + \delta(f, f, f))\\
&= (\partial(f, f, f)_{\mathfrak{g}}, \partial(f, f, f)_{\rho}, \partial(f, f, f)_{\mu}, \mathrm{d}_{\mathrm{reg}}\theta + \delta(f, f, f)).
\end{aligned}$$

By (2.1), we have $\partial(f, f, f)_{\mathfrak{g}} = \mathrm{d}_{\mathrm{reg}} f$. By straightforward computation, we have $\partial(f, f, f)_{\rho} = \mathrm{d}_{\mathrm{reg}} f$ and $\partial(f, f, f)_{\mu} = \mathrm{d}_{\mathrm{reg}} f$. The detail is given in Appendix. Thus, we obtain that

$$\mathcal{D}(f, f, f, \theta) = (\mathrm{d}_{\mathrm{reg}} f, \mathrm{d}_{\mathrm{reg}} f, \mathrm{d}_{\mathrm{reg}} f, \mathrm{d}_{\mathrm{reg}}\theta + \delta(f, f, f)) = i(\mathrm{d}_{\mathrm{reg}} f, \mathrm{d}_{\mathrm{reg}}\theta + \delta(f, f, f)),$$

which implies that $\big(\bigoplus_{n=0}^{+\infty} \mathrm{Im}^n(i), \mathcal{D}\big)$ is a subcomplex. ∎

Define the projection $p : \mathrm{Im}^n(i) \to C^n(\mathfrak{g}, D)$ by

$$p(f, f, f, \theta) = (f, \theta) \quad \forall f \in \mathrm{Hom}(\wedge^{n-1}\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g}),\ \theta \in \mathrm{Hom}(\wedge^{n-2}\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g}).$$

Then, for $n \geq 1$, we define $\bar{\mathcal{D}} : C^n(\mathfrak{g}, D) \to C^{n+1}(\mathfrak{g}, D)$ by $\bar{\mathcal{D}} = p \circ \mathcal{D} \circ i$. More precisely, we have

$$\bar{\mathcal{D}}(f, \theta) = (\mathrm{d}_{\mathrm{reg}} f, \mathrm{d}_{\mathrm{reg}}\theta + \Omega f) \quad \forall f \in \mathrm{Hom}(\wedge^{n-1}\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g}),\ \theta \in \mathrm{Hom}(\wedge^{n-2}\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g}),$$

where $\Omega : \mathrm{Hom}(\wedge^{n-1}\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g}) \to \mathrm{Hom}(\wedge^{n-1}\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$ is given by

$$\begin{aligned}\Omega(f)(x_1, \ldots, x_n) = (-1)^{n-2}(\Sigma_{i=1}^{n} f(x_1, \ldots, x_{i-1}, D(x_i), x_{i+1}, \ldots, x_n) \\ - D(f(x_1, \ldots, x_n))).\end{aligned}$$

Theorem 4.10. *With the above notation,* $\big(\bigoplus_{n=0}^{+\infty} C^n(\mathfrak{g}, D), \bar{\mathcal{D}}\big)$ *is a cochain complex, that is,* $\bar{D} \circ \bar{D} = 0$.

Proof. By $\bar{\mathcal{D}} = p \circ \mathcal{D} \circ i$ and $i \circ p = \mathrm{Id}$, we have

$$\bar{D} \circ \bar{D} = p \circ D \circ i \circ p \circ D \circ i = p \circ D \circ D \circ i = 0.$$

This finishes the proof. ∎

Definition 4.11. The cohomology of the cochain complex $\big(\bigoplus_{n=1}^{+\infty} C^n(\mathfrak{g}, D), \bar{\mathcal{D}}\big)$ is taken to be the cohomology of the regular pre-LieDer pair $((\mathfrak{g}, \cdot), D)$. The corresponding cohomology group is denoted by $H^n(\mathfrak{g}, D)$.

5. Abelian extensions of regular pre-LieDer pairs

In this section, first, we introduce the notion of representations of regular pre-LieDer pairs. Then, we give cohomologies of regular pre-LieDer pairs with coefficients in an arbitrary representation. Finally, we study abelian extensions of regular pre-LieDer pairs using this cohomological approach. We show that abelian extensions are classified by the second cohomology group.

Definition 5.1. A *representation* of a regular pre-LieDer pair $((\mathfrak{g}, \cdot), D)$ on a vector space V with respect to linear map $K : V \to V$ consists of a pair $(\widetilde{\rho}, \widetilde{\mu})$, where $(V; \widetilde{\rho}, \widetilde{\mu})$ is a representation of the pre-Lie algebra $(\mathfrak{g}, \cdot)$ such that for all $x \in \mathfrak{g}, u \in V$, the following equalities are satisfied:

$$K(\widetilde{\rho}(x)u) = \widetilde{\rho}(x)K(u) + \widetilde{\rho}(D(x))u, \tag{5.1}$$

$$K(\widetilde{\mu}(x)u) = \widetilde{\mu}(x)K(u) + \widetilde{\mu}(D(x))u. \tag{5.2}$$

We denote a representation by $(V, K, \widetilde{\rho}, \widetilde{\mu})$. Let $\widetilde{L}, \widetilde{R} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ be linear maps, where $\widetilde{L}_x y = x \cdot y$, $\widetilde{R}_x y = y \cdot x$. Then $(\mathfrak{g}, D, \widetilde{L}, \widetilde{R})$ is called a regular representation.

We define a bilinear operation $\cdot_\ltimes : \otimes^2(\mathfrak{g} \oplus V) \to (\mathfrak{g} \oplus V)$ by

$$(x + u) \cdot_\ltimes (y + v) := x \cdot y + \widetilde{\rho}(x)(v) + \widetilde{\mu}(y)(u) \quad \forall x, y \in \mathfrak{g},\ u, v \in V$$

and a linear map $D + K : \mathfrak{g} \oplus V \to \mathfrak{g} \oplus V$ by

$$(D + K)(x + u) := D(x) + K(u) \quad \forall x \in \mathfrak{g},\ u \in V.$$

Proposition 5.2. *With the above notation,* $((\mathfrak{g} \oplus V, \cdot_\ltimes), D + K)$ *is a regular pre-LieDer pair, which is denoted by* $(\mathfrak{g} \ltimes_{(\widetilde{\rho},\widetilde{\mu})} V, D + K)$ *and called the* semi-direct product *of the regular pre-LieDer pair* $((\mathfrak{g}, \cdot), D)$ *and the representation* $(V, K, \widetilde{\rho}, \widetilde{\mu})$.

Proof. Since $(V; \widetilde{\rho}, \widetilde{\mu})$ is a representation of the pre-Lie algebra $(\mathfrak{g}, \cdot)$, it is obvious that $(\mathfrak{g} \oplus V, \cdot_\ltimes)$ is a pre-Lie algebra. For all $x, y \in \mathfrak{g}, u, v \in V$, by (5.1) and (5.2), we have

$$\begin{aligned}
&(D + K)((x + u) \cdot_\ltimes (y + v)) \\
&\quad = D(x \cdot y) + K(\widetilde{\rho}(x)v) + K(\widetilde{\mu}(y)u) \\
&\quad = D(x) \cdot y + x \cdot D(y) + \widetilde{\rho}(x)K(v) + \widetilde{\rho}(D(x))v + \widetilde{\mu}(y)K(u) + \widetilde{\mu}(D(y))u \\
&\quad = (x + u) \cdot_\ltimes (D + K)(y + v) + (D + K)(x + u) \cdot_\ltimes (y + v),
\end{aligned}$$

which implies that $D + K$ is a derivation of the pre-Lie algebra $(\mathfrak{g} \oplus V, \cdot_\ltimes)$. Thus, $((\mathfrak{g} \oplus V, \cdot_\ltimes), D + K)$ is a regular pre-LieDer pair. ∎

Let $(V, K, \widetilde{\rho}, \widetilde{\mu})$ be a representation of the regular pre-LieDer pair $((\mathfrak{g}, \cdot), D)$. Define the set of 0-cochains $C^0(\mathfrak{g}, D; V, K, \widetilde{\rho}, \widetilde{\mu})$ to be 0, and define the set of 1-cochains $C^1(\mathfrak{g}, D; V, K, \widetilde{\rho}, \widetilde{\mu})$ to be $\mathrm{Hom}(\mathfrak{g}, V)$. For $n \geq 2$, define the set of n-cochains $C^n(\mathfrak{g}, D; V, K, \widetilde{\rho}, \widetilde{\mu})$ by

$$C^n(\mathfrak{g}, D; V, K, \widetilde{\rho}, \widetilde{\mu}) := \mathrm{Hom}(\wedge^{n-1}\mathfrak{g} \otimes \mathfrak{g}, V) \oplus \mathrm{Hom}(\wedge^{n-2}\mathfrak{g} \otimes \mathfrak{g}, V).$$

For all $f \in \mathrm{Hom}(\wedge^{n-1}\mathfrak{g} \otimes \mathfrak{g}, V)$ and $\theta \in \mathrm{Hom}(\wedge^{n-2}\mathfrak{g} \otimes \mathfrak{g}, V)$, define the coboundary operator $\mathcal{D}_{(\widetilde{\rho},\widetilde{\mu})} : C^n(\mathfrak{g}, D; V, K, \widetilde{\rho}, \widetilde{\mu}) \to C^{n+1}(\mathfrak{g}, D; V, K, \widetilde{\rho}, \widetilde{\mu})$ by

$$\mathcal{D}_{(\widetilde{\rho},\widetilde{\mu})}(f, \theta) = (\mathrm{d}_{(\widetilde{\rho},\widetilde{\mu})} f, \mathrm{d}_{(\widetilde{\rho},\widetilde{\mu})}\theta + \Omega_{(\widetilde{\rho},\widetilde{\mu})} f),$$

where $\mathrm{d}_{(\widetilde{\rho},\widetilde{\mu})}$ is the coboundary operator of the pre-Lie algebra $(\mathfrak{g},\cdot)$ with coefficients in the representation $(V,\widetilde{\rho},\widetilde{\mu})$ and $\Omega_{(\widetilde{\rho},\widetilde{\mu})}:\mathrm{Hom}(\wedge^{n-1}\mathfrak{g}\otimes\mathfrak{g},V)\to\mathrm{Hom}(\wedge^{n-1}\mathfrak{g}\otimes\mathfrak{g},V)$ is defined by

$$\begin{aligned}\Omega(f)(x_1,\ldots,x_n)=(-1)^{n-2}(\Sigma_{i=1}^{n}f(x_1,\ldots,x_{i-1},D(x_i),x_{i+1},\ldots,x_n)\\ -K(f(x_1,\ldots,x_n))).\end{aligned}$$

Theorem 5.3. *With the above notation,* $\big(\bigoplus_{n=0}^{+\infty}C^n(\mathfrak{g},D;V,K,\widetilde{\rho},\widetilde{\mu}),\mathcal{D}_{(\widetilde{\rho},\widetilde{\mu})}\big)$ *is a cochain complex, that is,* $\mathcal{D}_{(\widetilde{\rho},\widetilde{\mu})}\circ\mathcal{D}_{(\widetilde{\rho},\widetilde{\mu})}=0$.

Proof. By Proposition 5.2, we obtain that $(\mathfrak{g}\ltimes_{(\rho,\mu)}V,D+K)$ is a regular pre-LieDer pair. By Theorem 4.10, $\big(\bigoplus_{n=0}^{+\infty}C^n(\mathfrak{g}\oplus V,D+K),\bar{\mathcal{D}}\big)$ is a cochain complex. It is straightforward to deduce that $\big(\bigoplus_{n=0}^{+\infty}C^n(\mathfrak{g},D;V,K,\widetilde{\rho},\widetilde{\mu}),\mathcal{D}_{(\widetilde{\rho},\widetilde{\mu})}\big)$ is a subcomplex of $\big(\bigoplus_{n=0}^{+\infty}C^n(\mathfrak{g}\oplus V,D+K),\bar{\mathcal{D}}\big)$. Thus, we obtain that $\mathcal{D}_{(\widetilde{\rho},\widetilde{\mu})}\circ\mathcal{D}_{(\widetilde{\rho},\widetilde{\mu})}=0$. ∎

Definition 5.4. The cohomology of $\big(\bigoplus_{n=1}^{+\infty}C^n(\mathfrak{g},D;V,K,\widetilde{\rho},\widetilde{\mu}),\mathcal{D}_{(\widetilde{\rho},\widetilde{\mu})}\big)$ is called the cohomology of the regular pre-LieDer pair $((\mathfrak{g},\cdot),D)$ with coefficients in the representation $(V,K,\widetilde{\rho},\widetilde{\mu})$. The corresponding cohomology group is denoted by $H^n(\mathfrak{g},D;V,K,\widetilde{\rho},\widetilde{\mu})$.

Definition 5.5. Let $((\mathfrak{g},\cdot),D)$ and $((V,\cdot_V),K)$ be two regular pre-LieDer pairs. An *extension* of $((\mathfrak{g},\cdot),D)$ by $((V,\cdot_V),K)$ is a short exact sequence of pre-LieDer pair morphisms

$$\begin{array}{ccccccccc}0&\longrightarrow&V&\xrightarrow{\iota}&\widehat{\mathfrak{g}}&\xrightarrow{p}&\mathfrak{g}&\longrightarrow&0\\&&\downarrow{\scriptstyle K}&&\downarrow{\scriptstyle\widehat{D}}&&\downarrow{\scriptstyle D}&&\\0&\longrightarrow&V&\xrightarrow{\iota}&\widehat{\mathfrak{g}}&\xrightarrow{p}&\mathfrak{g}&\longrightarrow&0,\end{array}$$

where $((\widehat{\mathfrak{g}},\cdot_{\widehat{\mathfrak{g}}}),\widehat{D})$ is a regular pre-LieDer pair.

It is called an *abelian extension* if $(V,\cdot_V)$ is an abelian pre-Lie algebra, that is, for all $u,v\in V,u\cdot_V v=0$.

Definition 5.6. A *section* of an extension $((\widehat{\mathfrak{g}},\cdot_{\widehat{\mathfrak{g}}}),\widehat{D})$ of a regular pre-LieDer pair $((\mathfrak{g},\cdot),D)$ by $((V,\cdot_V),K)$ is a linear map $s:\mathfrak{g}\to\widehat{\mathfrak{g}}$ such that $p\circ s=\mathrm{Id}_{\mathfrak{g}}$.

Let $((\widehat{\mathfrak{g}},\cdot_{\widehat{\mathfrak{g}}}),\widehat{D})$ be an abelian extension of a regular pre-LieDer pair $((\mathfrak{g},\cdot),D)$ by $((V,\cdot_V),K)$ and $s:\mathfrak{g}\to\widehat{\mathfrak{g}}$ a section. For all $x,y\in\mathfrak{g}$, define linear maps $\theta:\mathfrak{g}\otimes\mathfrak{g}\to V$ and $\xi:\mathfrak{g}\to V$, respectively, by

$$\begin{aligned}\theta(x,y)&=s(x)\cdot_{\widehat{\mathfrak{g}}}s(y)-s(x\cdot y),\\ \xi(x)&=\widehat{D}(s(x))-s(D(x)).\end{aligned}$$

And for all $x, y \in \mathfrak{g}, u \in V$, define $\widetilde{\rho}, \widetilde{\mu} : \mathfrak{g} \to \mathfrak{gl}(V)$, respectively, by

$$\begin{aligned}\widetilde{\rho}(x)(u) &= s(x) \cdot_{\widehat{\mathfrak{g}}} u,\\ \widetilde{\mu}(x)(u) &= u \cdot_{\widehat{\mathfrak{g}}} s(x).\end{aligned}$$

Obviously, $\widehat{\mathfrak{g}}$ is isomorphic to $\mathfrak{g} \oplus V$ as vector spaces. Transfer the regular pre-LieDer pair structure on $\widehat{\mathfrak{g}}$ to that on $\mathfrak{g} \oplus V$, we obtain a regular pre-LieDer pair $((\mathfrak{g} \oplus V, \diamond), \phi)$, where $\diamond$ and ϕ are given by

$$\begin{aligned}(x+u) \diamond (y+v) &= x \cdot y + \theta(x,y) + \widetilde{\rho}(x)(v) + \widetilde{\mu}(y)(u) \quad && \forall x, y \in \mathfrak{g},\ u, v \in V,\\ \phi(x+u) &= D(x) + \xi(x) + K(u) && \forall x \in \mathfrak{g},\ u \in V.\end{aligned}$$

Theorem 5.7. *With the above notation, $(V, K, \widetilde{\rho}, \widetilde{\mu})$ is a representation of the regular pre-LieDer pair $((\mathfrak{g}, \cdot), D)$. Moreover, this representation is independent of the choice of sections.*

Proof. For all $x, y \in \mathfrak{g}, u \in V$, by the definition of a pre-Lie algebra, we have

$$\begin{aligned}0 &= (x \diamond y) \diamond u - x \diamond (y \diamond u)\\ &\quad - (y \diamond x) \diamond u + y \diamond (x \diamond u)\\ &= (x \cdot y + \theta(x,y)) \diamond u - x \diamond \widetilde{\rho}(y)(u)\\ &\quad - (y \cdot x + \theta(y,x)) \diamond u + y \diamond \widetilde{\rho}(x)(u)\\ &= \widetilde{\rho}(x \cdot y)u - \widetilde{\rho}(x)\widetilde{\rho}(y)u - \widetilde{\rho}(y \cdot x)u + \widetilde{\rho}(y)\widetilde{\rho}(x)u,\end{aligned}$$

which implies that

$$\widetilde{\rho}([x,y]_C) = \widetilde{\rho}(x) \circ \widetilde{\rho}(y) - \widetilde{\rho}(y) \circ \widetilde{\rho}(x). \tag{5.3}$$

Similarly, we have

$$\widetilde{\mu}(y) \circ \widetilde{\mu}(x) - \widetilde{\mu}(x \cdot y) = \widetilde{\mu}(y) \circ \widetilde{\rho}(x) - \widetilde{\rho}(x) \circ \widetilde{\mu}(y). \tag{5.4}$$

For all $x \in \mathfrak{g}, u \in V$, we have

$$\begin{aligned}0 &= \phi(x \diamond u) - \phi(x) \diamond u - x \diamond \phi(u)\\ &= \phi(\widetilde{\rho}(x)u) - (D(x) + \xi(x)) \diamond u - x \diamond K(u)\\ &= K(\widetilde{\rho}(x)u) - \widetilde{\rho}(D(x))u - \widetilde{\rho}(x)K(u),\end{aligned}$$

which implies that

$$K(\widetilde{\rho}(x)u) = \widetilde{\rho}(x)K(u) + \widetilde{\rho}(D(x))u. \tag{5.5}$$

Similarly, we have

$$K(\widetilde{\mu}(x)u) = \widetilde{\mu}(x)K(u) + \widetilde{\mu}(D(x))u. \tag{5.6}$$

Thus, by (5.3), (5.4), (5.5) and (5.6), we obtain that $(V, K, \widetilde{\rho}, \widetilde{\mu})$ is a representation.

Let s' be another section and $(V, K, \widetilde{\rho}', \widetilde{\mu}')$ be the corresponding representation of the regular pre-LieDer pair $((\mathfrak{g}, \cdot), D)$. Since $s(x) - s'(x) \in V$, then we have

$$\widetilde{\rho}(x)u - \widetilde{\rho}'(x)u = (s(x) - s'(x)) \cdot_{\widehat{\mathfrak{g}}} u = 0,$$

which implies that $\widetilde{\rho} = \widetilde{\rho}'$. Similarly, we have $\widetilde{\mu} = \widetilde{\mu}'$. Thus, this representation is independent of the choice of sections. ∎

Theorem 5.8. *With the above notation, (θ, ξ) is a 2-cocycle of the regular pre-LieDer pair $((\mathfrak{g}, \cdot), D)$ with coefficients in the representation $(V, K, \widetilde{\rho}, \widetilde{\mu})$.*

Proof. For all $x, y, z \in \mathfrak{g}$, by the definition of a pre-Lie algebra, we have

$$\begin{aligned}
0 &= (x \diamond y) \diamond z - x \diamond (y \diamond z) - (y \diamond x) \diamond z + y \diamond (x \diamond z) \\
&= (x \cdot y + \theta(x, y)) \diamond z - x \diamond (y \cdot z + \theta(y, z)) \\
&\quad - (y \cdot x + \theta(y, x)) \diamond z + y \diamond (x \cdot z + \theta(x, z)) \\
&= \theta(x \cdot y, z) + \widetilde{\mu}(z)\theta(x, y) - \theta(x, y \cdot z) - \widetilde{\rho}(x)\theta(y, z) \\
&\quad - \theta(y \cdot x, z) - \widetilde{\mu}(z)\theta(y, x) + \theta(y, x \cdot z) + \widetilde{\rho}(y)\theta(x, z) \\
&= -\mathrm{d}_{(\widetilde{\rho}, \widetilde{\mu})}\theta(x, y)
\end{aligned}$$

which implies that $\mathrm{d}_{(\widetilde{\rho}, \widetilde{\mu})}\theta = 0$. For all $x, y \in \mathfrak{g}, u, v \in V$, we have

$$\begin{aligned}
0 &= \phi((x + u) \diamond (y + v)) - \phi(x + u) \diamond (y + v) - (x + u) \diamond \phi(y + v) \\
&= \phi(x \cdot y + \theta(x, y) + \widetilde{\rho}(x)v + \widetilde{\mu}(y)u) - (D(x) + \xi(x) + K(u)) \diamond (y + v) \\
&\quad - (x + u) \diamond (D(y) + \xi(y) + K(v)) \\
&= K(\theta(x, y)) + \xi(x \cdot y) - \widetilde{\mu}(y)\xi(x) - \theta(D(x), y) - \widetilde{\rho}(x)\xi(y) - \theta(x, D(y)) \\
&= -(\mathrm{d}_{(\widetilde{\rho}, \widetilde{\mu})}\xi + \Omega_{(\widetilde{\rho}, \widetilde{\mu})}\theta)(x, y),
\end{aligned}$$

which implies that $\mathrm{d}_{(\widetilde{\rho}, \widetilde{\mu})}\xi + \Omega_{(\widetilde{\rho}, \widetilde{\mu})}\theta = 0$.

Thus, we obtain that $\mathcal{D}_{(\widetilde{\rho}, \widetilde{\mu})}(\theta, \xi) = 0$, which implies that (θ, ξ) is a 2-cocycle of the regular pre-LieDer pair $((\mathfrak{g}, \cdot), D)$ with coefficients in the representation $(V, K, \widetilde{\rho}, \widetilde{\mu})$. The proof is finished. ∎

Definition 5.9. Let $((\widehat{\mathfrak{g}}_1, \cdot_{\widehat{\mathfrak{g}}_1}), D_{\widehat{\mathfrak{g}}_1})$ and $((\widehat{\mathfrak{g}}_2, \cdot_{\widehat{\mathfrak{g}}_2}), D_{\widehat{\mathfrak{g}}_2})$ be two abelian extensions of a regular pre-LieDer pair $((\mathfrak{g}, \cdot), D)$ by $((V, \cdot_V), K)$. They are said to be *isomorphic* if there exists a regular pre-LieDer pair isomorphism $\zeta : ((\widehat{\mathfrak{g}}_1, \cdot_{\widehat{\mathfrak{g}}_1}), D_{\widehat{\mathfrak{g}}_1}) \to ((\widehat{\mathfrak{g}}_2, \cdot_{\widehat{\mathfrak{g}}_2}), D_{\widehat{\mathfrak{g}}_2})$ such that the following diagram is commutative.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V & \xrightarrow{\iota_1} & \widehat{\mathfrak{g}}_1 & \xrightarrow{p_1} & \mathfrak{g} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \zeta} & & \| & & \\
0 & \longrightarrow & V & \xrightarrow{\iota_2} & \widehat{\mathfrak{g}}_2 & \xrightarrow{p_2} & \mathfrak{g} & \longrightarrow & 0
\end{array}$$

Lemma 5.10. *Let* $((\hat{\mathfrak{g}}_1, \cdot_{\hat{\mathfrak{g}}_1}), D_{\hat{\mathfrak{g}}_1})$ *and* $((\hat{\mathfrak{g}}_2, \cdot_{\hat{\mathfrak{g}}_2}), D_{\hat{\mathfrak{g}}_2})$ *be two isomorphic abelian extensions of a regular pre-LieDer pair* $((\mathfrak{g}, \cdot), D)$ *by* $((V, \cdot_V), K)$. *Then they give rise to the same representation of* $((\mathfrak{g}, \cdot), D)$.

Proof. Let $s_1 : \mathfrak{g}_1 \to \hat{\mathfrak{g}}_1$ and $s_2 : \mathfrak{g}_2 \to \hat{\mathfrak{g}}_2$ be two sections of $((\hat{\mathfrak{g}}_1, \cdot_{\hat{\mathfrak{g}}_1}), D_{\hat{\mathfrak{g}}_1})$ and $((\hat{\mathfrak{g}}_2, \cdot_{\hat{\mathfrak{g}}_2}), D_{\hat{\mathfrak{g}}_2})$, respectively. By Theorem 5.7, we obtain that $(V, K, \widetilde{\rho}_1, \widetilde{\mu}_1)$ and $(V, K, \widetilde{\rho}_2, \widetilde{\mu}_2)$ are their representations, respectively. Define $s_1' : \mathfrak{g}_1 \to \hat{\mathfrak{g}}_1$ by $s_1' = \zeta^{-1} \circ s_2$. Since $\zeta : ((\hat{\mathfrak{g}}_1, \cdot_{\hat{\mathfrak{g}}_1}), D_{\hat{\mathfrak{g}}_1}) \to ((\hat{\mathfrak{g}}_2, \cdot_{\hat{\mathfrak{g}}_2}), D_{\hat{\mathfrak{g}}_2})$ is a pre-LieDer pair isomorphism satisfying the commutative diagram in Definition 5.9, by $p_2 \circ \zeta = p_1$, we have

$$p_1 \circ s_1' = p_2 \circ \zeta \circ \zeta^{-1} \circ s_2 = \mathrm{Id}_{\mathfrak{g}}.$$

Thus, we obtain that s_1' is a section of $((\hat{\mathfrak{g}}_1, \cdot_{\hat{\mathfrak{g}}_1}), D_{\hat{\mathfrak{g}}_1})$. For all $x \in \mathfrak{g}, u \in V$, we have

$$\widetilde{\rho}_1(x)(u) = s_1'(x) \cdot_{\hat{\mathfrak{g}}_1} u = (\zeta^{-1} \circ s_2)(x) \cdot_{\hat{\mathfrak{g}}_1} u = \zeta^{-1}(s_2(x) \cdot_{\hat{\mathfrak{g}}_2} u) = \widetilde{\rho}_2(x)(u),$$

which implies that $\widetilde{\rho}_1 = \widetilde{\rho}_2$. Similarly, we have $\widetilde{\mu}_1 = \widetilde{\mu}_2$. This finishes the proof. ∎

So in the sequel, we fixed a representation $(V, K, \widetilde{\rho}, \widetilde{\mu})$ of a regular pre-LieDer pair $((\mathfrak{g}, \cdot), D)$ and consider abelian extensions that induce the given representation.

Theorem 5.11. *Abelian extensions of a regular pre-LieDer pair* $((\mathfrak{g}, \cdot), D)$ *by* $((V, \cdot_V), K)$ *are classified by the second cohomology group* $H^2(\mathfrak{g}, D; V, K, \widetilde{\rho}, \widetilde{\mu})$.

Proof. Let $((\hat{\mathfrak{g}}, \cdot_{\hat{\mathfrak{g}}}), \widehat{D})$ be an abelian extension of a regular pre-LieDer pair $((\mathfrak{g}, \cdot), D)$ by $((V, \cdot_V), K)$. Choosing a section $s : \mathfrak{g} \to \hat{\mathfrak{g}}$, by Theorem 5.8, we obtain that $(\theta, \xi) \in Z^2(\mathfrak{g}, D; V, K, \widetilde{\rho}, \widetilde{\mu})$. Now we show that the cohomological class of (θ, ξ) does not depend on the choice of sections. In fact, let s and s' be two different sections. Define $\varphi : \mathfrak{g} \to V$ by $\varphi(x) = s(x) - s'(x)$. Then for all $x, y \in \mathfrak{g}$, we have

$$\begin{aligned}
\theta(x, y) &= s(x) \cdot_{\hat{\mathfrak{g}}} s(y) - s(x \cdot y) \\
&= (s'(x) + \varphi(x)) \cdot_{\hat{\mathfrak{g}}} (s'(y) + \varphi(y)) - s'(x \cdot y) - \varphi(x \cdot y) \\
&= s'(x) \cdot_{\hat{\mathfrak{g}}} s'(y) + \widetilde{\rho}(x)\varphi(y) + \widetilde{\mu}(y)\varphi(x) - s'(x \cdot y) - \varphi(x \cdot y) \\
&= \theta'(x, y) + \mathrm{d}_{(\widetilde{\rho}, \widetilde{\mu})}\varphi(x, y),
\end{aligned}$$

which implies that $\theta - \theta' = \mathrm{d}_{(\widetilde{\rho}, \widetilde{\mu})}\varphi$. Similarly, we have $\xi - \xi' = \Omega_{(\widetilde{\rho}, \widetilde{\mu})}\varphi$.

Therefore, we obtain that $(\theta - \theta', \xi - \xi') = (\mathrm{d}_{(\widetilde{\rho}, \widetilde{\mu})}\varphi, \Omega_{(\widetilde{\rho}, \widetilde{\mu})}\varphi) = \mathcal{D}_{(\widetilde{\rho}, \widetilde{\mu})}\varphi$, (θ, ξ) and (θ', ξ') are in the same cohomological class.

Now we go on to prove that isomorphic abelian extensions give rise to the same element in $H^2(\mathfrak{g}, D; V, K, \widetilde{\rho}, \widetilde{\mu})$. Assume that $((\hat{\mathfrak{g}}_1, \cdot_{\hat{\mathfrak{g}}_1}), D_{\hat{\mathfrak{g}}_1})$ and $((\hat{\mathfrak{g}}_2, \cdot_{\hat{\mathfrak{g}}_2}), D_{\hat{\mathfrak{g}}_2})$ are two isomorphic abelian extensions of a regular pre-LieDer pair $((\mathfrak{g}, \cdot), D)$ by $((V, \cdot_V), K)$, and $\zeta : ((\hat{\mathfrak{g}}_1, \cdot_{\hat{\mathfrak{g}}_1}), D_{\hat{\mathfrak{g}}_1}) \to ((\hat{\mathfrak{g}}_2, \cdot_{\hat{\mathfrak{g}}_2}), D_{\hat{\mathfrak{g}}_2})$ is a pre-LieDer pair isomorphism satisfying the commutative diagram in Definition 5.9. Assume that $s_1 : \mathfrak{g} \to \hat{\mathfrak{g}}_1$ is a section of $\hat{\mathfrak{g}}_1$. By $p_2 \circ \zeta = p_1$, we have

$$p_2 \circ (\zeta \circ s_1) = p_1 \circ s_1 = \mathrm{Id}_{\mathfrak{g}}.$$

Thus, we obtain that $\zeta \circ s_1$ is a section of $\widehat{\mathfrak{g}}_2$. Define $s_2 = \zeta \circ s_1$. Since ζ is an isomorphism of pre-LieDer pair and $\zeta \mid_V = \mathrm{Id}_V$, for all $x, y \in \mathfrak{g}$, we have

$$\begin{aligned}\theta_2(x, y) &= s_2(x) \cdot_{\widehat{\mathfrak{g}}_2} s_2(y) - s_2(x \cdot y) \\ &= (\zeta \circ s_1)(x) \cdot_{\widehat{\mathfrak{g}}_2} (\zeta \circ s_1)(y) - (\zeta \circ s_1)(x \cdot y) \\ &= \zeta(s_1(x) \cdot_{\widehat{\mathfrak{g}}_1} s_1(y) - s_1(x \cdot y)) \\ &= \theta_1(x, y).\end{aligned}$$

Similarly, we have $\xi_1 = \xi_2$. Thus, isomorphic abelian extensions give rise to the same element in $H^2(\mathfrak{g}, D; V, K, \widetilde{\rho}, \widetilde{\mu})$.

Given two 2-cocycles (θ_1, ξ_1) and (θ_2, ξ_2), we construct two abelian extensions $((\mathfrak{g} \oplus V, \diamond_1), \phi_1)$ and $((\mathfrak{g} \oplus V, \diamond_2), \phi_2)$. If $(\theta_1, \xi_1), (\theta_2, \xi_2) \in H^2(\mathfrak{g}, D; V, K, \widetilde{\rho}, \widetilde{\mu})$, then there exists $\varphi : \mathfrak{g} \to V$ such that $\theta_1 = \theta_2 + \mathrm{d}_{(\widetilde{\rho}, \widetilde{\mu})}\varphi$ and $\xi_1 = \xi_2 + \Omega_{(\widetilde{\rho}, \widetilde{\mu})}\varphi$. We define $\zeta : \mathfrak{g} \oplus V \to \mathfrak{g} \oplus V$ by

$$\zeta(x + u) = x + u + \varphi(x) \quad \forall x \in \mathfrak{g},\ u \in V.$$

For all $x, y \in \mathfrak{g}, u, v \in V$, by $\theta_1 = \theta_2 + \mathrm{d}_{(\widetilde{\rho}, \widetilde{\mu})}\varphi$, we have

$$\begin{aligned}&\zeta((x + u) \diamond_1 (y + v)) - \zeta(x + u) \diamond_2 \zeta(y + v) \\ &\quad = \zeta(x \cdot y + \theta_1(x, y) + \widetilde{\rho}(x)(v) + \widetilde{\mu}(y)(u)) \\ &\qquad - (x + u + \varphi(x)) \diamond_2 (y + v + \varphi(y)) \\ &\quad = \theta_1(x, y) + \varphi(x \cdot y) - \theta_2(x, y) - \widetilde{\rho}(x)\varphi(y) - \widetilde{\mu}(y)\varphi(x) \\ &\quad = \theta_1(x, y) - \theta_2(x, y) - \mathrm{d}_{(\widetilde{\rho}, \widetilde{\mu})}\varphi(x, y) = 0,\end{aligned}$$

and for all $x \in \mathfrak{g}, u \in V$, by $\xi_1 = \xi_2 + \Omega_{(\widetilde{\rho}, \widetilde{\mu})}\varphi$, we have

$$\begin{aligned}&\zeta \circ \phi_1(x + u) - \phi_2 \circ \zeta(x + u) \\ &\quad = \zeta(D(x) + \xi_1(x) + K(u)) - \phi_2(x + u + \varphi(x)) \\ &\quad = \xi_1(x) + \varphi(D(x)) - \xi_2(x) - K(\varphi(x)) \\ &\quad = \xi_1(x) - \xi_2(x) - \Omega_{(\widetilde{\rho}, \widetilde{\mu})}\varphi(x) = 0.\end{aligned}$$

Thus, ζ is a pre-LieDer pair isomorphism from $((\mathfrak{g} \oplus V, \diamond_1), \phi_1)$ to $((\mathfrak{g} \oplus V, \diamond_2), \phi_2)$. Moreover, it is obvious that the diagram in Definition 5.9 is commutative. This finishes the proof. ∎

A. The proof of Proposition 4.9

For all $x_1, \ldots, x_{n+1} \in \mathfrak{g}$, we have

$$\partial(f, f, f)_\rho(x_1, \ldots, x_{n+1})$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}(-1)^{i+1} f(x_1,\dots,\widehat{x_i},\dots,x_n,x_i)\cdot x_{n+1}\\
&\quad + \sum_{i=1}^{n}(-1)^{i+1} x_i \cdot f(x_1,\dots,\widehat{x_i},\dots,x_{n+1})\\
&\quad - \sum_{i=1}^{n}(-1)^{i+1} f(x_1,\dots,\widehat{x_i},\dots,x_n,x_i\cdot x_{n+1})\\
&\quad + \sum_{1\le i<j\le n}(-1)^{i+j} f([x_i,x_j]_C,x_1,\dots,\widehat{x_i},\dots,\widehat{x_j},\dots,x_{n+1})\\
&= \mathrm{d}_{\mathrm{reg}} f(x_1,\dots,x_{n+1})
\end{aligned}
$$

and

$$
\begin{aligned}
&\partial(f,f,f)_\mu(x_1,\dots,x_{n+1})\\
&\quad = (-1)^{n-1}(x_n\cdot f(x_1,\dots,x_{n-1},x_{n+1}) + f(x_1,\dots,x_n)\cdot x_{n+1}\\
&\qquad\qquad - f(x_1,\dots,x_{n-1},x_n\cdot x_{n+1}))\\
&\qquad - \sum_{i=1}^{n-1}(-1)^{i+1} f(x_1,\dots,\widehat{x_i},\dots,x_n,x_i\cdot x_{n+1})\\
&\qquad + \sum_{i=1}^{n-1}(-1)^{i+1} x_i\cdot f(x_1,\dots,\widehat{x_i},\dots,x_{n+1})\\
&\qquad - \sum_{i=1}^{n-1}(-1)^{i+1} f(x_1,\dots,\widehat{x_i},\dots,x_{n-1},x_i\cdot x_n,x_{n+1})\\
&\qquad + \sum_{i=1}^{n-1}(-1)^{i+1} f(x_1,\dots,\widehat{x_i},\dots,x_n,x_i)\cdot x_{n+1}\\
&\qquad + \sum_{i=1}^{n-1}(-1)^{i+1} f(x_1,\dots,\widehat{x_i},\dots,x_{n-1},x_n\cdot x_i,x_{n+1})\\
&\qquad + \sum_{1\le i<j\le n-1}(-1)^{i+j} f([x_i,x_j]_C,x_1,\dots,\widehat{x_i},\dots,\widehat{x_j},\dots,x_{n+1})\\
&\quad = \mathrm{d}_{\mathrm{reg}} f(x_1,\dots,x_{n+1}).
\end{aligned}
$$

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